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DOI:
10.1112/jlms.12210

Document Version
Accepted author manuscript

Citation for published version (APA):
Robertson, D., & Chaika, J. (2019). Ergodicity of skew products over linearly recurrent IETs. *Journal of the London Mathematical Society*, *100*(1), 223-248. https://doi.org/10.1112/jlms.12210

Published in:
Journal of the London Mathematical Society

Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Ergodicity of skew products over linearly recurrent IETs

Jon Chaika and Donald Robertson

Abstract

We prove that the skew product over a linearly recurrent interval exchange transformation defined by almost any real-valued, mean-zero linear combination of characteristic functions of intervals is ergodic with respect to Lebesgue measure.

1. *Introduction*

Let T be an ergodic measure-preserving transformation on a probability space $(X, \mathscr{B}, \mu)$. Given a measurable function $f : X \to \mathbb{R}$ one can consider the **skew product** T_f on $X \times \mathbb{R}$ defined by $T_f(x,t) = (Tx, t + f(x))$ for all $x \in X$ and all $t \in \mathbb{R}$. It is immediate that T_f preserves $\mu \otimes \nu$ where ν is Lebesgue measure on $\mathbb{R}$. Atkinson [Atk76] proved that T_f is recurrent with respect to $\mu \otimes \nu$ if and only if f has zero mean and Schmidt [Sch77, Theorem 5.5] proved that T_f is conservative if and only if f has zero mean. It is therefore natural to ask whether T_f is ergodic with respect to $\mu \otimes \nu$.

In this paper we are interested in the situation where T is an interval exchange transformation. We remind the reader that an **interval exchange transformation** is specified by a permutation π of $\{1, \dots, b\}$ for some $b \in \mathbb{N}$ and by positive lengths $\lambda_1, \dots, \lambda_b$ that sum to 1. Given such data one defines a map $T : [0,1) \to [0,1)$ by

$$Tx = x - \sum_{j<i} \lambda_j + \sum_{\pi j < \pi i} \lambda_j$$

for all $x \in I_i$ where $I_i = [\lambda_0 + \cdots + \lambda_{i-1}, \lambda_0 + \cdots + \lambda_i)$ for each $1 \le i \le b$ and $\lambda_0 = 0$. All interval exchange transformations preserve Lebesgue measure on $[0,1)$. A permutation π on $\{1, \dots, b\}$ is **irreducible** if there is not $1 \le k < b$ such that $\pi(\{1, \dots, k\}) = \{1, \dots, k\}$. Throughout, we only consider interval exchange transformations defined by permutations that are irreducible.

For interval exchange transformations with $b = 2$ (i.e. circle rotations) and varying classes of skewing function f the associated skew product T_f was shown to be ergodic by Oren [Ore83], Hellekalek and Larcher [HL86], Pask [Pas90] and Conze and Piękniewska [CP14]. For special IETs on more intervals Conze and Frączek [CF11] proved ergodicity for skew products by certain piecewise linear functions. Negative results also occur, as Frączek and Ulcigrai [FU14] showed typical non-ergodicity for a family of IETs with skewing functions that depend on the intervals of the IETs (which were considered for their relation to certain billiards).

In this paper we prove that, for linearly recurrent interval exchange transformations (an analogue of badly approximable rotations for interval exchange transformations) the skew product T_f is ergodic for almost every step function f with zero mean. (By a **step function** we mean a linear combination of characteristic function of intervals.)

2000 *Mathematics Subject Classification* 37A05 (primary), 37A40 (secondary)..

The first author is supported in part by NSF grants DMS-135500 and DMS-1452762, the Sloan foundation and a Warnock chair. The second author is grateful for the support of the NSF via grants DMS-1246989 and DMS-1703597.

THEOREM 1.1. *Let T be a linearly recurrent interval exchange transformation. For almost every mean-zero step function $f : [0, 1) \to \mathbb{R}$ the skew product T_f is ergodic.*

The terms “linear recurrent” and “almost every” in Theorem 1.1 require some explanation. We first recall the definition of linear recurrence for interval exchange transformations. Let T be an interval exchange transformation and let $\beta_i = \lambda_1 + \cdots + \lambda_i$ for all $1 \le i \le b-1$. Put $D = \{\beta_1, \ldots, \beta_{b-1}\}$. One says that T satisfies the **infinite distinct orbits condition** if $D \cap (T^n)^{-1}D = \varnothing$ for all $n \in \mathbb{N}$. An interval exchange transformation T satisfying the infinite distinct orbits condition is said to be **linearly recurrent** if

$$c_3 = \inf\{n\eta(n) : n \in \mathbb{N}\} > 0$$

where $\eta(n)$ is the length of the smallest interval in the partition $D \cup \cdots \cup (T^n)^{-1}D$ of $[0, 1)$. Linear recurrence implies the following statement: that there are constants $c_1, c_2 > 0$ such that every finite orbit $x, Tx, \ldots, T^{n-1}x$ is c_1/n dense and c_2/n separated.

The condition “almost every” in Theorem 1.1 refers to a particular parameterization of mean-zero step functions we now describe. Every step function $f : [0, 1) \to \mathbb{R}$ with $d > 0$ discontinuities can be written in the form

$$f = y_1 1_{[0,x_1)} + \cdots + y_{d+1} 1_{[x_1+\cdots+x_d,1)} \tag{1.2}$$

for some $y_1, \ldots, y_{d+1}$ in $\mathbb{R}$ with $y_i \neq y_{i+1}$ for all $1 \le i \le d$ and some $x_1, \ldots, x_{d+1}$ in $(0, 1)$ that sum to 1. By the **jumps** of any such f we mean the values $y_2 - y_1, \ldots, y_{d+1} - y_d$. The manifold

$$\mathcal{C}_d = \left\{ (x_1, \ldots, x_{d+1}, y_1, \ldots, y_{d+1}) \in (0, 1)^{d+1} \times \mathbb{R}^{d+1} : \begin{aligned} x_1y_1 + \cdots + x_{d+1}y_{d+1} &= 0 \\ x_1 + \cdots + x_{d+1} &= 1 \\ y_i \neq y_{i+1} \text{ for all } 1 \le i &\le d \end{aligned} \right\}$$

parameterizes all mean-zero step functions $f : [0, 1) \to \mathbb{R}$ with d discontinuities. We equip $\mathcal{C}_d$ with the metric d induced by the ℓ^∞ metric on $\mathbb{R}^{2d+2}$. In Theorem 1.1 and throughout the paper, a statement is true for almost every mean-zero step function f if, for every $d \in \mathbb{N}$ the points in $\mathcal{C}_d$ for which the statement is false is a null set for the Lebesgue measure class on $\mathcal{C}_d$.

Our methods also apply to mean-zero step functions $f : [0, 1) \to \mathbb{Z}$. The necessary modifications are to (i) replace $\mathbb{R}^{d+1}$ with $\mathbb{Z}^{d+1}$ in the definition of $\mathcal{C}_d$ and equip $(0, 1)^{d+1} \times \mathbb{Z}^{d+1}$ with the natural Lebesgue measure; (ii) redefine nudges (cf. Section 4.2 below) to remain $\mathbb{Z}$ valued. As well as being of intrinsic interest, the resulting skew products are related with $\mathbb{Z}$ covers of compact translation surfaces. Given such a cover $p : \tilde{M} \to M$ and a direction θ the first return map on any line segment Λ transverse to θ is an interval exchange transformation T on Λ. The first return dynamics on $p^{-1}(\Lambda)$ is equivalent to a skew-product T_f where $f : \Lambda \to \mathbb{Z}$ is a step function. It follows from [KW04] that, for a zero Lebesgue measure but full Hausdorff dimension set of θ the corresponding interval exchange transformation T is linearly recurrent. Given such a direction θ the induced skew product T_f and in turn the flow on $\tilde{M}$ in the direction θ are both recurrent provided f has mean zero (cf. [HW12]).

J. Chaika is supported in part by NSF grants DMS-135500 and DMS-1452762, the Sloan foundation and a Warnock chair. D. Robertson is grateful for the support of the NSF via grants DMS-1246989 and DMS-1703597. This project (in fact a more ambitious one) began in 2005 as a joint project between the first named author and Pascal Hubert. We thank Pascal Hubert for many helpful conversations.

2. *Proving ergodicity*

Fix a minimal interval exchange transformation T on $[0, 1)$ and a measurable function $f : [0, 1) \to \mathbb{R}$. As with all skew products, T_f is said to be **recurrent** if, for every $B \in \mathscr{B}$ with

$\mu(B) > 0$ and every $\epsilon > 0$ one has

$$\mu(B \cap (T^n)^{-1} B \cap \{x \in [0,1) : T_f^n(x,0) \in [0,1) \times (-\epsilon, \epsilon)\}) > 0$$

for some $n \in \mathbb{Z} \setminus \{0\}$. Atkinson [Atk76] proved that T_f is recurrent if and only if f has zero mean. Since T is minimal it then follows from [Sch77, Theorem 5.5] that if f has zero mean then T_f is conservative. It is therefore reasonable to ask – assuming f has zero mean – whether T_f is ergodic with respect to Lebesgue measure m on $[0,1) \times \mathbb{R}$. To answer this question one considers the $\mathbb{R}$ action V defined on $[0,1) \times \mathbb{R}$ by $V^v(x,t) = (x, t+v)$, which preserves m and commutes with T_f. Write $\mathscr{L}_f$ for the σ-algebra of T_f invariant Borel subsets of $[0,1) \times \mathbb{R}$. As a consequence of [Sch77, Theorem 5.2] and [Sch77, Corollary 5.4] the measure m is ergodic for T_f if and only if the closed subgroup

$$\mathsf{Ess}(f) = \{v \in \mathbb{R} : \mathsf{m}(B \triangle (V^v)^{-1} B) = 0 \text{ for all } B \in \mathscr{L}_f\}$$

is all of $\mathbb{R}$. The members of $\mathsf{Ess}(f)$ are the **essential values** of f. Our main result is therefore a consequence of the following theorem.

