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DOI:
10.1016/j.jnt.2020.04.020

Document Version
Accepted author manuscript

ON PERFECT POWERS THAT ARE SUMS OF CUBES OF A SEVEN TERM ARITHMETIC PROGRESSION

ALEJANDRO ARGÁEZ-GARCÍA AND VANDITA PATEL

Abstract. We prove that the equation $(x-3r)^3+(x-2r)^3+(x-r)^3+x^3+(x+r)^3+(x+2r)^3+(x+3r)^3=y^p$ only has solutions which satisfy $xy=0$ for $1 \le r \le 10^6$ and $p \ge 5$ prime. This article complements the work on the equations $(x-r)^3+x^3+(x+r)^3=y^p$ in [2] and $(x-2r)^3+(x-r)^3+x^3+(x+r)^3+(x+2r)^3=y^p$ in [1]. The methodology in this paper makes use of the Primitive Divisor Theorem due to Bilu, Hanrot and Voutier for a complete resolution of the Diophantine equation.

1. Introduction

Finding perfect powers that are sums of terms in an arithmetic progression has received much interest lately; recent contributions can be found in [1], [2], [3], [4], [5], [6], [9], [10], [14], [15], [17], [18], [19], [21],[23], [24], [26], [27]. This paper aims to demonstrate an application of Theorem 1 in [16] to the problem of finding perfect powers in sums of like powers, thus adding to the diverse range of methodologies currently being used to tackle such problems.

In this paper, we prove the following:

Theorem 1.1. *Let $p \ge 5$ be a prime. The equation*

$$(x-3r)^3+(x-2r)^3+(x-r)^3+x^3+(x+r)^3+(x+2r)^3+(x+3r)^3=y^p \tag{1}$$

with $x, r, y, p \in \mathbb{Z}$, $\gcd(x,r)=1$ and $0<r\le 10^6$ only has solutions which satisfy $xy=0$.

The restriction $\gcd(x,r)=1$ is natural one, for otherwise it is easy to construct artificial solutions by scaling.

This paper follows ideas presented in [14], [5], [2] and [1], to find perfect powers that are sums of terms in an arithmetic progression, whereby the main techniques used to resolve such equations include a result of Mignotte based on linear form in logarithms [12], the method of Chabauty ([20, 22, 11]), the theorem due to Bilu, Hanrot and Voutier on primitive divisors ([7]), as well as various elementary techniques. The novel approach in this paper is that we are able to apply a computationally efficient technique using Lehmer pairs and sequences due to recent work by the second author [16], thus leading to a full resolution of (1) for $1 \le r \le 10^6$ and $p \ge 5$ a prime. We note here that the approaches taken in [2, 1] are not sufficient to resolve these cases, as the Thue equations have large coefficients and/or

Date: April 1, 2020.

2010 *Mathematics Subject Classification.* Primary 11D61, Secondary 11D41, 11D59, 11J86.

Key words and phrases. Exponential equation, Chabauty, Thue equations, Lehmer sequences, primitive divisors.

large degree. Similarly, the approach taken in [16] is insufficient by itself. A full resolution of this problem requires a combination of techniques from [2, 1] and [16].

Acknowledgement. We would like to thank the referee for a careful reading of the paper and for suggesting several improvements.

2. Background

Here, we record some essential theorems and lemmas which are necessary to carry out the computations in section 4 in order to prove Theorem 1.1. We emphasise that there is nothing new in this section, and we briefly outline key results and techniques here. Full proofs can be found in [14, 5, 2].

We first apply a descent to equation (1) in Section 3. We are left with equations of the form:

$$aw_2^p - bw_1^{2p} = cr^2 \tag{2}$$

where p is an odd prime and a, b, c are positive integers satisfying $\gcd(a, b, c) = 1$.

Now, we state a theorem due to Mignotte [12], which is essential in providing an upper bound for our prime exponent p, thus enabling us to perform a finite computation.

