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DOI:
10.1016/j.automatica.2021.109570

Document Version
Accepted author manuscript

Link to publication record in Manchester Research Explorer

Citation for published version (APA):

Published in:
Automatica

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Applying negative imaginary systems theory to non-square systems with polytopic uncertainty

Parijat Bhowmick and Alexander Lanzon

Abstract

This paper aims to apply Negative Imaginary (NI) systems theory to non-square LTI systems, for example, flexible structure systems with redundant sensors and actuators. The paper proposes static pre- and post-compensation schemes to transform stable fat (i.e. no. of inputs more than the no. of outputs) and tall (i.e. no. of outputs more than the no. of inputs) LTI plants into the class of Strongly Strict Negative Imaginary (SSNI) systems, a subset of Strictly Negative Imaginary (SNI) systems. The proposed compensators can also stabilise a non-square or non-NI square plant in a positive feedback loop. Moreover, the compensators can be utilised to design a simple constant reference tracking scheme exploiting integral controllability of SSNI systems. Subsequently, observer-based pre- and post-compensators are developed to handle the cases when some of the system states are not available for direct measurement. Then, the compensation schemes are extended to stable, non-square or non-NI square systems with polytopic uncertainty. Finally, the post-compensation technique is applied to stabilise a class of tall/square uncertain LTI plants preceded by a slope-restricted nonlinearity.

Key words: Negative imaginary systems; DC-gain; Non-square plants; Polytopic uncertainty; LMIs; Observer-based control; Decentralised integral controllability.

1 Introduction

Negative Imaginary (NI) systems theory has attracted interest of control practitioners and control researchers over the past twelve years due to its simple internal stability condition for interconnected systems that depends only on the DC loop gain [22], and its wide applicability in solving variety of real-world engineering problems. NI systems property and the associated internal stability results were introduced in [22] for robust control of flexible structures with colocated position sensors and force actuators being motivated by the ‘positive position feedback control’ of similar systems [12]. Internal stability results of interconnected NI systems find potential applications in control of cantilever beams [1], flexible structures with free body motion (e.g. flexible spacecrafts, flexible robotic arms [30]), large vehicle platoons [6], nano-positioning systems (e.g. in atomic force microscope [31]), hard disk drives [25], etc. An NI system (say \( R(s) \)) is a square, Lyapunov-stable system that satisfies the point-wise frequency-domain condition \( j[R(j\omega) - R(j\omega)^*] \geq 0 \) for all \( \omega \in (0, \infty) \) such that \( j\omega \) is not a pole of \( R(s) \). A strict subclass of NI systems is known as Strictly NI (SNI) systems that satisfy a strict frequency-domain condition \( j[R(j\omega) - R(j\omega)^*] \geq 0 \) for all \( \omega \in (0, \infty) \). Another strict subset of SNI systems, called Strongly Strict NI (SSNI) systems, was introduced in [23] which satisfies two additional frequency-domain constraints at zero and infinite frequencies apart from the SNI conditions. The NI property is defined for systems with improper and non-rational transfer functions in [15, 13] and NI theory has also been recently extended to discrete-time LTI systems [14, 28, 29].

Although NI-research has witnessed continued progress in both theory and applications over the past thirteen years since its inception, a major setback of NI theory is that it remains inapplicable to non-square systems and systems having relative degree more than two. Due to this concern, all NI/SNI synthesis techniques available so far in the literature operate only on square systems. For instance, the articles [37, 26, 25, 11, 20, 33, 34, 9, 2, 3, 4] have pursued significant research in designing static and
dynamic output feedback controller synthesis framework for square LTI systems employing NI/SNI property in closed-loop but the synthesis results are meant for only square plants. This motivates us to think about a realistic and numerically tractable scheme by which NI results can still be used for non-square systems. With this objective in mind, in this paper, we introduce static pre- and post-compensation schemes to render stable non-square or (non-NI) square LTI systems into the SNI class. The post-compensator does judicious blending of the sensor signals, termed as ‘sensor-blending’, to make the number of inputs \((u)\) and outputs \((z)\) of the post-compensated system equal, as depicted in Fig. 1(a). Whereas the pre-compensator is required for systems having more inputs than outputs to distribute (or allocate) the control inputs, known as ‘control-allocation’, to make the pre-compensated system square with respect to the new input \(\bar{u}\) and the output \(y\) as shown in Fig. 1(b).

The contributions of the paper are summarized below:

- LMI-based static pre- and post-compensator design methodologies are proposed in this paper to transform stable non-square or (non-NI) square LTI plants into SNI class and to stabilise them in a unity positive feedback loop;
- The proposed compensators facilitate a fault-tolerant constant reference tracking scheme for stable non-square or (non-NI) square LTI plants exploiting integral controllability (IC) and decentralised integral controllability (DIC) properties of SSNI systems;
- A full-order observer-based pre- and post-compensation scheme is also designed to avoid the dependence on specific structure of the post-compensator (\(H\)), the scheme guarantees feasible solution for any stable, tall/square, LTI plant with a minimal state-space realization and a full-rank \(B\) matrix.

![Fig. 1. (a) Post-compensator required for tall/square plants \((r \leq l)\); (b) Pre-compensator required for fat plants \((r > l)\).](image)

**Notation**

The notation is standard throughout. \(\mathbb{R}\) and \(\mathbb{C}\) denote the sets of all real and complex numbers respectively. \(\mathbb{R}^{m \times n}\) and \(\mathbb{C}^{m \times n}\) denote the sets of real and complex matrices respectively of dimensions \((m \times u)\). \(A^\top\), \(A^*\) and \(\bar{A}\) denote respectively the transpose, complex conjugate transpose and complex conjugate of a matrix \(A\). \(A^{-\top}\) and \(A^{-\top}\) represent shorthand for \((A^{-1})^\top\) and \((A^{-1})^\top\) respectively. \((\cdot)^\top\) denotes the vector transpose. \(\mathcal{R}\mathcal{H}_{\infty}^{m \times n}\) denotes the set of all real, rational, proper and asymptotically stable transfer function matrices having dimension \((m \times n)\). We denote \(M(j\omega) = M(-j\omega)^\top\) and \(M^- (s) = M(s)^\top\). \((A, B, C, D)\) represents a state-space realization of a system given by \(M(s) = C(sI - A)^{-1}B + D\). \(\text{Co}\{\cdot\}\) denotes the convex hull.

**2 Technical preliminaries**

We start by defining the classes of NI and SNI systems followed by a lemma to describe SSNI systems. After that we recall the internal stability result for a positive feedback interconnection of stable NI and SNI systems.

**Definition 1** (NI System) \([30, 21]\) Let \(M(s)\) be the real, rational and proper transfer function matrix of a finite-dimensional, square, LTI system. Then, \(M(s)\) is said to be NI if

1. \(M(s)\) has no poles in \(\{s \in \mathbb{C} : \Re[s] > 0\}\);
2. \(jM(j\omega) - M(j\omega)^\top \geq 0\) for all \(\omega \in (0, \infty)\) except the values of \(\omega\) where \(s = j\omega\) is a pole of \(M(s)\);
3. \(\text{If } s = j\omega_0\) with \(\omega_0 \in (0, \infty)\) is a pole of \(M(s)\), then it is at most a simple pole and the residue matrix \(\lim_{s \to j\omega_0} (s - j\omega_0)jM(s)\) is Hermitian and positive semidefinite.

In the literature, there are extensions of the NI definition to allow poles at the origin \([30, 21]\) and to improper, non-rational systems \([15, 13, 27]\). However in this paper, we may find potential applications in vibration control of such complex systems. Another aspect of this work is that it can be used as a Static Output Feedback Control (SOFC) technique for non-square systems. Most of the SOFC design problems involve Bilinear Matrix Inequalities (BMI) and hence require rigorous mathematical manipulations to convert the BMIs into a set of sufficient-type LMI conditions (e.g. \([24, 8]\)). On the contrary, in the proposed post-compensation scheme, an auxiliary output \((z = Hx)\) is designed to be used as feedback instead of the actual plant output \((y)\) and due to the specific structure of the post-compensator \((H)\), the scheme guarantees feasible solution for any stable, tall/square, LTI plant with a minimal state-space realization and a full-rank \(B\) matrix.
restrict our attention to only real, rational, proper NI systems without poles at the origin as per Definition 1.

The following lemma, referred to as the NI Lemma, provides a state-space characterisation for NI systems without poles at the origin.

**Lemma 1 (NI Lemma) [38]** Let \( M(s) \) be the real, rational and proper transfer function matrix of a square and causal system having a minimal state-space realisation \((A, B, C, D)\). Then, \( M(s) \) is NI without poles at the origin if and only if \( \det(A) \neq 0 \), \( D = D^\top \) and there exists a real matrix \( Y = Y^\top > 0 \) such that

\[
AY + YA^\top \leq 0 \quad \text{and} \quad B + AYC^\top = 0. \tag{1}
\]

**Definition 2 (SNI System) [32]** Let \( M(s) \) be the real, rational and proper transfer function matrix of a finite-dimensional, square, LTI system. Then, \( M(s) \) is said to be SNI if \( M(s) \) has no poles in \( \{s \in \mathbb{C} : \Re[s] \geq 0 \} \) and \( jM(j\omega) - M(j\omega)^* \geq 0 \) for all \( \omega \in (0, \infty) \).

SSNI systems is a particular subset of the SNI class [38] that satisfy two additional frequency-domain criteria in the neighbourhood of \( \omega = 0 \) and \( \omega = \infty \). In a SISO setting, these two extra conditions indicate the patterns of departure from \( \omega = 0 \) and arrival at \( \omega = \infty \) of the Nyquist plot of an SSNI transfer function.

