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Phase limitations of Zames-Falb multipliers

Shuai Wang, Joaquin Carrasco, and William P. Heath

Abstract—Phase limitations of both continuous-time and discrete-time Zames-Falb multipliers and their relation with the Kalman conjecture are analysed. A phase limitation for continuous-time multipliers given by Megretski is generalised and its applicability is clarified; its relation to the Kalman conjecture is illustrated with a classical example from the literature. It is demonstrated that there exist fourth-order plants where the existence of a suitable Zames-Falb multiplier can be discarded and for which simulations show unstable behavior. A novel phase-limitation for discrete-time Zames-Falb multipliers is developed. Its application is demonstrated with a second-order counterexample to the Kalman conjecture. Finally, the discrete-time limitation is used to show that there can be no direct counterpart of the off-axis circle criterion in the discrete-time domain.

I. INTRODUCTION

The absolute stability of a negative feedback interconnection between an LTI system G and a nonlinearity ϕ with a slope restriction k has aroused the interests of many researchers. The stability tests include the circle criterion, Popov criterion [1], [2], and off-axis circle criterion [3], [4] in continuous time and the circle criterion [5], Tsytkin criterion [6] and Jury-Lee criterion [7], [8] in discrete time. For a recent discussion, see [9] and [10]. Apart from these, loop transformation and multiplier theory are both important tools to establish the stability of feedback interconnections. The Zames-Falb multipliers are a class of multipliers with the property of preserving the positivity of monotone and bounded nonlinearities, and hence of slope-restricted nonlinearities after loop transformation. The class of Zames-Falb multipliers can be defined in either continuous time [11], [12] or discrete time [13], [14]. Specifically, after loop transformation, the stability of the negative interconnection between an LTI system G and a nonlinearity ϕ with a slope restriction k is guaranteed if there exists a Zames-Falb multiplier M such that

$$\operatorname{Re}\{M(1+kG)\} > 0, \quad (1)$$

with M and G evaluated over all frequencies. That is to say, at $j\omega$, $\omega \in \mathbb{R}$ for continuous-time systems and at $e^{j\omega}$, $\omega \in [0, 2\pi]$ for discrete-time systems.

The Zames-Falb multipliers may be considered a classical tool [15]. Nevertheless, there has been considerable recent interest, largely sparked by the availability of numerical searches

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TABLE I
VARIOUS SLOPE RESTRICTIONS DISCUSSED IN THE TEXT.

\hat{k}_{ZF}	Maximum slope for which a Zames-Falb multiplier is known
k_{ZF}	Maximum slope for which there exists a Zames-Falb multiplier
k_S	Maximum slope for which the Lur'e system is absolutely stable
k_{PL}	Minimum slope for which phase limitation implies there is no Zames-Falb multiplier
\hat{k}_C	Minimum slope for which a counterexample to absolute stability is known
k_O	Slope for direct discrete-time counterpart off-axis circle criterion (which is false)
k_{RO}	Slope for Reduced Off-axis circle criterion in [42]
k_N	Nyquist value

([16], [17], [18], [19], [20], [21], [22], for continuous time; [23], [24] for discrete time) and their encapsulation within an IQC (integral quadratic constraint) framework [25], [26], [27], [28]. There has also been interest in generalising the class, both to MIMO (multi-input, multi-output) nonlinearities [29], [30], [31], [32], [33] and to nonlinearities outside the original classes considered by Zames and Falb [34], [35], [28], [36]. In addition to determining stability conditions, they can be used to analyse performance [37], [38]; further, they can be used to obtain tighter versions of the Popov criterion [39]. Applications of Zames-Falb multipliers range from input-constrained model predictive control [40] to first order numerical optimisation algorithms [41].

Although both continuous-time and discrete-time Zames-Falb multipliers are defined with similar conditions, there are clear distinctions between their properties. In discrete time the Zames-Falb multipliers are the full set of multipliers preserving the positivity of monotone and bounded nonlinearities, besides direct phase substitutions [14], [43]. In continuous time matters are more nuanced, but the class of Zames-Falb multipliers remains the widest known class

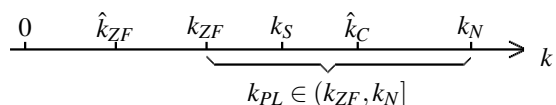


Fig. 1. Relations between slope restrictions discussed in the text. Conjecture I.2 is that $k_{ZF} = k_S$ and hence $k_S < k_{PL}$.

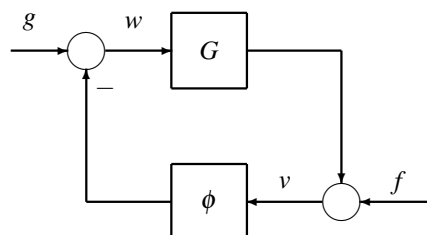


Fig. 2. Lur'e problem

of multipliers preserving the positivity of monotone and bounded nonlinearities, up to phase equivalence [39], [44]. For a tutorial introduction to the phase properties of continuous-time Zames-Falb multipliers, phase-equivalence results and the issues associated with causality, see [45]. Phase properties are essential to our understanding of Zames-Falb multipliers.

For example, if $k < k_N$ (see Table I for various slope restrictions discussed in this paper), then the phase of $(1+kG)$ lies between -180° and 180° . Meanwhile multipliers must be positive so are restricted to lie between -90° and 90° . But as the Kalman conjecture is false, any set of suitable multipliers must be restricted by some further fundamental limitations. This follows from the obvious but important fact:

Fact I.1 *If the system is not absolutely stable, there can be no appropriate Zames-Falb multiplier.*

However, only a few papers discuss such limitations. Megretski [46] shows that there exists a phase limitation for continuous-time Zames-Falb multipliers. Another phase limitation of Zames-Falb multipliers is given by Jönsson and Laiou [47], [48]. Such limitations are often ignored when new searches for multipliers are presented (see for example [45] and references therein). Often only k_N is provided as an upper limit for the slope restriction.

We discuss Megretski's phase restriction [46] with respect to a fourth-order continuous-time plant whose phase drops from $+180^\circ$ to -180° ; similarly with k sufficiently big the phase of $1+kG$ drops from above $+90^\circ$ degrees to below -90° . The limitation cannot be applied to first, second or third-order plants whose phase is in the range $(-180, 90)$ degrees or $(-90, 180)$ degrees; this agrees with the well-known result that the Kalman conjecture is true for such plants [49].

To the best of authors' knowledge, no similar limitation has been developed in the discrete-time domain. Since there exist second-order discrete-time counterexamples to the Kalman conjecture whose phase is in the range $(-180, 0)$ degrees [50], [51] one might expect a *simpler* limitation for discrete-time multipliers; this turns out to be indeed the case.

The contribution of this paper is for both continuous-time and discrete-time multipliers. We generalise Megretski's limitation [46] for continuous-time multipliers to a wider choice of frequency intervals. Further, we show that Megretski's limitation [46] only applies for the class of Zames-Falb multipliers which do not require the odd condition on the nonlinearity; we provide the corresponding result when the nonlinearity is odd. We discuss the limitation's numerical calculation and demonstrate its application in the context of a classical example due to O'Shea [11], [45]. In particular we demonstrate a fourth-order counterexample of the Kalman conjecture for which the constraint is active. A further contribution of the paper is the development of a phase limitation for discrete-time Zames-Falb multipliers. This limitation is fundamentally different to Megretski's limitation as it only requires the phase of $(1+kG)$ to be either in the interval $(90, 180)$ degrees or in the interval $(-180, -90)$ degrees. The limitation is easy to compute, and is active for a second-order discrete-time counterexample of the Kalman conjecture. This close link between the preclusion of a Zames-Falb multiplier and unstable behaviour leads us to the

following conjecture as the counterpart to Fact I.1; however no proof (or counterexample) is offered in this paper:

Conjecture I.2 *If there is no appropriate Zames-Falb multiplier, the system is not absolutely stable.*

One direct application of the phase limitation is to show there can be no direct discrete-time counterpart of the off-axis circle criterion. The continuous-time off-axis circle criterion is a useful graphical stability test and is shown to be a less conservative criterion compared to the circle criterion [3], [4]. The derivation is based on the phase properties of RL/RC multipliers. The direct discrete-time counterpart of the off-axis circle criterion is sometimes assumed to be true in the literature (e.g. [52], [53]). However only a highly restrictive discrete-time version is proposed in [42], without discussion as to whether the direct discrete-time counterpart off-axis circle criterion is true or false. In this paper, we show that in some cases there are no Zames-Falb multipliers with the requisite phase properties for its derivation - i.e. the direct counterpart off-axis circle criterion cannot be derived using multiplier theory. The invalidation is completed by counterexample.

