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Nonlinear and linearised analyses of a generic rotor on single-pad foil-air bearings using Galerkin Reduction with different applied air film conditions

Ibrahim Ghalayini, Philip Bonello,
Department of Mechanical, Aerospace and Civil Engineering, University of Manchester, United Kingdom

Abstract

The compressible Reynolds Equation is typically integrated within a fully-coupled dynamical foil-air bearings (FAB)-rotor system via spatial discretisation transformation e.g. Finite Difference (FD), Finite Element (FE). This paper presents a novel application of an arbitrary-order Galerkin Reduction (GR) method, which does not involve spatial discretisation, to both nonlinear and linearised analyses of rotor systems supported by single-pad FABs. The novel aspects are: application to generic flexible rotors; Jacobian-based linearisation of the GR-transformed system for the extraction of the full mode set and linear stability map; the facility to apply a pressure constraint at a circumferential location and/or the Gumbel condition. These developments are comprehensively verified on two previously considered systems by comparing alternative (GR, FD-based) simulations. The overall features of the experimental nonlinear phenomena are predicted satisfactorily. Transient nonlinear dynamic analysis (TNDA) with GR is found to be less prone to numerical divergence than with FD. For static equilibrium, stability and modal analysis (SESMA), GR is found to be over twice as fast, requiring 70-90 times less memory for saving the results.

Keywords: Foil-air bearings; nonlinear analysis; linearisation; Galerkin Reduction; order reduction
### List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLE/FTE</td>
<td>Clamped leading edge/free trailing edge</td>
</tr>
<tr>
<td>EO</td>
<td>Engine Order</td>
</tr>
<tr>
<td>FAB</td>
<td>Foil-air bearing</td>
</tr>
<tr>
<td>FD</td>
<td>Finite Difference</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FFSMM</td>
<td>Full foil structure modal model</td>
</tr>
<tr>
<td>FLE/CTE</td>
<td>Free leading edge/clamped trailing edge</td>
</tr>
<tr>
<td>GR</td>
<td>Galerkin Reduction</td>
</tr>
<tr>
<td>LFCM</td>
<td>Linear force coefficients method</td>
</tr>
<tr>
<td>OIS</td>
<td>Onset of instability speed</td>
</tr>
<tr>
<td>RE</td>
<td>Reynolds Equation</td>
</tr>
<tr>
<td>REB</td>
<td>Rolling-element bearing</td>
</tr>
<tr>
<td>SEFM</td>
<td>Simple elastic foundation model</td>
</tr>
<tr>
<td>SEP</td>
<td>Static equilibrium position</td>
</tr>
<tr>
<td>SESMA</td>
<td>Static equilibrium and stability analysis</td>
</tr>
<tr>
<td>TNDA</td>
<td>Transient nonlinear dynamic analysis</td>
</tr>
</tbody>
</table>

### 1. Introduction

Foil-air Bearings (FABs), also known as gas foil bearings, enable low-maintenance and efficient oil-free operation of high speed turbomachinery. Presently, they are adopted, but not limited to, the air cycling machines for aircraft, clean air turbocompressors, and distributed power generators [1]. FABs operate on hydrodynamic pressure, which is induced by a generated flow between the spinning journal and the top foil (Figure 1). The dynamic pressure deforms the elastic structure and an optimal film thickness is achieved due to the self-acting behaviour of the bearing. The analysis of FAB-rotor systems therefore presents a challenging problem that nonlinearly couples the air film domain with the rotor and foil structure domains.
Such analysis is essential to predict undesirable nonlinear phenomena that manifest themselves as large amplitude sub-synchronous vibrations caused by self-excitation (linear instability) or rotational unbalance [2, 3].

As mentioned in [4], the traditional practice for reducing the computational burden presented by nonlinear FAB-rotor system analysis has been to adopt an approximate non-simultaneous (also referred to as “time lagging”) solution approach wherein the air film, foil and rotor equations were decoupled e.g. [5]. A rotor-FAB system was first solved in a simultaneous (or “fully coupled”) fashion by Bonello and Pham [6, 7], who expressed the nonlinear model in the generic state space format \( \mathbf{s}' = \mathbf{\chi}(\tau, \mathbf{s}) \). This is the standard dynamical system representation [4], where \( (\cdot)' \) denotes differentiation with respect to the time variable \( \tau \), \( \mathbf{s} \) is the state vector, containing the state variables from all three domains (air film, foil and rotor), and \( \mathbf{\chi} \) is a vector of nonlinear functions of \( \tau \) and \( \mathbf{s} \). A transformation is needed to convert the compressible Reynolds Equation (RE) (governing the air film domain) into a subset of time-based differential equations that fits into the above dynamical system representation. This is typically done via spatial discretisation involving a grid or a mesh using Finite Difference (FD) [2, 6] or Finite Element (FE) techniques [3], but a transformation that does not involve spatial discretisation is also possible, namely Galerkin Reduction (GR) [6], which is the subject of the present paper.

Having expressed the dynamical system in the above format, it can be analysed for the following:

1) The nonlinear vibration response under rotor unbalance (or no unbalance) using transient nonlinear dynamic analysis (TNDA), involving numerical integration from prescribed initial conditions \( \mathbf{s}(\tau = 0) \), typically using a stiff solver [8];

2) The static equilibrium configuration \( \mathbf{s} = \mathbf{s}_E \) at each rotational speed \( \Omega \) over a range, by setting \( \mathbf{s}' = \mathbf{0} \);

3) Free small perturbations (linearised vibration) about the static equilibrium configuration \( \mathbf{s} = \mathbf{s}_E \) at given \( \Omega \), to determine the onset of instability speed (OIS) [6] and the Campbell diagrams [9, 10].

The result of the analysis in 2) will correspond to the final result of 1) for no unbalance and when the equilibrium configuration is stable to small perturbations. The results of the analysis in 3) should be consistent with TNDA (i.e. the result of 1)) for low amplitudes (i.e. in the vicinity of \( \mathbf{s} = \mathbf{s}_E \)). Consistency
is perfectly achieved by basing the linearisation on the state Jacobian $\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{s}}|_{s=s_E}$, which is a technique introduced by Bonello and Pham for rotor-FAB systems [6]. The OIS is determined as the speed at which the real part of the leading eigenvalue of the Jacobian changes sign. The traditional approach for linearisation, based on the linearised force coefficients method (LFCM) [11], is widely reported to have some issues with achieving the aforementioned consistency with TNDA for FAB application, particularly with regard to predicting the OIS e.g. [5, 12, 13], and these are attributed to its deviation from the fully coupled dynamical system representation [9].

The original simultaneous solution work of [6, 7] used two alternative approaches (FD, GR) for transformation of the RE, but was limited in a number of aspects, the main ones being:

a) The analysis was restricted to a theoretical symmetric rigid rotor system.

b) The linearisation process was restricted to the determination of the OIS i.e. focused only on the least-damped mode (associated with the leading eigenvalue), rather than the full mode set constituting the Campbell diagram.

c) Apart from the pressure constraint at the open ends of the FAB, no other pressure constraints were considered.

d) The foil structure was a viscoelastic compliance based on the simple elastic foundation model (SEFM) (Winkler foundation) [14] i.e. it was modelled as a continuum of linear springs and parallel viscous dampers, without elastic coupling between points, and neglected top foil detachment at air film regions of sub-atmospheric pressure, as also done in [3,5,12,13] for example.

The above limitations were addressed in subsequent works by Bonello and co-researchers [15,9,10,2,4], but always using FD transformation for the RE. Likewise, other researchers who have taken up the simultaneous solution approach (e.g. [3,12,13,17-21]) have used spatial discretisation for the air film i.e. FD e.g. [19] and the Bubnov-Galerkin FE method e.g. [3] (not to be confused with Galerkin Reduction).

Bonello and Pham [15] extended their previous analysis [6] to a flexible multi-modal rotor on FABs, using component modes for the rotor. Larsen and Santos [3] successfully correlated experimental test results from a rigid rotor rig on two FABs with TNDA simulations. The FABs each had three pads and were
modelled with SEFM for the foil, and FE for the air film, with atmospheric pressure constraints applied to the circumferential ends of each pad. Moreover, the Gümbel condition was applied wherein sub-atmospheric pressures are excluded from the computation of the air film forces of the journal, to correct for foil detachment when this is not considered by the foil model. Nielsen and Santos [17] later modelled top foil detachment by introducing a bilinear SEFM model of the foil, consisting of a modal model for the top foil (based on clamped-free shell modes) which could detach from the SEFM bump foil. By considering a simple rigid rotor with single-pad (i.e. 360°) FAB, and comparing the TNDA simulation from the bilinear-SEFM model with that from the SEFM-Gümbel, close agreement was evident when the FAB was operated with a clamped leading edge and free trailing edge (CLE/FTE). On the other hand, for a free leading edge and clamped trailing edge (FLE/CTE), simulations by Nielsen [18] showed that the film supports sub-atmospheric pressures. Hence, Nielsen [18] concluded that the often-used Gümbel condition may not be applicable for single-pad FAB with FLE/CTE, which is the manufacturer-recommended operating mode for single-pad FABs [2] (multi-pad FABs can be operated in either mode [2]). Bin Hassan and Bonello [16] introduced the full foil structure modal model (FFSMM) that considered elastic coupling between bumps, and their inertia, but ignored the top foil and its detachment. This linear foil model was then used by the same researchers in [2] to simulate the dynamic response of a test rig. In view of the uncertainty in the air film conditions at the circumferential location of the weld (where the leading and trailing edges of the top foil meet, see Figure 1), two FD-based air film models were considered in [2], referred to as “infinite (continuous) \( \theta \)” and “finite \( \theta \)”. In the former, no atmospheric pressure constraint was imposed at the weld, meaning that the pressure there is wholly determined by the dynamics; in the latter, the atmospheric pressure constraint was imposed. For the case in [2] (where the weld was located at the topmost position, as in Figure 1), the difference in predictions between the two models was relatively minor, with the continuous \( \theta \) considered to be more appropriate for high levels of unbalance. The main effect on the predictions in [2] was the Gümbel condition, and simulations without this condition achieved superior correlation with experimental results, thus lending experimental validation to the aforementioned finding from Nielsen regarding single-pad FLE/CTE. Another finding of the work [2] was that the observed nonlinear effects
were satisfactorily predicted using a theoretical model in which the air film was the only source of nonlinearity (as was also the case in [3]). This contrasted with the findings by previous researchers [22] who attributed similar nonlinear effects to nonlinearities in the foil structure and neglected air film effects. Bonello [9,10] addressed the limitation in b) above by developing the Jacobian-based linearisation process into a full eigenvalue analysis in order to extract the Campbell diagram, thus bypassing the traditional LFCM. Since s contained grid point state variables associated with the air film, the Jacobian \( \frac{\partial \mathbf{x}}{\partial \mathbf{s}} \bigg|_{\mathbf{s} = \mathbf{s}_E} \) had a multitude of eigenvalues. Thus, a filtering technique was devised in [9,10] to extract only those modes that involved significant rotor vibration. The resulting Campbell diagram (modal frequencies vs speed) and associated information, including modal damping ratios, mode shapes and whirl directions, provide a full picture of linearised behaviour of the rotor-FAB system (including linear stability and its loss, critical speeds,\ldots etc) that is an important tool in interpreting the wider nonlinear response [9]. Such a static equilibrium, stability and modal analysis (SESMA) approach was later adopted by Von Osmanski et al [13] as the benchmark for assessing the accuracy of LFCM. They established that previously reported discrepancies associated with LCFM were artificially large due to improper equivalence of foil damping levels in the methods being compared. A similar conclusion was reached by Pronobis and Liebich [23]. However, both [13] and [23] report that discrepancies persist and evidence is presented in [13] to suggest that this is due to the way the foil compliance is considered in LCFM (as hypothesised earlier in [12]). Bonello [4] later used SESMA to extract the Campbell diagrams of a rotor-FAB system with an advanced foil model comprising a modal beam top foil that can detach from discrete bumps according to a smoothed bilinear model. This model confirmed Nielsen’s model finding [18] that foil detachment in FLE/CTE operation does not prevent sub-atmospheric pressures. However, comparisons between the Campbell diagram from the advanced foil model with FLE/CTE and that from the simple foil model (SEFM) with/without Gümbel showed that the application of Gümbel, even though inappropriate in principle, actually led to improved results for the symmetric rigid rotor-FAB system considered in [4]. Hence, the applicability of Gümbel can be considered a source of uncertainty for FLE/CTE and it is important to have the option of applying it.
As far as foil damping is concerned, all above simultaneous solution approaches use a linear viscous damping model adapted from a structural damping loss factor model (as explained in [4]). Von Osmanski et al. [24] considered dry friction in the simultaneous solution scheme, but did not capture the unbalance response, thus reverting to linear damping in subsequent work e.g. [13]. It is noted that advanced foil models considering Coulomb friction as proposed by Arghir and Benchekroun [25] are outside the scope of this paper since it is not yet clear how these can be fitted into the simultaneous solution scheme.

