ON CERTAIN GROUPS WITH FINITE CENTRALISER DIMENSION

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In this thesis, we study certain groups with finite centraliser dimension. These are groups in which there is a finite bound on the length of any chain of centralisers. We study locally finite groups of finite centraliser dimension and stable groups. In particular, we consider a special kind of stable groups called groups of finite Morley rank.

The motivation to our work arises from two major conjectures in the topic of groups of finite Morley rank. One of them—the Cherlin–Zilber Conjecture—states that infinite simple groups of finite Morley rank are isomorphic to linear algebraic groups over algebraically closed fields. The other conjecture—the Principal Conjecture—is due to Ehud Hrushovski; it states that if an infinite simple group of finite Morley rank G admits a generic automorphism α, then the fixed point subgroup $C_G(\alpha)$ is pseudofinite. It is known, by results of Zoé Chatzidakis and Hrushovski and results of Hrushovski alone, that the Cherlin–Zilber Conjecture implies the Principal Conjecture; some evidence suggests that the converse also holds.

Chapters 2, 3 and 5 contain no original results. In Chapters 2 and 3, we provide the necessary group-theoretic and model-theoretic background material required in later chapters. In Chapter 5, we discuss groups of finite Morley rank and give a compendium of advanced results of the topic needed in Chapter 7.

Our results are proven in Chapters 4, 6 and 7.

Chapter 4 is devoted to the study of locally finite groups of finite centraliser dimension. We prove a general result describing the structure of such groups. Moreover, we prove that definably simple locally finite groups of finite centraliser dimension are simple groups of Lie type over locally finite fields.

In Chapter 6, we introduce a finitary automorphism group A. The definition of a finitary automorphism group A arises as we isolate certain properties of the group of Frobenius maps of an algebraically closed field K of positive characteristic. We prove that an infinite definably simple stable group admitting a finitary automorphism group A contains an infinite locally finite elementary subgroup. Then, we identify this locally finite elementary subgroup using our results in Chapter 4. Consequently, we classify infinite definably simple stable groups admitting a finitary automorphism group A as Chevalley groups over algebraically closed fields of positive characteristic.

In Chapter 7, we continue the path designated by Pınar Uğurlu. In her PhD thesis, Uğurlu developed a strategy towards proving the expected equivalence between the Cherlin–Zilber Conjecture and the Principal Conjecture. We first give Uğurlu's definition of a tight automorphism α of an infinite simple group of finite Morley rank G. Then, we prove that, under suitable assumptions, a "small" infinite simple group of finite Morley rank admitting a tight automorphism α is isomorphic to $PSL_2(K)$ over some algebraically closed field K of characteristic different from 2.

Keywords: Groups of finite centraliser dimension, locally finite groups, stable

groups, Frobenius maps, pseudofinite groups, groups of finite Morley rank, the Cherlin–Zilber Conjecture, the Principal Conjecture

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

A group G is said to be of a finite centraliser dimension if there is a finite bound on the length of any chain of centralisers in G.

In this thesis, we prove a general theorem describing the structure of locally finite groups of finite centraliser dimension. Moreover, we prove the two classifications below—note that, it is well-known that stable groups possess a finite centraliser dimension, and therefore, both of the two classifications below are for certain groups of finite centraliser dimension.

- Infinite definably simple locally finite groups of finite centraliser dimension are simple groups of Lie type over locally finite fields.
- Infinite definably simple stable groups admitting a finitary automorphism group A (a group A of special kind of automorphisms which, in particular, have finite fixed point subgroups) are Chevalley groups over algebraically closed fields of positive characteristic.

We also partially classify the following groups.

- Small (i.e., groups with Prüfer 2-rank 1) infinite simple groups of finite Morley rank admitting a tight automorphism α (an automorphism α which resembles the non-standard Frobenius automorphism of an algebraically closed field K) whose fixed point subgroup is pseudofinite.

Our structural result on locally finite groups of finite centraliser dimension as well as the two classifications specified above depend on the celebrated Classification of

Finite Simple Groups. However, we also prove that, in the restricted case of infinite simple groups of finite Morley rank admitting a finitary automorphism group A, our classification result does *not* depend on the Classification of Finite Simple Groups. The reader should note that groups of finite Morley rank are stable, and therefore, the classification of infinite simple groups of finite Morley rank admitting a finitary automorphism group A (using the Classification of Finite Simple Groups) follows from the classification of infinite definably simple stable groups admitting a finitary automorphism group A.

In terms of techniques, this thesis has a strong group-theoretic flavour. However, the original motivation to our work arises from the recently developed approach towards the classification of infinite simple groups of finite Morley rank—these groups arise naturally in the studies of model theory. We start by explaining the motivation that the reader (or at least a reader familiar with model theory) may wish to keep in mind throughout the text.

Model theory is a branch of mathematical logic which concerns the interplay between mathematical structures (e.g. groups, fields etc.) and the language of logic which is used to describe them. It is often thought that modern model theory started from the results of Michael Morley [62] in 1965. Answering positively a question asked by Jerzy Łós [67], Morley proved that if a complete countable first-order theory T is κ-categorical (i.e., has, up to isomorphism, a unique model of cardinality κ) for *some* uncountable cardinal κ, then T is κ-categorical for *all* uncountable cardinals—such theories are called *uncountably categorical*. In order to prove his famous Categoricity Theorem, Morley introduced a notion that we today call the *Morley rank*—a dimension-like function that assigns an ordinal number (rank) to each definable set of models of uncountably categorical theories. The Morley rank can be thought of as an abstract model-theoretic generalisation of the *Zariski dimension* (a classical notion of dimension from algebraic geometry, see Section 2.3).

After the seminal work of Morley, in the late 60's, Saharon Shelah [77] initiated a far-reaching program attempting to either 'classify' models of an arbitrary first-order theory T, up to isomorphism, or to show that such a 'classification' is not possible. In order to implement his programme, Shelah developed different combinatorial notions which a structure may or may not possess—the most important one being *stability*.

Today we know that there are many equivalent definitions for stable structures—one of them goes as follows; *stable* structures are structures in which no first-order formula can totally order arbitrarily large sets of tuples.

Groups of finite Morley rank appeared for the first time in the strictly model-theoretic context as groups whose first-order theory is uncountably categorical. These groups can be thought of as a model-theorist's approach to algebraic groups over algebraically closed fields. Indeed, in 1971, Angus Macintyre [59] proved that an infinite field K is of finite Morley rank if and only if K is algebraically closed. Furthermore, in the late 70's, Gregory Cherlin [30] and Boris I. Zilber [30] independently conjectured the following.

Conjecture 1 (The Cherlin–Zilber Conjecture)**.** *Infinite simple groups of finite Morley rank are linear algebraic groups over algebraically closed fields.*

The reader should note that there are no infinite simple abelian groups of finite Morley rank (see Section 5.2), and therefore, the Cherlin–Zilber Conjecture should not be too surprising keeping in mind that the Morley rank can be viewed as an abstract model-theoretic generalisation of the Zariski dimension. Today, after 40 years of effort towards solving the Cherlin–Zilber Conjecture, certain substantial parts are solved, however, the conjecture remains open.

So far, the most successful approach to the Cherlin–Zilber Conjecture has been the so-called Borovik Programme. The idea of this approach, suggested by Alexandre V. Borovik, is to prove "the analogue of the Classification of Finite Simple Groups" in the context of infinite simple groups of finite Morley rank. Infinite simple groups of finite Morley rank can be split into three cases, based on the well-defined connected component S° of a maximal 2-subgroup S of a group of finite Morley rank G; *even* type, *odd* type and *degenerated* type (see Section 5.2). The greatest achievement of the Borovik Programme is the result by Tuna Altınel, Borovik and Cherlin [5] that essentially proves "half" of the Cherlin–Zilber Conjecture. Namely, they proved that infinite simple groups of finite Morley rank of even type are linear algebraic groups over algebraically closed fields of characteristic 2. Though the Borovik Programme has been very successful in the past, for some time now, there has been no serious progress on the Programme. Therefore, there is a need of new approaches towards the Cherlin–Zilber Conjecture—below we explain an approach that has emerged recently.

An infinite group is called *pseudofinite* if it satisfies every first-order sentence that is true of all finite groups or, equivalently, if it is elementarily equivalent to an ultraproduct of finite groups.

Let G be a linear algebraic group defined over an algebraically closed field K and α be an automorphism of G induced by the *generic* automorphism of K. Then the fixed point subgroup $C_G(\alpha)$ is known to be a pseudofinite group by results of Macintyre [60] (note that here one identifies G with the K-rational points, $G(K)$, and therefore the subgroup $C_G(\alpha)$ coincides with the K-rational points, $G(\mathrm{Fix}_K(\alpha))$, fixed by α). In his proof, Macintyre uses the beautiful purely algebraic characterisation of pseudofinite fields given by James Ax [9]. In [50], Ehud Hrushovski showed that fixed points of generic automorphisms of the structures with certain nice model-theoretic properties are pseudo-algebraically closed with small Galois groups. Further, Hrushovski proved that any fixed point subgroup arising this way admits a certain kind of measure similar to a non-standard probabilistic measure on pseudofinite groups (one should note here that it is not known how to algebraically characterise pseudofinite groups). In the particular case of infinite simple groups of finite Morley rank, the aim is to prove that the group of fixed points of a generic automorphism is a pseudofinite group. Indeed, in her PhD thesis [85], Pınar Uğurlu formulated, from the results and observations of Hrushovski, the following conjecture.

Conjecture 2 (The Principal Conjecture)**.** *Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Then the fixed point subgroup $C_G(\alpha)$ is pseudofinite.*

One should note that, if the Cherlin–Zilber Conjecture holds, then a generic automorphism α of an algebraically closed field K, over which an infinite simple group of finite Morley rank G is defined, induces an automorphism on G whose fixed point subgroup $C_G(\alpha)$ is pseudofinite.

It follows from results of Zoé Chatzidakis and Hrushovski [28] and Hrushovski alone [51], that the Cherlin–Zilber Conjecture implies the Principal Conjecture. It is natural to expect that these two conjectures are actually equivalent—this expectation is supported by important results of Uğurlu in [86]. Indeed, in [85, 86], Uğurlu developed a strategy towards proving that the Cherlin–Zilber Conjecture and the Principal Conjecture are equivalent. Below we briefly introduce this strategy.

Since the Cherlin–Zilber Concjecture implies the Principal Conjecture, in her work, Uğurlu assumed that the Principal Conjecture holds and worked towards the Cherlin–Zilber Conjecture. In order to work in a purely algebraic context, she defined a *tight* automorphism α of an infinite simple group of finite Morley rank G—an automorphism which resembles the non-standard Frobenius automorphism of an algebraically closed field K. Uğurlu proved that if an infinite simple group of finite Morley rank G admits a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite, then $C_G(\alpha)$ contains a normal definable (in $C_G(\alpha)$) pseudofinite Chevalley or twisted Chevalley subgroup S such that G does not have any proper definable subgroups containing S (in other words, the definable closure $\overline{S}$ of S equals G).

With the path designated by Uğurlu in mind (towards proving the expectation that the Cherlin–Zilber Conjecture and the Principal Conjecture are equivalent), one naturally wishes to identify infinite simple groups of finite Morley rank admitting a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite with Chevalley groups over algebraically closed fields. In this thesis we start with a different natural approach; we define a *finitary automorphism group* A—the definition of a finitary automorphism group A arose as we isolated certain properties of the group of Frobenius maps of an algebraically closed field K of positive characteristic. We study infinite simple groups of finite Morley rank, and further, infinite definably simple stable groups, admitting a finitary automorphism group A. This naturally leads us to study infinite definably simple locally finite groups of finite centraliser dimension, which in turn, naturally leads us to study locally finite groups of finite centraliser dimension. Moreover, we also continue the work of Uğurlu and study infinite simple groups of finite Morley rank admitting a tight automorphism α whose fixed point subgroup is pseudofinite.

Let us be a bit more precise. To understand generic automorphisms (in the sense of Chatzidakis and Hrushovski [28]) and their pseudofinite groups of fixed points, it is desirable to understand possible analogues of Frobenius maps of an algebraically closed field K of positive characteristic. Therefore, we isolate certain conditions of these maps and, as a result, form a group of automorphisms of a group of finite Morley rank that resembles the group of Frobenius maps of an algebraically closed field K of positive characteristic—as mentioned earlier, such an automorphism group is called a finitary automorphism group A. Then, we classify infinite definably simple stable groups (and thus, in particular, infinite simple groups of finite Morley rank) admitting a finitary automorphism group A. Our classification result relies on the fact (Theorem 6.1.6) that any infinite structure admitting a finitary automorphism group A has an infinite locally finite elementary substructure, which we call the *locally finite core*. To identify the locally finite core of an infinite definably simple stable group admitting a finitary automorphism group A, we classify infinite definably simple locally finite groups of finite centraliser dimension. Our classification result of infinite definably simple locally finite groups of finite centraliser dimension requires a good understanding of locally finite groups of finite centraliser dimension. To this end, we also prove a general result describing the structure of locally finite groups of finite centraliser dimension. As mentioned above, we also continue the work of Uğurlu and study infinite simple groups of finite Morley rank admitting a tight automorphism α whose fixed point subgroup is pseudofinite. We prove that if G is an infinite simple group of finite Morley rank with a pure group structure and with Prüfer 2-rank 1 admitting a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite, and, if a maximal split torus T of the pseudofinite Chevalley group $S \leqslant C_G(\alpha)$ contains an involution i, then G is isomorphic to $PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.

1.1 Structure of this thesis

We now summarise the structure of this thesis and precisely state the obtained results and the main original definition.

Our results are proven in Chapters 4, 6 and 7. In Chapters 2 and 3, we fix the notation and provide all of the group-theoretic and model-theoretic background material the reader should be familiar with in order to read this thesis. As the well-studied theory of groups of finite Morley rank is heavily used in Chapter 7, groups of finite Morley rank deserve a chapter of their own. Therefore, in Chapter 5, we discuss groups of finite Morley rank and give a compendium of advanced results of the topic.

Chapter 4 is devoted for the study of locally finite groups of finite centraliser dimension.

In Section 4.1, we prove the following theorem.

Theorem 4.1.1 (Borovik and Karhumäki [20, Theorem 1]; see also Buturlakin [25])**.** *Let G be a locally finite group of centraliser dimension k. Then G has a normal series*

$$1 \trianglelefteq S \trianglelefteq L \trianglelefteq G,$$

where

(a) *S is a solvable group of derived length bounded by a function of k.*

(b) *$\overline{L} = L/S$ is a direct product $\overline{L} = \overline{L}_1 \times \cdots \times \overline{L}_m$ of finitely many non-abelian simple groups.*

(c) *Each $\overline{L}_i$ is either finite or a simple group of Lie type over a locally finite field.*

(d) *The factor group G/L is finite.*

Theorem 4.1.1 is joint work between Borovik and the author; it is published in [20]. Similar result to Theorem 4.1.1 was independently proven by Alexandre A. Buturlakin, see [25]. We should mention that Theorem 4.1.1 (as well as Buturlakin's result in [25]) depends on the Classification of Finite Simple Groups.

In Section 4.2, we restrict our attention to infinite definably simple locally finite groups of finite centraliser dimension. In [83], Simon Thomas proved that infinite simple locally finite groups of finite centraliser dimension are Chevalley groups or twisted Chevalley groups over locally finite fields—we wish to mention that, Thomas's result depends on the Classification of Finite Simple Groups. We give a natural generalisation of Thomas's classification.

Theorem 4.2.1 (Karhumäki [55, Theorem 1.1])**.** *An infinite definably simple locally finite group of finite centraliser dimension is a simple group of Lie type over a locally finite field.*

Theorem 4.2.1 heavily uses Theorem 4.1.1 and is therefore also dependent on the Classification of Finite Simple Groups. It is published in [55].

In Chapter 6, we define a finitary automorphism group A and classify infinite definably simple stable groups admitting such an automorphism group.

In Section 6.1, we start by giving the main original definition of this thesis.

Definition 6.1.1 (Karhumäki [55]). *Let $\mathcal{M}$ be an infinite structure with the underlying set M. We say that an infinite group A of automorphisms of $\mathcal{M}$ is* finitary*, if the following hold:*

(1.) *For every $\alpha \in A \setminus \{1\}$, the substructure of fixed points $\mathrm{Fix}_M(\alpha)$ is finite, and;*

(2.) *If $X \neq \emptyset$ is a definable subset in M which is invariant under the action of some non-trivial automorphism $\alpha \in A$, then there exists an element $x \in X$ with a finite orbit x^A. Equivalently, $[A : \mathrm{Stab}_A(x)] < \infty$.*

Further, we explain the usefulness of Definition 6.1.1—we prove that the presence of a finitary automorphism group A allows us to find infinite locally finite elementary substructures of infinite structures admitting A.

Theorem 6.1.5 (Karhumäki [55, Theorem 4.2.]). *Let $\mathcal{M}$ be an infinite structure with the underlying set M and A be a finitary automorphism group of $\mathcal{M}$. Then $\mathcal{M}$ contains an infinite locally finite elementary substructure $\mathcal{M}_*$ with the underlying set $M_* = \{\, m \in M : \text{the orbit } m^A \text{ is finite.}\}$.*

The classification of infinite definably simple stable groups admitting a finitary automorphism group A is given in Section 6.2. As stable groups possess a finite centraliser dimension, we may invoke Theorem 4.2.1 to prove our classification result.

Theorem 6.2.1 (Karhumäki [55, Theorem 1.2]). *Every infinite definably simple stable group G admitting a finitary automorphism group A is a Chevalley group over an algebraically closed field of positive characteristic.*

Again, we wish to emphasise that, since Theorem 6.2.1 uses Theorem 4.2.1, the Classification of Finite Simple Groups is present in the background of our proof.

In Section 6.3, we restrict our attention to the specific case of infinite simple groups of finite Morley rank admitting a finitary automorphism group A. Groups of finite Morley rank are stable, and therefore, Theorem 6.2.1 in particular implies the classification of infinite simple groups of finite Morley rank admitting a finitary automorphism group A. However, we explain that, due to the tame nature of groups of finite Morley rank, the existence of an infinite locally finite elementary substructure essentially proves the classification of infinite simple groups of finite Morley rank admitting a finitary automorphism group A. Therefore, we prove the analogue of Theorem 6.2.1, in the restricted case of groups of finite Morley rank, which does not depend on the Classification of Finite Simple Groups.

Theorem 6.3.1 (Karhumäki [55, Theorem 1.3]—does not depend on CFSG)**.** *Every infinite simple group of finite Morley rank G admitting a finitary automorphism group A is a Chevalley group over an algebraically closed field of positive characteristic.*

Note that Theorem 6.3.1 in particular proves that the Cherlin–Zilber Conjecture holds in the specific case in which an infinite simple group of finite Morley rank G admits a finitary automorphism group A.

Theorems 6.1.5, 6.2.1 and 6.3.1 are published in [55].

Finally, in Chapter 7, we continue the work of Uğurlu in [86]. We consider an infinite simple group of finite Morley rank G with a pure group structure and with Prüfer 2-rank 1 admitting a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite. By results of Uğurlu, G contains a pseudofinite Chevalley group $S \cong PSL_2(F)$, where F is a pseudofinite field of characteristic different from 2, as a normal subgroup of the pseudofinite fixed point subgroup $C_G(\alpha)$. We prove that if a maximal split torus T of S contains an involution i, then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2. That is, we prove the following theorem which is joint work between Uğurlu and the author.

Theorem 7.2.1 (Karhumäki and Uğurlu)**.** *Let G be an infinite simple group of finite Morley rank with a pure group structure admitting a tight automorphism α. Assume that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. Assume further that* $\mathrm{pr}_2(G) = 1$ *and that a maximal split torus T of the pseudofinite Chevalley group $S \cong PSL_2(F)$ contains an involution i. Then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.*

Chapter 2

Group-theoretic background material

In this chapter, we fix the group-theoretic notation used throughout this thesis and give a compendium of results that will be needed in Chapter 4.

Section 2.1 contains a review of elementary group theory; it is written to set-up our group-theoretic notation.

All groups considered in this thesis are of finite centraliser dimension. Therefore, in Section 2.2, we explain this notion and give examples.

In Section 2.3, we introduce linear algebraic groups and, in particular, we briefly discuss the classification of simple algebraic groups.

Locally finite groups are introduced in Section 2.4. Again, we pay some extra attention to simple locally finite groups.

2.1 Notation and elementary group theory

Let G be a group. Given a subset $X \subseteq G$, X^* stands for $X \setminus \{1\}$. We denote subgroups (resp. proper subgroups) of G by $H \leqslant G$ (resp. $H < G$) and normal subgroups (resp. proper normal subgroups) of G by $N \trianglelefteq G$ (resp. $N \triangleleft G$). Two subgroups H and K of a group G are said to be *disjoint* if $H \cap K = \{1\}$. If H and K are subgroups of a group G such that $H \trianglelefteq K$, then the factor group K/H is called a *section* of G.

Given a subset $X \subseteq G$, $\langle X \rangle$ denotes the subgroup of G generated by X, that is, the intersection of all subgroups of G containing X.

Given a subset $X \subseteq G$ and a subgroup $H \leqslant G$, we define the *centraliser* of X in H as the group $C_H(X) = \{h \in H : hx = xh \text{ for all } x \in X\}$. Likewise, given another subgroup $K \leqslant G$, we define the *normaliser* of K in H as the group $N_H(K) = \{h \in H : K^h = K\}$.

Let $x, y \in G$ be any two elements of a group G. Then, we denote $x^y = yxy^{-1}$ and $[x, y] = x^{-1}y^{-1}xy$. An element of the form $[x, y]$ is called a *commutator* of G. Given a subset $A \subseteq G$, we let $x^A = \{x^a : a \in A\}$. If $H \leqslant G$ and $x^h = y$ for some element $h \in H$, then x and y are said to be H-*conjugate*. Given two subsets X and Y of G, we let $[X, Y] = \langle\{[x, y] : x \in X, y \in Y\}\rangle$.

The subgroups G^n and $G^{(n)}$ are defined inductively as follows.

$$G^0 = G^{(0)} = G, \;\; G^1 = G^{(1)} = [G, G],$$

$$G^{n+1} = [G^n, G], \;\; G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

The subgroup G^1 which is generated by commutators of G is called the *commutator subgroup* of G and will be denoted by G'. A group G is called *nilpotent* if $G^n = 1$ for some n and the smallest such n is called the *nilpotency class* of G. It is well-known that a nilpotent group G satisfies the *normaliser condition*, that is, if $H < G$ then $H < N_G(H)$. Similarly, a group G is called *solvable* if $G^{(n)} = 1$ for some n and the smallest such n is called the *solvability class* or the *derived length* of G.

A subgroup H of a group G is called a *characteristic* subgroup if it is invariant under all automorphisms of G. Subgroups G^n and $G^{(n)}$ of G are characteristic subgroups.