Theorem 2.1. *(Mignotte) Assume that the exponential Diophantine inequality*

$$\mid ax^n - by^n \mid \leq c, \qquad \text{with } a, b, c \in \mathbb{Z}_{\geq 0} \text{ and } a \neq b$$

has a solution in strictly positive integers x and y with $\max\{x, y\} > 1$. *Let* $A = \max\{a, b, 3\}$. *Then*

$$n \leq \max\left\{3\log(1.5 \mid c/b \mid), \frac{7400 \log A}{\log\left(1 + \log A / \mid \log(a/b) \mid\right)}\right\}.$$

2.1. Criteria for eliminating equations of signature $(p, 2p, 2)$.

Steps to rule out (2) having a nontrivial solution for a fixed prime $p \geq 5$ was previously presented in [14, 5, 2]. We start with the following simple yet effective lemma, inspired from the work of Sophie Germain (see [14] for more history and context around the work of Sophie Germain on Fermat's Last Theorem) that provides a computationally efficient criteria to check in order to deduce the nonexistence of solutions. In the majority of cases, this lemma is highly successful in deducing the nonexistence of solutions.

Lemma 2.2. *Let $p \geq 3$ be a prime. Let a, b and c be positive integers such that* $\gcd(a, b, c) = 1$. *Let $q = 2kp + 1$ be a prime that does not divide a. Define*

$$\mu(p, q) = \{\eta^{2p} \ : \ \eta \in \mathbb{F}_q\} = \{0\} \cup \{\zeta \in \mathbb{F}_q^* \ : \ \zeta^k = 1\} \tag{3}$$

and

$$B(p, q) = \left\{\zeta \in \mu(p, q) \ : \ ((b\zeta + c)/a)^{2k} \in \{0, 1\}\right\}.$$

If $B(p, q) = \emptyset$, then equation (2) does not have integral solutions.

2.2. **Local Solubility.** In this section, we outline a classical local solubility method, which when applied, can conclude the nonexistence of solutions for a particular tuple (a, b, c, p) in equation (2).

Recall the condition $\gcd(a, b, c) = 1$ in (2). Let $g = \mathrm{Rad}(\gcd(a, c))$ and suppose that $g > 1$. Then $g \mid w_1$, and we can write $w_1 = g w_1'$. Thus

$$a w_2^p - b g^{2p} {w_1'}^{2p} = c.$$

Removing a factor of g from the coefficients, we obtain

$$a' {w_2}^p - b' {w_1'}^{2p} = c',$$

where $a' = a/g$ and $c' = c/g < c$. Similarly, if $h = \gcd(b, c) > 1$, we obtain

$$a' {w_2'}^p - b' {w_1}^{2p} = c',$$

where $c' = c/h < c$. Applying these operations repeatedly, we arrive at an equation of the form

$$A\rho^p - B\sigma^{2p} = C \tag{4}$$

where A, B, C are now pairwise coprime. A necessary condition for the existence of solutions is that for any odd prime $q \mid A$, the residue $-BC$ modulo q is a square. Besides this basic test, we also check for local solubility at the primes dividing A, B, C, and all primes $q \leq 19$.

2.3. **Descent.** If local techniques previously presented fail to rule out solutions to equation (2) for particular coefficients and exponent (a, b, c, p) then we may perform a further descent to rule out solutions. With A, B, C as in (4) we let

$$B' = \prod_{\mathrm{ord}_q(B) \text{ is odd}} q.$$

Thus $BB' = v^2$. Write $AB' = u$ and $CB' = mn^2$ with m squarefree. Rewrite (4) as

$$(v\sigma^p + n\sqrt{-m})(v\sigma^p - n\sqrt{-m}) = u\rho^p.$$

Let $K = \mathbb{Q}(\sqrt{-m})$ and $\mathcal{O}$ be its ring of integers. Let $\mathfrak{S}$ contain the prime ideals of $\mathcal{O}$ that divide u or $2n\sqrt{-m}$. Clearly $(v\sigma^p + n\sqrt{-m})K^{*p}$ belongs to the "p-Selmer group"

$$K(\mathfrak{S}, p) = \{\epsilon \in K^*/K^{*p} \ : \ \mathrm{ord}_{\mathcal{P}}(\epsilon) \equiv 0 \mod p \text{ for all } \mathcal{P} \notin \mathfrak{S}\}.$$

