NONLINEAR OPTIMAL STOPPING PROBLEMS

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The main contribution of this thesis is the derivation of solutions to nonlinear optimal stopping problems, both statically and dynamically. The results extend the results of [28] and the methods rely on standard optimal stopping theory because the nonlinear optimal stopping problem can be reduced to a family of linear problems by using the method of Lagrange multipliers.

Chapter 2 deals with standard optimal stopping theory. It presents the general setup and outlines the free-boundary approach that is used to solve the linear optimal stopping problem.

Chapter 3 and Chapter 4 are devoted to solving the mean-standard deviation optimal stopping problems. Chapter 3 makes precise how the problem is to be solved and what constitutes a solution to these problems. The square-root of the variance makes the problem more difficult and this case has not been studied in [28]. Chapter 4 solves the two constrained mean-standard deviation optimal stopping problems.

Chapter 5 and Chapter 6 present a study of mean-variance optimal stopping problems for the CEV process. These problems are interesting because the CEV process is one of the early alternatives to the geometric Brownian motion for modelling the asset price. It accounts for the implied volatility smile and leverage effect and makes the mean-variance model more practical. In Chapter 5 we show how to solve the standard optimal stopping problem for the CEV process and how to use the result to solve the nonlinear optimal stopping problems. Chapter 6 deals with the remaining constrained optimal stopping problems.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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I dedicate this work to my beloved parents, Qingxiang Chu and Xiulan Xu, with my deepest appreciation for their selfless and endless love since day one of my life. I feel guilty that I could not often accompany them in person during the past 10 years. They have done their utmost to give me the best of the world; I will make my greatest efforts to return the best to them.

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Chapter 1

Introduction

The objective of optimal stopping problems is to search for random times at which the observed random processes should be stopped, with the aim to maximise the gain and minimise the loss in discrete and continuous time. The simple example is to consider an investor who owns a stock which he wishes to sell so as to maximise the profit and minimise his risk. It is obvious that he will own the stock with the hope that selling it later can yield a bigger reward. In this thesis, we build on the results of [28] which solves mean-variance optimal stopping problems for geometric Brownian motion. The method of proof relies on [30] which considers the standard optimal stopping theory.

The origins of optimal stopping theory date back to Wald's sequential analysis [39] in 1947. Snell [38] formulated a general optimal stopping problem for the discrete-time stochastic process, and he characterized the value function as the smallest supermartingale in 1952. Dynkin [11] in 1963 described the key principle of the optimal stopping theory for continuous time Markov processes which states that the value function of an optimal stopping problem is the smallest superharmonic function dominating the gain function. Mikhalevich [25] in 1958 and McKean [24] in 1965 used that an optimal stopping problem can be converted to free-boundary problems from mathematical analysis. The book by Shiryaev [36] in 1978 provided the definite results of general optimal stopping theory. Most of the further theoretical developments and examples above were presented and thoroughly studied by Peskir and Shiryaev in [30].

The idea of maximising expected return and minimising variance in finance is dating back to Markowitz [23] in 1952. This classical mean-variance portfolio has been widely used in economics and financial applications and has become the foundation of modern finance theory. The mean-variance optimal stopping problem is a nonlinear optimal stopping problem. Therefore, we cannot use the results and methods of standard optimal stopping theory from [30] and [36] directly. Solving nonlinear optimal stopping problems is a rather new field. The paper [10] considered the optimal stopping problem of maximising the profit when the stock price follows a geometric Brownian motion. Pedersen [27] in 2011 first considered the optimal stopping problem where the objective function is involving a nonlinear term coming from the variance. The classic optimal stopping problems maximise the expectation whereas he maximises the variance which represents an upper bound for the risk. The principle of smooth fit cannot be applied directly in this context. Furthermore, in this paper he did not only consider the geometric Brownian motion, but also his diffusions include the square-root process and Jacobi diffusion, which can be used as a model of population growth and a model in genetics. Then the mean-variance optimal stopping problem is solved for geometric Brownian motion in [28]. The dynamic optimality used there which corresponds to solving infinitely many optimal stopping problems has not been considered in the nonlinear optimality before. Gad and Pedersen [13] in 2015 extended the results of [27] which considered the optimal stopping problem of maximising the variance for a geometric Lévy process.

In this thesis, because our method for solving nonlinear optimal stopping problem relies on solving linear constrained optimal stopping problems, Chapter 2 aims at briefly introducing what is a classical optimal stopping problem and how to use a free-boundary approach to solve it. This chapter is based on [30].

Chapter 3 extends the mean-variance problem to the mean-standard deviation optimal stopping problem of the form

$$V(x) = \sup_{\tau} \left[\mathbb{E}_x(X_\tau) - c\sqrt{\mathbb{V}_x(X_\tau)} \right] \tag{1.1}$$

where the process X is a geometric Brownian motion and the supremum is taken over all stopping times τ with respect to the natural filtration of X. This problem is inherently two-dimensional and hence more challenging to solve. Compared to the mean-variance problem, the mean-standard deviation optimal stopping problem can overcome some drawbacks. For example, the investor wants to minimise his risk when selling a stock. If he identifies the risk with the standard deviation, profit and risk have the same dimension. This chapter introduces the static optimization in contrast to dynamic optimization where the optimal stopping boundary depends on the starting point of X. The key to solve the problem is to use the Lagrange approach, reducing the nonlinear optimal stopping problem to a family of linear optimal stopping problems. In the results we show that this problem is much more complicated and the drift coefficient and diffusion coefficient need to satisfy two conditions in order to find the value function V.

Chapter 4 shows how to use the results from Chapter 3 to solve the constrained mean-standard deviation optimal stopping problems

$$V_1(x) = \sup_{\tau:\ \sqrt{\mathbb{V}_x(X_\tau)}\le m} \mathbb{E}_x(X_\tau) \tag{1.2}$$

$$V_2(x) = \inf_{\tau:\ \mathbb{E}_x(X_\tau)\ge n} \sqrt{\mathbb{V}_x(X_\tau)} \tag{1.3}$$

given $m, n > 0$. This is due to the fact that the constrained problems can be transformed into a problem of the form (1.1). Our approach to solving these problems is through another use of Lagrange multipliers.

The main contribution of Chapter 5 is to solve the mean-variance optimal stopping problem $V(x) = \sup_\tau \big[\mathbb{E}_x(X_\tau) - c\mathbb{V}_x(X_\tau)\big]$ for the Constant Elasticity of Variance (CEV) process $X = (X_t)_{t\ge 0}$ solving

$$dX_t = \mu X_t dt + \sigma X_t^{1+\beta} dB_t \tag{1.4}$$

where B is a standard Brownian motion. If $\beta = 0$, then X is a geometric Brownian motion and the CEV process can be considered as a natural extension of the geometric Brownian motion. For $\beta < 0$ this process was originally considered by Cox [7] in 1975 to describe the pricing of European call options. Then the case $\beta > 0$ was studied by Emanuel and MacBeth [12] in 1982. Davydov and Linetsky [9] in 2001 showed that the CEV model captures the implied volatility smile, therefore implying a better fit than the Black-Scholes model where the underlying asset follows a geometric Brownian motion. In this thesis, we only consider the situation $-\frac{1}{2} < \beta < 0$, where 0 is an exit boundary so the process is killed the first time it reaches 0. Likewise, in order to solve the nonlinear optimal stopping problem, we need to solve the linear optimal stopping problem for the CEV process first. When X is a GBM, the free-boundary problem can be solved directly; however, the problem is more complicated when X is a CEV process. The value function involves the lower incomplete Gamma function which is studied in [1]. Because the results are not intuitive we present an example and find that the solutions to the linear optimal stopping problem for the GBM and the CEV processes look similar, which is reasonable.

Finally in Chapter 6, we solve the constrained mean-variance optimal stopping problems for the CEV process. These results rely on the results from Chapter 5.

Chapter 2

Linear Optimal Stopping Problems

Although the results and methods of the standard optimal stopping theory are not directly applicable because the objective function is involving a nonlinear term, we can use the methods of Lagrange multipliers to reduce the nonlinear optimal stopping problem to a family of linear optimal stopping problems. Therefore the standard optimal stopping theory plays an important role in the methodology developed. In this chapter, I first briefly show what is an optimal stopping problem and how to use a free-boundary approach to solve it.

2.1 Markovian approach

Most of the optimal stopping theory presented in this section is taken from Peskir and Shiryaev [30]. Here we only consider the optimal stopping problem when time is continuous and the process is Markovian. Let $X = (X_t)_{t\geq 0}$ be a strong Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ and taking values on a measurable space $(E, \mathcal{B})$ such that $E = \mathbb{R}^n$ for some $n \geq 1$ and $\mathcal{B}$ is the associated Borel algebra. Here $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration satisfying the usual conditions, $\mathbb{P}_x$ denotes the probability measure where the stochastic process starts at $x \in E$ and the sample paths of X are right-continuous and left-continuous over stopping time. Let $G : E \to \mathbb{R}$ be a given measurable function;

it always can be interpreted as the gain function. We assume that G satisfies

$$\mathbb{E}_x\Big(\sup_{t\geq 0}|G(X_t)|\Big) < \infty$$

for all $x \in E$. We want to find an optimal stopping time τ_* of X that aims to maximise the performance so that

$$\mathbb{E}_x\big(G(X_{\tau_*})\big) = \sup_{\tau}\mathbb{E}_x\big(G(X_\tau)\big) \tag{2.1}$$

for $x \in E$ and the supremum is taken over all stopping times τ of X. Furthermore, we want to find the corresponding optimal expected reward, which is called the value function

$$V(x) = \mathbb{E}_x\big(G(X_{\tau_*})\big) \tag{2.2}$$

for $x \in E$. Therefore the optimal stopping problem means that there are two things to be solved. Firstly, from (2.1) we want to determine an optimal stopping time at which the supremum is attained. Secondly, from (2.2) we aim to compute the value function $V(x)$. The optimal stopping time may not exist or be unique.

In real life we can regard X_t as the price of a stock. At each time t we have the option of selling the stock and obtain the reward $G(X_t)$, or keeping it with the anticipation that there will exist a larger return in the future. Thus the state space E splits into the continuation set C

$$C = \{x \in E : V(x) > G(x)\} \tag{2.3}$$

and the stopping set D

$$D = \{x \in E : V(x) = G(x)\}. \tag{2.4}$$

When the stopping time $\tau_* = 0$, we stop X immediately and have $V(x) = G(x)$, therefore it is clear that $V(x) \geq G(x)$ for all $x \in E$. Note that if V is lower semi-continuous and G is upper semi-continuous, then C is open and D is closed. When X enters the set D, the observation should be stopped, and the optimal stopping time is the first entry time of the stopping set given by

$$\tau_D = \inf\{t \geq 0 : X_t \in D\} \tag{2.5}$$

2.2 The free-boundary problem

Chapter 1 of [30] shows that the original problems (2.1) and (2.2) are equivalent to finding the smallest superharmonic function $\hat{V}$ that dominates the gain function G on the state space E. Once $\hat{V}$ is found it follows that $V = \hat{V}$ and $\tau_D = \inf\{t \geq 0 : X_t \in D\}$ is optimal. Therefore we have the representation

$$V(x) = \mathbb{E}_x\big(G(X_{\tau_D})\big) \tag{2.6}$$

for $x \in E$. There are two traditional ways to find $\hat{V}$: Iterative procedure and free-boundary problem. Here, we only consider the optimal stopping problem as a free-boundary problem.

A free-boundary problem is formed by a partial differential equation that is defined in a domain whose boundary is unknown in advance. In order to solve the PDE, at the unknown boundary we must add additional conditions. So it is important that solving such a problem means not only to solve the differential equation but also to find the unknown boundary. The basic idea is that $\hat{V}$ and C (or D) should solve the free-boundary problem

$$\mathbb{L}_x\hat{V} \leq 0, \tag{2.7}$$

$$\hat{V} \geq G \ (\hat{V} > G \text{ on C and } \hat{\text{V}} = \text{G on D}). \tag{2.8}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X. Here both $\hat{V}$ and C are unknown. Condition (2.7) states that $\hat{V}$ is the smallest superharmonic function. It is important to select the optimal boundary ∂C because only special C (or D) will qualify to meet the two important properties "smallest"and "superharmonic".

Assuming that G is smooth in the neighbourhood of ∂C and that ∂C is sufficiently regular so that X after starting at ∂C enters immediately into D, the condition (2.7) splits into

$$\mathbb{L}_X\hat{V} = 0 \text{ in C}, \tag{2.9}$$

$$\left.\frac{\partial \hat{V}}{\partial x}\right|_{\partial C} = \left.\frac{\partial \hat{G}}{\partial x}\right|_{\partial C} \text{(smooth fit)}. \tag{2.10}$$

On the other hand, if X after starting at ∂C does not enter into D immediately, then the condition (2.7) splits into

$$\mathbb{L}_X \hat{V} = 0 \; in \; C, \tag{2.11}$$

$$\hat{V}\big|_{\partial C} = G\big|_{\partial C} \text{(continuous fit)}. \tag{2.12}$$

We consider the optimal stopping problem

$$V(x) = \inf_{\tau} \mathbb{E}_x\big(G(X_\tau)\big) \tag{2.13}$$

for $x \in E$ where the infimum is taken over all stopping times τ of X. This optimal stopping problem is equivalent to finding the largest subharmonic function $\hat{V}$ that solves the free-boundary problem

$$\mathbb{L}_X \hat{V} \geq 0, \tag{2.14}$$

$$\hat{V} \leq G \; (\hat{V} < G \text{ on C and } \hat{\text{V}} = \text{G on D}), \tag{2.15}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X. A more precise meaning of the conditions can be found in Chapter IV of [30].

Chapter 3

Mean-Standard Deviation Stopping Problems

3.1 Problem formulation

After addressing standard linear optimal stopping problems, we now consider nonlinear problems. First, we assume that a stock price X follows a geometric Brownian motion (GMB) with drift coefficient $\mu \in \mathbb{R}$ and diffusion coefficient $\sigma > 0$, i.e. that X is the unique strong solution to the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{3.1}$$

for $t \geq 0$. Here $X_0 = x$ for $x > 0$ and B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well known that this solution is given by

$$X_t^x = x \exp\left(\sigma B_t + (\mu - \frac{\sigma^2}{2})t\right) \tag{3.2}$$

for $t \geq 0$ under $\mathbb{P}$. The infinitesimal generator of X is given by

$$\mathbb{L}_X = \mu x \frac{d}{dx} + \frac{\sigma^2}{2} x^2 \frac{d^2}{dx^2}. \tag{3.3}$$

Pedersen and Peskir [28] consider the optimal stopping problem

$$V(x) = \sup_\tau \left[\mathbb{E}_x(X_\tau) - c\mathbb{V}_x(X_\tau)\right] \tag{3.4}$$

for $x > 0$. Motivated by the fact that the return (measured by the expectation $\mathbb{E}_x(X_\tau)$) and the risk (measured by the variance $\mathbb{V}_x(X_\tau)$) are not of the same order, here we study mean-standard deviation optimal stopping problems

$$V(x) = \sup_\tau \left[\mathbb{E}_x(X_\tau) - c\sqrt{\mathbb{V}_x(X_\tau)}\right] \tag{3.5}$$

where $c > 0$ is a given and fixed constant and the supremum is taken over all stopping time τ of X such that $\mathbb{E}_x(X_\tau^2) < \infty$. As discussed in Chapter 2, the problem is to find an optimal stopping time τ_* at which the supremum is attained, and to compute the value function

$$V(x) = \mathbb{E}_x(X_{\tau_*}) - c\sqrt{\mathbb{V}_x(X_{\tau_*})}. \tag{3.6}$$

It is evident that the problem seeks to maximise the expectation of X_τ and minimise the standard deviation of X_τ. Because the second term

$$\sqrt{\mathbb{V}_x(X_\tau)} = \sqrt{\mathbb{E}_x(X_\tau^2) - (\mathbb{E}_x(X_\tau))^2}$$

defines a nonlinear function of $\mathbb{E}_x(X_\tau)$, we see that V cannot be written in the form

$$V(x) = \sup_\tau \mathbb{E}_x\big(G(X_\tau)\big) \tag{3.7}$$

which moves the problem outside the scope of standard optimal stopping theory. Below we shall show how to use the Lagrange multiplier method to reduce the nonlinear optimal stopping problem to a family of linear optimal stopping problems. Solving the latter problems we find that the optimal stopping boundary depends on the initial point x of X in an essential way.

3.2 Static and dynamic optimality

In this section we recall definitions of the static and dynamic optimality from reference [28]. Recall that solving the linear optimal stopping problem means finding the optimal stopping time $\tau_D = \inf\{t \geq 0 : X_t \in D\}$ and calculating the value function $V(x) = \mathbb{E}_x\big(G(X_{\tau_D})\big)$. The optimal stopping time τ_D does not depend on the initial starting point x because τ_D is the first time when X enters the stopping set D when the value function equals to the gain function.

However, as we shall see, it turns out that the optimal stopping time τ_* in the static optimal stopping problems depends on the starting point $x > 0$. We know that at any later time $t > 0$, the stock price X_t will change to some other values that are different from x with probability one. So the investor needs to decide which optimality he will use to evaluate the performance and there are two different situations. If the investor fixes an initial starting point $X_0 = x$, then he solves only one optimal stopping problem according to static optimality.

Definition 3.2.1 **(Static optimality).** *A stopping time τ_* is statically optimal in (3.5), for $x > 0$ given and fixed, if for every stopping time σ we have*

$$\mathbb{E}_x(X_\sigma) - c\sqrt{\mathbb{V}_x(X_\sigma)} \leq \mathbb{E}_x(X_{\tau_*}) - c\sqrt{\mathbb{V}_x(X_{\tau_*})}. \tag{3.8}$$

On the other hand, if the investor continuously re-evaluates and solves infinitely many optimal stopping problems until he stops, this is referred to as dynamic optimality. Each new position of the process yields a new optimal stopping problem to be solved. At each time the investor has the option to stop (if no other stopping time could give a better reward) or to continue (if such a stopping time exists).