Given two subsets A and B of a group G, AB denotes the *setwise product* of A and B, that is, $AB = \{ab : a \in A, b \in B\}$. If $A \lhd G$ and $B \leqslant G$, then $AB = \langle AB\rangle$. A group G is said to be a *central product* of subgroups $H_1, \ldots, H_n$ if $G = \langle H_1, \ldots, H_n\rangle$ and if $[H_i, H_j] = 1$ for all $i \neq j$. In this case, $G = H_1 \cdots H_n$, $H_i \lhd G$ and $H_i \cap H_j \leqslant Z(G)$, where $Z(G)$ denotes the *center* of G, that is, $Z(G) = \{g \in G : [g, G] = 1\}$. A central product $H_1, \ldots, H_n$ is a *direct product* if, for all $i \neq j$, $H_i \cap H_j = \{1\}$. In this case we write $H_1 \times \ldots \times H_n$.

Let H and K be two groups and let

$$\phi : H \longrightarrow \mathrm{Aut}(K), \;\; h \mapsto \phi_h$$

be a group homomorphism from H to the automorphism group of K. Then we may construct a new group $K \rtimes H$; the Cartesian product $K \times H$ is the set with the operation on it defined as follows

$$(k, h)(k', h') = (k \cdot \phi_h(k'), hh').$$

Let $G = K \rtimes H$. In this case we say that G is a *semidirect product* of K by H. Then, $G = HK$ with $K \lhd G$, $H \leqslant G$ and $H \cap K = 1$. Conversely, if a group G has any two subgroups H and K satisfying these properties, then $G \cong K \rtimes H$ where $\phi_h(k) = hkh^{-1}$. Given a semidirect product $G = K \rtimes H$, one may easily observe that the following things hold:

- If $H \leqslant M \leqslant G$, then $M = (K \cap M) \rtimes H$. Also if $K \leqslant M \leqslant G$, then $M = K \rtimes (H \cap M)$.
- If $H_1 \leqslant H$, then $N_K(H_1) = C_K(H_1)$.

A group G is called *simple* if it has no proper non-trivial normal subgroups. A simple group G is called *uniformly simple* if for any two non-trivial elements g and h, the length of expression of g, as a finite product of conjugates of h or the inverse of h, is uniformly bounded. Further, a group G is called *quasisimple* if $G = G'$ and $G/Z(G)$ is a non-abelian simple group.

A *normal series* of a group G is a finite sequence of normal subgroups A_i such that

$$1 = A_0 \trianglelefteq A_1 \ldots \trianglelefteq A_{n-1} \trianglelefteq A_n = G.$$

A *subnormal series* is a subgroup series in which each member of the series is normal in the next one containing it. That is, an ascending series

$$H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G$$

of subgroups of G is a subnormal series if $H_i \trianglelefteq H_{i+1}$ for all $i = 0, \ldots n-1$. A *composition series* of a group G is a subnormal series where all the factor groups of successive terms are simple groups. That is, the subnormal series above is a composition series if the factor group H_{i+1}/H_i is simple for all $i = 0, \ldots n-1$.

A group G is called *perfect* if $G = G'$ or, equivalently, if G has no non-trivial abelian factor groups.

If N is a normal subgroup of a group H such that $G \cong H/N$, or equivalently, if there is a short exact sequence of groups

$$1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1,$$

then H is called an *extension* of G by N. Given a perfect group G, a *stem extension* E of G by Z is an extension such that $Z \leqslant Z(E)$ and E is perfect. A stem extension is called a *maximal stem extension* if it maps onto any other stem extension by a homomorphism which is unique subject to commuting with the identity map on G. It is well-known that there can be at most one maximal stem extension of a perfect group G. If a stem extension E of G by Z is the maximal stem extension of G, then E is called the *Schur covering group* of G and Z is called the *Schur multiplier* of G.

A group G is called *n-divisible* if for every element $g \in G$ there is an element $y \in G$ such that $y^n = g$. If G is n-divisible for all positive integers n, then we say that G is *divisible*.

The *order* of an element $x \in G$ is the smallest positive integer such that $x^n = 1$ (if there is no such positive integer, we say that x has an infinite order). An element of order 2 is called an *involution* of G. Also, an automorphism of G of order 2 is called an *involutory automorphism*. Let i be an involutory automorphism of G. Then we say that i *inverts* G if $igi = g^{-1}$ for all elements $g \in G$. If every element of a group G has order smaller or equal to n, for some positive integer n, then G is said to have a *bounded exponent* and the smallest such n is called the *exponent* of G. Note that the word 'order' in group theory has a twofold meaning; we also have that the *order* of a (finite) group G means the cardinality of G and this is denoted by $|G|$. If two finite groups G_1 and G_2 have co-prime orders then this is denoted by $(|G_1|, |G_2|) = 1$.

Let $\pi \neq \emptyset$ be a set of primes. A *π-number* is an integer whose prime factors lie in π. A *π-element* is an element of a group G whose order is a π-number. We call a group G *π-torsion* if its elements are π-elements; *π-divisible* if G is n-divisible for any non-trivial π-number n; and *π-torsion-free* if G has no non-trivial π-elements. If π is the set of all primes we drop the π- prefix and talk about *torsion/periodic*, divisible and *torsion-free* groups respectively. If $\pi = \{p\}$ we replace the prefix π- by p- and so a group is called a *p-group* if all of its elements have orders of powers of p. An *elementary p-group* is a group in which all elements have order p. A maximal π-subgroup of a group G is called a *Hall π-subgroup* and a maximal p-subgroup of G is called a *Sylow p-subgroup*.

The complementary set of primes to π is denoted by π'. Similarly as for π and p, we also have notions of π'- and p'-elements and π'- and p'-groups. For example, a $2'$-element is just a element of odd order. Further, we call a group with no non-trivial p-elements a $p^{\perp}$-group.

Let P_1 and P_2 be any two properties of groups (e.g. being finite, nilpotent, solvable, divisible, etc.). Then we say that a group G is *locally* P_1 if every finitely generated subgroup of G has the property P_1. Further, G is called a *P_1-by-P_2* group if it has a normal subgroup H such that H and G/H has properties P_1 and P_2.

A group G is said to have the *descending chain condition* on subgroups with property P_1 if there are no infinite strictly descending chains of subgroups with the property P_1.

Needed results from finite group theory We conclude this section by stating two results from finite group theory which will be useful to us in Chapter 4.

A group action $\iota : G \times X \longrightarrow X$ is called *faithful* if there are no non-trivial group elements $g \in G$ such that $gx = x$ for all $x \in X$.

Fact 2.1.1 (Khukhro [58, Lemma 3])**.** *If an elementary abelian p-group E of order p^n acts faithfully on a finite nilpotent p'-group Q, then there is a series of subgroups*

$$E = E_0 > E_1 > E_2 > \cdots > E_n = 1$$

such that all the inclusions in the series

$$C_Q(E_0) < C_Q(E_1) < C_Q(E_1) < \cdots < C_Q(E_n)$$

are strict.

Fact 2.1.2 ([53, Corollary 3.28])**.** *Let A and G be finite groups and suppose that $(|A|, |G|) = 1$. Let A act via automorphisms on G and let $N \triangleleft G$ be A-invariant.*

Assume that at least one of the groups A or N is solvable. Then, writing $\overline{G} = G/N$, we have $C_{\overline{G}}(A) = \overline{C_G(A)}$.

2.2 Groups with finite centraliser dimension

As mentioned in the introduction, a group G is said to be of *finite centraliser dimension* if there is a finite bound on the length of any chain of centralisers in G. Let us now give the definition more formally. By a proper descending chain of centralisers of length k in a group G we mean a chain which has the following form:

$$G = C_G(1) > C_G(x_1) > C_G(x_1, x_2) > \cdots > C_G(x_1, \ldots, x_k) = Z(G).$$

Definition 2.2.1. *Let G be a group and k be a positive integer. Then G has* centraliser dimension k *if the longest proper descending chain of centralisers in G has length k. If such a k exists then G is said to be of* finite centraliser dimension*.*

Notice that, given a group G, since $C_G(C_G(C_G(A))) = C_G(A)$ for any subset $A \subseteq G$, all descending chains of centralisers are finite if and only if all ascending chains of centralisers are finite.

From now on, if a group G has centraliser dimension k, then this is denoted by $\mathrm{cd}(G) = k$.

Note that if a group G has a finite centraliser dimension and $X \subseteq G$ is any subset of G, then there exists a finite subset $A \subseteq G$ such that $C_G(X) = C_G(A)$: If such a finite subset A did not exist then we could construct an infinite centraliser chain by choosing successive elements $x_1, x_2, \ldots$ of X and forming centralisers $C_G(x_1, \ldots, x_k)$, for increasing k.

Clearly, groups of finite centraliser dimension satisfy descending chain condition on centralisers. The converse fails; in [23, Section 4] Roger M. Bryant constructed an example of a group which satisfies descending chain condition on centralisers but does not have a finite centraliser dimension.

One can easily see that the class of groups of finite centraliser dimension is closed under the formation of subgroups and finite direct products (see e.g. [63, Lemma 2.2]). Also, this class is closed under the formation of finite extensions (the proof of this fact is similar to the proof of [56, Lemma 3.20]). However, one immediate difficulty encountered in the study of any groups of finite centraliser dimension is that the descending chain condition for centralisers is not inherited by factor groups or sections. This happens even in the class of periodic nilpotent groups of finite centraliser dimension, as demonstrated in the striking example by Bryant in [23, Section 4].

In Section 4.1, we will see a sufficiently strong property, provided to us by a result of Evgeny I. Khukhro, that is true in any locally finite group of finite centraliser dimension H and is inherited by sections of H.

Let us next see some examples of groups with finite centraliser dimension.

(1.) Abelian groups. For an abelian group A we obviously have $\mathrm{cd}(A) = 0$. Further, it is easy to observe that any group G with $\mathrm{cd}(G) = 1$ is an abelian group and therefore $\mathrm{cd}(G) = 0$.

(2.) A group G is called a CT-group if the centraliser of any non-trivial element of G is an abelian subgroup. It was proven in [41, Proposition 3.9.1] that, if $\mathrm{cd}(G) = 2$ and $Z(G) = 1$ then G is a CT-group and, conversely, if G is a CT-group then $\mathrm{cd}(G) = 2$ and $Z(G) = 1$. Thus, non-abelian free groups F_n serve as natural examples of groups with centraliser dimension 2.

(3.) General linear groups $GL_n(K)$ over a field K have centraliser dimension at most $n^2 + 1$ (see e.g. [63, Proposition 2.1]). Therefore, linear algebraic groups (see Section 2.3) are of finite centraliser dimension. More generally, as the class of groups of finite centraliser dimension is closed under the formation of finite direct products, general linear groups $GL_n(R)$, where R is a finite direct product of fields, are of finite centraliser dimension.

(4.) Stable groups; see Subsection 3.3.

(5.) Groups of finite Morley rank being stable are of course of finite centraliser dimension. Actually, something stronger is true, namely, groups of finite Morley rank have no proper infinite descending chains of definable subgroups; see Section 5.1.

Note that one may express in a first-order way that a group G has centraliser dimension k, for *fixed* k (that is, there exists a first-order sentence σ in the language of groups such that σ holds in a group G if and only if $\mathrm{cd}(G) = k$)—this will be explained in Subsection 3.2. However, the descending chain condition for centralisers cannot be expressed by a first-order sentence: Consider the ultraproduct (see Section 3.4) of groups with increasing centraliser dimensions.

In this thesis we consider locally finite groups of finite centraliser dimension (Chapter 4), stable groups (Chapter 6) and groups of finite Morley rank (Chapters 6 and 7). Therefore, all our results are for certain groups of finite centraliser dimension.

2.3 Linear algebraic groups

In this subsection we briefly introduce linear algebraic groups. For any further detail we refer the reader to either of the books [52] or [17].

Let K be an algebraically closed field. The set $K^n = K \times \cdots \times K$ is called an *affine n-space*. Let I be any subset of $K[\overline{T}] = K[T_1, \ldots, T_n]$, and let

$$V(I) = \{\bar{x} \in K^n : f(\bar{x}) = 0 \text{ for all } f \in I\}.$$

Note that, as $V(I) = V(\langle I \rangle)$, where $\langle I \rangle$ is the ideal of $K[\overline{T}]$ generated by I, we may assume that I is an ideal. Conversely, since $K[\overline{T}]$ is a Noetherian ring, every ideal of $K[\overline{T}]$ is finitely generated. Thus, one may also think of I as a finite set. A set of the form $V(I)$ is called an *affine variety* over K^n.

One topologises an affine n-space by declearing that the *closed sets* are precisely the affine varieties. This topology is called the *Zariski topology*, and so, affine varieties over K^n are also called *Zariski closed subsets* over K^n. A Boolean combination of Zariski closed subsets of K^n is called a *constructible set*. Note that, since $K[\overline{T}]$ is Noetherian, any descending chain of Zariski closed subsets of K^n is stationary. It follows that, for any subset $X \subseteq K^n$, there exists the smallest closed subset $\overline{X}^{Zar}$ of K^n containing X. Such $\overline{X}^{Zar}$ is called the *Zariski closure* of X. If the Zariski closure $\overline{X}^{Zar}$ of X is equal to K^n, then X is called *Zariski dense*.

We now give one of the classical notions of dimension from algebraic geometry called the *Zariski dimension*. The Zariski dimension of an affine variety X, denoted by $\dim_K(X)$, is the transcendence degree of $K[X]$ over K. This definition naturally extends to constructible sets: Let $Y \subseteq K^n$ be a constructible set. Then $\dim_K(Y) = \dim_K(\overline{Y}^{Zar})$

A *morphism* between two arbitrary affine varieties $V \subseteq K^n$ and $W \subseteq K^m$ is a polynomial map with coefficients in K given by m polynomials in n variables. That is, a map ϕ between $V \subseteq K^n$ and $W \subseteq K^m$ is a morphism if there are polynomials $f_1(\overline{T}), \ldots, f_m(\overline{T})$ in n variables $\overline{T}$ such that $\phi(\bar{v}) = (f_1(\bar{v}), \ldots, f_m(\bar{v}))$ for all $\bar{v} \in V$.

An *affine algebraic group* G is an affine variety such that the group operations, multiplication and inversion, are given by morphisms.

Example 2.3.1. *The general linear group $GL_n(K)$ is an affine algebraic group: Denote by $GL_n(K)$ the set of invertible $n \times n$ matrices with coefficients in K. The set $M_n(K)$ of all $n \times n$ matrices over K may be identified with K^{n^2}. But, clearly, the set $\{\bar{x} \in K^{n^2} : \det(\bar{x}) \neq 0\}$ is not a Zariski closed set. However, we can identify the underlying set of $GL_n(K)$ with*

$$X = \{(\bar{x}, y) \in K^{n^2+1} : \det(\bar{x})y = 1\}.$$

Determinant of an $n \times n$ matrix is a polynomial function in n^2 variables and therefore X is the zero set of a polynomial in $n^2 + 1$ variables. Thus, X is a Zariski closed set. Moreover, multiplication and inversion of $n \times n$ matrices are given by polynomial maps, and so, $GL_n(K)$ is an affine algebraic group.

Given Example 2.3.1, it is easy to construct further examples of affine algebraic groups. Indeed, if G is an affine algebraic group $G \subseteq K^n$, then a *closed subgroup* H of G is a subgroup $H \leqslant G$ such that H, seen as a subset of K^n, is a Zariski closed subset. That is, a closed subgroup of an affine algebraic group is again an affine algebraic group. Thus, we know that a closed subgroup of $GL_n(K)$ is an affine algebraic group. Actually, the converse also holds:

Fact 2.3.2 (Humphreys [52, Subsection 8.6])**.** *Let G be an affine algebraic group. Then G is isomorphic to a closed subgroup of some $GL_n(K)$.*

Fact 2.3.2 explains why affine algebraic groups are often called *linear algebraic groups*. However, throughout this thesis, we drop both of these possible prefixes. So, from now on, whenever we say an *algebraic group* we simply mean a closed subgroup of some $GL_n(K)$.

Given an arbitrary subfield k of K one wants to know what are the closed subgroups of $GL_n(K)$ which are defined by polynomials with coefficients in k. An affine variety $X \subseteq K^n$ is said to be *k-closed* if X is the set of zeros of some collection of polynomials with coefficients in k. Note that, to say that $X = V(I)$ for some ideal I generated by its intersection with $k[\overline{T}]$ is *not* to say that the radical ideal $J(X)$ has this property. In the case in which $J(X)$ is generated by polynomials with coefficients in k, we say that X is *defined* over k. In general, being defined over k is a stronger notion than the notion of being k-closed, however, in the case of *perfect fields* (e.g. finite fields, algebraically closed fields or pseudofinite fields), these notions coincide (see e.g. [52, Subsection 34.1]).

Throughout this thesis we work with perfect fields and so, for us, "defined over k" and "k-closed" have the same meaning. Thus, if G is an algebraic group, we say that G is defined over k if it is defined over k as an affine variety and if the group operation maps are given by polynomial maps with coefficients in k. If this is the case, then we denote k-rational points of G by $G(k)$, which is a subgroup of G.

2.3.1 Groups of Lie type

In what follows, we briefly discuss the classification of simple algebraic groups. We will not go into deep detail—instead, we refer the reader unfamiliar with the topic to either of the books [79] or [27].

An algebraic group G is said to be *connected* if it has no proper closed subgroups of finite index. Let G be a connected algebraic group. Then G is called *simple*, if it has no non-trivial proper connected closed normal subgroups. A simple algebraic group G has finite center $Z(G)$ and the factor group $G/Z(G)$ is *abstractly simple*, that is, $G/Z(G)$ has no non-trivial proper normal subgroups. Simple algebraic groups over algebraically closed fields of arbitrary characteristic were classified, up to isomorphism, by Claude Chevalley [33, 34]. We now discuss this classification, very briefly.

Each isomorphism class of finite-dimensional simple Lie algebras over $\mathbb{C}$ determines, and is determined by, a connected Dynkin diagram. Algebraic groups that belong to families $A_n (n \geqslant 1)$, $B_n (n \geqslant 2)$, $C_n (n \geqslant 3)$ and $D_n (n \geqslant 4)$ consisting of linear, orthogonal and symplectic groups are called *classical groups*. The rest of the groups belonging to families E_6, E_7, E_8, F_4, G_2 are called *exceptional groups*. In both of these lists, the subscripts denote the *Lie ranks* of the corresponding groups. For each type of classical and exceptional groups there exists *simply connected* and *adjoint* versions. The adjoint type groups have trivial centers, and so, they are abstractly simple.

Example 2.3.3. *Groups $SL_{n+1}(K)$ and $PSL_{n+1}(K) = PGL_{n+1}(K)$ over an algebraically closed field K are respectively simply connected and adjoint versions of Chevalley groups of type $A_n(K)$.*

Chevalley showed how to associate a group $X(k)$ to an arbitrary field k and a symbol X from the list A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2—groups constructed this way are called *Chevalley groups*. When the Dynkin diagram has a non-trivial symmetry and the field k satisfies suitable additional conditions, twisted analogues of Chevalley groups can be constructed. Such groups are called *twisted Chevalley groups*. Twisted Chevalley groups are also denoted by $X(k)$, where X denotes one of the symbols ${}^2A_n (n \geqslant 2)$, ${}^2D_n (n \geqslant 4)$, ${}^2B_2$, ${}^3D_4$, ${}^2E_6$, ${}^2F_4$, or ${}^2G_2$.

The groups of types ${}^2A_n$, ${}^2D_n$, ${}^3D_4$ and ${}^2E_6$ over appropriate finite fields were constructed by Robert Steinberg [78]. The so-called Suzuki and Ree groups, that is, groups of type ${}^2F_4$, ${}^2G_2$, and ${}^2B_2$ we constructed by Michio Suzuki [81] and Rimhak Ree [72, 73]. Finally, Chevalley groups and twisted Chevalley groups over finite fields were classified uniformly as groups of fixed points of some special endomorphisms of algebraic groups over the algebraic closures of the finite fields in concern by Steinberg [80].

Chevalley proved that a Chevalley group $X(k)$ is simple as an abstract group, except in the case of few small finite fields, namely, $A_1(\mathbb{F}_2)$, $B_2(\mathbb{F}_2)$, $G_2(\mathbb{F}_2)$, and $A_1(\mathbb{F}_3)$. Similarly, twisted Chevalley groups over the fields over which they exist are simple as abstract groups except for ${}^2A_2(\mathbb{F}_4)$, ${}^2B_2(\mathbb{F}_2)$, ${}^2F_4(\mathbb{F}_2)$, and ${}^2G_2(\mathbb{F}_3)$. In finite group theory, groups of types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, ${}^2A_n$, ${}^2D_n$, ${}^3D_4$, ${}^2E_6$, ${}^2B_2$, ${}^2F_4$, ${}^2G_2$ over finite fields, with the eight exceptions specified above, are called *simple groups of Lie type*. Throughout this thesis, we follow the finite group theory terminology and use this umbrella term "groups of Lie type" to all Chevalley groups, twisted Chevalley groups, and non-algebraic twisted groups of Lie type ${}^2B_2$, ${}^2F_4$, or ${}^2G_2$ over finite, locally finite, pseudofinite and algebraically closed fields.

Note that, a Chevalley group over an algebraically closed field K coincide with the K-rational points, $G(K)$, of an algebraic group G defined over K. Therefore, Chevalley groups over algebraically closed fields are algebraic groups by construction.

The following well-known fact will be useful to us in Section 4.2.

Fact 2.3.4 (Ellers, Gordeev and Herzog [43]). *Simple groups of Lie type are uniformly simple.*

2.3.2 Automorphisms of Chevalley groups

There are four types of automorphisms of a Chevalley group $X(F)$. They are called *inner*, *diagonal*, *field* and *graph* automorphisms, see e.g. [27] or [80].

Let $X(F)$ be a Chevalley group defined over a field F. Throughout this thesis we will denote by $\mathrm{Aut}(X)$, $\mathrm{Inn}(X)$, $\mathrm{Diag}(X)$, $\mathrm{Aut}(F)$ and $\mathrm{Grp}(X)$ the group of automorphisms, inner automorphisms, diagonal automorphisms, field automorphisms and graph automorphisms of the group $X(F)$, respectively. Further, the group of outer diagonal automorphisms, which is isomorphic to $\mathrm{Diag}(X)/\mathrm{Inn}(X)$, is denoted by $\mathrm{OutDiag}(X)$.

The four types of automorphisms of $X(F)$ can be described very briefly as follows.

- Inner automorphisms of $X(F)$ are induced by conjugation by the elements of the group. It is routine to check that $X(F) \cong \mathrm{Inn}(X)$ when $X(F)$ is a simple group.