This is an $\mathbb{F}_p$-vector space of finite dimension which can be computed by `Magma` using the command `pSelmerGroup`. Let

$$\mathcal{E} = \{\epsilon \in K(\mathfrak{S}, p) \ : \ \mathrm{Norm}(\epsilon)/u \in \mathbb{Q}^{*p}\}.$$

It follows that

$$v\sigma^p + n\sqrt{-m} = \epsilon\eta^p, \tag{5}$$

where $\eta \in K^*$ and $\epsilon \in \mathcal{E}$.

Lemma 2.3. *Let $\mathfrak{q}$ be a prime ideal of K. Suppose one of the following holds:*

(i) $\mathrm{ord}_{\mathfrak{q}}(v)$, $\mathrm{ord}_{\mathfrak{q}}(n\sqrt{-m})$, $\mathrm{ord}_{\mathfrak{q}}(\epsilon)$ *are pairwise distinct modulo p;*
(ii) $\mathrm{ord}_{\mathfrak{q}}(2v)$, $\mathrm{ord}_{\mathfrak{q}}(\epsilon)$, $\mathrm{ord}_{\mathfrak{q}}(\overline{\epsilon})$ *are pairwise distinct modulo p;*
(iii) $\mathrm{ord}_{\mathfrak{q}}(2n\sqrt{-m})$, $\mathrm{ord}_{\mathfrak{q}}(\epsilon)$, $\mathrm{ord}_{\mathfrak{q}}(\overline{\epsilon})$ *are pairwise distinct modulo p.*

Then there is no $\sigma \in \mathbb{Z}$ and $\eta \in K$ satisfying (5).

Lemma 2.4. *Let $q = 2kp + 1$ be a prime. Suppose $q\mathcal{O} = \mathfrak{q}_1\mathfrak{q}_2$ where $\mathfrak{q}_1$, $\mathfrak{q}_2$ are distinct, and such that $\mathrm{ord}_{\mathfrak{q}_j}(\epsilon) = 0$ for $j = 1, 2$. Let*

$$\chi(p, q) = \{\eta^p \ : \ \eta \in \mathbb{F}_q\}.$$

Let

$$C(p, q) = \{\zeta \in \chi(p, q) \ : \ ((v\zeta + n\sqrt{-m})/\epsilon)^{2k} \equiv 0 \text{ or } 1 \mod \mathfrak{q}_j \text{ for } j = 1, 2\}.$$

Suppose $C(p, q) = \emptyset$. Then there is no $\sigma \in \mathbb{Z}$ and $\eta \in K$ satisfying (5).

2.4. **Thue equations.** For the handful of remaining equations where we are unable to deduce nonexistence of solutions, we let $\sigma = w_2$ and $\tau = w_1^2$ in (2) to get:

$$a\sigma^p - b\tau^p = cr^2$$

For a fixed value of r, we note that this is a *Thue equation*. We use `Magma's` Thue solver [8] and `PARI/GP's` *thueinit*, *thue* commands [13, 25], as the final test to determine whether the equation has solutions. It is important to remark that when we specifically used *thueinit* and *thue* the result may depend on *GRH*.

3. The initial descent and twelve cases

We rewrite equation (1) as $7x(x^2 + 12r^2) = y^p$. Since $7 \mid y$, we let $y = 7w$ to obtain:

$$x(x^2 + 12r^2) = 7^{p-1}w^p.$$

We note that $\gcd(x, x^2 + 12r^2) \in \{1, 2, 3, 4, 6, 12\}$ depending on whether $2, 3, 4, 6$ and 12 divides x or not. This leads us to consider twelve cases. We apply a simple descent argument in each case and the results are summarised in the following table.