Some preliminary results related with Theorem IV.3 part (i) were presented in [54].

II. NOTATION AND PRELIMINARY RESULTS

A. Signal spaces

For continuous-time signals let $\mathcal{L}_2[0, \infty)$ be the Hilbert space of square integrable and Lebesgue measurable functions $f: [0, \infty) \rightarrow \mathbb{R}$ and let \mathcal{L}_2 be defined similarly for $f: \mathbb{R} \rightarrow \mathbb{R}$. Let $\mathcal{L}_{2e}[0, \infty)$ be the extended space of $\mathcal{L}_2[0, \infty)$ [43].

For discrete-time signals let \mathbb{Z} and \mathbb{Z}^+ be the set of integer numbers and positive integer numbers including 0, respectively. Let ℓ be the space of all real-valued sequences, $h: \mathbb{Z}^+ \rightarrow \mathbb{R}$ and let ℓ_2 denote the Hilbert space of all square-summable real sequences $f: \mathbb{Z}^+ \rightarrow \mathbb{R}$ (ℓ is the extended space of ℓ_2). Similarly, we can define the Hilbert space $\ell_2(\mathbb{Z})$ by considering real sequences $f: \mathbb{Z} \rightarrow \mathbb{R}$.

B. Lur'e problem and the Kalman conjecture

The feedback interconnection system is a Lur'e system represented in Fig. 2 with both G and ϕ mapping $\mathcal{L}_{2e}[0, \infty) \rightarrow \mathcal{L}_{2e}[0, \infty)$ (continuous time) or $\ell \rightarrow \ell$ (discrete time). The object G is assumed LTI stable and the object ϕ memoryless and slope-restricted (see below). The interconnection relationship is

$$\begin{cases} v = f + Gw, \\ w = -(\phi v) + g. \end{cases} \quad (2)$$

The system (2) is well-posed if the map $(v, w) \mapsto (g, f)$ has a causal inverse on $\ell \times \ell$, and this feedback interconnection is ℓ_2 -stable if for any $f, g \in \ell_2$, both $w, v \in \ell_2$.

Definition II.1 (*Memoryless slope-restricted nonlinearity*) *The nonlinearity $\phi: \mathcal{L}_{2e}[0, \infty) \rightarrow \mathcal{L}_{2e}[0, \infty)$ or $\phi: \ell \rightarrow \ell$ is said to be memoryless and slope-restricted in $S[0, k]$, if there*

is a function $N : \mathbb{R} \rightarrow \mathbb{R}$ such that $(\phi u)(t) = N(u(t))$ or $(\phi u)(k) = N(u(k))$, $N(0) = 0$, and

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq k, \quad \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2. \quad (3)$$

In addition, ϕ is said to be odd if N is odd, i.e. $N(x) = -N(-x)$, for all $x \in \mathbb{R}$.

We define the Nyquist value and state the Kalman conjecture for both continuous-time and discrete-time systems.

Definition II.2 (Nyquist value) Given a stable LTI system G , the Nyquist value k_N is the supremum of all the positive real numbers k such that $\tau k G$ satisfies the Nyquist Criterion for all $\tau \in [0, 1]$. It can also be expressed as:

$$k_N = \sup\{k > 0 : \inf_{\omega} \{|1 + \tau k G| > 0\}, \forall \tau \in [0, 1]\}, \quad (4)$$

with G evaluated over all frequencies (i.e. $\omega \in \mathbb{R}$ for continuous-time systems and $\omega \in [0, 2\pi]$ for discrete-time systems).

Conjecture II.3 (Kalman Conjecture, [55]) Let ϕ be a memoryless slope-restricted nonlinearity such that there exists a continuously differentiable $N : \mathbb{R} \rightarrow \mathbb{R}$ and $S > 0$ such that $\phi(v)(t) = N(v(t))$ (or $\phi(v)(k) = N(v(k))$) and

$$0 \leq \frac{dN(x)}{dx} \leq S, \quad \forall x \in \mathbb{R}. \quad (5)$$

Then the negative feedback interconnection of the continuous-time (or discrete-time) LTI systems $G \sim [A, B, C, 0]$ and ϕ (Fig 2) is globally asymptotically stable if $A - BCk$ is Hurwitz (Schur) for all $k \in [0, S]$.

There exist fourth-order continuous-time counterexamples to the Kalman conjecture [56], [49], [57] and second-order discrete-time counterexamples [50], [51].

C. Zames-Falb multipliers

The characteristics of continuous-time Zames-Falb multipliers is given in the following theorem that defines two different classes of multipliers.

Theorem II.4 (Continuous-time Zames-Falb multipliers, [12].) Consider the continuous-time feedback system in Fig. 2 with G a stable LTI system and ϕ memoryless and slope-restricted in $S[0, k]$. Suppose that there exists an LTI multiplier $M : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ whose transfer function has the form

$$M(s) = 1 - H(s) \quad (6)$$

such that the impulse response h of H satisfies

$$\int_{-\infty}^{\infty} |h(t)| dt < 1. \quad (7)$$

Moreover, let us assume that either ϕ is odd or $h(t) > 0$. Suppose further there is some $\delta > 0$ such that

$$\operatorname{Re}\{M(j\omega)(1 + kG(j\omega))\} \geq \delta \text{ for all } \omega \in \mathbb{R}. \quad (8)$$

Then the feedback interconnection (2) is \mathcal{L}_2 -stable. ■

Remark II.5 With some abuse of notation, we denote $h(t)$ as the addition of a real-valued function $h_a(t)$ and impulses at different instants, i.e.

$$h(t) = h_a(t) + \sum_{i=1}^{\infty} h_i \delta(t_i). \quad (9)$$

Definition II.6 The class of continuous-time Zames-Falb multipliers \mathcal{M}^c is defined as the LTI systems $M : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ whose transfer function has the form

$$M(s) = 1 - H(s) \quad (10)$$

such that the impulse response h of H satisfies that $h(t) \geq 0$ for all t and

$$\int_{-\infty}^{\infty} h(t) dt < 1. \quad (11)$$

Definition II.7 The class of continuous-time ‘‘odd’’ Zames-Falb multipliers $\mathcal{M}_{\text{odd}}^c$ is defined as the LTI systems $M : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ whose transfer function has the form

$$M(s) = 1 - H(s) \quad (12)$$

such that the impulse response h of H satisfies

$$\int_{-\infty}^{\infty} |h(t)| dt < 1. \quad (13)$$

By definition, $\mathcal{M}^c \subset \mathcal{M}_{\text{odd}}^c$.

The counterpart result in discrete time is given in the following theorem and it also defines two different classes of multipliers:

Theorem II.8 (Discrete-time Zames-Falb multipliers, [14], [43]) Consider the discrete-time feedback system in Fig. 2 with G a stable LTI system and ϕ memoryless and slope-restricted in $S[0, k]$. Suppose that there exists an LTI multiplier $M : \ell_2 \rightarrow \ell_2$ whose transfer function has the form

$$M(z) = 1 - H(z) \quad (14)$$

such that the impulse response h of H satisfies that $h_0 = 0$ and

$$\sum_{i=-\infty}^{\infty} |h_i| < 1. \quad (15)$$

Moreover, let us assume that either the nonlinearity is odd or $h_i \geq 0$. Suppose further

$$\operatorname{Re}\{M(z)(1 + kG(z))\} > 0, \quad \forall |z| = 1. \quad (16)$$

Then the feedback interconnection (2) is ℓ_2 -stable. ■

Remark II.9 Inequality (8) is evaluated over $\omega \in \mathbb{R}$ whereas inequality (16) is evaluated over the frequency interval $\omega \in [0, 2\pi]$. Hence, by the Extreme Value Theorem [58], it is unnecessary to define any $\delta > 0$ for the discrete case corresponding to that used in the continuous case.

Similarly to the previous definitions, we can define the classes of multipliers \mathcal{M}^d and $\mathcal{M}_{\text{odd}}^d$.

D. Off-axis circle criterion

The continuous-time off-axis circle criterion is given.

Lemma II.10 (Off-axis circle criterion for continuous-time systems, [3]) Consider the feedback system in Fig. 2 with G LTI stable and ϕ is slope-restricted in $S[0, k]$. Suppose that the Nyquist plot of the linear part of the system $G(j\omega)$ lies entirely to the right of a straight line passing through the point $(-\frac{1}{k} + \delta, 0)$ where $\delta > 0$ and ϕ is monotonically increasing. Then the feedback interconnection (2) is \mathcal{L}_2 -stable.

For discrete time, only a highly restrictive version is proposed.