Definition 3.2.2 **(Dynamic optimality).** *A stopping time τ_* is dynamically optimal in (3.5) if there is no other stopping time σ such that*

$$\mathbb{P}_x\big(\mathbb{E}_{X_{\tau_*}}(X_\sigma) - c\sqrt{\mathbb{V}_{X_{\tau_*}}(X_\sigma)} > X_{\tau_*}\big) > 0 \tag{3.9}$$

for some $x > 0$.

For both static and dynamic optimality we look for the minimal optimal stopping time. It is obvious that when we find the solution to the static optimal stopping problem, it is easy to find the solution to the dynamic optimal stopping problem, because the dynamic optimal stopping problem means solving infinitely many static optimal stopping problems. Static optimality remembers the past while dynamic optimality ignores the past and keeps re-evaluating.

3.3 Solution to the problem

Theorem 3.3.1. *Assume X is a geometric Brownian motion solves (3.1) with $X_0 = x$ under $\mathbb{P}_x$ for $x > 0$. Consider the optimal stopping problem (3.5) for given and fixed constant $c > 0$. Then:*

(A) *If $\mu \leq 0$ then it is both statically and dynamically optimal to stop at once. The value function in (3.5) is given by $V(x) = x$ for $x > 0$.*

(B) *If $\mu \geq \sigma^2/2$ then it is both statically and dynamically optimal not to stop at all. The value function in (3.5) in given by $V(x) = \infty$ for $x > 0$.*

(C) *If $\mu \in (0, \sigma^2/2)$ and $\alpha = \mu/(\sigma^2/2)$ satisfy the two conditions*

$$\alpha > \sqrt{\frac{c^2}{1+c^2}} \tag{3.10}$$

and

$$D_\alpha^{\frac{\alpha}{1-\alpha}}\big(1 - c\sqrt{D_\alpha - 1}\big) > 1 \tag{3.11}$$

with

$$D_\alpha = 2\frac{(1+\frac{1}{\alpha})c^2 + 1 + \sqrt{(1-\frac{1}{\alpha^2})c^2 + 1}}{c^2(1+\frac{1}{\alpha})^2},$$

then the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq b(x)\} \tag{3.12}$$

is statically optimal for $x > 0$, where

$$b(x) = D_\alpha^{\frac{1}{1-\alpha}} x. \tag{3.13}$$

The function $x \to b(x)$ is strictly increasing on $(0, \infty)$ and the value function in (3.5) is given by

$$V(x) = b(x)^\alpha x^{1-\alpha}\big(1 - c\sqrt{(\frac{b(x)}{x})^{1-\alpha} - 1}\big) \tag{3.14}$$

for $x \in (0, b(x))$. *If*

$$\alpha \leq \sqrt{\frac{c^2}{1+c^2}}$$

or α *cannot meet the condition (3.11) then it is optimal to stop at once for all* $x > 0$ *and the value function is given by* $V(x) = x$.

(D) *If* $\mu \in (0, \sigma^2/2)$ *and* $\alpha = \mu/(\sigma^2/2)$ *satisfy the two conditions (3.10) and (3.11), then it is dynamically optimal not to stop at all and for the value function we have* $V(x) = \infty$.

In order to prove the theorem we first solve the standard optimal stopping problem, which will then be used in the mean-standard deviation optimal stopping problem (3.5).

Lemma 3.3.1. *Assume that* $\mu \in (0, \sigma^2/2)$ *and* $\alpha = \mu/(\sigma^2/2) \in (0, 1)$. *Given* $\lambda > 0$ *and the gain function* $x \to G_\lambda(x) = x^2 - \lambda x$, *consider the standard optimal stopping problem*

$$V_\lambda(x) = \inf_\tau \mathbb{E}_x\big(G_\lambda(X_\tau)\big) \tag{3.15}$$

for $x > 0$. *If* $x \geq \lambda/2$, *the optimal stopping time is*

$$\tau_*^\lambda = \inf\{t \geq 0 \mid X_t \leq \lambda/2\}, \tag{3.16}$$

and if $x \in (0, \lambda/2)$, *the optimal stopping time is*

$$\tau_*^\lambda = \tau_b = \inf\{t \geq 0 \mid X_t \geq b\}, \tag{3.17}$$

where

$$b = \frac{\lambda\alpha}{1+\alpha}, \tag{3.18}$$

and the value function is given by

$$V_\lambda(x) = \begin{cases} (b^{1+\alpha} - \lambda b^\alpha)x^{1-\alpha} & if \quad x \in (0, b), \\ x^2 - \lambda x & if \quad x \in [b, \frac{\lambda}{2}], \\ -\frac{\lambda^2}{4} & if \quad x > \frac{\lambda}{2}. \end{cases} \tag{3.19}$$

Proof of Lemma 3.3.1. The gain function $G_\lambda(x) = (x - \lambda/2)^2 - \lambda^2/4$ is convex on

$(0, \infty)$. The value $G_\lambda(x)$ decreases on $(0, \lambda/2)$ and increases to ∞ as $x \to \infty$. It attains its unique minimum at $\lambda/2$ and the value is $-\lambda^2/4$. Because $\mu \in (0, \sigma^2/2)$ we know that $X_t \to 0$ $\mathbb{P}$-a.s. as $t \to \infty$ and we have two situations depending on the initial value of x. If $x \geq \lambda/2$, the optimal stopping time in (3.15) is

$$\tau_*^\lambda = \inf\{t \geq 0 \mid X_t \leq \lambda/2\} \tag{3.20}$$

and $V_\lambda(x) = -\lambda^2/4$. On the other hand, we conjecture that it is optimal to stop at once in (3.15) for $x \in [b, \lambda/2]$, where $b \in (0, \lambda/2)$ is the optimal stopping point for X starting at $x \in (0, b)$ to be determined. Therefore the optimal stopping time is

$$\tau_*^\lambda = \inf\{t \geq 0 \mid X_t \geq b\} \tag{3.21}$$

for $x \in (0, \lambda/2)$. Recalling that the value function is the largest subharmonic function dominated by the gain function, the optimal stopping problem can be reduced to the following free-boundary problem which can be solved explicitly

$$\mathbb{L}_X V_\lambda = 0 \text{ in } (0, \text{b}) \tag{3.22}$$

$$V_\lambda(0+) = 0 \tag{3.23}$$

$$V_\lambda(b) = G_\lambda(b) = b^2 - \lambda b \tag{3.24}$$

$$V_\lambda'(b) = G_\lambda'(b) = 2b - \lambda \text{ (smooth fit)} \tag{3.25}$$

where

$$\mathbb{L}_X = \mu x d/dx + (\sigma^2/2)x^2 d^2/dx^2$$

is the infinitesimal generator of the geometric Brownian motion X. From (3.22) we have

$$\mu x \frac{dV_\lambda}{dx} + \frac{\sigma^2}{2} x^2 \frac{d^2 V_\lambda}{dx^2} = 0. \tag{3.26}$$

So $V_\lambda(x) = c_1 + c_2 x^{1-\alpha}$ where $\alpha = \mu/(\sigma^2/2) \in (0, 1)$, and c_1 and c_2 are two constants to be determined. From (3.23) we calculate $c_1 = 0$ and $V_\lambda(x) = c_2 x^{1-\alpha}$. From (3.24) we see that $c_2 = b^{1+\alpha} - \lambda b^\alpha$. So the value function can be written as

$$V_\lambda(x) = (b^{1+\alpha} - \lambda b^\alpha) x^{1-\alpha} \tag{3.27}$$

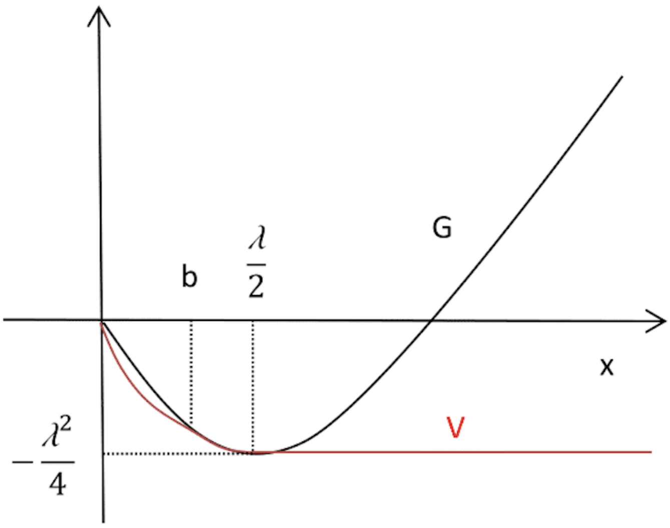

Figure 3.1: A computer drawing of the gain function G_λ and value function V_λ in the standard optimal stopping problem.

for $x \in (0, b]$. Then $V'_\lambda(b) = (b - \lambda)(1 - \alpha)$ and from (3.25) we find that

$$b = \frac{\lambda\alpha}{1 + \alpha} \tag{3.28}$$

for given $\lambda > 0$. Combining these two situations we find that the stopping times (3.20) and (3.21) are optimal in (3.15) and we get the value function given by

$$V_\lambda(x) = \begin{cases} (b^{1+\alpha} - \lambda b^\alpha)x^{1-\alpha} & \text{if} \quad x \in (0, b), \\ x^2 - \lambda x & \text{if} \quad x \in [b, \frac{\lambda}{2}], \\ -\frac{\lambda^2}{4} & \text{if} \quad x > \frac{\lambda}{2}. \end{cases} \tag{3.29}$$

Here the continuation set $C = (0, b) \cup (\frac{\lambda}{2}, \infty)$ where $V_\lambda(x) < G_\lambda(x)$ and the stopping set $D = [b, \lambda/2]$ where $V_\lambda(x) = G_\lambda(x) = x^2 - \lambda x$ as claimed. We see that $\mathbb{L}_X V_\lambda(x) = 0$ for $x \in (\frac{\lambda}{2}, \infty)$ so $\mathbb{L}_X V_\lambda(x) = 0$ holds for all $x \in C$. Clearly the optimal stopping time does not depend on the initial point $x > 0$, which is given and fixed.

The Figure 3.1 shows the graphs of V_λ and G_λ. Notice that $V_\lambda \leq G_\lambda$ is the first crucial requirement for the value function V_λ. Using the fact that V_λ defined by (3.29) is C^2 everywhere but at b and $\lambda/2$ where it is C^1, by using the Itô formula we have:

$$V_\lambda(X_t) = V_\lambda(x) + \int_0^t V_\lambda'(X_s)dX_s + \frac{1}{2}\int_0^t V_\lambda''(X_s)d\langle X, X\rangle_s$$
$$= V_\lambda(x) + \int_0^t \mathbb{L}_X V_\lambda(X_s)ds + M_t \tag{3.30}$$

for $x > 0$ where

$$M_t = \sigma \int_0^t X_s V_\lambda'(X_s)dB_s$$

is a continuous local martingale for $t \geq 0$. Moreover, we have

$$\mathbb{L}_X G_\lambda(x) = \mu x G_\lambda'(x) + \frac{\sigma^2}{2}x^2 G_\lambda''(x)$$
$$= \mu x(2x - \lambda) + \sigma^2 x^2. \tag{3.31}$$

Then $\mathbb{L}_X G_\lambda(x) \geq 0$ if and only if

$$x \geq \lambda\mu/(\sigma^2 + 2\mu) = \lambda\alpha/\big(2(1+\alpha)\big).$$

Because $b = \lambda\alpha/(1+\alpha) \geq \lambda\alpha/\big(2(1+\alpha)\big)$, we have $\mathbb{L}_X G_\lambda(x) = \mathbb{L}_X V_\lambda(x) \geq 0$ when x in the stopping set $[b, \lambda/2]$. Together with the fact that $\mathbb{L}_x V_\lambda(x) = 0$ for x in the continuation set C, we have

$$\mathbb{L}_X V_\lambda(x) \geq 0 \tag{3.32}$$

for $x \in (0, \infty) \setminus \{b, \lambda/2\}$. Because the time that X spends at b and $\lambda/2$ has Lebesgue measure zero, from (3.30) and (3.32) we get

$$V_\lambda(x) + M_t \leq V_\lambda(X_t) \leq G_\lambda(X_t)$$

for $t \geq 0$ and $x > 0$. Let τ_n be a localizing sequence of bounded stopping times for M with $n \geq 1$. Then for any stopping time τ of X such that $\mathbb{E}_x[X_\tau^2] < \infty$ we have

$$V_\lambda(x) + M_{\tau\wedge\tau_n} \leq G_\lambda(X_{\tau\wedge\tau_n}) \tag{3.33}$$

for $x > 0$. Taking expectations on the both sides of (3.33) and using the optional sampling theorem we obtain

$$V_\lambda(x) \le \mathbb{E}_x\big(G_\lambda(X_{\tau\wedge\tau_n})\big) \tag{3.34}$$

for $x > 0$ and $n \ge 1$ because $\mathbb{E}_x(M_{\tau\wedge\tau_n}) = 0$. Letting $n \to \infty$ and using Fatou's lemma we see that

$$V_\lambda(x) \le \limsup_{n\to\infty} \mathbb{E}_x\big(G_\lambda(X_{\tau\wedge\tau_n})\big) \le \mathbb{E}_x\big(\limsup_{n\to\infty} G_\lambda(X_{\tau\wedge\tau_n})\big) = \mathbb{E}_x\big(G_\lambda(X_\tau)\big).$$

Since this holds for any stopping time τ we have

$$V_\lambda(x) \le \inf_\tau \mathbb{E}_x\big(G_\lambda(X_\tau)\big) \tag{3.35}$$

for $x > 0$. Taking $\tau_*^\lambda := \inf\{t \ge 0 \mid X_t \le \lambda/2\}$ for $x \ge \lambda/2$ and $\tau_*^\lambda := \inf\{t \ge 0 \mid X_t \ge b\}$ for $x \in (0, \lambda/2)$ we can get $V_\lambda(x) = \mathbb{E}_x\big(G_\lambda(X_{\tau_*^\lambda})\big)$. □

Proof of Theorem 3.3.1. (A) If $\mu \le 0$ then X is a positive supermartingale, by Fatou's lemma and the optional sampling theory we see that

$$\mathbb{E}_x(X_\tau) = \mathbb{E}_x(\lim_{N\to\infty} X_{\tau\wedge N}) \le \liminf_{N\to\infty} \mathbb{E}_x(X_{\tau\wedge N}) \le \liminf_{N\to\infty} \mathbb{E}_x(X_0) = x$$

for all stopping time τ and $x > 0$. This shows that no stopping time will give a better mean value so it is optimal to stop at once with $V(x) = x$.

(B) If $\mu \ge \sigma^2/2$ then $\limsup_{t\to\infty} X_t = \infty$ $\mathbb{P}_x$-a.s. for all $x > 0$ and therefore the stopping time $\tau_N = \inf\{t \ge 0 | X_t \ge N\}$ is a finite valued $\mathbb{P}_x$-a.s. for $N \ge x > 0$. So $\mathbb{E}_x(X_{\tau_N}) = N$ and $\mathbb{V}_x(X_{\tau_N}) = \mathbb{E}\big(X_{\tau_N} - \mathbb{E}(X_{\tau_N})\big)^2 = 0$ from where we see that $V(x) \ge N$ for all $N \ge x > 0$. Letting $N \to \infty$ we get $V(x) = \infty$ so it is optimal not to stop at all.