Case	Conditions on x	Descent equations	Ternary Equation
1	$7 \nmid x$ and $12 \nmid x$	$x = w_1^p$ $x^2 + 12r^2 = 7^{p-1}w_2^p$	$7^{p-1}w_2^p - w_1^{2p} = 12r^2$
2	$7 \nmid x$ and $2 \nmid x$, $3 \mid x$	$x = 3^{p-1}w_1^p$ $x^2 + 12r^2 = 3 \cdot 7^{p-1}w_2^p$	$7^{p-1}w_2^p - 3^{2p-3}w_1^{2p} = 4r^2$
3	$7 \nmid x$ and $4 \mid x$, $3 \nmid x$	$x = 2^{p-2}w_1^p$ $x^2 + 12r^2 = 4 \cdot 7^{p-1}w_2^p$	$7^{p-1}w_2^p - 2^{2p-6}w_1^{2p} = 3r^2$
4	$7 \nmid x$ and $12 \mid x$	$x = 2^{p-2} \cdot 3^{p-1}w_1^p$ $x^2 + 12r^2 = 12 \cdot 7^{p-1}w_2^p$	$7^{p-1}w_2^p - 2^{2p-6} \cdot 3^{2p-3}w_1^{2p} = r^2$
5	$7 \nmid x$ and $2 \mid x$, $3, 4 \nmid x$	$x = 2^{p-1}w_1^p$ $x^2 + 12r^2 = 2 \cdot 7^{p-1}w_2^p$	$7^{p-1}w_2^p - 2^{2p-3}w_1^{2p} = 6r^2$
6	$7 \nmid x$ and $6 \mid x$, $4 \nmid x$	$x = 6^{p-1}w_1^p$ $x^2 + 12r^2 = 6 \cdot 7^{p-1}w_2^p$	$7^{p-1}w_2^p - 6^{2p-3}w_1^{2p} = 2r^2$
7	$7 \mid x$ and $12 \nmid x$	$x = 7^{p-1}w_1^p$ $x^2 + 12r^2 = w_2^p$	$w_2^p - 7^{2p-2}w_1^{2p} = 12r^2$
8	$7 \mid x$ and $2 \nmid x$, $3 \mid x$	$x = 3^{p-1} \cdot 7^{p-1}w_1^p$ $x^2 + 12r^2 = 3w_2^p$	$w_2^p - 3^{2p-3} \cdot 7^{2p-2}w_1^{2p} = 4r^2$
9	$7 \mid x$ and $4 \mid x$, $3 \nmid x$	$x = 2^{p-2} \cdot 7^{p-1}w_1^p$ $x^2 + 12r^2 = 4w_2^p$	$w_2^p - 2^{2p-6} \cdot 7^{2p-2}w_1^{2p} = 3r^2$
10	$7 \mid x$ and $12 \mid x$	$x = 2^{p-2} \cdot 3^{p-1} \cdot 7^{p-1}w_1^p$ $x^2 + 12r^2 = 12w_2^p$	$w_2^p - 2^{2p-6} \cdot 3^{2p-3} \cdot 7^{2p-2}w_1^{2p} = r^2$
11	$7 \mid x$ and $2 \mid x$, $3, 4 \nmid x$	$x = 2^{p-1} \cdot 7^{p-1}w_1^p$ $x^2 + 12r^2 = 2w_2^p$	$w_2^p - 2^{2p-3} \cdot 7^{2p-2}w_1^{2p} = 6r^2$
12	$7 \mid x$ and $6 \mid x$, $4 \nmid x$	$x = 6^{p-1} \cdot 7^{p-1}w_1^p$ $x^2 + 12r^2 = 6w_2^p$	$w_2^p - 6^{2p-3} \cdot 7^{2p-2}w_1^{2p} = 2r^2$

We note that cases 5, 6, 11 and 12 all lead to an immediate contradiction; by checking the valuation of 2 on both sides of the ternary equation and noting the fact that $p \geq 5$, we contradict our assumption of $\gcd(x, r) = 1$.