Lemma II.11 (Reduced off-axis circle criterion for discrete-time systems, [42]) Let the Nyquist plot of $G(e^{j\omega})$ for all $0 \leq \omega \leq \pi$ lie entirely to the right of a straight line, whose slope k is nonnegative passing through $(-\frac{1}{K_2}, 0)$. Let ω_0 be such that $\text{Re } G(e^{j\omega_0}) = -\frac{1}{K_2}$ and $\text{Re } G(e^{j\omega}) \geq -\frac{1}{K_2}$ for $\omega \geq \omega_0$ and $\text{Im } G(e^{j\omega}) \leq 0$ for $\omega_0 \geq \omega \geq 0$. Then the system is asymptotically stable for all monotone ϕ with slope restriction K_2 in the feedback path if

$$\theta \leq -\frac{1}{2}\omega_0 + \frac{\pi}{2}, \quad (17)$$

where θ is the angle made by the straight line and the imaginary axis, i.e., $\theta = \cot^{-1} k$. If $\text{Im } G(e^{j\omega}) \geq 0$ for $\omega_0 \geq \omega \geq 0$, the same argument can be used to prove the asymptotic stability of the system with nonpositive k and

$$\theta \geq \frac{1}{2}\omega_0 - \frac{\pi}{2}. \quad (18)$$

E. Further mathematical notation

For the convenience of solving potential numerical issues, the notation of $O(\cdot)$ is given.

Definition II.12 The condition

$$f(t) = g(t) + O(t^n), \text{ as } t \rightarrow 0. \quad (19)$$

means that there exist M and t_0 such that

$$|f(t) - g(t)| \leq Mt^n \text{ on } [0, t_0]. \quad (20)$$

The floor function, denoted by $\lfloor v \rfloor$, is defined by

$$\lfloor v \rfloor = \max\{m \in \mathbb{Z} \mid m \leq v\}. \quad (21)$$

III. CONTINUOUS PHASE LIMITATIONS AND THE KALMAN CONJECTURE

Megretski presents in [46] a phase limitation for continuous-time Zames-Falb multipliers. In this section we generalise the result to a wider set of frequency intervals, and derive separate results for both \mathcal{M}^c and \mathcal{M}_{odd}^c . Although it is stated in [46] that the result there is valid for \mathcal{M}_{odd}^c (in the terminology of this paper) we show by counterexample that it is in fact valid for \mathcal{M}^c only. Finally, we bridge the limitation of [46] with the Kalman conjecture; this is the key motivation to develop a different set of phase limitations for the discrete-time Zames-Falb multipliers.

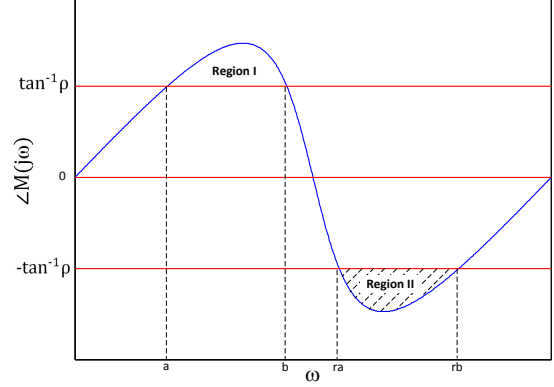


Fig. 3. Illustration of Theorem III.4 with the choice of interval from [46]: $\kappa = 1$, $c = ra$ and $d = rb$. The result is given in terms of ρ while the phase of the multiplier M is $\tan^{-1}\rho$.

A. Phase limitations

Definition III.1 Let $0 < a < b < c < d$, $\kappa > 0$, $\lambda > 0$ and $\mu > 0$. Define

$$\rho^c = \sup_{t>0} \frac{|\psi(t)|}{\phi(t)}, \quad (22)$$

and

$$\rho_{odd}^c = \sup_{t>0} \frac{|\psi(t)|}{\tilde{\phi}(t)}, \quad (23)$$

where

$$\psi(t) = \frac{\lambda \cos(at)}{t} - \frac{\lambda \cos(bt)}{t} - \frac{\mu \cos(ct)}{t} + \frac{\mu \cos(dt)}{t}, \quad (24)$$

$$\phi(t) = \lambda(b-a) + \kappa\mu(d-c) + \phi_1(t), \quad (25)$$

and

$$\tilde{\phi}(t) = \lambda(b-a) + \kappa\mu(d-c) - |\phi_1(t)|, \quad (26)$$

with

$$\phi_1(t) = \frac{\lambda \sin(at)}{t} - \frac{\lambda \sin(bt)}{t} + \frac{\kappa\mu \sin(ct)}{t} - \frac{\kappa\mu \sin(dt)}{t}. \quad (27)$$

Lemma III.2 If λ and μ are chosen such that

$$\frac{\lambda}{\mu} = \frac{d^2 - c^2}{b^2 - a^2}, \quad (28)$$

then ρ^c and ρ_{odd}^c in Definition III.1 are well-defined; that is to say $\rho^c < \infty$ and $\rho_{odd}^c < \infty$.

Proof: See Appendix. ■

Remark III.3 The direct calculation of the ratios $\psi(t)/\phi(t)$ and $\psi(t)/\tilde{\phi}(t)$ is numerically ill-conditioned for small t since, with the choice (28), we have $\psi(t) = 0 + O(t^3)$, $\phi(t) = 0 + O(t^2)$ and $\tilde{\phi}(t) = 0 + O(t^2)$, all as $t \rightarrow 0$. Nevertheless, the same construction ensures we can write

$$\frac{\psi(t)}{\phi(t)} = \gamma t + O(t^3) \text{ and } \frac{\psi(t)}{\tilde{\phi}(t)} = \gamma t + O(t^3) \text{ as } t \rightarrow 0, \quad (29)$$

with

$$\gamma = -\frac{1}{4} \frac{\lambda(b^4 - a^4) - \mu(d^4 - c^4)}{\lambda(b^3 - a^3) + \kappa\mu(d^3 - c^3)}. \quad (30)$$

We use this relation for small t in the numerical examples below.

Theorem III.4 (Continuous-time phase limitations) Let M be a continuous-time Zames-Falb multiplier. Suppose

$$\text{Im}(M(j\omega)) > \rho \text{Re}(M(j\omega)) \text{ for all } \omega \in [a, b], \quad (31)$$

and

$$\text{Im}(M(j\omega)) < -\kappa \rho \text{Re}(M(j\omega)) \text{ for all } \omega \in [c, d], \quad (32)$$

for some $\rho > 0$. Then under the conditions of Lemma III.2.

- (i) $\rho < \rho^c$ if $M \in \mathcal{M}^c$,
- (ii) $\rho < \rho_{odd}^c$ if $M \in \mathcal{M}_{odd}^c$.

Proof: See Appendix. ■

Lemma III.2 and Theorem III.4, with the choice $\kappa = 1$, $c = ra$, $d = rb$ and hence $\lambda/\mu = r^2$, are in [46]. An interpretation of Theorem III.4 with these values is illustrated in Fig. 3 (see also [45]). According to the constraints on the coefficients of continuous Zames-Falb multipliers, if the phase is simultaneously greater than $\tan^{-1} \rho$ on $\omega \in [a, b]$ (in Region I) and smaller than $-\tan^{-1} \rho$ on $\omega \in [ra, rb]$ (in Region II), then $\rho < \rho^c$ if $M \in \mathcal{M}^c$ and $\rho < \rho_{odd}^c$ if $M \in \mathcal{M}_{odd}^c$.

Remark III.5 It is straightforward to produce the counterpart of Theorem III.4 with (31) and (32) replaced by $\text{Im}(M(j\omega)) < -\rho \text{Re}(M(j\omega))$ for all $\omega \in [a, b]$ and $\text{Im}(M(j\omega)) > \kappa \rho \text{Re}(M(j\omega))$ for all $\omega \in [c, d]$ respectively.

Remark III.6 In [46], Megretski uses a positive sign in the exponential of the Laplace transform:

$$M(j\omega) = 1 - \int_{-\infty}^{\infty} e^{j\omega t} h(t) dt. \quad (33)$$

This is the standard convention in the Physics literature (see for example [59]) but opposite to that used in [12]. The apparent discrepancy has no significant consequence for the analysis of phase limitations since if $M(s)$, with impulse response $m(t)$, is a Zames-Falb multiplier then $M(-s)$, with impulse response $m(-t)$, is also a Zames-Falb multiplier.