(C) If $\mu \in (0, \sigma^2/2)$ then $X_t \to 0$ $\mathbb{P}_x$-a.s. but $\mathbb{E}_x(X_t) = xe^{\mu t} \to \infty$ as $t \to \infty$ for $x > 0$. Note that the objective function reads

$$\mathbb{E}_x(X_\tau) - c\sqrt{\mathbb{V}_x(X_\tau)} = \mathbb{E}_x(X_t) - c\sqrt{\mathbb{E}_x(X_\tau^2) - (\mathbb{E}_x(X_\tau))^2}.$$

The second term is the main difficulty and we overcome it by conditioning on the size of $\mathbb{E}_x(X_\tau)$

$$\begin{aligned}
V(x) &= \sup_{M\geq 0} \sup_{\tau:\mathbb{E}_x(X_\tau)=M} \Big[\mathbb{E}_x(X_\tau) - c\sqrt{\mathbb{V}_x(X_\tau)}\Big] \\
&= \sup_{M\geq 0} \sup_{\tau:\mathbb{E}_x(X_\tau)=M} \Big[\mathbb{E}_x(X_\tau) - c\sqrt{\mathbb{E}_x(X_\tau^2) - (\mathbb{E}_x(X_\tau))^2}\Big] \\
&= \sup_{M\geq 0} \Big[M - c\sqrt{\inf_{\tau:\mathbb{E}_x(X_\tau)=M} \mathbb{E}_x(X_\tau^2) - M^2}\Big] \qquad (3.36)
\end{aligned}$$

for $x > 0$. So if we want to solve (3.36) and thus (3.5) we need to solve the constrained problems

$$V_M(x) = \inf_{\tau:\mathbb{E}_x(X_\tau)=M} \mathbb{E}_x(X_\tau^2) \qquad (3.37)$$

for $x > 0$ and $M > 0$ given and fixed where τ is a stopping time of X. We use the Lagrange approach and define the Lagrangian by setting

$$L_x(\tau, \lambda) = \mathbb{E}_x(X_\tau^2) - \lambda\big[\mathbb{E}_x(X_\tau) - M\big] \qquad (3.38)$$

for $\lambda > 0$. This leads to the standard optimal stopping problem

$$\inf_\tau L_x(\tau, \lambda), \qquad (3.39)$$

and we assume that the optimal stopping time τ_*^λ in (3.39) exists. Suppose moreover that there exists $\lambda = \lambda(M, x) > 0$ such that

$$\mathbb{E}_x(X_{\tau_*^\lambda}) = M. \qquad (3.40)$$

From (3.38)-(3.40) it follows that

$$\mathbb{E}_x(X_{\tau_*^\lambda}^2) = L_x(\tau_*^\lambda, \lambda) \leq L_x(\tau, \lambda) = \mathbb{E}_x(X_\tau^2) \qquad (3.41)$$

for all stopping time τ such that $\mathbb{E}_x(X_\tau) = M$. So the optimal stopping time τ_*^λ in (3.39) satisfies the conditions (3.40) with $\lambda = \lambda(M, x)$ is optimal in (3.37). Rewriting (3.38) as $L_x(\tau, \lambda) = \mathbb{E}_x(X_\tau^2 - \lambda X_\tau) + \lambda M$ so we just need to consider the optimal stopping problem

$$V_\lambda(x) = \inf_\tau \mathbb{E}_x(X_\tau^2 - \lambda X_\tau) \qquad (3.42)$$

for $x > 0$ which we have already solved in the Lemma 3.3.1. Now we want to find the specific $\lambda = \lambda(M, x) > 0$ that meets the condition (3.40). Taking $x \in (0, b)$ and recalling Doob's identity

$$\mathbb{P}\Big(\sup_{t\geq 0}(B_t - ct) \geq d\Big) = e^{-2cd}$$

for $c, d > 0$ we find that

$$\begin{aligned}\mathbb{P}_x(\tau_b < \infty) &= \mathbb{P}_x\Big(\sup_{t\geq 0} X_t \geq b\Big)\\ &= \mathbb{P}_x\Big(\sup_{t\geq 0}\big[x\exp\big(\sigma B_t + (\mu - \frac{\sigma^2}{2})t\big)\big] \geq b\Big)\\ &= \mathbb{P}_x\Big(\exp\big(\sup_{t\geq 0}\big[\sigma B_t + (\mu - \frac{\sigma^2}{2})t\big]\big) \geq \frac{b}{x}\Big)\\ &= \mathbb{P}_x\Big(\sup_{t\geq 0}\big[\sigma B_t + (\mu - \frac{\sigma^2}{2})t\big] \geq \ln(\frac{b}{x})\Big)\\ &= \Big(\frac{x}{b}\Big)^{1-\alpha}. \end{aligned} \tag{3.43}$$

Because $X_t \to 0$ $\mathbb{P}_x$-a.s. as $t \to \infty$, we obtain

$$\mathbb{E}_x(X_{\tau_b}) = b\mathbb{P}_x(\tau_b < \infty) + 0\mathbb{P}_x(\tau_b = \infty) = b\Big(\frac{x}{b}\Big)^{1-\alpha} = b^{\alpha}x^{1-\alpha} \tag{3.44}$$

for $x \in (0, b]$. From (3.40) and (3.44) we have

$$M = \mathbb{E}_x(X_{\tau_b}) = b^{\alpha}x^{1-\alpha}. \tag{3.45}$$

This gives

$$b = M^{1/\alpha}x^{1-1/\alpha} \tag{3.46}$$

for $x \in (0, b]$. By substituting (3.46) into (3.18) we find

$$\lambda = \frac{b(1+\alpha)}{\alpha} = \frac{1+\alpha}{\alpha}M^{1/\alpha}x^{1-1/\alpha} \tag{3.47}$$

for $x \in (0, b]$. Also note that from (3.37), (3.43) and (3.46) we have

$$\begin{aligned}V_M(x) = \mathbb{E}_x(X_{\tau_b}^2) &= b^2\mathbb{P}_x(\tau_b < \infty) + 0\mathbb{P}_x(\tau_b = \infty) = b^{1+\alpha}x^{1-\alpha}\\ &= \big(M^{1/\alpha}x^{1-1/\alpha}\big)^{1+\alpha}x^{1-\alpha} = M^{1+1/\alpha}x^{1-1/\alpha}\end{aligned} \tag{3.48}$$

for $x \in (0, b]$. By rewriting (3.45) we have $b/x = (M/x)^{1/\alpha}$ so when $x \in (0, b]$ we must have $M \geq x$ in this case of λ. Recalling if $x > \lambda/2$ then the optimal stopping time in (3.15) is

$$\tau_*^\lambda := \inf\{t \geq 0 \mid X_t \leq \lambda/2\}$$

so we set $M = \lambda/2 < x$. In this situation because $\lim_{t\to\infty} X_t = 0$, we have $\mathbb{P}_x(\tau_*^\lambda < \infty) = 1$ and (3.37) is given by

$$V_M(x) = \mathbb{E}_x\big(X^2_{\tau_*^\lambda}\big) = M^2 \tag{3.49}$$

if $x > \lambda/2$. By (3.48) and (3.49) we have

$$V_M(x) = \begin{cases} M^{1+1/\alpha}x^{1-1/\alpha} & if \quad x \leq M, \\ M^2 & if \quad x > M. \end{cases} \tag{3.50}$$

Inserting it into (3.36) the value function

$$\begin{aligned} V(x) &= \sup_{M\geq x}\Big[M - c\sqrt{M^{1+1/\alpha}x^{1-1/\alpha} - M^2}\Big] \vee \sup_{M<x} M \\ &= \sup_{M\geq x}\Big[M - c\sqrt{M^{1+1/\alpha}x^{1-1/\alpha} - M^2}\Big] \end{aligned} \tag{3.51}$$

for all $x > 0$ where the second equality holds since $\sup_{M<x} M = x$ and we can take $M = x$ in the first supremum. Setting

$$F(M) := M - c\sqrt{M^{1+1/\alpha}x^{1-1/\alpha} - M^2} \tag{3.52}$$

for $M \geq x$ with $x > 0$ given and fixed. Recalling that $\alpha \in (0, 1)$, we see that the power $1 + 1/\alpha > 2$. Then $F(M)$ converges to $-\infty$ as $M \to \infty$ because

$$F(M) = M - c\sqrt{M^{1+1/\alpha}(x^{1-1/\alpha} - M^{1-1/\alpha})} = M - cM^{\frac{1}{2}(1+1/\alpha)}x^{\frac{1}{2}(1-1/\alpha)}$$

as $M \to \infty$. Therefore the function $M \mapsto F(M)$ attains its maximum on $[x, \infty)$ at some $M \geq x$. To find this M we can solve $F'(M) = 0$ which reads

$$1 - \frac{1}{2}c\frac{(1 + \frac{1}{\alpha})M^{1/\alpha}x^{1-1/\alpha} - 2M}{\sqrt{M^{1+1/\alpha}x^{1-1/\alpha} - M^2}} = 0.$$

Rearranging it and we get

$$2\sqrt{M^{1+1/\alpha}x^{1-1/\alpha}-M^2}=c\big[(1+\frac{1}{\alpha})M^{1/\alpha}x^{1-1/\alpha}-2M\big]$$
$$c^2(1+\frac{1}{\alpha})^2M^{2/\alpha}x^{2-2/\alpha}-4\big[(1+\frac{1}{\alpha})c^2+1\big]M^{1+1/\alpha}x^{1-1/\alpha}+4(c^2+1)M^2=0. \tag{3.53}$$

From (3.53) we see that $F'(x)<0$ so $F(M)$ decreases first when it starts at x. Because $M>0$ and setting $A=1/\alpha-1(0<A<\infty)$ in (3.53) we have

$$c^2(A+2)^2x^{-2A}M^{2A}-4\big[(A+2)c^2+1\big]x^{-A}M^A+4(c^2+1)=0.$$

Here we get

$$M^A=2\frac{(A+2)c^2+1\pm\sqrt{(-A^2-2A)c^2+1}}{c^2(A+2)^2}x^A. \tag{3.54}$$

Solving (3.54) we only consider the two positive solutions

$$\begin{aligned} M_1&=\Big(2\frac{(1+\frac{1}{\alpha})c^2+1-\sqrt{(1-\frac{1}{\alpha^2})c^2+1}}{c^2(1+\frac{1}{\alpha})^2}\Big)^{\frac{\alpha}{1-\alpha}}x,\\ M_2&=\Big(2\frac{(1+\frac{1}{\alpha})c^2+1+\sqrt{(1-\frac{1}{\alpha^2})c^2+1}}{c^2(1+\frac{1}{\alpha})^2}\Big)^{\frac{\alpha}{1-\alpha}}x. \end{aligned} \tag{3.55}$$

From (3.55) we see that the discriminant is

$$\Delta=(1-\frac{1}{\alpha^2})c^2+1$$

and according to its value we have three different situations.
(i) We have $\Delta>0$ if and only if α and c satisfy

$$\alpha>\sqrt{\frac{c^2}{1+c^2}}, \tag{3.56}$$

the function has two critical points. Moreover, we see that $\lim_{M\to x}F'(M)<0$, so after starting at x and exhibiting a decrease initially, $F(M)$ goes up and after passing the maximum point M_2 it is strictly decreasing and converges to $-\infty$ as $M \to \infty$. Therefore if we want to find $V(M)$ we need to compare the initial value $F(M) = F(x) = x$ with the value $F(M_2)$ of the maximum point M_2, where

$$F(M_2) = M_2 - c\sqrt{M_2^{1+\frac{1}{\alpha}} x^{1-\frac{1}{\alpha}} - M_2^2}. \qquad (3.57)$$

Inserting $M_2 = D_\alpha^{\frac{\alpha}{1-\alpha}} x$ into $F(M_2)$, then $F(M_2) > x$ if and only if

$$D_\alpha^{\frac{\alpha}{1-\alpha}} \left(1 - c\sqrt{D_\alpha - 1}\right) > 1 \qquad (3.58)$$

where $D_\alpha = 2\frac{(1+\frac{1}{\alpha})c^2+1+\sqrt{(1-\frac{1}{\alpha^2})c^2+1}}{c^2(1+\frac{1}{\alpha})^2}$ and (3.58) is an inequality of α and c. Figure 3.2 presents an example of $\alpha > \sqrt{\frac{c^2}{1+c^2}}$. This is a computer drawing of $M \mapsto F(M)$ for $c = 1$, $\alpha = 4/5$ and $x = 1/2$.

Therefore if $\alpha > \sqrt{\frac{c^2}{1+c^2}}$ and α satisfies the condition (3.58), then F has

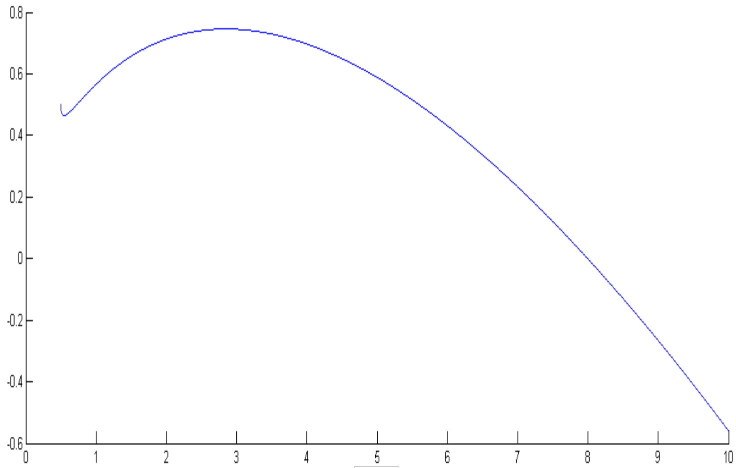

Figure 3.2: A computer drawing of $M \mapsto F(M)$ for $c = 1$, $\alpha = 4/5$ and $x = 1/2$.

a maximum point at $M_2 = D_\alpha^{\frac{\alpha}{1-\alpha}} x$ where $D_\alpha > 1$. The value function is given by

$$V(x) = M_2 - c\sqrt{M_2^{1+\frac{1}{\alpha}} x^{1-\frac{1}{\alpha}} - M_2^2} = D_\alpha^{\frac{\alpha}{1-\alpha}} \left(1 - c\sqrt{D_\alpha - 1}\right) x. \qquad (3.59)$$

Actually as α gets closer to 1, the value $F(M_2)$ of the maximum point M_2 becomes larger than x. So if α satisfies the condition (3.58), it is optimal to stop at $b(x)$ and we get the maximum value (3.59). Recalling (3.46) we see that

$$b(x) = M_2^{1/\alpha} x^{1-1/\alpha} = D_\alpha^{\frac{1}{1-\alpha}} x \tag{3.60}$$

and from (3.59) we get

$$V(x) = b(x)^\alpha x^{1-\alpha}\big(1 - c\sqrt{(\frac{b(x)}{x})^{1-\alpha} - 1}\big). \tag{3.61}$$

These facts show that if α and c satisfy the condition (3.56) and (3.58) then the stopping time $\tau_* = \inf\{t \geq 0 \mid X_t \geq b(x)\}$ is optimal in (3.5) for $x < b(x)$ where $b(x)$ equals (3.60) as claimed. If α and c do not meet these two conditions, then it is optimal to stop at once and the value function is given by $V(x) = x$.

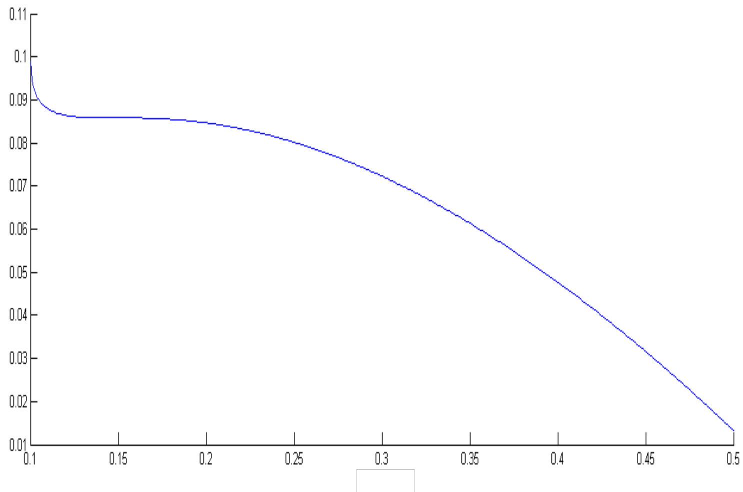

Figure 3.3: A computer drawing of $M \mapsto F(M)$ for $c = 1$, $\alpha = \sqrt{2}/2$ and $x = 1/10$.

(ii) We have $\Delta = 0$ if and only if α and c satisfy the condition

$$\alpha = \sqrt{\frac{c^2}{1+c^2}} \tag{3.62}$$

and we have one critical point

$$M = \Big(2\frac{(1+\frac{1}{\alpha})c^2 + 1}{c^2(1+\frac{1}{\alpha})^2}\Big)^{\frac{\alpha}{1-\alpha}} x. \tag{3.63}$$

But when we approach it, $F'(M) < 0$ and then as we cross it, $F'(M)$ is still negative. So the value of $F(M)$ is decreasing and it is optimal to stop at $M = x$. Figure 3.3 is a computer drawing of $M \mapsto F(M)$ for $c = 1$, $\alpha = \sqrt{2}/2$ and $x = 1/10$.
(iii) If $\Delta < 0$ then $F(M)$ is strictly decreasing on (x, ∞) and we also need to stop at once. Figure 3.4 is a computer drawing of $M \mapsto F(M)$ for $c = 1$, $\alpha = 1/5$ and $x = 1/2$.

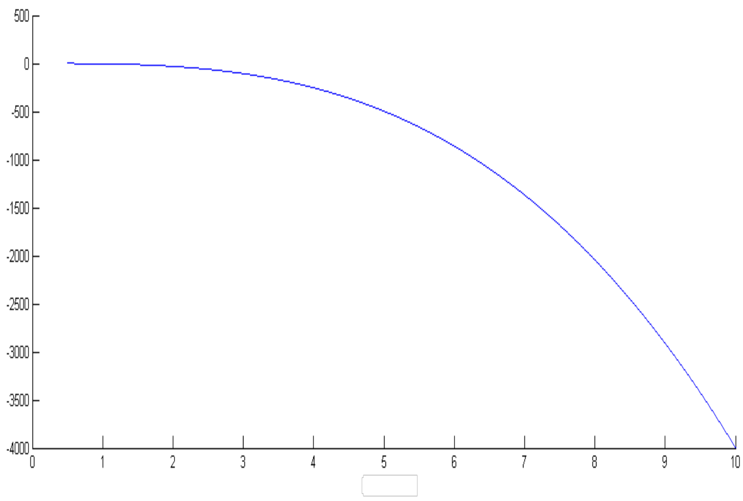

Figure 3.4: A computer drawing of $M \mapsto F(M)$ for $c = 1$, $\alpha = 1/5$ and $x = 1/2$.

(D) From (3.60) we see that

$$b'(x) = \Big(2\frac{(1+\frac{1}{\alpha})c^2 + 1 + \sqrt{(1-\frac{1}{\alpha^2})c^2 + 1}}{c^2(1+\frac{1}{\alpha})^2}\Big)^{\frac{1}{1-\alpha}} \tag{3.64}$$

where α and c satisfy the conditions (3.56) and (3.58) so $b'(x) > 1$ for $x > 0$. Hence we see that $x \mapsto b(x)$ is strictly increasing on $(0, \infty)$ so that it is dynamically optimal not to stop at all, because we will always have a larger $b(x)$ for all $x > 0$. This completes the proof.

