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The Scaling of Nonlinear Structural Dynamic Systems

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ABSTRACT

A new scaling theory has appeared in the recent literature that has the potential to transform current approaches to scaled experimentation. The new theory introduces new similitude rules that hitherto did not exist and significantly extends the classical definition of similitude underpinned by dimensional analysis. Each new similitude rule is tied to the number of scaled experiments, and in theoretical terms there is no limit to the number of scaled experiments involved.

The focus of this paper is on one and two scaled experiments applied to nonlinear structural dynamics, which is a field of study and application that gives rise to significant difficulties for scaled experimentation. The highly nonlinear behaviour common to these systems means that only very precise scaled-model designs can feasibly achieve acceptable outcomes. It is shown in the paper how the new exact similitude rules provided by the new theory deliver the precision necessary for mathematically exact replication of behaviours. It is demonstrated further how the limited scope provided by a single scaled experiment can be significantly extended by application of two properly designed scaled experiments. Through the analysis of carefully selected structural systems of one and two degrees of freedom involving nonlinear springs, dashpots and friction, the benefits of two scaled experiments are demonstrable for a range of loading conditions. Exact similitude for two scaled experiments is confirmed providing exact replication of behaviours with conformity recorded over long timescales.

Keywords: scaling, finite similitude, trial experimentation, nonlinear dynamic systems.
1. INTRODUCTION

Modern structural dynamics makes use of a range of analysis and investigative approaches which in broad terms can be classified as analytical, numerical, experimental, along with combinations of the three. The interminable need to achieve greater accuracy and deal with increased complexity and sophistication of structural systems necessitates the need for new improved investigative and analysis approaches. A particular convenient type of investigation, where accuracy can readily be assured, is those performed under laboratory conditions. Rather unfortunately, however, laboratory experiments have limitations, and one concern is the constraint imposed by size, where for structural dynamics in particular, large structures are the norm. Dealing with large experimental structures can be time consuming and expensive when contrasted against direct analytical and numerical investigations. For complex systems however an overreliance on these direct approaches can itself be problematic and not recommended as uncertainties can make model predictions questionable and invariably some form of experimental support is often required [1–3].

In principle, the solution to the size constraint faced by laboratory experimentation is scaling, where structures are scaled to a fraction of their original size for testing purposes. This is the approach investigated in this paper, but the downside of scaled experimentation is well understood by academic and industrial communities. The issue is one of scale effects [4,5], where the behaviour of the scaled model fails to be representative of the full-scale system. This is certainly the situation prior to the arrival of finite similitude [6,7] but has the arrival of this theory significantly changed the situation? Questions about the usefulness of the new similitude identities remain and an objective of this work is to discover whether nonlinear systems can be assessed with this new approach. It is recognised of course that it is only through the application of the new similitude rules to realistic systems can the benefits be truly quantified.

Over the many decades, following on from the inspirational work of Buckingham [8] and Rayleigh [9], techniques have been developed to quantify the conditions needed to representatively scale down and up systems. These techniques are termed similitude methods, which facilitate (where possible) the accurate representation of a system by preserving dynamic behaviour with the help of scaling laws [2,10–12]. Most recent approaches for the analysis of mechanically based dynamic systems tend to rely heavily on computer-based simulation, as computational models can readily be applied over a range of scales [13,14]. Modern numerical approaches are advancing at pace,
with robust approaches for analysis, with the ability to scrutinise the entire behaviour of dynamic systems, in response to changes in boundary conditions, properties, system properties, and so forth [15–17]. Despite the advances being made however, simplifications are required to make practicable complex models, and this can be achieved with discrete-element representations that can be employed effectively in investigative studies. It is recognised of course that simplified models require experimental validation to justify the many simplifying assumptions involved in practical modelling [11]. Discrete systems find commonplace usage in the study of structural engineering systems and take the appearance of a network of lumped masses, springs, and dashpots to capture complex dynamic and dissipative behaviours.

Spring-mass-damper discrete systems find common usage but an area of interest, where large-scale systems are involved, is earthquake mechanics [18–21]. Examining building behaviours subjected to seismic excitations is the focus here, where amongst the many possible options, dampers are often employed as part of a passive control system. One of the most commonly used energy dissipation devices in building and building related structures are fluid viscous dampers (FVDs) [22]. The reason behind their widespread adoption is due to their capacity to significantly dampen accelerations [23–27], and consequently increase the seismic operation of the many non-structural parts. Although the operating principles of viscous dampers are relatively straightforward, the design and fabrication of modern FVDs is a multidisciplinary task, requiring considerable expertise in a variety of fields of science [28]. The advantage of nonlinear force-velocity relationships is the focus of much research for FVDs [29–32], since best designs can limit the peak damper force at high structural velocities, but at the same time provide adequate supplementary damping. Understanding the details is very much needed in scaling to gauge the performance and behaviour of these in scaled experiments. In addition to dampers, the other significant energy dissipation mechanisms common to structures is friction and wear in the area where two bodies come into contact. The prevalence of friction in technology and everyday life justifies the extensive scientific research directed at it. A thorough understanding of friction [33] is a necessity, and the particular behaviour of interest here is that arising from stick-slip systems [34]. The advantage of discrete systems in this regard is that dominant physics (such as friction) can be captured in an efficient manner but as mentioned above experimental evidence is needed to justify and support what might appear to be controversial simplifications [2,35]. With such discrete approaches, the fundamental nonlinear aspects can be captured, and consequently is it of interest to understand their response
under scaling. Understanding the scaled nonlinear behaviour of the individual elements (e.g., FVDs, springs, contacting surfaces) is critical, but also in their entirety in relation to the systems under consideration.

The importance of the theory dimensional analysis in scaling is well recognised as it aids in the design of scaled experiments. The theory is founded on a well-known invariance principle, which basically states that dimensionless governing equations remain invariant with scale [36]. This statement confers importance on dimensionless forms, which have distinct advantages. In particular, they can be used to characterise dominant physics by means of a dominant subset of dimensionless parameters (i.e., the Pi groups) [8,36,37]. The biggest issue with dimensional analysis in scaling is connected to the underpinning invariance principle itself, as this is rarely applicable to all but the most basic of systems [11,38]. In the presence of scale effects, by definition, this invariance principle breaks down, and the importance of dimensional analysis and scaled experimentation is undoubtedly diminished by this [39].

To overcome these shortcomings with dimensional analysis a new theory has recently emerged called finite similitude [6,40–46]. This approach is not underpinned by dimensionless forms but assumes that scaling can be viewed as an imagined process in which space itself is contracted or expanded. The process is “metaphysical” in the sense that it cannot be achieved physically but nevertheless can be imagined and defined in very precise mathematical terms. Central to the new approach is the projection of the governing physics, described on a scaled space, onto the original full-scale space. This projection has the effect of exposing all possible scale dependencies (either explicitly or implicitly), transforming the scaling problem into one where the objective is to reveal hidden scale dependencies. The revealing of hidden dependencies is efficiently achieved with the application of scale invariances and unlike dimensional analysis there exists more than one invariance. There exists in fact a countable infinite number of scale invariances but with each invariance linked to the number of scaled experiments involved there exists practical limits on the approach.

A novel aspect of the study presented in this paper is the examination of the application of the new approach to nonlinear structural systems. The focus on discrete element modelling is particularly advantageous to the finite similitude theory as it provides a means to set free variables that exist with the theory. It is very important to appreciate that similitude rules do not constrain behaviours
in the scaled experiments but provide the means to connect experiments across the scales. It is necessary therefore to have a means to examine the systems involved prior to applying the similitude rules to the physical experiments. Discrete representations are shown to provide a highly efficient approach to achieve this and allow for the initial exploration of the benefits of similitude in any experimental study.

The finite similitude theory although explained elsewhere [6,47] is re-examined in brief in Sec. 2 for the sake of readability, but also to bring into focus the limitations of the theory. It is important to appreciate that scale dependencies as previously defined under dimensional analysis can cease to be scale dependencies under the new theory. Linear dynamic systems are briefly assessed in Sec. 3 to provide an illustration how the scaling theory is applied to confirm that replica scaling with a single scaled experiment is not representative. Examined in Sec. 4 are aspects relating to the scaling of nonlinear FVDs, which are critical items in structures where passive control is a requirement. Friction is the focus of Sec. 5 and on the stick-slip phenomenon and its analysis through scaled experimentation. Nonlinear springs are introduced in Sec. 6, where it is revealed just how flexible the invariance founded on two scaled experiments can be. Overall, the paper demonstrates that the new similitude rule can offer practical experimental solutions to problems that cannot be tackled by one scaled experiment. The paper ends with a set of conclusions.

2. **FINITE SIMILITUDE IN BRIEF**

The finite similitude theory brings together several concepts to provide a generic scaling theory that in principle can be applied to all physics. The starting point is space scaling and the idea that space itself can be distorted, and that through space distortion objects can be scaled. Such objects might include such things as cities, buildings, laboratories, machinery, experimental rigs down to individual components and specimens. Once space scaling is quantified mathematically, attention turns to the effect such a process has on the underpinning physics of interest. The theory is a little unusual in this respect in that it requires that transport equations in their integral form [48] are considered. The reason for this is that this form immediately involves geometrical measures (e.g., volume and area), which are first and foremost central to any scaling effect that takes place. Point based formulations founded on partial or ordinary differential equations do not provide this feature and consequently are not suitable. Ultimately however the theory does provide point-based identities although imperceptibly connecting spatial points in scaled and unscaled spaces. The
approach necessitates a very precise description of control volume movement and the formulation presented below follows the approach first formulated in reference [49]. With transport forms defined on a scaled space a critically important projection to the physical full-scale space is considered. This projection is key to the whole concept as it provides the means to describe all scale dependencies on the physical space. Scale invariances can then be applied and integrated to link scaled experiments, whose number depends on the invariance chosen. The whole approach is exact involving no approximations and provides new similitude rules for experimental design.

