Abstract—This paper considers the fully distributed tracking consensus problem for general nonlinear multi-agent systems with a leader whose control input is nonzero and bounded. First, a new class of distributed state observer for the leader is proposed without the knowledge of the upper bound of the leader’s input. Then, the situations that followers are affected by disturbances with unknown upper bound or disturbances generated by exosystems are investigated. Specifically, two distributed control protocols based on the distributed state observer, neural networks, and adaptive laws are proposed. Finally, simulation examples are provided to illustrate the theoretical results.

Index terms—tracking consensus, general nonlinearity, external disturbances, unknown upper bound.

I. INTRODUCTION

In the past few decades, collective behaviors of multi-agent systems have been ascending into a hot issue because of its widespread application in many fields, such as robotic teams [1], satellite clusters [2], unmanned air vehicles [3], distributed sensor networks [4], and so on. The focal point of the study on the collective behaviors of multi-agent systems lies in analyzing the essence that local information interaction among agents leads to globally collective behaviors.

Consensus problem of multi-agent systems is one of the most widely studied topics about collective behaviors of multi-agent systems [5]-[23]. Consensus problem means that, for a group of agents, design some proper distributed control protocols for each agent relying only on local information of the agent and its neighbors, so that these agents can reach an agreement. Generally, consensus problem can be divided into two categories, one is theleaderless consensus problem (without a leader) [5] [13] [14], the other is the leader-follower consensus (with one or more leaders) problem [30]-[34]. One generally studied model in multi-agent systems is the integral chain systems. [5] and [22] considered the consensus and bipartite consensus problems of first-order multi-agent systems with both undirected and directed communication topologies, respectively. [9] analyzed the tracking consensus problem for first-order multi-agent systems. [19] and [20] further considered the finite-time and fixed-time consensus problems for first-order multi-agent systems, respectively. [13] provided some necessary and sufficient conditions for achieving second-order asymptotical consensus. A further consideration on the consensus problem of second-order nonlinear multi-agent systems was given in [14], [15]-[17] considered the finite-time consensus and tracking consensus problems for second-order multi-agent systems, respectively. One can refer to [24]-[27] for more consensus results about integral chain multi-agent systems.

Another extensively researched model for multi-agent systems is the general linear time-invariant systems. During the last decade, plentiful works about the consensus problem of linear multi-agent systems have been reported [28]-[35]. [30] and [31] studied the fully distributed tracking consensus problem for a class of linear multi-agent systems under undirected and directed graphs, respectively. In these two works, the leader was assumed having no input. The situation that the leader has bounded input under general directed graphs was further investigated in [32] and [33]. The relative state and output information were utilized in designing fully distributed control protocols in [32] and [33], respectively. [35] considered the distributed containment control problem for a class of multi-agent systems with more than one leader under directed graphs. However, in most real control systems, nonlinearity is ubiquitous. In view of this, the consensus problem of general nonlinear multi-agent systems was discussed in [36]-[39]. [36] and [37] investigated the tracking consensus problem of general nonlinear multi-agent systems under undirected and general directed graphs, respectively. Both [36] and [37] assumed that the leader had bounded input, and the nonlinearity existed in followers. Neural network was introduced to estimate and counteract nonlinearities, which resulted in that the tracking errors were uniformly ultimately bounded. While in [38] and [39], both the leader and the followers could have Lipschitz-type nonlinearities and the leader had no input, and no disturbance existed.

As discussed above, there are few works concerning about the tracking consensus problem of general nonlinear multi-agent systems that the leader could have unknown bounded input, and the followers have general nonlinear dynamics and could be affected by external disturbances. The difficulty of such kind problems mainly lies in the coupling between external disturbances and nonlinearity, and the unknown upper bound of leader’s input and disturbances. The main contributions of this work are two-fold. First, a fully distributed state observer for the leader under general directed communication topologies is addressed, where the upper bound of...
the leader’s input is unknown to all the followers. To the best of the authors’ knowledge, there are very few works having solved such kind of problem. Second, for each follower with disturbances (disturbances with unknown upper bound or disturbances generated by exosystems) and nonlinearity, neural networks are introduced to estimate and compensate the nonlinearity. Even the disturbances and nonlinearity are coupled together, the tracking problem can still be solved with the help of the proposed distributed state observer.

The rest of this paper is arranged as follows. In Section II, some preliminaries and the tracking problem are introduced. In Section III, a new class of distributed state observer for the leader is proposed. The tracking consensus problem with bounded disturbances and disturbances generated by exosystems are studied in Section IV and V, respectively. Some simulations are presented to verify the theoretical results in Section VI, and Section VII concludes this paper.

Notations: $\lambda_{\min}(E)$ ($\lambda_{\max}(E)$) means the minimal (maximal) eigenvalue of the symmetric real matrix $E$. $||\cdot||$ denotes the Euclidean norm and $||\cdot||_1$ denotes the 1-norm. For any matrix $A \in \mathbb{R}^{m \times n}$, let $||A||_F = \sqrt{\text{tr}(A^T A)}$, where $\text{tr}()$ denotes the trace of a square matrix. Besides, for a vector $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$, denote $||x||_\infty = \max_{i=1,...,N} |x_i|$. Furthermore, $1_N$ means an $N$-dimensional column vector with all elements being 1. 0 is used to represent the matrix block of appropriate dimensions with all elements being 0. Denote $\text{sym}(H) = H + H^T$ with $H$ being a square matrix. For a real symmetric matrix $E$, $E > 0$ means that $E$ is positive definite. sign function is defined as: $\text{sign}(x) = 1$, if $x > 0$; $\text{sign}(x) = -1$, if $x < 0$; and $\text{sign}(0) = 0$.