THEOREM 2.1. *Let T be a linearly recurrent interval exchange transformation. For almost every mean-zero step function $f : [0,1) \to \mathbb{R}$ each of its jumps is an essential value of T_f.*

Proof Proof of Theorem 1.1 assuming Theorem 2.1. For almost every mean-zero step function f its jumps generate a dense subgroup of $\mathbb{R}$. Therefore $\mathsf{Ess}(f)$ is dense in $\mathbb{R}$. □

Other works, for example [CP14], also prove that the size of the jumps of the step function are essential values. They consider step function skew products over rotations R of the circle, for which one can always find infinitely many times q_n such that

- (Denjoy-Koksma) $\left| \sum_{i=0}^{q_n - 1} f(R^i x) \right| \leq \mathsf{Var}(f)$
- $\lim_{n \to \infty} d(R^{q_n} x, x) = 0$

both hold for all x. One then seeks to show there are pairs of level sets of $\sum_{i=0}^{q_n-1} f(R^i x)$ of definite measure where the values of $\sum_{i=0}^{q_n-1} f(R^i x)$ differ by the size of particular jump discontinuities of f. In short, one obtains invariance by looking at sets of definite measure at particular times. We do not suspect that something like the Denjoy-Koksma inequality holds in our context. As a substitute, we show that the size of the jumps of the skewing function are essential values by following a pair of nearby points whose values under the skew differ by the size of a jump discontinuity of f for a defnite proportion of an orbit segment. This approach is outlined below and carried out in Section 3. Such arguments go back at least to Ratner [Rat83].

In order to prove Theorem 2.1 we study for some $B > 0$ the transformation $S_{f,B}$ induced by T_f on the space $X_B := [0,1) \times [-B, B]$. This is defined almost everywhere because T_f is recurrent. Normalized Lebesgue measure m_B on X_B is $S_{f,B}$ invariant, so m_B almost every point (x,t) is generic for an ergodic $S_{f,B}$ invariant measure $\mu_{f,B,(x,t)}$ on X_B. The following theorem (proved in Appendix A) relates vertical invariance of the measures $\mu_{f,B,(x,t)}$ with the essential values of T_f.

THEOREM 2.2. *Let T be an ergodic interval exchange transformation. Fix $B > 0$. Suppose that there is $b \in \mathbb{R}$ such that a m_B positive measure set of (x,t) in X_B is generic for an $S_{f,B}$ invariant probability measure $\mu_{f,B,(x,t)}$ that is not singular with respect to $V^b \mu_{f,B,(x,t)}$. Then every T_f invariant set is V^b invariant.*

We now describe how Theorem 2.2 will be used to prove Theorem 2.1. Fix an interval exchange transformation T. Given a pair $(x,t) \in [0,1) \times \mathbb{R}$, a mean-zero step function $f : [0,1) \to \mathbb{R}$ and $B > 0$, say that (x,t) and f are **right friends** at B if there are constants $\beta > 0$, $\delta > 0$ such that, for every discontinuity p of f there is $K \subset \mathbb{N}$ infinite such that all of the following properties hold.

F1. For all $k \in K$ and all $0 \le i < 2^k$ the transformation T^i is continuous on $[x, x + \frac{2\delta}{2^k}]$.

F2. For all $k \in K$ the family $\{T^i[x, x + \frac{2\delta}{2^k}] : 0 \le i < 2^k\}$ of intervals is pairwise disjoint.

F3. There is $0 \le i < 2^{k-1}$ with $p \in T^i[x, x + \frac{\delta}{2^k}]$.

F4. No other discontinuity of f belongs to $\displaystyle\bigcup_{i=0}^{2^k-1} T^i[x, x + \frac{2\delta}{2^k}]$.

F5. One has

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B,B]} \left(t + \sum_{i=0}^{n-1} f(T^i x) \right) \ge \beta \sum_{n=0}^{2^k-1} 1_{[-B,B]} \left(t + \sum_{i=0}^{n-1} f(T^i x) \right) \tag{2.3}$$

for all $k \in K$.

We say that (x,t) and f are **left friends** at B if the above is true with all occurrences of $[x, x + \frac{2\delta}{2^k}]$ and $[x, x + \frac{\delta}{2^k}]$ replaced by $[x - \frac{2\delta}{2^k}, x]$ and $[x - \frac{\delta}{2^k}, x]$ respectively. Declare (x,t) and f to be **friends** at B if they are either left friends at B or right friends at B. Theorem 2.2 is now a consequence of the following results.

Theorem 2.4. *Let T be an ergodic interval exchange transformation and let f be a mean-zero step function whose jump discontinuities generate a dense subgroup of $\mathbb{R}$. If*

$$\mathsf{m}_B(\{(x,t) \in X_B : f \text{ and } (x,t) \text{ are friends at } B\}) \to 1$$

as $B \to \infty$ then T_f is ergodic with respect to Lebesgue measure on $[0,1) \times \mathbb{R}$.

We prove this in Section 3, by showing that for all large enough b the measure $\mu_{f,B,(x,t)}$ is not singular with respect to $V^b \mu_{f,B,(x,t)}$ for a positive measure set of (x,t). By Theorem 2.2 this implies Theorem 2.4.

The next theorem guarantees that the hypothesis of Theorem 2.4 are satisfied when T is a linearly recurrent interval exchange transformation - together with Theorem 2.4 it concludes the proof of Theorem 1.1.

Theorem 2.5. *Fix a linearly recurrent interval exchange transformation T. For almost every mean-zero step function f we have*

$$\mathsf{m}_B(\{(x,t) \in X_B : f \text{ and } (x,t) \text{ are friends at } B\}) \to 1 \tag{2.6}$$

as $B \to \infty$.

The proof of Theorem 2.5 is based on a density points argument with the following steps. The details are given in Section 4.

1. (Section 4.1) Properties **F1** and **F2** always hold either on the left or the right.
2. (Section 4.2) For any x, f and k we can perturb f to satisfy **F3** and **F4**.
3. (Section 4.3) For all f and almost every (x,t) (with $|t| < B$) there exists infinitely many k satisfying **F5**.
4. (Section 4.4) Small perturbations in f and x preserve **F5**.
5. (Section 4.5) The preceding steps imply Theorem 2.5.

We mention here the following questions, in which we are very interested.

QUESTION 2.7. Is Theorem 2.5 true with the assumption of linear recurrence weakened to unique ergodicity (or maybe even just minimality)?

QUESTION 2.8. Let $f = 1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1)}$. Is T_f ergodic as a $\mathbb{Z}$-valued skew product for almost every T?

QUESTION 2.9. Let $f(x) = \cos(2\pi x)$. Is T_f ergodic for almost every interval exchange transformation on at least three intervals?

We conclude this section with some preparatory remarks that will be used implicitly in the proofs of the above results. Firstly, given $d \in \mathbb{N}$ and $D \in \mathbb{N}$ write $\mathcal{C}_{d,D}$ for the set of points in $\mathcal{C}_d$ with $|y_i| \le D$ for all $1 \le i \le d+1$.

LEMMA 2.10. *The map*

$$\begin{aligned} \mathcal{C}_{d,D} \times [0,1) \times \mathbb{R} &\to [0,1) \times \mathbb{R} \\ (f,(x,t)) &\mapsto T_f(x,t) \end{aligned} \tag{2.11}$$

is measurable, and

$$(f,(x,t)) \mapsto \phi(T_f^n(x,t))$$

is measurable for every n in $\mathbb{Z}$ and every measurable function ϕ on $[0,1) \times \mathbb{R}$.

Proof. Writing $x_0 = 0$, the map

$$\begin{aligned} [0,1)^{d+1} \times \mathbb{R}^{d+1} \times [0,1) \times \mathbb{R} &\to [0,1) \times \mathbb{R} \\ (x_1,\dots,x_{d+1},y_1,\dots,y_{d+1},x,t) &\mapsto \left(Tx, t + \sum_{i=0}^{d} y_i \cdot 1_{[x_0+\cdots+x_i, x_0+\cdots+x_{i+1})}(x) \right) \end{aligned}$$

is measurable and (2.11) is simply its restriction to $\mathcal{C}_{d,D} \times [0,1) \times \mathbb{R}$. □

Secondly, define

$$G_2(B,K) = \bigcup_{\beta>0} \bigcap_{k\in K} \{(f,(x,t)) \in \mathcal{C}_d \times X_B : (2.3) \text{ holds}\}$$

for all $B > 0$ and all $K \subset \mathbb{N}$. Define also

$$G_3(B) = \{(f,(x,t)) \in \mathcal{C}_d \times X_B : (x,t) \text{ is generic for an } S_{f,B} \text{ invariant probability measure on } X_B\}$$

for all $B > 0$. Measurability of the sets $G_2(B,K)$ and $G_3(B)$ follows from Lemma 2.10.

Lastly, for each $B > 0$ and each $\epsilon > 0$ note that, by Egoroff's theorem, there is set $G_4(B,\epsilon) \subset X_B$ with m_B-measure at least $1-\epsilon$ on which the sequence

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{S^n_{f,B}(x,t)}$$

of measures converges uniformly to $\mu_{f,B,(x,t)}$.

3. *Proof of Theorem 2.4*

This section is a modification of a now standard argument that goes back at least to Ratner [Rat83]. Fix an interval exchange transformation T and a mean-zero step function

$f : [0,1) \to \mathbb{R}$ with coordinates $(x_1, \ldots, x_{d+1}, y_1, \ldots, y_{d+1})$. Assume the jump discontinuities $y_{i+1} - y_i$ of f generate a dense subgroup of $\mathbb{R}$. Fix $\epsilon > 0$. Choose $B > 2\max\{y_1, \ldots, y_{d+1}\}$ so large that

$$\mathsf{m}_B(\{(x,t) \in X_B : f \text{ and } (x,t) \text{ are friends at } B\}) \geq 1 - \epsilon$$

holds. Write Φ_B for the m_B almost-surely defined map $\Phi_B(x,t) = \mu_{f,B,(x,t)}$ and let L be a compact subset of X_B with measure at least $1-\epsilon$ on which Φ_B is continuous. Let b be one of the jump discontinuities of f. Thus $b = y_{i+1} - y_i$ in accordance with (1.2) for some $0 \leq i \leq d$.