2.1. A brief recap on space scaling

As mentioned above the starting point of the finite similitude theory is the concept of space scaling, which happens to be a physically intuitive approach. The structural system of interest is tethered to the space it resides in, in the sense that it is immediately affected, i.e., is contracted/expanded by the contraction/expansion of space. The starting point of any analysis in structural dynamics is the stipulation of suitable inertial frames for both the physical and trial spaces. The full-scale system of interest sits in the physical space with the scaled experiment residing in the trial space. With subscripts “ts” and “ps” denoting trial and physical spaces, respectively the assumed orthogonal inertial coordinate systems are labelled by $x_{ts}$ and $x_{ps}$. With Newtonian physics assumed, two absolute temporal measures for time are introduced in each space and are labelled by $t_{ps}$ and $t_{ts}$, and are related by the differential relationship $dt_{ts} = gdt_{ps}$, where $g$ is a positive parameter. Space scaling is easy to define mathematically and for isotropic scaling and is defined by a temporally invariant affine map $x_{ps} \mapsto x_{ts}$, which in differential terms takes the form $dx_{ts} = \beta dx_{ps}$ (i.e. $dx_{ts} \approx \beta dx_{ps}$), where $\beta$ is a positive parameter. Space contraction, which tends to be of principle interest is provided by $0 < \beta < 1$, with no scaling if $\beta = 1$ and expansion for $1 < \beta$. Note from references [7,44,50,51] that the theory of finite similitude is underpinned by physics described on synchronised moving controls. Depicted in Fig. 1 is the motion of control volume $\Omega_{ts}$ in the trial-space described mathematically by a velocity field $v_{ts}$. All motion is quantified with reference to something else and in this case the motion is relative to a reference control volume $\Omega_{ts}^{ref}$ (the set of coordinate points $x_{ts}$). The synchronised motion of the control volumes $\Omega_{ts}$ and $\Omega_{ps}$ in the trial and physical spaces are depicted in Fig. 1, where coordinate point $x_{ts}$ moves with $\Omega_{ts}$ with
velocity $v^*_ts$ and coordinate point $x^*_ps$ moves with $\Omega^*_ps$ with velocity $v^*_ps$. The control volumes (being regions of space) are affected by space scaling and consequently are related by the map $dx^*_ts = \beta dx^*_ps$ and since $dt^*_ts = gd^*_ps$, the field-velocity relationship $v^*_ts = g^{-1} \beta v^*_ps$ applies.

![Diagram](image)

Figure 1: 2D schematic of synchronous control volumes $\Omega^*_ts$ and $\Omega^*_ps$ moving while containing a moving dashpot.

2.2. Projected structural dynamics in transport form

Four transport equations are of interest in finite similitude for structural dynamics, which are those for conservation of volume, mass and momentum, and the non-conserved movement equation introduced by Davey and Darvizeh [48],

$$\frac{D^*}{D^*t^*_ts} \int d\Omega^*_ts - \int v^*_ts \cdot n^*_tn^*_ts d\Gamma^*_ts = 0$$

(1a)
\[ \frac{D^*}{D^* t_{ts}} \int_{\Omega_{ts}} \rho_{ts} \, d\Omega^*_{ts} + \int_{\Gamma^*_{ts}} \rho_{ts} \left( \mathbf{v}_{ts} - \mathbf{v}^*_{ts} \right) \cdot \mathbf{n}_{ts} \, d\Gamma^*_{ts} = 0 \]  \hspace{1cm} (1b)

\[ \frac{D^*}{D^* t_{ts}} \int_{\Omega_{ts}} \rho_{ts} \mathbf{v}_{ts} \, d\Omega^*_{ts} + \int_{\Gamma^*_{ts}} \rho_{ts} \mathbf{v}_{ts} \left( \mathbf{v}_{ts} - \mathbf{v}^*_{ts} \right) \cdot \mathbf{n}_{ts} \, d\Gamma^*_{ts} - \int_{\Omega_{ts}} \mathbf{\sigma}_{ts} \cdot \mathbf{n}_{ts} \, d\Gamma^*_{ts} - \int_{\Omega_{ts}} \mathbf{b}_{ts} \, d\Omega^*_{ts} = 0 \]  \hspace{1cm} (1c)

\[ \frac{D^*}{D^* t_{ts}} \int_{\Omega_{ts}} \rho_{ts} \mathbf{u}_{ts} \, d\Omega^*_{ts} + \int_{\Gamma^*_{ts}} \rho_{ts} \mathbf{u}_{ts} \left( \mathbf{v}_{ts} - \mathbf{v}^*_{ts} \right) \cdot \mathbf{n}_{ts} \, d\Gamma^*_{ts} - \int_{\Omega_{ts}} \rho_{ts} \mathbf{v}_{ts} \, d\Omega^*_{ts} = 0 \]  \hspace{1cm} (1d)

where \( \rho_{ts} \) is mass density, \( \mathbf{u}_{ts} \) is material displacement, \( \mathbf{v}_{ts} \) is material velocity, \( \mathbf{\sigma}_{ts} \) is the Cauchy stress tensor, \( \mathbf{b}_{ts} \) is a body force (force per unit mass), and \( \mathbf{n}_{ts} \) is an outward pointing unit normal on boundary \( \Gamma^*_{ts} \) of the control volume \( \Omega^*_{ts} \).

Note that the temporal derivative \( \frac{D^*}{D^* t_{ts}} \) in Eq. (1) signifies that the control volume \( \Omega^*_{ts} \) is moving relative to \( \Omega^*_{ts}^{ref} \) as depicted in Fig. 1, and this explains the presence of the velocity field \( \mathbf{v}^*_{ts} \) appearing in these equations. The most significant equation is Eq. (1c), and for most practical problems in structural mechanics, this equation can be sufficient. However, other considerations necessitate the inclusion of additional equations with finite similitude. Eq. (1a) is somewhat unusual and never features in structural dynamics because it has no field associated with it but is considered here nevertheless to enforce the synchronous velocity field relationship \( \mathbf{v}^*_{ts} = g^{-1} \beta \mathbf{v}^*_{ps} \).

Similarly, Eq. (1b), the continuity equation, has little role to play in most practical structural problems as density is invariably set to a constant but with finite similitude and physical modelling there exists the possibility that materials could be changed in the scaled models and some account must be made to accommodate this possibility. The equation for non-conserved movement, Eq. (1d), was first introduced by Davey and Darvizeh in reference [48], and has the effect of making displacement explicit in solid-mechanics type analysis. It is important to appreciate that there is no barrier to including more equations (see reference [52] for inclusion of a transport equation for energy) as the physics dictates in the problems under study.

The most important transformation critical to the whole approach can now be applied with the projection of Eqs. (1) onto the physical space. This has the effect of exposing all scale dependencies and involves the substitution of \( d\Omega_{ps} = \beta^3 d\Omega_{ts}^* \), \( \mathbf{n}_{ps} d\Gamma_{ps}^* = \beta^2 \mathbf{n}_{ts} d\Gamma_{ts}^* \), \( dt_{ps} = g dt_{ts} \) into Eqs. (1). Additionally each equation is multiplied throughout by non-zero scaling parameters.
\( \alpha_0^1, \alpha_0^\sigma, \alpha_0^\rho \) and \( \alpha_0^v \), respectively (whose role will be made clear) and time scalar \( g \). This procedure produces the following equations:

\[
\alpha_0^1 \Gamma_0^1 (\beta) = \frac{D^*}{D^* t^* \Omega_p^*} \int_{r_p^*} \alpha_0^1 \beta^3 d\Omega_p^* - \int_{r_p^*} \alpha_0^1 \beta^3 v_p^* \cdot n_p^* d\Gamma_p^* = 0 \tag{2a}
\]

\[
\alpha_0^\sigma \Gamma_0^\sigma (\beta) = \frac{D^*}{D^* t^* \Omega_p^*} \int_{r_p^*} \alpha_0^\sigma \beta^3 \rho_n d\Omega_p^* + \int_{r_p^*} \alpha_0^\sigma \beta^3 \rho_n (V_p^* - v_p^*) \cdot n_p^* d\Gamma_p^* = 0 \tag{2b}
\]

\[
\alpha_0^\rho \Gamma_0^\rho (\beta) = \frac{D^*}{D^* t^* \Omega_p^*} \int_{r_p^*} (\alpha_0^\rho g^{-2} \beta) \beta^3 \rho_n V_p^* d\Omega_p^* + \int_{r_p^*} (\alpha_0^\rho g^{-2} \beta) \beta^3 \rho_n (V_p^* - v_p^*) \cdot n_p^* d\Gamma_p^*
\]

\[-\int_{r_p^*} \Sigma_p \cdot n_p^* d\Gamma_p^* - \int_{r_p^*} B_{ts} d\Omega_p^* = 0 \tag{2c}
\]

\[
\alpha_0^v \Gamma_0^v (\beta) = \frac{D^*}{D^* t^* \Omega_p^*} \int_{r_p^*} (\alpha_0^v \beta) \beta^3 \rho_n U_p^* d\Omega_p^* + \int_{r_p^*} (\alpha_0^v \beta) \beta^3 \rho_n (V_p^* - v_p^*) \cdot n_p^* d\Gamma_p^*
\]

\[-\int_{r_p^*} (\alpha_0^v \beta) \beta^3 \rho_n V_p^* d\Omega_p^* = 0 \tag{2d}
\]

where \( V_p^* = g^{-1} v_p^* \), \( U_p^* = g^{-1} u_p^* \), \( \Sigma_p = \alpha_0^v g \beta^2 \sigma_p \), and \( B_{ts} = \alpha_0^v g \beta^3 \rho_n b_p^* \).

The significance of these equations is that they expose all possible scale dependencies that can feature in scaled structural dynamics. Note the explicit exposure of geometrical measures with the appearance of \( \beta^3 \) and \( \beta^2 \), but also exposed are other hidden dependencies with the fields \( V_p^* (\beta) \), \( U_p^* (\beta) \), \( \Sigma_p (\beta) \) and \( B_{ts} (\beta) \) being dependent on \( \beta \). The ability to relate fields and measures to \( \beta \) on the physical space means that the scaling problem has effectively been transformed into one where discovering the behaviour of hidden-field dependencies is the principal focus. In this way the finite similitude theory embraces the presence of scale effects as opposed to ignoring them as in the theory of dimensional analysis (see refs. [6,7,41,44]). The revealing of hidden dependencies can be readily achieved by the application of scale invariances, which are similitude rules that connect information across scaled experiments.

2.3. Scale invariances
Eqs. (2) are of the form \( \alpha_0^\psi T_0^\psi = 0 \), with \( \psi \) set to 1, \( \rho \), \( v \) and \( \mu \), and where \( \alpha_0^\psi \) are scalars that are functions of \( \beta \). These scalars play a critical role and facilitate the unified application of invariances to all the transport equations, which would not be possible without their presence as is shown below. The simplest scale invariance possible is to assume that all the transport equations \( \alpha_0^\psi T_0^\psi (\beta) = 0 \) do not in fact depend on \( \beta \), which of course is unlikely in practice. However, this assumption is readily captured by the identity,

\[
\frac{d}{d\beta} (\alpha_0^\psi T_0^\psi) \equiv 0
\]

where the equality sign “\( \equiv \)” means identically zero and infers that the transport equations vanish under the derivative.

The application details surrounding this identity termed zeroth-order finite similitude can be found in references \([7,41,44]\). The scaling parameters \( \alpha_0^\psi \) are being set to eliminate \( \beta \) from \( \alpha_0^\psi T_0^\psi (\beta) = 0 \) to satisfy Eq. (3). In the volume transport equation Eq. (2a) for example this is achieved with \( \alpha_0^1 = \beta^{-3} \) and note that \( \alpha_0^1 (1) = 1 \), which is a requirement imposed on all the scalars \( \alpha_0^\psi \) to ensure that \( \alpha_0^\psi T_0^\psi (1) = T_{ps}^\psi \). The field identities arising out of this Eqs. (2) are presented in Table 1 but also \([6,41]\), so are not considered further here.

