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New conditions for independence of events

by

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Abstract: It is well known that two events \( A \) and \( B \) are independent if and only if \( P(AB) = P(A)P(B) \). Here we derive a condition for independence not well known. We then extend the condition for independence of \( n \) events.

Keywords: Complement, Generalization, Intersection

1 Introduction

Suppose \( A \) and \( B \) are two events. Standard undergraduate textbooks (for example, Bertsekas and Tsitsiklis (2002)) give the following conditions for independence \( A \) and \( B \):

\[
P(AB) = P(A)P(B),
\]

where \( AB \) denotes the intersection of \( A \) and \( B \); \( P(A \mid B) = P(A) \), where \( P(A \mid B) \) denotes the probability of \( A \) conditioned on \( B \); \( P(B \mid A) = P(B) \), where \( P(B \mid A) \) denotes the probability of \( B \) conditioned on \( A \).

A condition for independence not well known is the following. We denote the complement of \( A \) by \( \overline{A} \).

**Proposition 1** \( A \) and \( B \) are independent if and only if

\[
P(AB)P(B\overline{A}) = P(AB)\{1 - P(B) - (A\overline{B})\}
\]

or equivalently

\[
P(AB)P(B\overline{A}) = P(AB)P(\overline{AB}).
\]

**Proof:** Note that

\[
P(A)P(B) = P(AB)
\]

\[
\iff [P(AB) + P(A) - P(AB)][P(AB) + P(B) - P(AB)] = P(AB)
\]

\[
\iff P(AB)P(B) + [P(A) - P(AB)][P(AB) + P(B) - P(AB)] = P(AB)
\]

\[
\iff P(AB)P(B) + P(AB)[P(A) - P(AB)] + [P(A) - P(AB)][P(B) - P(AB)] = P(AB)
\]

\[
\iff [P(A) - P(AB)][P(B) - P(AB)] = P(AB)\{1 - P(B) - [P(A) - P(AB)]\}
\]

\[
\iff P(AB)P(B\overline{A}) = P(AB)\{1 - P(B) - (A\overline{B})\}.
\]

The result follows.

The condition (2) has been noted before in the last four lines in Section 2 of Joarder and Al-Sabah (2002). The condition (1) appears to be new.
The aim of this note is to generalize Proposition 1. Section 2 generalizes it for \( n \) events. Section 3 notes some particular cases of the result of Section 2.

## 2 Main result

Suppose there are \( n \) events. For simplicity of notation, let \( P(i), i = 1, 2, \ldots, n \) denote the probability that the \( i \)th event occurs. Let \( V = P(123 \cdots n) \) denote the probability of the intersection of the \( n \) events. Let \( \bar{P}(i) = P(i) - V \) for \( i = 1, 2, \ldots, n \).

Theorem 1 generalizes Proposition 1 for \( n \) events.

**Theorem 1** With the notation as above, the \( n \) events are independent if and only if

\[
\bar{P}(1)\bar{P}(2)\cdots\bar{P}(n) = V \left\{ 1 - P(2)P(3)\cdots P(n) - \bar{P}(1)P(3)P(4)\cdots P(n) \right. \\
- \bar{P}(1)\bar{P}(2)P(4)P(5)\cdots P(n) - \cdots - \bar{P}(1)\bar{P}(2)\cdots \bar{P}(n-1) \right\},
\]

**Proof:** Note that

\[
P(1)P(2)P(3)\cdots P(n) = V
\]

\[
\iff [V + P(1) - V][V + P(2) - V][V + P(n) - V] = V
\]

\[
\iff VP(2)P(3)\cdots P(n) + [P(1) - V][V + P(2) - V]P(3)P(4)\cdots P(n) = V
\]

\[
\iff VP(2)P(3)\cdots P(n) + V[P(1) - V]P(3)P(4)\cdots P(n)
\]

\[
\quad + [P(1) - V][P(2) - V][V + P(n) - V] = V
\]

\[
\iff VP(2)P(3)\cdots P(n) + V[P(1) - V]P(3)P(4)\cdots P(n)
\]

\[
\quad + V[P(1) - V][P(2) - V][P(n) - V] = V
\]

\[
\iff [P(1) - V][P(2) - V][P(n) - V] = V \left\{ 1 - P(2)P(3)\cdots P(n) \\
- [P(1) - V]P(3)P(4)\cdots P(n) - [P(1) - V][P(2) - V]P(4)P(5)\cdots P(n) - \cdots \\
- [P(1) - V][P(2) - V][P(n) - V] \right\}.
\]

Hence the result.
3  Particular cases

Here, we state particular cases of Theorem 1 for \( n = 2, 3, 4 \). When \( n = 2 \), Theorem 1 reduces to

\[
\tilde{P}(1)\tilde{P}(2) = P(12) \left\{ 1 - P(2) - \tilde{P}(1) \right\},
\]

which is equivalent to Proposition 1. When \( n = 3 \), Theorem 1 reduces to

\[
\tilde{P}(1)\tilde{P}(2)\tilde{P}(3) = P(123) \left\{ 1 - P(2)P(3) - \tilde{P}(1)P(3) - \tilde{P}(1)\tilde{P}(2) \right\}.
\]

Finally, when \( n = 4 \), Theorem 1 reduces to

\[
\tilde{P}(1)\tilde{P}(2)\tilde{P}(3)\tilde{P}(4) = P(1234) \left\{ 1 - P(2)P(3)P(4) - \tilde{P}(1)P(3)P(4) - \tilde{P}(1)\tilde{P}(2)P(4) - \tilde{P}(1)\tilde{P}(2)\tilde{P}(3) \right\}.
\]

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