4. Solving cases 1–4

In this section, we apply the Theorems and Lemmas of Section 2 which fully resolve the descent cases 1–4 of equation (1) in order to prove Theorem 1.1.

4.1. **Case 1.** For $p = 5$ we have $7^4 w_2^5 - w_1^{10} = 3(2r)^2$. Letting $X = w_2/w_1^2$ and $Y = 6r/w_1^5$, we obtain the hyperelliptic curve

$$Y^2 = 3 \cdot 7^4 X^5 - 3$$

whose Jacobian has rank 1. The Chabauty implementation give us $C(\mathbb{Q}) = \{\infty\}$.

By Theorem 2.1, for $|r| \leq 4.9 \times 10^{1502}$, we can bound $p \leq 20775$. Thus, for $7 \leq p \leq 20775$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	37679	4077	3	0
11	9930	5375	0	0
13	3298	1405	0	0
17	461	253	0	0
19	1507	936	0	0
23	13	3	0	0
29	8	5	0	0
31	29	21	0	0
$37 \leq p \leq 20775$	0	0	0	0

4.2. **Case 2.** For $p = 5$ we have $7^4 w_2^5 - 3^7 w_1^{10} = (2r)^2$. Letting $X = w_2/w_1^2$ and $Y = 2r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 7^4 X^5 - 3^7$$

whose Jacobian has rank 2, and we are unable to use Chabauty techniques.

By Theorem 2.1, for $|r| \leq 1.9 \times 10^{1427}$, we can bound $p \leq 19734$. Thus, for $5 \leq p \leq 19734$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
5	102681	38771	819	819
7	24526	2400	0	0
11	9159	4629	0	0
13	3439	1804	0	0
17	80	51	0	0
19	3136	1012	0	0
23	11	3	0	0
29	1	0	0	0
31	4	2	0	0
$37 \leq p \leq 19734$	0	0	0	0

Using `PARI/GP`, we resolved the 819 equations using the commands *thueinit* and *thue*. No integer solutions were found.

4.3. **Case 3.** For $p = 5$ we have $7^6 w_2^5 - 2^4 w_1^{10} = 3r^2$. Letting $X = w_2/w_1^2$ and $Y = 3r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 3 \cdot 7^6 X^5 - 2^4 \cdot 3$$

whose Jacobian has rank 0. Applying Chabauty gives $C(\mathbb{Q}) = \{\infty\}$.

By Theorem 2.1, for $|r| \leq 1.5 \times 10^{2105}$, we can bound $p \leq 29101$. Thus, for $7 \leq p \leq 29101$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	29213	2969	0	0
11	6484	2332	0	0
13	1715	786	0	0
17	369	206	0	0
19	538	262	0	0
23	1	0	0	0
29	2	1	0	0
31	5	4	0	0
$37 \leq p \leq 29101$	0	0	0	0

4.4. **Case 4.** For $p = 5$ we have $7^6 w_2^5 - 2^4 \cdot 3^7 w_1^{10} = r^2$. Letting $X = w_2/w_1^2$ and $Y = r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 7^6 X^5 - 2^4 \cdot 3^7$$

whose Jacobian has rank 1. The Chabauty implementation gives $C(\mathbb{Q}) = \{\infty\}$.

By Theorem 2.1, for $|r| \leq 1.37 \times 10^{4664}$, we can bound $p \leq 64461$. Hence, for $7 \leq p \leq 64461$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	18908	1940	0	0
11	1384	434	0	0
13	479	177	0	0
17	366	173	0	0
19	365	184	0	0
23	5	1	0	0
29	3	3	0	0
31	14	9	0	0
37	1	0	0	0
$41 \leq p \leq 64461$	0	0	0	0

5. Solving cases 7–10

For cases 7–10, we rely on techniques developed in [16]. Using theorems and lemmas of Section 2 lead to resolving Thue equations with extremely large coefficients and/or high degree. In order to make computations tractable, we now outline an alternative approach to deal with cases 7–10.