**Lemma 2 (SSNI Lemma) [23]** Let \((A, B, C, D)\) be a state-space realisation of a \((m \times n)\) real, rational and proper transfer function matrix \( M(s) \). Suppose, \( [M(s) - M^-(s)] \) has full normal rank \( m \) and the pair \((A, C)\) is observable. Then, \( A \) is Hurwitz and \( M(s) \) is SNI with

\[
\lim_{\omega \to \infty} j\omega [M(j\omega) - M(j\omega)^*] > 0 \quad \text{and} \quad \lim_{\omega \to 0} j\omega [M(j\omega) - M(j\omega)^*] > 0
\]

and there exists a real matrix \( Y = Y^\top > 0 \) such that \( AY + YA^\top < 0 \) and \( B + AYC^\top = 0 \).

The SSNI lemma will be invoked later to prove the main results of this paper. Note that in case of SSNI systems the full normal rank constraint on \( [M(s) - M^-(s)] \) is implied by the condition \( AY + YA^\top < 0 \) when \( B \) has full column rank. This notion is further generalised in the following lemma.

**Lemma 3** Let \( M(s) \in \mathbb{R}^{m \times m} \) have a state-space realisation \((A, B, C, D)\) with \( \rank(B) = m \leq n \). Assume there exists a real matrix \( Y = Y^\top > 0 \) such that \( AY + YA^\top < 0 \) and \( B + AYC^\top = 0 \). Then \( [M(s) - M^-(s)] \) has full normal rank \( m \).

**Proof.** Since \( AY + YA^\top < 0 \), there exists a square and non-singular matrix \( L \) such that \( AY + YA^\top = -L^\top L \). For this choice of \( L \) and \( Y \), define the transfer function matrix \( N(s) = LY^{-1}A^{-1}(sI - A)^{-1}B \). \( N(s) \) acquires full column rank at \( s = j\omega \) for all \( \omega \in \mathbb{R} \) because \( A \) is Hurwitz, \( \rank(B) = m \) (via assumption) and \( \rank[LY^{-1}A^{-1}] = n \). Now, \( M(s) \) and \( N(s) \) are related via the expression \( s[M(s) - M^-(s)] = s^2N^-(s)N(s) \), as derived in [38, Corollary 1]. Hence,

\[
j\omega [M(j\omega) - M(j\omega)^*] = \omega^2 N(j\omega)^* N(j\omega) > 0 \tag{2}
\]

for all \( \omega \in (0, \infty) \) and \( [M(0) - M(0)^*] = 0 \) since \( M(0) = CYC^\top + D = M(0)^\top \). This, in turn, implies that there does not exist any continuum frequency interval \( \omega \in \mathbb{R} \) such that \( \det[M(j\omega) - M(j\omega)^*] = 0 \). This guarantees that \( [M(s) - M^-(s)] \) must have full normal rank. \( \blacksquare \)

Note that SSNI systems readily follow this lemma. Also note that minimality is not required in the proof above.

We now present an internal stability condition for a stable NI system interconnected with an SNI system via positive feedback. Please see [21] for updated internal stability results of NI-SNI interconnections.

**Theorem 1 [22]** Let \( M(s) \) be an NI system without poles at the origin and \( N(s) \) be an SNI system. Let either \( M(\infty) = 0 \), or else, \( let \ M(\infty)N(\infty) = 0 \) and \( N(\infty) \geq 0 \). Then, the positive feedback interconnection of \( M(s) \) and \( N(s) \), shown in Fig. 2, is internally stable if and only if \( \lambda_{\max} [N(0)M(0)] < 1 \).

**Fig. 2.** Interconnection of NI systems with positive feedback.

### 3 Design of static pre- and post-compensators to transform non-square/square LTI systems into stable NI or SSNI class

This paper primarily focuses on designing static pre- and post-compensators for stable, non-square (i.e., fat and tall) or (non-NI) square plants to transform them into either SSNI or stable NI class. While a pre-compensator is required for a fat plant, a post-compensator is employed for a tall/square plant. The role of a pre- or post-compensator is to first equalise the number of inputs and outputs of the compensated plant as shown in Fig. 1(a) and Fig. 1(b) and then to enforce stable NI/SSNI property of the compensated system. The LMI-based design methodology renders easy implementation of the proposed schemes and pave the path for NI theory towards non-square systems. Later, these compensators are also used to design a simple constant reference tracking for stable, non-square or (non-NI) square, LTI systems.
3.1 Post-compensator design for tall/square plants using output feedback

A static post-compensator design technique is presented here to transform a stable, tall/square LTI system into the stable NI class by defining an auxiliary output $z = Hy$ that depends on output feedback. Theorem 2 gives a set of LMI conditions to find a static post-compensator $H ∈ ℜ^{r×l}$ for a given plant. To implement this scheme in real applications, we assume that appropriate sensors are available to measure the outputs.

![Fig. 3. An output feedback post-compensation scheme for tall/square plants using stable NI property.](image)

Theorem 2 Let $(A, B, C, 0)$ be a minimal state-space realization of a real, rational and proper transfer function matrix $G(s) ∈ ℜ^{' l×r}$ where $r ≤ l ≤ n$. Suppose there exist real symmetric matrices $P > 0$, $Q_o > 0$ and a matrix $H ∈ ℜ^{r×l}$ such that

\[
\begin{bmatrix}
PA + A^TP & PB - A^TC^TH^T \\
B^TP - HCA & -HCB - B^TC^TH^T
\end{bmatrix} ≤ 0, \tag{3a}
\]

\[
\begin{bmatrix}
Q_oA + A^TQ_o & (HC)^T \\
HC & -I
\end{bmatrix} < 0, \tag{3b}
\]

\[
\begin{bmatrix}
I & HC \\
C^TH^T & P
\end{bmatrix} > 0. \tag{3c}
\]

Then the post-compensator $H$ renders the combined system $G(s) = HG(s) = HC(sI - A)^{-1}B$ from $u$ to $z$ stable NI and also stabilises the closed-loop system shown in Fig. 3 via unity positive feedback.

Proof. We begin the proof by showing that the combined system $G(s) = HG(s) = HC(sI - A)^{-1}B$ will retain minimality. Inequality (3b) is equivalent to

\[
Q_oA + A^TQ_o + (HC)^THC < 0 \tag{4}
\]

which represents the observability Gramian inequality [5] and confirms that the pair $(A, HC)$ is completely observable. Since $G(s)$ is stable, the LMI condition (3a) ensures stable NI property of the compensated system $G(s)$ subjected to a feasible set of solution matrices $P > 0$ and $H ∈ ℜ^{r×l}$. Finally, inequality (3c) is equivalent to $I - HCP^{-1}(HC)^T > 0 ⇔ G(0) < I$. There it can be readily concluded that the unity positive feedback interconnection of $G(s)$ is closed-loop stable. This completes the proof. ■

Note that the post-compensator introduced in Theorem 2 relies on a set of sufficient-type LMI conditions (3a)–(3c) and hence, may not yield feasible solutions for all tall/square plants. To conquer this limitation, we will now propose a state feedback based post-compensator design technique which offers a particular structure of the post-compensator matrix $(H)$ such that any stable, tall/square, LTI plant with a full-rank $B$ matrix can be transformed into the SSNI class.

3.2 Post-compensator design for tall/square plants using state feedback

![Fig. 4. A state feedback post-compensation scheme for tall/square plants using SSNI property.](image)

In this subsection, we will introduce another static post-compensation scheme, shown in Fig. 4, which depends on the state feedback in contrast to output feedback as used in Section 3.1. The post-compensator $H ∈ ℜ^{r×n}$ has now a specific structure because of which it guarantees a feasible solution for any stable, tall or square plant with a full-rank $B$ matrix. Since the post-compensation scheme in Fig. 4 relies on the states $x$ (instead of the output $y$), we assume that the states are available for direct measurement. Then, to extract the states of $G(s)$, an auxiliary plant $\hat{G}(s) = (sI - A)^{-1}B ∈ ℜ^{' n×r}$ with i/p $U(s)$ and o/p $X(s)$ is constructed based on the original plant $G(s)$. $\hat{G}(s)$ has the full state vector $x$ as its output, which is then fed to the post-compensator $H$. The concept of this auxiliary plant $\hat{G}(s)$ will be used in all the subsequent post-compensation schemes (that rely on state feedback, rather than output feedback) proposed in this paper.

Theorem 3 Let $G(s) ∈ ℜ^{l×r}$ have a minimal state-space realization $(A, B, C, D)$ with rank$[B] = r$ where $r ≤ l ≤ n$. Suppose there exist real symmetric matrices $Y > 0$ and $Q_o > 0$ such that

\[
AY + YA^T < 0, \tag{5a}
\]

\[
\begin{bmatrix}
I & (A^{-1}B)^T \\
A^{-1}B & Y
\end{bmatrix} > 0, \tag{5b}
\]

\[
\begin{bmatrix}
Q_oA + A^TQ_o & I \\
I & -Y
\end{bmatrix} ≤ 0. \tag{5c}
\]

Let $z = Hx$ with $H = -B^TA^{-T}Y^{-1} ∈ ℜ^{r×n}$ be defined as an auxiliary output of the system. Then the post-compensator $H$ makes the compensated plant $\hat{G}(s)$
\( H(sI - A)^{-1}B \) SSNI and also stabilizes \( \bar{G}(s) \) in a unity positive feedback loop shown in Fig. 4.