PROPOSITION 3.1. *If*

- (x,t) *and* f *are friends at* B,
- $t \in [-B+b, B-b]$,
- $(x,t) \in G_4(B+b, \epsilon)$ *and*
- (x,t) *is a density point of* $L \cap G_4(B,\epsilon) \cap \Big([0,1) \times \{t\}\Big)$

then $V^{-b}\Phi_B(x,t)$ *is not singular with respect to* $\Phi_B(x,t)$.

The following lemmas will be used to prove Proposition 3.1. Write

$$N((x,t),m) = \sum_{i=0}^{m-1} 1_{[0,1)\times[-B,B]}(T_f^i(x,t))$$

for all $m \in \mathbb{N}$.

LEMMA 3.2. *Under the assumptions of the proposition, there exists* $\beta' > 0$ *depending only on* (x,t), *a sequence* x_i *converging to* x *and* $k_i \in \mathbb{N}$ *so that*

- $(x_i, t) \in L \cap G_4(B,\epsilon)$
- *there exists* $U_i = \{n_1, \ldots, n_r\} \subset \{0, \ldots, N((x,t), 2^{k_i})\}$ *with cardinality at least* $\beta' N((x,t), 2^{k_i - 1})$
- *there exists* $U_i' = \{n_1, \ldots, n_r\} \subset \{0, \ldots, N((x_i,t), 2^{k_i})\}$ *with density at least* $\beta' N((x_i,t), 2^{k_i-1})$
- *there exists a bijection* $\psi_i : U_i \to U_i'$

so that if x *and* f *are left friends at* B *then*

$$d_{[0,1)\times\mathbb{R}}(S^j_{f,B}(x,t), V^b S^{\psi_i(j)}_{f,B}(x_i,t)) = |x_i - x|$$

for all $j \in U_i$ *and if* x *and* f *are right friends at* B *then*

$$d_{[0,1)\times\mathbb{R}}(S^j_{f,B}(x,t), V^{-b} S^{\psi_i(j)}_{f,B}(x_i,t)) = |x_i - x|$$

for all $j \in U_i$.

Proof. We assume (x,t) and f are right friends at B; the proof in the case that they are left friends at B is similar and omitted. By **F3** we may choose $x_i \in [x, x + \frac{2\delta}{2^{k_i}}]$ so that there is a discontinuity between $T^{\ell_i}(x)$ and $T^{\ell_i}(x_i)$ for some $0 \leq \ell_i < 2^{k_i - 1}$. Since a positive proportion of points in $[x, x+\frac{2\delta}{2^{k_i}}]$ have this property (because $[x + \frac{\delta}{2^{k_i}}, x + \frac{2\delta}{2^{k_i}}]$ is a definite proportion of $[x, x + \frac{2\delta}{2^{k_i}}]$) and we are assuming that (x,t) is a density point for $L \cap G_4(B,\epsilon) \cap ([0,1) \times \{t\})$, for all large enough i we may assume that $(x_i, t) \in L \cap G_4(B,\epsilon)$ as well. Let

$$\hat{U}_i = \{0 \leq j \leq 2^k : j > \ell_i \text{ and } T_f^j(x,t) \in [-B+b, B-b]\}$$

and define U_i to be the set of j' for which $S^{j'}_{f,B}(x,t) = T^j_f(x,t)$ for some $j \in \hat{U}_i$. Define U_i' to be the set of j'' so that $S^{j''}_{f,B}(x_i,t) = T^j(x_i,t)$ for some $j \in \hat{U}_i$. Let ψ_i be the map that sends j' to j''.

The lemma is completed if we establish the cardinality of U_i'.

Claim 1. *If $(x,t) \in G_4(B+b)$ and $t \in [-B+b, B-b]$, then there exists β' so that for all large enough i we have that $N((x_i,t),2^{k_i}) > (1+\beta')N((x_i,t),2^{k_i-1})$.*

Proof. First note that

$$N((x_i,t),2^{k_i}) \leq \sum_{j=0}^{2^{k_i}-1} 1_{[0,1)\times[-B-b,B+b]}(T_f^j(x,t))$$

holds. We have $(x,s) \in G_4(B+b)$ so **F5** gives $C > 0$ such that

$$\sum_{j=0}^{2^{k_i}-1} 1_{[0,1)\times[-B-b,B+b]}(T_f^j(x,t)) < C \sum_{j=2^{k_i-1}}^{2^{k_i}-1} 1_{[0,1)\times[-B+b,B-b]}(T_f^j(x,t)) \leq |U_i| = |U_i'|$$

for all large enough i. □

This concludes the proof of the lemma. □

Corollary 3.3. *Under the assumptions of Proposition 3.1 there exists $c > 0$ so that*

$$\int g \circ V^{-b} \, \mathrm{d}\Phi_B(x,t) > c \int g \, \mathrm{d}\Phi_B(x,t). \tag{3.4}$$

for all positive $g \in \mathsf{C_c}(X_{B-2b})$. The constant $c > 0$ does not depend on g (though it is allowed to depend on (x,t)).

Proof. As in the proof of Lemma 3.2, $(x_i,t) \in L \cap G_4(B,\epsilon)$ and that $(x,t) \in K \cap G_4(B)$, for any $\epsilon > 0$ and $g \in \mathsf{C}_c(X_{B+2b})$ there exists N so that for all large enough i we have that

$$\left| \frac{1}{N} \sum_{j=0}^{N-1} (g \circ V^{-b})(S_{f,B}^j(x_i,t)) - \int g \circ V^{-b} \, \mathrm{d}\Phi_B(x_i,t) \right| < \epsilon$$

holds. Further we may assume that for any such N we have

$$\left| \frac{1}{N} \sum_{j=0}^{N-1} (g \circ V^{-b})(S_{f,B}^j(x,t)) - \int g \circ V^{-b} \, \mathrm{d}\Phi_B(x,t) \right| < \epsilon$$

as well. Any such g is uniformly continuous and so by the assumption of the previous lemma we have for any $\delta > 0$ that, if i is large enough, then

$$|g(V^b S_f^{\psi_i(j)}(x_i,t)) - g(S_f^j(x,t))| < \delta$$

for all $j \in U_i$. Since $\psi(U_i) = U_i'$ is at least a β' proportion of $\{0,\dots,N((x_i,t),2^{k_i})\}$ and contains every j so that $S_{f,B}^j(x_i,t) \in \mathsf{supp}(g \circ V^{-b})$, it follows that

$$\left| \sum_{j \in U_i'} g \circ V^{-b} S_{f,B}^j f(x_i,t) - \int g \circ V^{-b} \, \mathrm{d}\Phi_B(x_i,t) \right| <$$

$$2\beta'^{-1} \left(\left| \frac{1}{N((x_i,t),2^{k_i})} \sum_{j=0}^{N((x_i,t),2^{k_i})-1} g \circ V^{-b} S_{f,B}^j(x_i,t) - \int g \circ V^{-b} \, \mathrm{d}\Phi_B(x_i,t) \right| - \right.$$

$$\left. \left| \frac{1}{N((x_i,t),2^{k_i-1})} \sum_{j=0}^{N((x_i,t),2^{k_i-1})-1} g \circ V^{-b} S_{f,B}^j(x_i,t) - \int g \circ V^{-b} \, \mathrm{d}\Phi_B(x_i,t) \right| \right)$$

holds. Provided ϵ is small enough and i is so large that $N < N((x_i,t),2^{k_i-1})$ we have the corollary. □

LEMMA 3.5. *If $\Phi_B(x,t)$ and $V^b\Phi_B(x,t)$ are mutually singular, then for all $\epsilon > 0$ there exists positive $g \in \mathsf{C_c}(X_{B-2b})$ so that*

$$\int g \circ V^b \,\mathrm{d}\Phi_B(x,t) < \epsilon \int g \,\mathrm{d}\Phi_B(x,t) \tag{3.6}$$

holds.

Proof. If μ_1 and μ_2 are mutually singular Borel probability measures on X_B then there exist disjoint compact sets K_1, K_2 so that $\mu_i(K_i) > 1-\epsilon$ and $\mu_i(K_{1-i}) = 0$. There also exists open sets U_i so that $K_i \subset U_i$ and $\mu_i(U_{1-i}) < \epsilon$. We may choose a continuous, positive function $0 \le g \le 1$ supported on U_1 with $g|_{K_1} = 1$. It is straightforward that $\int g \,\mathrm{d}\mu_1 \ge \mu_1(K_1) \ge 1-\epsilon$ and $\int g\,\mathrm{d}\mu_2 \le \mu_2(U_1) \le \epsilon$. The lemma follows from this. □

Proof Proof of Proposition 3.1. In view of (3.6) inequality (3.4) establishes that these measures can not be mutually singular. □

4. *Proof of Theorem 2.5*

Fix throughout this section a linearly recurrent interval exchange transformation T and attendant constants c_1, c_2 and c_3. Fix $\delta = c_2/4$.

4.1. *Conditions **F1** and **F2** always hold either on the left or the right*

LEMMA 4.1. *For all $x \in [0,1)$ and $n \in \mathbb{N}$ at least one of the the following two possibilities hold.*
- *$\{T^i[x, x+\frac{2\delta}{n}] : 0 \le i < n\}$ consists of n disjoint intervals.*
- *$\{T^i[x-\frac{2\delta}{n}, x] : 0 \le i < n\}$ consists of n disjoint intervals.*

The fact that these are disjoint follows from the next result of Boshernitzan. (Indeed if T^iA are disjoint sets for $p \le i \le q$ then T^jA are disjoint sets for $0 \le j \le q-p$.)

LEMMA 4.2 [Bos88, Lemma 4.4]. *If T satisfies the Keane condition and the distance between any discontinuities of T^{n+1} is s then for any interval J with measure at most s there exist integers $p \le 0 \le q$ (which depend on J) such that*
(1) $q - p \ge n$
(2) T^i acts continuously on J for $p \le i \le q$
(3) $T^i(J) \cap T^j(J) = \varnothing$ for $p \le i < j \le q$.