- Diagonal automorphisms of $X(F)$ are induced by conjugation by some elements which can be represented by diagonal matrices with respect to the Chevalley basis.

- Field automorphisms of $X(F)$ are induced by automorphisms of F.

- Graph automorphisms of $X(F)$ are induced by the symmetries of the Dynkin diagram.

We conclude this section by presenting a result which describes the automorphism group Aut(X) of a Chevalley group $X(k)$ defined over an arbitrary perfect field k.

Fact 2.3.5 (Gorenstein et al. [48]). *Let $X(k)$ be a Chevalley group over a perfect field k and $\alpha \in \mathrm{Aut(X)}$. Then, $\alpha = idfg$ where $i \in \mathrm{Inn}(X)$, $d \in \mathrm{Diag}(X)$, $f \in \mathrm{Aut}(k)$ and*

$g \in \mathrm{Grp}(X)$. *Moreover,*

$$\mathrm{Aut}(X) = \mathrm{Inn}(X)\mathrm{Diag}(X) \rtimes \mathrm{Aut}(k)\mathrm{Grp}(X),$$

and $\mathrm{OutDiag}(X)$ *is either cyclic of order bounded in terms of the rank or isomorphic to the elementary abelian group of order 4.*

2.4 Locally finite groups

We already defined locally finite groups in Section 2.1—they are groups in which every finitely generated subgroup is finite. The topic of locally finite groups lies in the, not too well-defined, cross-roads of finite group theory and general (infinite) group theory. One one hand, the class of locally finite groups is restricted enough for many interesting results on finite groups to carry over. On the other hand, this class is wide enough so that most of the arithmetic results on finite groups do not carry over, and so, new phenomenon arise. For the general theory of locally finite groups, see [56].

It immediately follows from the definition that subgroups and factor groups of locally finite groups are locally finite. It is also obvious that every locally finite group has to be periodic, however, the converse fails; even a periodic group of finite exponent need not to be locally finite, see [1, 2, 3]. Further, extensions of locally finite (resp. periodic) groups by locally finite (resp. periodic) groups are locally finite (resp. periodic), see [56, 1.A.2 Lemma].

Let us see some examples of locally finite groups.

(1.) Finite groups are obviously locally finite.

(2.) It follows from the fundamental theorem of finitely generated abelian groups that periodic abelian groups are locally finite. Therefore, periodic solvable groups and periodic locally solvable groups are locally finite.

(3.) The quasicyclic group $Z(p^\infty) = \{x \in \mathbb{C} : x^{p^n} = 1 \,\mathrm{for\,some}\, n \in \mathbb{N}\}$ called the *Prüfer p-group* is a locally finite divisible abelian group; here all subgroups are of the form $(\frac{1}{p^n}\mathbb{Z})/\mathbb{Z}$ with p^n elements.

(4.) ω-categorical groups (see Subsection 3.3); this follows easily from the well-known Ryll–Nardzewski Theorem, see e.g. [82, Theorem 4.3.1].

(5.) Groups of Lie type over fields $\overline{\mathbb{F}}_p$, where $\overline{\mathbb{F}}_p$ denotes the algebraic closure of the prime field $\mathbb{F}_p$. Note that $\overline{\mathbb{F}}_p$ is a locally finite field; every finite set of elements lies in a finite subfield (and of course locally finite fields are precisely the subfields of such algebraic closures). Since a finite number of matrices have only a finite number of entries between them, we see that $GL_n(\overline{\mathbb{F}}_p)$ is a locally finite group which has a *local system* (see [49, Definition 1.2]) consisting of groups $GL_n(\mathbb{F}_q)$, where q ranges over powers of p.

2.4.1 Frattini Argument for locally finite groups of finite centraliser dimension

One of the most important results in finite group theory is Sylow's theorem stating that, for any prime p, the Sylow p-subgroups of a finite group G are conjugate. A simple consequence of Sylow's theorem called the *Frattini Argument* is of fundamental importance: *Let G be a finite group, $H \lhd G$ be a normal subgroup and S be a Sylow p-subgroup of H. Then $G = HN_G(S)$.* In finite group theory, the Frattini Argument is frequently used in the following strengthened form.

Fact 2.4.1. (Frattini Complement) *Let G be a finite group and $H \lhd G$. Then $G = HM$ for some subgroup M such that $H \cap M$ is nilpotent.*

Proof. This fact is just a repeated application of the Frattini Argument to prime divisors of $|G|$. □

The Fattini Argument itself (assuming that one can prove the analogue of Sylow's theorem) of course carries over to the locally finite case (or to any other case) and the proof of the locally finite case is identical to the well-known proof of the finite case (see e.g. [47, Theorem 3.7]). Below we give the Frattini Argument in the locally finite case.

Fact 2.4.2 (Frattini Argument)**.** *Let H be a locally finite group and $K \lhd H$ be a normal subgroup. Assume that the Hall π-subgroups of K are conjugate in K and that P is one of them. Then $H = KN_H(P)$.*

In Chapter 4, we study locally finite groups of finite centraliser dimension. Therefore, the value of the Frattini Argument for us is obvious because of the following two important results.

Fact 2.4.3 (Bryant [23, Theorem B]). *In a locally finite group H of finite centraliser dimension Sylow p-subgroups are conjugate for all primes p.*

Fact 2.4.4 (Bryant and Hartley [24, Theorem 1.6]). *In a periodic solvable group H of finite centraliser dimension Hall π-subgroups are conjugate for all sets of primes π.*

2.4.2 Derived lengths of solvable subgroups of locally finite groups of finite centraliser dimension

In Subsection 4.1.1, we explain how, in the case of locally finite groups of finite centraliser dimension, one may overcome the problem that the descending chain condition for centralisers is not inherited in general by factor groups or sections. We will see that a sufficiently strong property which holds in every locally finite group of finite centraliser dimension and is inherited by its sections is provided to us by the following result due to Khukhro (the reader might also want to see [24]).

Fact 2.4.5 (Khukhro [58, Theorem 1(a)]). *Periodic locally solvable groups of centraliser dimension k are solvable and have derived lengths bounded by a function of k.*

Clearly, Fact 2.4.5 is saying that in a locally finite group G with $\mathrm{cd}(G) = k$ derived lengths of solvable subgroups are bounded and the bound depends on k only.

In [58], Khukhro also proved another result which is of our interest:

Fact 2.4.6 (Khukhro [58, Theorem 2]). *Suppose that a group G is elementarily equivalent to an ultraproduct of finite solvable groups. If G has centraliser dimension k, then G is solvable of k-bounded derived length.*

2.4.3 Simple locally finite groups

We turn our attention to simple locally finite groups. An excellent reference for simple locally finite groups is [49].

Though in this thesis we are mainly interested in infinite simple locally finite groups, we start by mentioning the famous Classification of Finite Simple Groups—which is denoted by "CFSG" from now on. CFSG states that *every* finite simple group is isomorphic to one of the following.

- Cyclic group of prime order.
- Alternating group Alt_n, for $n \geqslant 5$.
- Simple group of Lie type over some finite field.
- One of the 26 sporadic groups.

It seems very hard to make progress in the study of (infinite) simple locally finite groups without the use of CFSG—indeed, CFSG is used in both of our theorems on locally finite groups; Theorems 4.1.1 and 4.2.1. Given CFSG it would be natural to pursue classification results for simple locally finite groups. However, there is no general classification for such groups (and further, such a classification seems to be out of reach, see [49, Section 1]).

Let us briefly discuss simple groups in the general theory of (locally) finite groups. In the case of finite groups, the famous Jordan–Hölder Theorem for composition series states that any two composition series of a finite group G have the same length and, with respect to a suitable ordering of the composition factors, the corresponding factors are isomorphic. Therefore, it is natural to argue that simple groups are the building-blocks of finite group theory. In the case of locally finite groups, the Jordan-Hölder Theorem is no longer true. The following example of the situation where the Jordan–Hölder Theorem breaks is due to Brian Hartley.

Example 2.4.7 (Hartley [49, pages 2-3])**.** *Let S be a countably infinite abstractly simple locally finite group, e.g., $S \cong PSL_2(K)$, where K is an infinite algebraic extension of a finite field. Write $S \cong G/H$, where G is countable, locally finite and* residually finite*, i.e., the intersection of all the subgroups of finite index in G is $\{1\}$. The series $G > H > 1$ can be refined to a composition series and the factor G/H cannot be further refined. Therefore, S is a composition factor in this composition series. At the same time, since G is residually finite and countable, it has a series of normal subgroups*

$$G = G_0 > G_1 > G_2 > \cdots ,$$

where each G_i have a finite index in G and $\bigcap_{i=0}^{\infty} G_i = \{1\}$. Clearly, the factors of a composition series refining this are finite.

Example above describes a situation where the Jordan–Hölder Theorem strikingly breaks down; it is by no means the only example of the failure of the Jordan-Hölder Theorem in the context of locally finite groups. For example, it is well-known that there are countably infinite locally finite simple groups for which the following holds: G possess a series

$$\dots G_{-2} \lhd G_{-1} \lhd G_0 \lhd G_1 \lhd G_2 \dots$$

of proper subgroups of G where the factors are finite and $G = \bigcup_{i=1}^{\infty} G_i$ (see [49, Theorem 1.27] and [61]).

In Section 4.1, we tackle the issue of the absence of the Jordan–Hölder Theorem, in the case of locally finite groups of finite centraliser dimension, using the following result by Buturlakin and Andrei V. Vasil'ev.

Fact 2.4.8 (Buturlakin and Vasil'ev [26])**.** *Let G be a locally finite group of centraliser dimension k. Then the number of non-abelian composition factors of G is less than $5k$.*

Useful classification results We conclude this section with the following two classification results, both of which are based on CFSG.

Fact 2.4.9 (Hartley [49, Theorem 2.6])**.** *Let G be an infinite simple locally finite group. Then the following conditions are equivalent.*

(i) G is linear.

(ii) G is of Lie type over a locally finite field.

(iii) Some finite group is not involved in G.

(iv) One of the following holds:

(a) G has a Kegel cover *(see [49, Section 2]) of fixed classical type with bounded rank parameters.*

(b) G has a Kegel cover of fixed exceptional type.

Fact 2.4.10 (Thomas [83, Theorem 2]). *An infinite simple locally finite group of finite centraliser dimension is a Chevalley group or a twisted Chevalley group over a locally finite field.*

Chapter 3

Some model theory

This chapter serves as a brief introduction to model theory. We also fix our model-theoretic notation. For further introduction, we refer the reader to a classical book by Chen C. Chang and H. Jerome Keisler [29] or to a more recent book by Katrin Tent and Martin Ziegler [82].

First, in Section 3.1, we introduce first-order structures and first-order theories. Then, in Section 3.2, we introduce the most crucial concept of model theory—a definable set. We give several examples of definable subsets of a group G in the language of groups.

Stable structures and, more precisely, stable groups are introduced in Section 3.3. In particular, we present and prove the well-known result by John T. Baldwin and Jan Saxl stating that stable groups are of finite centraliser dimension.

In Section 3.4, we introduce ultraproducts and pseudofinite structures.

3.1 Languages, structures and theories

Model theory is a branch of mathematical logic which concerns the interplay between mathematical structures (such as groups, fields etc.) and the first-order language which is used to describe these structures.

We define a *language* $\mathcal{L}$ as follows: $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, where $\mathcal{R}$ is the set of relation symbols of given arity, $\mathcal{F}$ is the set of function symbols of given arity and $\mathcal{C}$ is the set of constant symbols. For example, we may consider the language of groups $\mathcal{L}_{groups} = (\,\cdot\,, {}^{-1}, e)$, where $\cdot \in \mathcal{F}$ is a binary (has arity 2) function symbol, ${}^{-1} \in \mathcal{F}$ is a unary (has arity 1) function symbol and the identity element $e \in \mathcal{C}$.

Given a first-order language $\mathcal{L}$ we define an $\mathcal{L}$-*structure* $\mathcal{M}$ as follows:

$$\mathcal{M} = (M, R^{\mathcal{M}}, f^{\mathcal{M}}, c^{\mathcal{M}} : R \in \mathcal{R}, f \in \mathcal{F}, \text{ and } c \in \mathcal{C}),$$

where M is the underlying set of $\mathcal{M}$ and $R^{\mathcal{M}}, f^{\mathcal{M}}$, and $c^{\mathcal{M}}$ are the *interpretations* of the symbols $R \in \mathcal{R}$, $f \in \mathcal{F}$ and $c \in \mathcal{C}$. For example, $\mathcal{G} = (G, \cdot^{\mathcal{G}}, ^{-1^{\mathcal{G}}}, e^{\mathcal{G}})$ is an $\mathcal{L}_{groups}$-structure equipped with a binary function symbol $\cdot^{\mathcal{G}}$ denoting the group multiplication operation, a unary function symbol $^{-1^{\mathcal{G}}}$ denoting the inversion operation and a constant symbol $e^{\mathcal{G}}$ denoting the identity element e of the group.

In model theory, one is interested in maps between structures that preserve the interpretations of the language—this way we may define substructures and extensions: Let $\mathcal{M} = (M, \ldots)$ and $\mathcal{N} = (N, \ldots)$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $\iota : \mathcal{M} \to \mathcal{N}$ is a one-to-one map which preserves the interpretation of each of the symbols of the language $\mathcal{L}$. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, then $\mathcal{M}$ is said to be a substructure of $\mathcal{N}$ and $\mathcal{N}$ is said to be an extension of $\mathcal{M}$.

Given an $\mathcal{L}$-structure $\mathcal{M}$, an $\mathcal{L}$-*formula* is a finite string of symbols which is formed in a natural way using

- symbols of the language $\mathcal{L}$,
- variables $v_1, \ldots, v_n$ denoting the elements of the underlying set M of $\mathcal{M}$,
- equality symbol $=$,
- logical connectives $\vee$, $\wedge$ and $\neg$,
- quantifiers $\forall$ and $\exists$, and,
- parentheses (,).

An $\mathcal{L}$-*sentence* is an $\mathcal{L}$-formula with no *free variables*, that is, an $\mathcal{L}$-sentence is an $\mathcal{L}$-formula in which all variables are bound by a quantifier. $\mathcal{L}$-formulas and $\mathcal{L}$-sentences have a natural notion of "truth" in an $\mathcal{L}$-structure $\mathcal{M}$: Let $\phi(\bar{x})$ be an $\mathcal{L}$-formula with variables $\bar{x}$ and let $\bar{a} \in M^n$. Then, we write $\mathcal{M} \models \phi(\bar{a})$ if the tuple $\bar{a}$ satisfy ϕ in $\mathcal{M}$. For example, given a group G and an $\mathcal{L}_{groups}$-formula $\phi(x) := \forall y(xy = yx)$, we have $G \models \phi(a)$ if and only if $a \in Z(G)$. Similarly, for an $\mathcal{L}$-sentence σ we write

$\mathcal{M} \models \sigma$ if σ holds in $\mathcal{M}$. For example, given a group H and an $\mathcal{L}_{groups}$-sentence $\sigma := \forall y \forall x(xy = yx)$, we have $H \models \sigma$ if and only if H is an abelian group. Notice that each $\mathcal{L}$-sentence is either true or false in an $\mathcal{L}$-structure $\mathcal{M}$. However, given an $\mathcal{L}$-formula ϕ with free variables $x_1, \ldots, x_n$, we think of ϕ as expressing a property of elements of M^n.

The notion of *elementary equivalence* arises as one considers structures that satisfy the same $\mathcal{L}$-sentences. Indeed, we say that two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementary equivalent* if, for all $\mathcal{L}$-sentences σ, $\mathcal{M} \models \sigma$ if and only if $\mathcal{N} \models \sigma$. If this is the case, then we write $\mathcal{M} \equiv \mathcal{N}$. Elementary equivalence is a strictly weaker notion than isomorphism—indeed, it is easy to see that isomorphism implies elementary equivalence but the converse clearly fails; $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ as rational numbers and real numbers have the same order structure but $(\mathbb{Q}, <) \not\cong (\mathbb{R}, <)$ as $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

One often wants to consider the more restrictive class (than just $\mathcal{L}$-embeddings) of maps that preserve all $\mathcal{L}$-formulas: Given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ and an $\mathcal{L}$-embedding $e : \mathcal{M} \to \mathcal{N}$, e is called an *elementary embedding* if it is an inclusion map such that

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \models \phi(e(a_1), \ldots, e(a_n))$$

for all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and for all $a_1, \ldots, a_n \in M^n$. If $\mathcal{M}$ is a substructure of $\mathcal{N}$ and an $\mathcal{L}$-embedding $e : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then $\mathcal{M}$ is called an *elementary substructure* of $\mathcal{N}$ and $\mathcal{N}$ is called an *elementary extension* of $\mathcal{M}$. In this case we write $\mathcal{M} \preceq \mathcal{N}$. Obviously, if $\mathcal{M} \preceq \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$.

We now present a well-known result, called the *Tarski–Vaught Test*, which offers one a tactic to check whether a given substructure $\mathcal{M}$ of a structure $\mathcal{N}$ is an elementary substructure.

Fact 3.1.1 (Tarski–Vaught Test, see e.g. [82, Theorem 2.1.2]). *Let $\mathcal{M} = (M, \ldots)$ be a substructure of $\mathcal{N} = (N, \ldots)$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if, for any formula $\phi(v, w_1, \ldots w_n)$ and $\bar{a} \in M^n$, if there exists $b \in N$ such that $\mathcal{N} \models \phi(b, \bar{a})$, then there exits $c \in M$ such that $\mathcal{N} \models \phi(c, \bar{a})$.*

An $\mathcal{L}$*-theory* T is a set of $\mathcal{L}$-sentences. A *model* $\mathcal{M}$ of an $\mathcal{L}$-theory T is an $\mathcal{L}$-structure in which all of the $\mathcal{L}$-sentences of T hold. If $\mathcal{M}$ is a model of a theory T, then we write $\mathcal{M} \models T$. If a theory T has a model, then we say that T is *consistent* or *satisfiable*. A theory T is called *complete* if it is a maximal consistent set of sentences. Note that the easiest way of forming complete theories is to consider a theory of given $\mathcal{L}$-structure $\mathcal{M}$, that is, $T = Th(\mathcal{M}) = \{\sigma : \mathcal{M} \models \sigma\}$ is a complete theory of $\mathcal{M}$. One may easily observe that $\mathcal{M} \equiv \mathcal{N}$ if and only if $Th(\mathcal{M}) = Th(\mathcal{N})$.

A class of $\mathcal{L}$-structures $\mathcal{C}$ is called an *elementary class* if there is an $\mathcal{L}$-theory T such that $\mathcal{C} = \{\mathcal{M} : \mathcal{M} \models T\}$. Typically in model theory, instead of working with the full theory $Th(\mathcal{M})$ of an $\mathcal{L}$-structure $\mathcal{M}$, one has a class of structures in mind and tries to write down a set of sentences, called *axioms* of the elementary class, of properties T describing these structures. One can easily write down axioms of linear orders, graphs, groups, abelian groups, ordered abelian groups, rings, fields, differential fields, etc. Clearly, if the class of structures with property P is an elementary class, then P is an elementary property (that is, P is preserved under elementary equivalences). We now state an important result by Thomas stating that being a Chevalley group $X(K)$ of fixed Lie type X over a field K is an elementary property.

Fact 3.1.2 (Thomas [84, Theorem 18])**.** *For each Chevalley group of fixed type X, the class $\{X(K) : K$ is a field $\}$ is finitely axiomatisable.*

3.2 Definable sets and interpretability

The most crucial concept of model theory, a *definable set*, the solution set of a first-order formula in a structure, can be seen as an abstraction of the notion of algebraic variety. The formal definition goes as follows.

Definition 3.2.1. *Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure. A subset $X \subseteq M^n$ is said to be* definable *if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and a tuple $\bar{b} \in M^m$ such that*

$$X = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}.$$

In this case, we say that $\phi(\bar{v}, \bar{b})$ defines *X.*

When considering a definable set $X \subseteq M^n$, one often needs to clarify what is the parameter set A that X is defined over. We say that X is *A-definable*, or *definable over A*, if (in the definition above) there exists a formula $\varphi(\bar{v}, w_1, \ldots, w_\ell)$ and a tuple

$\bar{b} \in A^\ell$ such that $\varphi(\bar{v}, \bar{b})$ defines X. If $A = \emptyset$ we say that X is definable without parameters.

Given the definition of a definable set, we may now give one of the key definitions of Sections 4.2 and 6.2: A *definably simple* group G is a group without proper non-trivial definable normal subgroups.

In what follows we express some definable subgroups of a group G in the language of groups.

(1.) $\phi_Z(y) := \forall x(xy = yx)$ is a formula without parameters and with a free variable y which defines the center $Z(G)$ of G.

(2.) $\phi_C(x) := [x, x_1] = 1 \wedge \cdots \wedge [x, x_n] = 1$ is a formula with parameters $x_1, \ldots, x_n$ and a free variable x which defines the centraliser $C_G(x_1, \ldots, x_n)$.

(3.) Let $A \subseteq G$ be a definable subset defined by a formula $\phi_A(x)$. Then

$$\phi_{C(A)}(y) := \forall x(\phi_A(x) \to [x, y] = 1)$$

is the formula defining the centraliser $C_G(A)$. Similarly,

$$\phi_{N(A)}(y) := \forall x(\phi_A(x) \to \phi_A(x^y))$$

is the formula defining the normaliser $N_G(A)$.

Note that, given a group G of finite centraliser dimension, $C_G(A)$ is definable for all (not only definable) subsets $A \subseteq G$. This follows as there exists a finite, and therefore definable, subset $B \subseteq G$ such that $C_G(B) = C_G(A)$.

Next we briefly discuss the expressibility and limitations of first-order logic. Limitations arise as we can only quantify over *elements* of a structure, not over relations, functions or even natural numbers if they are not part of the structure. Given this, let us see, via some examples, what can be expressed and what cannot be expressed in a first-order way.

Again, let G be a group in the language of groups.

(1.) $\sigma_Z := \forall x \forall y(xy = yx)$ is a sentence stating commutativity. Therefore, we may express in a first-order way that G is abelian.

(2.) We can express in a first-order way that G is solvable (resp. nilpotent) of derived length (resp. nilpotency class) at most n, for *fixed* n. However, we cannot express that G is solvable (resp. nilpotent) in general. For example, the sentence

$$\sigma_{S_2} := \forall x \forall y \forall z \forall h([[x,y],[z,h]] = 1)$$

expresses that G is solvable of derived length at most 2, and the sentence

$$\sigma_{N_2} := \forall x \forall y \forall z([[x,y],z] = 1)$$

expresses that G is nilpotent of nilpotency class at most 2.

(3.) The property that a group has a finite number of elements can not be expressed in a first-order way. Likewise, we cannot express that a group is finitely generated. However, for a *fixed* n, the sentence

$$\sigma_n := \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge \ldots \wedge x_{n-1} \neq x_n)$$

expresses that G has *at least* n many elements.