□

Chapter 4

Constrained Mean-Standard Deviation Stopping Problems

4.1 Problem formulation

Assuming that X follows a geometric Brownian motion, there are also two constrained problems we wish to investigate. Given $m, n > 0$, the constrained optimal stopping problems are

$$V_1(x) = \sup_{\tau:\ \sqrt{\mathbb{V}_x(X_\tau)} \leq m} \mathbb{E}_x(X_\tau) \tag{4.1}$$

$$V_2(x) = \inf_{\tau:\ \mathbb{E}_x(X_\tau) \geq n} \sqrt{\mathbb{V}_x(X_\tau)} \tag{4.2}$$

for $x > 0$. The equation (4.1) corresponds to the situation that we do not only aim to maximise the expectation of X_τ over all stopping time τ of X, but also improve that the standard deviation of X_τ is bounded above by a positive constant m. Indeed, the investor wants to maximise his return, and at the same time he wishes to keep the risk below an acceptable level. Similarly, the equation (4.2) means that the investor aims to minimise the risk such that his return is bounded below by a positive constant n. By solving the problem (3.5), we are able to solve the constrained problems (4.1) and (4.2) through another use of the Lagrangian method. We will show that the constrained problem can be transformed into a problem of the form (3.5). Here we also consider static optimality and dynamic optimality respectively from [28].

Definition 4.1.1 (Static optimality). *A stopping time τ_* is statically optimal in (4.1) for $x > 0$ given and fixed, if $\sqrt{\mathbb{V}_x(X_{\tau_*})} \le m$ and for every stopping time σ satisfying $\sqrt{\mathbb{V}_x(X_\sigma)} \le m$ we have*

$$\mathbb{E}_x(X_\sigma) \le \mathbb{E}_x(X_{\tau_*}). \tag{4.3}$$

A stopping time τ_* is statically optimal in (4.2) for $x > 0$ given and fixed, if $\mathbb{E}_x(X_{\tau_*}) \ge n$ and for every stopping time σ satisfying $\mathbb{E}_x(X_\sigma) \ge n$ we have

$$\sqrt{\mathbb{V}_x(X_\sigma)} \ge \sqrt{\mathbb{V}_x(X_{\tau_*})}. \tag{4.4}$$

Definition 4.1.2 (Dynamic optimality). *A stopping time τ_* is dynamically optimal in (4.1) if there is no other stopping time σ such that*

$$\mathbb{P}_x\Big(\sqrt{\mathbb{V}_{X_{\tau_*}}(X_\sigma)} \le m\Big) = 1 \tag{4.5}$$

and

$$\mathbb{P}_x\big(\mathbb{E}_{X_{\tau_*}}(X_\sigma) > X_{\tau_*}\big) > 0 \tag{4.6}$$

for some $x > 0$. The dynamic optimality in the problem (4.2) is trivial and we will derive it in Theorem 4.2.2.

4.2 Solution to the problem

Theorem 4.2.1. *Consider the constrained standard deviation problem (4.1) where the process X solves (3.1) with $X_0 = x$ under $\mathbb{P}_x$ for $x > 0$.*

(A) *If $\mu \le 0$ then it is both statically and dynamically optimal to stop at once. The value function in (4.1) is given by $V_1(x) = x$ for $x > 0$.*

(B) *If $\mu \ge \sigma^2/2$ then it is both statically and dynamically optimal not to stop at all. The value function in (4.1) in given by $V_1(x) = \infty$ for $x > 0$.*

(C) *If $\mu \in (0, \sigma^2/2)$ then the stopping time*

$$\tau_* = \inf\{t \ge 0 \mid X_t \ge b_m(x)\} \tag{4.7}$$

is statically optimal for $x > 0$ where $b_m(x)$ is the unique solution to

$$b_m(x)^{1+\alpha}x^{1-\alpha} - b_m(x)^{2\alpha}x^{2-2\alpha} = m^2 \tag{4.8}$$

in $(0, \infty)$ with $\alpha = \mu/(\sigma^2/2)$ when α satisfies the conditions

$$\alpha > \sqrt{\frac{c^2}{1+c^2}} \tag{4.9}$$

and

$$D_\alpha{}^{\frac{\alpha}{1-\alpha}}\left(1 - c\sqrt{D_\alpha - 1}\right) > 1 \tag{4.10}$$

with

$$c = \frac{\sqrt{(\frac{b_m(x)}{x})^{1-\alpha} - 1}}{\frac{1}{2}(1+\frac{1}{\alpha})(\frac{b_m(x)}{x})^{1-\alpha} - 1}$$

and

$$D_\alpha = (\frac{b_m(x)}{x})^{1-\alpha}.$$

The value function in (4.1) is given by

$$V_1(x) = x^{1-\alpha}b_m^\alpha(x) \tag{4.11}$$

for $x \in (0, b_m(x)]$ and $V_1(x) = x$ for $x \geq b_m(x)$. If α cannot meet the condition (4.9) and (4.10) then it is optimal to stop at once and $V_1(x) = x$ for $x > 0$.

(D) *If $\mu \in (0, \sigma^2/2)$ and $\alpha = \mu/(\sigma^2/2)$ satisfy (4.9) and (4.10), then it is dynamically optimal not to stop at all.*

Proof. (A) If $\mu \leq 0$ they by part (A) in the proof of the Theorem 3.3.1 we know that it is optimal to stop at once in the unconstrained problem. Therefore for the constrained problem the same result holds.

(B) If $\mu \geq \sigma^2/2$ then by part (B) in the proof of Theorem 3.3.1 we know that the stopping time $\tau_N = \inf\{t \geq 0 | X_t \geq N\}$ yields an infinite value as $N \to \infty$ implying also that it is optimal not to stop at all in the unconstrained problem. Since this stopping time yields zero standard deviation it follows therefore that the same conclusions holds for the constrained problem.

(C) If $\mu \in (0, \sigma^2/2)$ then in the proof of Theorem 3.3.1 we know that in the constrained problem (4.1) we can define the Lagrangian by setting

$$L_x(\tau, c) = \mathbb{E}_x(X_\tau) - c\big[\sqrt{\mathbb{V}_x(X_\tau)} - m\big] \tag{4.12}$$

for $x > 0$ and $c > 0$. Here we first assume that $\alpha = \mu/(\sigma^2/2)$ satisfies the conditions (3.10) and (3.11). By the result of Theorem 3.3.1 we know that the stopping time $\tau_*^c = \inf\{t \geq 0 \mid X_t \geq b(x)\}$ is optimal in the unconstrained problem

$$L_x(\tau_*^c, c) := \sup_\tau L_x(\tau, c) \tag{4.13}$$

for $x > 0$ and $c > 0$ where

$$b(x) = D_\alpha{}^{\frac{1}{1-\alpha}} x$$

with

$$D_\alpha = 2\frac{(1+\frac{1}{\alpha})c^2 + 1 + \sqrt{(1-\frac{1}{\alpha^2})c^2 + 1}}{c^2(1+\frac{1}{\alpha})^2}.$$

Suppose moreover that there exist $c = c(m, x) > 0$ such that

$$\sqrt{\mathbb{V}_x(X_{\tau_*^c})} = m \tag{4.14}$$

for $x > 0$. It then follows that

$$\mathbb{E}_x(X_{\tau_*^c}) = L_x(\tau_*^c, c) \geq \mathbb{E}_x(X_\tau) - c[\sqrt{\mathbb{V}_x(X_\tau)} - m] \geq \mathbb{E}_x(X_\tau) \tag{4.15}$$

for all stopping times τ such that $\sqrt{\mathbb{V}_x(X_\tau)} \leq m$ with $x > 0$. This shows that the stopping time τ_*^c satisfying (4.14) with $c = c(m, x)$ is statically optimal in (4.1) for $x > 0$. Since

$$\sqrt{\mathbb{V}_x(X_{\tau_*^c})} = \sqrt{\mathbb{V}_x(X_{\tau_b})} = \sqrt{\mathbb{E}_x(X_{\tau_b}^2) - (\mathbb{E}_x(X_{\tau_b})^2)}$$

recall from (3.43) and (3.44) we have

$$\mathbb{E}_x(X_{\tau_b}^2) = b^2 \mathbb{P}_x(\tau_b < \infty) + 0\mathbb{P}_x(\tau_b = \infty) = b^2\Big(\frac{x}{b}\Big)^{1-\alpha} = b^{\alpha+1}x^{1-\alpha}. \tag{4.16}$$

Then

$$\sqrt{\mathbb{V}_x(X_{\tau_b})} = \sqrt{b^{1+\alpha}x^{1-\alpha} - b^{2\alpha}x^{2-2\alpha}} \tag{4.17}$$

for $x > 0$ given and fixed. To meet the condition (4.14) we need to identify (4.17) with m, which yields

$$b^{1+\alpha}x^{1-\alpha} - b^{2\alpha}x^{2-2\alpha} = m^2. \tag{4.18}$$

Because $\alpha \in (0,1)$ we can rewrite this identity as follows

$$(b^{2\alpha})^p x^{1-\alpha} - b^{2\alpha}x^{2(1-\alpha)} = m^2 \tag{4.19}$$

where $p = (1+\alpha)/2\alpha > 1$. Setting

$$F(b) = (b^{2\alpha})^p x^{1-\alpha} - b^{2\alpha}x^{2(1-\alpha)} \tag{4.20}$$

for $b \geq 0$ with $x > 0$ given and fixed, we see that F is convex on $[0,\infty)$ with $F(0) = 0$, and after exhibiting a strict decrease initially and reaching its unique minimum point, F exhibits a strict increase afterwards with $F(x) = 0$ and $F(\infty) = \infty$. It follows therefore that there exists a unique point $b = b_m(x)$ such that $F(b) = m^2 > 0$. This shows that we can find a unique $b_m(x)$ where (4.18) is satisfied and hence (4.14) holds too with τ_*^c from (3.12) where $b(x) = b_m(x)$. Note that the arguments above also show that $b_m(x) > x$ for all $x > 0$. Recalling further that

$$b_m(x) = D_\alpha{}^{\frac{1}{1-\alpha}} x$$

in the Theorem 3.3.1 so we have

$$D_\alpha = (\frac{b_m(x)}{x})^{1-\alpha}.$$

Because

$$D_\alpha = 2\frac{(1+\frac{1}{\alpha})c^2 + 1 + \sqrt{(1-\frac{1}{\alpha^2})c^2 + 1}}{c^2(1+\frac{1}{\alpha})^2}$$

so after finding the unique point $b_m(x)$ in (4.18) we can find c satisfying

$$2\frac{(1+\frac{1}{\alpha})c^2+1+\sqrt{(1-\frac{1}{\alpha^2})c^2+1}}{c^2(1+\frac{1}{\alpha})^2}=(\frac{b_m(x)}{x})^{1-\alpha}. \tag{4.21}$$

Therefore $c>0$ is given by (see Appendix A.1)

$$c=c(m,x)=\frac{\sqrt{(\frac{b_m(x)}{x})^{1-\alpha}-1}}{\frac{1}{2}(1+\frac{1}{\alpha})(\frac{b_m(x)}{x})^{1-\alpha}-1} \tag{4.22}$$

which is uniquely determined. Inserting c and

$$D_\alpha=(\frac{b_m(x)}{x})^{1-\alpha}$$

into the conditions (3.10) and (3.11), if α meets these two conditions, then the stopping time τ_*^c exists and recalling (3.44) we have

$$V_1(x)=\mathbb{E}_x(X_{\tau_*^c})=\mathbb{E}_x(X_{\tau_b})=x^{1-\alpha}b_m^\alpha(x). \tag{4.23}$$

If α cannot meet these two conditions it is optimal to stop at once so $\sqrt{\mathbb{V}_x(X_\tau)}=0\le m$ and $V_1(x)=x$.

(D) We have established above that when α satisfying the two conditions we always have $b_m(x)>x$ for all $x>0$. This implies that it is dynamically optimal not to stop at all. For every x given and fixed, we have a larger $b_m(x)$ and this shows that x cannot be a dynamically optimal stopping point. This completes the proof.

□

It is easily seen that we can rewrite $V_1(x)$ as

$$V_1(x)=\sup_{\tau:\mathbb{V}_x(X_\tau)\le m^2}\mathbb{E}_x(X_\tau). \tag{4.24}$$

Setting $h=m^2$, we can use Corollary 5 of [28]. If $\mu\in(0,\sigma^2/2)$ then the stopping time $\tau_*=\inf\{t\ge 0\mid X_t\ge b_h(x)\}$ is statically optimal for $x>0$ where $b_h(x)$ is the unique solution to $b_h(x)^{1+\alpha}x^{1-\alpha}-b_h(x)^{2\alpha}x^{2-2\alpha}=hin(0,\infty)$ with $\alpha=\mu/(\sigma^2/2)$. We can also get the value function $V_1(X)=x^{1-\alpha}b_h^\alpha(x)$. The result is similar to Theorem 4.2.1 but the difference in Theorem 4.2.1 is that α needs to satisfy the two conditions (4.9) and (4.10).

Theorem 4.2.2. *Consider the constrained standard deviation problem (4.2), where the process X solves (3.1) with $X_0 = x$ under $\mathbb{P}_x$ for $x > 0$. Then:*

(A) If $\mu \leq 0$ then it is statically optimal to stop at once for $x \geq n$. The value function in (4.2) is given by $V_2(x) = 0$ for $x \geq n$. If $x \in (0, n)$ then the problem has no solution in the static sense.

(B) If $\mu \geq \sigma^2/2$ then the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq n\} \tag{4.25}$$

is statically optimal. The value function in (4.2) is given by $V_2(x) = 0$ for $x > 0$.

(C) If $\mu \in (0, \sigma^2/2)$ then it is statically optimal to stop at once for $x \geq n$ and $V_2(x) = 0$. If $x \in (0, n)$ then the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq n^{1/\alpha} x^{1-1/\alpha}\} \tag{4.26}$$

is statically optimal with $\alpha = \mu/(\sigma^2/2)$ when α satisfies the two conditions

$$\alpha > \sqrt{\frac{1}{1+c^2}} \tag{4.27}$$

and

$$D_\alpha^{\frac{\alpha}{1-\alpha}} \Big(1 - \frac{1}{c}\sqrt{D_\alpha - 1}\Big) > 1 \tag{4.28}$$

with

$$c = \frac{\frac{1}{2}(1+\frac{1}{\alpha})(\frac{n}{x})^{\frac{1}{\alpha}-1} - 1}{\sqrt{(\frac{n}{x})^{\frac{1}{\alpha}-1} - 1}}$$

and

$$D_\alpha = (\frac{n}{x})^{\frac{1}{\alpha}-1}.$$

The value function in (4.2) is given by

$$V_2(x) = \sqrt{n^{1+1/\alpha} x^{1-1/\alpha} - n^2} \tag{4.29}$$

for $x \in (0, n)$ and $V_2(x) = 0$ for $x \geq n$. If α cannot meet (4.27) and (4.28) then the problem has no solution in the static sense.

(D) The dynamic optimality is consistent with the static optimality (A)-(C).

Proof. (A) If $\mu \leq 0$ and $x \geq n$ then stopping at once we meet them constrained and obtain zero standard deviation so that this stopping time is statically optimal and $V_2(x) = 0$ as claimed. If $x \in (0, n)$ then by the supermartingale argument of part (A) in the proof of Theorem 3.3.1 we know that $\mathbb{E}_x(X_\tau) \leq x < n$ for all stopping time τ and $x > 0$. So we can not meet the constraint $\mathbb{E}_x(X_\tau) \geq n$ in the constrained optimal stopping problem $V_2(x)$ and hence the problem has no solution.

(B) If $\mu \geq \sigma^2/2$ then by part (B) in the proof of Theorem 3.3.1 we know that the stopping time τ_* from (4.25) satisfies $\mathbb{E}_x(X_{\tau_*}) = n$ for $x \in (0, n)$ and $\mathbb{E}(X_{\tau_*}) = x$ for $x \geq n$ and $\sqrt{\mathbb{V}_x(X_{\tau_*})} = 0$ in both cases. This shows that τ_* is the statically optimal stopping time in (4.2) and the value function $V_2(x) = 0$ for $x > 0$ as claimed.

