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New conditions for independence of events

by

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Abstract: It is well known that two events $A$ and $B$ are independent if and only if $P(AB) = P(A)P(B)$. Here we derive a condition for independence not well known. We then extend the condition for independence of $n$ events.

Keywords: Complement, Generalization, Intersection

1 Introduction

Suppose $A$ and $B$ are two events. Standard undergraduate textbooks (for example, Bertsekas and Tsitsiklis (2002)) give the following conditions for independence $A$ and $B$:

$$P(AB) = P(A)P(B),$$

where $AB$ denotes the intersection of $A$ and $B$; $P(A \mid B) = P(A)$, where $P(A \mid B)$ denotes the probability of $A$ conditioned on $B$; $P(B \mid A) = P(B)$, where $P(B \mid A)$ denotes the probability of $B$ conditioned on $A$.

A condition for independence not well known is the following. We denote the complement of $A$ by $\bar{A}$.

**Proposition 1** $A$ and $B$ are independent if and only if

$$P(A\bar{B})P(B) = P(AB)\{1 - P(B) - (A\bar{B})\}$$

or equivalently

$$P(A\bar{B})P(B) = P(AB)P(A\bar{B}).$$

**Proof:** Note that

$$P(A)P(B) = P(AB)$$

$$\iff [P(AB) + P(A) - P(AB)][P(AB) + P(B) - P(AB)] = P(AB)$$

$$\iff P(AB)P(B) + [P(A) - P(AB)][P(AB) + P(B) - P(AB)] = P(AB)$$

$$\iff P(AB)P(B) + P(AB)[P(A) - P(AB)] + [P(A) - P(AB)][P(B) - P(AB)] = P(AB)$$

$$\iff [P(A) - P(AB)][P(B) - P(AB)] = P(AB)\{1 - P(B) - [P(A) - P(AB)]\}$$

$$\iff P(A\bar{B})P(B) = P(AB)\{1 - P(B) - (A\bar{B})\}.$$

The result follows.

The condition (2) has been noted before in the last four lines in Section 2 of Joarder and Al-Sabah (2002). The condition (1) appears to be new.
Theorem 1

Let \( \tilde{P}(i) = P(i) - V \) for \( i = 1, 2, \ldots, n \).

Hence the result.

Section 2 generalizes it for \( n \) events. Section 3 notes some particular cases of the result of Section 2.

2 Main result

Suppose there are \( n \) events. For simplicity of notation, let \( P(i), i = 1, 2, \ldots, n \) denote the probability that the \( i \)th event occurs. Let \( V = P(123 \cdots n) \) denote the probability of the intersection of the \( n \) events. Let \( \tilde{P}(i) = P(i) - V \) for \( i = 1, 2, \ldots, n \).

Theorem 1 generalizes Proposition 1 for \( n \) events.

**Theorem 1** With the notation as above, the \( n \) events are independent if and only if

\[
\tilde{P}(1)\tilde{P}(2) \cdots \tilde{P}(n) = V \left\{ 1 - P(2)P(3) \cdots P(n) - \tilde{P}(1)P(3)P(4) \cdots P(n) - \tilde{P}(1)\tilde{P}(2)P(5) \cdots P(n) - \cdots - \tilde{P}(1)\tilde{P}(2) \cdots \tilde{P}(n-1) \right\},
\]

**Proof:** Note that

\[
P(1)P(2)P(3) \cdots P(n) = V
\]

\[
\iff [V + P(1) - V][V + P(2) - V][V + P(n) - V] = V
\]

\[
\iff VP(2)P(3) \cdots P(n) + [P(1) - V][V + P(2) - V]P(3)P(4) \cdots P(n) = V
\]

\[
\iff VP(2)P(3) \cdots P(n) + V[P(1) - V]P(3)P(4) \cdots P(n)
\]

\[
+ [P(1) - V][P(2) - V][V + P(n) - V] = V
\]

\[
\iff VP(2)P(3) \cdots P(n) + V[P(1) - V]P(3)P(4) \cdots P(n)
\]

\[
+ V[P(1) - V][P(2) - V]P(4) \cdots P(n)
\]

\[
+ V[P(1) - V][P(2) - V] \cdots [P(n-2) - V]P(n)
\]

\[
+ [P(1) - V][P(2) - V] \cdots [V + P(n) - V] = V
\]

\[
\iff VP(2)P(3) \cdots P(n) + V[P(1) - V]P(3)P(4) \cdots P(n)
\]

\[
+ V[P(1) - V][P(2) - V]P(4) \cdots P(n)
\]

\[
+ V[P(1) - V][P(2) - V] \cdots [P(n-2) - V]P(n)
\]

\[
+ V[P(1) - V][P(2) - V] \cdots [P(n-1) - V]
\]

\[
+ [P(1) - V][P(2) - V] \cdots [P(n) - V] = V
\]

\[
\iff [P(1) - V][P(2) - V] \cdots [P(n) - V] = V \left\{ 1 - P(2)P(3) \cdots P(n) - [P(1) - V]P(3)P(4) \cdots P(n) - \cdots - [P(1) - V][P(2) - V] \cdots [P(n-1) - V] \right\}.
\]

Hence the result.
3 Particular cases

Here, we state particular cases of Theorem 1 for \( n = 2, 3, 4 \). When \( n = 2 \), Theorem 1 reduces to

\[
\tilde{P}(1)\tilde{P}(2) = P(12) \left\{ 1 - P(2) - \tilde{P}(1) \right\},
\]

which is equivalent to Proposition 1. When \( n = 3 \), Theorem 1 reduces to

\[
\tilde{P}(1)\tilde{P}(2)\tilde{P}(3) = P(123) \left\{ 1 - P(2)P(3) - \tilde{P}(1)P(3) - \tilde{P}(1)\tilde{P}(2) \right\}.
\]

Finally, when \( n = 4 \), Theorem 1 reduces to

\[
\tilde{P}(1)\tilde{P}(2)\tilde{P}(3)\tilde{P}(4) = P(1234) \left\{ 1 - P(2)P(3)P(4) - \tilde{P}(1)P(3)P(4) - \tilde{P}(1)\tilde{P}(2)P(4) - \tilde{P}(1)\tilde{P}(2)\tilde{P}(3) \right\}.
\]

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References