II. PRELIMINARIES AND MODEL DESCRIPTION

A. Preliminaries

In this subsection, some basic concepts about graph theory and some useful lemmas are introduced.

A directed graph (or digraph) $G=(\mathcal{V}, \mathcal{E}, E)$ involves a set of nodes which is defined by $\mathcal{V} = \{v_1, v_2, ..., v_N\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $G = (a_{ij})_{N \times N}$. Here, an edge $\mathcal{E}_{ij}$ in $G$ means a pair of nodes $(v_i, v_j)$ representing that the information flow goes from node $v_j$ to node $v_i$. $G$ records the coupling configuration information of all the edges in $G$. The weight $a_{ij}$ is defined as: $a_{ij} > 0 \iff (v_i, v_j) \in \mathcal{E}$. In this paper, self loop is not considered, i.e. $a_{ii} = 0$, $i = 1, 2, ..., N$.

Definition 1: [6][14] A directed path from node $v_j$ to $v_i$ is a sequence of directed edges $(v_k, v_{k+1})$, $(v_{k+1}, v_{k+2})$, ..., $(v_{m-1}, v_m)$ in the directed network with distinct nodes $v_k$, $k = 1, 2, ..., l$. A directed network $G$ is strongly connected if between any pair of distinct nodes $v_i$ and $v_j$ in $G$, there exists a directed path from $v_j$ to $v_i$, $i, j = 1, 2, ..., N$. A directed network $G$ is said to contain a directed spanning tree, if there exists a node named root that, this node has directed paths to all the other nodes of this network $G$.

The Laplacian matrix $L = (l_{ij})_{N \times N}$ for a directed graph $G$ is denoted by [6]:

$$l_{ii} = \sum_{j, j \neq i} a_{ij}; \quad l_{ij} = -a_{ij}, \quad i \neq j.$$  

Lemma 1: [41] For any matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$, it holds that $|\text{tr}(AB)| \leq ||A||_F \cdot ||B||_F$. Besides, $||A|| \leq ||A||_F$.

B. Model Description

In this subsection, the tracking problem will be formulated. Consider a group of $N+1$ agents with unknown heterogeneous nonlinear dynamics, consisting of one leader and $N$ followers. Suppose that each agent has the following general nonlinear dynamics:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + f_i(x_i(t)) + d_i(t), \quad i = 1, 2, ..., N,$$

where $x_0(t) \in \mathbb{R}^m$ and $u_0(t) \in \mathbb{R}^n$ represent the leader’s state and control input, $x_i(t) \in \mathbb{R}^m$, $u_i(t) \in \mathbb{R}^n$, $f_i : \mathbb{R}^m \to \mathbb{R}^m$, and $d_i(t) \in \mathbb{R}^m$ represent the state, the control input, the heterogeneous smooth nonlinearity, and the disturbance of the $i$th follower, respectively, $i = 1, ..., N$. $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$ are constant matrices. In the following, $x_i$ and $u_i$ are used for simplification.

Definition 2: The tracking consensus is said to be achieved if and only if there exist some proper control inputs $u_i$, $i = 1, 2, ..., N$, such that:

$$\lim_{t \to \infty} ||x_i - x_0|| = 0, \quad i = 1, ..., N.$$  

III. A CLASS OF FULLY DISTRIBUTED STATE OBSERVER FOR THE LEADER

In this section, a new class of fully distributed observer will be constructed to estimate the state of the leader. First of all, some assumptions about the dynamics of the leader and the communication network among the agents are given as follows.

Assumption 1: The pair $(A, B)$ is controllable.

Lemma 2: [47][40] With Assumption 1, $\forall \beta > 0$, there exists a solution $P > 0$ to the following Riccati inequality

$$PA + A^TP - 2\beta PB B^TP + \beta I_n < 0.$$  

Assumption 2: There exists a constant $\omega > 0$ such that $||u_i|| \leq \omega$. Besides, the upper bound $\omega$ is unknown to all the followers.

Assumption 3: The state of the leader is bounded.

Remark 1: The leader could be a real agent or a virtual signal with predefined trajectory so as to render the multi-agent system to fulfill specific control objectives. In real applications, the workspace is usually with limited space, hence it could be reasonable to ensure that the leader has bounded state (by designing $u_0$ in advance). However, each follower still has no knowledge about $u_0$.

Assumption 4: The communication network of the $N + 1$ agents contains a directed spanning tree with the leader as the root node.

Let $L_f$ be the Laplacian matrix corresponding to the subgraph among all the followers, and denote $\Psi = \text{diag}(\xi_1, ..., \xi_N)$. Define $L = L_f + \Psi$. The following lemma provides an useful property of $L$.

Lemma 3: [7] Suppose that Assumption 4 is satisfied, then there exists a diagonal matrix $\Xi$ such that $\Xi = \text{diag}(\xi_1, ..., \xi_N) > 0$ and $\Xi L + L^T \Xi > 0$.

Before showing the distributed state observer design, the structure of the dynamics of the leader needs to be further analyzed.