(C) If $\mu \in (0, \sigma^2/2)$ and $x \geq n$ then it is optimal to stop at once since we meet the condition and $V_2(x) = 0$. On the other hand if $x \in (0, n)$ note that the Lagrangian for the constrained problem (4.2) can be defined by

$$L_x(\tau, c) = \sqrt{\mathbb{V}_x(X_\tau)} - c\big[\mathbb{E}_x(X_\tau) - n\big] \tag{4.30}$$

for $c > 0$. To connect to the results of Theorem 3.3.1 we observe that

$$\begin{aligned}\inf_\tau L_x(\tau, c) &= \inf_\tau \left(\sqrt{\mathbb{V}_x(X_\tau)} - c[\mathbb{E}_x(X_\tau) - n]\right) \\ &= c \sup_\tau \left(\mathbb{E}_x(X_\tau) - \frac{1}{c}\sqrt{\mathbb{V}_x(X_\tau)}\right) + cn. \end{aligned} \tag{4.31}$$

From (4.31) we first assume that α satisfies the conditions (3.10) and (3.11) with $1/c$ in place of c which reads

$$\alpha > \sqrt{\frac{1}{1+c^2}} \tag{4.32}$$

$$D_\alpha{}^{\frac{\alpha}{1-\alpha}}\left(1 - \frac{1}{c}\sqrt{D_\alpha - 1}\right) > 1 \tag{4.33}$$

with

$$D_\alpha = 2c^2 \frac{(1+\frac{1}{\alpha})\frac{1}{c^2} + 1 + \sqrt{(1-\frac{1}{\alpha^2})\frac{1}{c^2} + 1}}{(1+\frac{1}{\alpha})^2}.$$

Then by the result of Theorem 3.3.1 we know that the stopping time $\tau_*^{1/c} = \inf\{t \geq 0 \mid X_t \geq b(x)\}$ is optimal in the unconstrained problem

$$L_x(\tau_*^{1/c}, c) := \inf_{\tau} L_x(\tau, c) \tag{4.34}$$

for $x > 0$ and $c > 0$ where $b(x) = D_\alpha^{\frac{1}{1-\alpha}} x$. Moreover, we suppose that there exists $c = c(n, x) > 0$ such that

$$\mathbb{E}_x(X_{\tau_*^{1/c}}) = n \tag{4.35}$$

for $x > 0$. It follows that

$$\sqrt{\mathbb{V}_x(X_{\tau_*^{1/c}})} = L_x(\tau_*^{1/c}, c) \leq \sqrt{\mathbb{V}_x(X_\tau)} - c[\mathbb{E}_x(X_\tau) - n] \leq \sqrt{\mathbb{V}_x(X_\tau)} \tag{4.36}$$

for all stopping times τ such that $\mathbb{E}_x(X_\tau) \geq n$ with $x > 0$. This shows that the stopping time $\tau_*^{1/c}$ satisfies (4.35) with $c = c(n, x)$ is statically optimal in (4.2) for $x > 0$. To realise (4.35) recall from (3.44) that

$$\mathbb{E}_x(X_{\tau_b}) = b^\alpha x^{1-\alpha} \tag{4.37}$$

for $x > 0$ given and fixed. Setting this expression equals to n yields

$$b = b_n(x) = n^{1/\alpha} x^{1-1/\alpha} \tag{4.38}$$

and it also shows that $b_n(x) > x$ because $x \in (0, n)$. It follows that (4.35) holds with $\tau_*^{1/c} = \inf\{t \geq 0 \mid X_t \geq n^{1/\alpha} x^{1-1/\alpha}\}$. Recalling further that in Theorem 3.3.1 $b_n(x) = D_\alpha{}^{\frac{1}{1-\alpha}} x$, we have $D_\alpha = (\frac{n}{x})^{\frac{1}{\alpha}-1}$. Because

$$D_\alpha = 2c^2 \frac{(1 + \frac{1}{\alpha})\frac{1}{c^2} + 1 + \sqrt{(1 - \frac{1}{\alpha^2})\frac{1}{c^2} + 1}}{(1 + \frac{1}{\alpha})^2},$$

after finding the unique point $b_n(x)$ in (4.38), we can find c satisfying

$$2c^2 \frac{(1 + \frac{1}{\alpha})\frac{1}{c^2} + 1 + \sqrt{(1 - \frac{1}{\alpha^2})\frac{1}{c^2} + 1}}{(1 + \frac{1}{\alpha})^2} = (\frac{b_n(x)}{x})^{1-\alpha}. \tag{4.39}$$

This c is given by (see Appendix A.2)

$$c = c(n,x) = \frac{\frac{1}{2}(1+\frac{1}{\alpha})(\frac{b_n(x)}{x})^{1-\alpha} - 1}{\sqrt{(\frac{b_n(x)}{x})^{1-\alpha} - 1}} = \frac{\frac{1}{2}(1+\frac{1}{\alpha})(\frac{n}{x})^{\frac{1}{\alpha}-1} - 1}{\sqrt{(\frac{n}{x})^{\frac{1}{\alpha}-1} - 1}} \tag{4.40}$$

which is uniquely determined. Recalling that $\alpha \in (0,1)$ and $x \in (0,n)$, it is easily verified that $c > 0$. Inserting c and D_α into the conditions (4.32) and (4.33), if α meets these two conditions, then the stopping time $\tau_*^{1/c}$ exists. Invoking the optimality established in Theorem 3.3.1, we can conclude that the stopping time is statically optimal in (4.2) for $x \in (0,n)$. To calculate the value function we need to recall (3.44) and combine with (4.37). This yields

$$\begin{aligned} V_2(x) &= \sqrt{\mathbb{V}_x(X_{\tau_b})} = \sqrt{\mathbb{E}_x(X_{\tau_b}^2) - (\mathbb{E}_x(X_{\tau_b})^2)} \\ &= \sqrt{b^{1+\alpha}x^{1-\alpha} - b^{2\alpha}x^{2-2\alpha}} = \sqrt{n^{1+1/\alpha}x^{1-1/\alpha} - n^2} \end{aligned} \tag{4.41}$$

for $x \in (0,n)$. If α cannot meet these two conditions, then we need to stop at once but $x < n$ so we cannot meet the constrain $\mathbb{E}_x(X_\tau) \geq n$ and the problem has no solution in the static sense.

(D) In (4.2) it is dynamically optimal to stop when the process X enters the set $[n,\infty)$. Since when X enters this set we meet the condition $\mathbb{E}_x(X_\tau) = n$ and $V_2(x) = 0$ we cannot find a smaller value by continuing beyond the initial point. So if the process X can enter $[n,\infty)$ then we stop it immediately but if the process X cannot enter this set the problem (4.2) has no solution. This completes the proof.

□

We can rewrite $V_2(x)$ as

$$V_2(x) = \inf_{\tau:\ \mathbb{E}_x(X_\tau)\geq n} \sqrt{\mathbb{V}_x(X_\tau)} = \sqrt{\inf_{\tau:\ \mathbb{E}_x(X_\tau)\geq n} \mathbb{V}_x(X_\tau)}. \tag{4.42}$$

Then

$$\big(V_2(x)\big)^2 = \inf_{\tau:\ \mathbb{E}_x(X_\tau)\geq n} \mathbb{V}_x(X_\tau) \tag{4.43}$$

and we can use Corollary 7 of [28] directly. Because Corollary 7 already solved the constrained problem $\inf_{\tau:\ \mathbb{E}_x(X_\tau)\geq n} \mathbb{V}_x(X_\tau)$. If $\mu \in (0, \sigma^2/2)$ and $x \in (0, n)$ then the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq n^{1/\alpha} x^{1-1/\alpha}\}$$

is statically optimal and we have the value function

$$V_2(x) = \sqrt{n^{1+1/\alpha} x^{1-1/\alpha} - n^2}$$

for $x \in (0, n)$. The result is similar to Theorem 4.2.2 but the difference in Theorem 4.2.2 is that α should satisfy two conditions (4.27) and (4.28).

Example 4.2.1. *We want to show an example of Theorem 4.2.2. Setting* $n = 1$, $\alpha = 4/5$, *we want to solve the constrained problem*

$$V_2(x) = \inf_{\tau:\ \mathbb{E}_x(X_\tau)\geq 1} \sqrt{\mathbb{V}_x(X_\tau)}. \tag{4.44}$$

According to Theorem 4.4.2 the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq x^{-\frac{1}{4}}\} \tag{4.45}$$

is statically optimal if α, $D_\alpha = x^{-\frac{1}{4}}$ *and* $c = \frac{\frac{9}{8}x^{-\frac{1}{4}}-1}{\sqrt{x^{-\frac{1}{4}}-1}}$ *satisfy the conditions*

$$\alpha > \sqrt{\frac{1}{1+c^2}}, \tag{4.46}$$

$$D_\alpha{}^{\frac{\alpha}{1-\alpha}}\big(1 - \frac{1}{c}\sqrt{D_\alpha - 1}\big) > 1. \tag{4.47}$$

If setting $x = \frac{1}{16} \in (0, 1)$ *we can calculate* $D_\alpha = 2$ *and* $c = \frac{5}{4}$ *which satisfy the conditions (4.46) and (4.47), then from Theorem 4.4.2 we have the value function* $V_2(x) = 1$.

However, if setting $x = (\frac{4}{5})^4 \in (0, 1)$ *we can calculate* $D_\alpha = \frac{5}{4}$ *and* $c = \frac{13}{16}$. *We still meet the condition (4.46) but*

$$D_\alpha{}^{\frac{\alpha}{1-\alpha}}\big(1 - \frac{1}{c}\sqrt{D_\alpha - 1}\big) \approx 0.94 < 1.$$

So we cannot meet the condition (4.47) and the problem has no solution in the static sense.

Chapter 5

Mean-Variance Stopping Problems for the CEV Process

In the previous chapter and in [28], it is assumed that the stock price X follows a geometric Brownian motion. There is another process that can be used to model the asset price, which is called the Constant Elasticity of Variance (CEV) process. In this chapter I show how to solve the mean-variance optimal stopping problem for the CEV process.

5.1 Problem formulation

We assume that the asset price X follows a constant elasticity of variance (CEV) process which is a one-dimensional diffusion process that solves a stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t^{1+\beta} dB_t \tag{5.1}$$

with initial value $X_0 = x$ for $x > 0$, where the drift parameter satisfies $\mu > 0$, the volatility coefficient satisfies $\sigma > 0$, the elasticity parameter satisfies $-\frac{1}{2} < \beta < 0$, and B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The infinitesimal generator of X is given by

$$\mathbb{L}_X = \mu x \frac{d}{dx} + \frac{\sigma^2}{2} x^{2+2\beta} \frac{d^2}{dx^2}. \tag{5.2}$$

Consider the optimal stopping problem

$$V(x) = \sup_{\tau} \big[\mathbb{E}_x(X_\tau) - c\mathbb{V}_x(X_\tau)\big], \tag{5.3}$$

where $c > 0$ is a given fixed constant and the supremum is taken over all stopping times τ of X such that $\mathbb{E}_x(X_\tau^2) < \infty$ for $x > 0$.

For $\beta = 0$, the CEV process reduces to geometric Brownian motion and the mean-variance optimal stopping problems have already been solved in [28]. For $\beta \neq 0$, this model was first considered by Cox [7] for $\beta < 0$, and later Emanuel and MacBeth [12] extended to the case $\beta > 0$. The CEV process can be used to model the prices of commodities and equities (see [20] and [14]). If $\beta < 0$, then the volatility and price are inversely related, which means the volatility increases when the price declines. On the other hand, if $\beta > 0$, then the volatility increases as the price increases.

The optimal stopping problem (5.3) is to find an optimal stopping time τ_* at which the supremum is attained, and moreover to compute the value function

$$V(x) = \mathbb{E}_x(X_{\tau_*}) - c\mathbb{V}_x(X_{\tau_*}). \tag{5.4}$$

As discussed in Chapter 2, because the second term

$$\mathbb{V}_x(X_\tau) = \mathbb{E}_x(X_\tau^2) - (\mathbb{E}_x(X_\tau))^2$$

defines a nonlinear function of $\mathbb{E}_x(X_\tau)$, we see that (5.3) is also a nonlinear optimal stopping problem. We consider the static optimality and dynamic optimality from [28] respectively.

Definition 5.1.1 (Static optimality). *A stopping time τ_* is statically optimal in (5.3) for $x > 0$ given and fixed, if for every stopping time σ we have*

$$\mathbb{E}_x(X_\sigma) - c\mathbb{V}_x(X_\sigma) \leq \mathbb{E}_x(X_{\tau_*}) - c\mathbb{V}_x(X_{\tau_*}). \tag{5.5}$$

Definition 5.1.2 (Dynamic optimality). *A stopping time τ_* is dynamically optimal in (5.3) if there is no other stopping time σ such that*

$$\mathbb{P}_x\big(\mathbb{E}_{X_{\tau_*}}(X_\sigma) - c\mathbb{V}_{X_{\tau_*}}(X_\sigma) > X_{\tau_*}\big) > 0 \tag{5.6}$$

for some $x > 0$.

5.2 Solution to the problem

Theorem 5.2.1. *Assume X is a constant elasticity of variance process solves (5.1) with $X_0 = x$ under $\mathbb{P}_x$ for $x > 0$. Consider the optimal stopping problem (5.3) for given and fixed constant $c > 0$. If $\mu > 0$, $\sigma > 0$ and $-\frac{1}{2} < \beta < 0$, we look for all admissible b which satisfy the inequality*

$$\frac{b^{2\beta}}{\alpha} \geq \frac{\frac{1}{2}\alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}}} \tag{5.7}$$

where $\alpha = \mu/(\sigma^2/2) > 0$ and

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt$$

is the lower incomplete Gamma function. Then the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq b(x)\} \tag{5.8}$$

is optimal for $x > 0$ where $b(x)$ is the unique solution to

$$\Big[1 - b(x)\frac{S'(b(x))}{S(b(x))}\Big]\Big[2cb(x)\frac{S(x)}{S(b(x))} - cb(x) + 1\Big] = cb(x) \tag{5.9}$$

where

$$S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}} dy.$$

The value function in (5.3) is given by

$$V(x) = b(x)\frac{S(x)}{S(b(x))} + cb(x)^2\Big(\frac{S(x)^2}{S(b(x))^2} - \frac{S(x)}{S(b(x))}\Big) \tag{5.10}$$

for $x \in (0, b(x)]$ and $V(x) = x$ for $x \geq b(x)$ which holds if and only if

$$(cx^2 + x)S'(x) \geq S(x)$$

where $S'(x) = e^{\frac{\alpha}{2\beta}x^{-2\beta}}$.

If $(cx^2 + x)S'(x) < S(x)$ for all $x > 0$, then it is dynamically optimal not to *stop at all. If* $(cx^2 + x)S'(x) \geq S(x)$*, then the stopping time*

$$\tau_* := \inf\{t \geq 0 \mid X_t \geq x_*\} \tag{5.11}$$

is dynamically optimal for $x > 0$ *where* x_* *is the unique solution to*

$$({cx_*}^2 + x_*)S'(x_*) = S(x_*) \tag{5.12}$$

with $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}} dy$.

In order to prove the theorem we first solve the standard optimal stopping problem, which will then be used in the mean-variance optimal stopping problem.

Lemma 5.2.1. *Assume that* X *is a constant elasticity of variance process that solves (5.1) with* $X_0 = x$ *under* $\mathbb{P}_x$ *for* $x > 0$*, where* $\mu > 0$*,* $\sigma > 0$*,* $\alpha = \mu/(\sigma^2/2) > 0$ *and* $-\frac{1}{2} < \beta < 0$*. Given* $\lambda > 0$ *and the gain function* $x \to G_\lambda(x) = x^2 - \lambda x$*, consider the standard optimal stopping problem*

$$V_\lambda(x) = \inf_\tau \mathbb{E}_x\big(G_\lambda(X_\tau)\big) \tag{5.13}$$

for $x > 0$*. If* $x \geq \lambda/2$*, then the optimal stopping time is*

$$\tau_*^\lambda = \inf\{t \geq 0 \mid X_t \leq \lambda/2\}. \tag{5.14}$$

If $x \in (0, \lambda/2)$*, and* b *satisfies the inequality*

$$\frac{b^{2\beta}}{\alpha} \geq \frac{\frac{1}{2}\alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}}}, \tag{5.15}$$

then the optimal stopping time is

$$\tau_*^\lambda = \tau_b = \inf\{t \geq 0 \mid X_t \geq b\} \tag{5.16}$$

where b *is the unique solution to*

$$\alpha(b^2 - \lambda b)\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}} = (2b - \lambda)\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}). \tag{5.17}$$

The value function is given by

$$V_\lambda(x) = \begin{cases} \frac{b^2-\lambda b}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}x^{-2\beta}) & \text{if} \quad x \in (0, b), \\ x^2 - \lambda x & \text{if} \quad x \in [b, \frac{\lambda}{2}], \\ -\frac{\lambda^2}{4} & \text{if} \quad x > \frac{\lambda}{2}. \end{cases} \tag{5.18}$$

Proof of Lemma 5.2.1. The gain function $G_\lambda(x) = (x - \lambda/2)^2 - \lambda^2/4$ is convex on $(0, \infty)$. The value $G_\lambda(x)$ decreases on $(0, \lambda/2)$ and increases to ∞ as $x \to \infty$. It attains its unique minimum at $\lambda/2$ and the value is $-\lambda^2/4$. Later we will show that $\lim_{t\to\infty} X_t = 0$ so $\tau_*^\lambda := \inf\{t \geq 0 \mid X_t \leq \lambda/2\}$ is the optimal stopping time in (5.13) for $x \geq \lambda/2$ and $V_\lambda(x) = -\lambda^2/4$.