**Proof.** Since \( B \) has full column rank and \( AY + YA^T < 0 \) via (5a), \( \bar{G}(s) - \bar{G}^{-1}(s) \) has full normal rank via Lemma 3. Note that the LMI constraint (5b) is equivalent to

\[
Y - A^{-1}BB^TA^{-T} > 0 \tag{6}
\]

by applying Schur-complement lemma [5]. The new pair \((A,H)\) remains completely observable since (5c) implies the observability Gramian condition [5] as derived below

\[
\begin{bmatrix} Q_oA + A^TQ_o & I \\ I & -Y \end{bmatrix} \leq 0
\]

\( \Leftrightarrow Q_oA + A^TQ_o + Y^{-1} \leq 0 \) [applying Schur complement]

\( \Leftrightarrow Y[Q_oA + A^TQ_o]Y + Y \leq 0 \)

\( \Rightarrow Y[Q_oA + A^TQ_o]Y + A^{-1}BB^TA^{-T} \leq 0 \) [since \( A^{-1}BB^TA^{-T} < Y \) via (6)]

\( \Leftrightarrow Q_oA + A^TQ_o + Y^{-1}A^{-1}BB^TA^{-T}Y^{-1} \leq 0 \)

\( \Leftrightarrow Q_oA + A^TQ_o + H^TH \leq 0 \)

on noting that \( H = -B^TA^{-T}Y^{-1} \). It can be readily verified that \( AYH^T = AY(-B^TA^{-T}Y^{-1}) = -B \). Therefore the compensated system \( \bar{G}(s) = H(sI - A)^{-1}B \) satisfies all the required properties of an SSNI system according to Lemma 2. Furthermore, the unity positive feedback interconnection of \( \bar{G}(s) \) shown in Fig. 4 is closed-loop stable via exploiting the NI-SNI stability condition (Theorem 1) since \( \bar{G}(0) < I \Leftrightarrow H^TYH < I \Leftrightarrow (A^{-1}B)^TY^{-1}(A^{-1}B) < I \Leftrightarrow (5b) \). This completes the proof.  

**Remark 1** To avoid getting ill-conditioned solution matrices \( Y \) and \( Q_o \) from the set of LMI conditions (5a)–(5c) in Theorem 3 due to computational issues caused by the SDP solver packages, we may impose some upper and lower bounds on \( Y \) and \( Q_o \). The lower bound on \( Q_o \) can be selected as the observability Gramian \( W_0 \) of the uncompensated plant \( G(s) \). The upper bound on \( Y \) plays a crucial role in determining the maximum eigenvalue of \( G(0) = (A^{-1}B)^TY^{-1}(A^{-1}B) \). A high upper bound on \( Y \) causes an overall reduction of the eigenvalues of \( Y^{-1} \) which in turn makes \( \lambda_{\text{max}}[G(0)] \) very small. These bounds can appropriately be chosen on the basis of the desired performance specifications. Note here that \( \lambda_{\text{max}}[G(0)] \) cannot be negative or zero since \( G(0) = HYH^T > 0 \) and \( H = -B^TA^{-T}Y^{-1} \) has full rank as \( B \) has full rank.

**Remark 2** Note that while Theorem 3 renders a tall/square plant SSNI, Theorem 2 makes the same plant stable NI. Although SSNI is more restrictive than the stable NI class, the post-compensator design methodology given in Theorem 3 guarantees a feasible solution for any stable, tall or square LTI plant with a full-rank \( B \) matrix owing to the specific structure \( H = -B^TA^{-T}Y^{-1} \).

Below, we study a physically motivated example to show an application of the post-compensation scheme.

**Example 1** Consider the two-mode flexible structure taken in [19] with two inputs and four outputs having non-colocated actuators and sensors where the inputs are generalized forces and the outputs are rates. The nominal plant model is given by

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -0.1 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0 & 0 \\ 0.1 & -0.01 \\ 0 & 0 \\ -0.2 & 0.05 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 0.8913 & 0 & 0.9218 \\ 0.7621 & 0.7382 \\ 0.4565 & 0.1763 \\ 0.0185 & 0.4057 \end{bmatrix}
\]

and \( D = 0_{4 \times 2} \) where \((A,B,C,D)\) is minimal and \( A \) is Hurwitz. Discarding the output equation \( y = Cx + Du \) and applying Theorem 3 to this system, the set of LMIs (5a)–(5c) yields

\[
Y = \begin{bmatrix} 0.0228 & -0.0009 & 0.00 & 0.00 \\ -0.0009 & 0.0225 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.0063 & -0.0005 \\ 0.00 & 0.00 & -0.0005 & 0.0250 \end{bmatrix} > 0 \text{ and}
\]

\[
Q_o = \begin{bmatrix} 1342.46 & 666.47 & 0.00 & 0.00 \\ 666.47 & 1342.26 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1954.10 & 25.86 \\ 0.00 & 0.00 & 25.86 & 488.81 \end{bmatrix} > 0
\]

using the CVX toolbox [17]. It is easy to verify that

\[
AY + YA^T < 0, \quad B = -AYH^T \text{ and the new pair } (A,H) \text{ is completely observable. The post-compensator } H = -B^TA^{-T}Y^{-1} \text{ is now obtained as }
\]

\[
H = \begin{bmatrix} 4.3908 & 0.1931 & -7.9968 & -0.1599 \\ -0.4391 & -0.0193 & 1.9992 & 0.0400 \end{bmatrix}.
\]

Therefore, the post-compensated system \( \bar{G}(s) = H(sI - A)^{-1}B \) is SSNI according to Theorem 3 and ensures closed-loop stability of the scheme shown in Fig. 4 due to satisfying the DC loop gain condition \( \lambda_{\text{max}}[\bar{G}(0)] = 
\]

\[
\lambda_{\text{max}} \begin{bmatrix} 0.8389 & -0.1439 \\ -0.1439 & 0.0294 \end{bmatrix} = 0.8637 < 1.
\]

**3.3 Pre-compensator design for stable fat LTI plants**

Theorem 4 presents an LMI-based static pre-compensator \((N \in \mathbb{R}^{r \times l} \text{ where } r > l)\) design technique by which the
cascaded plant $\bar{G}(s) = G(s)N$ is rendered SSNI with $G(0) > 0$ and automatically stabilises the unity positive feedback loop according to the NI stability result [22].

![Fig. 5. A static pre-compensation scheme for fat plants.](image)

**Theorem 4** Let $(A, B, C, 0)$ with rank[$B$] = $r$ be a minimal state-space realization of a real, rational and proper transfer function matrix $G(s) \in \mathbb{R}_{\infty}^{l 	imes r}$ where $l \leq r \leq n$. Suppose there exist a full-rank matrix $N \in \mathbb{R}^{r 	imes l}$ and the real symmetric matrices $Y > 0$, $Q_c > 0$ such that

\[
\begin{align*}
AY + YA^T &< 0, \quad (7a) \\
BN & = -AYC^T, \quad (7b) \\
CYC^T &< I, \quad (7c) \\
\begin{bmatrix} AQ_c + Q_cA^T & BN \\ N^TB^T & -I \end{bmatrix} &< 0. \quad (7d)
\end{align*}
\]

Then the pre-compensator $N$ makes the cascaded system $\bar{G}(s) = G(s)N = C(sI - A)^{-1}BN$ from $u$ to $y$ SSNI and also stabilises the closed-loop scheme with unity positive feedback as shown in Fig. 5.

**Proof.** Note that rank[$BN$] = $l$ applying the matrix rank inequality [18] rank[$B$] + rank[$N$] - $r$ \leq rank[$BN$] \leq \min\{rank[B], rank[N]\} via the assumption rank[$B$] = $r$. Utilising the full column rank property of $BN$, conditions (7a) and (7b) guarantee via Lemma 3 that $[\bar{G}(s) - \bar{G}^{-}(s)]$ has full normal rank. Since $(A, B, C, 0)$ is minimal, $(A, BN, C, 0)$ remains completely observable and via satisfying the controllability Gramian [5] condition (7d), the new pair $(A, BN)$ retains complete controllability. Hence, the pre-compensator $N$ renders the compensated system $\bar{G}(s) = G(s)N$ SSNI satisfying (7a) and (7b), and also ensures closed-loop stability of the scheme shown in Fig. 5 applying the DC-gain condition $\bar{G}(0) = CYC^T < I$. This completes the proof. \qed

The LMI conditions (7a) and (7b) used in Theorem 4 may sometimes yield an $N$ matrix having very small elements due to numerical computational issues caused by the SDP solver packages. To prevent that, we have included the controllability Gramian inequality (7d) and in practice, a lower bound may also be imposed on $Q_c$, which can be selected as the controllability Gramian $W_c > 0$ of the uncompensated system. On the other hand, a sufficiently high lower bound of $Y > 0$ is required here, contrary to Theorem 3, to prevent $\lambda_{\text{max}}[\bar{G}(0) = CYC^T]$ becoming too small.

---

**Example 2** Let us consider a $(1 \times 2)$ stable, LTI plant

\[
G(s) \text{ with } A = \begin{bmatrix} -5 & -6.25 \\ 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}.
\]

Now, the set of conditions (7a)–(7d) is solved for this system and it gives

\[
Y = \begin{bmatrix} 0.0997 & -0.0371 & 0.5653 \\
-0.0371 & 0.0392 & -0.1862 \\
0.5653 & -0.1862 & 4.1087
\end{bmatrix} > 0,
\]

\[
Q_c = \begin{bmatrix} 21.1281 & 9.0864 & -0.1676 \\
9.0864 & 47.1168 & -0.0618 \\
-0.1676 & -0.0618 & 17.1542
\end{bmatrix} > 0,
\]

and $N = \begin{bmatrix} 0.0245 \\ -4.3320 \end{bmatrix}$ by using the CVX toolbox [17].