Definition 3: [42] The pairs $(A, B)$ and $(A', B')$ are said feedback equivalent (or F-equivalent) iff there exist matrices $C \in \mathbb{R}^{m \times m}$, $D \in \mathbb{R}^{n \times n}$, and $Q \in \mathbb{R}^{m \times n}$, with $C$ and $D$ being nonsingular, such that $A' = C^{-1}(A + BQ)C$, and $B' = C^{-1}BD$.
Lemma 4: [42] Suppose that (A, B) is controllable. Denote $r_0 = \text{rank}(B)$. Then, there exist $m$ integers $p_1, \ldots, p_m$ satisfying $0 \leq p_i \leq n$, $p_1 \geq p_2 \geq \cdots \geq p_m > 0$, $p_i = 0$, for $i > r_0$, and $\sum_{i=1}^{r_0} p_i = n$, such that $(A, B)$ is F-equivalent with a decoupled system of $r_0$ integrators:

$$\dot{y}_{k,i+1} = y_{k,i+2}, \ldots, \dot{y}_{k,i+1-r_0} = y_{k,i+1-r_0}, \quad \dot{y}_{k+1} = u_{k+1},$$

$$i = 0, \ldots, r_0 - 1, k_0 = 0, k_s = \sum_{j=1}^{s} p_j, \quad s = 1, \ldots, r_0.$$  \hfill (4)

According to Lemma 4, there exist matrices $C$, $D$, and $Q$, with the state transformation $y_0 = C^{-1}x_0$, such that $\dot{y}_0 = A'y_0 + B'u_{0}$ has the form

$$\dot{y}_{0,k+1} = y_{0,k+2}, \ldots, \dot{y}_{0,k+1-r_0} = y_{0,k+1-r_0}, \quad \dot{y}_{0,k+1} = d_0, t_1,$$

$$l = 0, \ldots, r_0 - 1, k_0 = 0, k_s = \sum_{j=1}^{s} p_j, \quad s = 1, \ldots, r_0,$$  \hfill (5)

where $y_0 = [y_{01}, \ldots, y_{0n}]^T$, $A' = C^{-1}(A + BQC)$, $B' = C^{-1}BD$, $u_{0} = D^{-1}(-Qx_0 + u_0)$, and $B'u_{0} = [0_{1 \times p_1 - 1}, a_{01}, 0_{1 \times p_2 - 1}, \alpha_{02}, \ldots, 0_{1 \times p_r - 1}, a_{0r}]^T$. With Assumptions 2 and 3, one has $d_0, d_{r_0}$ all bouned, and the upper bounds are unknown to all the followers. Suppose that $[d_{|0|} \leq D_0, l = 1, \ldots, r_0$, with $D_0$ being a finite positive number.

To this end, it is ready to present the construction of the fully distributed state observer. Specifically, for follower $i$, consider the following distributed state observer:

$$\dot{\hat{y}}_{i,k+1} = y_{i,k+2} - e_{i,k+1},$$

$$\ldots$$

$$\dot{\hat{y}}_{i,k+1-r_0} = y_{i,k+1-r_0} - e_{i,k+1-r_0},$$

$$\dot{\hat{y}}_{i,k+1} = -(\alpha_{i,1} + \beta_{i,1})[e_{i,k+1} + \text{sign}(e_{i,k+1})],$$

$$l = 0, \ldots, r_0 - 1, k_0 = 0, k_s = \sum_{j=1}^{s} p_j, \quad s = 1, \ldots, r_0,$$  \hfill (6)

where $e_{i,k} = \sum_{j=1}^{N} a_{ij}(y_{i,j} - y_{ij}) + \alpha_{0}(y_{i,j} - y_{i0}), k = 1, \ldots, n$, $\beta_{i,1} = (1 + \sum_{j=1}^{N} a_{ij}e_{i,j})$, and $\alpha_{i,1}$ is an adaptive control gain designed as follows:

$$\alpha_{i,1} + \beta_{i,1} = \sum_{j=1}^{N} e_{i,j} + \text{sign}(e_{i,j}),$$

$$\alpha_{i,1}(0) > 0, \quad l = 0, \ldots, r_0 - 1.$$  \hfill (7)

Theorem 1: Suppose that Assumptions 1-4 are satisfied. Then the distributed state observer (6) and (7) can estimate the state of the leader asymptotically.

Proof: Observing that in (6), the $r_0$ integrator-like components are decoupled from each other, one only needs to prove that each component in (6) can cooperatively estimate the states of the corresponding integrator in (5). Without loss of generality, only the first component (first $p_1$ rows of (6) will be analyzed in the following.

Denote $e_k = [e_{1k}, \ldots, e_{Nk}]^T$, $\alpha_1 = \text{diag}(\alpha_{11}, \ldots, \alpha_{N1})$, $\beta_1 = \text{diag}(\beta_{11}, \ldots, \beta_{N1})$. Then one can obtain the dynamics of $[e_{1T}, \ldots, e_{p_1T}]^T$ by subtracting (5) from (6):

$$\dot{e}_{1} = e_{2} - L_{e_{1}}$$

$$\ldots$$

$$\dot{e}_{p_1-1} = e_{p_1} - L_{e_{p_1-1}}$$

$$\dot{e}_{p_1} = -\tilde{L}(\alpha_1 + \beta_1)(e_{p_1} + \text{sign}(e_{p_1})) - \tilde{L}_1N d_0,$$  \hfill (8)

Consider the following Lyapunov function

$$V_o = \sum_{i=1}^{N} \xi_i(\alpha_{i1} + \beta_{i1}) \frac{1}{2} \sum_{i=1}^{N} \xi_i(\alpha_{i1} + \beta_{i1}) e_{p_1} + \text{sign}(e_{p_1})),$$  \hfill (9)

where $\xi_i$ is defined in Lemma 3, and $\bar{\alpha}$ is a positive constant to be designed.