On the other hand, we conjecture that it is optimal to stop at once in (5.13) for $x \in [b, \lambda/2]$, where $b \in (0, \lambda/2)$ is the optimal stopping point for X starting at $x \in (0, b)$ to be determined. Then the optimal stopping problem can be reduced to the following free-boundary problem which can be solved explicitly:

$$\mathbb{L}_X V_\lambda = 0 \text{ in } (0, \mathrm{b}) \tag{5.19}$$

$$V_\lambda(0+) = 0 \tag{5.20}$$

$$V_\lambda(b) = G_\lambda(b) = b^2 - \lambda b \tag{5.21}$$

$$V'_\lambda(b) = G'_\lambda(b) = 2b - \lambda \tag{5.22}$$

where

$$\mathbb{L}_X = \mu x \frac{d}{dx} + \frac{\sigma^2}{2} x^{2+2\beta} \frac{d^2}{dx^2}$$

is the infinitesimal generator of X. From (5.19) we have

$$\mu x \frac{dV_\lambda}{dx} + \frac{\sigma^2}{2} x^{2+2\beta} \frac{d^2 V_\lambda}{dx^2} = 0. \tag{5.23}$$

Here set

$$U = \frac{dV_\lambda}{dx}$$

and insert this into (5.23) we see that

$$\mu x U + \frac{\sigma^2}{2} x^{2+2\beta} U' = 0. \tag{5.24}$$

By rewriting this, we have

$$U' + \alpha x^{-1-2\beta} U = 0, \tag{5.25}$$

where $\alpha = \mu/(\sigma^2/2) > 0$. So the integrating factor is

$$e^{\int \alpha x^{-1-2\beta} dx} = e^{-\frac{\alpha}{2\beta}x^{-2\beta}} \tag{5.26}$$

and multipling it against both sides of (5.25) we find that

$$\begin{aligned}
&\left(Ue^{-\frac{\alpha}{2\beta}x^{-2\beta}}\right)' = 0 \\
&U = Ce^{\frac{\alpha}{2\beta}x^{-2\beta}} \\
&\frac{dV_\lambda}{dx} = Ce^{\frac{\alpha}{2\beta}x^{-2\beta}} && (5.27)\\
&V_\lambda(x) = C\int e^{\frac{\alpha}{2\beta}x^{-2\beta}} dx + D. && (5.28)
\end{aligned}$$

So we obtain

$$V_\lambda(x) = c_1\Gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}x^{-2\beta}) + c_2, \tag{5.29}$$

where c_1 and c_2 are two constants and

$$\Gamma(a, x) = \int_x^\infty e^{-t}t^{a-1}dt$$

is the upper incomplete Gamma function (see e.g. reference [1], pp. 260-263, for a summary of the incomplete Gamma function). Since $-\frac{1}{2} < \beta < 0$, we have $-\frac{1}{2\beta} > 1$. From (5.20) we see that

$$V_\lambda(0) = c_1\Gamma(-\frac{1}{2\beta}, 0) + c_2 = 0,$$

so $c_2 = -c_1\Gamma(-\frac{1}{2\beta})$ and

$$V_\lambda(x) = c_1\Gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}x^{-2\beta}) - c_1\Gamma(-\frac{1}{2\beta}) \tag{5.30}$$

$$V_\lambda(x) = -c_1\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}x^{-2\beta}) \tag{5.31}$$

for $x \in (0, b]$. Here

$$\gamma(a, x) = \Gamma(a) - \Gamma(a, x) = \int_0^x e^{-t}t^{a-1}dt$$

is the lower incomplete Gamma function. From (5.21) and (5.31) we find that

$$c_1 = \frac{\lambda b - b^2}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}$$

so that

$$V_\lambda(x) = \frac{b^2 - \lambda b}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}x^{-2\beta}) \tag{5.32}$$

for $x \in (0, b]$. Using the fact that

$$\frac{d\gamma(a,x)}{dx} = x^{a-1}e^{-x}$$

and substituting $y = -\frac{\alpha}{2\beta}x^{-2\beta}$ into (5.32) we find that

$$\begin{aligned} V'_\lambda(x) &= \frac{b^2 - \lambda b}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}\frac{d\gamma(-\frac{1}{2\beta}, y)}{dy}\frac{dy}{dx} \\ &= \frac{b^2 - \lambda b}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}\big(-\frac{\alpha}{2\beta}x^{-2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}x^{-2\beta}}\alpha x^{-2\beta-1} \\ &= \frac{\alpha(b^2 - \lambda b)}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}x^{-2\beta}}. \end{aligned} \tag{5.33}$$

Finally from (5.22) we obtain that

$$\frac{\alpha(b^2 - \lambda b)}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta})}\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}b^{-2\beta}} = 2b - \lambda \tag{5.34}$$

$$\alpha(b^2 - \lambda b)\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}b^{-2\beta}} = (2b - \lambda)\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) \tag{5.35}$$

is a function of b and λ. Rewriting (5.35), we have

$$\lambda = \frac{2\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}b^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}b^{-2\beta}}}b. \tag{5.36}$$

We can calculate λ as a function of b and vice versa. Given $\lambda > 0$, we can calculate a unique b which satisfies (5.35). Combining these two situations we find that the stopping times (5.14) and (5.16) are optimal in (5.13) and we can compute the value function $V_\lambda(x)$:

$$V_\lambda(x) = \begin{cases} \frac{b^2-\lambda b}{\gamma(-\frac{1}{2\beta},-\frac{\alpha}{2\beta}b^{-2\beta})}\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}x^{-2\beta}) & \text{if} \quad x \in (0,b), \\ x^2 - \lambda x & \text{if} \quad x \in [b, \frac{\lambda}{2}], \\ -\frac{\lambda^2}{4} & \text{if} \quad x > \frac{\lambda}{2}. \end{cases} \tag{5.37}$$

Here the continuation set $C = (0,b) \cup (\frac{\lambda}{2}, \infty)$ where $V_\lambda(x) < G_\lambda(x)$ and the stopping set $D = [b, \lambda/2]$ where $V_\lambda(x) = G_\lambda(x) = x^2 - \lambda x$ as claimed. We see that $\mathbb{L}_X V_\lambda(x) = 0$ holds for all $x \in C$. Clearly, the optimal stopping time does not depend on the initial point $x > 0$, which is given and fixed.

The function V_λ defined by (5.37) is C^2 everywhere but at b and $\lambda/2$ where it is C^1. Then applying the Itô formula we obtain

$$V_\lambda(X_t) = V_\lambda(x) + \int_0^t \mathbb{L}_X V_\lambda(X_s)ds + M_t \tag{5.38}$$

for $x > 0$ where

$$M_t = \sigma \int_0^t X_s^{1+\beta} V_\lambda'(X_s) dB_s$$

is a continuous local martingale for $t \geq 0$. Moreover, we have

$$\begin{aligned} \mathbb{L}_X G_\lambda(x) &= \mu x G_\lambda'(x) + \frac{\sigma^2}{2} x^{2+2\beta} G_\lambda''(x) \\ &= \mu x(2x - \lambda) + \sigma^2 x^{2+2\beta} \end{aligned} \tag{5.39}$$

for $x > 0$. We need to prove $\mathbb{L}_X G_\lambda(x) = \mathbb{L}_X V_\lambda(x) \geq 0$ for $x \in D$:

$$2\mu x + \sigma^2 x^{1+2\beta} - \mu\lambda \geq 0 \tag{5.40}$$

$$\alpha x + x^{1+2\beta} \geq \frac{1}{2}\alpha\lambda, \tag{5.41}$$

where $0 < 1 + 2\beta < 1$ and $\alpha = \mu/(\sigma^2/2) > 0$. From (5.41) we only need to show

$$\alpha b + b^{1+2\beta} \geq \frac{1}{2}\alpha\lambda. \tag{5.42}$$

Substituting (5.36) into (5.42), we need to prove that

$$\alpha + b^{2\beta} \geq \frac{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \frac{1}{2}\alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b^{-2\beta}}}\alpha. \tag{5.43}$$

Rewriting (5.43), we find that it is equivalent to

$$\frac{b^{2\beta}}{\alpha} \geq \frac{\frac{1}{2}\alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}b^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b^{-2\beta}) - \alpha b\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1}e^{\frac{\alpha}{2\beta}b^{-2\beta}}}. \tag{5.44}$$

By (5.15) we know that this inequality holds so $\mathbb{L}_X G_\lambda(x) = \mathbb{L}_X V_\lambda(x) \geq 0$ for $x \in D$. Recalling that $\mathbb{L}_X V_\lambda(x) = 0$ for $x \in (0, b) \cup (\lambda/2, \infty)$ this gives us that

$$\mathbb{L}_X V_\lambda(x) \geq 0 \tag{5.45}$$

for $x \in (0, \infty) \setminus \{b, \lambda/2\}$. Because the time that X spends at b and $\lambda/2$ has Lebesgue measure zero, from (5.38) and (5.45) we get

$$V_\lambda(x) + M_t \leq V_\lambda(X_t) \leq G_\lambda(X_t)$$

for $t \geq 0$ and $x > 0$. Let τ_n be a localizing sequence of bounded stopping times for M with $n \geq 1$. Then for any stopping time τ of X such that $\mathbb{E}_x[X_\tau^2] < \infty$ we have

$$V_\lambda(x) + M_{\tau\wedge\tau_n} \leq G_\lambda(X_{\tau\wedge\tau_n}) \tag{5.46}$$

for $x > 0$. Taking expectations on the both sides of (5.46) and using the optional sampling theorem we obtain

$$V_\lambda(x) \leq \mathbb{E}_x\big(G_\lambda(X_{\tau\wedge\tau_n})\big) \tag{5.47}$$

for $x > 0$ and $n \geq 1$ because $\mathbb{E}_x(M_{\tau\wedge\tau_n}) = 0$. Letting $n \to \infty$ and using Fatou's lemma, we see that

$$V_\lambda(x) \leq \limsup_{n\to\infty} \mathbb{E}_x\big(G_\lambda(X_{\tau\wedge\tau_n})\big) \leq \mathbb{E}_x\big(\limsup_{n\to\infty} G_\lambda(X_{\tau\wedge\tau_n})\big) = \mathbb{E}_x\big(G_\lambda(X_\tau)\big).$$

Since this holds for any stopping time τ we have

$$V_\lambda(x) \leq \inf_\tau \mathbb{E}_x\big(G_\lambda(X_\tau)\big) \tag{5.48}$$

for $x > 0$. Taking

$$\tau_*^\lambda := \inf\{t \geq 0 \mid X_t \leq \lambda/2\}$$

for $x \geq \lambda/2$ and

$$\tau_*^\lambda := \inf\{t \geq 0 \mid X_t \geq b\}$$

for $x \in (0, \lambda/2)$ where b and α meet the condition (5.44) we can get $V_\lambda(x) = \mathbb{E}_x\big(G_\lambda(X_{\tau_*^\lambda})\big)$. □

In our problem we have $-\frac{1}{2} < \beta < 0$. If we consider the situation when $\beta < -\frac{1}{2}$, then the origin is a regular boundary point which means that the SDE (5.1) cannot specify the process uniquely and we need a boundary condition at the origin. For example, when we change the condition (5.20) to $V_\lambda''(0+) = 0$ (Feller condition) corresponding to X being absorbed when reaching 0. From (5.29) we have

$$V_\lambda'(x) = -\alpha(-\frac{\alpha}{2\beta})^{-\frac{1}{2\beta}-1} c_1 e^{\frac{\alpha}{2\beta}x^{-2\beta}}, \tag{5.49}$$

$$V_\lambda''(x) = \alpha^2(-\frac{\alpha}{2\beta})^{-\frac{1}{2\beta}-1} c_1 e^{\frac{\alpha}{2\beta}x^{-2\beta}} x^{-2\beta-1}. \tag{5.50}$$

If $V_\lambda''(0+) = 0$, then $c_1 * 0 = 0$ and therefore the initial free-boundary problem has no solution. Similarly if we change the condition to $V_\lambda'(0+) = 0$ (Neumann condition) or $V_\lambda'(0) = V_\lambda''(0+)$ (Feller condition), there is no solution either. The boundary characterization can be found in the book [3] (pp. 133-134)

Example 5.2.1. *We want to show an example of Lemma 5.2.1. Setting $\alpha = 1/2$, $\beta = -1/8$ and $\lambda = 1$, we want to solve the optimal stopping problem $V_\lambda(x) = \inf_\tau \mathbb{E}_x(x^2 - x)$. According to Lemma 5.2.1, if $x \in (0, \lambda/2)$, b is the unique solution to*

$$4(b^2 - b)e^{-2b^{\frac{1}{4}}} = (2b-1)\gamma(4, 2b^{\frac{1}{4}}). \tag{5.51}$$

Solving (5.51) we get $b \approx 0.203$ and b satisfies the inequality (5.15). Substituting it into (5.18) we have

$$V_\lambda(x) = \frac{b^2 - b}{\gamma(4, 2b^{\frac{1}{4}})}\gamma(4, 2x^{\frac{1}{4}}) = -0.568\gamma(4, 2x^{\frac{1}{4}}) \tag{5.52}$$

for $x \in (0, b]$. Here $[b, \frac{1}{2}]$ is the optimal stopping set with $V_\lambda(x) = G_\lambda(x) = x^2 - x$ as claimed. After the optimal stopping point $b \approx 0.203$, $V_\lambda(x) = G_\lambda(x)$ and $V_\lambda(x) = -1/4$ when $x \geq 1/2$.

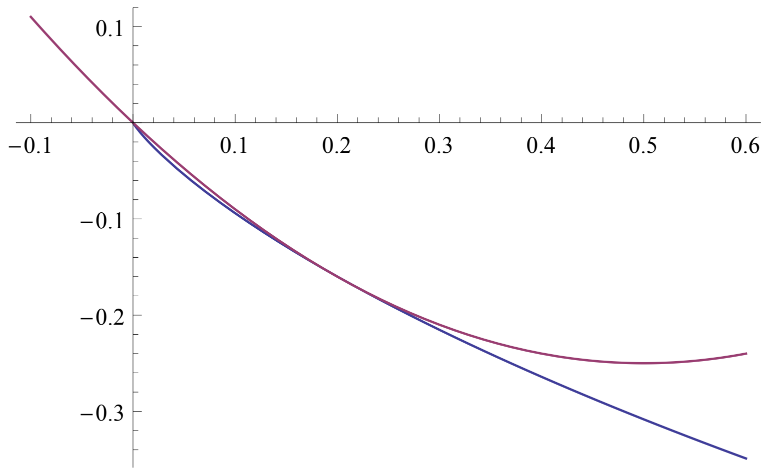

Figure 5.1: A computer drawing of G_λ and V_λ from Lemma 5.2.1 for $\alpha = 1/2$, $\beta = -1/8$ and $\lambda = 1$.

Proof of Theorem 5.2.1. From [20], we see that the CEV process can be reduced to a Bessel process. Because the CEV process with drift ($\mu \neq 0$) can be obtained from the process without drift ($\mu = 0$) via a scale and time change

$$X_t = e^{\mu t} X^{(0)}_{\tau(t)} = e^{\mu t}(\sigma|\beta|Z^{(v)}_{\tau(t)})^{-\frac{1}{\beta}}, \tag{5.53}$$

$$\tau(t) = \frac{e^{2\mu\beta t} - 1}{2\mu\beta} \tag{5.54}$$

where $Z^{(v)}_t$ is a standard Bessel process of order $v = 1/2\beta$. Recalling that $-\frac{1}{2} < \beta < 0$, we have $v < -1$. From [3] (pp. 133-134) we know that when $v < -1$, 0 is an exit boundary which means the process is killed the first time it reaches 0. From (5.54) we see that $\tau(t) \to -1/2\mu\beta$ as $t \to \infty$ and by rewriting it we obtain

$$e^{\mu t} = (1 + 2\mu\beta\tau(t))^{\frac{1}{2\beta}}. \tag{5.55}$$

Inserting (5.55) into (5.53) we have

$$\lim_{t\to\infty} X_t = \lim_{t\to\infty}(1 + 2\mu\beta\tau(t))^{\frac{1}{2\beta}}(\sigma|\beta|Z^{(v)}_{\tau(t)})^{-\frac{1}{\beta}} = 0. \tag{5.56}$$

The objective function in (5.3) reads

$$\mathbb{E}_x(X_\tau) - c\mathbb{V}_x(X_\tau) = \mathbb{E}_x(X_\tau) + c\big(\mathbb{E}_x(X_\tau)\big)^2 - c\mathbb{E}_x(X_\tau^2). \tag{5.57}$$

The key difficulty is the second term and we overcome it by conditioning on the size of $\mathbb{E}_x(X_\tau)$

$$\begin{aligned} V(x) &= \sup_{M\geq 0} \sup_{\tau:\mathbb{E}_x(X_\tau)=M} \Big[\mathbb{E}_x(X_\tau) - c\mathbb{V}_x(X_\tau)\Big] \\ &= \sup_{M\geq 0} \sup_{\tau:\mathbb{E}_x(X_\tau)=M} \Big[\mathbb{E}_x(X_\tau) + c\big(\mathbb{E}_x(X_\tau)\big)^2 - c\mathbb{E}_x(X_\tau^2)\Big] \\ &= \sup_{M\geq 0} \Big[M + cM^2 - c \inf_{\tau:\mathbb{E}_x(X_\tau)=M} \mathbb{E}_x(X_\tau^2)\Big] \end{aligned} \tag{5.58}$$

for $x > 0$. If we want to solve (5.58) we need to solve the constrained problem

$$V_M(x) = \inf_{\tau:\mathbb{E}_x(X_\tau)=M} \mathbb{E}_x(X_\tau^2) \tag{5.59}$$

for $x > 0$ and $M > 0$ given and fixed where τ is a stopping time of X. As shown in Chapter 3, we use the method of Lagrange multipliers. So the optimal stopping time τ_*^λ in the unconstrained problem

$$L_x(\tau_*^\lambda, \lambda) := \inf_\tau \Big(\mathbb{E}_x(X_\tau^2) - \lambda\big[\mathbb{E}_x(X_\tau) - M\big]\Big) \tag{5.60}$$

satisfying the condition

$$\mathbb{E}_x(X_{\tau_*^\lambda}) = M \tag{5.61}$$

with $\lambda = \lambda(M, x)$ is optimal in the constrained problem (5.59). From (5.60) we just need to consider the optimal stopping problem

$$V_\lambda(x) = \inf_\tau \mathbb{E}_x(X_\tau^2 - \lambda X_\tau) \tag{5.62}$$

for $x > 0$, which we have already solved in the Lemma 5.2.1. Now we want to find the specific $\lambda = \lambda(M, x) > 0$ that meets the condition (5.61). Recalling that the scale function of X is given by

$$S(x) = \int_0^x exp\Big(- \int_0^y \frac{\mu z}{\sigma^2 z^{2+2\beta}/2} dz\Big) dy = \int_0^x e^{\frac{\alpha}{2\beta} y^{-2\beta}} dy. \tag{5.63}$$

Thus we have

$$P_x(\tau_b < \infty) = \frac{S(x) - S(0)}{S(b) - S(0)} = \frac{\int_0^x e^{\frac{\alpha}{2\beta} y^{-2\beta}} dy}{\int_0^b e^{\frac{\alpha}{2\beta} y^{-2\beta}} dy} \tag{5.64}$$

for $x \in (0, b]$. Therefore

$$P_x(\tau_b < \infty) = \frac{S(x)}{S(b)} \tag{5.65}$$

for $x \in (0, b]$. Because $X_t \to 0$ P_x-a.s. as $t \to \infty$, it follows that

$$\mathbb{E}_x(X_{\tau_b}) = bP_x(\tau_b < \infty) + 0P_x(\tau_b = \infty) = b\frac{S(x)}{S(b)} \tag{5.66}$$

$$\mathbb{E}_x(X_{\tau_b}^2) = b^2 P_x(\tau_b < \infty) + 0P_x(\tau_b = \infty) = b^2\frac{S(x)}{S(b)} \tag{5.67}$$

for $x \in (0, b]$. To realize (5.61) we need to identify (5.66) with M and obtain

$$M = \mathbb{E}_x(X_{\tau_b}) = b\frac{S(x)}{S(b)} \tag{5.68}$$

for $x \in (0, b]$. For $M > 0$ given and fixed, b can be defined by the equation (5.68) uniquely and it shows that b is a function of M and x for $x \in (0, b]$. It then follows from (5.36) and (5.68) that we can calculate λ as a function of M and x for $x \in (0, b]$. Inserting (5.68) into (5.67) we obtain

$$V_M(x) = \mathbb{E}_x(X_{\tau_b}^2) = b^2\frac{S(x)}{S(b)} = bM \tag{5.69}$$

for $x \in (0, b]$. Because b is a function of M and x so $V_M(x)$ is actually a function of M and x for $x \in (0, b]$. Note that $x \le b$ if and only if $M \ge x$ in this case.