It is now opportune to develop the GR method for some of the developments that followed the preliminary simultaneous solution work of [6,7]. The motivation is the computational benefit that accrues from the drastic condensation of the problem following the elimination of the two-dimensional grid or mesh used for the air film in FD or FE. Bonello and Pham’s preliminary application of GR to a rotor-FAB system [6,7] was developed from the 1960’s work of Cheng and Pan [26], who applied GR to gas-lubricated plain cylindrical journal bearings. In GR, basis functions in the axial (ξ) and circumferential (θ) directions are used to approximate the RE’s air film state variable as a truncated series weighted by time-varying coefficients. Substitution into the RE leads to a residual function which is then minimised by multiplication by each basis function and integrating with respect to ξ, θ, thus yielding a set of first order differential equations with time as the independent variable. Such pre-solution spatial integrations constitute the main challenge in GR transformation since they are numerous for arbitrarily high-order GR [6]. However, Bonello and Pham [6] overcame this by identifying the fundamental structure of the integrals, enabling the use of an arbitrarily high-order GR transformation. To the authors’ knowledge, since the 2014 work of [6] there has been no simultaneous solution rotor-FAB work that used GR of any form, except the recent work of Baum et al. [27]. However, the latter work is better described as “hybrid GR/FD”, since discretisation was eliminated only in the axial direction. This was done using a single base function that was purposely optimised. The transformed RE, which still had a function of θ and time, was then transformed into a set of first order differential equations in time using FD approximations for the partial derivatives with respect to θ. The reduced order model was verified against two-dimensional FD by comparing the respective pressure fields for a set of five prescribed journal states, rather than via solution of the rotor-bearing
problem. Another interesting development of [27] is the replacement of the linear spring term in the SEFM by a nonlinear spring, while retaining linear damping and neglect of cross-coupling. This was done through a term $e^{w/b}$ where $w$ is the radial deflection and $b$ a compliance constant found by fitting the proposed force-deflection characteristic to static force-deflection data obtained by pushing a non-rotating shaft relative to the foil. While it is suggested in [27] that such a model could cover the case of top foil detachment through its different behaviour for $w < 0$, this effect was not considered and $b$ was determined from published data with $w > 0$ only.

The novel contribution of the present paper is the development of the arbitrary-order GR method to address above-listed limitations a), b), c) of the preliminary GR work [6,7] i.e.

- Application to generic flexible (multi-modal) rotor on FABs;
- Jacobian-based linearisation of the GR-transformed system for the extraction the full mode set constituting the Campbell diagram and linear stability map;
- The facility to accommodate a pressure constraint at a circumferential location and/or Gümbel.

The above developments enable a comprehensive validation on two previously considered systems using alternative FD-based simulations (both TNDA and SESMA) and experimental data (from [2]).

Above-listed limitation d) is not addressed at this stage, except for showing how a nonlinear spring function, as recently proposed in [27], can be readily included.

![Figure 1. Single-pad bump-type FAB with a free leading edge (LE)/clamped trailing edge (TE) i.e. FLE/CTE (B is the centre of the bearing shell, J is the centre of the journal).]
2. MODELLING AND COMPUTATION

In sections 2.1-2.3 the GR transformation for a single-pad FAB is developed from [6] into a generic formulation that can accommodate different basis functions according to the applied pressure constraints. In section 2.4 the GR-transformed equations of the FAB are applied to a generic flexible rotor and the solution process is presented.

2.1 GR transformation of air film equation

With reference to Figure 1, the FAB is of length $L$ and radius $R$, and the angular coordinate $\theta$ is measured from the weld location. The Reynolds Equation (RE) is expressed as follows:

$$\frac{1}{B} \left\{ \frac{\partial}{\partial \theta} \left[ \psi \left( \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \tilde{h}}{\partial \theta} \right) \right] + \frac{1}{\tilde{L}^2} \frac{\partial}{\partial \xi} \left[ \psi \left( \frac{\partial \psi}{\partial \xi} - \psi \frac{\partial \tilde{h}}{\partial \xi} \right) \right] \right\} - \frac{\partial \psi}{\partial \theta} - \frac{d \psi}{d \tau} = 0$$

(1)

In eq. (1): $\xi = \pi z_{local}/L$ (where $z_{local}$ is the local axial coordinate); $\psi \equiv \tilde{p} \tilde{h}$ (where $\tilde{p} = p/p_a$ and $\tilde{h} = h/c$ are the non-dimensional air film absolute pressure and thickness respectively at a position $(\xi, \theta)$, $p_a$ being the atmospheric pressure and $c$ the radial clearance); $B$ is the bearing number [9]; $\tilde{L} = L/(\pi R)$; $\tau = \Omega t/2$ defines non-dimensional time (where $\Omega$ is the rotational speed).

The RE state variable $\psi$ is written as [6]:

$$\psi = \tilde{h} + \varphi$$

(2)

where $\varphi = \tilde{p}_g \tilde{h}$ is that part of $\psi$ that is associated with the gauge pressure $p_g = \tilde{p}_g p_a$. The GR transformation of $\varphi$ is expressed as

$$\varphi = \varphi^T_b(\xi, \theta) \varphi_{co}(\tau)$$

(3)

where

$$\varphi_b(\xi, \theta) = \varphi_{b\xi}(\xi) \ast \varphi_{b\theta}(\theta) \equiv \text{diag} \left( \varphi_{b\xi}(\xi) \right) \varphi_{b\theta}(\theta)$$

(4)
\[ \varphi_{b\xi}(\xi) = \begin{bmatrix} \varphi_{b\xi e}^T \end{bmatrix}(:), \quad \varphi_{b\theta}(\theta) = \begin{bmatrix} \varphi_{b\theta e}^T \end{bmatrix} \] (5a,b)

In eqs. (3), (4), (5a,b), \( \varphi_b \) is the vector of basis functions of \( \varphi \) in two-dimensional space \((\xi, \theta)\) and \( \varphi_{co} \) is the vector of time dependent coefficients of \( \varphi \). \( \varphi_{b\xi e}, \varphi_{b\theta e} \) are vectors of elementary basis functions of \( \varphi \) in \( \xi \) and \( \theta \) respectively, of respective lengths \( l_{\varphi_{b\xi e}}, l_{\varphi_{b\theta e}} \). The following two operators are used:

- \( \cdot \cdot \cdot \) to denote term by term multiplication;
- \( \{ \cdot \cdot \cdot \}(:) \) to denote the vectorisation of a matrix \( \{ \cdot \cdot \cdot \} \) i.e.

\[
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}(:) = [a_{11} \quad a_{21} \quad a_{12} \quad a_{22}]^T
\]

Hence, in eqs. (3), (4), (5a,b), \( \varphi_b, \varphi_{co}, \varphi_{b\xi}, \varphi_{b\theta} \) are each of dimension \( l_{\varphi_{b\xi e}}l_{\varphi_{b\theta e}} \times 1 \).