Taking the derivative of $V_o$, one gets

$$\dot{V}_o = \sum_{i=1}^{N} \xi_i(\alpha_{i1} + \beta_{i1}) \dot{\beta}_{i1} + \sum_{i=1}^{N} \xi_i(\beta_{i1} \dot{\alpha}_{i1} + \sum_{i=1}^{N} (\alpha_{i1} + \bar{\alpha}) \dot{\alpha}_{i1}$$

$$= -\tilde{L}(\alpha_1 + \beta_1)(e_{p_1} + \text{sign}(e_{p_1})) - \tilde{L}_1N d_0,$$

$$+ \sum_{i=1}^{N} \xi_i(\alpha_{i1} + \beta_{i1}) \dot{\alpha}_{i1} - \bar{\alpha} \sum_{i=1}^{N} \xi_i \dot{\alpha}_{i1} \leq -\lambda_0 e_{p_1} + \text{sign}(e_{p_1}) \dot{e}_{p_1} + \text{sign}(e_{p_1})$$

$$+ \sum_{i=1}^{N} D_0 a_{i0} \xi_i(\alpha_{i1} + \beta_{i1}) \text{sign}(e_{p_1})$$

$$+ \sum_{i=1}^{N} \xi_i(\alpha_{i1} + \beta_{i1}) \text{sign}(e_{p_1})$$

$$- \bar{\alpha} \sum_{i=1}^{N} \xi_i \text{sign}(e_{p_1}) \leq -\lambda_0 e_{p_1} + \text{sign}(e_{p_1}) \leq \frac{1}{2} \sum_{i=1}^{N} (\alpha_{i1} + \beta_{i1}) \text{sign}(e_{p_1})$$

$$+ \sum_{i=1}^{N} \xi_i(D_0 a_{i0} + 1)(\alpha_{i1} + \beta_{i1}) \text{sign}(e_{p_1})$$

$$- \bar{\alpha} \sum_{i=1}^{N} \xi_i \text{sign}(e_{p_1})$$  \hfill (10)

where $\lambda_0 = \lambda_{\min} \left( \frac{1}{2}(L^T + L) \right)$, and the facts that $1 \xi_i(\alpha_{i1} + \beta_{i1}) \text{sign}(e_{p_1}) = (e_{p_1} + \text{sign}(e_{p_1}))^T (\alpha_{i1} + \beta_{i1}) \Xi$, $\tilde{L}_1N = [a_{i0}, \ldots, a_{N0}]^T$, and the boundedness of $a_{i0}$ are applied.

Choose $\bar{\alpha}$ such that $\bar{\alpha} \geq \max_{i=1,\ldots,N} \{ \frac{1}{2} \sum_{i=1}^{N} (D_0 a_{i0} + 1) \}$. Noticing that $|e_{p_1} + \text{sign}(e_{p_1})| \geq 1$ whenever $|e_{p_1}| > 0$, it follows that $|e_{p_1} + \text{sign}(e_{p_1})|^2 \geq |e_{p_1} + \text{sign}(e_{p_1})|$. In view of this property as well as invoking the basic inequality, one has

$$\dot{V}_o \leq -\frac{\lambda_0}{2} \sum_{i=1}^{N} (\alpha_{i1} + \beta_{i1})^2 e_{p_1} + \text{sign}(e_{p_1})^2$$

$$- \sum_{i=1}^{N} \sqrt{2\lambda_0 \bar{\alpha}) \xi_i(\alpha_{i1} + \beta_{i1}) e_{p_1} + \text{sign}(e_{p_1})^2}$$

$$+ \sum_{i=1}^{N} \xi_i(D_0 a_{i0} + 1)(\alpha_{i1} + \beta_{i1}) \text{sign}(e_{p_1})$$

$$\leq -\frac{\lambda_0}{2} \sum_{i=1}^{N} (\alpha_{i1} + \beta_{i1})^2 e_{p_1} + \text{sign}(e_{p_1})^2$$

$$\leq -\frac{\lambda_0}{2} \sum_{i=1}^{N} (\alpha_{i1}(0) \text{sign}(e_{p_1})^2,$$  \hfill (11)
where the last inequality is resulted by the facts that 1) $\alpha_{i1}(t)$ is nondecreasing; 2) $\beta_{i1} \geq 0$; 3) $|e_{ip1} + \text{sign}(e_{ip1})|^2 + 2|e_{ip1} + \text{sign}(e_{ip1})|^2 \geq |\text{sign}(e_{ip1})|^2$.

As long as the tracking consensus is not achieved, there must exist $j \in \{1, ..., N\}$, such that $e_{ip1} \neq 0$. Here, $j$ could vary from time. Denote $\alpha_o = \frac{1}{\min_{s=1,...,N}\{\alpha_{i1}(0)\}}$. Then, before the tracking consensus is achieved, one has

$$\dot{V}_o \leq -\alpha_o,$$  

(12)

which follows that $e_{ip1, i = 1, ..., N}$ would reach to zero no later than $T_o = \frac{V_o(0)}{\alpha_o}$, which means that $y_{ip1}$ can estimate the real value of $y_{ip1}$ as $t \to \infty$.