Proof Proof of Lemma 4.1. Linear recurrence implies the discontinuities of T^n are $\frac{c_2}{n}$ separated. Writing $\beta_1,\dots,\beta_r$ for the discontinuities of T, it follows that $T^i[\beta_j, \beta_j+\frac{c_2}{n}) \cap \{\beta_1,\dots,\beta_r\} = \varnothing$ and $T^i[\beta_j-\frac{c_2}{n}, \beta_j) \cap \{\beta_1,\dots,\beta_r\} = \varnothing$ for all $1 \le i \le n$ and all $1 \le j \le r$. Now consider $T^i(x-\frac{c_2}{n}, x+\frac{c_2}{n})$. If it is not an interval then some discontinuity β of T belongs to $T^j(x-\frac{c_2}{n}, x+\frac{c_2}{n})$ for some $0 \le j < i$. If $\beta \in [T^jx - \frac{c_2}{2n}, T^jx)$ then by above we have that $T^i[x, x+\frac{c_2}{2n}) \cap \{\beta_1,\dots,\beta_r\} = \varnothing$ for all $0 \le i < n$. Similarly, if $\beta \in [T^jx, T^jx+\frac{c_2}{2n})$ we have the other possibility. □

4.2. *Conditions **F3** and **F4** are obtainable by nudging*

Fix f in $\mathcal{C}_{d,D}$ and $x \in [0,1)$. Let $(x_1, \dots, x_{d+1}, y_1, \dots, y_{d+1})$ be the coordinates of f as in (1.2). Put $\xi = \min\{x_1, \dots, x_{d+1}\}$. We verify in this subsection that, by perturbing f, we can assume conditions **F3** and **F4** are true.

We wish to choose $g \in \mathcal{C}_d$ close enough to f such that **F3** and **F4** hold. We construct g by nudging the locations of the discontinuities of f. Specifically, if we wish to move the location of a discontinuity of f to the left or to the right we adjust the values taken by f on the interval to the right of the discontinuity in such a way that the resulting step function still has zero mean. Explicitly, to nudge f by moving its ith discontinuity from $x_1 + \cdots + x_i$ to $x_1 + \cdots + x_i + \zeta$ we replace f with the step function $\mathsf{nudge}(f, i, \zeta)$ having coordinates

$$\left(x_1, \dots, x_i + \zeta, x_{i+1} - \zeta, \dots, x_{d+1}, y_1, \dots, y_i, \frac{y_{i+1}x_{i+1} - \zeta y_i}{x_{i+1} - d}, \dots, y_{d+1} \right)$$

which makes sense provided $|\zeta| < \frac{\xi}{2}$. Recall that d denotes the ℓ^∞ metric on the data $(x_1, \dots, y_{d+1})$.

LEMMA 4.3. *If* $|\zeta| < \frac{\xi}{2}$ *then* $\mathsf{d}(f, \mathsf{nudge}(f, i, \zeta)) < \max\{|\zeta|, 8|\zeta|D/3\xi\}$.

Proof. The values of the skewing function have changed by

$$\left| \frac{y_{i+1}x_{i+1} - \zeta y_i}{x_{i+1} - \zeta} - y_{i+1} \right| = \left| \frac{\zeta y_{i+1} - \zeta y_i}{x_{i+1} - \zeta} \right| \le \frac{|\zeta| 2D}{|x_{i+1} - \zeta|} \le \frac{|\zeta| 8D}{3x_{i+1}} \le \frac{|\zeta| 8D}{3\xi} \tag{4.4}$$

in carrying out the nudge. □

PROPOSITION 4.5. *Fix* f *in* $\mathcal{C}_{d,D}$, $1 \le i \le d$ *and* $x \in [0,1)$. *For every* k *in* $\mathbb{N}$ *with*

$$\frac{c_1 + \delta}{2^{k-1}} < \frac{\xi}{4} \tag{4.6}$$

there is g *in* $\mathcal{C}_d$ *with*

$$\mathsf{d}(f, g) \le \frac{2c_1 + 3\delta}{2^k}(d+1) \max\left\{1, \frac{8D}{3\xi}\right\} \tag{4.7}$$

such that $\mathsf{d}(g, h) < \frac{\delta}{3 \cdot 2^k}$ *implies* x *and* h *satisfy **F1** through **F4** on the left or on the right.*

Proof. Fix $k \in \mathbb{N}$ satisfying (4.6). Assume that the first possibility in Lemma 4.1 is true for $n = 2^k$. (The alternative is treated similarly.) There are two cases to consider, according to whether there is a time $0 \le \ell < 2^{k-1}$ at which $T^\ell[x, x + \frac{\delta}{2^k}]$ contains the ith discontinuity of f. Note that our assumption on k guarantees that each such interval contains at most one discontinuity of f.

Case 1: There is such an ℓ. Nudge the discontinuity in $T^\ell[x, x + \frac{\delta}{2^k}]$ by at most $\frac{\delta}{2^k}$ so that it lies in $T^\ell[x + \frac{\delta}{3 \cdot 2^k}, x + \frac{2\delta}{3 \cdot 2^k}]$. For each $0 \le j < 2^k$ with $j \ne \ell$ and $T^j[x, x + \frac{3\delta}{2^k}]$ containing a discontinuity of f we nudge the discontinuity of f by at most $\frac{3\delta}{2^k}$ so that it lies in $T^j[x + \frac{3\delta}{2^k}, x + \frac{4\delta}{2^k}]$. For the resulting function g we have

$$\mathsf{d}(f, g) \le \frac{3\delta}{2^k}(d+1) \max\left\{1, \frac{8D}{3\xi}\right\} \tag{4.8}$$

by Lemma 4.3.

Case 2: There is no such ℓ. For each $0 \le j < 2^k$ with $T^j[x, x + \frac{3\delta}{2^k}]$ containing a discontinuity of f we nudge the discontinuity of f by at most $\frac{3\delta}{2^k}$ so that it lies in $T^j[x + \frac{3\delta}{2^k}, x + \frac{4\delta}{3 \cdot 2^k}]$. By linear recurrence $\{T^j x : 0 \le j < 2^{k-1}\}$ is within a distance of at most $\frac{c_1}{2^{k-1}}$ from the ith discontinuity of f. We nudge it by at most $\frac{c_1}{2^{k-1}} + \frac{\delta}{2^k}$ to lie within some $T^j[x + \frac{\delta}{3 \cdot 2^k}, x + \frac{2\delta}{3 \cdot 2^k}]$ with $0 \le j < 2^{k-1}$. For the resulting function g we have

$$\mathsf{d}(f,g) \le \frac{3\delta + 2c_1}{2^k}(d+1)\max\left\{1, \frac{8D}{3\xi}\right\}$$

by Lemma 4.3.

In both cases we have constructed a function g with

$$\mathsf{d}(f,g) \le \frac{3\delta + 2c_1}{2^k}(d+1)\max\left\{1, \frac{8D}{3\xi}\right\}$$

and the following properties:
- there is only one discontinuity of g in $\cup\{T^j[x, x + \frac{\delta}{2^k}] : 0 \le i < 2^k\}$;
- there is $0 \le j < 2^{k-1}$ such that $T^j[x, x + \frac{\delta}{2^k}]$ contains a discontinuity of g.

Moreover, by the construction every h in $\mathcal{C}_d$ with $\mathsf{d}(g,h) < \frac{\delta}{3 \cdot 2^k}$ also satisfies the above properties. Thus x and any such h satisfy **F1** through **F4** on the right (for this k). □

4.3. *Condition **F5** holds almost surely*

Using the material of Appendix B we prove the following theorem.

THEOREM 4.9. *Fix $f \in \mathcal{C}_d$. For every $B > 0$ and every $t \in (-B, B)$ there is $\beta > 0$ such that almost every x satisfies*

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right) \ge \beta \sum_{n=0}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$

for infinitely many k.

The first step is to prove the following quantitative version of Atkinson's theorem.

THEOREM 4.10. *Let T be an aperiodic measure-preserving transformation on a probability space $(X, \mathscr{B}, \mu)$ and let $f : X \to \mathbb{R}$ in $\mathrm{L}^1(X, \mathscr{B}, \mu)$ have zero mean. Further assume that there exists $0 < \gamma < 1$ and $N_0 \in \mathbb{N}$ such that*

$$\left|\sum_{n=0}^{N-1} f(T^n x)\right| < N^\gamma \tag{4.11}$$

for all $x \in X$ and all $N \ge N_0$. Then for every $\epsilon > 0$ and almost every $x \in X$ we have

$$\sum_{n=0}^{N-1} 1_{(-\epsilon,\epsilon)}\left(\sum_{i=0}^{n} f(T^i x)\right) > N^{1-\gamma-\epsilon}$$

for arbitrarily large N.

Proof. First, we claim that it suffices to prove for every $\eta > 0$ that there is arbitrarily large $N \in \mathbb{N}$ with

$$\left|\left\{1 \le L \le N : \sum_{m=0}^{N} 1_{(-\epsilon,\epsilon)}\left(\sum_{n=0}^{m} f(T^i T^L x)\right) > N^{1-\gamma-\epsilon}\right\}\right| > (1-\eta)N \tag{4.12}$$

for all x. Denote the subset of $\{1,\dots,N\}$ at left by H_x (with the dependence on N implicit). Let

$$E_N = \left\{ x \in [0,1) : \sum_{n=0}^{N-1} 1_{(-\epsilon,\epsilon)} \left(\sum_{i=0}^{n} f(T^i x) \right) > N^{1-\gamma-\epsilon} \right\}$$

and notice that if $j \in H_x$ then $T^j x \in E_N$. Now, if we have (4.12) then

$$\int_X \sum_{i=1}^{N} 1_{E_N}(T^i x)\,\mathrm{d}\mu \geq (1-\eta)N$$

and so $\mu(E_N) \geq \frac{(1-\eta)N}{N}$.