**Table 1.** Zeroth-order relationships arising out of Eq. (3).

<table>
<thead>
<tr>
<th>Quantity/ Equations</th>
<th>Scalar identities</th>
<th>Field relationships</th>
<th>Transfer terms</th>
<th>Source terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume Eq.(2a)</td>
<td>( \alpha_0^1 = \beta^{-3} )</td>
<td>( v_{ps}^* = g \beta^{-1} v_{ss}^* )</td>
<td>( \rho_{ps} = \alpha_0^\psi \beta^3 \rho_{is} )</td>
<td>( V_{ps} = v_{ps} = g \beta^{-1} v_{is} )</td>
</tr>
<tr>
<td>Mass Eq.(2b)</td>
<td></td>
<td></td>
<td>( \rho_{ps} = \alpha_0^\psi \beta^3 \rho_{is} )</td>
<td>( V_{ps} = v_{ps} = g \beta^{-1} v_{is} )</td>
</tr>
<tr>
<td>Momentum Eq.(2c)</td>
<td>( \alpha_0^\psi = g \beta^{-1} \alpha_0^\rho )</td>
<td>( V_{ps} = v_{ps} = g \beta^{-1} v_{is} )</td>
<td>( \Sigma_{ps} = \sigma_{ps} = \alpha_0^\psi g \beta^3 \sigma_{is} )</td>
<td>( B_{ps} = \rho_{ps} b_{ps} = \alpha_0^\psi g \beta^3 \rho_{is} b_{is} )</td>
</tr>
<tr>
<td>Movement Eq.(2d)</td>
<td>( \alpha_0^\psi = \beta^{-1} \alpha_0^c )</td>
<td>( U_{ps} = u_{ps} = \beta^{-1} u_{is} )</td>
<td>( V_{ps} = v_{ps} = g \beta^{-1} v_{is} )</td>
<td></td>
</tr>
</tbody>
</table>
The following definition provides a recursive relationship that facilitates the creation of higher forms of similitude rules:

2.1. Definition (High-order finite similitude)

The finite similitude rule of $k^{th}$ order is identified with the lowest order derivative that satisfies,

$$T_{k+1} = \frac{d}{d\beta} \left( \alpha_k' T_k \right) \equiv 0$$

(4)

for all $\beta > 0$, and where $\alpha_0' T_0$ is defined by Eqs. (2) and the scalars $\alpha_k'$ are functions of $\beta$ with $\alpha_1'(1) = 1$, and where the symbol “$\equiv$” signifies identically zero in Eq. (4).

The motivation for definition 2.1 is the expectation that higher derivatives are involved in similitude rules involving more than one scaled experiment along with the requirement that lower order rules are contained in higher-order forms. If Eq. (3) is not satisfied, then consideration is given to the identity,

$$\alpha_1' T_1 = \alpha_1' \frac{d}{d\beta} \left( \alpha_0' T_0 \right)$$

(5)

where $\alpha_1'(1) = 1$ and similarly in accordance with Eq. (4) the similitude condition

$$T_2 = \frac{d}{d\beta} \left( \alpha_1' T_1 \right) = \frac{d}{d\beta} \left( \alpha_1' \frac{d}{d\beta} \left( \alpha_0' T_0 \right) \right) \equiv 0$$

(6)

provides the identity for first-order finite similitude involving not one but two derivatives.

Since the similitude rules are nested it is possible to take forward the zeroth-order identities

$$\rho_{ps} = \alpha_0' \rho_{st}, \quad \alpha_0' = g \beta^{-1} \alpha_0'' \quad \text{and} \quad \alpha_0'' = \beta^{-1} \alpha_0'''$$

and substitute them in Eqs. (3) to obtain

$$\alpha_0' T_0' (\beta) = \frac{D^*}{D t_{ps}} \int \rho_{ps} d\Omega_{ps}^{s} + \int_{\gamma_{ps}} \rho_{ps} (V_{ps} - v_{ps}^*) \cdot n_{ps} d\Gamma_{ps}^{s} = 0 \quad (7a)$$

$$\alpha_0' T_0' (\beta) = \frac{D^*}{D t_{ps}} \int \rho_{ps} d\Omega_{ps}^{s} + \int_{\gamma_{ps}} \rho_{ps} V_{ps} (v_{ps} - v_{ps}^*) \cdot n_{ps} d\Gamma_{ps}^{s}$$

$$- \int_{\gamma_{ps}} \Sigma_{st} \cdot n_{ps} d\Gamma_{ps}^{s} - \int_{\Omega_{ps}} B_{st} d\Omega_{ps}^{s} = 0 \quad (7b)$$
\[
\alpha_0^\nu T_0^\nu (\beta) = \frac{D^s}{D^s \Omega^s_{ps}} \int_{\Omega^s_{ps}} \rho_{ps} U_{\nu s} d\Omega^s_{ps} + \int_{\Gamma^s_{ps}} \rho_{ps} U_{\nu s} (v_{ps} - v^*_s) \cdot n_{ps} d\Gamma^s_{ps} - \int_{\Omega^s_{ps}} \rho_{ps} V_{\nu s} d\Omega^s_{ps} = 0
\]  
(7c)

where \( \Sigma_{ps} = \alpha_0^\nu g \beta^2 \sigma_{ps}, \) \( B_{is} = \alpha_0^\nu g \beta^3 \rho_{is} \) \( b_{is} = g^2 \beta^{-1} b_{is} \) and observe that Eq. (2a) satisfies Eq. (4), and therefore plays no part in first-order theory.

All the theory presented thus far is exact apart from the substitution of the zeroth-order term \( (v_{ps} - v^*_s) \cdot n_{ps} \) in place of \( (V_{ps} - v^*_s) \cdot n_{ps} \) in equations Eq. (7b) and Eq. (7c). This reflects the fact that the convective term \( V_{ps} (V_{ps} \cdot n_{ps}) \) is typically negligible in solid mechanics but also conveniently sidesteps the need to higher forms of similitude.

### 2.4. First-order fields

Exact integration of Eq. (6) can be readily achieved using divided differences with the application of a mean-value theorem, which provides:

\[
\alpha_i^\nu T_i^\nu (\beta_2) = \alpha_i^\nu (\beta_2) \frac{\alpha_0^\nu T_0^\nu (\beta_1) - \alpha_0^\nu T_0^\nu (\beta_2)}{\beta_1 - \beta_2} \quad (8a)
\]

\[
\alpha_i^\nu T_i^\nu (\beta_0) = \alpha_i^\nu (\beta_0) \frac{\alpha_0^\nu T_0^\nu (\beta_0) - \alpha_0^\nu T_0^\nu (\beta_1)}{\beta_0 - \beta_1} \quad (8b)
\]

where \( \beta_2 \leq \beta_2' \leq \beta_1 \) and \( \beta_1 \leq \beta_1' \leq \beta_0 \), and where \( \beta_2 \) and \( \beta_1 \) are the scales for the trial-space experiments with \( \beta_0 = 1 \) signifying the physical space (i.e., full scale experimentation).

Note that the direct integration of Eq. (6) between the limits \( \beta_2' \) and \( \beta_1' \) provides the identity

\[
\alpha_i^\nu T_i^\nu (\beta_0) = \alpha_i^\nu T_i^\nu (\beta_2')
\]

and consequently on substitution of Eqs. (8) gives rise to the key first-order similitude rule for scaled transport equation, which is

\[
\alpha_0^\nu T_0^\nu (\beta_0) = \alpha_0^\nu T_0^\nu (\beta_1) + R_i^\nu \left( \alpha_0^\nu T_0^\nu (\beta_1) - \alpha_0^\nu T_0^\nu (\beta_2) \right)
\]  
(9)

where

\[
R_i^\nu = \left( \frac{\alpha_i^\nu (\beta_2')}{\alpha_i^\nu (\beta_1')} \right) \frac{\beta_0 - \beta_1}{\beta_1 - \beta_2}
\]  
(10)
which leads immediately to the field relationships listed in Table 2, but note that \( R_i^u \) is in the form of a parameter due to indeterminacy of \( \alpha_i^u \) and consistent velocity expressions necessitates that \( R_i = R_i^\rho = R_i^v = R_i^\alpha \) imposed by \( \alpha_i^\rho = \alpha_i^v = \alpha_i^\alpha \).

Table 2. Zeroth and first order field relationships

<table>
<thead>
<tr>
<th>Quantity/Equations</th>
<th>Zeroth order field relationships</th>
<th>First order field relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass Eq.(7a)</td>
<td>[ \rho_{ps} = \left( \alpha_0^\rho \beta_1 \rho_{\alpha} \right) (\beta_1) ] [ V_{ps} = v_{ps} = \left( \rho_{\beta}^{-1} v_{ts} \right) (\beta_1) ]</td>
<td>[ v_{ps} = V_{ps} (\beta_1) + R_i^\rho \left( V_{ps} (\beta_1) - V_{ps} (\beta_2) \right) ]</td>
</tr>
<tr>
<td>Momentum Eq.(7b)</td>
<td>[ \Sigma_{ps} (\beta_1) = \sigma_{ps} = \left( \alpha_0^\rho \beta_1 \rho_{\sigma} \right) (\beta_1) ] [ V_{ps} (\beta_1) = v_{ps} = \left( \rho_{\beta}^{-1} v_{ts} \right) (\beta_1) ]</td>
<td>[ \sigma_{ps} = \Sigma_{ps} (\beta_1) + R_i^\rho \left( \Sigma_{ps} (\beta_1) - \Sigma_{ps} (\beta_2) \right) ] [ b_{ps} = B_{ps} (\beta_1) + R_i^\rho \left( B_{ps} (\beta_1) - B_{ps} (\beta_2) \right) ] [ v_{ps} = V_{ps} (\beta_1) + R_i^\rho \left( V_{ps} (\beta_1) - V_{ps} (\beta_2) \right) ]</td>
</tr>
<tr>
<td>Movement Eq.(7c)</td>
<td>[ U_{ps} (\beta_1) = u_{ps} = \left( \beta_1^{-1} u_{ts} \right) (\beta_1) ] [ V_{ps} (\beta_1) = v_{ps} = \left( \rho_{\beta}^{-1} v_{ts} \right) (\beta_1) ]</td>
<td>[ u_{ps} = U_{ps} (\beta_1) + R_i^\rho \left( U_{ps} (\beta_1) - U_{ps} (\beta_2) \right) ] [ v_{ps} = V_{ps} (\beta_1) + R_i^\rho \left( V_{ps} (\beta_1) - V_{ps} (\beta_2) \right) ]</td>
</tr>
</tbody>
</table>

A feature of the finite similitude approach is that constitutive laws do not feature in its formulation since all the fields required for the physical space can be obtained from existing fields. Shown in Table 3 are typical fields of interest in structural dynamics and note how relationships for both stress and strain exist for small strain theory. Constitutive laws can play a role in the setting of free parameters \( g_1, g_2 \) and \( R_i \) as required, but in order to apply the new theory to structural dynamics some understanding of the nonlinearities involved is required.

Table 3. Zeroth and first order field relationships for practical use.