On the other hand, since $\tilde{L}$ is Hurwitz, one can observe from (8) that, the system of $e_{p1-1}$ is input-to-state stable (ISS) with respect to $e_{p1}$. Then, one has $e_{p1-1} \to 0$ as $e_{p1} \to 0$, which means that $y_{ip1-1} \to y_{ip1-1}$ as $t \to \infty$. With similar arguments, one can conclude that $y_{ik} \to y_{0k}, k = 1, ..., p_1$, as $t \to \infty$. This completes the proof. \hfill \blacksquare

Remark 2: With the differential inequality (12), one can only claim that the tracking error $y_{ip1} - y_{0p1}$ will reach zero asymptotically. In fact, the evolution of each follower depends on how $d_{01}$ evolves. For one thing, $V_o$ is possible to stop decreasing before reaching to zero if the tracking error $y_{ip1} - y_{0p1}$ has reached zero before $T_o$. This statement can be observed from the discussion of $V_o$. For another, if $d_{01}$ evolves very slowly and its absolute value reaches maximum after a long time. While $\alpha_{i1}$, the main component of the coefficient of $\text{sign}(e_{ip1})$, increases quickly. Then $\alpha_{i1} + \beta_{i1} \text{sign}(e_{ip1})$ can dominate the effect caused by $d_{01}$. During this time interval, the tracking error will reach zero fast. When $d_{01}$ evolves above the domination of $\alpha_{i1} + \beta_{i1} \text{sign}(e_{ip1})$, $\alpha_{i1}$ increases quickly and makes the followers track the leader. So (12) should be understood as that the distributed observer (6) and (7) has a fast response with respect to the leader’s evolution.

Remark 3: Theoretically, $\alpha_{i,j+1}$ is bounded. While it is not the truth in real applications. To ensure the boundedness of adaptive gains in practical implementations, one can use a sufficiently small value $\epsilon$ to adjust the ideal adaptive law to practical one:

$$\dot{\alpha}_{i,j+1} = \begin{cases} \left| e_{i,k,i+1} + \text{sign}(e_{i,k,i+1}) \right|, & |e_{i,k,i+1}| \geq \epsilon, \\ 0, & |e_{i,v,i}| < \epsilon. \end{cases}$$

Remark 4: One should notice that if the leader is originally a higher-order integrator, then the boundedness requirement of the leader’s state is no more needed.

IV. TRACKING CONSENSUS OF GENERAL NONLINEAR MULTI-AGENT SYSTEMS WITH BOUNDED DISTURBANCES

In this section, consider the tracking problem of a group of $N + 1$ agents described by (1), where the disturbance $d_i(t)$ is bounded.

Assumption 5: The disturbance of each follower is bounded, i.e.:

$$\|d_i(t)\|_{\infty} \leq \omega_i, \quad i = 1, 2, ..., N,$$  

(13)

where $\omega_i > 0$ are some finite but unknown constants.

First of all, take a further look at the distributed state observer (6). Noticing the special structure of $A' = C^{-1}(A + BQ)C, B' = C^{-1}BD$, (6) can be written as

$$\dot{y}_i = A'y_i + B'\dot{u}_i + \tau_i,$$  

(14)

where $y_i = [y_{i1}, ..., y_{in}]^T, u_i = -\left[\left(\alpha_{i1} + \beta_{i1}\right)e_{ik_1} + \text{sign}(e_{ik_1}), ..., (\alpha_{i10} + \beta_{i10})e_{ik_{10}} + \text{sign}(e_{ik_{10}})\right]^T, 0, ..., 0]^T$ with $m - r_0$ zeros, $\tau_{i1} = [\tau_{i11}, ..., \tau_{i1r}]^T$, and $\tau_{i1+l} = -[e_{ik_{11}}, ..., e_{ik_{1+l}}, -1]^T, l = 0, ..., r_0 - 1$. Let $\hat{x}_i = Cy_i$, one has

$$\dot{\hat{x}}_i = A\hat{x}_i + B(Q\hat{x}_i + D\hat{u}_i) + C\tau_i.$$  

(15)

Then, consider the dynamics of each follower. Owing to the approximation property of neural networks (NNs) [43], the smooth function $f_i(x_i)$ can be approximated on a compact set as

$$f_i(x_i) = W_i^T \phi_i(x_i) + \varepsilon_i,$$  

(16)

where $\phi_i(\cdot): \mathbb{R}^n \to \mathbb{R}^r$ is a known activation function, $W_i \in \mathbb{R}^{r \times m}$ is a constant real matrix representing the ideal NN weight matrix, and $\varepsilon_i$ is the bounded approximation error vector satisfying $\|\varepsilon_i\|_{\infty} \leq \tilde{\varepsilon}_i$. Usually, $\tilde{\varepsilon}_i$ is unknown and hence it will not be used in the following controller design. Suppose that $\phi_i(\cdot)$ is bounded on compact sets.

For each follower, design the controller as follows:

$$u_i = Q\hat{x}_i + D\hat{u}_i - \hat{W}_i \phi_i(x_i) - K\hat{x}_i - \mu_i \text{sign}(K\hat{x}_i),$$  

(17)

of which $\hat{W}_i$ is the estimation of $W_i$ and is utilized to compensate the nonlinearity $f_i(x_i), K = B^TP$ with $P$ being the solution of (3) ($\beta = 1$), $\hat{x}_i = \hat{x}_i - \tilde{x}_i$, and $\mu_i$ is an adaptive gain designed as

$$\dot{\mu}_i = ||K\hat{x}_i||_1.$$  

(18)

Furthermore, $\hat{W}_i$ is designed as

$$\dot{\hat{W}}_i = m_i \phi_i(x_i)\tilde{x}_i^TPB - m_i h_i(\hat{W}_i - \hat{W}_i),$$  

(19)

where $m_i, n_i$, and $h_i$ are positive factors.

Theorem 2: Suppose that Assumptions 1-5 are satisfied. Then, with the controller (17)-(19), each follower can track the trajectory of the leader asymptotically.