Recalling that if $x > \lambda/2$ then the optimal stopping time in (5.62) is

$$\tau_*^\lambda := \inf\{t \ge 0 \mid X_t \le \lambda/2\}$$

so we set $M = \lambda/2 < x$. In this situation because $\lim_{t\to\infty} X_t = 0$, we have $\mathbb{P}_x(\tau_*^\lambda < \infty) = 1$ and (5.59) is given by

$$V_M(x) = \mathbb{E}_x\big(X_{\tau_*^\lambda}^2\big) = M^2. \tag{5.70}$$

Inserting $V_M(x)$ into the original problem (5.58), it follows that

$$\begin{aligned} V(x) &= \sup_{M\ge 0}\Big[M + cM^2 - cV_M(x)\Big] \\ &= \sup_{M\ge x}\Big[M + cM^2 - cbM\Big] \vee \sup_{M<x} M \\ &= \sup_{M\ge x}\Big[M + cM^2 - cbM\Big] \end{aligned} \tag{5.71}$$

for all $x > 0$ and that b is function of M and x which is defined by the equation (5.66) uniquely. The third equation holds since $\sup_{M<x} M = x$ and we can take $M = x = b$ in the first supremum above.

From (5.68) we know that $M = bS(x)/S(b)$ where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}}dy$ for $x \in (0, b]$, $b = b(M, x) \geq M$ since $S(x)/S(b) \leq 1$ for $x \in (0, b]$. Substituting (5.68) into (5.71) we get

$$\begin{aligned} V(x) &= \sup_{b \geq x} \Big[b\frac{S(x)}{S(b)} + cb^2 \frac{S(x)^2}{S(b)^2} - cb^2 \frac{S(x)}{S(b)} \Big] \\ &= \sup_{b \geq x} \Big[b\frac{S(x)}{S(b)} + cb^2 \Big(\frac{S(x)^2}{S(b)^2} - \frac{S(x)}{S(b)} \Big) \Big]. \end{aligned} \tag{5.72}$$

Set

$$F(b) = b\frac{S(x)}{S(b)} + cb^2 \Big[\frac{S(x)^2}{S(b)^2} - \frac{S(x)}{S(b)} \Big] \tag{5.73}$$

for $b \geq 0$ with $x > 0$ given and fixed. Because $S(b) \to \int_0^\infty e^{\frac{\alpha}{2\beta}y^{-2\beta}}dy$ as b moves to ∞, $S(b)$ tends to a finite number as b moves to ∞. For example, if $\alpha = 1/2$ and $\beta = -1/8$, then $S(b) = 3/2$ as b moves to ∞. To find the maximum point we need to differentiate $F(b)$ and get

$$\begin{aligned} F'(b) &= \frac{S(x)}{S(b)} - bS'(b)\frac{S(x)}{S(b)^2} + 2cb\Big[\frac{S(x)^2}{S(b)^2} - \frac{S(x)}{S(b)} \Big] \\ &+ cb^2 S'(b) \Big[-\frac{2S(x)^2}{S(b)^3} + \frac{S(x)}{S(b)^2} \Big] \\ &= \frac{S(x)}{S(b)} \Big\{ \Big[1 - b\frac{S'(b)}{S(b)} \Big] \Big[2cb\frac{S(x)}{S(b)} - cb + 1 \Big] - cb \Big\}, \end{aligned} \tag{5.74}$$

where $S'(b) = e^{\frac{\alpha}{2\beta}b^{-2\beta}}$. From (5.74) we see that it is important for us to know the value of $F'(x)$, which is given by

$$F'(x) = 1 - (cx^2 + x)\frac{S'(x)}{S(x)}. \tag{5.75}$$

We have $F'(0+) = 1$. There are two differential situations. When $F'(x) > 0$, which holds if and only if $(cx^2 + x)S'(x) < S(x)$ for all $x > 0$, we have $b > x$ for all x and

$$0 < \frac{S(x)}{S(b)} < 1$$

where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}}dy$. Then it follows that

$$(\frac{S(x)}{S(b)})^2 - \frac{S(x)}{S(b)} < 0$$

so after starting at 0 and exhibiting a strict increasing initially, we see that $F(b)$ converges to $-\infty$ as b moves to ∞ for all $x > 0$. Thus the function $F(b)$ attains its maximum at some $b > x > 0$. To find the maximum point we need to solve $F'(b) = 0$ which reads

$$\Big[1 - b\frac{S'(b)}{S(b)}\Big]\Big[2cb\frac{S(x)}{S(b)} - cb + 1\Big] = cb \tag{5.76}$$

where $S(b) = \int_0^b e^{\frac{\alpha}{2\beta}y^{-2\beta}}dy$. We know that the function $F(b)$ goes up, and then it goes down to $-\infty$, so that there exists a smallest local maximum point. If we can prove that the function is concave, then this point is also the unique maximum point. We illustrate this by an example. Setting $\alpha = 1/2$, $\beta = -1/8$ and $c = 1/100$. We have $F'(x) > 0$ for all $x > 0$ so $b > x$ as needed. Figure 5.2 shows $F'(x)$.

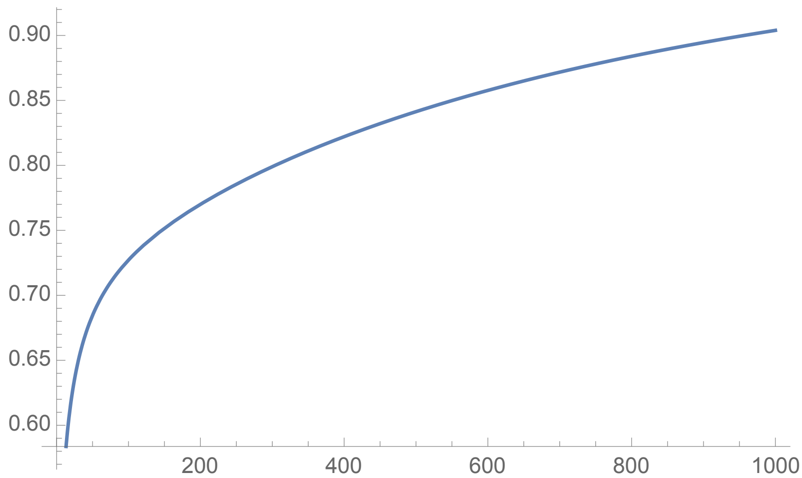

Figure 5.2: A computer drawing of $F'(x)$ when $\alpha = 1/2$, $\beta = -1/8$ and $c = 1/100$.

If we set $x = 1/10$, then

$$F(b) = b\frac{S(1/10)}{S(b)} + \frac{1}{100}b^2\Big[\frac{S(1/10)^2}{S(b)^2} - \frac{S(1/10)}{S(b)}\Big],$$

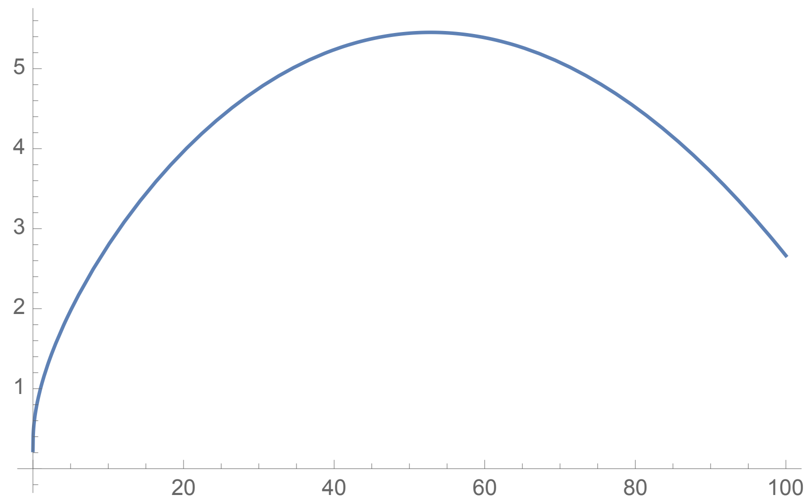

Figure 5.3: A computer drawing of $F(b)$ when $\alpha = 1/2$, $\beta = -1/8$, $c = 1/100$ and $x = 1/10$.

where $S(b) = \int_0^b e^{-2y^{\frac{1}{4}}} dy$.

Combining the concavity of F with the optimality in Lemma 5.2.1, these facts show that the stopping time

$$\tau_* := \inf\{t \geq 0 \mid X_t \geq b(x)\}$$

is optimal in (5.3) where $b(x)$ solves (5.76) for all $x > 0$ (in my example $b(1/10) \approx 52.834$). We can calculate $b(0+) \approx 43.6336$ and Figure 5.4 shows the optimal boundary $b(x)$, it is easily seen that $b(x) > x$ for all $x > 0$ and $b(x)$ is strictly increasing on $(0, \infty)$.

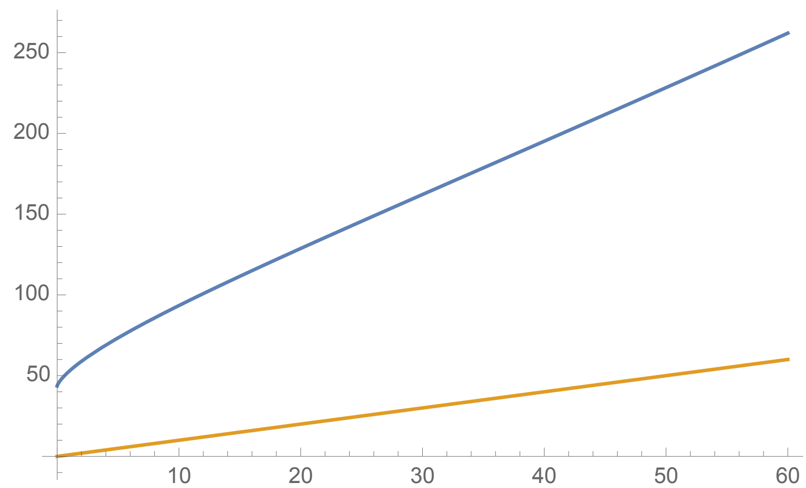

Figure 5.4: A computer drawing of $b(x)$ and x when $\alpha = 1/2$, $\beta = -1/8$ and $c = 1/100$.

The value function is given by

$$V(x) = b(x)\frac{S(x)}{S(b(x))} + cb(x)^2\Big(\frac{S(x)^2}{S(b(x))^2} - \frac{S(x)}{S(b(x))}\Big) \tag{5.77}$$

where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}}dy$ and $b(x)$ solves (5.76) uniquely for all $x > 0$.

Recalling $F'(0+) > 0$, if we can find $x_* > 0$ such that $F'(x_*) = 0$, then $F'(x) > 0$ or equivalently $(cx^2 + x)S'(x) < S(x)$ for $x \in (0, x_*)$. Here $b > x$ for $x \in (0, x_*)$ so we have

$$\Big(\frac{S(x)}{S(b)}\Big)^2 - \frac{S(x)}{S(b)} < 0$$

again and $b \to F(b)$ converges to $-\infty$ as b moves to ∞. Therefore the optimal stopping time is optimal where $b(x)$ solves (5.76) uniquely and we have the value function (5.78) for $x \in (0, x_*)$. For example, setting $\alpha = 1/2$, $\beta = -1/8$ and $c = 1$. We calculate that $x_* \approx 0.44$ where $F'(x_*) = 0$. Figure 5.5 shows $F'(x)$.

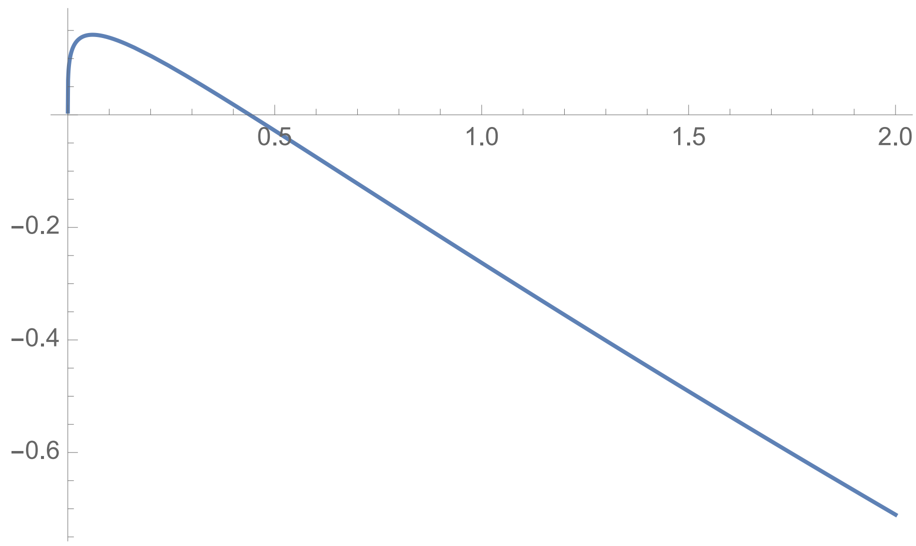

Figure 5.5: A computer drawing of $F'(x)$ when $\alpha = 1/2$, $\beta = -1/8$ and $c = 1$.

Therefore $b > x$ for $x \in (0, x_*)$ and we can find the supremum $F(b)$ for $x \in (0, x_*)$ given and fixed. In our example, if we set $x = 3/10$ then

$$F(b) = b\frac{S(3/10)}{S(b)} + b^2\Big[\frac{S(3/10)^2}{S(b)^2} - \frac{S(3/10)}{S(b)}\Big]$$

and through solving (5.76) we have $b(3/10) \approx 0.37$. Figure 5.6 shows the optimal stopping boundary $b(x)$ and x. It is clear that $b(x) = x$ at $x = x_*$ and

$b(x)$ is strictly increasing. If $x \geq x_*$ or equivalently $(cx^2 + x)S'(x) \geq S(x)$, then $F'(x) \leq 0$ so the final supremum in (5.72) is attained at $b = x$ with $V(x) = x$ for $x \geq b$. This shows that it is optimal to stop at once when $x \geq x_*$.

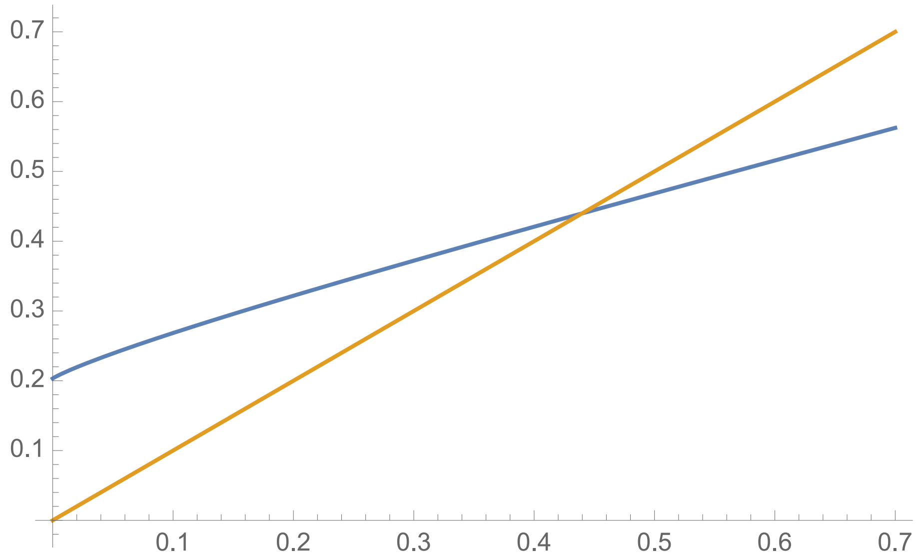

Figure 5.6: A computer drawing of $b(x)$ and x when $\alpha = 1/2$, $\beta = -1/8$ and $c = 1$.

Combining these two situations together we see that $F'(x) > 0$ if and only if $(cx^2 + x)S'(x) < S(x)$ or equivalently $x < b(x)$ where $b(x)$ solves (5.76) uniquely and the value function is given by (5.78). On the other hand $F'(x) \leq 0$ if and only if $(cx^2 + x)S'(x) \leq S(x)$ or equivalently $x \geq b(x)$, we have $V(x) = x$ and it is optimal to stop at once.

We know that $x \to b(x)$ is increasing on $(0, \infty)$. If $F'(x) > 0$ for all $x > 0$ we will always have $b(x) > x$ then it is dynamically optimal not stop at all. If $F'(x) > 0$ for $x \in (0, x_*)$ we see that $b(x) = x$ at $x = x_*$. Therefore $b(x) > x$ for $x \in (0, x_*)$ and $b(x) < x$ for $x > x_*$ with $b(x_*) = x_*$. So the dynamic optimal stopping time is $\tau_* := \inf\{t \geq 0 \mid X_t \geq x_*\}$ where x_* is the unique solution to $(c{x_*}^2 + x_*)S'(x) = S(x_*)$. $\square$

Chapter 6

Constrained Mean-Variance Stopping Problems for the CEV Process

6.1 Problem formulation

Assuming that X follows a constant elasticity of variance (CEV) process, we wish to investigate the constrained problems. Given $m, n > 0$, the constrained optimal stopping problems are

$$V_1(x) = \sup_{\tau:\ \mathbb{V}_x(X_\tau) \leq m} \mathbb{E}_x(X_\tau) \tag{6.1}$$

$$V_2(x) = \inf_{\tau:\ \mathbb{E}_x(X_\tau) \geq n} \mathbb{V}_x(X_\tau) \tag{6.2}$$

for $x > 0$, where τ is a stopping time of X. In Chapter 5, we saw if $\beta = 0$, then X reduces to the geometric Brownian motion and the constrained optimal stopping problems (6.1) and (6.2) have already been solved in [28]. Solving (5.3) we will therefore be able to solve (6.1) and (6.2). In this chapter we will show how to solve the constrained problems (6.1) and (6.2) by using the Theorem 5.2.1. We recall the definitions of the static and dynamic optimality from [28].