The non-dimensional film thickness is given by

\[
\tilde{h} = 1 + \mathbf{e}^T \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix} + \tilde{w}
\] (6)

where \( \mathbf{e} = [\varepsilon_x \quad \varepsilon_y]^T \) is the vector of non-dimensional Cartesian displacements of \( \mathbf{J} \) relative to \( \mathbf{B} \) (Figure 1) and \( \tilde{w} \) is the radial deflection of the foil pad. Neglecting variation of \( \tilde{w} \) in the axial direction, as in [6], its GR transformation is expressed as

\[
\tilde{w} = \tilde{w}_b(\theta)\tilde{w}_{co}(\tau)
\] (7)

where \( \tilde{w}_b \) is the vector of basis functions of \( \varphi \) in one-dimensional space \( \theta \) and \( \tilde{w}_{co} \) the vector of time dependent coordinates. The non-dimensional film thickness \( \tilde{h} \) can then be written as

\[
\tilde{h} = h_\theta(\theta)h_{co}
\] (8)

where \( h_\theta \) is a vector of functions of \( \theta \) and \( h_{co} \) is the vector of corresponding time-dependent coefficients that is composed of the elements of \( \mathbf{e} \) and \( \tilde{w}_{co} \):

\[
h_{co} = h_{co}(\mathbf{e}(\tau), \tilde{w}_{co}(\tau))
\] (9)
Hence, defining

\[ u(\xi, \vartheta) = \begin{bmatrix} \Phi_b(\xi, \vartheta) \\ \hat{h}_\vartheta(\vartheta) \end{bmatrix}, \quad \psi_{co}(\tau) = \begin{bmatrix} \varphi_{co}(\tau) \end{bmatrix} \] (10, 11)

then, from eqs. (2), (3) and (8), \( \psi \) can be written as

\[ \psi = u(\xi, \vartheta)^T \psi_{co}(\tau) \] (12)

The Galerkin transformation of eq. (1) is achieved by minimising the residual [6]

\[ \int_{\vartheta=0}^{2\pi} \int_{\xi=-\pi/2}^{\pi/2} R_{\text{air}} \Phi_b(\xi, \vartheta) \, d\xi \, d\vartheta = 0 \] (13)

where the residual function \( R_{\text{air}} \) is the left hand side of eq. (1), which is rewritten as

\[ R_{\text{air}} = \frac{1}{B} \left( \hat{h}^2 + \psi \frac{\partial^2 \hat{h}}{\partial \vartheta^2} - \psi \frac{\partial \psi \, \partial \hat{h}}{\partial \vartheta} - \psi^2 \frac{\partial^2 \hat{h}}{\partial \vartheta^2} + \frac{\hat{h}}{L^2} \left( \frac{\partial^2 \psi}{\partial \xi^2} \right)^2 + \frac{\psi \hat{h}}{L^2} \right) - \frac{\partial \psi}{\partial \vartheta} - \frac{\partial \psi}{\partial \tau} \] (14)

Substituting eqs. (8) and (12) into eqs. (14) and (13), along with eq. (3), gives the following system of equations:

\[ \frac{1}{B} \mathbf{K}_r \mathbf{r}(\tau) - \mathbf{K}_\psi \psi_{co}(\tau) - \mathbf{K}_{\psi_d} \psi_{co}(\tau) = 0 \] (15)

where \( \mathbf{r}(\tau) \) is given by

\[ \mathbf{r}(\tau) = \left\{ \psi_{co}^T \right\}^T \hat{h}_{co}^T \] (16)

and the constant matrices \( \mathbf{K}_r, \mathbf{K}_\psi \) and \( \mathbf{K}_{\psi_d} \) are determined from the following expressions.

\[ \mathbf{K}_r = \]

\[ \int_{\vartheta=0}^{2\pi} \int_{\xi=-\pi/2}^{\pi/2} \Phi_b \begin{bmatrix} \left( \frac{\partial u \, \partial u^T}{\partial \vartheta} \right)(\cdot) \hat{h}_{\vartheta}(\cdot) \end{bmatrix}^T \, d\xi \, d\vartheta + \int_{\vartheta=0}^{2\pi} \int_{\xi=-\pi/2}^{\pi/2} \Phi_b \begin{bmatrix} \left( \frac{\partial^2 u \, \partial u^T}{\partial \vartheta^2} \right)(\cdot) \hat{h}_{\vartheta}(\cdot) \end{bmatrix}^T \, d\xi \, d\vartheta \]
\[
\begin{align*}
&\int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \left\{ \left( \frac{\partial \textbf{u}^T}{\partial \phi} \right) \left( \frac{\partial \hat{\textbf{h}}_{\phi}}{\partial \phi} \right) \right\}^T \ d\xi d\phi - \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \left\{ \left( \text{uu}^T \right) \left( \frac{\partial^2 \hat{\textbf{h}}_{\phi}}{\partial \phi^2} \right) \right\}^T \ d\xi d\phi \\
&+ \frac{1}{L^2} \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \left\{ \left( \frac{\partial \textbf{u}}{\partial \xi} \frac{\partial \textbf{u}^T}{\partial \xi} \right) \right\}^T \ d\xi d\phi + \frac{1}{L^2} \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \left\{ \left( \text{uu}^T \right) \left( \frac{\partial^2 \textbf{u}}{\partial \xi^2} \right) \right\}^T \ d\xi d\phi 
\end{align*}
\]

(17)

\[
\begin{align*}
\textbf{K}_\psi &= \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \frac{\partial \textbf{u}^T}{\partial \phi} d\xi d\phi \\
\textbf{K}_{\psi,d} &= \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \textbf{u}^T d\xi d\phi 
\end{align*}
\]

(18, 19)

Substituting eq. (10) into eq. (19) shows that

\[
\textbf{K}_{\psi,d} = \textbf{D}_{\varphi_{\text{b}}} + \textbf{E}
\]

(20a)

where

\[
\begin{align*}
\textbf{D}_{\varphi_{\text{b}}} &= \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \varphi_{\text{b}}^T d\xi d\phi \\
\textbf{E} &= \int_{\phi = 0}^{\pi} \int_{\xi = 0}^{\pi/2} \varphi_{\text{b}} \hat{\textbf{h}}_{\phi}^T d\xi d\phi 
\end{align*}
\]

(20b,c)

\(\textbf{D}_{\varphi_{\text{b}}}\) is a diagonal matrix due to the orthogonality of the elementary basis functions in \(\varphi_{\text{b}}\xi\) and \(\varphi_{\text{b}}\theta\) (eqs. (4), (5)) Substituting eqs. (20a,b,c) into eq. (15), and rearranging, gives the GR-transformed RE equation of the air film:

\[
\varphi'_{\text{co}} = \chi_{\varphi'_{\text{co}}} \left( \varphi_{\text{co}}, \bar{\text{w}}_{\text{co}}, \varepsilon, \bar{\text{w}}'_{\text{co}}, \varepsilon' \right) = \textbf{D}_{\varphi_{\text{b}}}^{-1} \left\{ \frac{1}{B} \textbf{K}_r \textbf{r} - \textbf{K}_{\psi} \varphi_{\text{co}} - \textbf{E} \hat{\textbf{h}}'_{\text{co}} \right\}
\]

(21)
With reference to Figure 1, the expression for the Cartesian components of the air film forces on the journal is:

\[ f_J(\mathbf{\varphi}_{co}, \mathbf{\bar{w}}_{co}, \mathbf{\varepsilon}) = p_a LR \int_{\vartheta=0}^{\vartheta=2\pi} \left\{ \tilde{p}_{g\xi_{av}}(\vartheta, \mathbf{\varphi}_{co}, \mathbf{\bar{w}}_{co}, \mathbf{\varepsilon}) \left[ \begin{array}{c} \sin \vartheta \\ -\cos \vartheta \end{array} \right] \right\} d\vartheta \]  

(22)

where \( \tilde{p}_{g\xi_{av}} \) is the average non-dimensional gauge pressure along the axial direction for a given angular position \( \vartheta \). From eqs. (3), (8):

\[ \tilde{p}_{g\xi_{av}}(\vartheta, \mathbf{\varphi}_{co}, \mathbf{\bar{w}}_{co}, \mathbf{\varepsilon}) = \frac{\text{diag}(\mathbf{\Phi}_b \xi) \mathbf{\phi}_{b\theta}(\vartheta) \mathbf{\varepsilon}^T \mathbf{\varphi}_{co}}{\mathbf{h}_{\vartheta}^T(\vartheta) \mathbf{h}_{co}} \]  

(23)

where

\[ \mathbf{\Phi}_b \xi = \frac{1}{\pi} \int_{\xi=-\pi}^{\xi=\pi} \mathbf{\Phi}_b(\xi) d\xi \]  

(24)

Evaluation of eq. (22) requires equations governing \( \mathbf{\varphi}_{co}, \mathbf{\bar{w}}_{co}, \mathbf{\varepsilon} \) which are respectively given by eq. (21), and the equations in the subsequent sections 2.2, 2.3.

### 2.2 GR transformation of SEFM foil equations

In the simple elastic foundation model, top foil detachment is neglected and the foil pad structure is defined by a continuously distributed stiffness per unit area \( k \) and corresponding equivalent viscous damping coefficient per unit area \( k\eta/\omega_{ref} \). In the latter expression, \( \eta \) is the hysteretic damping factor and \( \omega_{ref} \) is the reference frequency used to change the damping model to frequency-independent viscous damping to enable arbitrary time domain analysis [4]. \( \omega_{ref} \) is fixed for a given speed and, in most works e.g. [2,3] \( \omega_{ref} = \Omega \). As an alternative, the speed-dependent frequency of the mode causing self-excitation can be used for \( \omega_{ref} \), as first done by von Osmanski [13], and later by Bonello [4] who used both options and discussed the reasoning behind both. The dynamical equation of the SEFM is therefore
\[
\frac{2\tilde{\omega}_{\text{ref}}}{\eta} \left\{ \tilde{p}_{g,av} \frac{k}{\tilde{k}} - \tilde{w} \right\} - \frac{d\tilde{w}}{dt} = 0
\] (25)

where \( \tilde{\omega}_{\text{ref}} = \omega_{\text{ref}}/\Omega \), and the non-dimensional stiffness per unit area \( \tilde{k} = kc/p_a \). In line with most previous works and the reasoning in [4], the present work takes \( \tilde{\omega}_{\text{ref}} = 1 \) i.e. \( \omega_{\text{ref}} = \Omega \). The Galerkin transformation of eq. (25) is achieved by minimising the residual

\[
\int_{\theta=0}^{\theta=2\pi} R_{\text{foil}} \tilde{W}_b(\theta) d\theta = 0
\] (26)

where \( R_{\text{foil}} \) is the left hand side of eq. (25) with \( \tilde{w} \) given by eq. (7). The GR transformation of eq. (25) is therefore

\[
\tilde{W}_c' = X_{\tilde{W}_c'}(\varphi_{co}, \tilde{W}_{co}, \varepsilon) = \frac{2}{\eta} \left\{ D_{\tilde{w}_b}^{-1} f_g - \tilde{W}_{co} \right\}
\] (27)

where

\[
 f_g(\varphi_{co}, \tilde{W}_{co}, \varepsilon) = \int_{\theta=0}^{\theta=2\pi} \left\{ \tilde{p}_{g,av}(\theta, \varphi_{co}, \tilde{W}_{co}, \varepsilon) \tilde{W}_b(\theta) \right\} d\theta
\] (28)

and \( D_{\tilde{w}_b} \) is the diagonal matrix

\[
D_{\tilde{w}_b} = \int_{\theta=0}^{\theta=2\pi} \tilde{W}_b^T(\theta) \tilde{W}_b(\theta) d\theta
\] (29)

If the pad deflection at a given angular location depends only on the (axially-averaged) air pressure at that location (no elastic cross coupling), the same \( \theta \)-boundary conditions should then apply for both \( \tilde{w} \) and \( \varphi \).