Proof: Noticing (1), (15), and (17), one obtains the dynamics of $\hat{x}_i$:

$$\dot{\hat{x}}_i = A\hat{x}_i + B[\hat{W}_i^T \phi_i(x_i) + \tilde{d}_i - K\hat{x}_i - \mu_i \text{sign}(K\hat{x}_i)] - C\tau_i,$$  

(20)

where $\tilde{W}_i = \hat{W}_i - \hat{W}_i$, and $\tilde{d}_i = \varepsilon_i + d_i$. Then one has $\|\tilde{d}_i\|_{\infty} \leq \omega_i + \tilde{\varepsilon}_i$.

Consider the following Lyapunov function candidate

$$V_i = \tilde{x}_i^TP\tilde{x}_i + \frac{1}{m_i} \text{tr}(\tilde{W}_i^T \tilde{W}_i) + \frac{1}{m_i} \text{tr}(\hat{W}_i^T \hat{W}_i) + (\mu_i - \mu_i)^2,$$  

(21)

of which $\tilde{W}_i = W_i - \hat{W}_i, \mu$ is a parameter to be designed. Taking the derivative of $V_i$ gives

$$\dot{V}_i = 2\tilde{x}_i^TP\tilde{x}_i + \frac{2}{m_i} \text{tr}(\tilde{W}_i^T \tilde{W}_i) - \frac{2}{m_i} \text{tr}(\hat{W}_i^T \hat{W}_i) + 2(\mu_i - \mu_i)^2.$$  

(22)

On the one hand,

$$\frac{1}{m_i} \text{tr}(\tilde{W}_i^T \tilde{W}_i) = \text{tr}(\tilde{W}_i^T \phi_i(x_i)\tilde{x}_i^TPB - h_i \tilde{W}_i^T (\hat{W}_i - \hat{W}_i)) = \tilde{x}_i^TPB\tilde{W}_i^T \phi_i(x_i) - h_i \text{tr}(\tilde{W}_i^T (\hat{W}_i - \hat{W}_i)).$$
On the other hand,
\[
\frac{1}{n_i} \text{tr}(\hat{W}_i^T \hat{W}_i) = h_i \text{tr}(\hat{W}_i^T (\hat{W}_i - W_i)).
\]
It follows that
\[
\dot{V}_i = \hat{x}_i^T (PA + A^T P - 2PB^T P) \hat{x}_i + 2\hat{x}_i^T PD \hat{d}_i
- 2\mu ||K \hat{x}_i||_1 - 2\hat{x}_i^T PC \tau_i
- 2h_i \text{tr}((\hat{W}_i - W_i)^T (\hat{W}_i - W_i))
\leq - ||\hat{x}_i||^2 + 2(\omega_i + \varepsilon_i - \mu)||K \hat{x}_i||_1 - 2\hat{x}_i^T PC \tau_i,
\]
where \(\hat{x}_i^T PB\text{sign}(K \hat{x}_i) = ||K \hat{x}_i||_1\) and \(\hat{W}_i - W_i = -\hat{W}_i + W_i\) are applied in the first equality. Choose \(\mu > \omega_i + \varepsilon_i\). Noticing that \(-2\hat{x}_i^T PC \tau_i \leq \varphi ||\hat{x}_i||^2 + \frac{1}{\varphi} ||PC\tau_i||^2\) with \(0 < \varphi < 1\), one further has
\[
\dot{V}_i \leq -(1 - \varphi)||\hat{x}_i||^2 + \frac{1}{\varphi} ||PC\tau_i||^2.
\]
According to Theorem 1, \(\tau_i \to 0\) as \(t \to \infty\). Observing from (11), one can also obtain that \(\hat{V}_i \leq -\lambda_0 \sum_{i=1}^{N} \alpha_{i}(0)||e_{i,p_i}||^2\), which follows that \(\int_{0}^{\infty} ||e_{i,p_i}||^2 dt\) exists and is finite. Furthermore, one can conclude from (8) that \(\int_{0}^{\infty} ||e_{i}|^2 dt, \ldots, \int_{0}^{\infty} ||e_{i,n_{i-1}}||^2 dt\) exist and are finite. With similar arguments, one can claim that \(\int_{0}^{\infty} ||\hat{e}_{i,n_{i-1}}||^2 dt\) exist and are finite. Noticing the definition of \(\tau_i\), one can further get that \(\int_{0}^{\infty} ||PC\tau_i||^2 dt\) exists and is finite. It follows that \(V_i\) is bounded and \(\int_{0}^{\infty} ||\hat{x}_i||^2 dt\) exists and is finite. By invoking (21), \(\hat{x}_i, W_i, \mu_i\) are bounded. Then \(\hat{x}_i^T \hat{x}_i\) is bounded by noticing (20). According to Barbalat Lemma [46], one has \(\hat{x}_i \to 0\), as \(t \to \infty\). This completes the proof.

Remark 5: Inspired by [44] and [45], the so-called \(\sigma-\)modification method can be applied for the adaptive gain \(\mu_i\) to avoid high amplitude. Specifically, (18) can be modified as:
\[
\dot{\mu}_i = ||K \hat{x}_i||_1 - \hat{h}_i(\mu_i - \hat{\mu}_i),
\]
where \(\hat{h}_i\) is a positive factor. In fact, the design of (19) has similar effects. Different from the work in [44], the upper bound of the neural network approximation error as well as the disturbance is not required to be known.

V. TRACKING CONSENSUS OF GENERAL NONLINEAR MULTI-AGENT SYSTEMS WITH DISTURBANCES GENERATED BY EXOSYSTEM

In this section, consider the tracking problem of a group of \(N + 1\) agents described by (1), and the disturbance \(d_i(t)\) is generated by an exosystem:
\[
\begin{aligned}
\dot{\xi}_i &= W \xi_i, \\
\dot{d}_i &= V \xi_i,
\end{aligned}
\]
of which \(\xi_i\) is the state of the exosystem (24), \(W\) and \(V\) are with appropriate dimensions.