It is natural to ask whether a phase limitation over a single frequency range can be constructed in a similar manner. This is not possible in continuous time, as any corresponding definition of ρ^c or ρ_{odd}^c would be unbounded as t approaches 0. Loosely speaking, we can generate a multiplier in \mathcal{M}^c with phase arbitrarily close to $\pm 90^\circ$ over an arbitrarily large frequency interval by selecting $h(t) = (1 - \varepsilon)\delta(t - t^*)$ with $\varepsilon > 0$ arbitrarily close to 0 and t^* arbitrarily close to 0. But we construct such phase limitations for discrete-time multipliers below, in Section IV.

B. Numerical example

Here we illustrate Theorem III.4 with a numerical example. Let $a = 1.6$ and $b = 2.25$. Let $\kappa = 1$, $c = ra$ and $d = rb$ with $r = 2.1$. Then a sweep over time intervals followed by local numerical search gives

$$\rho^c \approx 0.6069, \quad \tan^{-1} \rho^c \approx 31.25^\circ, \quad (34)$$

and

$$\rho_{odd}^c \approx 1.4928, \quad \tan^{-1} \rho_{odd}^c \approx 56.18^\circ. \quad (35)$$

Now consider the multiplier

$$M(j\omega) = 1 - \int_{-\infty}^{\infty} e^{-j\omega t} h(t) dt \quad (36)$$

with $h(t) = -0.9\delta(t + 1)$. Figure 4 shows that the relations (31) and (32), or equivalently

$$\angle M(j\omega) > \tan^{-1} \rho \text{ over the interval } [a, b], \quad (37)$$

and

$$\angle M(j\omega) < -\tan^{-1} \rho \text{ over the interval } [ra, rb], \quad (38)$$

are satisfied simultaneously for $\rho = \rho^c$ but not for $\rho = \rho_{odd}^c$. This is consistent with Theorem III.4 as $M \in \mathcal{M}_{odd}^c$ but $M \notin \mathcal{M}^c$. It is a counterexample to the false claim in [46] that the phase limitation of Theorem III.4 part (i) is applicable to the wider class $M \in \mathcal{M}_{odd}^c$.

Remark III.7 Both in this numerical example and at the end of Section III-A we consider multipliers where h takes the form $h(t) = (1 - \varepsilon)\delta(t - \tau)$ for some τ . There is a close link with Theorem III.4. Specifically if $|\psi(\tau)|/\phi(\tau) = \rho^c$ and $\varepsilon \rightarrow 0$ then

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt = \rho^c[\lambda(b-a) + \kappa\mu(d-c)] + \rho^c \int_{-\infty}^{\infty} \phi_1 h(t) dt. \quad (39)$$

Compare (80) in the proof of Theorem III.4. Similarly if $|\psi(\tau)|/\tilde{\phi}(\tau) = \rho_{odd}^c$ then

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt = \rho_{odd}^c[\lambda(b-a) + \kappa\mu(d-c)] + \rho_{odd}^c \int_{-\infty}^{\infty} \phi_1 h(t) dt. \quad (40)$$

We discuss the corresponding relations at greater length in Section IV-B for the discrete-time case.

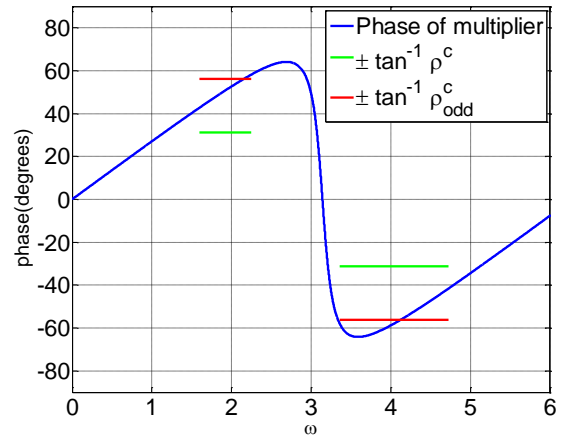


Fig. 4. Phase of the multiplier (36) and the continuous phase limitations $\pm \tan^{-1} \rho^c \approx \pm 31.25^\circ$ and $\pm \tan^{-1} \rho_{odd}^c \approx \pm 56.18^\circ$ evaluated on $[1.6, 2.25]$ and $[3.36, 4.725]$.

C. Counterexamples to Kalman conjecture via phase limitations

It is instructive to interpret the phase limitations of Theorem III.4 in a manner consistent with known results about the Kalman conjecture.

On the one hand, it is well-known that first, second and third order plants hold the Kalman conjecture [49]. The phase of such plants cannot reach both Regions I and II in Fig. 3. So the phase limitations cannot apply to these plants. First-order plants do not require a dynamic multiplier, second-order plants require a dynamics multiplier with a tunable zero and a pole at infinity, i.e. a Popov multiplier, and third order plant requires both a tunable pole and zero, i.e. first order RL/RC multipliers. In all these cases, only a first-order multiplier is required, and we know that there is no phase limitation in the selection of such multipliers [39].

Remark III.8 *Although the off-axis circle criterion is also based on RL/RC multipliers, it is not sufficient to show all third-order plants hold the Kalman conjecture. For example, the off-axis circle criterion with the plant*

$$G = \frac{s^2}{s^3 + 1.002s^2 + s + 0.998}, \quad (41)$$

guarantees stability with $k < 3.928$, whereas the multiplier $M = (s + 1)/(s + \varepsilon)$ guarantees stability for any positive k with a sufficiently small value $\varepsilon > 0$.

On the other hand the phase limitations may be applied to fourth-order plants, and these in turn may be counterexamples to the Kalman conjecture by: a) showing numerically that a phase limitation can be applied to a well-known plant, and b) showing that the Lur'e system with this plant and a slope-restricted nonlinearity may be unstable.

Specifically we will consider the phase limitation of Theorem III.4 part (i) with $\kappa = 1$, $c = ra$, and $d = rb$; that is to say the original result of [46] applied to \mathcal{M}^c . A particularly suitable example to show this limitation is O'Shea example [11], [45]:

$$G(s) = \frac{s^2}{(s^2 + 2\xi s + 1)^2}, \quad (42)$$

since the symmetry of the problem simplifies the selection of the parameters. In this example, O'Shea showed that there is a Zames-Falb multiplier for any k if $\xi > 0.5$. The following result shows that it is not possible to reach an arbitrary large k for any $\xi \leq 0.25$. For the case $\xi = 0.25$, the phase of $G(s)$ is above 177.98° over the interval $[a, b]$ where $a = 0.02249$ and $b = 0.03511$; hence it is below -177.98° over the interval $[1/b, 1/a]$ by using the symmetry of the plant. Then a suitable Zames-Falb multiplier for this plant would require a phase below -87.98° over the interval $[a, b]$ and above 87.98° over the interval $[1/b, 1/a]$. The phase limitation ensures that there is no Zames-Falb multiplier with such a phase characteristic, since $\tan^{-1} \rho^c \approx 87.79^\circ$. Strictly speaking, we have used the counterpart of Theorem III.4 mentioned in Remark III.5.

Although numerical reliability can be problematic in the discussion of the Kalman conjecture [57], simulations of the plant with asymmetrical saturation show a time evolution that

does not appear to settle to zero, supporting the validity of Conjecture I.2. The simulation shown in Figure 5 has been run in MATLAB R2013, using the solver ode45, with maximum step size of 0.0001 s, and relative tolerance of 10^{-3} . The nonlinearity ϕ is described by the nonlinear function

$$N(x) = \begin{cases} -1000 & x < -1; \\ 1000x & -1 \leq x \leq 0; \\ 0 & x > 0; \end{cases} \quad (43)$$

the input g is given by

$$g(t) = \begin{cases} 100 & t \leq 20s; \\ 0 & t > 20s; \end{cases} \quad (44)$$

and $f(t) = 0$. The relevance of this counterexample to the Kalman conjecture is that we can show that there is no Zames-Falb multiplier with $h(t) \geq 0$ for the system. The asymmetry of the nonlinearity seems to be a key factor as simulations with symmetric saturations show stable behaviour. The importance of asymmetry in the stability of Lur'e systems with saturation has been discussed recently [36], [60].

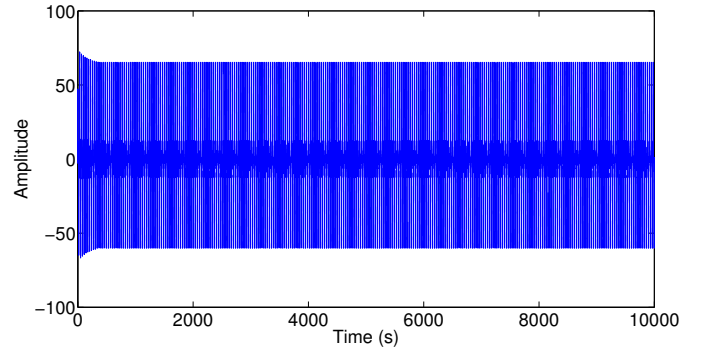


Fig. 5. Amplitude of the signal v in a simulation of the feedback interconnection depicted in Fig. 2 where G is given by (42) with $\xi = 0.25$, ϕ is described by (43), $g(t)$ is given by (44), and $f(t) = 0$.