Hence, we can take the basis functions of \( \tilde{w} \) to be identical to the elementary basis functions of \( \varphi \) in the \( \theta \) direction i.e.

\[
\tilde{W}_b(\theta) \equiv \varphi_{b\theta}(\theta)
\] (30)

Notice that, if one wishes to replace the spring element of the SEFM by a nonlinear spring, as done recently by Baum et al. [27], the term \( \tilde{w} \) within the brackets in eq. (25) would be replaced by \( \left( 1/\tilde{k} \right) \left( e^{k\tilde{w}} - 1 \right) \) (this can be verified by expanding \( e^{k\tilde{w}} \) as an infinite series and truncating nonlinear terms to recover the
expression in eq. (26)). Considering eqs. (7), (26), term \( \hat{\mathbf{w}}_{\text{co}} \) on the left hand of eq. (27) would then be replaced by
\[
D_{\hat{\mathbf{w}}_{\text{b}}}^{-1} \int_{\theta=0}^{\theta=2\pi} \left( \frac{1}{\tilde{k}} e^{\tilde{k}\hat{\mathbf{w}}_{\text{b}}^T(\theta)\hat{\mathbf{w}}_{\text{co}}} - 1 \right) \hat{\mathbf{w}}_{\text{b}}(\theta) d\theta.
\]
Such a term can be evaluated for given \( \hat{\mathbf{w}}_{\text{co}} \) during the course of the solution process (section 2.4), as already done for \( \mathbf{f}_g \) (eq. (28)), thus adding little extra computational cost.

2.3 Elementary basis functions for FAB equations

This section gives expressions for the vectors of elementary basis functions \( \varphi_{\text{b}\xi} \), \( \varphi_{\text{b}\theta} \). Since the FAB is open at both ends, the basis functions of \( \varphi_{\text{b}\xi} \) are symmetric about \( \xi = 0 \) i.e. \( \varphi_{\text{b}\xi} \) is the \( N \times 1 \) vector

\[
\varphi_{\text{b}\xi} = \begin{bmatrix} \cos \xi & \cos 3 \xi & \cdots & \cos (2N - 1) \xi \end{bmatrix}^T
\]

In eq. (31), \( N \) is the order of \( \varphi_{\text{b}\xi} \) and its length \( l_{\varphi_{\text{b}\xi}} = N \).

2.3.1 No pressure constraint at weld location

This is the case previously considered in the GR of [6,7] and its FD equivalent is referred to in [2] as the “infinite (continuous) \( \theta \)” pad FD model. \( \varphi_{\text{b}\theta} \) is the \( (1 + 2M) \times 1 \) vector

\[
\varphi_{\text{b}\theta} = \begin{bmatrix} 1 & \cos \theta & \sin \theta & \cos 2\theta & \sin 2\theta & \cdots & \cos M\theta & \sin M\theta \end{bmatrix}^T
\]

\( M \) is the order of \( \varphi_{\text{b}\theta} \) and its length \( l_{\varphi_{\text{b}\theta}} = 1 + 2M \). Expressing the elements of \( \varphi_{\text{co}} \) (eq. (3)) as follows

\[
\varphi_{\text{co}}(\tau) = \begin{bmatrix} [C_1 & A_{1,1} & B_{1,1} & \cdots & A_{1,M} & B_{1,M}] & \cdots & [C_N & A_{N,1} & B_{N,1} & \cdots & A_{N,M} & B_{N,M}] \end{bmatrix}^T
\]

and substituting eq. (31), (32), (33) into eq. (3), the expression for \( \varphi \) is

\[
\varphi = \sum_{n=1}^{N} \cos (2n - 1) \xi \left[ C_n(\tau) + \sum_{m=1}^{M} \left[ A_{n,m}(\tau) \cos m\theta + B_{n,m}(\tau) \sin m\theta \right] \right]
\]

Noting from eq. (30), \( \hat{\mathbf{w}}_{\text{b}}(\theta) \equiv \varphi_{\text{b}\theta}(\theta) \), the vector \( \hat{\mathbf{w}}_{\text{co}} \) in eq. (7) is expressed as

\[
\hat{\mathbf{w}}_{\text{co}}(\tau) = \begin{bmatrix} W_0 & W_{c,1} & W_{s,1} & \cdots & W_{c,M} & W_{s,M} \end{bmatrix}^T
\]

and substituting into eq. (7) gives
\[ \tilde{w} = W_0(\tau) + \sum_{m=1}^{M}[W_{c,m}(\tau)\cos \theta + W_{s,m}(\tau)\sin \theta] \]  

From eq. (6), the expressions of \( \tilde{h}_\theta \) and \( \tilde{h}_c \) in eq. (8) are taken as:

\[ \tilde{h}_\theta = \tilde{w}_b(\theta) \equiv \varphi_{b\theta e}(\theta) \]  
\[ \tilde{h}_c = [(W_0 + 1) (W_{c,1} - \epsilon_y) (W_{s,1} + \epsilon_x) W_{c,2} W_{s,2} \ldots W_{c,M} W_{s,M}]^T \]  

2.3.2 Atmospheric pressure constraint at weld location i.e. \( \tilde{p}_g = 0 \) at \( \theta = 0, 2\pi \)

This case has not been previously considered for GR of FABs and corresponds to the “finite \( \theta \)” pad FD model in [2]. \( \varphi_{b\theta e} \) is the \( M \times 1 \) vector

\[ \varphi_{b\theta e} = [\sin(\theta/2) \sin(\theta/2) \ldots \sin(M(\theta/2))]^T \]  

\( M \) is the order of \( \varphi_{b\theta e} \) and its length \( l_{\varphi_{b\theta e}} = M \). This choice of the elementary basis functions ensures that \( \varphi = \tilde{p}_g \tilde{h} = 0 \) at \( \theta = 0, 2\pi \) without imposing a similar constraint at intermediate values of \( \theta \). Expressing the elements of \( \varphi_{co} \) in eq. (3) as follows

\[ \varphi_{co} = [[B_{1,1} \ldots B_{1,M}] \ldots [B_{N,1} \ldots B_{N,M}]]^T \]  

and substituting (31), (39), (40) into eq. (3), the expression for \( \varphi \) is

\[ \varphi = \sum_{n=1}^{N} [\cos(2n - 1)\xi(\sum_{m=1}^{M} B_{n,m}(\tau)\sin(\theta/2))] \]  

Noting from eq. (30) that \( \tilde{w}_b(\theta) \equiv \varphi_{b\theta e}(\theta) \), the vector \( \tilde{w}_{co} \) in eq. (7) is expressed as

\[ \tilde{w}_{co}(\tau) = [W_{s,1} \ldots W_{s,M}]^T \]  

and substituting into eq. (7) gives

\[ \tilde{w} = \sum_{m=1}^{M} W_{s,m}(\tau)\sin(\theta/2) \]  

From eq. (6), the expressions of \( \tilde{h}_\theta \) and \( \tilde{h}_c \) in eq. (8) are taken as:

\[ \tilde{h}_\theta = [\tilde{w}_b^T(\theta) 1 \sin \theta - \cos \theta]^T = [\varphi_{b\theta e}^T(\theta) 1 \sin \theta - \cos \theta]^T \]
 Unless otherwise stated, it is assumed that the weld is located as in Figure 1. For other locations, the \( \sin \vartheta \), \( \cos \vartheta \) terms in eqs. (44), (22) and (6) are replaced by \( \sin(\alpha + \vartheta) \), \( \cos(\alpha + \vartheta) \) where \( \alpha \) is the angular position of the weld relative to topmost location and \( \vartheta \) is still measured from the weld location.

### 2.3.3 Note on computation of terms in constant coefficient matrices of eq. (21)

Having prescribed the air film elementary basis functions \( \varphi_{b\vartheta e} \) and the foil pad basis functions \( \tilde{w}_b \) \( (\equiv \varphi_{b\vartheta e} \) in present case), the constant coefficient matrices in eq. (21) can be computed by evaluating the integrals in eqs. (17-20). This is a one-off calculation for given GR order \( N, M \) and the most time consuming part is the evaluation of the terms in \( K_r \), which is of size

\[
(l_{\varphi_{b\vartheta e}} l_{\varphi_{b\vartheta e}}) \times \left( \left( l_{\varphi_{b\vartheta e}} l_{\varphi_{b\vartheta e}} + l_{h_\vartheta} \right)^2 l_{h_\vartheta} \right)
\]

where \( l_{h_\vartheta} \) is the length of \( \tilde{h}_\vartheta \) (eq. (37) or (44)). For the basis functions of section 2.3.1 \( l_{\varphi_{b\vartheta e}} = N, l_{\varphi_{b\vartheta e}} = l_{h_\vartheta} = 1 + 2M \) with typical values of \( N = 3, M = 8 \) (section 3), the size of \( K_r \) is \( 51 \times 78608 \). For the basis functions of section 2.3.2 \( l_{\varphi_{b\vartheta e}} = N, l_{\varphi_{b\vartheta e}} = M, l_{h_\vartheta} = M + 3 \) with typical values of \( N = 3, M = 16 \), the size of \( K_r \) is \( 48 \times 85291 \). Despite this enormous number of terms, the computation for \( K_r \) can be performed in a short time with appropriate coding and by noting from eq. (17) and eq. (10) that a typical term in each of the integral matrices of eq. (17) must be of the form

\[
\int_{\vartheta=0}^{\vartheta=2\pi} f(\vartheta)d\vartheta \int_{\xi=-\pi/2}^{\xi=\pi/2} g(\xi)d\xi
\]

where:

- \( f(\vartheta) \) is the product of four terms taken respectively from \( \varphi_{b\vartheta e}, \varphi_{b\vartheta e} \) or \( \tilde{h}_\vartheta, \varphi_{b\vartheta e} \) or \( \tilde{h}_\vartheta, \tilde{h}_\vartheta \) (or their derivatives, depending on which integral matrix of eq. (17) is being considered);
- \( g(\xi) \) is the product of up to three terms taken from \( \varphi_{b\xi e} \) (or its derivatives).
2.4 Combination with multi-modal rotor equations and solution process