<table>
<thead>
<tr>
<th>Physical quantity</th>
<th>Zeroth order field relationships</th>
<th>First order field relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>[ \rho_{ps} = \alpha_0^\rho \beta_1 \rho_{\rho_{\rho_1}} ]</td>
<td>[ v_{ps} = g_1 \beta_1^{-1} v_{\rho_{\rho_1}} + R_i^\rho \left( g_1 \beta_1^{-1} v_{\rho_{\rho_1}} - g_2 \beta_2^{-1} v_{\rho_{\rho_2}} \right) ]</td>
</tr>
<tr>
<td>Velocity</td>
<td>[ v_{ps} = g_1 \beta_1^{-1} v_{\rho_{\rho_1}} ]</td>
<td>[ v_{ps} = g_1 \beta_1^{-1} v_{\rho_{\rho_1}} + R_i^\rho \left( g_1 \beta_1^{-1} v_{\rho_{\rho_1}} - g_2 \beta_2^{-1} v_{\rho_{\rho_2}} \right) ]</td>
</tr>
<tr>
<td>Displacement</td>
<td>$u_{ps} = \beta_1^{-1}u_{ts1}$</td>
<td>$u_{ps} = \beta_1^{-1}u_{ts1} + R_1 \left( \beta_1^{-1}u_{ts1} - \beta_2^{-1}u_{ts2} \right)$</td>
</tr>
<tr>
<td>--------------</td>
<td>----------------</td>
<td>-----------------------------------------------------------------</td>
</tr>
<tr>
<td>Acceleration</td>
<td>$a_{ps} = g_1^2\beta_1^{-1}a_{ts1}$</td>
<td>$a_{ps} = g_1^2\beta_1^{-1}a_{ts1} + R_1 \left( g_1^2\beta_1^{-1}a_{ts1} - g_2^2\beta_2^{-1}a_{ts2} \right)$</td>
</tr>
<tr>
<td>Small strain</td>
<td>$\varepsilon_{ps} = \varepsilon_{ts1}$</td>
<td>$\varepsilon_{ps} = \varepsilon_{ts1} + R_1 \left( \varepsilon_{ts1} - \varepsilon_{ts2} \right)$</td>
</tr>
<tr>
<td>Strain rate</td>
<td>$\dot{\varepsilon}<em>{ps} = g_1\dot{\varepsilon}</em>{ts1}$</td>
<td>$\dot{\varepsilon}<em>{ps} = g_1\dot{\varepsilon}</em>{ts1} + R_1 \left( g_1\dot{\varepsilon}<em>{ts1} - g_2\dot{\varepsilon}</em>{ts2} \right)$</td>
</tr>
<tr>
<td>Stress</td>
<td>$\sigma_{ps} = \alpha_0^\psi g_1^2\sigma_{ts1}$</td>
<td>$\sigma_{ps} = \alpha_0^\psi g_1^2\sigma_{ts1} + R_1 \left( \alpha_0^\psi g_1^2\sigma_{ts1} - \alpha_0^\phi g_2^2\beta_2\sigma_{ts2} \right)$</td>
</tr>
<tr>
<td>Force</td>
<td>$F_{ps} = \alpha_0^\psi g_1F_{ts1}$</td>
<td>$F_{ps} = \alpha_0^\psi g_1F_{ts1} + R_1 \left( \alpha_0^\psi g_1F_{ts1} - \alpha_0^\phi g_2F_{ts2} \right)$</td>
</tr>
</tbody>
</table>

3. **LINEAR DYNAMIC SYSTEMS**

Prior to the analysis of nonlinear systems, it is useful to examine in brief, a linear mass-spring-damper system [36]. Consider then the “forces” $F_{is}^i = m_{is}a_{is}$, $F_{is}^d = -c_{is}v_{is}$ and $F_{is}^s = -k_{is}u_{is}$ in a series free-vibration arrangement satisfying the equation $F_{is}^i = F_{is}^d + F_{is}^s$, where in the trial space, $m_{is}$ is mass, $c_{is}$ is a damping coefficient and $k_{is}$ is a spring stiffness. To understand how this equation scales for a single scaled experiment it is necessary examine the zeroth-order relationship $F_{ps} = \alpha_0^\psi g F_{ts}$ in Table 3. The inertial relationship $F_{is}^i = m_{is}a_{is}$ on application of the identity $F_{ps} = \alpha_0^\psi g F_{ts}$ and substitution of $a_{ps} = g^2\beta^{-1}a_{ts}$ (see Table 3) and $\alpha_0^\psi = g\beta^{-1}\alpha_0^\phi$ (see Table 1) gives,

$$F_{ps}^i = m_{ps}a_{ps} = m_{ps}g^2\beta^{-1}a_{ts} = \alpha_0^\psi g F_{ts}^i = \alpha_0^\psi g m_{ts}a_{ts} = \alpha_0^\phi g^2\beta^{-1}m_{ts}a_{ts}$$

(11)

which necessitates that $\alpha_0^\phi m_{ts} = m_{ps}$.

Similarly, the spring force relationship $F_{is}^s = -k_{is}u_{is}$ on substitution of $u_{ps} = \beta^{-1}u_{ts}$ (see Table 3) gives,

$$F_{ps}^s = -k_{ps}u_{ps} = -k_{ps}\beta^{-1}u_{ts} = \alpha_0^\psi g F_{ts}^s = -\alpha_0^\psi g k_{ts}a_{ts} = -\alpha_0^\phi g^2\beta^{-1}k_{ts}u_{ts}$$

(12)

which requires $\alpha_0^\phi g^2k_{ts} = k_{ps}$ and finally $F_{is}^d = -c_{is}v_{is}$ on substitution of $v_{ps} = g\beta^{-1}v_{ts}$ (see Table 3) gives,

$$F_{ps}^d = -c_{ps}v_{ps} = -c_{ps}g\beta^{-1}v_{ts} = \alpha_0^\psi g F_{ts}^d = -\alpha_0^\psi g c_{ts}v_{ts} = -\alpha_0^\phi g^2\beta^{-1}c_{ts}v_{ts}$$

(13)
and the requirement $\alpha_0^p g c_{ts} = c_{ps}$.

In the situation where identical material used in the trial and physical models, then the zeroth-order relationship $\rho_{ps} = \alpha_0^p \beta^3 \rho_{ts}$ (see Table 2) with $\rho_{ps} = \rho_{ts}$ gives $\alpha_0^p = \beta^{-3}$ and therefore the relationships $\alpha_0^p m_{ts} = m_{ps}$, $\alpha_0^p g^2 k_{ts} = k_{ps}$ and $\alpha_0^p g c_{ts} = c_{ps}$ reduce to $m_{ts} = \beta^3 m_{ps}$, $k_{ts} = \beta^3 g^{-2} k_{ps}$ and $c_{ts} = \beta^3 g^{-1} c_{ps}$, respectively. To proceed further and determine the relationship between the time scalar $g$ and $\beta$ requires information about constitutive behaviour of the spring material and the damping fluid used in the dashpot. An elastic response is expected for the spring material satisfying a constitutive model of the form $\sigma_{ts} = C_{ts} : \epsilon_{ts}$ (Hooke’s law), where $C_{ts}$ is the elasticity tensor. Application of the identity $\sigma_{ps} = \alpha_0^p g \beta^2 \sigma_{ts}$ and $\epsilon_{ps} = \epsilon_{ts}$ from Table 3 provides,

$$\sigma_{ps} = C_{ps} : \epsilon_{ps} = C_{ps} : \epsilon_{ts} = \alpha_0^p g \beta^2 \sigma_{ts} = \alpha_0^p g \beta^2 C_{ts} : \epsilon_{ts} = \alpha_0^p g^2 \beta C_{ts} : \epsilon_{ts} = g^2 \beta^2 C_{ts} : \epsilon_{ts}$$

which requires $g^2 \beta^2 C_{ts} = C_{ps}$, and for an identical material must satisfy $C_{ts} = C_{ps}$, and therefore $g = \beta$ and the stiffness relationship is $k_{ts} = \beta k_{ps}$.

Substitution of $g = \beta$ into the dashpot relationship $c_{ts} = \beta^3 g^{-1} c_{ps}$ gives $c_{ts} = \beta^2 c_{ps}$, but this relationship provides a conflict. The reason for this is because the damping fluid might be anticipated to behave as a Newtonian fluid and satisfy a relationship of the form $\tau_{ts} = 2 \mu_{ts} \dot{\epsilon}_{ts}$, where the dash signifies a reduced tensor and with the assumption of incompressibility gives rise to $\tau_{ts} = 2 \mu_{ts} \dot{\epsilon}_{ts}$, where $\sigma_{ts} = -p_{ts} I + \tau_{ts}$, and where $\mu_{ts}$ is (shear) viscosity, $p_{ts}$ is hydrostatic pressure, and $I$ is a unit tensor. Application of the identity $\sigma_{ps} = \alpha_0^p g \beta^2 \sigma_{ts}$ and $\dot{\epsilon}_{ps} = g \dot{\epsilon}_{ts}$ from Table 3 provides,

$$\tau_{ps} = 2 \mu_{ps} \dot{\epsilon}_{ps} = 2 \mu_{ps} g \dot{\epsilon}_{ts} = \alpha_0^p g \beta^2 \tau_{ts} = \alpha_0^p g \beta^2 2 \mu_{ts} \dot{\epsilon}_{ts} = 2 \alpha_0^p g^2 \beta \mu_{ts} \dot{\epsilon}_{ts} = 2 g^2 \beta^2 \mu_{ts} \dot{\epsilon}_{ts}$$

which requires $\alpha_0^p g \beta \mu_{ts} = g \beta^2 \mu_{ts} = \mu_{ps}$, and for an identical material with $\mu_{ts} = \mu_{ps}$ suggests that $g = \beta^2$, which reduces the expression $c_{ts} = \beta^3 g^{-1} c_{ps}$ to $c_{ts} = \beta c_{ps}$, confirming that a single experiment with identical materials will not provide representative behaviour for the mass-spring-damper system.
4. SCALE DEPENDENCIES OF NONLINEAR FLUID VISCOUS DAMPERS

Fluid viscous dampers are recognised as common energy dissipation devices that provide a level of protection to structures from damage inflicted by seismic excitations. The use of this form of damping system is universal and finds use in a multitude of constructions, which includes bridges, low rise buildings to skyscrapers. A damper typically consists of a steel cylinder filled with silicon oil (for dissipation) and a piston. The whole arrangement can be connected between floors and/or beams [53] and are designed according to need. The basic schematic for a fluid viscous damper (FVD) is depicted in Fig. 2, where only the cylinder and piston head arrangement are shown. The nonlinear behaviour of a FVD can be approximated by a simple fractional velocity power law such as,

\[ f_D = c_\alpha \text{sgn}(v)|v|^{\alpha} \]  \hspace{1cm} (16)

where \( f_D \) is force, \( v \) is piston velocity relative to the cylinder, \( c_\alpha \) is an experimentally determined damping coefficient, \( \alpha \) is the velocity exponent, and the signum function \( \text{sgn} \) is either plus or minus unity depending on the sign of the relative velocity \( v \).