(4.) The property that a group has the descending chain condition for centralisers cannot be expressed in a first-order way. However, we can express in a first-order way that a group G has centraliser dimension k, for *fixed* k. This can be observed as follows. Let $\phi_{C_k}(x)$ be the formula defining $C_G(x_1, \cdots, x_k)$. Then, for fixed $k \geqslant 2$ (recall that if $k = 0$ or 1, then G is abelian), let σ_k be the following sentence.

$\forall x_1 \ldots \forall x_k \forall y_1 \ldots \forall y_{k-1}((\phi_{C_1}(y_1, x_1) \wedge \neg\phi_{C_2}(y_1, x_1, x_2) \wedge \phi_{C_2}(y_2, x_1, x_2) \wedge$

$\neg\phi_{C_3}(y_2, x_1, x_2, x_3) \wedge \cdots \wedge \phi_{C_{k-1}}(y_{k-1}, x_1, \ldots, x_{k-1}) \wedge \neg\phi_{C_k}(y_{k-1}, x_1, \ldots, x_k))$

$\rightarrow \forall x \forall y(\phi_{C_k}(x, x_1, \ldots, x_k) \rightarrow [x,y] = 1)$

We may observe that σ_k states that G has centraliser dimension less than or equal to k. Further, one can easily check that the sentence $\sigma_k \wedge \neg\sigma_{k-1}$ holds in a group G if and only if G has centraliser dimension k.

Let then F be a field and $\mathcal{L}_{rings} = (\cdot, +, -, 0, 1)$ be the language of rings with two binary function symbols $\cdot$ and $+$, one unary function symbol $-$ and two constant symbols 0 and 1. We now give some first-order properties of fields in the language of rings.

(1.) We can express is a first-order way that F has characteristic p, for p a prime. Let $\sigma_p := \forall x(p \cdot x = 0)$ for $p \neq 0$. Then $F \models \sigma_p$ if and only if F has characteristic p. Further, the infinite set of sentences $\{\neg\sigma_p : p > 0\}$ hold in F if and only if F has characteristic 0.

(2.) Let $\sigma_n := \forall x_0 \ldots \forall x_n(x_n \neq 0 \rightarrow \exists y(x_n y^n + x_{n-1}y^{n-1} + \ldots + x_1 y + x_0 = 0))$. Clearly σ_n says that every polynomial of degree n has a root, and therefore, F is algebraically closed if and only if the infinite set of sentences $\{\sigma_n : n \geqslant 2\}$ hold in F.

(3.) Let p be a prime and $\sigma_p := (p \cdot 1 \neq 0) \rightarrow \forall x \exists y(y^p = x)$. Further, let T be the theory consisting all the sentences σ_p. Then a field F is perfect if and only if F is a model of T.

Many times in model theory we find it useful to study structures which can be defined inside a given structure. Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}_0$-structure and $\mathcal{N} = (N, \ldots)$ be an $\mathcal{L}_1$-structure. Then, we say that $\mathcal{M}$ is *definably interpretable* in $\mathcal{N}$ if and only if, for some n, there exists an $\mathcal{L}_1$-definable $X \subseteq N^n$ and we can interpret symbols of $\mathcal{L}_0$ as $\mathcal{L}_1$-definable subsets and functions on X so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{M}$.

Example 3.2.2. *Let K be a field, $G = GL_2(K)$ and $X = \{(x, y, z, h) \in K^4 : xh - yz \neq 0\}$. Define $f : X^2 \rightarrow X$ by*

$$f((x_1, y_1, z_1, h_1), (x_2, y_2, z_2, h_2)) = (x_1x_2 + y_1z_2, x_1y_2 + y_1h_2, z_1x_2 + h_1z_2, z_1y_2 + h_1h_2).$$

Clearly, X and f are definable in $(K, +, \cdot)$. We also see that the set X with operation f is isomorphic to $GL_2(K)$, where the identity element of X is $(1, 0, 0, 1)$.

Example 3.2.2 shows that $(GL_n(K), \cdot, e)$ is definably interpretable in K.

3.2.1 The space of types

Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure with $A \subseteq M$. We obtain a new language $\mathcal{L}_A$ by adding to our language $\mathcal{L}$ new constant symbols c_a for each element $a \in A$. Now we may naturally view $\mathcal{M}$ as an $\mathcal{L}_A$-structure by interpreting c_a's in the obvious way. Denote by $Th_A(\mathcal{M})$ the set of all $\mathcal{L}_A$-sentences which hold in $\mathcal{M}$.

Definition 3.2.3. *Let p be the set of $\mathcal{L}_A$-formulas in free variables $x_1, \ldots, x_n$. Then p is called an* n-type *of $\mathcal{M}$ over A if $p \cup Th_A(\mathcal{M})$ is satisfiable. An n-type p is called a* complete n-type *if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables $x_1, \ldots, x_n$. We denote the set of all complete n-types by $S_n^{\mathcal{M}}(A)$.*

The space of all complete n-types $S_n^{\mathcal{M}}(A)$ is a compact Hausdorff space when endowed with a natural topology called the *Stone topology*, see e.g. [82, Section 4.2].

3.3 Stable structures

One crucially important concept in model theory, introduced by Shelah [77], is *stability*. In what follows, we briefly introduce stable theories and stable structures. For further introduction on stability theory, we refer the reader to [69].

A first-order theory T with only a single model, up to isomorphism, is called *categorical*. It is easy to see that a categorical theory T is complete, i.e., for every sentence σ of the appropriate language, either $T \models \sigma$ or $T \models \neg\sigma$. Further, it is well-known that no first-order theory with infinite models can be categorical (this is a consequence of a well-known theorem called the Upwards Löwenhein-Skolem Theorem, see e.g. [82, Theorem 2.3.1]). However, it turns out that the following weaker notion is extremely useful in the study of complete first-order theories. Given a cardinal κ and a complete first-order theory T with models of size κ, T is called κ*-categorical* if any two models of T of cardinality κ are isomorphic. Note that, like any categorical theory, also a κ-categorical theory T is complete. In 1965, Morley proved that if a complete countable first-order theory T is κ-categorical for *some* uncountable cardinal κ, then T is κ-categorical for *all* uncountable cardinals [62]—such theories are called *uncountably categorical*. Morley's result is today known as the *Morley's Categoricity Theorem*.

From now on, following the standard notation, the smallest infinite cardinal number $\aleph_0$ is denoted by ω.

Example 3.3.1. *The theory* ACF_p *of algebraically closed fields of fixed characteristic* p *(here* $p = 0$ *or* p *is a prime number) is uncountably categorical: Two algebraically closed fields are isomorphic if and only if they have the same characteristic and transcendence degree. An algebraically closed field of transcendence degree* λ *has cardinality* $\lambda + \omega$*. Given an uncountable cardinal* κ*, an algebraically closed field of cardinality* κ *has transcendence degree* κ *and therefore any two algebraically closed fields of the same characteristic and cardinality* κ *are isomorphic.*

A non-example of an uncountably categorical theory is DLO, the theory of dense linear order without endpoints—this theory is ω-categorical but not uncountably categorical (see e.g. [82, Section 2.3]).

Let T be an arbitrary first-order theory. In the late 60's, Shelah initiated a far-reaching program of attempting to either 'classify' models of T, up to isomorphism, or to show that such a 'classification' is not possible [77]. In order to implement his programme, Shelah developed different notions—the most important one being *stability*. Shelah proved that an *unstable* theory T has 2^λ non-isomorphic models of cardinality λ for all $\lambda > |T|$. So, from the point of view of Shelah's programme, unstable models cannot be 'classified'. We now give the definition of a κ-stable theory T.

Definition 3.3.2. *Let* T *be a complete theory in a countable language and* κ *be an infinite cardinal. Then* T *is called* κ-stable *if for all models* $\mathcal{M} \models T$ *with* $A \subseteq M$*, if* $|A| \leqslant \kappa$ *then* $|S_n^{\mathcal{M}}(A)| \leqslant \kappa$*.*

If there is some cardinal κ such that T is κ-stable, then T is called *stable*—otherwise T is called unstable.

We say that a structure $\mathcal{M}$ is (κ-)stable if $Th(\mathcal{M})$ is (κ-)stable.

Stable theories and stable structures have been of the interest of many people after Shelah defined them. It is well-known today that there exist many equivalent definitions for stable structures. One of such—a combinatorial one—goes as follows: Stable structures are structures in which no first-order formula can totally order arbitrarily large sets of tuples. Formally, this can be written down as follows.

Definition 3.3.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then $\mathcal{M}$ is called stable if there is no $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y}) \in Th(\mathcal{M})$ and tuples $\bar{a}_i$, $\bar{b}_j$ with $i, j < \omega$ such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_j)$ if and only if $i \leqslant j$.*

It is well-known that a structure interpretable in a stable structure is stable (see e.g. [82, Lemma 8.4.2]).

Note that, a fortiori, a stable structure does not possess any stronger properties (such as the independence property, see below) than the order property.

Definition 3.3.4. *A formula $\phi(x, y)$ is said to have the* independence property *if there are $a_i, i \in \omega$, such that for each $A \subseteq \omega$ the set $\{\phi(x, a_i) : i \in A\} \cup \{\neg\phi(x, a_i) : i \notin A\}$ is consistent.*

Before moving on to discuss stable groups, we present a well-known fact on infinite locally finite stable fields which will be useful to us in Section 6.2.

Fact 3.3.5 (Duret [42, Corollaire 6.6])**.** *Infinite locally finite stable fields are algebraically closed.*

3.3.1 Stable groups

By a stable group we mean a stable $\mathcal{L}$-structure $\mathcal{G} = (G, \cdot, ^{-1}, e, \ldots)$, where $(G, \cdot, ^{-1}, e)$ is a group. Throughout this thesis, as often done in literature, we abuse the notation and write $G = \mathcal{G}$, that is, G denotes both the $\mathcal{L}$-structure $\mathcal{G}$ and the underlying set G of $\mathcal{G}$.

In what follows we first give some examples of stable groups. Then, we prove the result due to Baldwin and Saxl stating that stable groups are of finite centraliser dimension—this result plays a crucial role in Section 6.2 where we classify infinite definably simple stable groups admitting a finitary automorphism group A.

A detailed discussion on stable groups can be found in either of the books by Bruno Poizat [71] or Frank O. Wagner [88].

Let us now see some examples of stable groups.

(1.) Perhaps the most typical example of stable groups is an algebraic group G over an algebraically closed field K. Such G is in particular a group of finite Morley rank and the Morley rank of G coincides with the Zariski dimension of G over K (see Chapter 5).

(2.) Pure abelian groups (that is, abelian groups in the language $\mathcal{L}_{groups}$), or more generally modules, and the pure abelian-by-finite groups. These structures have much stronger uniformity properties than those required by stability alone, see [71, Section 1.1].

(3.) Groups of finite Morley rank (see Chapter 5).

(4.) Non-abelian free groups F_n; it was recently proven by Zlil Sela that the elementary theory T_{fg} of non-abelian free groups F_n is stable [76].

A family H_i of subgroups of a group G is called *uniformly definable* if every H_i is the set of elements of G satisfying some formula $\phi(x, \bar{v}_i)$, where the formula $\phi(x, \bar{y})$ is fixed and only the tuple of parameters $\bar{v}_i$ varies. Using this notion, one may prove that stable groups are of finite centraliser dimension.

Fact 3.3.6 (Baldwin and Saxl [10]; or see [71, Chapter 1])**.** *Let G be a stable group. Then G has* icc*, the uniform chain condition on intersections of uniformly definable subgroups; for any formula $\phi(x, \bar{y})$, there is $n_\phi \in \mathbb{N}$ such that any descending chain of intersections of ϕ-definable subgroups has length at most n_ϕ. In particular, for any subset A of G, the centraliser $C_G(A)$ is a definable subgroup and there exists a finite bounded $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$, that is, G is of finite centraliser dimension.*

Proof. We prove the first part of the claim. The in particular part then follows as one can apply the first part to the quantifier-free formula $\phi(x, y) := xy = yx$.

Let G be a group without the independence property. We prove that every formula $\phi(x, \bar{y})$ is associated with a natural number n_ϕ such that the intersection of a finite family of uniformly definable subgroups $H_1, \ldots, H_m$ of G, defined by the formulas $\phi(x, \bar{v}_1), \ldots, \phi(x, \bar{v}_m)$, is the intersection of at most n_ϕ of them. Suppose towards a contradiction that for arbitrarily large m we can find a family $H_1, \ldots, H_m$ of such subgroups whose intersection is strictly contained in the intersection of every proper subfamily. Then, for each i, there exists an element b_i such that $b_i \notin H_i$ but $b_i \in H_j$ for $j \neq i$. Given an arbitrary subset J of the set $\{1, \ldots, m\}$, let b_J denote the product of all the b_i's for $i \in J$. Then, $\phi(b_I, \bar{v}_i)$ is satisfied if and only if i does not belong to J; since m is arbitrarily large, that yields the independence property—we have derived a contradiction.

As stable groups do not possess the independence property, the claim is proven. □

3.4 Ultraproducts and pseudofinite structures

Fix a countable language $\mathcal{L}$. Let I be a non-empty set. An *ultrafilter* $\mathcal{U}$ on I is a subset U of the powerset $P(I)$ which is closed under finite intersections and supersets, contains I and omits $\emptyset$, and is maximal subject to this. An ultrafilter $\mathcal{U}$ is called a *principal* ultrafilter if it has the form $U = \{X \subseteq I : i \in X\}$ for some $i \in I$, otherwise $\mathcal{U}$ is called *non-principal*. Let then $\{\mathcal{M}_i : i \in I\}$ be a family of $\mathcal{L}$-structures and $\mathcal{U}$ be a non-principal ultrafilter on I. We define $\mathcal{M}^* := \prod_{i\in I} \mathcal{M}_i$ to be the Cartesian product of $\mathcal{L}$-structures $\mathcal{M}_i$. We say that a property P holds *almost everywhere* if $\{i : P \text{ holds for } \mathcal{M}_i\} \in \mathcal{U}$. Now we define an equivalence relation $\sim_{\mathcal{U}}$ on $\mathcal{M}^*$ as follows

$$x \sim_{\mathcal{U}} y \text{ if and only if } \{i \in I : x_i = y_i\} \in \mathcal{U},$$

where $x, y \in \mathcal{M}^*$, and x_i and y_i denote the i^{th} coordinate of x and y, respectively. Finally, we fix $\mathcal{M} = \mathcal{M}^*/\sim_{\mathcal{U}}$. Relations of the language $\mathcal{L}$ are defined to hold of a tuple of $\mathcal{M}$ if they hold in the i^{th} coordinate for almost all i (i.e., almost everywhere) and functions and constants in $\mathcal{M}$ are interpreted similarly. This is well-defined, and the resulting $\mathcal{L}$-structure $\mathcal{M}$ is called the *ultraproduct* of the $\mathcal{L}$-structures $\mathcal{M}_i$ with respect to the ultrafilter $\mathcal{U}$. We denote $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$.

The following fundamental theorem on ultraproducts is due to Łoś.

Fact 3.4.1 (Łoś's Theorem, see e.g. [82, Exercise 1.2.4])**.** *Let $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ be the ultraproduct of the $\mathcal{L}$-structures $\mathcal{M}_i$ with respect to the ultrafilter $\mathcal{U}$ and let $\phi(v_i, \ldots, v_n)$ be an $\mathcal{L}$-formula. Then, $\mathcal{M} \models \phi((g_1/\sim), \ldots, (g_n/\sim))$ if and only if $\{i \in I : \mathcal{M}_i \models \phi(g_1(i), \ldots, g_n(i))\} \in \mathcal{U}$.*

We now give two equivalent definitions of pseudofinite structures which will demonstrate how ultraproducts and pseudofiniteness are linked together.

Definition 3.4.2. *An infinite structure is called* pseudofinite *if every first-order sentence* *true in it also holds in some finite structure or, equivalently, if it is elementarily equivalent to an ultraproduct of finite structures.*

Pseudofinite fields were studied, and characterised in purely algebraic terms, by Ax [9].

Fact 3.4.3 (Ax [9])**.** *A field F is pseudofinite if and only if all of the following hold:*

(i) F is perfect, that is, F is of characteristic 0 or F has characteristic p and every element has a pth root;

(ii) F is quasifinite, that is, inside a fixed algebraic closure, F has a unique extension of each finite degree;

(iii) F is pseudo-algebraically closed, that is, every absolutely irreducible variety which is defined over F has a F-rational point.

No such classification as for pseudofinite fields is expected for pseudofinite groups. However, there is a close relationship between simple pseudofinite groups and pseudofinite fields.

Fact 3.4.4 (Point [70])**.** *Let $\{X(F_i) : i \in I\}$ be a family of Chevalley or twisted Chevalley groups of a fixed type X over finite or pseudofinite fields F_i, and let $\mathcal{U}$ be a non-principal ultrafilter on the set I. Then,*

$$\prod_{i\in I} X(F_i)/\mathcal{U} \cong X\left(\prod_{i\in I} F_i/\mathcal{U}\right).$$

Using the result of Françoise Point above, John S. Wilson classified simple pseudofinite groups, up to elementary equivalence [90]: If G is a simple pseudofinite group, then G is elementary equivalent to a Chevalley group or a twisted Chevalley group over a pseudofinite field. Further, Mark J. Ryten proved the following in his PhD thesis.

Fact 3.4.5 (Ryten [75, Chapter 5])**.** *Let $G(\mathbb{F}_q)$ be finite groups of a fixed Lie type over varying finite fields $\mathbb{F}_q$. Then groups $G(\mathbb{F}_q)$ are uniformly bi-interpretable over parameters with $\mathbb{F}_q$, or, in the case of groups ${}^2B_2$, ${}^2F_4$, or ${}^2G_2$, with structures $(\mathbb{F}_q, \sigma)$, for an appropriate field automorphism σ.*

Putting together results of Wilson and Ryten we have the classification of simple pseudofinite groups, up to isomorphism.

Fact 3.4.6 (Wilson [90] + Ryten [75]). *Every simple pseudofinite group is isomorphic to a Chevalley or a twisted Chevalley group over a pseudofinite field.*

Chapter 4

Locally finite groups of finite centraliser dimension

In this chapter we prove the first results of this thesis. Namely, we prove two results on locally finite groups of finite centraliser dimension; Theorems 4.1.1 and 4.2.1.

In Section 4.1, we present and prove a general theorem on locally finite groups of finite centraliser dimension which describes the structure of such groups, Theorem 4.1.1. Our theorem depends on CFSG. Theorem 4.1.1 is a result of joint work between Borovik and the author; it is published in [20]. We should mention that a similar result to Theorem 4.1.1 was independently proven by Buturlakin in [25].

In Section 4.2, we classify infinite definably simple locally finite groups of finite centraliser dimension, that is, we prove Theorem 4.2.1. This result is a natural generalisation of the classification of infinite simple locally finite groups of finite centraliser dimension by Thomas (Fact 2.4.10). Theorem 4.2.1 heavily uses the main result of Section 4.1, Theorem 4.1.1, and therefore it also depends on CFSG. It is published in [55].

4.1 The structural theorem

Recall that $\mathrm{cd}(G) = k$ denotes that a group G has centraliser dimension k. The reader should also keep in mind that we use an umberella term "groups of Lie type" to all Chevalley groups, twisted Chevalley groups, and non-algebraic twisted groups of Lie type ${}^2B_2$, ${}^2F_4$, or ${}^2G_2$ over finite, locally finite and algebraically closed fields.

In this section we prove the following theorem.

Theorem 4.1.1 (Borovik and Karhumäki [20, Theorem 1]; see also Buturlakin [25])**.** *Let G be a locally finite group with* $\mathrm{cd}(G) = k$*. Then G has a normal series*

$$1 \trianglelefteq S \trianglelefteq L \trianglelefteq G,$$

where

(a) *S is a solvable group of derived length bounded by a function of k.*

(b) *$\overline{L} = L/S$ is a direct product $\overline{L} = \overline{L}_1 \times \cdots \times \overline{L}_m$ of finitely many non-abelian simple groups.*

(c) *Each $\overline{L}_i$ is either finite or a simple group of Lie type over a locally finite field.*

(d) *The factor group G/L is finite.*

In the following subsections we handle the proof of Theorem 4.1.1 issue by issue: We first prove that solvable subgroups of any section of a locally finite group G of centraliser dimension k have bounded derived lengths where the bound depends on k only. For such groups we may coherently define the *solvable radical*, that is, the maximal solvable normal subgroup—we prove that S in Theorem 4.1.1 is the solvable radical of G. Further, we shall use the standard terminology and call a quasisimple subnormal subgroup of a group H a *component* of H. The product of all the components of a group H is called the *layer* of H and is denoted by $L(H)$—we prove that L in Theorem 4.1.1 is the full preimage in G of the layer $L(G/S)$.

Throughout this section we work with a group G which satisfies the assumptions of Theorem 4.1.1.

4.1.1 Control of sections

Let G be a locally finite group with $\mathrm{cd}(G) = k$. As explained in the Section 2.4, subgroups of G also possess a descending chain condition for centralisers but this property is not inherited by sections of G. Therefore, we need a sufficiently strong property which holds in G and is inherited by its sections. Luckily, such a property is provided to us by the result of Khukhro already stated in Subsection 2.4.2 as Fact 2.4.5: *Periodic locally solvable groups of centraliser dimension k are solvable and have derived lengths bounded by a function of k.*

Fact 2.4.5 allows us to prove that solvable subgroups of sections of locally finite groups of centraliser dimension k have bounded derived lengths where the bounds depend on k only. To prove this we define the following.

Definition 4.1.2. *A group H is called* constrained *if derived lengths of its solvable subgroups are bounded.*

We now prove that sections of G are constrained.