Assuming there are two FABs, and using the subscripts “L”, “R” to denote the left and right hand FABs respectively, the dynamical system representation of a generic FAB-supported rotor system using the present GR transformation can be expressed as:

\[
\mathbf{s}' = \begin{bmatrix}
\varphi'_{\text{col}} \\
\dot{w}_{\text{col}} \\
\varphi'_{\text{cor}} \\
\dot{w}_{\text{cor}} \\
q \\
q'
\end{bmatrix} = \chi(\tau, s) = \begin{bmatrix}
\chi_{\varphi'_{\text{col}}}(\varphi_{\text{col}}, \dot{w}_{\text{col}}, \varepsilon_{L}, \dot{w}_{\text{col}}', \varepsilon_{L}') \\
\chi_{\dot{w}_{\text{col}}}(\varphi_{\text{col}}, \dot{w}_{\text{col}}, \varepsilon_{L}) \\
\chi_{\varphi'_{\text{cor}}}(\varphi_{\text{cor}}, \dot{w}_{\text{cor}}, \varepsilon_{R}, \dot{w}_{\text{cor}}', \varepsilon_{R}') \\
\chi_{\dot{w}_{\text{cor}}}(\varphi_{\text{cor}}, \dot{w}_{\text{cor}}, \varepsilon_{R}) \\
\chi_{\text{rotor}}(\tau, s)
\end{bmatrix}
\]

where the right hand sides of the equations of the air films and foils \(\chi_{\varphi'_{\text{col}}}, \chi_{\dot{w}_{\text{col}}}, \chi_{\varphi'_{\text{cor}}}, \chi_{\dot{w}_{\text{cor}}}, \chi_{\text{rotor}}\) are given by eqs. (21), (27) respectively, and the right hand side of the rotor equation is given by

\[
\chi_{\text{rotor}}(\tau, s) =
\begin{bmatrix}
\frac{4}{J^2} \left[ -\Lambda \mathbf{q} + H_{u}^T \mathbf{f}_u(\tau) + H_{s}^T \mathbf{f}_s + \frac{\alpha}{2} H_{g}^T \mathbf{H} \mathbf{q}' + H_{f_L}^T \mathbf{f}_{f_L}(\varphi_{\text{col}}, \dot{w}_{\text{col}}, \varepsilon_{L}) + H_{f_R}^T \mathbf{f}_{f_R}(\varphi_{\text{cor}}, \dot{w}_{\text{cor}}, \varepsilon_{R}) \right]
\end{bmatrix}
\]

A truncated series of \(H\) component modes is used to transform the degrees of freedom of the rotor to modal coordinates [9]. These component modes are the undamped free vibration modes of the given system at zero speed with the nonlinear elements (FABs) removed. In eq. (47), \(\mathbf{q}\) is the \(H \times 1\) column matrix (vector) of modal coordinates of the rotor and \(\Lambda\) is the \(H \times H\) diagonal matrix of squares of the circular frequencies of the component modes. \(\mathbf{f}_u\) is the vector of rotational unbalance forces and \(\mathbf{f}_s\) is the vector of the distributed weight of the rotor. \(\mathbf{f}_{f_L, R}\) are the \(2 \times 1\) vectors of air film pressure forces exerted by the respective FABs in the \(x, y\) directions, evaluated according to eq. (22). The matrix symbols in eq. (47) are well documented e.g. [4, 9]. The term \(H_{g}^T \mathbf{H} \mathbf{q}'\) takes into account the gyroscopic effect, and the modal matrices \(H_{f_L, R}\) are associated with the vectors \(\mathbf{f}_u, \mathbf{f}_s, \mathbf{f}_{f_L, R}\) respectively. It is noted that the vectors of the journal eccentricities at the respective bearings, \(\varepsilon_{L, R}\), are given by
\[ \varepsilon_{LR} = H_{q_{LR}} q(\tau)/c_{LR} \]  

(48)

where \( c_{LR} \) denote the corresponding radial clearances.

Transient nonlinear dynamic analysis (TNDA) of eq. (46) can be performed using the implicit time domain integrators in Matlab (ode23s or ode15s).

With regard to static equilibrium, stability and modal analysis (SESMA), this follows the scheme presented in [9], but with modification to the implementation to account for the state variables used in the present (GR) transformation of the air film and foil (i.e. \( \varphi_{coLR}, \tilde{w}_{coLR} \) in eq. (46)) being of a fundamentally different nature to the air film and foil state variables used within \( s \) in [9] (which referred to values of \( \psi \) and \( \tilde{w} \) at the FD grid and its circumferential locations respectively).

2.4.1 SESMA overview

Figure 2 gives a flowchart of the SESMA procedure. For the purpose of generating the Campbell diagram, the results of the eigenvalue analysis are filtered using tandem application of the minimum journal amplitude criterion (eq. (51) of reference [9]) and the maximum allowable modal damping ratio criterion [9]. The former criterion requires each eigenvector \( \rho^{(n)} (n = 1 \ldots N_o) \) to be scaled by a factor \( \kappa^{(n)} \) such that the FAB journal or foil pad vibration (whichever of these is bigger) is \( 100\sigma\% \) of the radial clearance.

The scaled mode is then rejected if its journal vibration is not at least \( 100C\sigma\% \) of the radial clearance (as in previous works [9, 4], \( \sigma = 0.2 \) and \( C = 0.1 \), while the upper limit for damping ratio is 0.7). The formula for \( \kappa^{(n)} \) is given by eqs. (42, 43) in reference [9] and requires the complex amplitudes of the corresponding modal vibrations of the L (left), R (right) journals \( \rho^{(n)}_{\varphi_{LR}} \) and the corresponding foil pads \( \rho^{(n)}_{w_{LR}} \). When working with GR, these are determined as follows.

- Partition the eigenvector in accordance to the partitioning of state-vector \( s \) (in eq. (46)):

\[ \rho^{(n)} = \begin{bmatrix} \rho^{(n)}_{\varphi_{coL}} & \rho^{(n)}_{w_{coL}} & \rho^{(n)}_{\varphi_{coR}} & \rho^{(n)}_{w_{coR}} & \rho^{(n)}_{\varphi} & \rho^{(n)}_{\varphi'} \end{bmatrix}^T \]  

(49)

- Using eqs. (48) and (7), and the sub-vectors in eq. (49), calculate the complex amplitudes of the corresponding modal vibrations of the journals and corresponding foil pads:
\[
\rho_{\text{FLR}}^{(n)} = H_{f\text{FLR}} \rho_q^{(n)}/c_{\text{LR}}
\]

\[
\rho_{\text{WL}}^{(n)} = \begin{bmatrix}
\tilde{w}_b^T(\dot{\theta}_1) \\
\vdots \\
\tilde{w}_b^T(\dot{\theta}_{N_R})
\end{bmatrix} \rho_{\text{col LR}}^{(n)}
\]

where \(\rho_{\text{WL}}^{(n)}\) are the vectors of complex amplitudes of the L, R foil pads at angular positions \(\dot{\theta}_1 \ldots \dot{\theta}_{N_R}\) covering the full extent of the respective pads.

\[\text{Determine static equilibrium condition } s = s_E \text{ at speed } \Omega \text{ by solving } \chi(0, s)|_{f_u=0} = 0\]

\[\text{Compute Jacobian } J = \partial \chi / \partial s |_{f_u=0, s=s_E}\]

\[\text{Eigenvalue analysis of } J\]

\[\text{Eigenvalues } \lambda_{n,\text{Re}} \pm j \lambda_{n,\text{Im}}, \lambda_{n,\text{Im}} > 0 \quad (n = 1 \ldots N_u)\]

\[\text{Damped natural frequencies } \sigma_{d,n} \text{ and damping ratios } \zeta_n \quad (\text{eqs. (41a-c) of [9]})\]

\[\text{Partitioning of } \rho^{(n)} \text{ and determination of modal vibrations at FABs (see eqs. (49-51))}\]

\[\text{Scaling factor } \kappa^{(n)} \quad (\text{eqs. (42, 43) of [9]})\]

\[\text{Scaled eigenvectors } \tilde{\rho}^{(n)} = \kappa^{(n)} \rho^{(n)}\]

\[\text{Scaled vibration amplitudes at left, right FABs } \hat{A}^{(n)}_{\text{FL}}, \hat{A}^{(n)}_{\text{FR}} \quad (\text{eqs. (50a,b) of [9]})\]

\[\text{Filtering criteria [9] satisfied?} \quad \begin{cases} \text{no} \\
\text{yes} \end{cases}\]

\[\text{Accept in Campbell diagram}\]

**Figure 2.** Flowchart of static equilibrium, stability and modal analysis (SESMA) procedure.
2.4.2 Note on Jacobian Computation

The system Jacobian (used in both TNDA and SESMA) is the derivative of the vector functions on the right hand side of eq. (46) with respect to the state variables within $s$. As in the more simple system case of [6], the most complicated part of the Jacobian evaluation is the differentiation of $r$ in eq. (21), where $r$ is given by eq. (16). As in [6], eq. (16) shows that it is easier to differentiate $r$ with respect to $\psi_{co}$. Since (from eqs. (11), (38) or (45)) $\psi_{co}$ is composed of subvectors $\varphi_{co}$, $\vec{w}_{co}$, $\epsilon$, the derivatives of $r$ with respect to the relevant state variables can then be extracted from $\partial r / \partial \psi_{co}$ for use in the computation of the required system Jacobian $\partial \chi / \partial s$. Appendix A gives an expression for $\partial r / \partial \psi_{co}$ that is developed from the more restricted form in [6] to account for either case in sections 2.3.1, 2.3.2.

2.4.3 Note on Gümbel Condition

The Gümbel condition is applied by truncating sub-atmospheric pressures when integrating for the air film forces on the journal [2,3]. Based on the form of eq. (22), the truncation is applied here to $\vec{p}_{g \xi_{av}}$ i.e. setting $\vec{p}_{g \xi_{av}}(\theta, \varphi_{co}, \vec{w}_{co}, \epsilon)$ to 0 for those values of $\theta$ where $\vec{p}_{g \xi_{av}}(\theta, \varphi_{co}, \vec{w}_{co}, \epsilon) < 0$. This is somewhat different to the way the condition is applied in FD-based computation (e.g. [2]), where truncation is applied to the 2-dimensional grid point pressures rather than the axially-averaged pressures. However, the authors did not detect any significant differences from the results obtained by the two alternative approaches.

3. APPLICATION TO TWO SYSTEMS: RESULTS AND DISCUSSION

The GR method is verified against FD for two different systems. In section 3.1, the basis functions used in the GR transformation are those of section 2.3.1 (no pressure constraint at weld location), and no Gümbel condition is applied. For this air film condition, GR was already verified against FD in [6,7] in a limited capacity using a symmetric rigid rotor-FAB system. The system considered in section 3.1 is therefore the experimental rig in [2] where simulations based on the “continuous $\vartheta$” pad FD model and no Gümbel proved to be most suitable for experimental validation purposes. Hence, with regard to the three aspects of
the contribution bulleted at the end of the Introduction, the results of section 3.1 serve to verify the first two aspects. On the other hand, the results of section 3.2 serve to verify the second and third aspects, particularly the development of GR for applied pressure condition at the weld (i.e. “finite $\vartheta$”, basis functions of 2.3.2), with or without Gümbel. For this purpose, the system considered in section 3.2 will be the symmetric rigid rotor-FAB system previously considered in works [6,10,4,13,17] where such air film conditions were applied using mesh-based air film models (FD [10,4] or FE [13,17]).