We now compute the pre-compensated system $\bar{G}(s) = G(s)N = \frac{4.945s^2 + 25.34s + 111.4}{s^3 + 10s^2 + 50s + 125}$, the Nyquist plot of which is shown in Fig. 6. It has been verified that $\bar{G}(s)$ satisfies the SSNI properties: $AY + YA^T < 0$, $BN = -AYC^T$ and retains minimality. Finally, we check that the DC-gain condition $\bar{G}(0) = 0.8909 < 1$ is also satisfied. Therefore, the unity positive feedback interconnection of $\bar{G}(s)$ is closed-loop stable via Theorem 4.

![Fig. 6. Nyquist plot of the pre-compensated SSNI system $\bar{G}(s) = \frac{4.945s^2 + 25.34s + 111.4}{s^3 + 10s^2 + 50s + 125}$ taken in Experiment 2.](image)

**Remark 3** It is sometimes possible that the set of sufficient conditions (7a)–(7d), given in Theorem 4, may not be feasible. For instance, in Experiment 3, there does not exist any feasible pre-compensator for the system that renders the compensated system $\bar{G}(s)$ into the SSNI class. Moreover, a necessary condition for bi-proper fat plants $G(s)$ to be transformed into an NI system is that $DN = N^TD^T$, which is an excessively restrictive assumption in practice. To handle such cases, a post-compensator $(H)$ with a specific structure can be used in addition to...
the pre-compensator (N) such that the condition BN = −AYH becomes easier to satisfy. This requires us to
discard the output equation y = Cx + Du and work only
with the state equation ̇x = Ax + Bu. Such a scheme with
combined pre-post compensator is discussed in the next
subsection.

3.4 A combined pre-post compensator for fat plants

![Fig. 7. A combined pre-post-compensation scheme for stable fat LTI plants utilising SSNI property.](image)

The theorem given below suggests a combined pre-post-
compensation scheme to transform any stable, fat plant
with a full-rank B matrix into an SSNI system leading
to closed-loop stabilization via unity positive feedback
as shown in Fig. 7.

**Theorem 5** Let G(s) ∈ ℜ × ℜ × ℜ × ℜ have a minimal state-
space realization (A, B, C, D) where l < r ≤ n and
rank[B] = r. Suppose there exist a full-rank matrix N ∈ ℜ × ℜ and the real symmetric matrices Y > 0, Q_o > 0
such that

\[
AY + YA^T < 0, \quad (8a)
\]

\[
\begin{bmatrix}
I & (A^{-1}BN)^T \\
A^{-1}BN & Y
\end{bmatrix} > 0, \quad (8b)
\]

\[
\begin{bmatrix}
Q_o A + A^T Q_o & I \\
I & -Y
\end{bmatrix} \leq 0. \quad (8c)
\]

Let z = Hx be defined as an auxiliary output of the
system where H = −N^T B^T A^T Y^(-1) ∈ ℜ × ℜ. Then the
compensated system \( \hat{G}(s) = H(sI - A)^{-1}BN \) from u to z
is SSNI and the closed-loop system is stabilized via unity
positive feedback as shown in Fig. 7.

**Proof.** The proof follows directly by combining Theorem
3 and Theorem 4 on noting that rank[BN] = l, the
pair (A, H) is completely observable via (8c), \( AYH^T = \)
\( AY(-N^T B^T A^T Y^(-1))^T = -BN \) and \( \hat{G}(0) < I \)
⟹ HYH^T < I ⟹ Y = (A^{-1}BN)(A^{-1}BN)^T > 0 \) (8b)
applying Schur-complement lemma.

Note that (8a) holds for any stable G(s) with a full-rank
B matrix and BN = −AYH holds automatically since
H = −N^T B^T A^T Y^(-1). Henceforth, the combined pre-
post compensator ensures that \( \hat{G}(s) \) from u to z is SSNI
for any stable fat plant with a full-rank B matrix. Note
that sufficiency of Theorem 5 arises due to satisfying the
other two conditions (8b) and (8c).

![Fig. 8. A non-colocated spring-mass-damper system with two inputs and one output considered in Experiment 3.](image)

**Example 3** Consider the minimal state-space representa-
tion

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-3 & -1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
2 & 0 & -3 & -1
\end{bmatrix}, B = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}, C = \begin{bmatrix}
0 & 0 & 1 & 0
\end{bmatrix}
\]

and \( N = \begin{bmatrix}
0.5834 \\
0.5746
\end{bmatrix} \). The post-compensator is then con-
structed as \( H = \begin{bmatrix}
0.7491 & 0.0946 & 0.7366 & 0.0936
\end{bmatrix} \) which
gives rise to the compensated system

\[
\hat{G}(s) = \frac{0.109s^3 + 0.9693s^2 + 1.405s + 4.301}{s^4 + 2s^3 + 7s^2 + 6s + 5}
\]

\( \hat{G}(s) \) satisfies all the criteria of an SSNI transfer func-
tion and retains both controllability and observability. The Nyquist plot of \( \hat{G}(s) \) is given in Fig. 9. Furthermore, the
DC-gain condition \( \hat{G}(0) = 0.8603 < 1 \) is also sat-
isfied. Therefore, it can now be concluded that the unity
positive feedback interconnection of \( \hat{G}(s) \) is closed-loop
stable via Theorem 5.

3.5 A post-compensation scheme for unstable plants

The pre- and post-compensation schemes developed
for stable plants can be extended to marginally-
stable/unstable plants. Such a scheme for tall/square
Section 4.1 presents an observer-based constant reference tracking scheme for stable (non-NI) square or ‘tall’ systems using the post-compensator. Note here that for tall plants, we need to choose an appropriate set of outputs depending on the inputs to make a square system mapping. Subsection 4.2 applies the same principle to design a tracking scheme for stable ‘fat’ plants.

4.1 A constant reference tracking scheme for square or tall plants using observer-based post-compensator

In the following analysis, we will use the state-space representation of the combined system $\hat{G}(s)$ from input $u$ to the auxiliary output $z$ (see Fig. 11) as derived below

$$\hat{G}(s) = \begin{bmatrix} A & 0 \\ 0 & (A - LC) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$

where $\hat{x} = \dot{x} - x$. We also introduce the following short-hands: $A_L = A - LC$, $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A_L \end{bmatrix}$, $\hat{B} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}$ and $\hat{C} = \begin{bmatrix} \hat{H} & H \end{bmatrix}$. Now the transfer function mapping from $u$ to $z$ becomes $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} = H(sI - A)^{-1}B$. Note that the state-space realization $(\hat{A}, \hat{B}, C, 0)$ is not minimal since the modes associated with the block $[A - LC]$ are not controllable.

Theorem 6 Let $(A, B, C, D)$ be a minimal state-space realization of a real, rational and proper transfer function matrix $G(s) \in \mathbb{R}^{l \times r}$, $l = r \leq n$, with $(A, B, C, 0)$ not minimal. Let an observer gain matrix $\hat{L} \in \mathbb{R}^{n \times l}$ be given such
that $A_L = A - LC$ is Hurwitz. Assume there exist real symmetric matrices $Y_{11}$, $Y_{22}$ and $Q_o > 0$ such that

\begin{align}
Y &= \begin{bmatrix} Y_{11} & -Y_{22} \\ -Y_{22} & Y_{22} \end{bmatrix} > 0, \\
\bar{A}Y + Y\bar{A}^T &< 0, \\
\begin{bmatrix} I & (A^{-1}B)^T \\ A^{-1}B & Y_{11} - Y_{22} \end{bmatrix} > 0, \\
\begin{bmatrix} Q_o\bar{A} + \bar{A}^TQ_o & I \\ I & -Y \end{bmatrix} &\leq 0.
\end{align}

(10a) implies that $[\bar{G}(s) - \bar{G}^-(s)]$ has full normal rank via exploiting Lemma 3. Therefore, $\bar{G}(s)$ satisfies all the pre-requisite criteria of an SSNI system. Subsequently, after showing that

\begin{align}
AYC^T &= \begin{bmatrix} A & 0 \\ 0 & A_L \end{bmatrix} \begin{bmatrix} Y_{11} & -Y_{22} \\ -Y_{22} & Y_{22} \end{bmatrix} H^T \\
&= \begin{bmatrix} A(Y_{11} - Y_{22})H^T \\ A_L(-Y_{22} + Y_{22})H^T \end{bmatrix} \\
&= \begin{bmatrix} A(Y_{22} - Y_{11})(Y_{22} - Y_{11})^{-1}A^{-1}B \\ 0 \end{bmatrix} \\
&= -B [as \bar{H} = B^T A^{-T}(Y_{22} - Y_{11})^{-1}],
\end{align}

we can confirm that $\bar{G}(s)$ is an SSNI system with $\bar{G}(0) = C\bar{Y}\bar{C}^T = H(Y_{11} - Y_{22})H^T > 0$ via Lemma 1.