Lemma 4.1.3. *Let G be a locally finite group of finite centraliser dimension and $\overline{H} = H/K$ be its section. Then $\overline{H}$ is constrained.*

Proof. It can be immediately observed from Fact 2.4.5 that locally finite groups of finite centraliser dimension are constrained, and so, G is constrained. Therefore, we may assume that every solvable subgroup in G has derived length at most d. Since $\overline{H}$ is a section of G it is locally finite. Our first observation is that it suffices to prove that an arbitrary *finite* solvable subgroup of $\overline{H}$ has solvability degree at most d. This can be observed as follows. Assume that all finite solvable subgroups in $\overline{H}$ have solvability degrees at most d and, towards a contradiction, that some infinite solvable subgroup I in $\overline{H}$ has a solvability degree $n > d$. Then, we may pick elements $x_1, x_2, \ldots, x_{2^d} \in I$ such that

$$[[\ldots[x_1, x_2], [x_3, x_4], \ldots],], [\ldots[x_{2^d-3}, x_{2^d-2}], [x_{2^d-1}, x_{2^d}]\ldots]] \neq 1.$$

Consider then a subgroup $J = \langle x_1, x_2, \ldots, x_{2^d}\rangle$ of I generate by the finitely many elements $x_1, x_2, \ldots, x_{2^d}$ of I for which the inequality above holds. Now J has a solvability degree $m > d$. At the same time, since J is a finitely generated solvable subgroup of a locally finite group $\overline{H}$, it is finite, and thus, J has a solvablity degree at most d. This contradiction shows that it suffices to prove that an arbitrary finite solvable subgroup $\overline{S}$ in $\overline{H}$ has solvability degree at most d. Pick representatives $s_1, \ldots s_n$ of cosets of $\overline{S}$ in $\overline{H}$ and generate by them a subgroup R. Since R is a finite group, by repeatedly applying the Frattini Argument of finite groups to the prime divisors of $|R|$ (see Fact 2.4.1), we know that R contains a subgroup P such that $P(R \cap K) = R$ and $P \cap K$ is a nilpotent group. Thus, P is a solvable subgroup of G, and therefore, P has a derived length at most d; but

$$PK/K = RK/K = \overline{S},$$

and hence the derived length of $\overline{S}$ is also at most d. □

Now we recall (a weaker version of) the classification by Hartley presented in Subsection 2.4.3 as Fact 2.4.9.

Fact 4.1.4 (Hartley [49])**.** *Let G be an infinite simple locally finite group. If some finite group is not a subgroup of G, then G is a group of Lie type over a locally finite field.*

As an immediate consequence of Fact 4.1.4 and Lemma 4.1.3 we may now present a partial generalisation of the classification of infinite simple locally finite groups of finite centraliser dimension by Thomas (Fact 2.4.10). Note that, like Thomas's result, the following theorem uses CFSG as it follows from Fact 4.1.4 which also depends on CFSG.

Theorem 4.1.5. *If L is an infinite simple section of a locally finite group of a finite centraliser dimension, then L is a group Lie type over a locally finite field.*

Proof. Since L is a section of a locally finite group of finite centraliser dimension it is constrained by Lemma 4.1.3. As a result, not every finite solvable group is a subgroup of L, and so, Fact 4.1.4 proves the claim. □

4.1.2 Quasisimple locally finite groups of Lie type

Recall that a group H is called quasisimple if $H = H'$ and $H/Z(H)$ is a non-abelian simple group.

Lemma 4.1.6. *If H is a quasisimple locally finite group and $\overline{H} = H/Z(H)$ is a simple group of Lie type, then $|Z(H)|$ is finite and bounded by a constant depending only on $H/Z(H)$.*

Proof. The simple group of Lie type $\overline{H} = H/Z(H)$ is defined over a locally finite field, say F. Taking groups of points of finite subfields of F, we can construct a sequence of subgroups

$$1 < \overline{H}_1 < \overline{H}_2 < \dots$$

such that $\overline{H}_i$'s are simple groups of the same Lie type as $\overline{H}$ and $\overline{H}_i$'s cover $\overline{H}$. That is, $\overline{H} = \bigcup_{i=1}^{\infty} \overline{H}_i$.

Now the derived subgroups of the preimages of $\overline{H}_i$'s in H form a sequence of subgroups

$$1 < H_1 < H_2 < \dots$$

of H such that $H_i = H_i'$ for all $i = 1, 2, \dots$.

Let then $Z = Z(H)$. The central extensions of groups of Lie type $\overline{H}_i$ are are controlled by their Schur multipliers, which have orders of bounded size (see [48, §6.1]). Therefore, the subgroups $Z(H_i) = H_i \cap Z$, which are factor groups of the Schur multipliers of groups of Lie type $\overline{H}_i$, have orders of bounded size, and thus, the sequence of groups

$$1 \leqslant Z(H_1) \leqslant Z(H_2) \leqslant \dots$$

stabilises at some finite group Z_* of bounded order. If Z is infinite, then there is an element $z \in Z \setminus Z_*$ which is written as a product of some commutators from H; $z = [h_1, h_2] \cdots [h_{2k-1}, h_{2k}]$ with elements $h_1, h_2, \dots, h_{2k}$ contained in one of the subgroup H_m. But then, $z \in H_m' = H_m$ and therefore z belongs to $Z(H_m) = H_m \cap Z \leqslant Z_*$— a contradiction. □

4.1.3 Proof of Theorem 4.1.1; the solvable radical and the layer

Let us at this point mention again the difficulty addressed in Subsection 2.4.3; the concept of a composition factor is somewhat vague in the case of locally finite groups because the classical Jordan–Hölder Theorem for composition series of finite groups is no longer true. We may overcome this difficulty by repeatedly applying Fact 2.4.8. Below we give a weaker formulation of Fact 2.4.8, which is more suitable to our purposes.

Fact 4.1.7 (Buturlakin and Vasil'ev [26]). *Let G be a locally finite group with $\mathrm{cd}(G) = k$ and let*

$$1 = G_0 < G_1 < G_2 < \cdots < G_l = G$$

be a finite subnormal series in G. Then the number of distinct non-solvable factors G_i/G_{i-1}, $i \geqslant 1$, is at most $5k$.

In what follows we apply Fact 4.1.7 repeatedly.

We start building the normal series

$$1 \trianglelefteq S \trianglelefteq L \trianglelefteq G$$

of Theorem 4.1.1.

For a locally finite constrained group H, we denote by $R(H)$ the maximal locally solvable normal subgroup of H. As H is constrained, $R(H)$ is solvable and therefore, following the standard terminology, we call $R(H)$ the solvable radical of H. Note that $H/R(H)$ has no non-trivial solvable normal subgroups as $R(H)$ is the maximal solvable normal subgroup of H. Further, we denote by $Q(H)$ the minimal normal subgroup of H with solvable factor $H/Q(H)$; since H is constrained, $Q(H)$ exists and coincides with the last term of the derived series of H. Also, $Q(H)$ has no non-trivial solvable factor groups. Observe that $R(H)$ and $Q(H)$ are characteristic subgroups of H. Now we need another definition.

Definition 4.1.8. *Let H be a constrained group. Then H is called* truncated *if $R(H) = 1$ and $Q(H) = H$.*

We start with the normal series

$$1 = G_0 \triangleleft G_1 = G,$$

and refine and re-build it and get the subsequent series

$$1 = G_0 \triangleleft G_1 < \cdots < G_l = G,$$

appropriately changing numeration at every step, in according to the rules below. The reader should recall here that Lemma 4.1.3 ensures that all sections of G are constrained.

- For every factor G_i/G_{i-1} that is not truncated, we insert subgroups

$$G_{i-1} \trianglelefteq G_j \trianglelefteq G_k \trianglelefteq G_i,$$

  where G_j is the full preimage of $R(G_i/G_{i-1})$ and G_k is the full preimage of $Q(G_i/G_{i-1})$.

- If G_i/G_{i-1} and G_j/G_{j-1}, $i < j-1$, are two truncated factors and there are no truncated factors G_k/G_{k-1} for $i < k < j$, then G_{j-1}/G_i is solvable and we can remove from the series its members $G_{i+1}, \ldots, G_{j-2}$.

- If G_i/G_{i-1} is a truncated factor and there is a normal subgroup $G_j \lhd G$ fitting into $G_{i-1} < G_j < G_i$, we insert it in the series—and repeat the process from the beginning.

In view of Fact 4.1.7, this process terminates after finitely many steps, producing a finite series $1 = G_0 \lhd G_1 < \cdots < G_\ell = G$ of normal subgroups, where every factor G_i/G_{i-1} is either truncated without non-trivial proper characteristic subgroups or solvable.

We again apply Fact 4.1.7:

Lemma 4.1.9. *In the series $1 = G_0 \lhd G_1 < \cdots < G_\ell = G$ produced above, the truncated factors G_i/G_{i-1} are finite direct products of isomorphic non-abelian simple groups which are either finite or of Lie type over locally finite fields.*

Proof. Consider the normal series in a truncated factor $H = G_i/G_{i-1}$; the number of non-solvable factors in each series is bounded by Fact 4.1.7. Consider a normal series

$$H = H_1 \rhd H_2 \rhd H_3 \cdots \rhd H_j \rhd \cdots$$

with a maximal possible number of non-solvable factors. If H_k/H_{k+1} is the last non-solvable factor, then H_{k+1} must be solvable, and therefore $H_{k+1} = 1$ since H is truncated. Moreover, if $H_k \rhd K \neq 1$ and $H \rhd K$, then H_k/K is solvable—as otherwise, the refinement

$$H = H_1 \rhd H_2 \rhd H_3 \cdots \rhd H_j \rhd K \rhd 1$$

would contain more non-solvable factors than the originally chosen series. Without a loss of generality, we can replace K by the last term of the derived series of H_k. Now K is truncated, and the same argument as the one applied to H_k shows that K contains no non-trivial proper subgroups normal in H, that is, K is a minimal normal subgroup in H.

Let $K_1, K_2, \ldots$ be distinct minimal normal subgroups in H and consider their product $\prod_j K_j$. Obviously, $\prod_j K_j$ is a characteristic subgroup of H, and so, $\prod_j K_j = H$ since the truncated group H does not have any non-trivial proper characteristic subgroups. Since all K_j's in the product $\prod_j K_j = H$ are minimal normal and non-abelian, any product $K_1 K_2 \cdots K_\ell$ of finitely many of them is a direct product,

$$K_1 K_2 \cdots K_\ell = K_1 \times K_2 \times \cdots \times K_\ell.$$

Obviously,

$$K_1 \lhd K_1 K_2 \lhd K_1 K_2 K_3 \lhd \cdots \lhd H;$$

this normal series in H extends to a subnormal series in G with non-solvable factors, and, in view of Fact 4.1.7, is finite. Therefore, $\prod_j K_j$ is a direct product $H = \oplus_j K_j$ of finitely many subgroups K_j. Since H is characteristically simple, all K_j's are isomorphic. If K_j is infinite, then it is of Lie type over a locally finite field by Theorem 4.1.5. □

Now we return to the proof of Theorem 4.1.1.

Recall that a quasisimple subnormal subgroup of a group H is called a component of H and that the product of all the components of a group H is called the layer of H and is denoted by $L(H)$.

We may now coherently define S as the solvable radical $R(G)$ of G and L as the full preimage in G of the layer $L(G/S)$. Therefore $\overline{L} = L/S$ is a direct product $\overline{L} = \overline{L}_1 \times \cdots \times \overline{L}_m$ of finitely many non-abelian simple groups, each of which is either finite or a group of Lie type over a locally finite field. This proves Theorem 4.1.1 as soon as we address the point (d), finiteness of G/L—see next subsections.

4.1.4 Action of G on G/S

We retain notation from the previous subsection. Further, we set $\overline{G} = G/S$.

Our first observation is that $C_{\overline{G}}(\overline{L}) \cap \overline{L} = 1$ and $C_{\overline{G}}(\overline{L}) \trianglelefteq \overline{G}$. Moreover, if $C_{\overline{G}}(\overline{L}) \neq 1$, then, applying analysis of subsection 4.1.3 to the full preimage of $C_{\overline{G}}(\overline{L})$ in G, we see that the group $C_{\overline{G}}(\overline{L})$ has subnormal non-abelian simple subgroups which are subnormal in $\overline{G}$ but do not belong to $\overline{L}$, which contradicts the way $\overline{L}$ was constructed. Hence, $C_{\overline{G}}(\overline{L}) = 1$.

Now the group G, in its action on $\overline{L}$ by conjugation, permutes simple subgroups $\overline{L}_i$; the kernel of this permutation action, say G°, is a normal subgroup of finite index in G. Without a loss of generality, we can assume that $G^\circ = G$ and each $\overline{L}_i \trianglelefteq \overline{G}$.

Let now $\overline{M} = \overline{L}_1 \times \cdots \times \overline{L}_k$ be the product of all *infinite* components of $\overline{G}$; if $\overline{M} = 1$, then $\overline{L}$ and $\overline{G}$ are finite. In this case the point (d) holds and Theorem 4.1.1 is proven.

Assume then that $\overline{M} \neq 1$. Denote by $\overline{N} = \overline{L}_{k+1} \times \cdots \times \overline{L}_m$ the product of all *finite* components of $\overline{G}$. If $\overline{N} \neq 1$, then $C_{\overline{G}}(\overline{N})$ is the kernel of the action of $\overline{G}$ on $\overline{N}$ by conjugation and therefore has finite index in $\overline{G}$. Again, we can assume without a loss of generality that $C_{\overline{G}}(\overline{N}) = \overline{G}$ and $\overline{N} = 1$, that is, all components of $\overline{G}$ are infinite simple groups of Lie type over (infinite) locally finite fields.

4.1.5 The factor group G/L is abelian-by-finite

Again, we retain our notation from previous subsections.

We next turn our attention to the action of $\overline{G} = G/S$ on $\overline{L} = \overline{L}_1 \times \cdots \times \overline{L}_m$.

It is well-known that every automorphism of a group of Lie type $X(F)$ over a locally finite field F is a product of inner, diagonal, field, and graph automorphisms. If Out(X) = Aut(X)/Inn(X) is the group of outer automorphisms of X, then images in Out(X) of diagonal and graph automorphisms of X generate a finite subgroup. Moreover, the image Γ in Out(X) of the group of field automorphisms is naturally isomorphic to the group $\mathrm{Aut}(F)$. It is well-known that $\mathrm{Aut}(F)$ is a factor group of $\widehat{\mathbb{Z}}$, the profinite completion of the additive group of integers (the latter is the Galois group of the algebraic closure $\overline{\mathbb{F}}_p$ of a finite prime field $\mathbb{F}_p$, that is, $\widehat{\mathbb{Z}} = \mathrm{Aut}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$).

We have a natural embedding

$$\overline{G}/\overline{L} \hookrightarrow \prod_{i=1}^{m} \mathrm{Out}(\overline{L}_i).$$

We see now that if $\overline{G}/\overline{L}$ is infinite, then it contains an abelian subgroup Δ of finite index which either centralises, or acts by field automorphisms on components $\overline{L}_i$—in the sense that elements from the preimage $\overline{D}$ of Δ induce on $\overline{L}_i$ automorphisms that are products of inner and field automorphisms.

Let D be the full preimage of $\overline{D}$ in G. Then, D has a finite index in G, that is, we have proven that G/L is abelian-by-finite. Therefore, for the rest of the proof we can assume, without a loss of generality, that G/L is an abelian group and that the outer automorphisms induced from $\overline{D}$ on Lie type subgroups $\overline{L}_i$ are field automorphisms.

4.1.6 The Frattini Argument

At this point it becomes essential to find a more efficient way around the fact that in general the descending chain condition for centralisers is not preserved under taking factor groups.

The general situation is the following: We have a group G of finite centraliser dimension and a subgroup $K \lhd G$; we wish to derive some information about $\widehat{G} = G/K$ without being able to prove directly that $\widehat{G}$ has a finite centraliser dimension. The idea is to calculate instead in an appropriate partial complement $M \leqslant G$ to K, that is, in a subgroup M such that $G = MK$ and M is sufficiently small for easier deduction of the desired facts about $G/K \cong M/(M \cap K)$.

The classical way to construct partial complements is the Frattini Argument; recall from Subsection 2.4.1 the Frattini Argument and conjugacy theorems for Hall π-subgroups in our context, Facts 2.4.2, 2.4.3 and 2.4.4.

We shall now start using the Frattini argument as a tool for carving out, from G, subgroups where we have better control of centralisers.

4.1.7 Towards finiteness of G/L

Let us now look at the group $\Delta < \overline{G}/\overline{L}$ constructed in Section 4.1.5 and its full preimage $\overline{D}$ in $\overline{G}$.

To prove that G/L is finite it will suffice to prove that Δ is finite. So let us assume that Δ is infinite and work towards a contradiction.

Among components $\overline{L}_i \lhd \overline{L}$, we pick one, say $\overline{K}$, such that $\overline{D}$ induces, in its action on $\overline{K}$, an infinite group of outer automorphisms. The group $\overline{K}$ is of Lie type over a locally finite field F; let p be the characteristic of F. Consider the natural homomorphism

$$\rho : \overline{D} \longrightarrow \overline{K} \rtimes \Gamma, \quad \text{where} \quad \Gamma = \mathrm{Aut}(F),$$

and take $\mathrm{E} = \mathrm{Im}\,\rho \cap \Gamma$.

The group Γ is the continuous image of $\widehat{\mathbb{Z}}$ (the profinite completion of $\mathbb{Z}$, the Galois group of the algebraic closure of the prime field $\mathbb{F}_p$), and E, as a locally finite subgroup of Γ, is locally cyclic and is a direct sum of finite cyclic groups of pairwise co-prime prime power orders; let $\epsilon_1, \epsilon_2, \dots,$ be generators of the cyclic direct summands of E. Then, we have an infinite decreasing sequence of fields

$$C_F(\langle \epsilon_1 \rangle) > C_F(\langle \epsilon_1, \epsilon_2 \rangle) > C_F(\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle) > \dots,$$

and, correspondingly, an infinite descending chain of centralisers

$$C_{\overline{K}}(\langle \epsilon_1 \rangle) > C_{\overline{K}}(\langle \epsilon_1, \epsilon_2 \rangle) > C_{\overline{K}}(\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle) > \dots.$$

This would produce a contradiction if we knew that $\overline{G}$ had a descending chain condition for centralises; but we don't, so we have to conduct further surgery on the group G, and, in particular, reduce E to a manageable size.

Pick in E elements $\alpha_1, \dots, \alpha_n$, of pairwise different prime orders $p_1, \dots, p_n$, none of which is p (where p is the characteristic of F), and none divides the order of the center of any quasisimple extension of $\overline{K}$ (see Lemma 4.1.6), so that n is bigger than the centraliser dimension k of G. Take their product $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ and its preimage $\bar{a}$ in $\overline{G}$; let a be some coset representative of $\bar{a}$ and $A = \langle a \rangle$ be the cyclic group generated by a; replacing a by another coset representative, we can ensure that prime divisors of $|A|$ are exactly $p_1, \dots, p_n$.

Now let K be the full preimage of $\overline{K}$; we can replace, without a loss of generality, G by KA; the solvable radical S could slightly grow up by absorbing part of A, but this does not affect our considerations; we still have the property that, for appropriate powers $a_1, \dots, a_n$ of a, we have a descending chain of centralisers of length $n > k$:

$$C_{\overline{K}}(\langle a_1 \rangle) > C_{\overline{K}}(\langle a_1, a_2 \rangle) > \cdots > C_{\overline{K}}(\langle a_1, a_2, \dots, a_n \rangle).$$

4.1.8 Trimming the solvable radical

Let U be a Sylow p-subgroup of S, where p is the characteristic of F, the underlying field of $\overline{K}$. Using the Frattini Argument, we replace G by $N_G(U)$ and assume, without a loss of generality, that $U \lhd G$. Now we take a Hall p'-subgroup Q in S and, applying the Frattini Argument again, replace G by $N_G(Q)$, thus assuming, without a loss of generality, that $Q \lhd G$. Now $S = U \times Q$.

Let now P be a Sylow p-subgroup in K, then its image $\overline{P}$ in $\overline{K}$ is a maximal unipotent subgroup in the group $\overline{K}$ of Lie type over an infinite field of characteristic p; hence $\overline{P}$ contains an infinite elementary abelian subgroup $\overline{V}$ (the center of any root subgroup can be used for this purpose). We denote by V a Sylow p-subgroup in the full preimage of $\overline{V}$ in K, then V acts on Q by conjugation and U belongs to the kernel of this action. The group VQ has finite centraliser dimension, and therefore has uniformly bounded lengths of chains of centralisers of subsets from $\overline{V}$ in Q. At this point, we can invoke the following lemma.

Lemma 4.1.10. *If an elementary abelian p-group W of order p^n acts faithfully on a locally finite solvable p'-group Q, then there exists a series of subgroups*

$$W = W_0 > W_1 > W_2 > \cdots > W_n = 1$$

such that

$$C_Q(W_0) < C_Q(W_1) < \cdots < C_Q(W_n).$$

Proof. The claim is proven by Khukhro (Fact 2.1.1) in the special case of Q being finite and nilpotent and, in particular, when Q is abelian. Reduction of the more general case to this one is easy. First of all, W acts on Q faithfully; therefore, for every element

$w \in W \setminus \{1\}$, we can pick an element $q_w \in Q$ such that $q_w^w \neq q^w$ and replace Q by a W-invariant finite subgroup $\langle q_w^W : w \in W \setminus \{1\}\rangle$ generated by the W-orbits q_w^W of elements q^w. So we can assume without a loss of generality that Q is finite. Now we can apply a much more general result:

Fact 4.1.11 (Hartley-Turull; see [53, Theorem 3.31])**.** *Let W act via automorphisms on Q, where W and Q are finite groups, and suppose that $(|W|, |Q|) = 1$. Assume also that at least one of W or Q is solvable. Then W acts via automorphisms on some abelian group R in such a way that every subgroup $U \subseteq W$ has the same number of fixed points on Q and on R.*

Now an application of Khukhro's lemma (Fact 2.1.1) to the action of W on R completes the proof. □

We can now return to proof of Theorem 4.1.1.

We see that $Y = C_V(Q)$ has finite index in V, and therefore the image $\overline{Y}$ of Y in $\overline{K}$ is infinite. Let $H = C_K(Q)$; obviously, $H \trianglelefteq K$ and contains Y, but $\overline{K}$ is simple; hence $\overline{K} = \overline{H}$ and so $\overline{H}$ contains $\overline{Y}$. Therefore, without a loss of generality, we can replace K by $H = C_K(Q)$ and then replace G by KA.

Now $Q \leqslant Z(K)$. At the last step of our trimming procedure, we replace K by the intersection of its derived series, making sure that $K = K'$, and then again replace G by KA.

Consider $\hat{K} = K/U$, and let $\hat{Q}$ be the image of Q in $\hat{K}$; then $\hat{Q} \leqslant Z(\hat{K})$. The centers of locally finite quasisimple groups of Lie type are finite by Lemma 4.1.6. Hence, $\hat{Q}$ and Q are finite. In the resulting normal series

$$1 \trianglelefteq U \trianglelefteq S \trianglelefteq K \trianglelefteq G,$$

the factor S/U is a finite abelian group in view of Lemma 4.1.6. In particular, only finitely many distinct prime numbers divide orders of elements in S.

4.1.9 End of the proof

We retain the notation of previous subsections and return to the chain of centralisers

$$C_{\overline{K}}(\langle a_1\rangle) > C_{\overline{K}}(\langle a_1, a_2\rangle) > \cdots > C_{\overline{K}}(\langle a_1, a_2, \ldots, a_n\rangle)$$

constructed in Subsection 4.1.7.