Definition 6.1.1 (Static optimality). *A stopping time τ_* is statically optimal in (6.1) for $x > 0$ given and fixed, if $\mathbb{V}_x(X_{\tau_*}) \leq m$ and for every stopping time σ*

satisfying $\mathbb{V}_x(X_\sigma) \leq m$ *we have*

$$\mathbb{E}_x(X_\sigma) \leq \mathbb{E}_x(X_{\tau_*}). \tag{6.3}$$

A stopping time τ_* *is statically optimal in (6.2) for* $x > 0$ *given and fixed, if* $\mathbb{E}_x(X_{\tau_*}) \geq n$ *and for every stopping time* σ *satisfying* $\mathbb{E}_x(X_\sigma) \geq n$ *we have*

$$\mathbb{V}_x(X_\sigma) \geq \mathbb{V}_x(X_{\tau_*}). \tag{6.4}$$

Definition 6.1.2 (Dynamic optimality). *A stopping time* τ_* *is dynamically optimal in (6.1) if there is no other stopping time* σ *such that*

$$\mathbb{P}_x\big(\mathbb{V}_{X_{\tau_*}(X_\sigma)} \leq m\big) = 1 \tag{6.5}$$

and

$$\mathbb{P}_x\big(\mathbb{E}_{X_{\tau_*}}(X_\sigma) > X_{\tau_*}\big) > 0 \tag{6.6}$$

for some $x > 0$.

6.2 Solution to the problem

Theorem 6.2.1. *Assume* X *is a constant elasticity of variance process solves (5.1) with* $X_0 = x$ *under* $\mathbb{P}_x$ *for* $x > 0$. *Consider the constrained optimal stopping problem (6.1). For* $\mu > 0$, $\sigma > 0$ *and* $-\frac{1}{2} < \beta < 0$, *assume that all admissible* b_m *satisfy the inequality*

$$\frac{b_m^{2\beta}}{\alpha} \geq \frac{\frac{1}{2}\alpha b_m\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b_m^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b_m^{-2\beta}) - \alpha b_m\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b_m^{-2\beta}}} \tag{6.7}$$

where $\alpha = \mu/(\sigma^2/2) > 0$ *and* $\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$ *is the lower incomplete Gamma function. Then the stopping time*

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq b_m(x)\} \tag{6.8}$$

is statically optimal for $x > 0$ where $b_m(x)$ is the unique solution to

$$b_m(x)^2 \frac{S(x)}{S(b_m(x))} - b_m(x)^2 \frac{S(x)^2}{S(b_m(x))^2} = m \tag{6.9}$$

where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta} y^{-2\beta}} dy$. The value function in (6.1) is given by

$$V_1(x) = b_m(x) \frac{S(x)}{S(b_m(x))} \tag{6.10}$$

for $x \in (0, b_m(x)]$ and $V_1(x) = x$ for $x \geq b_m(x)$. It is dynamically optimal not to stop at all.

Proof. In the proof of Theorem 5.2.1 we know that in the constrained problem (6.1) we can define the Lagrangian by setting

$$L_x(\tau, c) := \mathbb{E}_x(X_\tau) - c\big[\mathbb{V}_x(X_\tau) - m\big] \tag{6.11}$$

for $x > 0$ and $c > 0$. Here we assume that b satisfies the inequality (5.7). Therefore by the results of Theorem 5.2.1 we know that the stopping time

$$\tau_*^c = \inf\{t \geq 0 \mid X_t \geq b(x)\}$$

is optimal in the unconstrained problem

$$L_x(\tau_*^c, c) := \sup_\tau L_x(\tau, c) \tag{6.12}$$

for $x > 0$ and $c > 0$. Suppose moreover that there exists a $c = c(m, x) > 0$ such that

$$\mathbb{V}_x(X_{\tau_*^c}) = m \tag{6.13}$$

for $x > 0$. Then we have

$$\mathbb{E}_x(X_{\tau_*^c}) = L_x(\tau_*^c, c) \geq \mathbb{E}_x(X_\tau) - c[\mathbb{V}_x(X_\tau) - m] \geq \mathbb{E}_x(X_\tau) \tag{6.14}$$

for all stopping times τ such that $\mathbb{V}_x(X_\tau) \leq m$ with $x > 0$. This shows that the stopping time τ_*^c satisfying (6.13) with $c = c(m, x) > 0$ is statically optimal in (6.1) for $x > 0$.

Because $\mathbb{V}_x(X_{\tau_*^c}) = \mathbb{V}_x(X_{\tau_b}) = \mathbb{E}_x(X_{\tau_b}^2) - (\mathbb{E}_x(X_{\tau_b})^2)$ recall from (5.66) and (5.67) that we have

$$\mathbb{V}_x(X_{\tau_b}) = b^2 \frac{S(x)}{S(b)} - b^2 \frac{S(x)^2}{S(b)^2} \tag{6.15}$$

for $x > 0$ given and fixed, where

$$S(x) = \int_0^x e^{\frac{\alpha}{2\beta} y^{-2\beta}} dy.$$

To meet the condition (6.13) we need to identify (6.15) with m, which yields

$$b^2 \frac{S(x)}{S(b)} - b^2 \frac{S(x)^2}{S(b)^2} = m \tag{6.16}$$

$$b^2 \Big(\frac{S(x)}{S(b)} - \frac{S(x)^2}{S(b)^2} \Big) = m. \tag{6.17}$$

Set

$$F(b) = b^2 \Big(\frac{S(x)}{S(b)} - \frac{S(x)^2}{S(b)^2} \Big) \tag{6.18}$$

for $b \geq 0$ with $x > 0$ given and fixed, where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta} y^{-2\beta}} dy$. We see that if $x \geq b$, then

$$\frac{S(x)}{S(b)} - \frac{S(x)^2}{S(b)^2} \leq 0$$

so there is no such b satisfying $F(b) = m$. On the other hand if $x < b$, then

$$\frac{S(x)}{S(b)} - \frac{S(x)^2}{S(b)^2} > 0$$

and there exists a unique point $b = b_m(x) > 0$ such that $F(b) = m$. Then (6.16) is satisfied and hence (6.13) holds too with τ_c^* from (5.8) where $b(x) = b_m(x)$. The arguments above also show that $b_m(x) > x$ for all $x > 0$. Having that $b_m(x)$ satisfies the inequality (5.7) and solves (5.9) in Theorem 5.2.1, we can conclude that c is determined in (5.9).

We have established above that $b_m(x) > x$ for all $x > 0$. This implies that it is dynamically optimal not to stop at all. This completes the proof.

□

Theorem 6.2.2. *Assume a constant elasticity of variance process X solves (5.1) with $X_0 = x$ under $\mathbb{P}_x$ for $x > 0$. Consider the constrained optimal stopping problem (6.2). If $\mu > 0$, $\sigma > 0$ and $-\frac{1}{2} < \beta < 0$, it is statically optimal to stop at once for $x \geq n$ and $V_2(x) = 0$. For $x \in (0, n)$, we assume that all admissible b_n satisfy the inequality*

$$\frac{b_n^{2\beta}}{\alpha} \geq \frac{\frac{1}{2}\alpha b_n\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b_n^{-2\beta}}}{\gamma(-\frac{1}{2\beta}, -\frac{\alpha}{2\beta}b_n^{-2\beta}) - \alpha b_n\big(-\frac{\alpha}{2\beta}\big)^{-\frac{1}{2\beta}-1} e^{\frac{\alpha}{2\beta}b_n^{-2\beta}}} \tag{6.19}$$

with $\alpha = \mu/(\sigma^2/2) > 0$ and $\gamma(a, x) = \int_0^x e^{-t}t^{a-1}dt$ is the lower incomplete Gamma function. Then the stopping time

$$\tau_* = \inf\{t \geq 0 \mid X_t \geq b_n(x)\} \tag{6.20}$$

is statically optimal for $x > 0$ where $b_n(x)$ is the unique solution to

$$b_n(x)\frac{S(x)}{S(b_n(x))} = n \tag{6.21}$$

where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}}dy$. The value function in (6.2) is given by

$$V_2(x) = nb_n(x) - n^2 \tag{6.22}$$

for $x \in (0, b_n(x)]$.

Proof. If $x \geq n$ then it is optimal to stop at once since we meet the condition $\mathbb{E}_x(X_\tau) \geq n$ and $V_2(x) = 0$. On the other hand, if $x \in (0, n)$ note that in the constrained problem (6.2) we can define the Lagrangian by

$$L_x(\tau, c) := \mathbb{V}_x(X_\tau) - c\big[\mathbb{E}_x(X_\tau) - n\big] \tag{6.23}$$

for $x > 0$ and $c > 0$. To connect to the results of Theorem 5.2.1 we observe that

$$\begin{aligned}\inf_\tau L_x(\tau, c) &= \inf_\tau \big(\mathbb{V}_x(X_\tau) - c[\mathbb{E}_x(X_\tau) - n]\big)\\ &= c\sup_\tau \Big(\mathbb{E}_x(X_\tau) - \frac{1}{c}\mathbb{V}_x(X_\tau)\Big) + cn.\end{aligned} \tag{6.24}$$

If we assume that b satisfies the inequality (5.7) then the stopping time $\tau_*^{1/c} =$

$\inf\{t \geq 0 \mid X_t \geq b(x)\}$ is optimal in the unconstrained problem

$$L_x(\tau_*^{1/c}, c) := \inf_\tau L_x(\tau, c) \tag{6.25}$$

for $x > 0$ and $c > 0$. Moreover, we suppose that there exists $c = c(n, x) > 0$ such that

$$\mathbb{E}_x(X_{\tau_*^{1/c}}) = n \tag{6.26}$$

for $x > 0$. It follows that

$$\mathbb{V}_x(X_{\tau_*^{1/c}}) = L_x(\tau_*^{1/c}, c) \leq \mathbb{V}_x(X_\tau) - c[\mathbb{E}_x(X_\tau) - n] \leq \mathbb{V}_x(X_\tau) \tag{6.27}$$

for all stopping times τ such that $\mathbb{E}_x(X_\tau) \geq n$ with $x > 0$. This shows that the stopping time $\tau_*^{1/c}$ satisfying (6.22) with $c = c(n, x)$ is statically optimal in (6.2) for $x > 0$. To realise (6.22) recall from (5.66) that

$$\mathbb{E}_x(X_{\tau_b}) = b\frac{S(x)}{S(b)} \tag{6.28}$$

where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}} dy$ for $x > 0$ given and fixed. Setting this expression equals to n yields

$$b_n(x)\frac{S(x)}{S(b_n(x))} = n. \tag{6.29}$$

It follows that (6.26) holds with

$$\tau_*^{1/c} = \inf\{t \geq 0 \mid X_t \geq b_n(x)\}.$$

Having that $b_n(x)$ satisfies the inequality (5.7) and solves (5.9) with $1/c$ in place of c, we can conclude that $c = c(n, x)$ is determined by

$$\left[1 - b_n(x)\frac{S'(b_n(x))}{S(b_n(x))}\right]\left[\frac{2}{c}n - \frac{1}{c}b_n(x) + 1\right] = \frac{1}{c}b_n(x). \tag{6.30}$$

where $S(x) = \int_0^x e^{\frac{\alpha}{2\beta}y^{-2\beta}} dy$. Invoking the optimality established in Theorem 5.2.1 we can conclude that the stopping time (6.20) is statically optimal in (6.2) for

$x \in (0, n)$. Recalling from (5.66) and (5.67) we can calculate the value function

$$\begin{aligned} V_2(x) &= \mathbb{V}_x(X_{\tau_b}) = \mathbb{E}_x(X_{\tau_b}^2) - (\mathbb{E}(X_{\tau_b}))^2 \\ &= b_n(x)^2 \frac{S(x)}{S(b_n(x))} - b_n(x)^2 \frac{S(x)^2}{S(b_n(x))^2} \\ &= nb_n(x) - n^2. \end{aligned} \tag{6.31}$$

This completes the proof.

□

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Appendix A

Finding the Lagrange multiplier

A.1

We wish to show how to calculate $c = c(m, x) > 0$ in (4.22), which is uniquely determined. Recalling (4.21), we have

$$2\frac{(1+\frac{1}{\alpha})c^2 + 1 + \sqrt{(1-\frac{1}{\alpha^2})c^2 + 1}}{c^2(1+\frac{1}{\alpha})^2} = (\frac{b_m(x)}{x})^{1-\alpha}. \tag{A.1}$$

Here, we set $A = 1 + \frac{1}{\alpha}$, $B = 1 - \frac{1}{\alpha^2}$, $K = (1 + \frac{1}{\alpha})^2$ and $L = \frac{1}{2}(\frac{b_m(x)}{x})^{1-\alpha}$ and we can rewrite (A.1) as

$$\frac{Ac^2 + 1 + \sqrt{Bc^2 + 1}}{Kc^2} = L. \tag{A.2}$$

Rearranging gives

$$(A - LK)c^2 + \sqrt{Bc^2 + 1} + 1 = 0. \tag{A.3}$$

Setting $y^2 := Bc^2 + 1$ where $y > 0$, we have

$$\frac{A - LK}{B}(y^2 - 1) + y + 1 = 0. \tag{A.4}$$

Because

$$\frac{A - LK}{B} = \frac{1 - (1 + 1/\alpha)L}{1 - 1/\alpha}$$

it follows that

$$y = \frac{\frac{1}{\alpha} - (1 + \frac{1}{\alpha})L}{1 - (1 + \frac{1}{\alpha})L}. \tag{A.5}$$

So we have

$$\begin{aligned} c^2 &= \frac{y^2 - 1}{B} \\ &= \frac{\left(\frac{\frac{1}{\alpha} - (1+\frac{1}{\alpha})L}{1-(1+\frac{1}{\alpha})L}\right)^2 - 1}{1 - \frac{1}{\alpha^2}} \\ &= \frac{2L - 1}{\left(1 - (1 + \frac{1}{\alpha})L\right)^2}. \end{aligned} \tag{A.6}$$

Recalling that

$$L = \frac{1}{2}(\frac{b_m(x)}{x})^{1-\alpha},$$

we get

$$c = c(m, x) = \frac{\sqrt{(\frac{b_m(x)}{x})^{1-\alpha} - 1}}{\frac{1}{2}(1 + \frac{1}{\alpha})(\frac{b_m(x)}{x})^{1-\alpha} - 1}. \tag{A.7}$$

A.2

We wish to show how to calculate $c = c(n, x) > 0$ in (4.40), which is uniquely determined. Recalling (4.39), we have

$$2c^2 \frac{(1 + \frac{1}{\alpha})\frac{1}{c^2} + 1 + \sqrt{(1 - \frac{1}{\alpha^2})\frac{1}{c^2} + 1}}{(1 + \frac{1}{\alpha})^2} = (\frac{b_n(x)}{x})^{1-\alpha}. \tag{A.8}$$

Here we set $A = 1 + \frac{1}{\alpha}$, $B = 1 - \frac{1}{\alpha^2}$, $K = (1 + \frac{1}{\alpha})^2$ and $L = \frac{1}{2}(\frac{b_n(x)}{x})^{1-\alpha}$ and we can rewrite (A.8) as

$$c^2\left(1 + \sqrt{\frac{B}{c^2} + 1}\right) = KL - A. \tag{A.9}$$

Setting $y^2 := \frac{B}{c^2} + 1$ where $y > 0$, we have

$$\frac{B}{y^2 - 1}(1 + y) = KL - A$$

$$y = \frac{B}{KL - A} + 1 = \frac{(1 + \frac{1}{\alpha})L - \frac{1}{\alpha}}{(1 + \frac{1}{\alpha})L - 1}. \tag{A.10}$$

Here $c = \sqrt{\frac{B}{y^2 - 1}}$ combined with (A.10) yields

$$c = \frac{\frac{1}{2}(1 + \frac{1}{\alpha})(\frac{b_n(x)}{x})^{1-\alpha} - 1}{\sqrt{(\frac{b_n(x)}{x})^{1-\alpha} - 1}}. \tag{A.11}$$

Substituting $b_n(x) = n^{1/\alpha} x^{1-1/\alpha}$ we obtain

$$c = \frac{\frac{1}{2}(1 + \frac{1}{\alpha})(\frac{n}{x})^{\frac{1}{\alpha}-1} - 1}{\sqrt{(\frac{n}{x})^{\frac{1}{\alpha}-1} - 1}}. \tag{A.12}$$

Appendix B

Incomplete Gamma Function

Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad (\mathbb{R}z > 0) \tag{B.1}$$

Incomplete Gamma function

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \qquad (\mathbb{R}a > 0) \tag{B.2}$$

Lower incomplete Gamma function

$$\gamma(a, x) = P(a, x)\Gamma(a) = \int_0^x e^{-t} t^{a-1} dt \qquad (\mathbb{R}a > 0) \tag{B.3}$$

Upper incomplete Gamma function

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt \tag{B.4}$$

Differential equation

$$\frac{d\gamma(a, x)}{dx} = -\frac{d\Gamma(a, x)}{dx} = x^{a-1} e^{-x} \tag{B.5}$$