Remark III.9 *The magnitude of the response in Fig. 5 is bounded. Since the plant G is stable and ϕ is sector-bounded it follows that if $g \in \mathcal{L}_\infty$ then all signals must be in \mathcal{L}_∞ .*

IV. DISCRETE-TIME PHASE LIMITATION

In this section we develop phase limitations for discrete-time Zames-Falb multipliers. Their derivation is in the spirit of Megretski's limitation [46] and Theorem III.4 for continuous-time multipliers. However their properties are simpler and consistent with the existence of second-order discrete-time counterexamples to the Kalman conjecture [50], [51]. In particular, and by contrast with their continuous-time counterparts, they are concerned with properties over a single interval $\omega \in [a, b]$.

It is worth highlighting that if a discrete-time multiplier preserves the positivity of all monotone and bounded nonlinearities then either it is a Zames-Falb multiplier or there exists a Zames-Falb multiplier with the same phase [14], [43]. Hence any phase limitation on the discrete-time Zames-Falb multipliers is also a limitation for any discrete-time multiplier.

A. Phase limitations

Definition IV.1 Let $0 \leq a < b \leq \pi$. Define

$$\rho^d = \max_{n \in \mathbb{Z}^+} \frac{|\Psi_d(n)|}{\Phi_d(n)}, \quad (45)$$

and

$$\rho_{odd}^d = \max_{n \in \mathbb{Z}^+} \frac{|\tilde{\Psi}_d(n)|}{\tilde{\Phi}_d(n)}, \quad (46)$$

where

$$\Psi_d(n) = \frac{\cos(an)}{n} - \frac{\cos(bn)}{n}, \quad (47)$$

$$\Phi_d(n) = (b-a) + \phi_{d,1}(n), \quad (48)$$

and

$$\tilde{\Phi}_d(n) = (b-a) - |\phi_{d,1}(n)|, \quad (49)$$

with

$$\phi_{d,1}(n) = \frac{\sin(an)}{n} - \frac{\sin(bn)}{n}. \quad (50)$$

Lemma IV.2 Both ρ^d and ρ_{odd}^d in Definition IV.1 are well-defined; that is to say $\rho^d < \infty$ and $\rho_{odd}^d < \infty$.

Proof: See Appendix. ■

Theorem IV.3 (Discrete-time phase limitations) Let M be a discrete-time Zames-Falb multiplier. Suppose

$$\text{Im}(M(e^{j\omega})) > \rho \text{Re}(M(e^{j\omega})) \text{ for all } \omega \in [a, b], \quad (51)$$

for some $\rho > 0$. Then

- (i) $\rho < \rho^d$ if $M \in \mathcal{M}^d$,
- (ii) $\rho < \rho_{odd}^d$ if $M \in \mathcal{M}_{odd}^d$.

Proof: See Appendix. ■

An interpretation of Theorem IV.3 is illustrated in Fig. 6. According to the constraints on the coefficients of discrete-time Zames-Falb multipliers, if the phase is greater than $\tan^{-1} \rho$ on $\omega \in [a, b]$ (in Region A), then $\rho < \rho^d$ if $M \in \mathcal{M}^d$ and $\rho < \rho_{odd}^d$ if $M \in \mathcal{M}_{odd}^d$.

Remark IV.4 It is straightforward to produce the counterpart of Theorem IV.3 with (51) replaced by

$$\text{Im}(M(e^{j\omega})) < -\rho \text{Re}(M(e^{j\omega})) \text{ for all } \omega \in [a, b]. \quad (52)$$

An algorithm for finding the phase limitation in Theorem IV.3 part (i) for a second order plant is given in [54]. For a given stable plant G and a value of k such that $0 < k < k_N$ the phase of an ideal multiplier is obtained as

$$\angle M_d = \begin{cases} \angle(G+1/k) - 90 & \text{if } \angle(G+1/k) > 90 \\ \angle(G+1/k) + 90 & \text{if } \angle(G+1/k) < -90 \\ 0 & \text{otherwise.} \end{cases} \quad (53)$$

Then the algorithm increases k until the existence of such a multiplier can be discarded by using the limitation presented in Theorem IV.3.

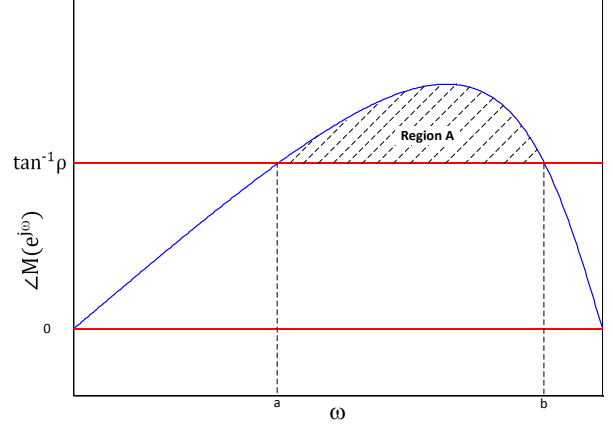


Fig. 6. Illustration of Theorem IV.3. The result is given in terms of ρ while the phase of the multiplier M is $\tan^{-1} \rho$.

B. Integral bound and sparsely parametrized multipliers

Theorem IV.3 gives relative bounds on the real and imaginary parts of a Zames-Falb multiplier's frequency response over an interval $[a, b]$. It is straightforward to derive a closely related result in terms of the integrals over the same interval.

Theorem IV.5 Let M be a discrete-time Zames-Falb multiplier. Suppose

$$\int_a^b \text{Im}(M(e^{j\omega})) d\omega > \rho \int_a^b \text{Re}(M(e^{j\omega})) d\omega \quad (54)$$

for some $\rho > 0$. Then

- (i) $\rho < \rho^d$ if $M \in \mathcal{M}^d$,
- (ii) $\rho < \rho_{odd}^d$ if $M \in \mathcal{M}_{odd}^d$.

Proof: See Appendix. ■

Remark IV.6 Theorem IV.5 is stronger than Theorem IV.3 in the sense that condition (51) is sufficient for condition (54) but not necessary. Theorem IV.3 may be derived as a Corollary of Theorem IV.5 by applying the Mean Value Theorem [58].

Theorem IV.5 gives a tight phase limitation in the sense that we can associate a set of sparsely parameterized multipliers with Theorem IV.5 as follows.

Proposition IV.7

- (i) For a given a and b , define the set $\mathcal{N}^d \subset \mathbb{Z}$ as the set of integers n such that $\Psi_d(n)/\Phi_d(n) = \rho^d$. Then multipliers of the form

$$M(z) = 1 - \sum_{n \in \mathcal{N}^d} h_n z^{-n} \quad (55)$$

with

$$h_0 = 0, h_n \geq 0 \text{ and } \sum_{n \in \mathcal{N}^d([a, b])} h_n = 1 - \varepsilon \quad (56)$$

satisfy (54) with ρ arbitrarily close to ρ^d in the limit as $\varepsilon \rightarrow 0$.

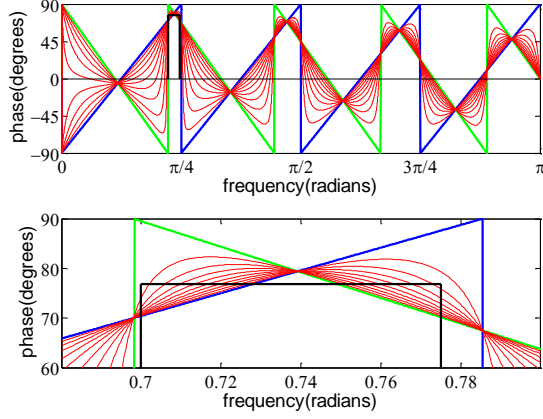


Fig. 7. Phases of the limiting cases $M(z) = 1 - z^8$, $M(z) = 1 - z^9$ and linear combinations of the form $M(z) = 1 - \lambda z^8 - (1 - \lambda)z^9$ with $0 < \lambda < 1$. The phase limitation $\tan^{-1} \rho^d \approx 76.8^\circ$ over the interval $[a, b] = [0.7, 0.77501]$ is also shown. The top figure shows the phase over the frequency range from 0 to π radians, while the bottom figure shows the same data in the frequency range from 0.68 to 0.8 radians.