Recall that A was constructed in a way that $a_i \in A$ have orders co-prime to the orders of elements in U and in S/U. These restrictions have been forced on A with the aim of lifting centralisers of subgroups $B \leqslant A$ in $\overline{K}$ to centralisers in K, that is, proving that $C_{\overline{K}}(B)$ is the image of $C_K(B)$ in $\overline{K}$. For that, we need a simple tool from finite group theory.

Fact 4.1.12. *Let B be a finite cyclic π-group of automorphisms of a finite group H, and let R be a B-invariant normal π'-subgroup of H. Then $C_{H/R}(B)$ is the image of $C_H(B)$ in H/R.*

Proof. This is a special case of Fact 2.1.2 presented in Section 2.1. □

Now we can expand, in a routine way, Fact 4.1.12 to locally finite groups.

Lemma 4.1.13. *Let B be a finite cyclic π-group of automorphisms of a locally finite group K, and let S be a B-invariant normal solvable π'-subgroup of K. Assume, in addition, that orders of elements from S are divisible only by finitely many different prime numbers. Then $C_{K/S}(B)$ is the image of $C_K(B)$ in K/S.*

Now, after lifting centralisers in the chain

$$C_{\overline{K}}(\langle a_1\rangle) > C_{\overline{K}}(\langle a_1, a_2\rangle) > \cdots > C_{\overline{K}}(\langle a_1, a_2, \ldots, a_n\rangle)$$

from $\overline{K}$ to K, we have the chain of centralisers

$$C_K(\langle a_1\rangle) > C_K(\langle a_1, a_2\rangle) > \cdots > C_K(\langle a_1, a_2, \ldots, a_n\rangle)$$

of length exceeding the centraliser dimension k of G. This contradiction completes the proof of Theorem 4.1.1. □

4.2 The definably simple case

In this section we prove Theorem 4.2.1, which classifies infinite definably simple locally finite groups of finite centraliser dimension as simple groups of Lie type over locally finite fields. Theorem 4.2.1 is a natural generalisation of Thomas's classification of infinite simple locally finite groups of finite centraliser dimension, Fact 2.4.10.

We prove the following theorem.

Theorem 4.2.1 (Karhumäki [55, Theorem 1.1]). *An infinite definably simple locally finite group of finite centraliser dimension is a simple group of Lie type over a locally finite field.*

In the following subsection we prove Theorem 4.2.1 by invoking our structural theorem on locally finite groups of finite centraliser dimension, Theorem 4.1.1, which depends on CFSG. Therefore, Theorem 4.2.1 also depends on CFSG.

4.2.1 Proof of Theorem 4.2.1

Throughout this subsection we work in the notation of Theorem 4.1.1, that is, we consider an infinite definably simple locally finite group G of finite centraliser dimension which has a normal series $1 \trianglelefteq S \trianglelefteq L \trianglelefteq G$ as in Theorem 4.1.1.

Before starting the proof of Theorem 4.2.1, let us recall some definitions and facts that will be used in our proof.

- A group G is called definably simple if it has no proper non-trivial definable normal subgroups.
- In a group of finite centraliser dimension all centralisers are definable (see Subsection 3.2).
- A simple group H is called uniformly simple if for any two non-trivial elements g and h, the length of expression of g, as a finite product of conjugates of h or the inverse of h, is uniformly bounded.
- Simple groups of Lie type are uniformly simple (Fact 2.3.4).

Now we begin to prove Theorem 4.2.1. We start by observing that $S = 1$.

Lemma 4.2.2. *$S = 1$.*

Proof. Towards a contradiction, assume that $S \neq 1$. In this case G contains a non-trivial abelian normal subgroup A. But then $A \leqslant Z(C_G(A))$ and $Z = Z(C_G(A))$ is a non-trivial definable abelian normal subgroup of G. Since G is definably simple, $Z = G$ and therefore G is abelian; being definably simple locally finite group, it is forced to be a cyclic group of prime order. But G is assumed to be infinite—a contradiction. □

Since $S = 1$, we have $\overline{L} = L/S = L$. Therefore, L is a normal finite index subgroup of G such that $L = L_1 \times \cdots \times L_m$ is a direct product of finitely many non-abelian simple groups L_i each of which is a group of Lie type over a locally finite field. We move on to prove that $L = G$.

Lemma 4.2.3. *$L = G$.*

Proof. Since G is definably simple, to prove the claim, it suffices to prove that L is definable in G.

Groups L_i being groups of Lie type are uniformly simple. Thus, every element in each L_i is a product of a bounded number of L_i-conjugates of some non-trivial element $x_i \in L_i$. Further, G-conjugates of x_i belong to the normal subgroup L of G, that is,

$$L_i = (x_i^{\pm 1})^{L_i} \cdots (x_i^{\pm 1})^{L_i} \subseteq (x_i^{\pm 1})^{G} \cdots (x_i^{\pm 1})^{G} \subseteq \langle x_i^G \rangle \leqslant L.$$

Take the maximal length of the product $(x_i^{\pm 1})^{L_i} \cdots (x_i^{\pm 1})^{L_i}$ (and thus, the maximal length of the product $(x_i^{\pm 1})^{G} \cdots (x_i^{\pm 1})^{G}$ as well) for each i. Then,

$$L = L_1 \times \cdots \times L_m \subseteq \left((x_1^{\pm 1})^{G} \cdots (x_1^{\pm 1})^{G}\right) \cdots \left((x_m^{\pm 1})^{G} \cdots (x_m^{\pm 1})^{G}\right) \subseteq L.$$

This proves that L is a definable subgroup of G, and thus, $L = G$ since G is definably simple. □

At this point we know that $G = L = L_1 \times \cdots \times L_m$ is a direct product of finitely many non-abelian simple groups L_i each of which is a group of Lie type over a locally finite field. Thus, to complete the proof of Theorem 4.2.1, it is enough to prove that $m = 1$. Towards a contradiction, assume that $m > 1$. The double centraliser $C_G(C_G(L_1))$ is a definable and normal subgroup of the definably simple group G of finite centraliser dimension. At the same time, $C_G(L_1) \nleqslant C_G(C_G(L_1))$ since $C_G(L_1)$

must contains L_2. It follows that $C_G(C_G(L_1))$ is a proper non-trivial definable normal subgroup of the definably simple group G—a contradiction. □

Remark 4.2.4. *One can also find a first-order formula without parameters defining L in G.*

Proof. Groups L_i, being groups of Lie type over locally finite fields, are unions of ascending chains of finite simple groups of the same Lie type over finite fields of the same characteristic, say p. Their orders are well-known (see e.g. the table [47, pp. 490]) and contain divisors of the form $p^k \pm 1$ for k growing without bounds as subgroups in the chains grow bigger. Now the classical theorem by Zsigmondy [92] ensures that numbers of the form $p^k \pm 1$ with unbounded k have arbitrarily large prime divisors, which means that each group L_i contains elements of arbitrarily large prime orders.

Let $\ell_i \in L_i$ be an element of a prime order t chosen in the way that $t > [G : L]$. Notice that all elements of order t in L belong to L_i. Similarly as in the proof of Lemma 4.2.3, every element in L_i is a product of a bounded number of L_i-conjugates of ℓ_i. That is, similarly as in the proof of Lemma 4.2.3, we may observe that L is definable, this time without parameters, in G. □

Chapter 5

Groups of finite Morley rank

In this chapter we introduce groups of finite Morley rank. In particular, we discuss infinite simple groups of finite Morley rank.

In Section 5.1, we define groups of finite Morley rank, explain some fundamental properties of these groups and give examples and non-examples.

In Section 5.2, we consider infinite simple groups of finite Morley rank. In particular, we introduce two major conjectures of the topic; the Cherlin–Zilber Conjecture and the Principal Conjecture.

Section 5.3 is written so that the reader might find from there all the necessary advanced background results on groups of finite Morley rank needed in the proofs of Theorems 6.3.1 and 7.2.1.

5.1 Introduction to groups of finite Morley rank

In this section we briefly introduce groups of finite Morley rank. The reader unfamiliar with the topic may find an excellent general introduction from either of the books by Borovik and Ali Nesin [21] or by Altınel, Borovik and Cherlin [5].

Groups of finite Morley rank can be thought of as a model-theorist's approach to algebraic groups over algebraically closed fields. Indeed, they appeared first time in the strictly model-theoretic context as groups whose first-order theory is uncountably categorical. In order to prove his famous Categoricity Theorem, Morley introduced a notion that we today call the *Morley rank*—a dimension-like function that assigns an ordinal number (rank) to each definable set of models of uncountably categorical theories. Formally, the Morley rank is defined inductively on definable sets of an $\mathcal{L}$-structure $\mathcal{M}$ as follows.

Definition 5.1.1. *Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure and $X \subseteq M^n$ be a definable set. The Morley rank of X, denoted by $rk(X)$, is defined as follows.*

(i) $rk(X) \geqslant 0$ if and only if X is non-empty.

(ii) For any ordinal α, $rk(X) \geqslant \alpha + 1$ if and only if there exist infinitely many pairwise disjoint definable sets $X_1, X_2, \ldots \subset X$ such that $rk(X_i) \geqslant \alpha$ for all $i = 1, 2, \ldots$.

(iii) For a limit ordinal α, $rk(X) \geqslant \alpha$ if and only if $rk(X) \geqslant \beta$ for all $\beta < \alpha$.

For a definable set X, $rk(X) = \alpha$ if $rk(X) \geqslant \alpha$ but $rk(X) \not\geqslant \alpha + 1$. Further, $rk(X) = -1$ when $X = \emptyset$. In the case $rk(X) \geqslant \alpha$ for all ordinals α, one says that X has an *unbounded* Morley rank.

Given the definition of the Morley rank of the definable set X, we may now also define the *Morley degree* of X: The Morley degree of X, denoted by $deg(X)$, is the largest $d \in \mathbb{N}$ such that X contains d-many pairwise disjoint definable sets each of which is of the same rank as X.

Morley rank can be seen as an abstract model-theoretic generalisation of the Zariski dimension. We have a good understanding of definable sets in algebraically closed fields: If K is an algebraically closed field and $X \subseteq K^n$, then X is definable if and only if X is constructible (this is what geometers call Chevalley's Theorem and logicians call Tarski's Theorem, see [82, Theorem 3.3.11]). Therefore, we may assign Morley ranks to constructible sets. Indeed, given an irreducible algebraic variety $V \subseteq K^n$, $rk(V) = \dim_K(V)$, where $\dim_K(V)$ denotes the Zariski dimension of V (see e.g. [82, Corollary 6.4.5]).

A group of finite Morley rank is an ω-stable $\mathcal{L}$-structure $\mathcal{G} = (G, \cdot, ^{-1}, e, \ldots)$, where $(G, \cdot, ^{-1}, e)$ is a group and $rk(G) < \omega$. Similarly as for stable groups, throughout this thesis, we abuse the notation and write $G = \mathcal{G}$, that is, G denotes both the $\mathcal{L}$-structure $\mathcal{G}$ and the underlying set G of $\mathcal{G}$.

Interestingly, groups of finite Morley rank can be defined equivalenty in purely group-theoretic terms via a *rank-function* defined by Borovik and Poizat (see [21] or [5]). Then, one defines a *definable universe* $\mathcal{U}$ to be a set of sets closed under taking singletons (one-element sets), Boolean operations, direct products, projections and quotients (by equivalence relations in the universe). Further, one calls a definable universe $\mathcal{U}$ *ranked* if it carries a rank-function. This means that one can assign, to every set $S \in \mathcal{U}$, a natural number which behaves as a "dimension" of the corresponding set. A group G is said to be *ranked* if its base set and graphs of multiplication and inversion belong to a ranked universe. It is well-known (see [71]) that a simple ranked group is ω-stable of finite Morley rank. We won't give the definition of the rank-function in this thesis but an interested reader may find it from either of the books [21] or [5].

We now state one of the most basic facts of groups of finite Morley rank:

Fact 5.1.2 ([21, Lemma 5.1]). *Let $K \leqslant H$ be definable subgroups of a group of finite Morley rank G. Then*

(i) $[H : K] = \infty$ *if and only if* $rk(K) < rk(H)$.

(ii) $[H : K] < \infty$ *if and only if* $rk(K) = rk(H)$. *In this case,* $deg(H) = [H : K]deg(K)$.

(iii) $H = K$ *if and only if* $rk(K) = rk(H)$ *and* $deg(K) = deg(H)$.

Since groups of finite Morley rank are ω-stable, and therefore stable, they are of finite centraliser dimension by Fact 3.3.6. Actually, something stronger is true. Similarly as algebraic groups do not have infinite descending chains of algebraic subgroups, groups of finite Morley rank do not have infinite descending chains of definable subgroups—this follows easily from Fact 5.1.2.

Let H be a definable subgroup of a group of finite Morley rank G. We can apply the descending chain condition to the set of definable subgroups of H of finite indices. We call this intersection the *connected component* of H and denote it by H°. Indeed, it is well-known that a group of finite Morley rank G contains a unique, minimal, definably characteristic, definable, normal and connected subgroup of finite index, called the connected component of G, and denoted by G° (see e.g. [21, Section 5.2]). A group of finite Morley rank G is said to be *connected* if $G = G^\circ$.

We may also apply the descending chain condition of definable subgroups to define the *definable closure* of any subset $X \subseteq G$ of a group of finite Morley rank G: The definable closure of X, denoted by $\overline{X}$, is the intersection of all definable subgroups of

G containing X. It is immediate from the definition that $\overline{X}$ is the smallest definable subgroup of G containing X. The properties of definable closures of subsets and subgroups of a group of finite Morley rank G are well-known—these properties play an important role in the proof of Theorem 7.2.1:

Fact 5.1.3 ([21, Lemmas 5.34 and 5.35] and [5, Lemma 2.15])**.** *Let G be a group of finite Morley rank and let $\overline{X}$ denote the definable closure of any subset X of G. Then the followings hold:*

(a) If $A \leqslant G$ is an abelian subgroup of G then $\overline{A}$ is abelian.

(b) If a subgroup $A \leqslant G$ normalises the set X, then $\overline{A}$ normalises $\overline{X}$.

(c) $C_G(X) = C_G(\overline{X})$.

(d) For a subgroup $A \leqslant G$, $\overline{N_G(A)} \leqslant N_G(\overline{A})$.

(e) For a subgroup $A \leqslant G$, $\overline{A^i} = \overline{A}^i$ and $\overline{A^{(i)}} = \overline{A}^{(i)}$.

(f) For subgroups $A, B \leqslant G$, $[\overline{A}, \overline{B}] = \overline{[A, B]}$.

(g) Let $A \leqslant B \leqslant G$ be subgroups of G. If A has finite index in B then $\overline{A}$ has finite index in $\overline{B}$ and if $A \trianglelefteq B$ then $\overline{A} \trianglelefteq \overline{B}$ and $\overline{B} = \overline{A}B$.

(h) If $A \leqslant G$ is solvable (resp. nilpotent) subgroup of class n, then $\overline{A}$ is also solvable (resp. nilpotent) of class n.

Let us next see some examples of groups of finite Morley rank.

(1.) Finite groups; they have Morley rank 0.

(2.) Any group G which is a model of an $\aleph_1$-categorical theory T is a group of finite Morley rank; this is how the subject started—Zilber was essentially trying to understand uncountably categorical structures after the seminal work of Morley.

(3.) Algebraic groups; an algebraic group G over an algebraically closed field K is a group of finite Morley rank and the Morley rank of G itself coincides with the Zariski dimension of G over K.

(4.) Abelian groups of bounded exponent.

(5.) Direct products of the above, e.g., $PSL_2(\mathbb{C}) \times PSL_2(\overline{\mathbb{F}}_p)$.

Non-examples of groups of finite Morley rank include the infinite cyclic group $\mathbb{Z}$ and free groups F_n: The group $(\mathbb{Z}, +, -, 0)$ is not of finite Morley rank because it has the following infinite descending chain of definable subgroups

$$\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \cdots .$$

Further, free groups F_n are not of finite Morley rank as they contain a definable subgroup isomorphic to $(\mathbb{Z}, +, -, 0)$. One should note that, as free groups F_n are stable, they demonstrate that the class of stable groups is much wider than the class of groups of finite Morley rank.

We give another important non-example of groups of finite Morley rank—to do so we need more definitions.

A theory T is said to have a *quantifier elimination* if for every formula ϕ there is a quantifier-free formula ψ such that $T \models \phi \Leftrightarrow \psi$.

Given a structure $\mathcal{M} \models T$, $\mathcal{M}$ is said to be *existentially closed* if whenever $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{N} \models T$ and $\mathcal{N} \models \exists \bar{v} \phi(\bar{v}, \bar{a})$, where $\bar{a} \in M$ and ϕ is a quantifier-free formula, then $\mathcal{M} \models \exists \bar{v} \phi(\bar{v}, \bar{a})$.

A theory T is *model complete* if every model of T is existentially closed. One says that the theory T' is the *model companion* of the theory T if T' is model complete, every model of T embeds as a substructure of a model of T', and every model of T' embeds as a substructure of a model of T. If, in addition, T' admits quantifier elimination then we say that T' is the *model completion* of T.

A *differential field* is a field K endowed with a derivation δ, i.e., an additive map $\delta : K \longrightarrow K$ satisfying the Leibniz rule $\delta(xy) = x\delta(y) + \delta(x)y$. The model completion of the theory of differential fields of characteristic 0 is the theory of *differentially closed fields* of characteristic 0, commonly denoted by DCF_0. It was proven by Lenore Blum in his PhD thesis [15] (see also [16]) that the theory DCF_0 is ω-stable.

Given a model (K, δ) of the theory DCF_0, the additive group K^+ is an ω-stable group with $rk(K^+) = \omega$.

5.2 Infinite simple groups of finite Morley rank

Let us consider the examples of groups of finite Morley rank given in the last section; one observes that algebraic groups over algebraically closed fields are the only infinite simple ones. Indeed, the only known infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields. In the late 70's, Cherlin [30] and Zilber [30] independently conjectured that these are the only examples; this conjecture is today known as the *Cherlin–Zilber Conjecture* or the *Algebraicity Conjecture*.

Conjecture 1 (The Cherlin–Zilber Conjecture). *Infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields.*

At this point we state one of the most important results in the area of ω-stable groups of finite Morley rank, which explains why in the Cherlin–Zilber Conjecture we hope to identify infinite simple groups of finite Morley rank with algebraic groups over *algebraically closed fields*.

Fact 5.2.1 (Macintyre's Theorem [59]; or see [21, Theorem 8.1])**.** *An infinite field of finite Morley rank is algebraically closed; its additive group and its multiplicative group are connected.*

Despite over 40 years of effort, the Cherlin–Zilber Conjecture remains open. One should note that infinite simple groups of Morley rank 3 were classified only very recently, in 2018, by Olivier Frécon [46]. This illustrates the difficulty of the problem we are dealing with.

So far, the most successful approach towards the Cherlin–Zilber Conjecture has been the so-called Borovik Programme. The idea of this approach, suggested by Borovik, is to prove "the analogue of CFSG" in the context of infinite simple groups of finite Morley rank. To understand the analogue of CFSG in the context of infinite simple groups of finite Morley rank, one should note that there are no infinite simple abelian groups of finite Morley rank (the infinite cyclic group $\mathbb{Z}$ is not ω-stable) and that analogues of (finite) alternating groups Alt_X on infinite sets X are not groups of finite Morley rank (they have the independence property and thus they are not even stable). Moreover, a field of finite Morley rank K does not have a definable non-trivial automorphism of finite order (therefore, there are no infinite twisted Chevalley groups of finite Morley rank). Finally, the hope is that infiniteness will smooth out the sporadic groups. This would leave us with Chevalley groups over algebraically closed fields, which are algebraic groups by construction—indeed, the classification of infinite simple groups definable in an algebraically closed field shows that they are isomorphic to Chevalley groups.

The Sylow 2-theory of groups of finite Morley rank is well-understood. In this context, Sylow 2-subgroups do not need to be definable but they are nilpotent-by-finite and conjugate. Moreover, they have well-defined notion of a connected component; given a Sylow 2-subgroup S of a group of finite Morley rank G, we have $S^\circ = S \cap \overline{S}^\circ$ (here $\overline{S}^\circ$ denotes the connected component of the definable closure $\overline{S}$ of S). Groups of finite Morley rank can be split into four cases based on the structure of the connected component S° of a Sylow 2-subgroup S of G:

(i) *Even type*: S° is non-trivial, definable, nilpotent, and of bounded exponent, i.e., S° is *2-unipotent.*

(ii) *Odd type*: S° is non-trivial, divisible and abelian i.e., S° is a *2-torus.*

(iii) *Mixed type*: S° is a central product of a non-trivial 2-unipotent group and a non-trivial 2-torus.

(iv) *Degenerated type*: S° is trivial.

The greatest achievements of the Borovik Programme are the following two result by Altınel, Borovik and Cherlin.

Fact 5.2.2 (Altınel, Borovik and Cherlin [5])**.** *Infinite simple groups of finite Morley rank of even type are Chevalley groups over algebraically closed fields of characteristic 2.*

Fact 5.2.3 (Altınel, Borovik and Cherlin [5])**.** *Infinite simple groups of finite Morley rank of mixed type do not exist.*

It is known that infinite simple groups of finite Morley rank of degenerated type would provide an immediate counter-example to the Cherlin–Zilber Conjecture—the existence of such groups is a widely open problem. At the same time, it is widely believed that infinite simple groups of finite Morley rank of odd type are Chevalley groups over algebraically closed fields of characteristic different from 2. Thus, the current aim of the Borovik Programme is to separately deal with odd and degenerated type groups.

Though the Borovik Programme has been very successful in the past, for some time now, there has been no serious progress on the Programme. Therefore, new approaches towards the Cherlin–Zilber Conjecture are needed. One of such, has emerged in recent years—this approach was systematically developed by Uğurlu in her PhD thesis [85]. Uğurlu's work, and the strategy of the programme initiated by her, follows from results and ideas of Hrushovski in [50]. The aim of this programme is to prove that another conjecture called the *Principal Conjecture*—which can be traced to Hrushovski's work in [50]—is equivalent to the Cherlin–Zilber Conjecture. To introduce the Principal Conjecture, we need one more definitions.

Let T be a complete theory with a quantifier elimination. An automorphism β of a structure $\mathcal{M} \models T$ is called *generic* if $(\mathcal{M}, \beta)$ is an existentially closed model of

$$T \cup \{\text{“}\beta \text{ is an automorphism”.}\}.$$

Generic automorphisms of algebraically closed fields were studied by Macintyre in [60] (see also [28]). He proved that in this context the fixed points of a generic automorphism form a pseudofinite field. Further, in [50], Hrushovski studied certain classes of structures with generic automorphisms. From his results an observations, Uğurlu [85] formulated the following conjecture.

Conjecture 2 (The Principal Conjecture). *Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Then the fixed point subgroup $C_G(\alpha)$ is pseudofinite.*

It follows from results of Chatzidakis and Hrushovski [28] and Hrushovski [51] that the Cherlin–Zilber Conjecture implies the Principal Conjecture. Furthermore, there is an expectation that these conjectures are actually equivalent—this expectation is supported by an important partial result by Uğurlu [86].