Figure 1:(colour online) 3-D configurations of fluid viscous damper (a) Discretised model using 4-noded linear tetrahedron element type (FC3D4) (b) Fluid flow inside the rigid body of dashpot
modelled as rigid walls. The number of elements for full-scale and trial models are identically 44105 and follow the rules of space scaling.

The nonlinearity of the damper is dictated by the velocity exponent $\alpha$, where linear behaviour is returned for $\alpha = 1$ and Eq. (10) in this case reduces to $f_D = c_1 v$. The velocity exponent $\alpha$ for seismic protection applications is usually in the range of $0.35 – 1$ [31,54] with typical curves relating force $f_D$ to velocity $v$ depicted in Fig. 3.

![Figure 3: (colour online) Schematic representation of constitutive behaviour of fluid viscous damper. The red and grey curves represent two different nonlinear behaviours of the FVD while the black curve is force-velocity curve for a linear FVD.](image)

The curves in Fig. 3 display the behaviour of the damper subjected to a sinusoidal excitation $x = x_0 \sin \omega t$, for different values of $\alpha$, where $\omega$ is angular frequency and $v = \dot{x}$. Note how the linear damper encounters high forces if exposed to large velocities, which can be an unwelcome feature. The damping force, for damper designs with velocity exponent in the range of $0 < \alpha < 1$, increases at a relatively lower rate than other conditions [53]. This is a desirable feature and makes nonlinear FVDs particularly attractive for the control and suppression of earthquake loads.

To investigate the scaling of a FVD and to capture more precisely nonlinear behaviours it is of interest to construct a numerical model. A previous study on FVDs detailed in Hou (2008) [55] is replicated here for scaling purposes and is performed in the commercial package ABAQUS/CFD [56]. A particular difficulty with this type of analysis is that it features fluid-solid interaction,
which is recognised to be problematic for some solvers. To avoid these problems the analysis performed here considers the piston head to be stationary (also excludes the piston rod) and allows the fluid to flow around it being sourced by the cylinder motion. The detailed presentation is depicted in Fig. 2, and an initial analysis is conducted to verify the results in comparison with those provided by Hou (2008) [55], as confirmed in Fig. 4. The velocity of the flow is defined as sinusoidal with respect to the equation \( v = v_{\text{max}} \sin \omega t \), where \( v_{\text{max}} \) is the maximum flow velocity, which is specified as \( v_{\text{max}} = \omega A \), where \( \omega \) and \( A \) present the angular frequency and oscillation amplitude [57], respectively.

The damper is filled with incompressible silicon oil with oil properties selected to emphasise nonlinear behaviour. The density is set as 100,000 kg/m\(^3\) and the dynamic viscosity is 1 kg/m/s, with the overall dimensions of the damper tabulated in Table 4. The design of dashpot considered (as depicted in Fig. 2) permits the movement of the fluid from one side of the chamber to the other, and the constitutive behaviour of the fluid is assumed Newtonian (see Sec. 3). An initial analysis based on Eq. (16) and the relationships

\[
0 \leq \frac{v}{v_{\text{max}}} \leq 1, \quad \frac{g}{\beta v_{\text{max}}} \leq 1,
\]

provides,

\[
f_{Ds} = c_{Dv} \text{sgn}(v_{\text{ps}}) |v_{\text{ps}}|^{\alpha_{Dv}} = c_{Dv} \text{sgn}(g \beta^{-1} v_{ts}) |g \beta^{-1} v_{ts}|^{\alpha_{Dv}} = c_{Dv} \left( g \beta^{-1} \right)^{\alpha_{Dv}} \text{sgn}(v_{ts}) |v_{ts}|^{\alpha_{Dv}}
\]

\[
= \alpha_{0}^v g f_{Dv} = \alpha_{0}^v g c_{Dv} \text{sgn}(v_{ts}) |v_{ts}|^{\alpha_{ts}}
\]

which requires that,

\[
c_{Dv} = \alpha_{0}^v g^{1-a_{Dv}} \beta^{a_{Dv}} c_{Dv} |v_{ts}|^{a_{Dv}} = \alpha_{0}^v g^{2-a_{Dv}} \beta^{a_{Dv}-1} c_{Dv} |v_{ts}|^{a_{Dv}-a_{Dv}} = g^{2-a_{Dv}} \beta^{a_{Dv}-4} c_{Dv} |v_{ts}|^{a_{Dv}-a_{Dv}}
\]

on substitution of \( \alpha_{0}^v = g \beta^{-1} \alpha_{0}^v \) (see Table 1) and on setting \( \alpha_{0}^v = \beta^{-3} \) for an identical fluid.

Note additionally from Sec. 3 that a Newtonian fluid requires that \( g = \beta^3 \), which provides the simplification of Eq. (18) to

\[
c_{Dv} = \beta^{-a_{Dv}} c_{Dv} |v_{ts}|^{a_{Dv}-a_{Dv}}.
\]

This reduces to the relationship for a linear dashpot (i.e. \( c_{Dv} = \beta c_{Dv} \)) on setting \( \alpha_{ts} = \alpha_{ps} = 1 \), and further infers [41] that for a scaled model incorporating an identical damping fluid, it can be anticipated that the velocity exponents are equal, i.e., \( \alpha_{ps} = \alpha_{ts} \). This provides a further simplification to the relationship

\[
c_{Dv} = \beta^{-a_{Dv}} c_{Dv} |v_{ts}|^{a_{Dv}-a_{Dv}},
\]

which reduces to \( c_{Dv} = \beta^{-a_{Dv}} c_{Dv} \) with \( \alpha_{ps} = \alpha_{ts} \), and is readily confirmed below with numerical analysis. Note that the expectation that \( \alpha_{ps} = \alpha_{ts} \) is a consequence of the behaviour of the dashpot being dominated by viscous flow being critically dependent on Reynolds number. Substitution of
\( \mu_{ps} = \alpha_0^0 g \beta \mu_{ts} \) (to satisfy Eq. (15)), \( \rho_{ps} = \alpha_0^0 \beta^3 \rho_{ts} \), \( V_{ps} = g \beta^{-1} V_{ts} \) (for some velocity) and \( D_{ps} = \beta^{-1} D_{ts} \) (for some diameter/length) into Reynolds number \( Re \) gives not unexpectedly,

\[
Re_{ps} = \frac{\rho_{ps} V_{ps} D_{ps}}{\mu_{ps}} = \left( \frac{\alpha_0^0 \beta^3 \rho_{ts}}{\alpha_0^0 g \beta \mu_{ts}} \right) \left( g \beta^{-1} V_{ts} \right) \left( \beta^{-1} D_{ts} \right) = \frac{\alpha_0^0 \beta^3 \beta^{-1}}{\alpha_0^0 g \beta} \frac{V_{ts} D_{ts}}{\mu_{ts}} = Re_{ts}
\]

(19)

and note that kinematic viscosity,

\[
v_{ps} = \frac{\mu_{ps}}{\rho_{ps}} = \frac{\alpha_0^0 g \beta \mu_{ts}}{\alpha_0^0 \beta^3 \rho_{ts}} = g \beta^{-2} V_{ts}
\]

(20)

which for \( g = \beta^2 \) reduces to \( v_{ps} = v_{ts} \) and for \( g = \beta \) gives \( v_{ps} = \beta^{-1} v_{ts} \).

Therefore, scaling with an identical damping fluid with \( g = \beta^2 \) satisfies zeroth-order rules, so non-linear behaviours at full and at scale are expected to correspond, i.e., \( \alpha_{ps} = \alpha_{ts} \).

Table 4: Physical dimensions of nonlinear damper model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Full-scale system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of piston head ( R ) (mm)</td>
<td>24.45</td>
</tr>
<tr>
<td>Width of piston head ( L ) (mm)</td>
<td>15</td>
</tr>
<tr>
<td>Width of orifice ( h ) (mm)</td>
<td>0.55</td>
</tr>
<tr>
<td>Radius of cylinder ( R_c ) (mm)</td>
<td>25</td>
</tr>
<tr>
<td>Length of cylinder ( L_c ) (mm)</td>
<td>110</td>
</tr>
<tr>
<td>Applied frequency ( \omega ) (Hz)</td>
<td>10</td>
</tr>
<tr>
<td>Amplitude ( x_0 ) (mm)</td>
<td>10</td>
</tr>
</tbody>
</table>
Figure 4:(colour online) Evaluation of the numerical results for the force-velocity of the non-linear FVD by comparison against FE analytical results presented in the literature.

The results of the FVD simulation presented in Fig. 4 confirm the nonlinearity in the force-velocity relationship and confirm likewise good agreement with the results of Hou (2008). The verified finite-element model provides a convenient vehicle to investigate the effects of scaling on damping. Both contraction and expansion are considered with scaling parameter $\beta$ taking on discrete values $1/2$ and $2$. Identical materials are used in all scaled models and the time scalar $g$ is set equal to $\beta^2$ in accordance with the analysis presented in Sec. 3. The damping coefficients and velocity exponents obtained from the nonlinear curve-fitting analysis at scale are tabulated in Table 5. The results confirm that the exponent components do not change with scale over the range of scales considered, i.e., $\alpha_p = \alpha_s$. Also confirmed by this table is the relationship $c_{\alpha_p} = \beta^{\alpha_p} c_{\alpha_p}$ (with $\alpha_p = \alpha_s$) and reaffirmed by the curves in Fig. 5 conforming to good accuracy to the form expected from Eq. (16).
Table 5: Nonlinear fluid viscous damper damping coefficient and velocity exponent for full-scale and trial models.

<table>
<thead>
<tr>
<th>Models</th>
<th>Damping coefficients (N.s/mm) ($c_{\alpha}$)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full-scale model $\beta_0 = 1$</td>
<td>0.0612</td>
<td>2.0834</td>
</tr>
<tr>
<td>Scale-down model $\beta_1 = \frac{1}{2}$</td>
<td>0.0144</td>
<td>2.0834</td>
</tr>
<tr>
<td>Scale-up model $\beta_1 = 2$</td>
<td>0.259</td>
<td>2.0834</td>
</tr>
</tbody>
</table>

Figure 5: (colour online) Size dependency of the nonlinear FVD damping coefficient for identical material selection. The markers present the FE results while the continuous lines demonstrate the analytical solution.