**Part II.** In this case, we assume $K_o = \frac{1}{2}I_m$ with $k > 0$. Note that $\bar{G}(s)$ being SSNI with $\bar{G}(0) > 0$ exhibits IC property and hence, there exists a finite $k^* > 0$ such that the closed-loop scheme in Fig. 11 remains asymptotically stable for any $k \in [0, k^*]$. Therefore, $\lim_{t \to \infty} e(t) = 0$ where $e(t) = \bar{r} - \bar{z}(t)$ and hence, $\lim_{t \to \infty} \bar{z}(t) = \bar{r}.$ However, it remains to be shown whether the physical output $y(t)$ will track the actual reference $r$ or not. Assuming faster observer dynamics than the underlying plant and relying on the closed-loop asymptotic stability, we have $\bar{x} = 0,$ $\bar{x} = 0,$ $y = \bar{y}$ and $x = \bar{x}$ after reaching the steady-state. This results in $x_{ss} = \lim_{t \to \infty} x(t) = -A^{-1}Bk_g e_{ss}$ denoting $e_{ss} = \lim_{t \to \infty} e(t).$ We then find $y_{ss} = \lim_{t \to \infty} y(t) = C_{x_{ss}} + DK_g e_{ss} = -CA^{-1}BK_{g_{e_{ss}}} + DK_g e_{ss} = G(0)K_{g_{e_{ss}}}$ and subsequently, $z_{ss} = \lim_{t \to \infty} z(t) = \bar{H}\bar{x}_{ss} = H_{x_{ss}} = -HA^{-1}BK_{g_{e_{ss}}} = -HA^{-1}(-A(Y_{11} - Y_{22})H^T)K_{g_{e_{ss}}} = H(Y_{11} - Y_{22})^T K_y e_{ss} = (\bar{G}(0)K_{g_{e_{ss}}})$. This in turn gives $e_{ss} = K_{g}^{-1}G(0)(0)z_{ss} = K_{g}^{-1}G(0)(0)\bar{r} = K_{g}^{-1}G(0)(0)\bar{r}$ since $\bar{r} = \bar{r}$ from Fig. 11. Now substituting the expression for $e_{ss}$ into the expression of $y_{ss}$, we get $y_{ss} = G(0)K_{g}K_{g}^{-1}G(0)\bar{r} = G(0)G(0)(0)(0)\bar{r} = r$ using $S = G(0)G(0)^{-1}$. Hence, $y(t) \to r$ as $t \to \infty$ while $z(t) \to \bar{r}$.

**Part III.** In this case, $K_g = \text{diag}\{k_1,k_2,\cdots,k_m\}$ with $k_i \in [0,k^*] \forall i \in \{1,2,\ldots,m\}$. In addition to IC property, $\bar{G}(s)$ also satisfies DIC property because of its SSNI property with $\bar{G}(0) > 0$. Therefore, closed-loop stability of the tracking scheme (Fig. 11) remains preserved when $k_i = 0$ for some $i \in \{1,2,\ldots,m\}$. $k_i = 0$ indicates that the $i^{th}$ input-output channel is faulty.

Parts I, II and III together complete the proof. ■
**Example 4** Consider a $(2 \times 2)$ stable LTI system $G(s)$ with a minimal state-space realization given by $A = \begin{bmatrix} -2.0 & -2.50 & 0 & 0 \\ 2.0 & 0 & 0 & 0 \\ 0 & 0 & -2.0 & -2.50 \\ 0 & 0 & 2.0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$, $C = \begin{bmatrix} 1.0 & 0.50 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \end{bmatrix}$, $D = 0_{2 \times 2}$ Now, applying Theorem 6 the LMI conditions (10a)–(10d) yield a feasible set of solution matrices $Y_{11} = \begin{bmatrix} 0.8279 & -0.0063 & 0.0059 & -0.0025 \\ -0.0063 & 0.6715 & -0.0024 & 0.0098 \\ 0.0059 & -0.0024 & 0.8435 & -0.0129 \\ -0.0025 & 0.0098 & -0.0129 & 0.6971 \end{bmatrix} > 0$, $Y_{22} = \begin{bmatrix} 0.0065 & -0.0056 & 0.0069 & -0.0025 \\ -0.0056 & 0.0150 & -0.0025 & 0.0098 \\ 0.0069 & -0.0025 & 0.0243 & -0.0123 \\ -0.0025 & 0.0098 & -0.0123 & 0.0407 \end{bmatrix} > 0$, and $Q_o > 0$. The post-compensator is then formed as $H = \begin{bmatrix} 0.0010 & 1.2186 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.0010 & 1.2186 \end{bmatrix}$ which makes the post-compensated plant $\hat{G}(s)$ SSNI with $\hat{G}(0) = \begin{bmatrix} 0.9749 & 0.00 \\ 0.00 & 0.9749 \end{bmatrix} > 0$. Now exploiting the IC property of $\hat{G}(s)$, we can design a simple constant reference tracking scheme as discussed in Theorem 6. The input-shaping matrix is constructed as $S = \hat{G}(0)G(0)^{-1} = \begin{bmatrix} 3.2497 & -1.6249 \\ -1.6249 & 3.2497 \end{bmatrix}$. Step simulation response is shown in Fig. 12 for which the references are chosen as $r_1 = 1$ and $r_2 = -1$. The figure suggests that both $y_1$ and $y_2$ are satisfactorily tracking the respective references. Integral gains were selected as $K_g = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}$. As mentioned earlier, SSNI systems also satisfy the DIC property and hence, the integral control scheme containing the compensated SSNI system will offer failure tolerance in the event of sensor/actuator faults.

**Remark 4** In this paper, being inspired by [3], IC property of the SSNI class of systems is exploited to show that an integral control scheme involving an SSNI system with positive definite DC-gain can be closed-loop stable via negative feedback and accordingly, this idea has been utilized to design a constant reference tracking scheme for stable non-square systems. Note that the proposed tracking scheme could also be established by applying [21, Corollary 32] and [30, Corollary 2], which dealt with closed-loop stability of a positive feedback NI-SNI interconnection containing a simple pole at the origin. However, the aforementioned results remain inapplicable when an SSNI system has positive definite DC-gain matrix. In our paper, all the compensated plants $\hat{G}(s)$ are designed to be SSNI with $\hat{G}(0) > 0$, which therefore restricts the use of the ideas given in [21] and [30]. Moreover, the proposed IC-based tracking scheme also unveils that a negative feedback interconnection of an SSNI system and a decentralised integrator system having positive gains can be closed-loop stable in contrast to most of the existing stability results in the NI literature those are applicable only to positive feedback cases.

**4.2 A reference tracking scheme for fat plants using observer-based combined pre-post compensator**

We will now put forward the complementary result of Theorem 6 which applies to stable fat systems. Fig. 13 depicts the constant reference tracking scheme for stable, fat plants employing an observer-based combined pre-post compensator. This scheme exploits IC and DIC properties of the compensated system $\hat{G}(s)$ owing to its SSNI property with $\hat{G}(0) > 0$. Theorem 7 is derived below to theoretically establish the aforementioned idea.

![Fig. 12](image1.png)

**Fig. 12.** Unit step tracking response of the negative feedback integral control scheme used in Experiment 4 when both the channels are operating in healthy condition.

**Fig. 13.** A constant reference tracking scheme for fat plants using an observer-based combined pre-post compensator.

**Theorem 7** Let $(A, B, C, D)$ be a minimal state-space realization of a real, rational and proper transfer function matrix $G(s) \in \mathbb{R}^{l \times r}$ with rank($B$) = $r$ where $1 < r \leq n$. Let an observer gain matrix $L \in \mathbb{R}^{n \times l}$ be given such
that $A_L = A - LC$ is Hurwitz. Assume there exist real symmetric matrices $Y_{11}, Y_{22}, Q_o > 0$ and a full-rank matrix $N \in \mathbb{R}^{r \times l}$ satisfying $\det[G(0)N] \neq 0$ such that

$$Y = \begin{bmatrix} Y_{11} & -Y_{22} \\ -Y_{22} & Y_{22} \end{bmatrix} > 0,$$  \hfill (14a)

$$AY + YA^T < 0,$$  \hfill (14b)

$$\begin{bmatrix} I & (A^{-1}BN)^T \\ A^{-1}BN & Y_{11} - Y_{22} \end{bmatrix} > 0,$$  \hfill (14c)

$$Q_oA + A^TQ_oI - Y \leq 0.$$  \hfill (14d)

Let $z = H\dot{z}$ with $H = N^T B^T A^{-T} (Y_{22} - Y_{11})^{-1} \in \mathbb{R}^{l \times n}$ in Fig. 13 and the input-shaping matrix be defined as $S = H(Y_{11} - Y_{22}) H^T [G(0)N]^{-1}$. Then there exists a finite $k^* > 0$ such that for any $k \in [0, k^*]$ and for each $i \in \{1, 2, \ldots, m\}$, the plant output $y(t)$ asymptotically tracks any constant reference $r \in \mathbb{R}^r$ while the auxiliary output $z(t)$ tracks the shaped reference $\tilde{r} = Sr$.

**Proof.** The proof proceeds similarly to that of Theorem 6 and the following shorthands are used to represent the state-space matrices of the combined system $\hat{G}(s)$:

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} BN \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} H & H \end{bmatrix}. $$

The LMI condition (14c) is equivalent to

$$I - (A^{-1}BN)^T (Y_{22} - Y_{11})^{-1} (A^{-1}BN) > 0$$

$$\Rightarrow I - H(Y_{11} - Y_{22}) H^T$$

[denoting $H = N^T B^T A^{-T} (Y_{22} - Y_{11})^{-1}$]

$$\Rightarrow CYC^T < I$$

$$\Rightarrow Y^{-1} - C^T C > 0$$  \hfill (15)

applying Schur-complement lemma [5]. Now, inequality (14d) is equivalent to $Q_oA + A^TQ_oI + Y^{-1} \leq 0$ which implies $Q_oA + A^TQ_o + C^T C < 0$ via (15). Therefore, $(A, C)$ remains completely observable. Note that

$$\text{rank}[\hat{B}] = \text{rank} \begin{bmatrix} BN \\ 0 \end{bmatrix} = l \text{ since } \text{rank}[B] = r$$

$$\text{rank}[N] = l \text{ via assumption. Note that } \hat{A}Y + YA^T < 0$$

via (14b) and $\hat{A}Y + YA^T = -\hat{B}$ holds automatically since $H = N^T B^T A^{-T} (Y_{22} - Y_{11})^{-1}$. Hence, the pre-post compensated system $\hat{G}(s)$ is SSNI with $\hat{G}(0) = CYC^T = (A^{-1}BN)^T (Y_{11} - Y_{22})^{-1} (A^{-1}BN) = H(Y_{11} - Y_{22}) H^T > 0$. Now, following Parts II and III of Theorem 6, it can be readily shown that $\hat{G}(s)$ satisfies both IC and DIC properties in a negative feedback loop. Therefore, the plant output $y(t)$ also tracks the reference $r$ in presence of the input-shaping matrix $S$ and the scheme remains closed-loop stable in the event of sensor/actuator faults. This completes the proof. ■