Now we prove (4.12). Fix $\eta > 0$. If N is large enough then $f(x) + \cdots + f(T^{N-1}x)$ belongs to $[-N^\gamma, N^\gamma]$ for all $1 \leq L \leq N$ and all $x \in X$. Fix an interval $J \subset [-N^\gamma, N^\gamma]$ of length ϵ. Let $0 \leq L_1 < \cdots < L_s \leq N$ be an enumeration of those $1 \leq L \leq N$ at which $f(x) + \cdots + f(T^{L-1}x)$ belongs to J. We have

$$\epsilon > \left| \sum_{n=0}^{L_j-1} f(T^n x) - \sum_{n=0}^{L_i-1} f(T^n x) \right| = \left| \sum_{n=0}^{L_j-L_i-1} f(T^n T^{L_i} x) \right|$$

for all $0 \leq i < j \leq s$. Therefore L_i belongs to H_x whenever $s - i \geq N^{1-\gamma-\epsilon}$ holds. Since we can cover $[-N^\gamma, N^\gamma]$ by at most $\lceil 2N^\gamma\epsilon^{-1} \rceil$ intervals of length ϵ, it follows that at most $\lceil 2N^\gamma\epsilon^{-1} \rceil N^{1-\gamma-\epsilon}$ of the $1 \leq L \leq N$ do not belong to H_x. For N large enough (and independent of x) we will therefore have $|H_x| > (1-\eta)N$. □

LEMMA 4.13. *Fix $0 < \alpha < 1$ and suppose $g : \mathbb{N} \to \mathbb{N}$ is non-decreasing and satisfies $n^\alpha \leq g(n) \leq n$ for infinitely many $n \in \mathbb{N}$. Then for every $0 < \beta < 1 - 2^{-\alpha}$ one has $g(2^k) - g(2^{k-1}) \geq \beta g(2^k)$ infinitely often.*

Proof. Fix $0 < \beta < 1 - 2^{-\alpha}$. Suppose the conclusion is false. Then there is $K \in \mathbb{N}$ such that $(1-\beta)g(2^k) < g(2^{k-1})$ for all $k \geq K$. Thus $(1-\beta)^l g(2^{k+l}) < g(2^k)$ for all $l \in \mathbb{N}$ and all $k \geq K$ by induction. Write n_j for the increasing sequence of times $n \geq 2^K$ at which $n^\alpha \leq g(n) \leq n$ holds. For each j fix $l_j \in \mathbb{N} \cup \{0\}$ with $2^{K+l_j} \leq n_j < 2^{K+l_j+1}$. We have

$$(1-\beta)^{-(1+\ell_j)} g(2^K) \geq g(2^{K+l_j+1}) \geq g(n_j) \geq n_j^\alpha \geq 2^{(K+l_j)\alpha}$$

for all $j \in \mathbb{N}$. But then

$$\Big((1-\beta)2^\alpha \Big)^{l_j} \leq \frac{2^K}{(1-\beta)2^{K\alpha}}$$

for all $j \in \mathbb{N}$. Taking j large enough gives the desired contradiction because $(1-\beta)2^\alpha > 1$ and so the left hand side goes to infinity with j while the right hand side is independent of j. □

Proof Proof of Theorem 4.9. Let B and t be given with $-B < t < B$. It suffices to show that for almost every x we have that there exists infinitely many k so that

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B-t,B-t]} \left(\sum_{i=0}^{n-1} f(T^i x) \right) \geq \beta \sum_{n=0}^{2^k-1} 1_{[-B-t,B-t]} \left(\sum_{i=0}^{n-1} f(T^i x) \right).$$

By Lemma 4.13 it suffices to show that there exists $\gamma > 0$ and an infinite sequence of N_j so that

$$\sum_{i=0}^{N_j} 1_{[-B-t,B-t]} \left(\sum_{i=0}^{n-1} f(T^i x) \right) > (N_j)^\gamma$$

holds. Letting $0 < \epsilon < |B| - |t|$ and invoking Theorem 4.10 (which we may do because Theorem B.1 shows Inequality (4.11) is satisfied) gives this condition. □

4.4. *Condition **F5** survives perturbations*

Here we prove that if (2.3) holds for a specific pair $(f,(x,t))$ then it also holds if either f or x is perturbed a little. Write

$$\mathsf{coll}(B,w) = [0,1) \times \Big([-B,-B+w] \cup [B-w,B]\Big)$$

for any $B > 0$ and any $w > 0$. Note that $\mathsf{coll}(B,B) = [0,1) \times [-B,B]$. We will make use of the following lemma, whose proof is omitted.

Lemma 4.14. *Let $n \mapsto a_n$ be a sequence that Cesàro converges to α. Fix $\epsilon > 0$ and $0 < \gamma < 1$. There is $K \in \mathbb{N}$ so large that*

$$\left| \alpha - \frac{1}{N-M} \sum_{n=M}^{N-1} a_n \right| < \epsilon$$

whenever $N > K$ and $N - M \geq \gamma N$.

Proposition 4.15. *Fix $C > 0$ and $B > C$ and $\theta > 0$. If $(f,(x,t)) \in G_2(B+C,K) \cap G_3(B+C)$ and*

$$\mu_{f,(x,t),B+C}\Big(\mathsf{coll}(B+C,2C)\Big) < \frac{1}{1+\theta} \tag{4.16}$$

then there is $\beta' > 0$ and $K' \subset K$ cofinite such that

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B+C,B-C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right) \geq \beta' \sum_{n=0}^{2^k-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right) \tag{4.17}$$

for all $k \in K'$.

Proof. Since $(f,(x,t)) \in G_2(B+C,K)$ there is $\beta > 0$ such that

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right) \geq \beta \sum_{n=0}^{2^k-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right) \tag{4.18}$$

for all $k \in K$. Since $(f,(x,t)) \in G_3(B+C)$ we can apply Lemma 4.14 to get $L \in \mathbb{N}$ so large that

$$\frac{1}{N-M} \sum_{n=M}^{N-1} 1_{\mathsf{coll}(B+C,2C)}\left(S^n_{f,B+C}(x,t)\right) \leq (1+\theta)\mu_{f,(x,t),B+C}\Big(\mathsf{coll}(B+C,2C)\Big)$$

whenever $N > L$ and $N - M \geq \beta N$. There is $K' \subset K$ cofinite such that

$$N := \sum_{n=0}^{2^k-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$

is at least L whenever $k' \in K$ and, choosing

$$M := \sum_{n=0}^{2^{k-1}-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$

it follows from (4.18) that $N - M \geq \beta N$. Therefore

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B-C,-B+C]\cup[B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$
$$\leq (1+\theta)\mu_{f,(x,t),B+C}\Big(\mathsf{coll}(B+C,2C)\Big)\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B-C,B+2C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$

for all $k \in K'$ because times n at which $t + \sum_{i=0}^{n-1} f(T^i x)$ belongs to $[-B-C,-B+C] \cup [B-C,B+C]$ are in bijective correspondence with the visits of the $S_{f,B+C}$ orbit of (x,t) to $\mathsf{coll}(B+C,2C)$. For all $k \in K'$ we therefore have

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B+C,B-C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$
$$\geq \Big(1-(1+\theta)\mu_{f,(x,t),B+C}\Big(\mathsf{coll}(B+C,2C)\Big)\Big)\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x)\right)$$

and, using (4.18) again, we have (4.17) with

$$\beta' = \Big(1-(1+\theta)\mu_{f,(x,t),B+C}\Big(\mathsf{coll}(B+C,2C)\Big)\Big)\beta$$

which is positive because of (4.16). □

Recall that if f in $\mathcal{C}_d$ has coordinates $(x_1,\ldots,x_{d+1},y_1,\ldots,y_{d+1})$ then ξ denotes $\min\{x_1,\ldots,x_{d+1}\}$.

THEOREM 4.19. *Fix $A>0$. Fix f in $\mathcal{C}_{d,D}$. Put $C = 6dDAc_1/c_2 + 2dD + 4dAc_1$. Fix $B > C$. Suppose*

$$(f,(x,t)) \in G_2(B+C,K)\cap G_3(B+C)$$

for some $\beta > 0$ and some infinite set $K \subset \mathbb{N}$. Suppose also that (4.16) holds. There is $\beta' > 0$ and $K' \subset K$ cofinite such that if $k \in K'$ and $g \in \mathcal{C}_{d,D}$ satisfies:

$$\mathsf{d}(f,g) \leq \min\left\{\frac{\xi}{4(d+1)}, \frac{A(c_1+\delta)}{2^k}\right\} \tag{4.20}$$

then

$$\sum_{n=0}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} g(T^i x)\right) \geq \beta' \sum_{n=2^{k-1}}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} g(T^i x)\right)$$

holds.

Proof. Let β' and K' be as in Proposition 4.15 for $(f,(x,t))$. Fix $k \in K'$. Fix g with coordinates $(\tilde{x}_1,\ldots,\tilde{x}_{d+1},\tilde{y}_1,\ldots,\tilde{y}_{d+1})$ satisfying (4.20).