The analysis performed thus far is for a FVD in isolation using identical materials, but no account is made of other structural elements. Considering elastic structural behaviour requires that the time scalar $g$ is set equal to $\beta$ (see Sec. 3) and consequently the FVD does not scale according to zeroth-order requirements. Substitution of $g = \beta$ into Eq. (18) provides $c_{\alpha_{ps}} = \beta^{-2}c_{\alpha_{ts}}\left|v_{ts}\right|_{\alpha_{ts} - \alpha_{ps}}$, which reduces further to $c_{\alpha_{ps}} = \beta^{-2}c_{\alpha_{ts}}$ under the assumption that $\alpha_{ps} = \alpha_{ts}$. This assumption (because of Eqs. (19) and (20)) requires a change in the damping fluid to ensure that zeroth-order
scaling rules apply, with kinematic viscosity satisfying the relationship $\nu_{ps} = \beta^{-1}\nu_{st}$. Numerical analysis supporting this assertion is shown in Fig. 6 with scaled FVDs conforming to the expected behaviours (i.e., $c_{\alpha_n} = \beta^2 c_{\alpha_{ps}}$ and $\alpha_{ps} = \alpha_{ts}$). This result is of practical value since, apart from giving special consideration to the properties of the damping fluid, it confirms that scaled FVDs can be readily incorporated into scaled models without major modification. It is demonstrated in reference [36] that structural damping follows the relationship $c_{ts}^{\text{struct}} = \beta^2 c_{ps}^{\text{struct}}$ (with $g = \beta$) and arranging for $c_{\alpha_n} = \beta^2 c_{\alpha_{ps}}$ for FVDs also is evidently desirable in scaled experimentation.
Figure 6: (colour online) Size dependency of the nonlinear FVD damping coefficient for different damper fluid selection ($\mu_r = \beta \mu_{ps}$). The FE simulation and analytical results based on Eq. (16) are demonstrated for distinct scale models, i.e. (a) scaling with $\beta = \frac{1}{2}$ and $\beta = 2$ and (b) large-scale up with $\beta = 10$.

4.1. Scaling a single-story building incorporating a nonlinear FVD

To investigate further the practical value of scaling when featuring a nonlinear fluid viscous damper, a single-story building is considered. The main purpose of this case study is to investigate what scale effects arise (if any) for the situation when the structure and connected dampers are scaled down. Since the aim of using nonlinear dampers is to reduce structural vibration and peak loading, it is important to capture the nonlinear behaviour, so that adequate scaled models can be created. The case study here involves a nonlinear FVD with velocity exponent $\alpha = \frac{1}{2}$. The model is exposed to the half-cycle sine shock, which is detailed in the Fig. 7 [58].
Figure 7: (colour online) Schematic and idealized representation of one-story structure equipped by nonlinear FVD under impulse loading. The curves depict the applied external impulse loads on the full-scale physical model ($\beta_0 = 1$) and the trial-space model ($\beta_1 = 1/4$).

The model parameters are detailed in Table 6, and the model behaviour is assumed to remain in the linear elastic region. The details of the model has been taken from reference [58], where the lump mass is 25,000 kg, the stiffness of the columns calculated with respect to the time period of 0.05 s (i.e., $T_n = 0.05s$). The inherent structural damping of the system is assumed to be 2%, and critical damping value is determined by $c_{cr} = 2m\omega_n$ and $\omega_n = \sqrt{\frac{k}{m}}$. The maximum acceleration is taken to be $60g$ (i.e., $\ddot{u}_0 = 60g$) with the total duration of 0.01s (i.e., $t_d = 0.01s$). As detailed in reference [40], the applied sine shock is a form of impulse load, where the ratio of the time-period to the impulse-load duration is smaller than 1/4. The structure depicted in Fig. 7 can be represented therefore as a single degree of freedom (SDF) system with mass $m$, stiffness $k$, structural damping $c$, and nonlinear viscous-fluid damping $c_a$, being exposed to half-cycle sine shock. The mathematical model representing the SDF system is,

$$m\ddot{u} + c\dot{u} + ku + c_a \text{sgn}(\dot{u})|\dot{u}|^\alpha = m\ddot{u}_0 \sin \omega t$$

(21)

where the nonlinear FDV is exposed to half-cycle sine shock $m\ddot{u}_0 \sin \omega t$. 
In applying the similitude rules to this model, the time scalar \( g \) is required to be set equal to \( \beta \) (i.e., \( g = \beta \)) as dictated by the elastic members as discussed in Sec. 3. However, as also mentioned above, this setting poses some difficulty for the FVD, where an identical damping fluid is to be employed in both the full size and scaled FVDs. The significance of this is shown in Fig. 8, where the projected-trial model signifies results transferred to the physical space by the zeroth-order relationships \( F_{ps} = \alpha_{ps} \beta^1 g \) and \( u_{ps} = \beta^1 u_{ts1} \) (see Table 3). It is evident that the transferred scaled behaviour is not representative of the full-scale behaviour. Replacing the damping fluid in the scaled model and noting the properties tabulated in Table 6, which are in accordance with the relationships (see Table 3) \( g_1 = \beta_1 \), \( k_{ts1} = \beta_1 k_{ps} \), \( c_{ts1} = \beta_1^2 c_{ps} \), \( c_{a_{ts1}} = \beta_1^2 c_{a_{ps}} \), \( m_{ts1} = \beta_1^3 m_{ps} \), and \( a_{ps} = \beta_1^2 \beta_1^{-1} a_{ts1} \), new predictions are provided in Figs. 9 and 10. In complete accordance with theory exact replication is returned between full scale and projected trial-scale models. Note from a practical perspective, the Taylor-devices manual [59] confirms that it is indeed possible to obtain or design a damper with the required zeroth-order requirements, i.e. one which provides \( c_{a_{ts1}} = \beta_1^2 c_{a_{ps}} \) and \( \alpha_{ts1} = \alpha_{ps} \).

Table 6: Physical properties of equivalent 1-DoF model for full-scale and scaled models.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Full-scale system</th>
<th>Scaled-down system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension scaling factor (( \beta ))</td>
<td>1</td>
<td>1/4</td>
</tr>
<tr>
<td>Time scaling factor (( g ))</td>
<td>1</td>
<td>1/4</td>
</tr>
<tr>
<td>Stiffness (( N/m ))</td>
<td>394784176</td>
<td>98696044</td>
</tr>
<tr>
<td>Damping coefficient ( c (N.s/m) )</td>
<td>10125663</td>
<td>632854</td>
</tr>
<tr>
<td>Nonlinear Damping coefficient ( c_\alpha (N.s/m) )</td>
<td>3760000</td>
<td>235000</td>
</tr>
<tr>
<td>Mass (kg)</td>
<td>25000</td>
<td>390.6</td>
</tr>
<tr>
<td>Peak acceleration (g)</td>
<td>60</td>
<td>240</td>
</tr>
</tbody>
</table>
Figure 8: (colour online) The orange line illustrates the replication of damping force-displacement behaviour of the equivalent 1-DOF model of one-story structure equipped by nonlinear FVD based on zeroth-order finite similitude. The blue curve depicts the full-scale model behaviour.

Figure 9: (colour online) The orange marker presents the replication of temporal behaviour of displacement of the projected trial-space equivalent model based on zeroth-order finite similitude, while the blue curve demonstrates the full-scale equivalent model behaviour.
Figure 10:(colour online) The numerical results show the exact match between the behaviour of the real model (blue line) and projected model (orange marker) replicated by first-order finite similitude.

5. SCALING OF FRICTION INDUCED STICK-SLIP BEHAVIOUR

A complex friction case study is considered in this section to ascertain whether it is possible to examine nonlinear friction behaviour via scaled experiments. The model under scrutiny here is the Burridge-Knopoff earthquake model, which is commonly applied in the study of stick-slip vibration, being recognised to be a significant phenomenon [60]. The model of interest is depicted Fig. 11 and consists of two masses connected by spring elements on a moving belt. This model is assumed to represent the movement of stone blocks during seismic excitation by introducing friction between the components. The system features two forms of friction, which are static friction and dynamic friction. A rough surface is assumed to exist between the belt and masses to give rise to dry friction at the mass-belt interface. The blocks stick when the maximum force applied by the springs to a block happens to be less than the maximum static friction force. When the spring forces exceed the static friction forces, slipping motion starts and it slips until the velocity
of the belt and masses are equal. The constant repetition of such movements creates a stick-slip oscillation.

![Schematic representation of the 2-DoF stick-slip vibrational system](image)

Figure 11: Schematic representation of the 2-DoF stick-slip vibrational system. $k_1$, $k_2$ and $k_c$ represent spring stiffnesses, $v_{dr}$ is the belt velocity and $m_1$, $m_2$ represent the mass of the blocks.

The system presented in Fig. 11 consists of two mass blocks $m_1$ and $m_2$ sitting on a moving belt which is traversing under the blocks with constant velocity $v_{dr}$ [61]. The two mass blocks are connected by a spring of stiffness $k_c$ and at the same time each block is connected individually to fixed constraints through springs of stiffness $k_1$ and $k_2$, respectively. The displacement of each mass block ($m_1$ and $m_2$) is quantified by $x_1$ and $x_2$, respectively. Since the blocks are in contact with the moving belt friction forces $F_1$ and $F_2$ are assumed to apply. The material properties and details relating to the full trial-scale models can be found in Table 7. The system with scaled models is depicted in Fig. 12 and are governed by the differential equations [46]

$$m_1\ddot{x}_1 + k_1 x_1 + k_c (x_1 - x_2) = F_1$$  \hspace{1cm} (21a)

$$m_2\ddot{x}_2 + k_2 x_2 + k_c (x_2 - x_1) = F_2$$  \hspace{1cm} (21b)

which can be readily solved with the assistance of the commercial Matlab software package, where smoothed analytical relationships for the frictional forces $F_1$ and $F_2$ are given by
\[
\begin{align*}
F_1 \leq F_{s,1} & \quad \text{if} \quad \frac{\sqrt{k_1 m_1}}{F_{s,1}} (\dot{x}_1 - v_{dr}) = 0 \\
F_1 = -F_{s,1} & \quad \text{if} \quad \frac{\sqrt{k_1 m_1}}{F_{s,1}} (\dot{x}_1 - v_{dr}) \neq 0 \\
F_2 \leq F_{s,2} & \quad \text{if} \quad \frac{\sqrt{k_2 m_2}}{F_{s,2}} (\dot{x}_2 - v_{dr}) = 0 \\
F_2 = -F_{s,2} & \quad \text{if} \quad \frac{\sqrt{k_2 m_2}}{F_{s,2}} (\dot{x}_2 - v_{dr}) \neq 0 
\end{align*}
\]

(22a)

(22b)

where, \( \gamma \) is the shape coefficient of the dynamic friction law, \( F_{s,1} = \mu_{s,1} m_1 G \) and \( F_{s,2} = \mu_{s,2} m_2 G \), represent static frictional forces, and where \( \mu_{s,1} \) and \( \mu_{s,2} \) are Coulomb coefficients of friction pertaining to each block, and \( G \) is acceleration due to gravity (9.81 m/s\(^2\)).

In order to solve the specified system, the initial conditions in reference [62], for the displacements and velocities are applied, i.e., \( x_1 = \frac{0.9 F_{s,1}}{k_1}, \quad x_2 = \frac{F_{s,2}}{k_2} \) and \( \dot{x}_1 = \dot{x}_2 = v_{dr} \). In addition, following again the suggestions of Galvanetto et al. [62], the following settings are also imposed: \( k_1 = 1.2 k \), \( \mu_{s,1} = \mu_s = 0.22 \), \( \mu_{s,2} = 1.3 \mu_s \), \( \gamma = 3 \), and \( v_{dr} = \frac{0.14 F_{s,1}}{\sqrt{k_1 m_1}} \). Presented in Fig. 13 are validation results, where the solution to Eqs. (22) is contrasted against that provided in reference [44]; complete agreement is shown.