Note that, if the Cherlin–Zilber Conjecture holds, then a pair (G, α) can be constructed as an ultraproduct of simple algebraic groups over fields of positive characteristic enriched by the Frobenius automorphisms of the underlying fields.

In Chapter 7, we explain Uğurlu's approach towards proving the expected equivalence between the Cherlin–Zilber Conjecture and the Principal Conjecture. Moreover, we prove a first of hopefully many results which continues Uğurlu's work in [86], namely, we prove Theorem 7.2.1. In Chapter 6, we use a different natural approach. In order to understand generic automorphisms (in the sense of Chatzidakis and Hrushovski [28]) and their pseudofinite groups of fixed points, it is desirable to understand possible analogues of Frobenius maps of an algebraically closed field K of positive characteristic. Therefore, we define an automorphism group of a group of finite Morley rank that resembles the group of such Frobenius maps and classify infinite simple groups of finite Morley rank admitting such an automorphism group, that is, we prove Theorem 6.3.1.

5.3 Useful facts

In this rather long section we give all the background results and definitions on groups of finite Morley rank the reader is required to know in order to read the proofs of Theorems 6.3.1 and 7.2.1. We wish to mention that, most of the results in this section are needed in the proof of Theorem 7.2.1—the only results that are needed in the proof of Theorem 6.3.1 (from this section) are Facts 5.3.1, 5.3.3 and 6.2.3.

It is obvious that if G_1 is a group (resp. structure) of finite Morley rank and G_2 is such that $G_1 \equiv G_2$ then the group (resp. structure) G_2 is also of finite Morley rank. However, we wish to mention that it is also known that a group G is ranked if and only if G is of finite Morley rank. This fact was proven by Poizat; see the introduction of [71].

Fact 5.3.1 (Poizat [71])**.** *A ranked group G is of finite Morley rank. Conversely, if G is a group of finite Morley rank then G is ranked. Therefore, if G is a ranked group and H is a group such that $G \equiv H$ then the group H is also a ranked group.*

5.3.1 Classification of infinite simple locally finite groups of finite Morley rank

We now briefly comment the classification of infinite simple locally finite groups of finite Morley rank.

Due to the result of Thomas (Fact 2.4.10) we know the following.

Fact 5.3.2 (Thomas [83]). *An infinite simple locally finite group of finite Morley rank is a Chevalley group over an algebraically closed field.*

Thomas's result above depends on CFSG. Later result by Borovik [18] removed the use of CFSG in the case of odd type groups:

Fact 5.3.3 (Borovik [18, Theorem 7.1]—does not depend on CFSG). *An infinite simple locally finite group of finite Morley rank and odd type is isomorphic to a simple algebraic group over an algebraically closed locally finite field of odd characteristic.*

We finish this subsection with the following fact.

Fact 5.3.4 ([5, Lemma 3.27]). *Let G be a locally solvable group of finite Morley rank. Then G is solvable.*

5.3.2 Few rank tools

Let G be a group of finite Morley rank. The following is well-known/well-defined.

(1.) A definable subset X of G is called *generic* if $rk(X) = rk(G)$.

(2.) Let x be any element of G. Then $rk(G) = rk(x^G) + rk(C_G(x))$ (see [21, Exercise 12 of Section 4.2]).

(3.) Following the terminology in [21], a *generalised centraliser* $C_G^*(x)$ of an element $x \in G$ is of the form

$$C_G^*(x) = \{g \in G : x^g = x \text{ or } x^g = x^{-1}\}.$$

We have $[C_G^*(x) : C_G(x)] = 1$ or 2 and therefore $rk(C_G^*(x)) = rk(C_G(x))$ (see [21, Section 10]). We wish to warn the reader that another tool, which is used in many places in the literature in the past, also named a 'generalised centraliser' is defined by Frécon in [45]. Though our terminology clashes with Frécon's terminology the reader should not think that these two notions coincide.

(4.) Let $f : X \longrightarrow Y$ be a definable surjective map between two definable subsets of G. For $i \in \mathbb{N}$, let $Y_i = \{y \in Y : rk(f^{-1}(y)) = i\}$. The sets Y_i are definable and Y is a disjoint union of them. Let $X_i = f^{-1}(Y_i)$. Then $rk(X_i) = rk(Y_i) + i$ and $rk(X) \leqslant rk(Y) + n$ where $n = \max_{y \in Y} rk(f^{-1}(y))$ (see [21, Exercise 14 of Section 4.2]).

5.3.3 Good tori

We now introduce Cherlin's [31] notion of a good torus of a group of finite Morley rank G.

Let G be a group of finite Morley rank. A definable connected divisible abelian subgroup (resp. p-subgroup) T of G is called a *torus* (resp. *p-torus*) of G. A torus T of G is called a *good torus* if every definable subgroup of T is the definable closure of its torsion. Moreover, we have a weakening of Cherlin's notion of a good torus: a torus T of G is called a *decent torus* if it is the definable closure of its torsion. Note that every good torus is a decent torus.

The following facts are needed in the proof of Theorem 7.2.1.

Fact 5.3.5 (Wagner [89]). *The multiplicative group K^* of a field of finite Morley rank K of positive characteristic is a good torus.*

Fact 5.3.6 ([5, Corollary 4.22]). *A connected definable subgroup of a finite product of good tori is a good torus.*

Fact 5.3.7 (Altınel and Burdges [6, Theorem 1]). *If T is a decent torus of a connected group G of finite Morley rank, then $C_G(T)$ is connected.*

Fact 5.3.8 ([5, Lemma 4.23]). *Let G be a group of finite Morley rank and T be a definable abelian subgroup of G. Assume that T° is a good torus. Then $N^\circ_G(T) = C^\circ_G(T)$.*

Fact 5.3.9 (Cherlin [31, Theorem 1]). *Any two maximal good tori in a group of finite Morley rank G are conjugate.*

Fact 5.3.10 (Altınel and Burdges [6, Lemma 6.6]). *Let G be a connected solvable group of finite Morley rank and $H \leqslant G$ be a definable connected subgroup of G such that $[N_G(H) : H] < \infty$. Then $N_G(H) = H$.*

5.3.4 The Fitting subgroup and Borel subgroups

We start this subsection by citing the following well-known result.

Fact 5.3.11 ([5, Lemma 5.1]). *Let H be a nilpotent group of finite Morley rank and P be an infinite normal subgroup of H. Then $P \cap Z(H)$ is infinite.*

The *Fitting subgroup* $F(G)$ of a group of finite Morley rank G is the subgroup generated by all normal nilpotent subgroups of G. It is well-known that $F(G)$ is a characteristic subgroup of G which is definable and nilpotent (see [21, Section 7.2]).

A subgroup B of a group of finite Morley rank G is called a *Borel subgroup* if it is a maximal definable connected and solvable subgroup of G (to compare the notion of a Borel subgroup in the contexts of algebraic groups and groups of finite Morley rank, see e.g. [21, Section 13]). It is well-known that if B is a Borel subgroup of a group of finite Morley rank G, then $N_G^\circ(B) = B$ (see [21, Exercise 2 of Section 13.1]).

Note that if M is a connected solvable algebraic group, then $M = A \rtimes H$ where A is the unipotent radical and H is a torus (see [52, Theorem 19.3.b]). In the setting of groups of finite Morley rank the situation is not as clear, however, among other things, the following property of connected solvable groups of finite Morley rank is known.

Fact 5.3.12 (Nesin [64]). *Let B be a connected solvable group of finite Morley rank. Then B' is nilpotent. Therefore, B' is contained in $F(B)$.*

5.3.5 The socle

A subgroup H of a group of finite Morley rank G is called *G-minimal* if it is an infinite definable normal subgroup of G which is minimal with respect to these properties. Similarly, a K-normal definable subgroup $H \leqslant G$ is called K-minimal, for some definable subgroup $K \leqslant G$, if there are no proper infinite definable subgroups of H which are K-normal. Note that G-minimal subgroups are necessarily connected. A *minimal normal subgroup* of G is either G-minimal or finite (see [21, Section 7.3]).

The *socle* of G, denoted by $S(G)$, is the subgroup generated by all minimal normal subgroups of G. Note that the socle of G is a characteristic subgroup. Moreover, $S(G)_\circ$ stands for the subgroup of G generated by all G-minimal subgroups. By *Zilber's Indecomposability Theorem* (see next subsection), it is clear that $S(G)_\circ$ is definable and connected.

The structure of the socle of a connected solvable group of finite Morley rank is known by the following result of Nesin.

Fact 5.3.13 (Nesin [65, Proposition 1]). *Let G be a connected solvable group of finite Morley rank with $Z(G) \cap G' = 1$. Then $S(G) = A_1 \oplus \cdots \oplus A_m$ for some finitely many G-minimal subgroups of G'. In particular, $S(G)$ is definable and connected. Further $S(G) \leqslant G'$.*

If G is a group of finite Morley rank such that G° is centerless, then $S(G)_\circ \leqslant S(G^\circ)$:

Fact 5.3.14 ([21, Theorem 7.8]). *Let G be a group of finite Morley rank. Assume that $Z(G^\circ) = 1$. Then $S(G)$ is definable and $S(G)^\circ = S(G)_\circ \leqslant S(G^\circ)$.*

5.3.6 Two theorem's by Zilber

We now state two important results by Zilber called Zilber's Indecomposability Theorem and *Zilber's Field Theorem*. Note that the former generalises a result from algebraic group theory (see [17, I.2.2]). We also state one of the numerous consequences of Zilber's Indecomposability Theorem, which is very important to us. We refer an interested reader who wishes to see more consequences to [21, Subsection 5.4].

Let Q be a definable subgroup of a group of finite Morley rank G. A definable subset X of G is called *Q-indecomposable* if whenever the cosets of Q partition X into more than one subset, then they partition X into infinitely many subsets. Further, a definable set X is called *indecomposable* if X is Q-indecomposable for all definable subgroups Q of G. Note that a definable subgroup of G is indecomposable if and only if it is connected (see [21, Section 5.4]).

Fact 5.3.15 (Zilber's Indecomposability Theorem [91]; or see [21, Theorem 5.27]). *Let $(A_i)_i$ be a family of indecomposable subsets of a group of finite Morley rank G. Assume that each A_i contains the identity element of G. Then the subgroup generated by the subsets of A_i is definable and connected.*

One of the most important corollaries of Zilber's Indecomposability Theorem is that simplicity is a first-order property in the context of groups of finite Morley rank:

Fact 5.3.16 ([21, Ex. 5 of Subsection 5.4]). *A non-abelian definably simple group of finite Morley rank is simple.*

We move on to present Zilber's Field Theorem.

Fact 5.3.17 (Zilber's Field Theorem [91]; or see [21, Theorem 9.1])**.** *Let $G = A \rtimes H$ be a group of finite Morley rank where A and H are infinite definable abelian subgroups and A is H-minimal. Assume that $C_H(A) = 1$. Then*

- *The subring $K = \mathbb{Z}[H]/\mathrm{ann}_{\mathbb{Z}[H]}(A)$ of $\mathrm{End}(A)$ is a definable algebraically closed field; in fact there is an integer ℓ such that every element of K can be represented as the endomorphism $\sum_{i=1}^{\ell} h_i (h_i \in H)$.*
- *$A \cong K^+$ and H is isomorphic to a subgroup T of K^* and H acts on A by multiplication.*
- *H acts freely on A, $K = T + \ldots + T$ (ℓ times) and $A = \{\prod_{i=1}^{\ell} a^{h_i} : h_i \in H\}$, or using the additive notion, $A = \{\sum_{i=1}^{\ell} h_i a : h_i \in H\}$, for any $a \in A^*$.*

5.3.7 Some Sylow 2-theory

Recall that a Prüfer 2-group is a group isomorphic to a quasicyclic group

$$Z(2^\infty) = \{x \in \mathbb{C} : x^{2^n} = 1 \,\text{for some}\, n \in \mathbb{N}\}.$$

The largest such k that $Z(2^\infty)^k$ embeds into G is called the *Prüfer 2-rank* of G and is denoted by $\mathrm{pr}_2(G)$.

The connected component of a Sylow 2-subgroup of a group of finite Morley rank G is a direct product of finitely many copies of the Prüfer 2-group $Z(2^\infty)$ (with this finite number of course being the Prüfer 2-rank of the group), see [22].

It is know that connected solvable groups of finite Morley rank have connected Sylow 2-subgroups:

Fact 5.3.18 ([21, Theorem 9.29])**.** *Let G be a connected solvable group of finite Morley rank. Then the Sylow p-subgroups of G are connected.*

By Fact 5.3.18, a Sylow 2-subgroup of a connected solvable group of finite Morley rank G of odd type with $\mathrm{pr}_2(G) = 1$ satisfies

$$S = S^\circ \cong Z(2^\infty).$$

Moreover, if G is a connected odd type group of finite Morley rank with $\mathrm{pr}_2(G) = 1$, then the structure of a Sylow 2-subgroup S of G is well-understood by the following result of Adrien Deloro and Éric Jaligot.

Fact 5.3.19 (Deloro and Jaligot [40, Proposition 27]). *Let G be a connected group of finite Morley rank of odd type and with $\mathrm{pr}_2(G) = 1$. Then there are exactly three possibilities for the isomorphism type of a Sylow 2-subgroup S of G.*

(1.) $S = S^\circ \rtimes \langle \omega \rangle$ for some involution ω which acts on S° by inversion.

(2.) $S = S^\circ \cdot \langle \omega \rangle$ for some element ω of order 4 which acts on S° by inversion.

(3.) $S = S^\circ$.

The following facts will be useful to us in Section 7.2.2.

Fact 5.3.20 ([21, Theorem 10.15]). *Let G be a group of finite Morley rank and i be an involution in G. Assume that the centraliser $C_G(i)$ is finite. Then G° is abelian and i inverts G°.*

Fact 5.3.21 (Deloro [38]). *Let G be a connected group of finite Morley rank of odd type and let $i \in G$ be an involution. Then $C_G(i)/C_G^\circ(i)$ has exponent dividing 2.*

Fact 5.3.22 (Deloro and Jaligot [40, Lemma 26]). *Let B be a connected solvable group of finite Morley rank of odd type with $\mathrm{pr}_2(B) = 1$ and let i be an involution of B. Then the set of involutions of B is exactly $i^{F^\circ(B)}$.*

Fact 5.3.23 (Borovik and Poizat [22]; or see [21, Theorem 10.11]). *Sylow 2-subgroups of a group of finite Morley rank G are conjugate to each other.*

Fact 5.3.24 (Borovik, Burdges and Cherlin [19]). *Let G be a connected group of finite Morley rank whose Sylow 2-subgroup is finite. Then G contains no involutions, that is, the Sylow 2-subgroup is trivial.*

5.3.8 Strongly embedded subgroups

A subgroup M of a group of finite Morley rank G is called a *strongly embedded* subgroup if it satisfies the following two conditions:

(1.) M contains an involution, and;

(2.) For every $g \in G \setminus M$, $M \cap M^g$ does not contain any involutions.

Fact 5.3.25 (Altınel [4, Proposition 3.4]). *Let G be a group on finite Morley rank and M be a strongly embedded subgroup of G. Then, if N is a definable subgroup of G such that $M \leqslant N < G$ then N is a strongly embedded subgroup of G.*

5.3.9 Some results on Frobenius groups and Zassenhaus groups of finite Morley rank

In this subsection we list some properties of split Frobenius groups and split Zassenhaus groups of finite Morley rank that will be useful to us in the proof of Theorem 7.2.1. We start by defining these groups.

A group B with a proper non-trivial subgroup T such that, for all $b \in B$, $T^b \cap T \neq 1$ implies that $b \in T$ is called a *Frobenius group* with a *Frobenius complement* T. Whenever $B = U \rtimes T$ for some $U \lhd B$, then B is said to be a *split* Frobenius group with a *Frobenius kernel* U.

A 2-transitive permutation group is called a *Zassenhaus group* if the stabiliser of any three distinct points is the identity. A Zassenhaus group H is called a *split* Zassenhaus group if a one-point stabiliser B of H is a split Frobenius group.

We now list some properties of Frobenius groups of finite Morley rank which we will need in the proof of Theorem 7.2.1. The phrase "$T < B$ is a Frobenius group" will mean that B is a Frobenius group with a Frobenius complement T.

Fact 5.3.26 ([21, Lemma 11.10 and Theorem 11.32]). *Let $B = U \rtimes T$ be a solvable split Frobenius group of finite Morley rank with a Frobenius complement T and a Frobenius kernel U. Then,*

(i) for all $t \in T^*$, $C_B(t) \leqslant T$, and,

(ii) for all $u \in U^*$, $C_B(u) \leqslant U$.

Fact 5.3.27 ([21, Lemma 11.21]). *Let $T < B$ be a Frobenius group of finite Morley rank. Assume that T has an involution. If B has a normal definable subgroup disjoint from T, then $T < B$ splits.*

Fact 5.3.28 ([21, Corollary 11.24]). *Let $B = U \rtimes T$ be a split Frobenius group of finite Morley rank. Let $X \leqslant U$ be a B-normal subgroup of U. Then, if T is infinite then X is definable and connected. In particular, U is connected.*

Fact 5.3.29 ([21, Theorem 11.32]). *Let B be a solvable split Frobenius group of finite Morley rank with a Frobenius complement T and Frobenius kernel U. Then, if $X \cap U = 1$ for some $X \leqslant B$ then X is conjugate to a subgroup of T.*

We conclude this chapter with an important identification result by Franz Delahan and Nesin; we will invoke this result to identify our G in Theorem 7.2.1.

Fact 5.3.30 (Delahan and Nesin [35]). *Let G be an infinite split Zassenhaus group of finite Morley rank. If the stabiliser of two distinct points contains an involution, then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.*

Chapter 6

Finitary automorphism group A

In this chapter we give the main original definition of this thesis, namely, we introduce a finitary automorphism group A. Further, we classify infinite definably simple stable groups admitting a finitary automorphism group A. All results in this chapter are published in [55].

A finitary automorphism group A of an infinite structure $\mathcal{M}$ is introduced in Section 6.1. Moreover, we prove that if an infinite structure $\mathcal{M}$ admits a finitary automorphism group A, then there exists an infinite locally finite elementary substructure $\mathcal{M}_*$ of $\mathcal{M}$.

In Section 6.2, we classify infinite definably simple stable groups admitting a finitary automorphis group A, that is, we prove Theorem 6.2.1. Theorem 6.2.1 depends on CFSG.

In Section 6.3, we prove the analogue of Theorem 6.2.1, in the restricted case of groups of finite Morley rank, without the use of CFSG. In particular, this confirms the Cherlin–Zilber Conjecture (without the use of CFSG) in the specific case in which an infinite simple group of finite Morley rank G admits a finitary automorphism group A.

6.1 The locally finite core

In this section we give the main original definition of this thesis, that is, we introduce a *finitary automorphism group* A of an infinite structure $\mathcal{M}$.

Let us very briefly recall the motivation behind our definition of the finitary automorphism group A.

Recall first that the Cherlin–Zilber Conjecture states that *infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields.*

In [28], Chatzidakis and Hrushovski studied algebraically closed fields with a generic automorphism—due to their results, together with results of Hrushovski in [51], it is known that the Cherlin–Zilber Conjecture implies the Principal Conjecture: *Let G be an infinite simple group of finite Morley rank and α be a generic automorphism of G. Then the fixed point subgroup $C_G(\alpha)$ is pseudofinite.*

To understand generic automorphisms (in the sense of Chatzidakis and Hrushovski [28]) and their pseudofinite groups of fixed points, it is desirable to understand possible analogues of Frobenius maps of an algebraically closed field K of positive characteristic. Indeed, the definition of a finitary automorphism group A arises from the attempt to identify conditions on a group of automorphisms of a group of finite Morley rank that make it resemble the group of Frobenius maps of an algebraically closed field K of positive characteristic.

Now we give the key definitions of this chapter.

Definition 6.1.1 (Karhumäki [55]). *Let $\mathcal{M} = (M, \ldots)$ be an infinite structure. We say that an infinite group A of automorphisms of $\mathcal{M}$ is* finitary*, if the following hold:*

(1.) *For every $\alpha \in A \setminus \{1\}$, the substructure of fixed points $\mathrm{Fix}_M(\alpha)$ is finite, and;*

(2.) *If $X \neq \emptyset$ is definable subset in M which is invariant under the action of some non-trivial automorphism $\alpha \in A$, then there exists an element $x \in X$ with a finite orbit x^A. Equivalently, $[A : \mathrm{Stab}_A(x)] < \infty$.*

Given Definition 6.1.1, we may further define the *locally finite core* M_* of the underlying set M of an infinite structure $\mathcal{M}$ admitting a finitary automorphism group A.

Definition 6.1.2 (Karhumäki [55]). *Let $\mathcal{M} = (M, \ldots)$ be an infinite structure admitting a finitary automorphism group A. Then, we define the* locally finite core M_* *of M to be the set of elements of M with finite A-orbits:*

$$M_* = \{\, m \in M \mid \text{the orbit } m^A \text{ is finite.}\}.$$

Notice that the assumption (2.) of Definition 6.1.1, applied to $X = M$, ensures that M_* is non-empty. Moreover, we may easily observe that M_* is infinite:

Lemma 6.1.3. *The locally finite core M_* is infinite.*

Proof. The claim follows immediately from the assumption (2) of Definition 6.1.1: Assume towards a contradiction that M_* is finite, that is, $M \setminus M_*$ is definable. Then, by the assumption (2) of Definition 6.1.1, A has a finite orbit x^A on the definable A-invariant subset $X = M \setminus M_*$ with x not belonging to M_*—this contradicts the way M_* was constructed. □

An important example (and also the only example the author is aware of) of finitary groups of automorphisms is provided by Frobenius maps:

Example 6.1.4. *Let G be a group of points over an algebraically closed field K of characteristic $p > 0$ of an algebraic group defined over the field $\mathbb{F}_q$, where q is some power of p, and let ϕ be the automorphism of G induced by the Frobenius map $x \mapsto x^q$ on K. Then the group $\langle \phi \rangle$ generated by ϕ is a finitary group of automorphisms.*

Remark 6.1.5. *Though we define a finitary automorphism group A as an infinite group of automorphisms of a given structure $\mathcal{M}$, one may think of this automorphism group also as* one *automorphism α such that $A = \langle \alpha \rangle$. Indeed, our only example (Example 6.1.4) of the finitary automorphism group A is of this form.*

Definitions 6.1.1 and 6.1.2 together with the Tarski-Vaught Test (Fact 3.1.1) for elementary substructures allow us to prove that the infinite locally finite core M_* is actually the underlying set of an infinite locally finite elementary substructure $\mathcal{M}_*$ of $\mathcal{M}$.