Let I_j be the interval with endpoints $x_1+\cdots+x_j$ and $\tilde{x}_1+\cdots+\tilde{x}_j$ for each $1 \leq j \leq d+1$. These intervals are disjoint by (4.20). We have $|f(x)-g(x)| \leq 2D$ on each such interval. Off these intervals we have $|f(x)-g(x)| \leq \mathsf{d}(f,g)$. Put $J_j = \{0 \leq i < 2^k : T^i x \in I_j\}$ for each $1 \leq j \leq d+1$ and $J = J_1 \cup \cdots \cup J_{d+1}$. The complement of the intervals I_i is a collection $I'_1,\ldots,I'_{d+1}$ of $d+1$ disjoint intervals with respective widths at most x_i. Linear recurrence implies

$$|J_j| \leq \lceil 2^k |I_j|/c_2 \rceil \leq \lceil 2^k \mathsf{d}(f,g)/c_2 \rceil$$

for all j and that the orbit $x, \ldots, T^{2^k-1}x$ is in I_i' at most $\lceil 2^k x_i / c_2 \rceil$ times. Now for each $0 < n \leq 2^k$ we estimate that

$$\begin{aligned}\left| \sum_{i=0}^{n-1} f(T^i x) - \sum_{i=0}^{n-1} g(T^i x) \right| &\leq \sum_{i \in J} |f(T^i x) - g(T^i x)| + \sum_{i \notin J} |f(T^i x) - g(T^i x)| \\ &\leq d\left(\frac{2^k \mathsf{d}(f,g)}{c_2} + 1 \right) 2D + \sum_{i=1}^{d+1} \left(\frac{2^k x_i}{c_2} + 1 \right) \mathsf{d}(f,g) \\ &\leq 2dD \frac{A(c_1+\delta)}{c_2} + 2dD + \frac{A(c_1+\delta)}{c_2} + (d+1)A(c_1+\delta) \\ &\leq 4dDA\frac{c_1}{c_2} + 2dD + 2A\frac{c_1}{c_2} + 4dAc_1 \\ &\leq 6dDA\frac{c_1}{c_2} + 2dD + 4dAc_1\end{aligned}$$

using (4.20) and $c_1 \geq c_2$ and $\delta = c_2/4$. This gives the implications

$$\begin{aligned} t + \sum_{i=0}^{n-1} f(T^i x) \in [-B+C, B-C] &\Rightarrow t + \sum_{i=0}^{n-1} g(T^i x) \in [-B, B] \\ t + \sum_{i=0}^{n-1} g(T^i x) \in [-B, B] &\Rightarrow t + \sum_{i=0}^{n-1} f(T^i x) \in [-B-C, B+C]\end{aligned}$$

for all $0 < n \leq 2^k$. Combining with (4.17) we get

$$\begin{aligned}\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} g(T^i x) \right) &\geq \sum_{n=2^{k-1}}^{2^k-1} 1_{[-B+C,B-C]}\left(t + \sum_{i=0}^{n-1} f(T^i x) \right) \\ &\geq \beta' \sum_{n=0}^{2^k-1} 1_{[-B-C,B+C]}\left(t + \sum_{i=0}^{n-1} f(T^i x) \right) \\ &\geq \beta' \sum_{n=0}^{2^k-1} 1_{[-B,B]}\left(\sum_{i=0}^{n-1} t + g(T^i x) \right)\end{aligned}$$

for all $k \in K'$ as desired. □

4.5. *Conditions **F1** though **F5** hold almost surely*

Fix $d \in \mathbb{N}$ and $D \in \mathbb{N}$. Given a mean-zero step function f in $\mathcal{C}_{d,D}$ put

$$A = 2(d+1)\frac{10\delta + 6c_1}{3\delta + 3c_1} \max\left\{1, \frac{8D}{3\xi}\right\} \tag{4.21}$$

where $\xi = \min\{x_1, \ldots, x_{d+1}\}$ and write $C = 6dDAc_1/c_2 + 2dD + 4dAc_1$.

Lemma 4.22. *For every f in $\mathcal{C}_{d,D}$ the m_B measure of the set of pairs $(x,t) \in X_B$ for which the following statement is true converges to 1 as $B \to \infty$: there are $\beta > 0$, $\theta > 0$ and $K \subset \mathbb{N}$ infinite such that the conditions*

***L1**. (x,t) is generic for an $S_{f,B+C}$ invariant probability $\mu_{f,B+C,(x,t)}$;*

***L2**. $\mu_{f,B+C,(x,t)}\Big(\mathsf{coll}(B+C, 2C)\Big) < \dfrac{1}{1+\theta}$;*

***L3**. **F5** with $B+C$ in place of B;*

***L4**. $\dfrac{A(c_1+\delta)}{2^k} \leq \dfrac{\xi}{4(d+1)}$ for all $k \in K$;*

L5. $\dfrac{c_1+\delta}{2^{k-1}} < \dfrac{\xi}{4}$ *for all* $k \in K$;
all hold.

Proof. For every $B > 0$ almost every $(x,t) \in X_B$ has the property that is generic for an $S_{f,B}$ invariant probability measure $\mu_{f,B,(x,t)}$. As $B \to \infty$ the m_B measure of the set of $(x,t) \in X_B$ for which there is $\theta > 0$ such that **L2** holds converges to 1. For any $B > 0$, Theorem 4.9 gives both $\beta > 0$ and $K \subset \mathbb{N}$ infinite such that **L3** is true. By removing finitely many points from K we get **L4** and **L5**. □

Proof Proof of Theorem 2.5. Fix f in $\mathcal{C}_{d,D}$. Fix $B > 0$ and choose (x,t) in X_B such that the statement in Lemma 4.22 is true. By **L1** through **L3** we can apply Theorem 4.19, by which there is $\beta' > 0$ and $K' \subset K$ cofinite with the following property: if $h \in \mathcal{C}_{d,D}$ satisfies

$$\mathsf{d}(f,h) \le \min\left\{\frac{\xi}{4(d+1)}, \frac{A(c_1+\delta)}{2^k}\right\} \tag{4.23}$$

for some $k \in K'$ then

$$\sum_{n=2^{k-1}}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} h(T^i x)\right) \ge \beta' \sum_{n=0}^{2^k-1} 1_{[-B,B]}\left(t + \sum_{i=0}^{n-1} h(T^i x)\right) \tag{4.24}$$

holds.

Fix $k \in K'$. By **L5** we can apply Proposition 4.5 to get $g \in \mathcal{C}_d$ with the properties therein. In particular, for any h in $\mathcal{C}_d$ with $\mathsf{d}(g,h) < \frac{\delta}{3\cdot 2^k}$ we have that h and (x,t) satisfy **F1** through **F4** on either the left or the right for our current value of k. Now

$$\begin{aligned}\mathsf{d}(f,h) &\le \frac{2c_1+3\delta}{2^k}(d+1)\max\left\{1, \frac{8D}{3\xi}\right\} + \frac{\delta}{3\cdot 2^k} \\ &\le \frac{A(c_1+\delta)}{2^k} \\ &\le \min\left\{\frac{\xi}{4(d+1)}, \frac{A(c_1+\delta)}{2^k}\right\}\end{aligned}$$

for any such h upon using (4.7), (4.21) and **L4**. By the previous paragraph, this implies (4.24) holds. So, for our current value of $k \in K'$ the pair (x,t) and the function h satisfy **F1** through **F5** either on the left or the right.

We have proved, for every $(x,t) \in X_B$ such that the statement in Lemma 4.22 is true, that there is for every $k \in K'$ a ball of radius $\frac{\delta}{3\cdot 2^k}$ whose centre is a distance of at most

$$\frac{3\delta}{2^k}(d+1)\max\left\{1, \frac{8D}{3\xi}\right\}$$

from f such that (x,t) and h are friends at B for every h in the ball. Since k can be arbitrarily large, the Lebesgue density theorem (applied to a chart of $\mathcal{C}_d$ containing f) implies f is not a density point for the set of step functions in $\mathcal{C}_d$ that are not friends with (x,t) at B. We conclude that (2.6) holds for almost every f as $B \to \infty$. □

Appendix A. *An ergodic decomposition*

In this appendix we prove Theorem 2.2. We will do this using an ergodic decomposition result for quasi-invariant measures due to Schmidt that we reproduce here for convenience.

THEOREM A.1 [Sch77, Theorems 6.6 and 6.7]. *Let $(X, \mathscr{B})$ be a measurable space and let $F : X \to X$ be a $\mathcal{B}$ measurable map. Fix an F quasi-invariant probability measure μ on $(X, \mathscr{B})$. There exists a standard Borel space $(Y, \mathscr{Y})$, a surjective, measurable map $\psi : X \to Y$, and a family $y \mapsto q_y$ of Borel probability measures on $(X, \mathscr{B})$ with the following properties.*

(i) *The map $y \mapsto q_y(A)$ is Borel for every $A \in \mathscr{B}$.*
(ii) *Against all members of $\mathscr{B}$ one has $\mu = \int q_y \, \mathrm{d}\rho(y)$ where $\rho = \psi\mu$.*
(iii) *All of the measures q_y are F quasi-invariant and ergodic.*
(iv) *$q_y(\psi^{-1}(y)) = 1$ for every $y \in Y$.*
(v) *Let $\mathscr{C}$ be the σ-algebra of F invariant sets. Let $\mathscr{C}^* = \{\psi^{-1}(B) : B \in \mathscr{Y}\}$. Then $\mathscr{C}$ and $\mathscr{C}^*$ are μ equivalent.*
(vi) *If there is another collection $(Y', \mathscr{Y}', \psi', q')$ satisfying 1. through 4. then there exists a measurable map $\Theta : Y \to Y'$ that is an isomorphism between $(Y, \mathscr{Y}, \rho)$ and $(Y', \mathscr{Y}', \rho')$ such that $q'_{\Theta(y)} = q_y$ for ρ almost every $y \in Y$.*

The following corollary is virtually the same as [Sch77, Theorem 6.9]. We reformulate it slightly for our purposes, giving a proof for completeness.

COROLLARY A.2 cf. [Sch77, Theorem 6.9]. *Let $(X, \mathscr{B})$ be a measurable space and let $F : X \to X$ be a $\mathscr{B}$ measurable map. Fix an F invariant σ-finite measure ν on $(X, \mathscr{B})$. There exists a Borel space $(Y, \mathscr{Y})$, a surjective Borel measurable map $\psi : X \to Y$, and a family $y \mapsto p_y$ of σ-finite Borel measures on $(X, \mathscr{B})$ with the following properties.*

(i) *The map $y \to p_y(A)$ is measurable for every $A \in \mathscr{B}$.*
(ii) *Against all members of $\mathscr{B}$ one has $\nu = \int p_y d\rho(y)$ where $\rho = \psi\nu$.*
(iii) *All of the measures p_y are F invariant and ergodic.*
(iv) *All of the measures p_y satisfy $p_y(X \setminus \psi^{-1}(y)) = 0$.*
(v) *Let $\mathscr{C}$ be the σ-algebra of F invariant sets. Let $\mathscr{C}^* = \{\psi^{-1}(B) : B \in \mathscr{Y}\}$. Then $\mathscr{C}$ and $\mathscr{C}^*$ are μ equivalent.*
(vi) *If there is another such collection $(Y', \mathscr{Y}', \psi', p')$ satisfying 1. through 5. then there is a measurable map $\Theta : Y \to Y'$ that is an isomorphism between $(Y, \mathscr{Y}, \rho)$ and $(Y', \mathscr{Y}', \rho')$ such that $p'_{\Theta(y)} = p_y$ for ρ almost every y.*