3.1 Experimental rig [2]

Figure 3 shows a schematic of the rotor of the test rig [2]. The rotor is driven by an air motor and full details of the setup are given in [2]. The rotor is supported by a high-speed self-aligning rolling-element bearing (REB) at the driven end, and by a FAB at the other end. Following [2], the effect of the coupling between the motor and rotor is neglected and the REB is considered as a rigidly-supported pivot. With reference to eq. (47), $H = 4$ rotor component modes are considered which define the free undamped motion of the rotor minus FAB at zero speed, comprising one rigid body mode and one bending mode (0 Hz, 479 Hz respectively) in each of the $xz$ and $yz$ planes [2] (the next bending mode occurs at 1.615 kHz, which is well beyond the highest achievable speed of the rig of 500 rev/s).

![Figure 3. Schematic of rotor set-up of test rig (full details in [2]).](image)

The details of the FAB, including the foil, are the same as those in [2] (Table 1). In the present analysis, for both the GR model and the comparative FD model of the air film, the foil model used is SEFM. On the
other hand, the work in [2] used the full foil structure modal model (FFSMM) and an FD model for the air film. The FFSMM neglects the effect of the top foil (like the SEFM) but considers the inertia of the bumps and elastic cross-coupling. Whereas the bump foil inertia is negligible for the speed range considered (the lowest mode of the bump foil is ~2.2 kHz, top speed <500 rev/s), elastic cross-coupling is not insignificant.

To minimise the difference between the two models, the stiffness per unit area $k$ used in eq. (26) was not calculated using the classical formula [14], which is based on the stiffness of one bump in isolation (free sliding ends). Instead, the structural effects of the complete bump foil were indirectly captured in $k$ by deriving it from a representative term of the static (0 Hz) receptance (compliance) matrix. This matrix is denoted as $R_{yyf}(0)$ in [16] and was derived from FE analysis in a cylindrical coordinate system to also account for the curvature of the bearing shell [16]. The stiffness per unit area at the apex of each of the 26 bumps was computed by taking the reciprocal of the respective direct receptance term in the leading diagonal of this matrix, and dividing by the projected area of one bump. The stiffness per unit area $k = 18.365$ GNm$^{-3}$ was then determined from the plateau region of the resulting graph (see Figure 4). It is noted that the magnitude of the cross receptance between adjacent bumps was only ~1% of the direct receptance. Despite this, a stiffness estimate based on a single isolated bump would have resulted in a serious underestimation of the stiffness (as observed in [16]).

**Table 1.** FAB parameters used in the analysis of the test rig of Figure 3 [2].

<table>
<thead>
<tr>
<th><strong>FAB parameters</strong></th>
<th><strong>Value</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Air viscosity ($\text{Nsm}^{-2}$)</td>
<td>$1.95 \times 10^{-5}$</td>
</tr>
<tr>
<td>Air atmospheric pressure (Pa)</td>
<td>101,325</td>
</tr>
<tr>
<td>Damping Loss factor</td>
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</tr>
<tr>
<td>Mean radial clearance (m)</td>
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</tr>
<tr>
<td>Bearing radius (m)</td>
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</tr>
<tr>
<td>Bearing length (m)</td>
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</tr>
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<td>Foil material</td>
<td>INCONEL$\text{®}$ alloy X-750</td>
</tr>
<tr>
<td>Foil Young’s Modulus (Nm$^{-2}$)</td>
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</tr>
<tr>
<td>Foil density (kgm$^{-3}$)</td>
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</tr>
<tr>
<td>Foil thickness (mm)</td>
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</tr>
<tr>
<td>No. of bumps</td>
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</tr>
<tr>
<td>Bump pitch (m)</td>
<td>$4.572 \times 10^{-3}$</td>
</tr>
<tr>
<td>Bump length (m)</td>
<td>$3.209 \times 10^{-3}$</td>
</tr>
<tr>
<td>Bump height (m)</td>
<td>$0.5 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
The order of the basis functions used here (eqs. (31), (32)) is \( N = 3, M = 8 \), which was already tested for convergence with a FAB of the same dimensions [6]. The effective size of the comparative FD grid is \( 7 \times 72 \) (covering one symmetric half of the FAB, excluding the open ends) [2]. With reference to the dynamical system of eqs. (46), since there is only one FAB, all terms with subscript “L” are omitted and the size of the rotor-FAB state vector \( \textbf{s} \) when using GR for the FAB is \( 76 \times 1 \). The size of the rotor-FAB state vector when using the comparative FD grid for the FAB is \( 584 \times 1 \).

**Figure 4.** Stiffness per unit area at the apex of each of the 26 bumps in the bump foil in GNm\(^{-3}\) per m\(^2\) (i.e. GNm\(^{-3}\)) obtained by taking the reciprocal of respective terms in leading diagonal of the 0 Hz receptance (compliance) matrix and dividing by projected area of one bump (4.572 mm \( \times \) 38.1 mm).

Figure 5 compares the loci of the static equilibrium positions (SEPs) of the FAB journal from 2 krpm to 40 krpm (in steps of 500 rpm), computed by GR and FD. Close agreement between GR and FD is evident. In both cases, the SEPs are unstable with respect to small perturbations from 9 krpm upwards. The results of Figure 5 agree with the corresponding results in Figure 9a of reference [2].
The results of the eigenvalue analysis for the modes of small (linearised) free vibration about the SEPs in Figure 5 are shown in Figure 6a for GR and Figure 6b for FD. At each speed there is a multitude of modes, but only one of these is responsible for SEP instability.

The eigenfrequency vs speed maps of Figures 6a, 6b are considerably different due to the sheer difference in size of the respective Jacobians (76 × 76 in case of GR versus 584 × 584 in case of FD). However, the application of the same filtering criterion to both maps in Figures 6a,b resulted in virtually identical Campbell diagrams and associated modal damping diagrams, as shown in Figures 7a,b.
At each speed beyond 5 krpm there are four whirl modes (below 5 krpm there is an extra mode that could be filtered out by lowering the damping ratio limit to 0.6). At each given speed, there is very little flexure of the rotor in mode nos. 1, 2 which are respectively in forward and reverse whirl. Mode 1 becomes unstable from 9 krpm onwards and Mode 2 is stable and highly damped. Mode nos. 3, 4 are flexural modes that are stable (very low positive damping), in reverse and forward whirl respectively, diverging in frequency with increasing speed due to the gyroscopic effect. Figure 7a shows that the first critical speed of the system

![Diagram](image-url)
occurs at 30 krpm and is flexural due to the intersection of the 1EO (engine order) excitation line with Mode 4 (the other frequency curves intersecting the 1EO line define reverse whirl modes and are thus not excited by rotor unbalance, which is the typical source of 1EO excitation). The mode shapes of Mode nos. 1, 2 and 4 at 9 krpm, as determined from the eigenvalue analysis, are respectively depicted in Figures 8, 9 and 10. The captions of these figures show the close correlation between the GR and FD values of damped frequency and viscous damping ratio.

**Figure 8.** Unstable forward whirl mode at 9 krpm taken from Campbell diagram in Figure 7(a) ($f_{d,n} = 65.44$ Hz, $\zeta_n = -0.00913$ using GR and $f_{d,n} = 65.32$ Hz, $\zeta_n = -0.01114$ using FD): (a) the 3D-view; (b) modal vibration at FAB ($z = 0.334$ m). (NB: for readers of digital version, temporal sequence of vibration is indicated by colour sequence black-red-green-magenta – same applies for other figures showing eigenmodes).

**Figure 9.** Stable reverse whirl mode at 9 krpm taken from Campbell diagram in Figure 7(a) ($f_{d,n} = 250.25$ Hz, $\zeta_n = 0.4448$ using GR and $f_{d,n} = 250.23$ Hz, $\zeta_n = 0.4446$ using FD): (a) the 3D-view; (b) modal vibration at FAB ($z = 0.334$ m).
The instability of Mode 1 at 9 krpm (Figure 8) was investigated further by performing TNDA at that speed and zero unbalance from an initial condition derived from the eigendata so that the system responds only in that mode in the vicinity of the SEP. With reference to section 2.4.1 such a condition is given by [9]:

\[ s(\tau = 0) = s_E + c_n \text{mod} \left\{ \rho^{(n)} \right\} \cdot \cos \left( \arg \left\{ \rho^{(n)} \right\} + \alpha_n \right) \]  

(52)

where \( \alpha_n = 0 \) and \( c_n = 1/20 \). Since \( \rho^{(n)} \) is scaled such that the greatest vibration in the FAB is 20% of the radial clearance \( c \), the initial perturbation is \( \sim 1\% \) of \( c \). Figure 11 shows the results of such a simulation over 400 shaft revolutions for both GR and FD. As Figure 11 shows, the journal orbits by both GR and FD initially diverge from the SEP, as expected from the eigenvalue analysis, but then settle down into a periodic oscillation (limit cycle) that is driven by the nonlinearity of the air film (self-excitation). Figure 11b shows the damped frequency and damping ratio values estimated from the initial stages of the time histories by measuring the period and logarithmic decrement respectively. These values agree with the eigendata given in the caption of Figure 8. Figure 11c shows that the frequency estimate of the limit cycle (\( \sim 71 \) Hz) is significantly different from that of the initial perturbation (\( \sim 65 \) Hz) i.e. the frequency of the time history drifts from 0.43 EO to \( \sim 0.5 \) EO for both GR and FD simulations. It is noted that a similar drift was also observed in [2] where the FAB model used was FD air film with FFSMM and the initial condition applied was random rather than mode-specific. The GR and FD results in Figure 11 show remarkable consistency,
and the phase shift between the trajectories is merely the result of the fact that the eigenvector \( \mathbf{\hat{p}}^{(n)} \) (used in the initial condition eq. (52)) is inherently different for GR and FD and the only tuning done to the respective eigenvectors was with regard to their moduli (to achieve similar size perturbations). The phase shift could be eliminated by multiplying the GR or FD eigenvector by an appropriate imaginary scalar of unit modulus, or by using a different phase angle \( \alpha_n \) with one of them in eq. (52).

**Figure 11.** TNDA journal trajectories and time histories at 9 krpm over 400 shaft revolutions using mode-specific initial conditions (eq. (52)) derived from the eigendata of the mode in Figure 8 using GR (—) and FD (---): (a) orbital trajectory; (b) initial part of time history showing estimates of \( f_{d,n}, \zeta_n \); (c) time history of limit cycle, showing its fundamental frequency.