(ii) For a given a and b , define the set $\mathcal{N}_{\text{odd}}^d \subset \mathbb{Z}$ as the set of integers n such that $\psi_d(n)/\tilde{\phi}_d(n) = \rho_{\text{odd}}^d$. Then multipliers of the form

$$M(z) = 1 - \sum_{n \in \mathcal{N}^d} h_n z^{-n} \quad (57)$$

with

$$h_0 = 0 \text{ and } \sum_{n \in \mathcal{N}([a, b])} h_n = 1 - \varepsilon \quad (58)$$

satisfy (54) with ρ arbitrarily close to ρ_{odd}^d in the limit as $\varepsilon \rightarrow 0$.

Proof: See Appendix. ■

Remark IV.8 It is, once again, straightforward to produce the counterpart of Theorem IV.5 with (54) replaced by

$$\int_a^b \text{Im}(M(e^{j\omega})) d\omega < -\rho \int_a^b \text{Re}(M(e^{j\omega})) d\omega. \quad (59)$$

Similarly for Theorem IV.7 with (i) $\Psi_d(n)/\phi_d(n) = -\rho^d$ and (ii) $\Psi_d(n)/\tilde{\phi}_d(n) = -\rho_{\text{odd}}^d$.

As an illustrative example, suppose $a = 0.7$ and $b = 0.77501$ (approx.). Then

$$\tan^{-1} \rho^d \approx 76.8^\circ, \quad (60)$$

and

$$\mathcal{N}^d = \{-8, 9\}. \quad (61)$$

Fig 7 shows the phases of the limiting cases $M(z) = 1 - z^8$, $M(z) = 1 - z^9$ and linear combinations of the form $M(z) = 1 - \lambda z^8 - (1 - \lambda)z^9$ with $0 < \lambda < 1$. It can be seen that the phases are near to $\tan^{-1} \rho^d$ over the interval $[a, b]$. However they always have values both above and below, indicating that Theorem IV.3 is not tight in the same sense as Theorem IV.5.

Remark IV.9 A similar analysis is possible for continuous-time multipliers. Compare Remark III.7.

C. Discrete-time counterparts of the off-axis circle criterion

The off-axis circle criterion [3] (Theorem II.10) is a useful frequency-based graphical stability test for continuous-time systems. It is sometimes assumed (e.g. in [52]) that its discrete-time counterpart is true. We state this as a conjecture:

Conjecture IV.10 Consider the feedback system in Fig. 2 with $G \in \mathbf{RH}_\infty$, and ϕ is slope-restricted in $S[0, k]$. Suppose that the Nyquist plot of the linear part of the system $G(e^{j\omega})$ lies entirely to the right of a straight line passing through the point $(-\frac{1}{k} + \delta, 0)$ where $\delta > 0$ and ϕ is monotonically increasing. Then the feedback interconnection (2) is ℓ_2 -stable.

A geometrical interpretation of both Theorem II.10 for continuous-time systems and Conjecture IV.10 for discrete-time systems is given in Fig 8.

The phase-limitation on discrete-time Zames-Falb multipliers carries the implication that there can be no multiplier construction corresponding to that for RL/RC multipliers of [42] used to prove Theorem II.10. We summarise the argument as follows:

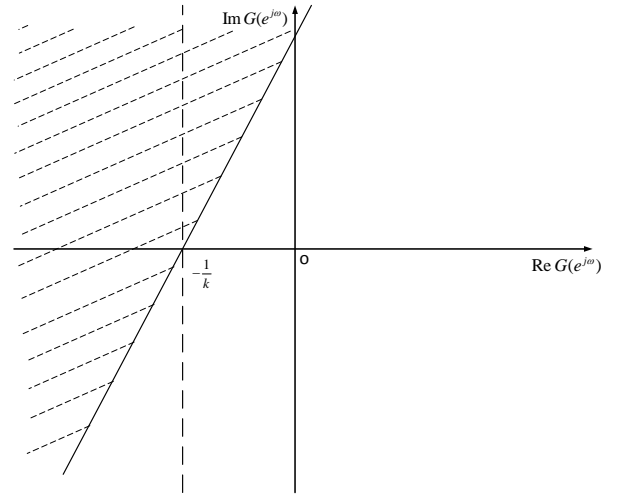


Fig. 8. Geometrical interpretation of the off-axis circle criterion considering the plant G (Theorem II.10 for continuous-time systems and Conjecture IV.10 for discrete-time systems). The Theorem for continuous-time systems is true but the Conjecture for discrete-time systems is false in general.

- 1) Under the conditions of Conjecture IV.10 there is some θ in $(-90, 90)$ degrees such that the phase of $1 + kG$ always lies in the interval $(-90 - \theta, 90 - \theta)$ degrees. Hence an ideal LTI multiplier with constant phase θ would render the real part of $M(1 + kG)$ positive over all frequencies.
- 2) In their proof of the continuous off-axis circle criterion Cho and Narendra [3] show that it is possible to construct RL/RC multipliers whose phase is arbitrarily close to some constant θ degrees over an arbitrarily large interval. We show that for some values of θ this may not be possible for any discrete-time LTI multiplier.
- 3) If a discrete-time LTI multiplier preserves the positivity of a slope-restricted nonlinearity then there is a Zames-Falb multiplier with the same phase [14], [43], so we can

limit our set of multipliers to the class of LTI Zames-Falb multipliers.

- 4) If $\theta > \tan^{-1}(2/\pi) \approx 32.48^\circ$ then Theorem IV.3 precludes any such construction of a Zames-Falb multiplier since if $a \rightarrow 0^+$ and $b \rightarrow \pi^-$ then $\rho^d \rightarrow \tan^{-1}(-2/\pi)$.

Hence the phase limitation can be used to invalidate Conjecture IV.10 when $\theta > \tan^{-1}(2/\pi)$. Smaller values can be obtained by using different values of a and b . It follows that any counterpart of the off-axis circle criterion in discrete-time must take into account specific information about frequency intervals. This is true of the more limited result originally derived by Narendra and Cho [42] (Theorem II.11). In fact it can be shown that the counterexamples to the Kalman conjecture of [50] and [51] are also counterexamples to Conjecture IV.10.

D. Finite search in discrete-time domain

Here we provide a result which simplifies the numerical implementation. Although the definitions of ρ^d and ρ_{odd}^d are given with an infinite number of terms, they can be calculated using a finite number $n = n_N$ given in Lemma IV.11.

Lemma IV.11 *Let $0 \leq a < b \leq \pi$, then*

$$\rho^d = \max_{1 \leq n \leq n_N} \frac{|\psi_d(n)|}{\phi_d(n)}, \quad (62)$$

and

$$\rho_{odd}^d = \max_{1 \leq n \leq n_N} \frac{|\psi_d(n)|}{\tilde{\phi}_d(n)}, \quad (63)$$

with

$$n_N = \lfloor \nu \rfloor, \quad (64)$$

where

$$\nu = \frac{2(b-a) - 2\sin b + 2\sin a - 2\cos b + 2\cos a}{(a-b)(\cos b - \cos a)}. \quad (65)$$

Proof: See Appendix. ■

Suppose we wish to find a phase limitation over the interval $\omega \in [0.7, 0.75]$. Applying Lemma IV.11 we find $\nu = 55.2$ and hence $|\psi_d(n)|/\phi_d(n) < |\psi_d(1)|/\phi_d(1)$ for all $n > n_N$, with

$$n_N = 55. \quad (66)$$

Hence it is sufficient to search over the integers $1 \leq n \leq 55$ for ρ^d . The numerical results shown in Fig. 9 demonstrate that $|\psi_d(n)|/\phi_d(n) < |\psi_d(1)|/\phi_d(1)$ for all $n > 18$. In fact the maximum occurs at $n = -9$.

E. Numerical example

Let us consider the negative feedback interconnection between the plant

$$G(z) = \frac{z}{z^2 - 1.8z + 0.81}, \quad (67)$$

and a slope-restricted nonlinearity. This second-order plant is a counterexample to the discrete-time Kalman conjecture as there is a periodic solution when $k = 2.1$ [50], and the Nyquist value is $k_N = 3.61$. Using the algorithm of [24] we find there

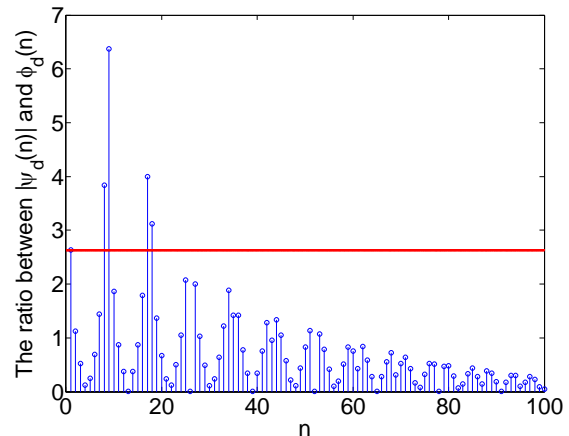


Fig. 9. The value of $f_d(n)$ with different value of n

exists a Zames-Falb multiplier for non-odd nonlinearities when $\hat{k}_{ZF} = 1.3028$.