Proof. Let μ be a probability measure that is equivalent to ν. Write $\mu = f\nu$ where f is a positive, measurable function. Theorem A.1 applied to $(X, \mathscr{B}, F, \mu)$ gives $(Y, \mathscr{Y}, \psi, q)$ satisfying 1. through 5. of Theorem A.1. Put $p_y = \frac{1}{f} q_y$. It is straightforward to verify that p is a disintegration of ν and is therefore measure-preserving satisfying 2. and 3. It inherits the other 4 properties from q_y. □

LEMMA A.3. *Let $(Y, \mathscr{Y}, \psi, p)$ and $(Y', \mathscr{Y}', \psi', p')$ satisfy 1. through 5. in Corollary A.2 and let Θ be as in 6. of Corollary A.2. Then $p'_{\psi'(x)} = p_{\psi(x)}$ for ν almost every x.*

Proof. Since $\rho = \psi\nu$ we have $p'_{\Theta(\psi(x))} = p_{\psi(x)}$ for ν almost every x. Fix x such that equality holds. Since $p_{\psi(x)}$ lives on $\psi^{-1}(\psi(x))$ and $p'_{\Theta(\psi(x))}$ lives on $(\psi')^{-1}(\Theta(\psi(x)))$ the intersection is conull for both measures. Therefore $p'_{\psi'(z)} = p'_{\Theta(\psi(x))}$ and $p_{\psi(z)} = p_{\psi(x)}$ for almost every z in the intersection. In other words $p'_{\psi'(z)} = p_{\psi(z)}$ for $p_{\psi(x)}$ almost every z. But this is true for ν almost every x. □

Proof Proof of Theorem 2.2. We apply Corollary A.2 with $X = [0,1) \times \mathbb{R}$ and $F = T_f$ and ν equal to m. Let $(Y, \mathscr{Y}, \psi, p)$ be the resulting disintegration. Define $p'_y = V^b p_y$ and $\psi'(x,t) = \psi(V^{-b}(x,t)) = \psi(x, t-b)$. We claim that $(Y, \mathscr{Y}, \psi', p')$ satisfies 1. through 5. of Corollary A.2. This is easily verified: we only check 2. here by observing that $\psi'\mathsf{m} = \psi\mathsf{m}$ and calculating

$$\iint 1_B \, \mathrm{d}p'_y \, \mathrm{d}\rho(y) = \iint 1_{V^{-b}B} \, \mathrm{d}p_y \, \mathrm{d}\rho(y) = \int 1_{V^{-b}B} \, \mathrm{d}\mathsf{m} = \int 1_B \, \mathrm{d}\mathsf{m}$$

for all B in $\mathscr{B}$. Therefore there is an automorphism $\Theta : Y \to Y$ such that $p'_{\Theta(y)} = p_y$ for ρ almost every y.

Next, note that

$$p_{\psi(x,t)} = p'_{\psi'(x,t)} = p'_{\psi(x,t-b)} = V^b p_{\psi(x,t-b)}$$

almost surely by Lemma A.3. For m almost every (x,t) the measure $p_{\psi(x,t)}$ is ergodic for T_f. Thus, for m almost every (x,t) the restriction $\mu_{x,t} := p_{\psi(x,t)}|X_B$ is ergodic for $S_{f,B}$. Therefore $(x,t) \mapsto \mu_{x,t}$ is an ergodic decomposition of $\mathsf{m}|X_B$ for $S_{f,B}$.

If $\mu_{x,t}$ is not singular with respect to $V^b\mu_{x,t}$ then $p_{\psi(x,t)}|X_B$ is not singular with respect to $V^b(p_{\psi(x,t)}|X_B)$. But $V^b p_{\psi(x,t)}$ is T_f ergodic because V^b commutes with T_f, and ergodic measures that are not mutually singular must be equal. Therefore $V^b p_{\psi(x,t)} = p_{\psi(x,t)}$ whenever $\mu_{x,t}$ and $V^b\mu_{x,t}$ are not mutually singular.

By the assumption of Theorem 2.2 we have that the exists a positive measure set of (x,t) so that $V^b\mu_{x,t}$ is not singular with respect to $\mu_{x,t}$. This implies that

$$A = \{(x,t) \in [0,1) \times \mathbb{R} : p_{\psi(x,t)} = V^b p_{\psi(x,t)}\}$$

has positive measure. Note that this set is measurable (because it is the set where two measurable maps agree) and almost surely invariant under the translations V^a (because V^a commutes with T_f for every $a \in \mathbb{R}$). It follows that $\{x : (x,t) \in A\}$ has positive measure for almost every t. Now A is clearly T_f invariant and since $(Tx,t) = V^{-f(x)}T_f(x,t)$ we have for almost every $(x,t) \in A$ that $(Tx,t) \in A$. By the previous sentence and the ergodicity of T we have for almost every t that $\{(x,t) \in A\}$ has full measure. This gives the theorem. □

Appendix B. *Quantitative unique ergodicity*

It follows from work of Boshernitzan [Bos85, Theorem 1.7] that every linearly recurrent interval exchange transformation is uniquely ergodic. In this section we prove the following quantitative version of Boshernitzan's result. Throughout this section we use b to denote the number of intervals of an interval exchange transformation.

THEOREM B.1. *Let T be a linearly recurrent interval exchange transformation and let $f : [0,1) \to \mathbb{R}$ be a mean-zero step function. Then there is $0 < \gamma < 1$ such that*

$$\left| \sum_{n=0}^{N-1} f(T^n x) \right| \leq N^\gamma \tag{B.2}$$

everywhere.

In fact, this result follows from Section 4 of [CC12]. Most of this section constitutes a self-contained proof of Theorem B.1, which we give for completeness. Our interest in Theorem B.1 is in deducing from it that Property **F5** holds for every mean-zero step function $f : [0,1) \to \mathbb{R}$ and almost every x. We begin with some notation for the induction scheme we will use throughout the proof of Theorem B.1.

Fix a linearly recurrent interval exchange transformation T with $b-1$ discontinuities. Write I_0 for $[0,1)$ and let $I_{0,1},\dots,I_{0,b}$ be the intervals of continuity of T. Define inductively $I_n = I_{n-1,1}$ and $I_{n,1},\dots,I_{n,b}$ as the intervals of the induced transformation $T|I_n$ on I_n. (See Figure B.1 for a schematic.) (Because we define linearly recurrent IETs to satisfy the Keane condition, this is also a b-IET.) Given $\ell > k \geq 0$ define

$$r_{k,\ell}(j) = \min\{n \in \mathbb{N} : (T|I_k)^n I_{\ell,j} \subset I_\ell\}$$

for all $1 \leq j \leq b$. This is the first time the $T|I_k$ orbit of $I_{\ell,j}$ returns to I_ℓ. Write

$$r_{k,\ell}(x) = \min\{n \in \mathbb{N} : (T|I_k)^n x \in I_\ell\}$$

for all $x \in I_k$. Note that $r_{k,\ell}(j) = r_{k,\ell}(x)$ for all $x \in I_{\ell,j}$. Define also for each $k \in \mathbb{N}$ a matrix B_k with entries

$$B_k(i,j) = \sum_{n=0}^{r_{k,k+1}(j)-1} 1_{I_{k,i}}\Big((T|I_k)^n I_{k+1,j}\Big)$$

for all $1 \leq i,j \leq b$ that count the number of visits of the $T|I_k$ orbit of $I_{k+1,j}$ to $I_{k,i}$ before the orbit visits I_{k+1}. Therefore $B_k(1,j) = 1$ for all $1 \leq j \leq b$. Note also that

$$\|B_k e_j\|_1 = B_k(1,j) + \cdots + B_k(b,j) = r_{k,k+1}(j)$$

for all $1 \leq j \leq b$ where $e_1,\dots,e_b$ is the standard basis of $\mathbb{R}^b$.

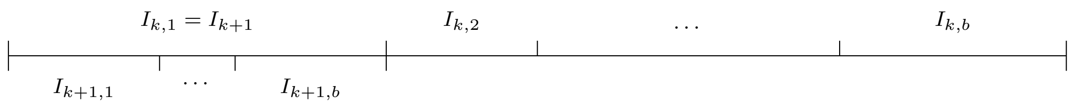

FIGURE B.1. *The interval I_k and some of its subintervals.*

The entries of the matrix $B_{k,r} := B_k B_{k+1} \cdots B_{k+r}$ are

$$B_{k,r}(i,j) = \sum_{n=0}^{r_{k,k+r+1}(j)-1} 1_{I_{k,i}}\Big((T|I_k)^n I_{k+r+1,j}\Big)$$

and they count the number of visits of $I_{k+r+1,j}$ to $I_{k,i}$ under $T|I_k$ before it returns to I_{k+r+1}. Therefore

$$\|B_{k,r} e_j\|_1 = B_{k,r}(1,j) + \cdots + B_{k,r}(b,j) = r_{k,k+r+1}(j)$$

for all $1 \leq j \leq b$. Our proof of Theorem B.1 relies on the following facts.