Assumption 6: The pair \((BV, W)\) is observable.
In the last section, the general smooth nonlinearity was considered. In this section, another class of nonlinearity will be discussed.

Assumption 7: Assume that \(f_i(x_i) = H_i^T v_i(x_i)\), where \(H_i \in \mathbb{R}^{n_x \times m}\) is the unknown constant coefficient matrix, and \(v_i(\cdot): \mathbb{R}^n \to \mathbb{R}^m\) is a known bounded function.

In the following, a class of disturbance-observer-based tracking control protocols is proposed to solve the tracking problem:
\[
\begin{aligned}
\dot{z}_i &= (W + FBV)(z_i - Fx_i) + FAx_i + FBu_i \ \\
&\quad + FB\hat{H}_i^T v_i(x_i), \\
\dot{d}_i &= V\hat{\xi}_i, \\
\hat{\xi}_i &= z_i - Fx_i, \\
\hat{u}_i &= Q \hat{x}_i + D \dot{\hat{u}}_i - \hat{H}_i^T v_i(x_i) - \hat{K} \hat{x}_i - \hat{d}_i,
\end{aligned}
\]
where \(z_i\) and \(d_i\) are the state and output of the disturbance observer for (24), \(F\) is chosen such that \(W + FBV\) is Hurwitz, \(K = B^T P\) with \(P\) being the solution of (3) (\(\beta = 1\)), \(\hat{x}_i, \hat{u}_i\) and \(\hat{\xi}_i\) are defined the same as the previous section. Assumption 6 ensures the existence of \(F\). Furthermore, \(\hat{H}_i\) is the estimation of \(H_i\) with the following dynamics:
\[
\dot{\hat{H}}_i = g_i v_i(x_i) \hat{x}_i^T PB - g_i \epsilon_i \hat{H}_i,
\]
of which \(g_i\) and \(\epsilon_i\) are positive factors.

Theorem 3: Suppose that Assumptions 1-4, 6 and 7 are satisfied. Then with the controller design (25) and (26), each follower can track to a bounded neighborhood of the leader’s state.

Proof: Denote \(e_{i\xi} = \xi_i - \hat{\xi}_i\), and \(\hat{H}_i = H_i - \hat{H}_i\). By invoking (15), (24), and (25), one can obtain the dynamics of \(e_{i\xi}\) and \(\dot{\hat{\xi}}_i\):
\[
\dot{e}_{i\xi} = (W + FBV)e_{i\xi} + FB\hat{H}_i^T v_i(x_i)
\]
\[
\dot{\hat{\xi}}_i = A\hat{x}_i + B\hat{H}_i^T v_i(x_i) - BK\hat{x}_i + BVe_{i\xi} - C\tau_i.
\]
Consider the following Lyapunov function candidate
\[
V_{ei} = \hat{x}_i^T P \hat{x}_i + \frac{1}{2} \text{tr}((\hat{H}_i^T \hat{H}_i) + \lambda e_{i\xi}^T P_1 e_{i\xi})
\]
where \(P_1\) is a positive definite matrix satisfying \(X = (W + FBV)^T P_1 + P_1(W + FBV) < 0\) and \(\lambda > 0\) is a parameter to be designed.
Taking the derivative of \(V_{ei}\), one has
\[
\begin{aligned}
\dot{V}_{ei} &= \hat{x}_i^T (PA + A^T P - 2PB^T P) \hat{x}_i
+ 2\hat{x}_i^T PBV e_{i\xi} - 2\hat{x}_i^T PC \tau_i + 2e_{i\xi} \text{tr}(\hat{H}_i^T \hat{H}_i)
+ \lambda e_{i\xi}^T X e_{i\xi} + 2\lambda e_{i\xi}^T P_1 FB\hat{H}_i^T v_i(x_i).
\end{aligned}
\]
Noticing that \(||\hat{H}_i||_F^2 = \text{tr}(\hat{H}_i^T \hat{H}_i)\), one has
\[
\begin{aligned}
2\hat{x}_i^T PBV e_{i\xi} &\leq \frac{1}{4} ||\hat{x}_i||^2 + 4\eta_1 ||e_{i\xi}||^2, \\
2\hat{x}_i^T PC \tau_i &\leq \frac{1}{4} ||\hat{x}_i||^2 + 4 ||PC\tau_i||^2, \\
2e_{i\xi} \text{tr}(\hat{H}_i^T \hat{H}_i) &\leq -e_{i\xi} ||\hat{H}_i||_F^2 + e_{i\xi} ||H_i||_F^2, \\
e_{i\xi}^T X e_{i\xi} &\leq -\eta_2 ||e_{i\xi}||^2, \\
2\epsilon_{i\xi}^T P_1 FB\hat{H}_i^T v_i(x_i) &\leq \frac{2}{\eta_3} ||v_i(x_i)||^2 ||P_1 FB||^2 ||\hat{H}_i||^2 + \frac{\eta_2}{4} ||e_{i\xi}||^2 \\
&\quad + \frac{\eta_2}{2} ||e_{i\xi}||^2 + \eta_3 ||\hat{H}_i||^2,
\end{aligned}
\]
\[ \lambda = \frac{(8 \eta_1 + 2\delta)}{\eta_2}, \quad \epsilon_i = \lambda \eta_3 + \bar{\delta}, \] where \( \delta \) and \( \bar{\delta} \) are two positive numbers. It follows that
\[ \dot{V}_{ei} \leq -\frac{1}{2} ||\tilde{x}_i||^2 + \theta \]
of which \( \kappa = \min(1/2, \delta, \bar{\delta}) \) and \( \theta = 4||PC\tau_i||^2 + \epsilon_i||H_i||^2 \).
Since \( \tau_i \to 0 \) as \( t \to \infty \), it follows from (30) that \( \tilde{x}_i \) will converge to the compact set \( \Omega = \{ \tilde{x}_i : \dot{V}_{ei} \leq -\frac{\theta}{2} \} \), where \( \dot{\theta} = \epsilon_i||H_i||^2 \).
This completes the proof.