5.1. **Primitive prime divisors of Lehmer sequences.**

Theorem 5.1. *Let $C_1 \geq 1$ be a squarefree integer and C_2 a positive integer. Write $C_1C_2 = cd^2$ where c is squarefree. We assume that $C_1C_2 \not\equiv 7 \pmod 8$. Let p be a prime for which*

$$C_1x^2 + C_2 = y^p, \quad x,\ y \in \mathbb{Z}^+, \quad \gcd(C_1x^2, C_2, y^p) = 1, \tag{6}$$

has a solution (x, y). Then either,

(i) $p \leq 13$, or
(ii) p divides the class number of $\mathbb{Q}(\sqrt{-c})$, or
(iii) $p \mid \left(q - \left(\frac{-c}{q}\right)\right)$, where q is some prime $q \mid d$ and $q \nmid 2c$.

We follow **Case I** in Section 6 of [16] which outlines effective methods to solve equation (6) for fixed values of C_1, C_2, p. Theorem 5.1 is crucial to gain good bounds for p. For the convenience of the reader, we shall reproduce the key computational steps used to prove Theorem 1.1 here. For the full exposition and intricate details of computations, see [16].

Let C_1, C_2 be positive integers, with C_1 squarefree. Let $\gcd(C_1, C_2) = 1$ and suppose that $C_1C_2 \not\equiv 7 \pmod 8$. We write $C_1C_2 = cd^2$ where c, d are positive integers and c is squarefree. Theorem 5.1 gives a list of possible odd prime exponents p for which (6) might have solutions.

Let $K = \mathbb{Q}(\sqrt{-3})$ be a number field with $\mathcal{O}_K$ it's ring of integers. We note here that $c = 3$ in the notation of Theorem 5.1. Furthermore, we note that the class number of K is 1. Let $p \geq 5$ be a prime. Let $\gamma \in \mathcal{O}_K$ such that $\gamma = a + b\sqrt{-3}$ for some integers a, b, and let $\bar{\gamma}$ be it's conjugate.

Fix a value of b dividing d. To determine the solutions we merely have to determine the possible values of a corresponding to each $b \mid d$. We write an explicit polynomial $g_b \in \mathbb{Z}[X]$ whose integer roots contain all the possible values of a corresponding to b.

Fix $s \mid d$. Since $-3 \not\equiv 1 \pmod 4$, we let

$$g_b(X) = \frac{(X + b\sqrt{-3})^p - (X - b\sqrt{-3})^p}{2b\sqrt{-3}} - \frac{d \cdot C_1^{(p-1)/2}}{b}. \tag{7}$$

Clearly $g_b \in \mathbb{Z}[X]$. Moreover,

$$g_b(a) = \frac{\gamma^p - \overline{\gamma}^p}{\gamma - \overline{\gamma}} - \frac{d \cdot C_1^{(p-1)/2}}{b} = 0.$$

Finding the integer roots $g_b(X)$ gives associated values of a to the specified b, hence γ is known. Since $y = \gamma\bar{\gamma}/C_1$, hence $y = (a^2 + 3b^2)/C_1$, the possible values for y are now easily known. Using equation (6), we finally determine possible values of x, hence all possible solutions to equation (6).

5.2. **Case 7.** One of the descent equations is:

$$x^2 + 12r^2 = w_2^p \tag{8}$$

We let $C_1 = 1$, $C_2 = 12r^2$ and we see that $C_1C_2 \equiv 0, 3, 4 \pmod 8 \not\equiv 7 \pmod 8$. Thus if (8) has a solution, then we have $5 \leq p \leq 13$ or $p \mid \left(q - \left(\frac{-3}{q}\right)\right)$, where q is some prime $q \mid r$ and $q \nmid 6$. For fixed values of $1 \leq r \leq 10^6$ (hence fixed C_1, C_2, p), finding the roots of the polynomial (7), leads to solutions (x, w_2, p). Thus to obtain solutions (x, y, p) to the original equation, we simply note that $y = 7w_1w_2$ where w_1 is easily deduced from the descent equation $x = 7^{p-1}w_1^p$ in Section 3. The computation yielded no such solutions.