Using the phase limitation result given in Theorem IV.3 part (i), it is possible to show that there is no Zames-Falb multiplier for any $k > k_{PL} = 1.4603$. Fig. 10 illustrates that the phase limitation results indicate there can be no appropriate Zames-Falb multiplier when $k = 1.5$. The phase limitation is given by $\tan^{-1} \rho^d = 66.7137^\circ$, where ρ^d is obtained using Definition IV.1 with $a = 0.7198$ and $b = 0.8996$. By contrast, Fig. 11 shows that this limitation is not active when $k = \hat{k}_{ZF}$; this is expected since we have been able to find a suitable Zames-Falb multiplier for this value of the gain. A complete list of slope restriction results of $G(z)$ in (67) is given in Table II. The result of the reduced off-axis circle criterion k_R shows conservativeness compared to all the other results in the Table. The (false) result from the direct discrete-time counterpart of the off-axis circle criterion is greater than the slope obtained by phase limitation, i.e. $k_O > k_{PL}$; this demonstrates that Conjecture IV.10 is false.

Finally, using combination of deadzone and saturation as nonlinearity, we are able to find periodic solution with $\hat{k}_C = 1.3666$. These results are consistent with Conjecture I.2, i.e. $\hat{k}_{ZF} < \hat{k}_C < k_{PL}$.

TABLE II
RESULTS OF DIFFERENT SLOPE RESTRICTIONS (NON-ODD NONLINEARITY)

k	k_{RO}	\hat{k}_{ZF}	\hat{k}_C	k_{PL}	k_O	k_N
Result	0.8962	1.3028	1.3666	1.4603	3.61	3.61

V. CONCLUSIONS

In this paper we have demonstrated the connection between phase limitations of Zames-Falb multipliers and the Kalman conjecture.

In continuous time, we have generalised a limitation proposed by Megretski, clarified its remit and illustrated its effect with a numerical example. In particular we show it can be applied to a fourth-order plant where the resulting numerical implementation shows instability. It remains open which

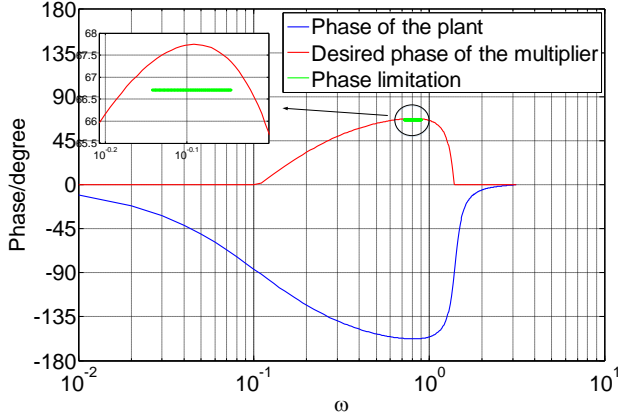


Fig. 10. Phase of $(1 + 1.5G)$, desired phase of the multiplier and the phase limitation

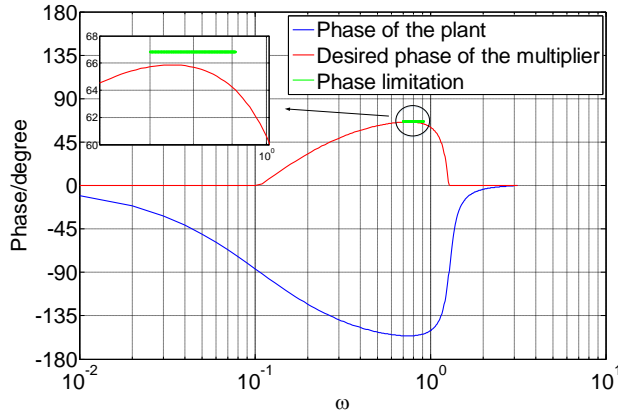


Fig. 11. Phase of $(1 + 1.3028G)$, desired phase of the multiplier and the phase limitation

choice of intervals $[a, b]$ and $[c, d]$ and scaling parameter κ in Theorem III.4 provides most insight.

Motivated by this connection and recent results on the Kalman conjecture in discrete time, we have derived a more simple phase limitation for discrete-time Zames-Falb multipliers. Numerical results in discrete time are easier to obtain and we show that the slope restriction obtained by using phase limitation theorems can be about 40% of the Nyquist value even for some second-order examples. Thus the phase limitation can be directly useful when forming benchmarks for searches over Zames-Falb multipliers. Further, the phase limitation can be used to show there can be no direct counterpart in discrete time (Conjecture IV.10) to the off-axis circle criterion for continuous-time systems (Theorem II.10).

Based on the results of this paper, we propose Conjecture I.2, which seems to be compatible with current state-of-the-art knowledge and results for both continuous and discrete-time domains.

There is plenty of scope for future work. It seems possible that the phase limitations might be used to provide a more computationally efficient search for appropriate multipliers.

Phase limitations for the class of Zames-Falb multipliers available when the nonlinearity is quasi-odd [36] require further research.

VI. ACKNOWLEDGEMENTS

We would like to thank the anonymous reviewers for their helpful suggestions.

VII. APPENDIX

A. Proof of Lemma III.2

Both the functions

$$f_1(\omega) = \omega - \frac{\sin \omega t}{t} \quad (68)$$

and

$$f_2(\omega) = \omega + \frac{\sin \omega t}{t} \quad (69)$$

are monotone non-decreasing in ω when $t > 0$. It follows that $\phi(t) > 0$ and $\tilde{\phi}(t) > 0$ when $t > 0$. In addition

$$\lim_{t \rightarrow \infty} \phi(t) = \lambda(b-a) + \kappa\mu(d-c) > 0, \quad (70)$$

and

$$\lim_{t \rightarrow \infty} \tilde{\phi}(t) = \lambda(b-a) + \kappa\mu(d-c) > 0. \quad (71)$$

Finally

$$\begin{aligned} \phi_1(t) = & -[\lambda(b-a) + \kappa\mu(d-c)] \\ & + \lambda \frac{(b^3 - a^3)t^2}{6} + \kappa\mu \frac{(d^3 - c^3)t^2}{6} + O(t^4) \text{ as } t \rightarrow 0, \end{aligned} \quad (72)$$

and

$$\begin{aligned} \psi(t) = & \lambda \frac{(b^2 - a^2)t}{2} - \mu \frac{(d^2 - c^2)t}{2} \\ & - \lambda \frac{(b^4 - a^4)t^3}{24} + \mu \frac{(d^4 - c^4)t^3}{24} + O(t^5) \text{ as } t \rightarrow 0, \end{aligned} \quad (73)$$

so the choice (28) ensures

$$\lim_{t \rightarrow 0} \frac{|\psi(t)|}{\phi(t)} = 0, \quad (74)$$

and

$$\lim_{t \rightarrow 0} \frac{|\psi(t)|}{\tilde{\phi}(t)} = 0. \quad (75)$$

B. Proof of Theorem III.4

Suppose (31) and (32) hold for some multiplier $M(j\omega) = 1 - H(j\omega)$. Then

$$\text{Im}(M(j\omega)) = \int_{-\infty}^{\infty} \sin(\omega t) h(t) dt, \quad (76)$$

and

$$\text{Re}(M(j\omega)) = 1 - \int_{-\infty}^{\infty} \cos(\omega t) h(t) dt, \quad (77)$$

where h is the impulse response of H . Hence integrating (31) and (32) over their respective intervals gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos(at) - \cos(bt)}{t} h(t) dt > \\ \rho(b-a) + \rho \int_{-\infty}^{\infty} \frac{\sin(at) - \sin(bt)}{t} h(t) dt, \end{aligned} \quad (78)$$

and

$$\int_{-\infty}^{\infty} \frac{\cos(ct) - \cos(dt)}{t} h(t) dt < -\kappa\rho(d-c) - \kappa\rho \int_{-\infty}^{\infty} \frac{\sin(ct) - \sin(dt)}{t} h(t) dt. \quad (79)$$

Summing the two inequalities, multiplied by λ and $-\mu$ respectively, gives

$$\int_{-\infty}^{\infty} \psi(t) h(t) dt > \rho[\lambda(b-a) + \kappa\mu(d-c)] + \rho \int_{-\infty}^{\infty} \phi_1 h(t) dt. \quad (80)$$

(i) If $M \in \mathcal{M}^c$ then $\|h\|_1 < 1$ and $h(t) \geq 0$ for all t . So

$$\rho[\lambda(b-a) + \kappa\mu(d-c)] > \int_{-\infty}^{\infty} \rho[\lambda(b-a) + \kappa\mu(d-c)] h(t) dt. \quad (81)$$

and hence we can write (80) as

$$\int_{-\infty}^{\infty} (\psi(t) - \rho\phi(t)) h(t) dt > 0. \quad (82)$$

But, since ϕ is an even function and ψ is an odd function,

$$\psi(t) - \rho\phi(t) \leq 0 \text{ for all } t. \quad (83)$$

Further, since ϕ is non-negative,

$$\psi(t) - \rho\phi(t) \leq 0 \text{ for all } t \text{ when } \rho \geq \rho^c. \quad (84)$$

Hence $\rho < \rho^c$.