Figures 12a-d compares waterfall diagrams of the predicted steady-state nonlinear response at the disc location for two levels of unbalance at the disc (0 gmm, 10 gmm) using GR (Figure 12a for 0 gmm and Figure 12b for 10 gmm) and FD (Figure 12c for 0 gmm and Figure 12d for 10 gmm). At each fixed speed TNDA was run over 450 shaft revolutions and the last 0.5 s of data retained to generate a frequency
spectrum of 0.5 Hz resolution. The initial conditions for TNDA at any given speed were taken from the final conditions at the previous speed (default initial conditions corresponding to zero journal eccentricity were used for lowest speed). The waterfall diagrams shown are for run-up, however those for run-down were found to be practically identical (as in the simulations of [2]). For 0 gmm (Figures 12a,c) the spectra start at 9 krpm (since there is no self-excitation below this speed) and exhibit the 0.5 EO tracking observed in the latter stages of the simulation of Figure 11. As seen from Figure 12c, for the case of 0 gmm, the FD approach could not produce data of the required duration at speeds above 12 krpm due to the Matlab integrator (whether ode23s or ode15s) encountering numerical convergence issues. Such numerical issues did not occur when using GR (Figure 12a) with the same solver and tolerance settings. A weak component at ~1EO that can be seen in Figures 12a,c is the second harmonic of the sub-harmonic (~0.5EO) fundamental due to the nonlinearity of the oscillation.

Figures 12b,d refer to 10 gmm unbalance predictions with GR (Figure 12b) and FD (Figure 12d) and are virtually identical to each other. The results of Figures 12b,d are distinguished from those of Figures 12a,c by the inclusion of a significant 1EO component that is caused by the unbalance excitation. It is noted that in both Figures 12b,d one finds short blank regions at a few high-end speeds due to numerical convergence issues as the predicted bending critical speed (30 kpm or 500 rev/s) is approached. However, in both cases, a result is achieved at 30 krpm, which is the bending critical speed predicted by the Campbell diagram (Figure 7a). As this speed is approached, the 0.5EO suddenly disappears and the amplitude of the 1EO component shoots up (comparing spectrum at 28.5 krpm with that at 30 krpm in Figure 12b). This effect is also evident in Figure 13, which compares the operating deflection shape of the rotor during steady-state vibration over 50 shaft revolutions as predicted by GR and FD for four different speeds: 15 krpm (a,b), 20 krpm (c,d), 25 krpm (e,f) and 30 krpm (g,h). The vibrating forms at 15, 20 and 25 krpm are dominated by the 0.5EO component and at the latter two speeds the vibration is aperiodic due to additional minor non-synchronous frequency components visible in Figures 12b,d. In Figure 13 one notices an increasing amount of shaft deformation as the speed is increased, and at the bending critical of 30 krpm, the vibrating form is entirely due to the 1EO and corresponds to the mode shape of Figure 10. It is important to note that in the
previous work on the same test rig [2], a Campbell diagram was not available to confirm the location of the bending critical of the coupled rotor-FAB system – this was estimated in [2] by considering that it would be very close to the first flexural component mode of the rotor minus FAB since that mode’s node coincided with the FAB location. Moreover, due to numerical convergence issues, the FD-FFSMM analysis in [2] could not achieve a simulation at 30 krpm, and the sudden disappearance of sub-synchronous frequencies at the bending critical speed was therefore not predicted in [2].

**Figure 12.** Waterfall diagrams of simulated steady-state vibration in y-direction at disc location of test rig in Figure 3 for two levels of unbalance: 0 gmm (GR – (a), FD – (c)); 10 gmm (GR – (b), FD – (d)).
Figure 13. Operating deflection shape of simulated steady-state unbalance vibration of test rig rotor in Figure 3 at 10 gmm unbalance for three different speeds 15 krpm (a,b), 20 krpm (c,d) 25 krpm (e,f), 30 krpm (g,h), computed using GR (a,c,e,g) and FD (b,d,f,h).
The new predictions of the present paper appear to be consistent with key phenomena observed in experimental data from [2], reproduced in Figure 14. This figure shows run-up to a top speed of 27.5 krpm, which was then held for ~3 s, after which the motor was switched off and the rotor allowed to coast down. The inability to exceed 27.5 krpm was attributed in [2] to excessive tilting at the FAB location causing rubbing at the foil edges. Figure 14 shows the absence of sub-synchronous frequencies at the top speed, the sudden increase in 1EO amplitude in the run-up to this speed (evident from the colour scale of the amplitude), the sudden reduction in 1EO amplitude upon coast down, and the simultaneous appearance of sub-synchronous activity. While these observations can now be correlated with the new predictions of this paper, some differences are evident. Considering the coast down part of Figure 14, the sub-synchronous activity appearing below the top speed starts roughly at 0.33EO and then drifts towards 0.5EO at ~16.5 krpm, maintaining a well-defined 0.5EO tracking until ~11.5 krpm, virtually disappearing at lower speeds.

On the other hand, the predictions in Figures 12b or 12d show 0.5EO tracking from just below the top speed until 9 krpm. The sub-synchronous activity predicted in the corresponding result in [2] (Figure 14f in [2]), based on FD with FFSMM for foil, consisted of: 0.5EO tracking from 9 to 19 krpm; 0.33EO, 0.67EO from 19.5-22 krpm; gradually drifting to 0.5EO at higher speeds. As seen in Figure 15 of the present paper, whereas the GR/FD orbits at 15 krpm (Figures 15a,b) are virtually identical to the corresponding prediction of [2] (reproduced in Figure 15c), the GR/FD orbits at 20 krpm (Figures 15d,e) are different from the prediction of [2] (Figure 15f) which has a 0.33EO component. It is noted however that the present model (GR/FD with SEFM for foil) can predict 0.33EO/0.67EO activity if the unbalance level is raised to 15gmm, as seen in Figure 16a, which shows better agreement with measured data at 10 gmm added unbalance (Figure 16b) than the simulation at 10 gmm (Figure 12b or d). This is might be attributed to the significant effect of residual unbalance in the test rig, which is evident from the data in Figure 14. Nonetheless, it should be noted that the predictions in Figure 16a have the same limitations as those in [2], particularly after 19 krpm, where it is seen that the 0.33EO is significantly weaker than measured, and its predicted harmonic (0.67EO) virtually invisible in the experimental results of Figure 16b (albeit visible in other experimental results in [2]). It is also noted that the achievable top speed in Figure 16b (~26 krpm) is lower...
than that in Figure 14 (27.5 krpm) due to the additional unbalance increasing vibrations when approaching the bending critical speed. Hence, the spike in the 1EO at 30 krpm predicted in Figure 16a is not visible in Figure 16b but is evident in Figure 14.

**Figure 14.** Spectrogram (frequency-amplitude vs time) of experimental steady-state vibration in y-direction at disc location of test rig in Figure 3 with residual unbalance, reproduced from [2].
Figure 15. Simulated steady-state unbalance vibration orbits at disc location of test rig rotor in Figure 3 with 10 gmm unbalance for two different speeds 15 krpm (a,b,c) and 20 krpm (d,e,f) computed using GR with simple elastic foundation model SEFM for foil (a,d), FD with SEFM (b,e), and FD with full foil structure modal model [2] (c,f).

Figure 16. Comparison of waterfall diagrams of steady-state vibration in y-direction at disc location of test rig in Figure 3 (a) GR-simulated with 15 gmm unbalance; (b) measurement with 10 gmm added unbalance, reproduced from [2].
3.2 Symmetric rigid rotor-FAB system [10]

As stated in the beginning of Section 3, for verification of the finite $\vartheta$ pad GR model, with or without Gümbel, the system considered is as shown in Figure 17, with the parameters in Table 2.

![Symmetric rigid rotor system](image)

**Figure 17.** Symmetric rigid rotor system considered for testing GR transformation with pressure constraint at weld (parameters in Table 2).

**Table 2.** Parameters of system in Figure 17 [10]

<table>
<thead>
<tr>
<th>Rotor</th>
<th>symmetrical rigid, total mass $M_{\text{tot}} = 2 \times 3.061$ kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air film (either bearing)</td>
<td></td>
</tr>
<tr>
<td>radial clearance $c$ ($\times 10^{-6}$ m)</td>
<td>32</td>
</tr>
<tr>
<td>viscosity (Nsm$^2$)</td>
<td>$1.95 \times 10^{-5}$</td>
</tr>
<tr>
<td>atmospheric pressure $p_a$ (Pa)</td>
<td>101325</td>
</tr>
<tr>
<td>Simple elastic foundation model for foil pad (either bearing)</td>
<td></td>
</tr>
<tr>
<td>stiffness per unit area $k$ (GNm$^{-3}$)</td>
<td>4.739</td>
</tr>
<tr>
<td>hysteretic damping loss factor $\eta$</td>
<td>0.25</td>
</tr>
</tbody>
</table>

With reference to the dynamical system of eqs. (46), although system is symmetric, both FABs are considered, in order to keep closer to the level of computational burden encountered in practical scenarios. $H = 2$ rotor component modes are considered, comprising one translational mode at 0 Hz defined by $\phi(z) = 1/\sqrt{M_{\text{tot}}}$ in each of the $xz$ and $yz$ planes. Unless otherwise stated, the order of the basis functions for finite $\vartheta$ (eqs. (31), (39)) is $N = 3, M = 16$. This is comparable to the typical order used for continuous $\vartheta$ ($N = 3, M = 8$) which uses approximately twice the number of terms in the $\vartheta$ direction (eq. (39) vs eq. (32)). The effective size of the comparative finite $\vartheta$ FD grid for each FAB is $7 \times 71$ (covering one
symmetric half of the FAB, excluding the open ends and the weld location) [2,10]. The size of the rotor-FAB state vector $\mathbf{s}$ is therefore $132 \times 1$ when using GR for the FABs and $1140 \times 1$ when using FD.

Figure 18 compares GR and FD results for the transient journal trajectory from initial conditions (corresponding to zero eccentricity) at 12 krpm, zero unbalance, as well as the final (steady-state) profile of the deformed pad. Figures 18a,b show the finite $\vartheta$ results with and without Gümbel respectively; the equivalent result for continuous $\vartheta$ without Gümbel is included for reference in Figure 18c (previously evaluated by GR/FD in [7]). The GR/FD results in each of these figures are virtually indistinguishable.

Although the trajectories in Figure 18a,b are well-documented (e.g. ref. [17] for Figure 18a, ref. [28] for Figure 18b, ref. [4] for both), this is the first time that these finite $\vartheta$ results have been achieved using a method (GR) that does not require a grid (FD) or mesh (FE).

The results in Figures 18a,b justify the choice of $N = 3, M = 16$ for the GR transformation in the finite $\vartheta$ case. This choice is further supported by the convergence study in Figure 19, which focuses on the case of Figure 18b. Figure 19a shows that for given $N (= 3)$, reducing $M$ from 16 to 12 produces a minor change when compared to reducing $M$ from 12 to 10. Figure 19b shows that, for given $M (= 16)$, increasing $N$ from 3 to 5 produces a negligible change.