## 5 Pre- and post-compensators design for LTI systems with polytopic uncertainty

In this section, we investigate the problem of designing pre- and post-compensators for stable, non-square or (non)-square LTI systems with uncertain parameters varying in a polytope invoking SSNI theory. Let a family of uncertain, LTI systems be described as

$$G_\rho : \begin{cases} \dot{x} = A(\rho)x + B(\rho)u, \\ y = C(\rho)x + D(\rho)u, \end{cases} \quad x(0) = x_0;$$  \hfill (16)

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^r$, $y = y(t) \in \mathbb{R}^l$ for all $t \geq 0$ and $\rho = [\rho_1, \rho_2, \ldots, \rho_k]^T \in \mathbb{R}^k$ denotes the vector of uncertain parameters where the individual parameter $\rho_i$ can take any fixed value within the range $[\rho_{i\min}, \rho_{i\max}]$ for each $i \in \{1, 2, \ldots, k\}$ and as a whole, $\rho$ varies within a polytope having $2^k$ vertices defined by the set $\mathcal{V}$ as introduced below:

$$\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_k) : \nu_i \in \{\rho_{i\min}, \rho_{i\max}\} \forall i\}. $$  \hfill (17)

The state-space matrices $A(\rho), B(\rho), C(\rho)$ and $D(\rho)$ depend affinely on the parameter vector $\rho$, which implies [16] that each member of the family of uncertain plants $G(\rho, s) = C(\rho)[sI - A(\rho)]^{-1} B(\rho) + D(\rho)$ belongs to the convex hull of the systems represented by

$$\mathcal{G} \{G_j(s) = C_j(sI - A_j)^{-1} B_j + D_j \}$$

where the index $j \in \{1, 2, \ldots, 2^k\}$ denotes the vertices in $\mathcal{V}$.

### 5.1 Post-compensator design for tall/square systems with polytopic uncertainty

![Fig. 14. A static post-compensation scheme for tall/square systems with polytopic uncertainty](image)

The following theorem presents an LMI-based methodology to design a static post-compensator $H$ for transforming a tall/square LTI system with polytopic uncertainty into the SSNI class. Note that the static post-compensator $H = -B^T A^{-T} Y^{-1} \in \mathbb{R}^{r \times n}$, proposed in Theorem 3, is no longer applicable in the present case since $A$ and $B$ matrices are uncertain.

**Theorem 8** Let $G(\rho, s) = C(\rho)[sI - A(\rho)]^{-1} B(\rho) + D(\rho) \in \mathcal{G}_{\mathcal{K}_{\infty}^s}$ represent a family of uncertain plants with rank $[B(\rho)] = r \forall \rho \in \mathcal{V}$, given in (16), where $r \leq l \leq n$. Assume there exist real symmetric matrices $Y > 0$, $Z > 0$ such that

$$\begin{bmatrix} Y & Z \\ Z^T & Y \end{bmatrix} > 0$$

This completes the proof.
For $Q_o > 0$ and a full-rank matrix $S \in \mathbb{R}^{r \times n}$ such that

\begin{align}
A_j Y + YA_j &< 0, \quad \text{(18a)} \\
B_j + A_j S^T &< 0, \quad \text{(18b)} \\
\begin{bmatrix} Y S^T \\ S I \end{bmatrix} &> 0, \quad \text{(18c)} \\
\begin{bmatrix} Q_o A_j + A_j^T Q_o & I \\ I & -Y \end{bmatrix} &\leq 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}. \quad \text{(18d)}
\end{align}

Let $z = Hx$ with $H = SY^{-1} \in \mathbb{R}^{r \times n}$ be defined as an auxiliary output. Then the post-compensator $H$ makes the family of compensated plants $G(\rho, s) = H[sI - A(\rho)]^{-1}B(\rho)$ SSNI for all $\rho \in \mathcal{Y}$ and stabilises each member of $G(\rho, s)$ in closed-loop via unity positive feedback shown in Fig. 14.

**Proof.** We begin the proof by noting that the assumption $\text{rank}[B(\rho)] = r$ along with the strict LMI condition (18a) implies that $[G_j(s) - G_j(-s)]^T$ has full normal rank for all $j \in \{1, 2, \ldots, 2^k\}$ via Lemma 3. Then the inequalities (18a), (18b) and (18c) jointly ensure that the set of compensated systems $\tilde{G}_j(s)$, evaluated at each of the $2^k$ vertices of the convex hull of systems $\text{Co}\{G_j(s)\}$, exhibits SSNI property. The inequality (18c) is also equivalent to satisfying the DC gain bound $G(0) < I$ of the set of compensated systems $\tilde{G}_j(s)$ for all $j$ as derived below

\[ \begin{bmatrix} Y S^T \\ S I \end{bmatrix} > 0 \]

\[ Y > S^T S > 0 \quad \text{[using $S = HY$ where $Y > 0$]} \]

\[ Y^{-1} - H^TH > 0 \]

\[ \begin{bmatrix} I & H \\ H^TY & Y^{-1} \end{bmatrix} > 0 \quad \text{[since $Y > 0$]} \]

\[ HYH^T < I \]

on noting that $G(0) = HA_j^{-1}B_j = -HA_j^{-1}(-A_j Y H^T) = HYH^T$. It is interesting to notice here that even though there are $2^k$ compensated systems, denoted by $\tilde{G}_j(s)$ for $j \in \{1, 2, \ldots, 2^k\}$, but due to enforcing the SSNI property, $G(0) = HYH^T$ is a fixed entity. Finally, inequality (18d) proves that each of the compensated systems remains completely observable as illustrated below. On taking Schur complement of (18d) with respect to $Y > 0$, we get

\[ Q_o A_j + A_j^T Q_o + Y^{-1} \leq 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}, \quad \text{(20)} \]

which implies

\[ Q_o A_j + A_j^T Q_o + H^TH \leq 0 \quad \forall j \in \{1, 2, \ldots, 2^k\} \quad \text{(21)} \]

via (19). Hence, the new pair $(A_j, H)$ remains completely observable for all $j$. Since it is already shown that each of the $2^k$ plants, $\tilde{G}_j(s)$, lying at the vertices of $\text{Co}\{G_j(s)\}$ satisfies the SSNI property upon satisfying (18a)–(18d), any $G(\rho, s) \in \text{Co}\{G_j(s)\}$ will also satisfy the SSNI property by exploiting the property of convex hull [16]. Hence, it can be concluded that a single post-compensator $H = SY^{-1}$ transforms the family of uncertain systems $G(\rho, s)$ into a set of strictly-proper SSNI systems $\tilde{G}(\rho, s) = H[sI - A(\rho)]^{-1}B(\rho)$ with a fixed DC-gain matrix $\tilde{G}(0) = HYH^T$ and the unity positive feedback interconnection of $\tilde{G}(\rho, s)$, as shown in Fig. 14, is closed-loop stable for any fixed value of $\rho$ within the polytope having $2^k$ vertices in $\mathcal{Y}$. This completes the proof. 

Below, we study an illustrative example of an overactuated spring-mass-damper system with non-colocated input-output pairs as shown in Fig. 15 to highlight the usefulness of Theorem 8. The viscous friction coefficients of the dampers are considered to be uncertain.

**Example 5** Consider the $(3 \times 1)$ system configuration shown in Fig. 15 having one input $u_1$ and three outputs $[y_1 \ y_2 \ y_3]^T$. A minimal state-space realisation of the above system is given by

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ -k & -b_1 & 0 \\ -k & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = 0. \]

$\begin{cases} \begin{bmatrix} m_1 \ b_1 \ k \end{bmatrix} \end{bcases}$ and $\begin{cases} \begin{bmatrix} m_2 \ b_2 \ k \end{bcases} \end{cases}$ are defined such that $m_1 = m_2 = 1$ Kg and $b_1 \in [0.8Ns/m, 1.2Ns/m]$ and $b_2 \in [0.8Ns/m, 1.2Ns/m]$. The set of LMI conditions (18a)–(18d) is found to be feasible at all vertices of the convex hull of systems $\text{Co}\{G_j(s) = C(sI - A_j)^{-1}B + D\}$.
where \( j \in \{1, 2, 3, 4\} \) and yields the solution matrices

\[
Y = \begin{bmatrix}
0.2007 & -0.0460 & 0.0915 & -0.0135 \\
-0.0460 & 0.4599 & -0.0139 & -0.0259 \\
0.0915 & -0.0139 & 0.2004 & -0.0458 \\
-0.0135 & -0.0259 & -0.0458 & 0.4598
\end{bmatrix} > 0,
\]

\[
Q_o = \begin{bmatrix}
323.5450 & 22.4586 & -86.9039 & 10.9539 \\
22.4586 & 86.7516 & 10.9575 & 18.8217 \\
-86.9039 & 10.9575 & 323.5450 & 22.4587 \\
10.9539 & 18.8217 & 22.4587 & 86.7508
\end{bmatrix} > 0,
\]

and \( S = \begin{bmatrix}
0.3333 & 0 & 0.1667 & 0
\end{bmatrix} \). The post-compensator is then formed as \( H = \begin{bmatrix}
1.6586 & 0.1728 & 0.1017 & 0.0687
\end{bmatrix} \).