Table 7: Material parameters of stick-slip system for full-scale and trial models

<table>
<thead>
<tr>
<th>Geometric/Material Properties</th>
<th>Full-scale Model</th>
<th>Trial Model I</th>
<th>Trial Model II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 ) (kg)</td>
<td>80</td>
<td>10</td>
<td>1.25</td>
</tr>
<tr>
<td>( m_2 ) (kg)</td>
<td>80</td>
<td>10</td>
<td>1.25</td>
</tr>
<tr>
<td>( k_1 ) (N/m)</td>
<td>1000</td>
<td>500</td>
<td>250</td>
</tr>
<tr>
<td>$k_z (N/m)$</td>
<td>1000</td>
<td>500</td>
<td>250</td>
</tr>
<tr>
<td>------------</td>
<td>------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>$G (m/s^2)$</td>
<td>9.81</td>
<td>9.81</td>
<td>9.81</td>
</tr>
<tr>
<td>$\mu_s$</td>
<td>0.22</td>
<td>0.22</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Figure 12: Full-scale ($\beta_0 = 1$) and scaled models for the stick-slip vibrational system. The geometrical scaling factors of scaled models 1 and 2 are $\beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{1}{4}$, respectively.

It is relatively easy to confirm for a scaled model consisting of identical materials as the full-scale system, that a single scaled model is insufficient to represent full-scale behaviour. To show this consider the zeroth-order force relationship in Table 3, i.e., $F_{ps} = \alpha_0^v g F_{ts}$ and consider further the static friction satisfying a relationship of the form $(F_s)_{ts} = -\left(\mu_s\right)_{ts} m_i G_{ts} \frac{v_s}{\left|v_s\right|}$. Multiplying both sides of this equation by $\alpha_0^v g$ provides after some manipulation,
\[ \alpha'_0 g \left( F'_s \right)_{ts} = -\left( \mu_s \right)_{ts} \left( \alpha'_0 g^{-1} \beta m_s \right) \left( \beta^{-1} g^2 G_{ts} \right) \left[ g \beta^{-1} v_s \right] \]

and recall that \( \alpha'_0 g^{-1} \beta m_s = \alpha'_0 m_s = m_{ps} \) and \( g \beta^{-1} v_s = v_{ps} \), so consequently to satisfy the relationship \( F_{ps} = \alpha'_0 g F_s \) with \( \left( F'_s \right)_{ps} = -\left( \mu_s \right)_{ps} m_{ps} G_{ps} v_{ps} \left[ F_{ps} \right] \) requires that \( \left( \mu_s \right)_{ps} = \left( \mu_s \right)_{ts} \), and somewhat problematically \( \beta^{-1} g^2 G_{ts} = G_{ps} \).

The last expression is problematic because elastic structural members necessitate that \( g = \beta \) (for identical spring materials), giving rise to a contradiction forcing the acceleration due to gravity to change!

Figure 13: (colour online) The curves represent a comparison between two in-phase solutions, presented in this work (current results) and literature (orange marker) where both masses move in the same direction.

Because of this contradiction attention now turns to first-order finite similitude with selected geometric scaling factors set to \( \beta_1 = \frac{1}{2} \) and \( \beta_2 = \frac{1}{4} \) for trial-models 1 and 2, respectively. Under the assumption that identical materials are involved at full size and at scale the time scalars are required to be set as \( g_1 = \beta_1 = \frac{1}{2} \) and \( g_2 = \beta_2 = \frac{1}{4} \). Prior to examining the first-order solution it is of interest to examine what is achieved with zeroth order. Presented in Fig. 14 therefore is the
behaviour of the full-scale model and the projected trial models 1 and 2, where significant disparities are apparent. The main issue is gravity, and it would require an additional mass approach or alteration to friction conditions or gravitational acceleration, for zeroth-order theory to apply. Rather than amending the physical problem, the first-order theory is applied, which combines of two projected zeroth-order solutions in accordance with first-order relationships in Table 3. Since gravity is an issue with zeroth order the identity for acceleration in Table 3 is employed in the determination of the unknown parameter $R_1$, being selected to provide the correct acceleration due to gravity by the virtual model (i.e., the combined projected models).

Figure 14: (colour online) The curves represent a comparison between the full-scale (blue curve) and two distinct trial models in-phase solutions. The response of masses $m_1$ and $m_2$ described by zeroth-order finite similitude for trial model 1 with $\beta_1 = \frac{1}{2}$ (orange curve) and trial model 2 with $\beta_1 = \frac{1}{4}$ (green curve).

The first-order equation of interest from Table 3 is acceleration in the form

$$G_{\mu} = \beta_1^{-1} g_1^2 G_{\mu 1} + R_1 \left( \beta_1^{-1} g_1^2 G_{\mu 1} - \beta_2^{-1} g_2^2 G_{\mu 2} \right)$$

(24)
which on substitution of $g_1 = \beta_1 = \frac{1}{2}$, $g_2 = \beta_2 = \frac{1}{4}$ and on setting $G_{ps} = G_{v1} = G_{v2}$, provides after a little algebraic manipulation,

$$R_i = 1 - \frac{\beta_1}{\beta_1 - \beta_2} = \frac{1 - 0.5}{0.5 - 0.25} = 2$$

(25)

The expression for displacement provided in Table 3, which for this case gives,

$$x_{1\,ps} = \beta_1^{-1} x_{1\,1\,1} + R_i \left( \beta_1^{-1} x_{1\,1\,1} - \beta_2^{-1} x_{1\,1\,2} \right)$$

(26a)

$$x_{2\,ps} = \beta_1^{-1} x_{2\,1\,1} + R_i \left( \beta_1^{-1} x_{2\,1\,1} - \beta_2^{-1} x_{2\,1\,2} \right)$$

(26b)

where $x_{1\,ps}$ and $x_{2\,ps}$ are the required displacements of the masses for the virtual model, and these are plotted in Fig. 15, where a perfect match with the full-scale model is obtained.

Figure 15: (colour online) The curves represent a comparison between the full-scale (blue curve) and first order projected virtual model (red marker) in-phase solutions. The response of the first-order virtual model is achieved by a linear combination of the result of two distinct trial models.

Further confirmation of the exactness of the predictions is provided in Fig. 16, where phase-space plots are provided for each of the lumped masses. Despite the complicated nature of the stick-slip behaviour, perfect replication is provided by the first-order theory.
Figure 16: (colour online) The phase diagrams for masses $m_1$ and $m_2$ for full-scale (blue and orange curves) and first-order virtual models (red and green markers), respectively.

6. SCALING OF A NONLINEAR SPRING-DAMPER-FRICTION SYSTEM

In this section, another nonlinear feature is added in the form of a nonlinear spring to investigate whether scaled experimentation is still applicable in this case. Moreover, cyclic loading is applied to the mechanical system under scrutiny to exacerbate any disparities that might occur over a long timescale. The system under consideration is presented in Fig. 17 along with scaled versions, and features both nonlinear dampers and springs, and again features mass on a moving belt. As in the system considered in Section 5, the friction between the mass block and the belt induces a stick-slip behaviour, which in combination with the other nonlinearities provides a highly nonlinear system for scaled experimentation. There is little possibility that such a system could be represented by a single scaled experiment without significantly modifying the physical problem. It is of interest therefore to examine whether the first-order theory has the capability to accurately describe the response of the system, which is governed by equation [63,64]
\[ m\ddot{x} + c\dot{x} + c_\alpha \text{sgn}(\dot{x})|\dot{x}|^\mu + P(L_0 - x) + \mu mG \text{sgn}(x - v_b) = F_0 \sin \left( \frac{\pi t}{t_d} \right) \] (27)

where for the full-scale model, the parameters of this equation are tabulated according to Table 8, and where the nonlinear function \( P \) is the force arising from a conical spring, \( v_b \) is belt velocity and \( t_d \) is the time period for the applied external force.

Figure 17: A schematic diagram of the full-scale and small-scale models of the 1-DoF vibrational system including nonlinear spring, nonlinear fluid viscous and Coulomb friction damper under cyclic loading.

Conical springs are in common usage [41,65] and present nonlinear behaviour arising from their geometry, so provide a good practical example of wide interest. The linear response of a conical spring under scaling is presented in ref [41] but the nonlinear response is of principal interest here and is reasonably well described by the relationship [65],

\[ P(L) = \left( \frac{K_1}{2} \right)^{3/2} \left\{ 1 - 2 \left[ 1 - \left( 1 + \frac{K_2}{K_1} \right)^{1/2} \right] \right\}^{1/2} \] (28)
where \( K_1 = K_3 - \frac{K_2}{3K_1}, \) \( K_2 = -\frac{K_1}{K_3}, \) and where, \( K_4 = \left( \frac{K_1 - L_a + L_r}{K_3} \right)^2 \) with

\[
K_3 = \left[ \frac{K_4}{16} + \left( \frac{K_4}{16} + \left( \frac{K_2}{3} \right)^3 \right)^{1/3} \right]
\]

(29a)

\[
K_5 = -\frac{2D_d^4 n_a}{Sd^4 (D_2 - D_1)}
\]

(29b)

\[
K_6 = -\frac{3}{8(D_2 - D_1)} \left( \frac{Sd^4 (L_a - L_r)^4}{n_a} \right)^{1/3}
\]

(29c)

\[
K_7 = (L_a - L_r) \frac{D_2}{D_2 - D_1}
\]

(29d)

where \( L_a = L_0 - n_d \) is the initial active length, \( L_r = \max\left\{0, \left( n_a, d \right)^2 - \frac{1}{2}(D_2 - D_1)^2 \right\} \) refers to the active coils solid length, and where the values considered are provided in the Table 9; it is readily confirmed that \( P_{\alpha} = \beta^2 P_{\alpha} \) for an identical spring material.