5.3. **Case 8.** One of the descent equations is:

$$x^2 + 12r^2 = 3w_2^p.$$

We let $x = 3X$ and obtain

$$3X^2 + 4r^2 = w_2^p. \tag{9}$$

We let $C_1 = 3$, $C_2 = 4r^2$ and we see that $C_1C_2 \equiv 0, 3, 4 \pmod 8 \not\equiv 7 \pmod 8$. Thus if (9) has a solution, then we have $5 \leq p \leq 13$ or $p \mid \left(q - \left(\frac{-3}{q}\right)\right)$, where q is some prime $q \mid r$ and $q \nmid 6$. For fixed values of $1 \leq r \leq 10^6$ (hence fixed C_1, C_2, p), finding the roots of the polynomial (7), leads to solutions (X, w_2, p). Thus to obtain solutions (x, y, p) to the original equation, we simply note that $x = 3X$ and $y = 7w_1w_2$ where w_1 is easily deduced from the descent equation $x = 3^{p-1}7^{p-1}w_1^p$ in Section 3. The computation yielded no such solutions.

5.4. **Case 9.** One of the descent equations is:

$$x^2 + 12r^2 = 4w_2^p.$$

We let $x = 2X$ and obtain

$$X^2 + 3r^2 = w_2^p. \tag{10}$$

We let $C_1 = 1$, $C_2 = 3r^2$ and we see that $C_1C_2 \equiv 0, 3, 4 \pmod 8 \not\equiv 7 \pmod 8$. Thus if (10) has a solution, then we have $5 \leq p \leq 13$ or $p \mid \left(q - \left(\frac{-3}{q}\right)\right)$, where q is some prime $q \mid r$ and $q \nmid 6$. For fixed values of $1 \leq r \leq 10^6$ (hence fixed C_1, C_2, p), finding the roots of the polynomial (7), leads to solutions (X, w_2, p). Thus to obtain solutions (x, y, p) to the original equation, we simply note that $x = 2X$ and $y = 7w_1w_2$ where w_1 is easily deduced from the descent equation $x = 2^{p-2}7^{p-1}w_1^p$ in Section 3. The computation yielded no such solutions.

5.5. **Case 10.** One of the descent equations is:

$$x^2 + 12r^2 = 12w_2^p.$$

We let $x = 6X$ and obtain

$$3X^2 + r^2 = w_2^p. \tag{11}$$

We let $C_1 = 3$, $C_2 = r^2$ and we see that $C_1C_2 \equiv 0, 3, 4 \pmod 8 \not\equiv 7 \pmod 8$. Thus if (11) has a solution, then we have $5 \leq p \leq 13$ or $p \mid \left(q - \left(\frac{-3}{q}\right)\right)$, where q is some prime $q \mid r$ and $q \nmid 6$. For fixed values of $1 \leq r \leq 10^6$ (hence fixed C_1, C_2, p), finding the roots of the polynomial (7), leads to solutions (X, w_2, p). Thus to obtain solutions (x, y, p) to the original equation, we simply note that $x = 6X$ and $y = 7w_1w_2$ where w_1 is easily deduced from the descent equation $x = 2^{p-2}3^{p-1}7^{p-1}w_1^p$ in Section 3. The computation yielded no such solutions.

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Facultad de Ingeniería Química, Universidad Autónoma de Yucatán. Periférico Norte Kilómetro 33.5, Tablaje Catastral 13615 Chuburna de Hidalgo Inn, Mérida, Yucatán, México. C.P. 97200 ; Escuela Nacional de Estudios Superiores - Mérida, UNAM, Calle 7-B No. 227 por 20 y 22-A Colonia Juan B. Sosa Mérida, Yucatán, México C.P. 97205

Email address: `alejandro.argaez@correo.uady.mx`

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom

Email address: `vandita.patel@manchester.ac.uk`