(ii) If $M \in \mathcal{M}_{odd}^c$ then we can only say $\|h\|_1 < 1$. Nevertheless,

$$\rho[\lambda(b-a) + \kappa\mu(d-c)] > \int_{-\infty}^{\infty} \rho[\lambda(b-a) + \kappa\mu(d-c)] |h(t)| dt. \quad (85)$$

and hence (80) leads to

$$\int_{-\infty}^{\infty} (|\psi(t)| - \rho\tilde{\phi}(t)) |h(t)| dt > 0. \quad (86)$$

But, since $\tilde{\phi}$ is also an even function and (as before) ψ is an odd function,

$$|\psi(t)| - \rho_{odd}^c \tilde{\phi}(t) \leq 0 \text{ for all } t. \quad (87)$$

Further, since $\tilde{\phi}$ is non-negative,

$$|\psi(t)| - \rho\tilde{\phi}(t) \leq 0 \text{ for all } t \text{ when } \rho \geq \rho_{odd}^c. \quad (88)$$

Hence $\rho < \rho_{odd}^c$.

C. Proof of Lemma IV.2

The result is immediate following a similar argument to the proof of Lemma III.2. In particular, as ϕ_d and $\tilde{\phi}_d$ are evaluated for discrete values of $n \geq 1$, their limiting behaviour as $n \rightarrow 0$ need not be considered.

D. Proof of Theorem IV.3

Suppose (51) holds for some multiplier $M(e^{j\omega}) = 1 - H(e^{j\omega})$. Then

$$\text{Im}(M(e^{j\omega})) = \sum_{n=-\infty}^{\infty} \sin(\omega n) h_n \quad (89)$$

and

$$\text{Re}(M(e^{j\omega})) = 1 - \sum_{n=-\infty}^{\infty} \cos(\omega n) h_n, \quad (90)$$

where h is the impulse response of H . Hence integrating (51) over the interval $[a, b]$ gives

$$\sum_{n=-\infty}^{\infty} \frac{\cos(an) - \cos(bn)}{n} h_n > \rho(b-a) + \rho \sum_{n=-\infty}^{\infty} \frac{\sin(an) - \sin(bn)}{n} h_n. \quad (91)$$

(i) If $M \in \mathcal{M}^d$ then $h_0 = 0$, $\|h\|_1 < 1$ and $h_n \geq 0$ for all n . So

$$\rho(b-a) > \sum_{n=-\infty}^{\infty} \rho(b-a) h_n, \quad (92)$$

and hence we can write (91) as

$$\sum_{n=-\infty}^{\infty} (\psi_d(n) - \rho\phi_d(n)) h_n > 0. \quad (93)$$

But, since ϕ_d is an even function and ψ_d is an odd function,

$$\psi_d(n) - \rho^d \phi_d(n) \leq 0 \text{ for all } n \geq 1. \quad (94)$$

Further, since ϕ_d is non-negative,

$$\psi_d(n) - \rho\phi_d(n) \leq 0 \text{ for all } n \geq 1 \text{ when } \rho \geq \rho^d. \quad (95)$$

Hence $\rho < \rho^d$.

(ii) If $M \in \mathcal{M}_{odd}^d$ then we can only say $h_0 = 0$ and $\|h\|_1 < 1$. Nevertheless,

$$\rho(b-a) > \sum_{n=-\infty}^{\infty} \rho(b-a) |h_n|, \quad (96)$$

and hence (91) leads to

$$\sum_{n=-\infty}^{\infty} (|\psi_d(n)| - \rho\tilde{\phi}_d(n)) |h_n| > 0. \quad (97)$$

But, since $\tilde{\phi}_d$ is also an even function and (as before) ψ_d is an odd function,

$$|\psi_d(n)| - \rho_{odd}^d \tilde{\phi}_d(n) \leq 0 \text{ for all } n \geq 1. \quad (98)$$

Further, since $\tilde{\phi}_d$ is non-negative,

$$|\psi_d(n)| - \rho\tilde{\phi}_d(n) \leq 0 \text{ for all } n \geq 1 \text{ when } \rho \geq \rho_{odd}^d. \quad (99)$$

Hence $\rho < \rho_{odd}^d$.

E. Proof of Theorem IV.5

Substituting

$$\text{Im}(M(e^{j\omega})) = \sum_{n=-\infty}^{\infty} \sin(\omega n) h_n \quad (100)$$

and

$$\text{Re}(M(e^{j\omega})) = 1 - \sum_{n=-\infty}^{\infty} \cos(\omega n) h_n, \quad (101)$$

into (54) leads to (91). The proof is then identical to that of Theorem IV.3.

F. Proof of Proposition IV.7

(i) Let $M_n(z) = 1 - z^{-n}$ with $n \in \mathcal{N}([a, b])$. Then

$$\text{Im}(M_n(e^{j\omega})) = \sin(\omega n), \quad (102)$$

$$\text{Re}(M_n(e^{j\omega})) = 1 - \cos(\omega n). \quad (103)$$

Integrating over the interval yields

$$\int_a^b \text{Im}(M(e^{j\omega})) d\omega = \rho^c \int_a^b \text{Re}(M(e^{j\omega})) d\omega. \quad (104)$$

Furthermore, if

$$M(z) = 1 - \sum_{n \in \mathcal{N}([a, b])} \lambda_n z^{-n}, \quad (105)$$

with

$$\lambda_n \geq 0 \text{ and } \sum_{n \in \mathcal{N}([a, b])} \lambda_n = 1, \quad (106)$$

then we may write

$$M(z) = \sum_{n \in \mathcal{N}([a, b])} M_n(z). \quad (107)$$

The proof follows straightforwardly.

(ii) Similar.

G. Proof of Lemma IV.11

Let

$$\begin{aligned} \varepsilon &= \frac{|\psi_d(1)|}{\phi_d(1)} = \frac{|\cos a - \cos b|}{b - a - (\sin b - \sin a)} \\ &= -\frac{\cos b - \cos a}{b - a - (\sin b - \sin a)} = -\frac{\psi_d(1)}{\phi_d(1)}, \end{aligned} \quad (108)$$

where we have used that $(x - \sin x)$ is a monotonically increasing function; and

$$\begin{aligned} \nu &= \frac{2}{(b-a)} \frac{1+\varepsilon}{\varepsilon} = \frac{2-2\psi_d(1)/\phi_d(1)}{-(b-a)\psi_d(1)/\phi_d(1)} \\ &= \frac{2(b-a) - 2\sin b + 2\sin a - 2\cos b + 2\cos a}{(a-b)(\cos b - \cos a)}. \end{aligned} \quad (109)$$

For $n > \nu$, we know $(b-a)n - 2 > 0$. In addition,

$$\varepsilon(b-a)n > 2 + 2\varepsilon, \quad (110)$$

so

$$\frac{|\psi_d(n)|}{\phi_d(n)} = \frac{|\cos(bn) - \cos(an)|}{(b-a)n - [\sin(bn) - \sin(an)]} < \frac{2}{(b-a)n - 2} < \varepsilon. \quad (111)$$

As a result,

$$\frac{|\psi_d(n)|}{\phi_d(n)} < \frac{|\psi_d(1)|}{\phi_d(1)} \quad \forall n > \nu. \quad (112)$$

Finally, it is easy to check that

$$\frac{|\psi_d(1)|}{\tilde{\phi}_d(1)} = \frac{|\psi_d(1)|}{\phi_d(1)} \quad (113)$$

and hence the same relation holds for $|\psi_d(n)|/\tilde{\phi}_d(n)$.

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