Table 8: The parameters of the vibrational system including a nonlinear damper and a nonlinear spring for full-scale and trial models.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Full-scale Model</th>
<th>Trial Model 1</th>
<th>Trial Model 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>1/2</td>
<td>1/4</td>
<td></td>
</tr>
<tr>
<td>( g )</td>
<td>1/2</td>
<td>1/4</td>
<td></td>
</tr>
<tr>
<td>( m ) (kg)</td>
<td>1000</td>
<td>125</td>
<td>15.625</td>
</tr>
<tr>
<td>( c ) (N.s/m)</td>
<td>189</td>
<td>47.25</td>
<td>11.8125</td>
</tr>
<tr>
<td>( c_a ) (s/m)^a</td>
<td>200</td>
<td>50</td>
<td>12.5</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.50</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.22</td>
<td>0.22</td>
<td>0.22</td>
</tr>
<tr>
<td>( v_b ) (m/s)</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( F_0 ) (N)</td>
<td>45000</td>
<td>11250</td>
<td>2812.5</td>
</tr>
</tbody>
</table>
Table 9: Characteristics parameters of conical springs for full-scale and trial models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Full-scale model $\beta_0 = 1$</th>
<th>Scaled-down model $\beta_1 = 1/2$</th>
<th>Scaled-up model $\beta_2 = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean diameter of the smallest active coil: $D_1$ (mm)</td>
<td>8.97</td>
<td>4.485</td>
<td>71.76</td>
</tr>
<tr>
<td>mean diameter of the largest active coil: $D_2$ (mm)</td>
<td>13.3</td>
<td>6.65</td>
<td>106.4</td>
</tr>
<tr>
<td>wire diameter: $d$ (mm)</td>
<td>1.2</td>
<td>0.6</td>
<td>9.6</td>
</tr>
<tr>
<td>free length: $L_0$ (mm)</td>
<td>37.2</td>
<td>18.6</td>
<td>297.6</td>
</tr>
<tr>
<td>parameter defining the influence of end coils on the difference between $L_0$ and $L_s$: $n_i$</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>total number of active coils: $n_a$</td>
<td>7.13</td>
<td>7.13</td>
<td>7.13</td>
</tr>
<tr>
<td>shear modulus: $S$ (MPa)</td>
<td>80000</td>
<td>80000</td>
<td>80000</td>
</tr>
</tbody>
</table>

Figure 18: (colour online) Verification of the FE numerical results for the force-displacement of the nonlinear conical spring by comparison against experimental and analytical results presented in the literature. The FE (ABAQUS) model of spring modelled as a 10-node quadratic tetrahedron element (C3D10) including 9641 elements and 18601 nodes.
In addition to the analytical expression provided by Eq. (28) an ABAQUS model for the conical spring under scrutiny is presented in Fig. 18 along with the mesh used for analysis. Also shown in this figure is the predicted nonlinear behaviour in the force-displacement curve and confirmed is excellent agreement between the finite element results and those of Rodriguez (2006) [65] and Eq. (28). The validated finite element model provides a convenient vehicle to study the behaviour of the nonlinear conical spring with scaling. The selected geometric scaling parameters for the contraction and expansion are set to $\beta_1 = \frac{1}{2}$ and $\beta_2 = 8$ to provide a good range of scale. Identical material is used for both the scaled-down and up models and consequently in accordance with the theory presented in Sec. 3, the time scaling factor $g$ is set equal to $\beta$. Note additionally that differentiation of Eq. (27) with respect to displacement $-x$ provides the nonlinear stiffness of the conical spring,

$$ k = \frac{dP}{d(-x)} = \frac{dP}{dL} \frac{dL}{d(-x)} = \frac{dP}{dL} $$

but recall the scaling relationship $P_{ts} = \beta^2 P_{ps}$ and consequently $k_{ts} = \beta k_{ps}$, which as discussed in Sec. 3 is the expected zeroth-order relationship.

The validity of this analysis is confirmed in Figs. 19, where the figure compares the curves for force against displacement obtained on the trial space but projected onto the physical space using the relationships for force ($F_{ps} = \alpha_0 g F_{ts} = \beta^2 F_{ts}$) and displacement ($u_{ps} = \beta^{-1} u_{ts}$) from Table 3.

With the spring behaviour confirmed to be well described by Eq. (28), and structural damping and nonlinear damper taking the form presented in Sec. 5, i.e., $c\ddot{x} + c_a \text{sgn}(\dot{x})|\dot{x}|^{\alpha}$, attention now turns to the solving of Eq. (27) and the application of first-order finite similitude.
Figure 19: (colour online) Response of scaled nonlinear-spring system subjected to compression force. The blue markers represent the full-scale model behaviour, while the zeroth-order projected trial models are the results of the projection of the scaled models for trial 1 and 2, purple and green lines, respectively.

Note that the first-order rule for force in Table 3 with $\alpha_0 \gamma = \beta^{-2}$ provides the following combination of Eq. (27):

$$
\beta_1^{-2} m_{tr1} \ddot{x}_{tr1} + R_1 \left( \beta_1^{-2} m_{tr1} \ddot{x}_{tr1} - \beta_2^{-2} m_{tr2} \ddot{x}_{tr2} \right) + \\
\beta_1^{-2} c_{tr1} \dot{x}_{tr1} + R_1 \left( \beta_1^{-2} c_{tr1} \dot{x}_{tr1} - \beta_2^{-2} m_{tr2} \dot{x}_{tr2} \right) + \\
\beta_1^{-2} c_{tr1} \text{sgn} \left( \dot{x}_{tr1} \right) \dot{x}_{tr1}^{\text{sgn}} + R_1 \left( \beta_1^{-2} c_{tr1} \text{sgn} \left( \dot{x}_{tr1} \right) \dot{x}_{tr1}^{\text{sgn}} - \beta_2^{-2} c_{tr2} \text{sgn} \left( \dot{x}_{tr2} \right) \dot{x}_{tr2}^{\text{sgn}} \right) \\
\beta_1^{-2} \mu_{tr1} m_{tr1} G \text{sgn} \left( \dot{x}_{tr1} - v_{tr1} \right) + R_1 \left( \beta_1^{-2} \mu_{tr1} m_{tr1} G \text{sgn} \left( \dot{x}_{tr1} - v_{tr1} \right) - \beta_2^{-2} \mu_{tr2} m_{tr2} G \text{sgn} \left( \dot{x}_{tr2} - v_{tr2} \right) \right) + \\
\beta_1^{-2} \mu_{tr1} m_{tr1} G \text{sgn} \left( \dot{x}_{tr1} - v_{tr1} \right) + R_1 \left( \beta_1^{-2} \mu_{tr1} m_{tr1} G \text{sgn} \left( \dot{x}_{tr1} - v_{tr1} \right) - \beta_2^{-2} \mu_{tr2} m_{tr2} G \text{sgn} \left( \dot{x}_{tr2} - v_{tr2} \right) \right) + \\
= \beta_1^{-2} F_{0tr1} \sin \left( \frac{\pi t_{tr1}}{t_{ds1}} \right) + R_1 \left( \beta_1^{-2} F_{0tr1} \sin \left( \frac{\pi t_{tr1}}{t_{ds1}} \right) - \beta_2^{-2} F_{0tr2} \sin \left( \frac{\pi t_{tr2}}{t_{ds2}} \right) \right)
$$

(31)
which provides a perfect match for the equation in the physical space on setting \( m_s = \beta^3 m_{ps} \),
\[
c_s = \beta^2 c_{ps}, \quad c_{au} = \beta^2 c_{a_{ps}}, \quad P_s (L_{0s} - x_s) = \beta^2 P_{ps} (L_{0ps} - x_{ps}), \quad\mu_s = \mu_{ps}, \quad\alpha_s = \alpha_{ps}, \quad R_i \text{ satisfying Eq. (25)},
\]
\[
F_{0s} = \beta^2 F_{0ps}, \quad t_s = \beta t_{ps} \quad \text{and} \quad t_{ds} = \beta t_{dps}.
\]

This is confirmed by the numerical simulation results presented in Figs. 20 and 21, respectively representing the displacement against time and the corresponding phase diagram. The details of the conical coils used is presented in Table 10 with other data found in Table 8. It is evident from the results in the figures that the first-order finite similitude theory can provide acceptable results in a situation where nonlinear behaviour is present in springs and dampers but also when friction is present.

**Table 10: Conical springs for full-scale and trial models used in scaled system**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Full-scale model ((\beta_0 = 1))</th>
<th>Trial model 1 ((\beta_1 = 1/2))</th>
<th>Trial model 2 ((\beta_2 = 1/4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_1) (mm)</td>
<td>98.67</td>
<td>49.335</td>
<td>24.6675</td>
</tr>
<tr>
<td>(D_2) (mm)</td>
<td>143</td>
<td>71.5</td>
<td>35.75</td>
</tr>
<tr>
<td>(d) (mm)</td>
<td>13.2</td>
<td>6.6</td>
<td>3.3</td>
</tr>
<tr>
<td>(L_0) (mm)</td>
<td>409.2</td>
<td>204.6</td>
<td>102.3</td>
</tr>
<tr>
<td>(n_i)</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>(n_a)</td>
<td>7.13</td>
<td>7.13</td>
<td>7.13</td>
</tr>
<tr>
<td>(S) (MPa)</td>
<td>80000</td>
<td>80000</td>
<td>80000</td>
</tr>
</tbody>
</table>
Figure 20: (colour online) The temporal behaviour of mass displacement of the full-scale model, projected trial model 1 and 2 designed based on the zeroth order theory which could not replicate the behaviour while the projected first order virtual model designed based on the first order finite similitude nearly captures the full-scale model response.
Figure 21: (colour online) The nonlinear spring phase response of the full-scale model, projected virtual model which is the combination of two distinct trial models (1 and 2) designed based on the zeroth order theory which could not replicate the behaviour while the first-order virtual model captures the global behaviour of the full-scale system response.

7. CONCLUSION

The paper aims to present the application of the finite-similitude theory to show how complex nonlinear mechanical dynamic systems, involving nonlinear viscous dampers, springs, and friction, can be scaled using the recently developed first-order finite similitude theory, where traditional scaling theories (e.g., dimensional analysis) are not able to satisfy complete similarity conditions. The high capability of this approach was illustrated for the proposed vibrational systems in which the response behaviour of the prototype was predicted with high accuracy by the combination of the response of two distinct scaled models without the requirement of any additional technique to be applied to the models. The resulting conclusions can be extracted from the proposed case studies including numerical and analytical analysis:

- The scale dependency of the nonlinear fluid viscous damper was investigated and a relationship \( c_{\alpha_u} = \beta^\alpha c_{\alpha_p} \) was obtained for the identical material selection. In addition, it is concluded that the desired damping coefficient \( c_{\alpha_u} = \beta^2 c_{\alpha_p} \) between the full-scale and scaled model is always possible by changing the viscous damping fluid.

- A single-story case study confirmed that zeroth-order finite similitude is satisfactory in the case of realisable fluid material selection for the trial model damper, and complete replica scaling was shown possible.

- Scaling of stick-slip friction system was studied in the second case study, and it was demonstrated that the zeroth-order finite similitude (and consequently dimensional analysis) failed to capture the behaviour of the prototype, in contrast, the first-order finite similitude, which provided complete similarity. The study illustrated that the combination of two distinct scaled models can replicate the response behaviour of the physical model with complete agreement. This can be contrasted against replica zeroth-order models (i.e.,
no additional mass etc.) which gave rise to large errors, i.e., 75% and 93.75% errors for trial model 1 and 2, respectively.

- The scaling behaviour of a nonlinear spring-damper-friction system was examined according to the third case study. It was found that a large deviation was created between the response behaviour of full-scale and singly applied replica small-scale models. For displacement, the full-scale model was predicted with errors up to 46.9% and 70.4% with dimensional scaling factors set respectively to 0.50 and 0.25. Exact replication (within numerical error) was provided with the first order finite similitude theory on combining the results of two distinct trial models and on correctly setting the extra independent degree of freedom $R_i$.

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