VI. SIMULATION EXAMPLES

In this section, two simulations are provided to verify the theoretical results. Consider that there are 5 agents with 1 leader labeled as 0, and 4 followers labeled from 1 to 4. The directed communication topology is depicted in Fig. 1 with all the weights indicated on the communication edges. The corresponding Laplacian matrix is
\[
L = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-2 & 2 & 0 & 0 & 0 \\
0 & -1.5 & 1.5 & 0 & 0 \\
0 & 0 & -1.5 & 1.5 & 0 \\
-2 & 0 & 0 & 0 & 2 \\
\end{pmatrix}
\]
and \( \tilde{L} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1.5 & 1.5 & 0 & 0 \\
0 & 0 & -1.5 & 1.5 & 0 \\
-2 & 0 & 0 & 0 & 2 \\
\end{pmatrix} \).

For each agent, denote \( x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix}, i = 0, 1, \ldots, 4 \). Take \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \), and \( B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).
In both examples, the leader’s input is chosen as
\[ u_0(t) = \tanh(x_{01}) + 3x_{02} + \sin(x_{03} - 1). \]
The initial values of the agents’ states are generated randomly.

A. Example 1: followers with bounded disturbance

In this example, the followers are assumed to be influenced by bounded disturbances and general nonlinear dynamics:
\[
\begin{align*}
    f_i(x_i) &= x_{i1} \cos(x_{i2}) + x_{i2} \cos(x_{i3}), \\
    d_i(t) &= \sin(x_{i1} + x_{i3}) + 2 \cos(1 + x_{i2}).
\end{align*}
\]
The distributed observer and the tracking controller are given in (6) and (17), respectively. The adaptive gain \( \alpha_i \) is designed according to Remark 3, and the adaptive control gain \( \mu_i \) is designed according to Remark 5 with \( h_i = 1 \). It is easy to see that \( r_{01} = 1 \). Hence, randomly set the initial value of the adaptive gain such that \( \alpha_i(0) > 0 \). In the simulation, a one-layer NN with 50 neurons is used for each follower. Sigmoid basis function \( \sigma(z) = \left(1 + e^{-z}\right)^{-1} \) is employed and the NN weights \( W_i \) are initialized as zero. Besides, the parameters in (19) are chosen as \( m_i = 1000, h_i = 0.1, n_i = 20 \). The initial states of each agent and observer are generated randomly.

Furthermore, solving the LMI (3) with \( \beta = 1 \) by using LMI toolbox of Matlab gives
\[
P = \begin{pmatrix}
5.1201 & 5.7017 & 2.1169 \\
5.7017 & 10.4176 & 4.3302 \\
2.1169 & 4.3302 & 3.0545
\end{pmatrix}.
\]
It follows that \( K = B^T P = \begin{pmatrix} 2.1169, 4.3302, 3.0545 \end{pmatrix} \).

Fig. 2 depicts the estimation error between the distributed observer and the leader, which shows that the proposed distributed observer has good performance. Fig. 3 shows the tracking error among each follower and the leader. One can see that with the observer based controller (17), each follower can track the leader’s state eventually.

B. Example 2: followers with disturbance generated by exosystems

In this example, each follower is assumed to be influenced by heterogeneous nonlinearity and external disturbances (24),

Fig. 3. Example 1: Trajectories of the tracking error \( x_i - x_0, i = 1, \ldots, 4 \).

Fig. 2. Example 1: The trajectories of the estimation error \( \hat{x}_i - x_0, i = 1, \ldots, 4 \).
The observer-based control protocol is designed as (25).

Choose $F = \begin{bmatrix} 0 & 0 & -19 \\ 0 & 0 & -24 \end{bmatrix}$, solving LMI $X = (W + FBV)^T P_1 + P_1 (W + FBV) < 0$ gives $P_1 = \begin{bmatrix} 3.3160 & 3.9092 & 0.2766 \\ 3.9092 & 8.0513 & 0.4082 \\ 0.2766 & 0.4082 & 0.1137 \end{bmatrix}$.

The tracking error for each follower is described in Fig. 6. One can observe that each follower can track to a bounded neighbor of the leader eventually.

VII. CONCLUSION

In this paper, the tracking consensus problem of general nonlinear multi-agent systems with external disturbances has been discussed. The leader is assumed to have bounded input. A new class of fully distributed state observer for the leader has been proposed under the assumption that the upper bound of the leader’s input is unknown to all the followers. Furthermore, two distributed observer-based control protocols have been designed for solving the tracking problems where each follower may be influenced by bounded disturbances (the upper bound is assumed unknown) or disturbances generated by exosystems, respectively. Neural networks and adaptive laws have been applied for each follower to compensate their own nonlinearity. One further work is to consider the fully distributed problems with both general nonlinearity and mismatched disturbances under directed communication topologies.

REFERENCES