It is easy to verify that the post-compensated systems \( G_j(s) = H(sI - A_j)^{-1}B \) is SSNI for all \( j \in \{1, 2, 3, 4\} \). Fig. 17 shows the Bode plots of \( G_j(s) = H(sI - A_j)^{-1}B \) for all \( j \) corresponding to each of the four vertices of the convex hull of systems \( \mathcal{G} \{ G_j(s) \} \). It is well observed from Fig. 17 that in each of the four cases the phase angle \( \phi_j(\omega) \in (0, \pi) \) for all \( \omega \in (0, \infty) \) and \( \phi_j(\omega) \to -\pi \) as \( \omega \to \infty \), which conforms to SSNI property. Finally, exploiting the property of convex hull as explained in Theorem 8, the static post-compensator \( H \) renders the family of uncertain systems \( G(\rho, s) \) into a set of SSNI systems \( \bar{G}(\rho, s) \) with a fixed DC-gain matrix \( H(0) = HYH^\top = 0.5608 < 1 \) and thereby ensures closed-loop stability of the scheme shown in Fig. 14.

5.2 Pre-compensator design for stable fat systems with polytopic uncertainty

![Fig. 16. A static pre-compensation scheme for stable, fat plants with polytopic uncertainty.](image)

Similar to the post-compensation scheme for uncertain tall/square plants, in this subsection, a pre-compensation scheme is provided here for stable, fat, LTI systems with polytopic uncertainty. The scheme is depicted in Fig. 16 where a static pre-compensator \( N \in \mathbb{R}^{s \times t} \) is designed to transform a family of uncertain plants \( G(\rho, s) \) into the SSNI class and to stabilise each member of \( G(\rho, s) \) in closed-loop via unity positive feedback. The uncertain parameter vector \( \rho = [\rho_1, \rho_2, \ldots, \rho_k]^\top \) can take any fixed value within the polytope having \( 2^k \) vertices in \( Y \) as described in (17). Theorem 9 gives an LMI-based design methodology of the pre-compensation scheme for fat plants with polytopic uncertainty.

**Theorem 9** Let \( G(\rho, s) = C(\rho)[sI - A(\rho)]^{-1}B(\rho) \in \mathbb{H}^{r \times t} \) represent a family of uncertain plants as described in (16) with \( \text{rank}[C(\rho)] = l \) for all \( \rho \in \mathcal{P} \), where \( l < r \leq n \). Assume there exist real symmetric matrices \( Y > 0, Q_o > 0, Q_c > 0 \) and a full-rank matrix \( N \in \mathbb{R}^{r \times t} \) such that

\[
A_jY + YA_j^\top < 0,
\]

\[
B_jN + A_jYC_j^\top = 0,
\]

\[
C_jYC_j^\top < I,
\]

\[
\begin{bmatrix}
Q_oA_j + A_j^\top Q_o & C_j^\top \\
C_j & -I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
A_jQ_c + Q_cA_j^\top & B_jN \\
N^TB_j^\top & -I
\end{bmatrix} < 0
\]
for all \( j \in \{1, 2, \ldots, 2^k\} \). Then the pre-compensator \( N \) makes the family of compensated plants \( G(p, s) = G(p, s)N = C(p)[sI - A(p)]^{-1}B(p)N \) SSNI for all \( p \in \mathcal{V} \) and stabilises each member of \( G(p, s) \) via unity positive feedback as shown in Fig. 16.

**Proof.** We begin the proof by showing that the family of pre-compensated plant \( \tilde{G}(p, s) \) retains its minimality. Inequality \((22e)\) is equivalent to the controllability Gramian condition

\[
A_jQ_c + Q_cA_j^T + B_jNN^TB_j^T < 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}
\]

which ensures that the pair \((A_j, B_jN)\) remains controllable at each of the \( 2^k \) vertices of the convex hull of systems \( \text{Co} \{G_j(s)\} \). Similarly, inequality \((22d)\) is equivalent to the observability Gramian condition

\[
Q_oA_j + A_j^TQ_o + C_j^TC_j < 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}
\]

and hence, the pair \((A_j, C_j)\) remains observable for all \( j \in \{1, 2, \ldots, 2^k\} \). It implies that each member of the uncertain family \( G(p, s) \) will satisfy the LMIs \((22d)\) and \((22e)\) owing to the properties of convex hull \((16)\) \( \forall \rho \in \mathcal{V} \). This guarantees that the set of compensated systems \( G(p, s) \) retains minimality. Now the assumption \( C(p) \) having full rank implies from \((22b)\) that \( B_jN \) also has full rank for all \( j \) since \( C_j \in C(p) \) \( \forall j \), \( A_j \) is Hurwitz and \( Y > 0 \). This result together with the LMI constraint \( A_j^TY + YA_j^T < 0 \) implies \( [G_j(s)N - N^TG_j(-s)^T] \) has full normal rank for all \( j \) via Lemma 3. Again exploiting the properties of convex hull, it can be assured that \( [G(p, s)N - N^TG(p, -s)^T] \) has full normal rank \( \forall \rho \in \mathcal{V} \). Therefore \( \tilde{G}(p, s) \) satisfies the pre-requisite criteria of SSNI systems. Now, the first two LMIs \((22a)\) and \((22b)\) together ensures that \( G_j(s) = C_j(sI - A_j)^{-1}B_jN \) is SSNI for all \( j \in \{1, 2, \ldots, 2^k\} \). Finally, \((22c)\) ensures the DC-gain condition, that is, \( G_j(0) < I \) for all \( j \), which eventually holds for the entire family of pre-compensated plants applying the argument of \((16)\). That is, \( \tilde{G}(0, 0) < I \). Therefore, it is now proved that the pre-compensator \( N \) makes the family of compensated (or transformed) systems SSNI along with stabilizing the closed-loop system via unity positive feedback (Fig. 16) \( \forall \rho \in \mathcal{V} \) within the polytope having its vertices in \( \mathcal{V} \) defined in \((17)\). ■

Note that controllability of the compensated system is not required for \( G(p, s) \) to be SSNI but it is imposed to prevent the elements of the pre-compensator \( N \) becoming too small due to numerical computational issues of the SDP solver packages as mentioned in Section 3 after the proof of Theorem 4.

**Example 6** Reconsider the fat \((1 \times 2)\), stable, LTI un-

\[
\begin{bmatrix}
-5 & -\rho_1 & 0 & 0 \\
\rho_2 & 0 & 0 & 0 \\
0 & 0 & -5 & 0 \\
6.21 & 1.563 & -1.00 & 0
\end{bmatrix}
\]

the uncertain parameter vector \( \rho = [\rho_1 \rho_2]^T \) where \( \rho_1 \) and \( \rho_2 \) vary within 10\% and 40\% with respect to their nominal values \( \rho_{nom} = 6.25 \) and \( \rho_{nom} = 4.0 \). To design a static pre-compensator following Theorem 9, the set of LMI conditions \((22a)-(22e)\) is applied at each of the four vertices of the convex hull of systems \( \text{Co} \{G_j(s)\} \) where \( j \in \{1, 2, 3, 4\} \) and yields a set of feasible solution matrices

\[
Q_o > 0 \quad \text{and} \quad N = \begin{bmatrix} 0.0377 \\ -3.7764 \end{bmatrix}
\]

It can be confirmed in a similar manner to \((16)\) that the LMI conditions will also hold for each member of the family of uncertain systems \( G(p, s) \in \text{Co} \{G_j(s)\} \) where \( j \in \{1, 2, 3, 4\} \) and hence any member of \( G(p, s) \) will be rendered into an SSNI system \( G(s) = C(sI - A(p))^{-1}BN \) by the same pre-compensator \( N \) with \( \tilde{G}(0) = CYC^T = 0.7930 < 1 \) when the parameter vector \( \rho \) is allowed to take any fixed value inside a rectangle (two dimensional polytope) defined by the corners \((5.625, 2.4)\), \((5.625, 5.6)\), \((6.875, 2.4)\) and \((6.875, 5.6)\). Hence, it can be concluded via Theorem 9 that the above uncertain system \( G(p, s) \) can be stabilised in closed-loop via unity positive feedback by the designed static pre-compensator \( N \) subject to any fixed value of the parameter \( \rho \) within the above rectangle. Fig. 19 and Fig. 20 confirm the SSNI property of the pre-compensated systems \( G_1(s) \), \( G_2(s) \), \( G_3(s) \) and \( G_4(s) \) evaluated at the four vertices of the convex hull. It is apparent that in each of the four cases the phase angle \( \phi_1(\omega) \in (0, -\pi) \) for all \( \omega \in (0, \infty) \) and \( \phi_2(\omega) \to -\frac{\pi}{2} \) as \( \omega \to \infty \), which conforms to SSNI property.

**Remark 5** As an immediate consequence of Theorem 8 and Theorem 9, the proposed pre- and post-compensators can be used together to execute the sensor-blending and control-allocation tasks for systems having a large number of redundant sensors and actuators. A set of linear matrix equality and inequality constraints can be derived to search for a feasible choice of compensators which makes the combined system SSNI maintaining both con-
Fig. 19. Bode plots of the pre-compensated SSNI systems \( G_1(s), G_2(s), G_3(s) \) and \( G_4(s) \) evaluated at the four vertices of the convex hull while \( G_{\text{nominal}}(s) \) denotes the compensated system corresponding to the nominal parameter values.

Fig. 20. Closed-loop impulse response of the pre-compensated SSNI systems \( G_1(s), G_2(s), G_3(s) \) and \( G_4(s) \) evaluated at the four vertices of the convex hull.

Trollability and observability properties along with satisfying the DC-gain condition to ensure closed-loop stability with unity positive feedback. Moreover, the scheme is robust against parametric uncertainties.