Figure 18. Verification of GR against FD for different applied air film conditions, showing TNDA results of journal trajectory and steady-state foil pad profile (system in Figure 17, 12 krpm, zero unbalance): (a) $p = p_a$ at weld and Gümbel condition (GR uses eqs. (31), (39) with $N = 3, M = 16$); (b) $p = p_a$ at weld and no Gümbel (GR uses eqs. (31), (39) with $N = 3, M = 16$); (c) (for reference) no pressure constraint at weld and no Gümbel (GR uses eqs. (31), (32) with $N = 3, M = 8$).
Table 3 compares the GR and FD predictions for the onset of instability speed (OIS) (which refers to free perturbations about the static equilibrium condition) for the three sets of air film conditions considered. As can be seen, the difference does not exceed 0.4%. For each case, the OIS was found by interpolating for the zero crossing of the modal damping curve associated with the Campbell diagram, which was computed in steps of 250 rpm.

Figures 20 and 21 show the process for the extraction of the GR/FD Campbell diagrams and modal damping diagrams for the finite $\vartheta$ with Gümbel case (first row of Table 3). As observed in Section 3.1, the stark contrast between the unfiltered eigenfrequency vs maps for GR and FD (Figures 20a,b respectively) is the
result of the sheer difference in the sizes of the respective Jacobians being analysed (132 × 132 vs 1140 × 1140).

Figure 20. Unfiltered eigenfrequency $\omega_{d,n}/(2\pi)$ vs speed map for system in Figure 17 with $p = p_a$ at weld and Gumbel condition: (a) GR with $N = 3$, $M = 16$; (b) FD with $7 \times 71$ grid (one symmetric half, excluding pressure constraint locations).

The application of the same filtering criterion to both maps in Figures 20a,b resulted in the Campbell diagrams, and associated modal damping diagrams, shown in Figures 21a,b respectively, where the GR results (extracted from Figure 20a) are plotted on the same axes as the FD results (extracted from Figure 20b). The degree of correlation between GR and FD is seen to be very good.
As already known for this system [10,13], mode 1 is in forward whirl over the entire speed range considered and is the mode that becomes unstable, whereas mode 2 is highly damped and is in reverse whirl for most of the speed range (from 8.75 krpm for GR and from 9 krpm for FD, with resolution of 250 rpm). Figures 22a,b respectively show modes 1 and 2 at 22 krpm, as computed from the GR-based eigenvector analysis. These modes are seen to be virtually identical to the corresponding results from FD-based eigenvector analysis which were previously presented in Figure 7 of reference [10] (which used identical parameters),

**Figure 21.** Campbell diagrams (a) and associated modal damping diagrams (b) extracted from the alternative eigenvalue analyses in Figures 20(a,b) i.e. GR, FD, indicating forward (Fwd.) and reverse (Rev.) whirl modes, and including inset of zoomed view of region of instability onset in mode no. 1 by each method.
and are also similar to those presented in Figure 4 of reference [13] (for a slightly different speed and parameters).

![Diagram](image_url)

**Figure 22.** GR-computed modes at 22 krpm taken from Campbell diagram in Figure 21: (a) mode 1 \( (f_{d,n} = 99.62 \text{ Hz}, \ zeta_n = 0.0020754, \text{ forward whirl}) \); (c) mode 2 \( (f_{d,n} = 135.30 \text{ Hz}, \ zeta_n = 0.1507, \text{ reverse whirl}) \).

### 3.3 Overall discussion

Comparing the last two entries of Table 3 it is seen that the atmospheric pressure condition at the weld moderately delays the OIS. Likewise, in the case of the test rig of section 3.1, the previous FD (with FFSMM foil) work in [2] showed that this condition shifted the OIS to 11 krpm (from 9 krpm). The same result is achieved by reworking the GR problem in section 3.1 using the finite \( \theta \) basis functions (2.3.2), causing a corresponding shift in the zero-crossing of the damping curve of mode 1 (Figure 7b) but otherwise relatively minor changes to the remainder of the curves in Figures 7a,b. As shown elsewhere [29], the pressure constraint at the weld becomes more influential if the location of the weld is shifted from that indicated in Figure 1.
Table 4. Comparison of CPU time and data storage for static equilibrium, stability and modal analysis by GR and FD.

<table>
<thead>
<tr>
<th>CASE</th>
<th>Computation</th>
<th>GR</th>
<th>FD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Total CPU time</td>
<td>Average data</td>
</tr>
<tr>
<td></td>
<td></td>
<td>over speed range</td>
<td>size per speed</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(seconds)</td>
<td>point (kB)</td>
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<td>section 3.1</td>
<td>eigenvalue analyses of Figures 6a,b and underpinning SEP loci</td>
<td>9.84</td>
<td>50</td>
</tr>
<tr>
<td>section 3.2</td>
<td>eigenvalue analyses of Figures 20a,b and underpinning SEP loci</td>
<td>40.87</td>
<td>120</td>
</tr>
</tbody>
</table>

Table 4 compares SESMA computation times for GR and FD. The sheer difference in Jacobian size for GR and FD means that the CPU time for GR is not only lower (less than half) but the results storage requirements are much less (between 70 to 90 times less). Both these considerations are important when performing multiple simulations as part of a design optimisation exercise.

With regard to TNDA, the computational advantage of GR is more nuanced for the cases considered. For example the CPU time to generate the trajectory in Figure 18a (over 20 shaft revolutions) using ode15s was found to be 4.8 s with GR and 5.2 s with FD, whereas with ode23s (which uses the Jacobian more frequently [8]) the CPU time increased to 49.4 s for GR and 43.2 s for FD. What is certain however is that GR was found to be less likely than FD to encounter numerical divergence when computing limit cycles or unbalance responses, as evident from comparing Figures 12a and 12c. This observation is consistent with nonlinear systems tending to become stiffer (and therefore more prone to numerical instability) as the number of equations comprising them is increased [8]. It is also considered that TNDA becomes unwieldy if high resolution FD/FE grids are necessary, especially when used with advanced foil models, since Jacobian computation (required by stiff solvers [8]) becomes unfeasible.
4. CONCLUSIONS

This paper has presented a novel application of the arbitrary-order GR method to both nonlinear and linearised analyses of rotor systems supported by single-pad FABs. The novel aspects are: application to generic flexible (multi-modal) rotors on FABs; Jacobian-based linearisation of the GR-transformed system for the extraction of the full mode set and associated linear stability map; the facility to accommodate a pressure constraint at a circumferential location and/or Gumbel. These developments were comprehensively validated on two previously considered systems by comparing alternative (GR, FD-based) simulations and experimental data. The first system was a test rig that was flexible at the upper end of the operating speed range and was previously analysed using FD for the air film and a full structural model for the bump foil. In the present case, an equivalent simple elastic foundation model (SEFM) was used with both the GR air film model and the comparative FD model. Static equilibrium, stability and modal analysis (SESMA) generated previously unavailable Campbell diagrams and associated modal damping curves by both GR and FD that were in excellent agreement, and which also served to further support the previous findings. Transient nonlinear dynamic analysis (TNDA) results by GR and FD were also in excellent agreement, with the former being less prone to numerical divergence. The predictions of the observed nonlinear phenomena were of a standard similar to those previously achieved by FD and alternative foil model. The second system was a symmetric rigid rotor rig and was used to successfully validate the GR transformation with pressure constraint at circumferential location, with/without Gumbel. This aspect of the development paves the way for GR transformation of multi-pad FABs. For both test rigs, comparison of SESMA results by GR and FD over the entire speed range provided a rigorous verification exercise which moreover showed GR to be over twice as fast, and requiring 70-90 times less memory for saving the results. Current efforts are being directed at applying GR to condense rotor-FAB problems that use advanced foil structure models, providing much-needed facilitation of TNDA.

Appendix A
Using \([ m \times n ]\) to denote a matrix of size \( m \times n \), \( I_{m \times m} \) to denote an \( m \)-square matrix, \( n_{m \times 1} \) to denote a vector of ones of length \( m \), the following matrices are defined [6]:

\[
B^{(1)}_{P,Q} = \begin{bmatrix} I_{P \times P} \\ \vdots \\ I_{P \times P} \end{bmatrix}_{PQ \times PQ}, \quad B^{(2)}_{P,Q} = \text{blkdiag}(n_{P \times 1}, \ldots, n_{P \times 1})_{PQ \times PQ}
\]

(A.1a,b)

Following [6], the derivative of \( \mathbf{r} \) (eq. (16)) with respect to \( \psi_{co} \) is then given by:

\[
\frac{\partial \mathbf{r}}{\partial \psi_{co}} = \text{diag}(B^{(2)}_{P_1,Q_1} \tilde{h}_{co})B^{(1)}_{P_1,Q_1} \frac{\partial (\psi_{co} \psi_{co}^T)(\cdot)}{\partial \psi_{co}} + \text{diag}(B^{(1)}_{P_1,Q_1} (\psi_{co} \psi_{co}^T)(\cdot))B^{(2)}_{P_1,Q_1} \frac{\partial \tilde{h}_{co}}{\partial \psi_{co}}
\]

(A.2)

where

- \( P_1, Q_1 \) are the respective lengths of \( \{\psi_{co} \psi_{co}^T\}(\cdot) \) and \( \tilde{h}_{co} \), which are as follows, using notation in 2.3.3:

\[
P_1 = \left( l_{\varphi_{be}} l_{\varphi_{b\theta_{e}}} + l_{\tilde{h}_{\theta}} \right)^2, \quad Q_1 = l_{\tilde{h}_{\theta}}
\]

(A.3)

- The partial derivatives in eq. (A.2) are given by

\[
\frac{\partial (\psi_{co} \psi_{co}^T)(\cdot)}{\partial \psi_{co}} = \text{diag}(B^{(2)}_{P_2,P_2} \psi_{co})B^{(1)}_{P_2,P_2} + \text{diag}(B^{(1)}_{P_2,P_2} \psi_{co})B^{(2)}_{P_2,P_2}
\]

(A.4)

where \( P_2 \) is the length of \( \psi_{co} \):

\[
P_2 = l_{\varphi_{be}} l_{\varphi_{b\theta_{e}}} + l_{\tilde{h}_{\theta}}
\]

(A.5)

\[
\frac{\partial \tilde{h}_{co}}{\partial \psi_{co}} = [0_{Q_1 \times (P_2 - Q_1)}]_{Q_1 \times Q_1}
\]

(A.6)

where \( 0_{m \times n} \) denotes a matrix of zeros of size \( m \times n \).

It is noted that the expression in reference [6] that is analogous to eq. (A.2) is missing the second term (due to a typographical error) and is also limited to the case where no pressure condition is applied at the weld.

**ACKNOWLEDGMENTS**
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DECLARATIONS OF INTEREST

None.

REFERENCES