FACT B.3. *There is a constant $D_1 > 0$ such that $\frac{1}{D_1} < \frac{r_{k,l}(i)}{r_{k,l}(j)} < D_1$ for all $k > l \geq 0$ and all $1 \leq i,j \leq b$.*

Proof. Fix $x \in I_l$. First note that

$$\min\{r_{m,k}(z) : z \in I_m\} r_{k,l}(x) \leq r_{m,l}(x) \leq \max\{r_{m,k}(z) : z \in I_m\} r_{k,l}(x) \tag{B.4}$$

for all $l > k > m$ because each step in the $T|I_k$ orbit of x involves a return of some point in I_k to I_k under $T|I_m$. Now

$$c_2 \leq r_{0,l}(x)|I_l| \leq c_1 \tag{B.5}$$

for all l by linear recurrence so

$$\frac{c_2/|I_l|}{c_1/|I_k|} \leq \frac{r_{0,l}(x)}{\max\{r_{0,k}(z) : z \in I_0\}} \leq \frac{r_{0,l}(x)}{\min\{r_{0,k}(z) : z \in I_0\}} \leq \frac{c_1/|I_l|}{c_2/|I_k|} \tag{B.6}$$

for all k and l. Taking $m = 0$ in (B.4) and combining with (B.6) shows that $D_1 = \dfrac{c_1^2}{c_2^2}$ works. □

Fact B.7. *There is a constant $D_2 > 0$ such that $\frac{1}{D_2} < \frac{|I_{k,j}|}{|I_{k,i}|} < D_2$ for all $1 \leq i, j \leq b$ and all $k \in \mathbb{N}$.*

Proof. Linear recurrence implies that the discontinuities of $T^{r_{0,k}(j)}$ are $c_1/r_{0,k}(j)$ dense and $c_2/r_{0,k}(j)$ separated. Since $T|I_k$ is continuous on the interior of $I_{k,j}$ and has discontinuities at its endpoints we must have

$$\frac{c_2}{\max\{r_{0,k}(j) : 1 \leq j \leq b\}} \leq |I_{k,j}| \leq \frac{c_1}{\min\{r_{0,k}(j) : 1 \leq j \leq b\}}$$

so Fact B.3 implies D_2 can be chosen to be $c_1 D_1/c_2$. □

Fact B.8. *There are constants $\rho_1, \rho_2 > 1$ such that $\rho_1 |I_{k,j}| \leq |I_k| \leq \rho_2 |I_{k,j}|$ for all $k \in \mathbb{N}$ and all $1 \leq j \leq b$. In particular $\rho_1 |I_{k+1}| \leq |I_k| \leq \rho_2 |I_{k+1}|$ for all $k \in \mathbb{N}$.*

Proof. We have

$$|I_k| = |I_{k,1}| + |I_{k,2}| + \cdots + |I_{k,b}| \leq |I_{k,j}| + (b-1)D_2 |I_{k,j}| = (1 + (b-1)D_2)|I_{k,j}| \tag{B.9}$$

and

$$|I_k| = |I_{k,1}| + |I_{k,2}| + \cdots + |I_{k,b}| \geq |I_{k,j}| + \frac{b-1}{D_2}|I_{k,j}| = \left(1 + \frac{b-1}{D_2}\right)|I_{k,j}|$$

by Fact B.7. Applying this when $j = 1$ we can take $\rho_1 = 1 + (b-1)/D_2$ and $\rho_2 = 1 + (b-1)D_2$. □

Fact B.10. *There is a constant $D_3 > 0$ such that $\|B_k\|_1 \leq D_3$ for all k.*

Proof. We have

$$B_k(i,j) \leq r_{k,k+1}(j) \leq \frac{r_{0,k+1}(j)}{\min\{r_{0,k}(l) : 1 \leq l \leq b\}} \leq \frac{c_1}{c_2}\frac{|I_k|}{|I_{k+1}|} \leq \frac{c_1}{c_2}\rho_2$$

by taking $m = 0$ and $l = k+1$ in (B.4) and then using (B.6) and Fact B.8. It then follows that $\|B_k\|_1 \leq b c_1 \rho_2 / c_2$ for all k. □

Fact B.11. *There is $r \in \mathbb{N}$ such that $B_k B_{k+1} \cdots B_{k+r}$ is positive for all $k \in \mathbb{N}$.*

Proof. We must produce $r \in \mathbb{N}$ such that, for every i, j, k the $T|I_k$ orbit of $I_{k+r+1,j}$ visits $I_{k,i}$ before returning to I_{k+r+1}. It is enough to find r such that, for every i, j, k the T orbit of any point $x \in I_{k+r+1,j}$ visits $I_{k,i}$ before time $r_{0,k+r+1}(j)$. By linear recurrence this is the case if $r_{0,k+r+1}(j)|I_{k,i}| \geq c_1$. Using Fact B.8 repeatedly and then (B.5) we have

$$r_{0,k+r+1}(j)|I_{k,i}| \geq r_{0,k+r+1}(j)\frac{|I_k|}{\rho_2} \geq r_{0,k+r+1}(j)|I_{k+r+1}|\frac{\rho_1^{r+1}}{\rho_2} \geq c_2\frac{\rho_1^{r+1}}{\rho_2}$$

and, independent of i, j, k, this will be at least c_1 if r is large enough. □

Fact B.12. *There are constants $D_4 \in \mathbb{R}$ and $\gamma < 1$ such that*

$$\Theta(B_{k,r}e_j, B_{k,r}e_\ell) < D_4\gamma^r$$

for all j, ℓ, k, r where Θ denotes the angle between two vectors in $\mathbb{R}^b$ and $e_1, \ldots, e_b$ is the standard basis of $\mathbb{R}^b$.

Proof. Because positive matrices of a fixed size act as definite contractions in the Hilbert projective metric, Fact B.11 implies that there exists $\rho < 1$ so that $\Theta(B_{k,k+r}v, B_{k,k+r}w) < \rho\Theta(v, w)$ for any $v, r \in \mathbb{R}^b_+$. Iterating we have $\Theta(B_{k,k+nr}v, B_{k,k+nr}w) < \rho^n$. Letting $\gamma = \rho^{\frac{1}{r}}$ and choosing $D_4 = \rho^{-1}$ we obtain the fact. □

We use these facts to prove the following lemmas.

Lemma B.13. *There exists $0 < \zeta_2$ and E_2 such that*

$$\left|\sum_{n=0}^{N-1} 1_{I_k}(T^n x) - 1_{I_k}(T^n y)\right| \leq E_2 \max\{1, (|I_k|N)^{\zeta_2}\}$$

for all $x, y \in [0, 1)$ and all $k, N \in \mathbb{N}$.

Proof. Let $r = \frac{1}{2}c\log(N|I_k|)$. We first consider $x \in I_{k+r}$. Let M be the maximal number so that $M < N$ and $T^M x = (T|I_{k+r})^a x \in I_{k+r}$. Put

$$v = \sum_{i=0}^{M} 1_{I_k}(T^i x) = |C_{i_1}(B_k \cdots B_{k+r})| + \cdots + |C_{i_{a-1}}(B_k \cdots B_{k+r})|$$

where i_j is defined by $(T|I_{k+r})^j x \in I_{k+r,i_j}$. By Fact B.12

$$\left|\frac{C_j(B_k \cdots B_{k+r})}{|C_j(B_k \cdots B_{k+r})|} - \frac{v}{M}\right|$$

is exponentially small (in $\log(N|I_k|)$) for each j. Now

$$v \leq \sum_{i=0}^{N-1} 1_{I_k}(T^i x) \leq |C_{\max}(B_k \cdots B_{k+r})| + v$$

and by our choice of r we have that $|C_{\max}(B_k \ldots B_{k+r})|/M$ is exponentially small in $\log(N|I_k|)$ so

$$\left|\frac{1}{N}\sum_{i=0}^{N-1} 1_{I_k}(T^i x) - \frac{C_j(B_k \cdots B_{k+r})}{|C_j(B_k \cdots B_{k+r}|}\right|$$

is exponentially small, establishing the lemma for $x \in I_{k+r}$. A general $x \in [0, 1)$ gives another error of at most $|C_{\max}(B_k \cdots B_{k+r})|$. □

The proof of the next lemma is similar and omitted.

Lemma B.14. *There is $0 < \zeta < 1$ and $E_3 > 0$ such that*

$$\left|\sum_{n=0}^{N-1} 1_{I_{k,j}}(T^n x) - 1_{I_{k,j}}(T^n y)\right| < E_3 \max\{1, (|I_{k,j}|N)^{\zeta}\}$$

for all k, j, N, x, y.

Corollary B.15. *There is $0 < \zeta < 1$ and $E_3 > 0$ such that*

$$\left|\sum_{n=0}^{N-1} 1_{T^s I_{k,j}}(T^n x) - 1_{T^s I_{k,j}}(T^n y)\right| < E_3 \max\{1, (|I_{k,j}|N)^\zeta\}$$

for all k, j, N, x, y, s.

Proof. This is immediate from Lemma B.14 because x and y therein can be any point. □

With these lemmas we can prove Theorem B.1.

Proof Proof of Theorem B.1. Fix a mean-zero step function (1.2). Set $a_0 = 0$ and $a_{d+1} = 1$. Put $a_i = x_1 + \cdots + x_i$ for all $1 \le i \le d$. For each $k \in \mathbb{N}$ the partition

$$\mathcal{P}_k = \{T^n I_{k,j} : 0 \le n < r_{0,k}(j), 1 \le j \le b\}$$

of $[0, 1)$ is $|I_k|$ dense.

Let ζ be as in Lemma B.14 and fix $\frac{1}{2} + \frac{\zeta}{2} < \gamma < 1$. Fix $N \in \mathbb{N}$ with $\rho_1^2 N^{2\gamma-1-\zeta} \ge 1$. For each $1 \le i \le d+1$ choose k minimal with $|I_k|\sqrt{N} < \sqrt{a_i - a_{i-1}}$. Write the interval $[a_{i-1}, a_i)$ as a union of at most $(a_i - a_{i-1})/|I_k|$ intervals from $\mathcal{P}_k$ together with an interval of length at most $|I_k|$ at each end. Fix $x, y \in [0, 1)$. By estimating the hits of x and y to the end intervals using linear recurrence, and by comparing hits of x and y to the other intervals using Corollary B.15, we obtain

$$\begin{aligned}\left|\sum_{n=0}^{N-1} 1_{[a_{i-1},a_i)}(T^n x) - \sum_{n=0}^{N-1} 1_{[a_{i-1},a_i)}(T^n y)\right| &\le \frac{4}{c_2}\sqrt{N}\sqrt{a_i - a_{i-1}} + \frac{a_i - a_{i-1}}{|I_k|} E_3 \max\{1, (|I_k|N)^{\zeta_3}\} \\ &\le \frac{4\sqrt{N}}{c_2} + \sqrt{a_i - a_{i-1}} \max\left\{\frac{\sqrt{N}}{\rho_1}, \sqrt{N^{\zeta_3+2}}\right\}\end{aligned}$$

for all $1 \le i \le d+1$ whenever $\rho_1^2 N \ge 1$. Combined with (1.2) we have (B.2) as desired. □

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