5.3 Stabilization of tall/square, uncertain LTI plants preceded by slope-restricted nonlinearity using post-compensation technique

\[ r = 0 \]

Fig. 21. A stabilizing static post-compensation scheme for a class of stable, uncertain, LTI plant \( G(r,s) \) preceded by a slope-restricted nonlinearity \( \Phi(\cdot) \).

We here investigate global asymptotic stability of a positive feedback loop containing a tall/square, uncertain, LTI plant preceded by a slope-restricted nonlinearity \( \Phi(\cdot) \) as shown in Fig. 21 using the techniques inspired from [10], where \( \Phi \) belongs to the class of time-invariant, diagonal, memoryless, slope-restricted nonlinearities, \( \Phi \), as defined below:

\[
\Phi(\cdot) = [\phi_1(\cdot), \phi_2(\cdot), \ldots, \phi_r(\cdot)]^T
\]

with \( \Phi(0) = 0 \), \( \phi_i \) is globally Lipschitz for all \( i \in \{1, 2, \ldots, r\} \),

\[
0 \leq \partial \phi_i < \mu_i \quad \forall i \in \{1, 2, \ldots, r\},
\]

\[
\mu_i \in (0, \infty) \quad \forall i \in \{1, 2, \ldots, r\},
\]

where \( \partial \phi_i \) denotes the slope of the individual scalar nonlinearities \( \phi_i \). The slope restriction \( [0, \mu] \) of a single valued, globally Lipschitz (with respect to its input) nonlinearity \( \phi : \mathbb{R} \to \mathbb{R} \) is recalled here following the notion of [35, 36] as

\[
0 \leq \frac{\phi(u) - \phi(v)}{u - v} < \mu \quad \forall u, v \in \mathbb{R}, \; u \neq v, \; 0 < \mu < \infty.
\]

The static post-compensator proposed in Subsection 5.1 is utilised here to stabilise a positive feedback loop comprised of a stable, tall/square, uncertain LTI plant \( G(r,s) \) preceded by a slope-restricted nonlinearity \( \Phi \in \Phi \). The scheme is shown in Fig. 21, where the post-compensated plant \( \tilde{G}(r,s) \) is designed to be SSNI along with satisfying a particular DC-gain condition that depends on the maximum slope \( \mu_i \) of the individual scalar nonlinearities \( \phi_i \). This problem can also be viewed as an absolute stability problem in the Lur’e framework of a non-square, LTI, uncertain system preceded by a slope-restricted nonlinearity, and absolute stability of the positive feedback loop is established by transforming the uncertain plant \( G(r,s) \) into the SSNI class \( \tilde{G}(r,s) \) via a static post-compensator \( (H) \) as shown in Fig. 21.

Theorem 10 proves global asymptotic stability of the scheme assuming wellposedness of the positive feedback loop for any fixed value of the uncertain parameter vector \( \rho = [\rho_1, \rho_2, \ldots, \rho_r]^T \) within a polytope having \( 2^k \) vertices in \( \mathcal{V} \) as defined in (17).

\[
\text{Theorem 10} \quad \text{Let } G(r,s) = C(r)[sI - A(r)]^{-1}B(r) + D(r) \in \mathbb{R}^{m \times r} \text{ be the family of uncertain plants where the parameter vector } \rho \text{ can take any fixed value within a polytope having } 2^k \text{ vertices in } \mathcal{V}, G(r,s) \text{ is preceded by a slope-restricted nonlinearity } \Phi \in \Phi, \text{ defined in (23), as shown in Fig. 21. Let } \text{rank}[B(r)] = r \quad \forall \rho \in \mathcal{V} \text{ where } r \leq l \leq n \text{ and } M = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_r\} > 0 \text{ be a given matrix with the diagonal elements being the slope bounds } \mu_i \text{ of the individual scalar nonlinearities } \phi_i \text{ for all } i \in \{1, 2, \ldots, r\}. \text{ Assume there exist a real symmetric matrix } \tilde{Y} > 0 \text{ and a full-rank matrix } S \in \mathbb{R}^{r \times n} \text{ such that}
\]

\[
\begin{bmatrix}
A_jY + YA_j^T & S^T \\
S & -I
\end{bmatrix} < 0,
\]

as defined in (25a).
for all $j \in \{1, 2, \ldots, 2^k\}$. Let $z = Hx$ with $H = SY^{-1}$ be defined as an auxiliary output of the system. Then the post-compensator $H$ ensures global asymptotic stability of the positive feedback interconnection of the slope-restricted nonlinearity $\Phi \in \Phi$ and the post-compensated system $G(\rho, s)$ as shown in Fig. 21.

**Proof.** We begin this proof by showing that the post-compensated system $G(\rho, s)$ is SSNI upon satisfying the LMI conditions (25a)–(25c) following Theorem 8. Inequality (25a) is equivalent to

$$A_jY + YA_j^T + S^T S < 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}$$

or $Y^{-1}$ $A_jY + YA_j^T Y^{-1} Y + YY^{-1} S^T Y^{-1} Y < 0$ or $Y^{-1} A_j Y + A_j^T Y^{-1} + Y^{-1} S^T Y^{-1}$ $Y < 0$ or $[Y^{-1} A_j + A_j^T Y^{-1} + Y^{-1} S^T Y^{-1}] Y < 0$ or $PA_j + A_j^T P + H^T H < 0$.

on denoting $P = Y^{-1} > 0$ and since $H = SY^{-1}$. This implies that the pair $(A_j, H)$ is observable for all $j$. Similarly, the LMI condition (25b) is equivalent to

$$A_jY + YA_j^T + A_j S^T S A_j^T < 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}$$

or $A_j Y + YA_j^T + (-A_j S^T)(-A_j S^T)^T < 0$ or $A_j Y + YA_j^T + B_j B_j^T < 0$ [via (25c)].

This implies that the pair $(A_j, B_j)$ is controllable for all $j$, (25a) also implies

$$A_j Y + YA_j^T < 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}.$$  

Now the assumption rank$[B(\rho)] = r$ along with the strict inequality condition (26) ensures that $[G_j(s) - G_j(-s)^T]^T$ has full normal rank for all $j$ via Lemma 3. Therefore, the inequalities (26) and (25c) together ensure that the post-compensated system $G_j(s)$ is SSNI for all $j$. Since the state-space matrices $A(\rho), B(\rho), C(\rho)$ and $D(\rho)$ depend affinely on the parameter vector $\rho$ where $\rho$ can take any fixed value within a polytope having $2^k$ vertices in $\theta$, exploiting the notion of [16], it can be guaranteed that the LMI conditions (25a)–(25c) will hold for each member of $G(\rho, s) = C(\rho)[sI - A(\rho)]^{-1} B(\rho) + D(\rho)$. This then confirms that the set of post-compensated systems $\bar{G}(\rho, s)$ will belong to the SSNI class $\forall \rho \in \theta$. Finally, (25d) is equivalent to the condition

$$M(0)^{-1} \bar{G}_j(0) > 0 \quad \forall j \in \{1, 2, \ldots, 2^k\}.$$  

Now, exploiting the fact that $\bar{G}_j(0) = HYH^T = SY^{-1}S = \bar{G}(\rho, 0)$ does not depend on $\rho$, it can be readily asserted that (27) is equivalent to

$$M(0)^{-1} - \bar{G}(\rho, 0) > 0,$$

which then guarantees via [10, Theorem 9] that the positive feedback interconnection of the slope-restricted nonlinearity $\Phi \in \Phi$ defined in (23) and the post-compensated system $\bar{G}(\rho, s) = H[sI - A(\rho)]^{-1} B(\rho)$ being SSNI with $\bar{G}(\rho, 0) < M(0)^{-1}$ is globally asymptotically stable for any fixed value of the uncertain parameter vector $\rho$ within the polytope. In other words, the static post-compensator $H$ transforms the family of LTI uncertain plants $G(\rho, s)$ into a family of SSNI systems $\bar{G}(\rho, s)$ and asymptotically stabilizes the positive feedback loop containing $\Phi \in \Phi$ and $\bar{G}(\rho, s)$ shown in Fig. 21 subject to any initial condition and any fixed value of $\rho$ within the defined polytope.

The aforementioned result may find potential applications in controlling nonlinear systems that can be modelled as an LTI subsystem preceded by a slope-restricted nonlinearity such as saturation, deadzone, etc. Moreover, the proposed scheme shown in Fig. 21 can be readily extended to design a constant reference tracking scheme for stable NI systems with actuator saturation exploiting the ideas presented in [11].

### 6 Conclusion

To expand the scope of NI systems theory to a more general class of LTI systems, in this paper, we propose LMI-based static pre- and post-compensator design techniques to transform non-square or non-NI square LTI systems into the SSNI class. While the pre-compensator is designed for ‘fat’ plants, the post-compensator guarantees a feasible solution for any stable ‘tall’ or square plant with a full-rank $B$ matrix. The proposed compensators also lead to a fault-tolerant constant reference tracking scheme for stable non-square or non-NI square plants by exploiting the IC and DIC properties of SSNI systems. Since full-state feedback may not always be available, observer-based pre- and post-compensators are developed to remove the explicit dependence on state feedback. The pre- and post-compensator design methodologies are then extended to non-square LTI systems with polytopic uncertainty in an affine form. Interestingly, the post-compensator can also be used to ensure global asymptotic stability of a positive feedback loop containing an uncertain, ‘tall’ or square LTI plant preceded by a slope-restricted nonlinearity. The same conclusion can be drawn in the scenario where a stable
‘fat’ plant having polytopic-uncertainty is connected to a similar slope-restricted nonlinearity.

Acknowledgement

The authors express their sincere thanks to the anonymous reviewers for giving constructive technical comments and valuable suggestions that have really helped us to improve the quality and clarity of the paper.

References