Note at this point that there is an obvious restatement of the Tarski-Vaught Test in terms of definable subsets:

> *Let $\mathcal{M} = (M, \ldots)$ be a substructure of $\mathcal{N} = (N, \ldots)$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if, for all non-empty M-definable subsets $S \subseteq N$, $S \cap M \neq \emptyset$.*

Theorem 6.1.6 (Karhumäki [55, Theorem 4.2])**.** *Let $\mathcal{M} = (M, \ldots)$ be an infinite structure and A be a finitary automorphism group of $\mathcal{M}$. Then $\mathcal{M}_* = (M_*, \ldots)$ is an infinite locally finite elementary substructure of $\mathcal{M}$.*

Proof. First we check that the infinite set M_* is the underlying set of an infinite locally finite substructure $\mathcal{M}_*$ of $\mathcal{M}$, that is, the operations and relations of $\mathcal{M}_* = (M_*, \ldots)$ are inherited from $\mathcal{M} = (M, \ldots)$.

Let $x_1, \ldots, x_n \in M_*$, and $X = \langle x_1, \ldots, x_n \rangle$ be the substructure of $\mathcal{M}$ generated by elements $x_1, \ldots, x_n$ in M, that is, X is the minimal subset of M which contains $x_1, \ldots, x_n$ and is closed under all operations from $\mathcal{M}$. By the definition of M_*, the elements $x_1, \ldots, x_n$ have finite A-orbits, and so, the stabilisers of $x_1, \ldots, x_n$ in A have finite indices in A. Moreover, the intersection

$$B = \mathrm{Stab}_A(x_1) \cap \cdots \cap \mathrm{Stab}_A(x_n)$$

of the stabilisers of $x_1, \ldots, x_n$ in A also has a finite index in A. Therefore, B is non-trivial. Clearly, B fixes every element in X. Therefore, A-orbits of elements in X are finite. We have observed that $X \subseteq M_*$, that is, M_* is closed under all operations from $\mathcal{M}$ and therefore the structure $\mathcal{M}_* = (M_*, \ldots)$ is a substructure of $\mathcal{M} = (M, \ldots)$. Moreover, $X \subseteq \mathrm{Fix}_M(B)$, and therefore, X is finite by the definition of the finitary automorphism group A. We have proven that $\mathcal{M}_*$ is an infinite locally finite substructure of $\mathcal{M}$.

Next, we shall show that every M_*-definable non-empty subset $Y \subseteq M$ has a point in M_*. This will prove that $\mathcal{M}_*$ is an elementary substructure of $\mathcal{M}$ by the Tarski-Vaught Test.

Any M_*-definable set Y is of the form

$$Y = \{y \in M \mid \varphi(y, \overline{m})\},$$

where $\overline{m} = (m_1, \ldots, m_k) \in M_*$. Each m_i has a finite A-orbit and therefore its stabiliser $S_i = \mathrm{Stab}_A(m_i)$ has finite index in A. It follows that the intersection

$$S = \mathrm{Stab}_A(m_1) \cap \cdots \cap \mathrm{Stab}_A(m_k)$$

of all S_i's also has a finite index in A. Clearly, S fixes all m_i's. Now the set Y is S-invariant and S has a finite orbit O in Y such that the point-wise stabiliser $\mathrm{Stab}_A(O)$ has a finite index in A, that is, A also has a finite orbit in Y. Elements in this orbit belong to M_*, that is, $Y \cap M_* \neq \emptyset$. □

In what follows, in Sections 6.2 and 6.3, when applying the definition of a finitary automorphism group A to groups, we shall follow the standard notation of this thesis and refer to stabilisers of points as centralisers and write $C_A(x)$ instead of $\mathrm{Stab}_A(x)$ and $C_X(\alpha)$ instead of $\mathrm{Fix}_X(\alpha)$.

6.2 Definably simple stable groups admitting A

In this section we classify infinite definably simple stable groups in the specific case in which these groups admit a finitary automorphism group A. Thus, we prove the following theorem.

Theorem 6.2.1 (Karhumäki [55, Theorem 1.2]). *Every infinite definably simple stable group G admitting a finitary automorphism group A is a Chevalley group over an algebraically closed field of positive characteristic.*

Note that Theorem 6.2.1 implies that, in the presence of a finitary automorphism group A, an infinite definably simple stable group is simple.

6.2.1 Proof of Theorem 6.2.1

Proof of Theorem 6.2.1 follows as we combine together our Theorems 4.2.1 and 6.1.6. Since we use Theorem 4.2.1, which depends on CFSG, Theorem 6.2.1 also uses CFSG.

We wish to remind the reader that we use an umberella term "groups of Lie type" to all Chevalley groups, twisted Chevalley groups, and non-algebraic twisted groups of Lie type ${}^2B_2$, ${}^2F_4$, or ${}^2G_2$ over locally finite and algebraically closed fields. Moreover, for readers convenience, we recall several results stated in Chapter 3:

- Given two groups $G_1 \preceq G_2$ such that G_1 is a group of fixed Lie type $G_1 = X(K_1)$ over a K_1 it follows that G_2 is a group of the same Lie type $G_2 = X(K_2)$ over some field K_2 (Fact 3.1.2).

- Stable groups possess a finite centraliser dimension (Fact 3.3.6).

- Infinite locally finite stable fields are algebraically closed (Fact 3.3.5).

In what follows we prove a slightly more general result than Theorem 6.2.1, which is actually just a corollary of our earlier results.

Theorem 6.2.2 (Karhumäki [55, Theorem 5.1]). *Every infinite definably simple group G of finite centraliser dimension admitting a finitary automorphism group A is a simple group of Lie type over a field of positive characteristic.*

Proof. Let G be an infinite definably simple group of finite centraliser dimension admitting a finitary automorphism group A. Then, by Theorem 6.1.6, G has an infinite locally finite elementary substructure G_*. Due to the elementary equivalence, G_* also has a finite centraliser dimension and is definably simple. Therefore, by Theorem 4.2.1, G_* is a simple group of Lie type over a locally finite field. Since G is elementary equivalent to G_*, by Fact 3.1.2, G is a simple group of the same Lie type over some field of positive characteristic. □

Theorem 6.2.2 almost immediately leads to our result about definably simple stable groups: We know that stable groups possess a finite centraliser dimension; note further that a field appearing in the statement of Theorem 6.2.2 is interpretable in the group G due to Facts 3.4.4 and 3.4.5. Therefore, Theorem 6.2.1 follows from Theorem 6.2.2 together with the fact that infinite locally finite stable fields are algebraically closed. We only need to recall that groups of Lie type over algebraically closed fields are Chevalley groups. This follows since algebraically closed locally finite fields are algebraic closures $\overline{\mathbb{F}}_p$ of finite prime fields $\mathbb{F}_p$ and for any positive prime p there do not exist twisted Chevalley groups with underlying field $\overline{\mathbb{F}}_p$. This completes the proof of Theorem 6.2.1. □

It is worth mentioning that there exist definably simple stable groups which are not simple. The first example was given by Sela in [76], where he proved that the elementary theory T_{fg} of non-abelian free groups F_n is stable. This provided an example since it was proven by several authors that a free group $F_n \models T_{fg}$ is definably simple (but of-course not simple).

Fact 6.2.3 (Bestiva and Feighn [14]; Kharlampovich and Myasnikov [57]; Perin, Pillay, Sklinos and Tent [68]). *Any proper definable subgroup of a non-abelian free group F_n is cyclic.*

Fact 6.2.3 immediately implies that a free group $F_n \models T_{fg}$ is definably simple since the normaliser of a cyclic subgroup in a torsion-free hyperbolic group coincides with its (cyclic) centraliser.

6.3 Groups of finite Morley rank admitting A

In this section we prove the analogue of Theorem 6.2.1, without the use of CFSG, in the restricted case of groups of finite Morley rank. That is, we prove the following theorem without invoking Theorem 4.1.5.

Theorem 6.3.1 (Karhumäki [55, Theorem 1.3]—does not use CFSG)**.** *Every infinite simple group of finite Morley rank G admitting a finitary automorphism group A is a Chevalley group over an algebraically closed field of positive characteristic.*

Note that Theorem 6.3.1 implies that the Cherlin–Zilber Conjecture holds in the specific case in which an infinite simple group of finite Morley rank admits a finitary automorphism group A.

6.3.1 Proof of Theorem 6.3.1

Theorem 6.3.1 almost immediately follows from the definition of the finitary automorphism group A together with Theorem 6.1.6. As explained in Section 5.3, many properties are preserved under elementary equivalence in the context of groups of finite Morley rank. In particular, we know the following two things:

- If G is a group of finite Morley rank and H is a group elementary equivalent to G then H is also of finite Morley rank (Fact 5.3.1).
- A non-abelian definably simple group of finite Morley rank G is simple (Fact 5.3.16).

Further, the following is proven *without the use of CFSG*:

- An infinite simple locally finite group of finite Morley rank is a Chevalley group over an algebraically closed locally finite field. This follows from Facts 5.3.3 and 5.2.3 as there are no infinite simple locally finite groups of finite Morley rank of degenerated type—this fact is explained below inside the proof of Theorem 6.3.1.

Given above, the proof of Theorem 6.3.1 is very easy, we only need to recall the result of Thomas stating that being a Chevalley group over a field K is a first-order axiomatisable property (Fact 3.1.2).

Now we start the proof of Theorem 6.3.1.

Let G be an infinite simple group of finite Morley rank admitting a finitary automorphism group A. In view of Theorem 6.1.6, there exists an infinite locally finite elementary subgroup G_* of G. As explained above, we may observe that:

- G_* is of finite Morley rank.
- G_* is simple.

At this point, we observe that G cannot be of degenerated type. Towards a contradiction, assume that G is of degenerate type. Now, by Fact 5.3.24, G contains no involutions, and thus, G_* is an infinite simple locally finite group of finite Morley rank which contains no involutions. However, this is a contradictions as one may observe that such groups do not exist: Each finitely generated subgroup of an infinite simple locally finite group of finite Morley rank H which contains no involutions is a solvable group by the famous Feit–Thompson Odd Order Theorem [44]. Therefore, such H is a locally solvable group of finite Morley rank and thus, by Fact 5.3.4, a solvable group—this contradicts the simplicity of H.

Since G can be assumed to be of odd or even type, we may observe, without invoking CFSG, that:

- G_* is a Chevalley group over an algebraically closed locally finite field of positive characteristic.

Clearly, Theorem 6.3.1 follows now from the elementary embedding $G_* \preceq G$ together with the fact that being a Chevalley group over a field K is a first-order axiomatisable property. □

We conclude this section with the following two remarks.

Remark 6.3.2. *Theorem 6.3.1 can be also proven using purely group-theoretic methods via the following strategy. First, one can shown that G_* is a simple locally finite group of finite Morley rank, using the notion of the rank-function. Thus, G_* is a Chevalley group over an algebraically closed locally finite field. Then, it is possible to expand a* *specific configuration of the Curtis–Phan–Tits–Lyons Theorem [48, Theorem 1.24.2] present in G_*, from G_* to G, and finally apply the Curtis–Phan–Tits–Lyons Theorem to G. The model-theoretic approach we gave here is shorter and more transparent.*

Remark 6.3.3. *Since infinite simple locally finite groups of finite Morley rank satisfy the Cherlin–Zilber Conjecture, and the properties of being simple non-abelian, being of finite Morley rank, and being a Chevalley group over a field are all elementary properties in this context, one might hope that the Cherlin–Zilber Conjecture could be proven by simply 'transfering down' to a locally finite elementary substructure. Unfortunately, this in general seems hopeless. The author is aware of only two specific cases where such 'locally finite transfer' has been used:*

- *Theorem 6.3.1, and;*
- *A* bad field *is a field K of finite Morley rank with a distinguished predicate T for a proper infinite multiplicative subgroup. Note that the existence of bad fields in positive characteristic is an open problem. However, it is known that there exists a bad field of characteristic 0 [11]. In [89], Wagner proved that if there exists a bad field of characteristic $p > 0$, then there exists a locally finite one. Using Wagners result, Poizat proved his 'linear theorem' on groups definable in $GL_n(K)$ [71].*

Chapter 7

The Hrushovski Programme for infinite simple groups of finite Morley rank

In this chapter we explain in detail 'The Hrushovski Programme for infinite simple groups of finite Morley rank' developed by Uğurlu from results and observations of Hrushovski. Further, we partially prove one of the aims of this programme, namely, we prove Theorem 7.2.1.

In Section 7.1, we thoroughly introduce the Hrushovski Programme and results of Uğurlu suggesting that the Cherlin–Zilber Conjecture and the Principal Conjecture are equivalent. In particular, we give Uğurlu's definition of a tight automorphims α and explain what are the specific aims of the programme developed by Uğurlu.

Theorem 7.2.1 is proven in Section 7.2. That is, we prove that an infinite simple group of finite Morley rank G with a pure group structure and with $\mathrm{pr}_2(G) = 1$ which admits a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite is isomorphic to $PSL_2(K)$ for some algebraically closed field K of characteristic different from 2, if there is an involution i in a maximal split torus T of the pseudofinite Chevalley group $S \cong PSL_2(F) \leqslant C_G(\alpha)$. This is joint work between Uğurlu and the author.

In Section 7.3, we outline a strategy of the future work regarding the Hrushovski Programme.

7.1 Results of Uğurlu and a tight automorphism α

For readers convenience we start by recalling the Cherlin–Zilber Conjecture and the Principal Conjecture one more time:

Conjecture 1 (The Cherlin–Zilber Conjecture). *Infinite simple groups of finite Morley rank are algebraic groups over algebraically closed fields.*

Conjecture 2 (The Principal Conjecture). *Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Then the fixed point subgroup $C_G(\alpha)$ is pseudofinite.*

As explained in Section 5.2, it is known that the Cherlin–Zilber Conjecture implies the Principal Conjecture. In her thesis [86], following observations of Hrushovski in [50], Uğurlu suggested a strategy towards proving the expected equivalence between these two conjectures. We call this strategy 'The Hrushovski Programme for infinite simple groups of finite Morley rank'. The aim of this programme is to prove that the Principal Conjecture implies the Cherlin–Zilber Conjecure—that is, to prove that these two conjectures are equivalent.

Recall at this point that for any subset X of a group of finite Morley rank G, $\overline{X}$ denotes the definable closure of X, that is, $\overline{X}$ is the smallest definable subgroup of G containing X.

In order to work in a purely algebraic context (instead of working with a generic automorphism which has a model-theoretic definition, see Section 5.2), Uğurlu introduced in her PhD thesis the notion of a *tight* automorphism α of an infinite simple group of finite Morley rank G:

Definition 7.1.1 (Uğurlu [85]). *An automorphism α of an infinite simple group of finite Morley rank G is called* tight *if, for any connected definable and α-invariant subgroup H of G, $\overline{C_H(\alpha)} = H$.*

As explained by Uğurlu, an important example of a tight automorphism is provided by a non-standard Frobenius automorphism α of an algebraically closed field $K = \prod_{p_i \in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U}$. Moreover, such an α is an example of a tight automorphism whose fixed point subgroup is pseudofinite:

Example 7.1.2 (Uğurlu [86, page 61]). *Let G be a simple algebraic group of adjoint type defined over the prime subfield of the algebraically closed field $K = \prod_{p_i \in I} \overline{\mathbb{F}}_{p_i}/\mathcal{U}$, where I is the set of all prime numbers p_i and $\mathcal{U}$ is a non-principal ultrafilter on I. Let α be a non-standard Frobenius automorphism of K, that is,*

$$\alpha = \prod_{p_i \in I} \phi_{p_i}/\mathcal{U},$$

where ϕ_{p_i} is the standard Frobenius map $x \mapsto x^{p_i}$. Then α induces a tight automorphism on G. Moreover, the fixed point subgroup $C_G(\alpha)$ is pseudofinite. This can be observed as follows: We have $C_G(\alpha) = G(\mathrm{Fix}_K(\alpha))$, where $\mathrm{Fix}_K(\alpha) = \prod_{p_i \in I} \mathbb{F}_{p_i}/\mathcal{U}$. By Łós's Theorem (Fact 3.4.1),

$$C_G(\alpha) = G(\prod_{p_i \in I} \mathbb{F}_{p_i}/\mathcal{U}) \cong \prod_{p_i \in I} G(\mathbb{F}_{p_i})/\mathcal{U},$$

and so, $C_G(\alpha)$ is pseudofinite. Further, given a closed subgroup H of G (recall that in the context of algebraic groups definable subgroups and closed subgroups coincide), we have $C_H(\alpha) = H(k)$, where k is a pseudofinite field. Since H is connected, $C_H(\alpha)$ is Zariski dense in H (see [74]). Moreover, since the definable closure of $C_H(\alpha)$ is Zariski closed, we have $\overline{C_H(\alpha)} = H$.

Uğurlu proved that whenever an infinite simple group of finite Morley rank G admits a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite, then $C_G(\alpha)$ contains a normal definable (in $C_G(\alpha)$) pseudofinite Chevalley or twisted Chevalley subgroup S such that the definable closure $\overline{S}$ of S equals G:

Fact 7.1.3 (Uğurlu [86, Theorem 3.1]). *Let G be an infinite simple group of finite Morley rank and α be a tight automorphism of G. Assume that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. Then there is a definable (in $C_G(\alpha)$) normal subgroup S of $C_G(\alpha)$ such that*

$$S \trianglelefteq C_G(\alpha) \leqslant \mathrm{Aut}(S),$$

where S is isomorphic to a Chevalley or a twisted Chevalley group over a pseudofinite field.

Fact 7.1.4 (Uğurlu [86, Lemma 3.3]). *Let G be an infinite simple group of finite*

Morley rank, α be a tight automorphism of G whose fixed point subgroup $C_G(\alpha)$ is pseudofinite and N be a non-trivial normal subgroup of $C_G(\alpha)$. Then $\overline{N} = G$.

Uğurlu's results above allow us to explain the strategy of the the Hrushovski Programme as follows: Let G be an infinite simple group of finite Morley rank admitting a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite. Then, to prove the desired equivalence between the Cherlin–Zilber Conjecture and the Principal Conjecture, one would need to prove the following two steps.

(1.) *Identification step*: By results of Uğurlu, G contains a pseudofinite (possibly twisted) Chevalley group $S = X(F)$ such that $\overline{S} = G$. Prove that this forces G to be a Chevalley group $X(K)$ over an algebraically closed field K.

(2.) *Step from the algebraic context to the model-theoretic context*: Prove that a generic automorphism (in the model theory sense) of an infinite simple group of finite Morley rank G is tight.

One should note that it is not known whether an infinite simple group of finite Morley rank G admits a generic automorphism. Indeed, the proof that the Cherlin–Zilber Conjecture and the Principal Conjecture are equivalent would imply that it is 'as hard' to prove that an infinite simple group of finite Morley rank G admits a generic automorphism than it is to prove the the Cherlin–Zilber Conjecture. Moreover, it is not known whether a tight automorphism α of an infinite simple group of finite Morley rank G is generic.

Let us now briefly comment the identification step specified above.

Let G be an infinite simple group of finite Morley rank. It was observed by Uğurlu in [86, Remark 3.2] that if G is a group of degenerated type then G cannot admit a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite. This observation can be explained as follows. Assume towards a contradiction that G is a degenerate type infinite simple group of finite Morley rank which admits a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite. Then, by Fact 5.3.24, the fixed point subgroup $C_G(\alpha)$ is a group with no involutions, that is, a sentence $\sigma := \forall x(x^2 = 1 \longrightarrow x = 1)$ holds in $C_G(\alpha)$. Since $C_G(\alpha)$ is pseudofinite, it is an ultraproduct of finite groups G_i such that σ holds in G_i for almost all i. Therefore, $C_G(\alpha)$ is an ultraproduct of finite groups of odd orders—such groups are solvable by the famous Feit–Thompson Odd Order Theorem [44]. Then, by Fact 2.4.6, the fixed point subgroup $C_G(\alpha)$ and its definable closure $\overline{C_G(\alpha)} = G$ are solvable—this contradicts the simplicity of G.

Given above, when working with infinite simple groups of finite Morley rank admitting a tight automorphism α whose fixed point subgroup is pseudofinite, one only has to consider groups of odd type. Therefore, the identification step above can be naturally split into three cases depending on the $\mathrm{pr}_2(G)$:

Conjecture 7.1.5 (Intermediate Conjecture; Uğurlu [86])**.** *Let G be an infinite simple group of finite Morley rank admitting a tight automorphism α. Assume that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. Assume further that*

(i) $\mathrm{pr}_2(G) = 1$. *Then G is isomorphic to the Chevalley group $PSL_2(K)$, where K is an algebraically closed field of characteristic different from 2.*

(ii) $\mathrm{pr}_2(G) = 2$. *Then G is isomorphic to one of the Chevalley groups $PSL_3(K)$, $PSp_4(K)$ or $G_2(K)$ over an algebraically closed field of characteristic different from 2.*

(iii) $\mathrm{pr}_2(G) \geqslant 3$. *Then G is isomorphic to a Chevalley group over an algebraically closed field of characteristic different from 2.*

7.2 Small groups of finite Morley rank admitting α

Throughout this section G is an infinite simple group of finite Morley rank with a pure group structure admitting a tight automorphism α and with $\mathrm{pr}_2(G) = 1$. We assume that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. We partially solve the part (i) of Conjecture 7.1.5, namely, we prove that if a maximal split torus T of the pseudofinite Chevalley group $S \cong PSL_2(F) \leqslant C_G(\alpha)$ contains an involution i, then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2. That is, we prove the following theorem which is a result of joint work between Uğurlu and the author.

Theorem 7.2.1 (Karhumäki and Uğurlu)**.** *Let G be an infinite simple group of finite Morley rank with a pure group structure admitting a tight automorphism α. Assume* *that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. Assume further that* $\mathrm{pr}_2(G) = 1$ *and that a maximal split torus T of the pseudofinite Chevalley group $S \cong PSL_2(F)$ contains an involution i. Then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.*

7.2.1 Our strategy

We consider an infinite simple group of finite Morley rank G with a pure group structure admitting a tight automorphism α and with $\mathrm{pr}_2(G)$=1. We assume that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. By results of Uğurlu (Facts 7.1.3 and 7.1.4), the pseudofinite group of fixed points $C_G(\alpha)$ contains a definable (in $C_G(\alpha)$) normal subgroup S which is isomorphic to $X(F)$, where F is a pseudofinite field of characteristic different from 2 and X denotes the (possibly twisted) Lie type of S. Since $\mathrm{pr}_2(G) = 1$, we may assume that $S \cong PSL_2(F)$, where F is a pseudofinite field of characteristic different from 2, that is, $F \equiv \prod_{i\in\mathbb{N}} F_{p^i}/\mathcal{U}$ where $\mathcal{U}$ is an ultrafilter over natural numbers and p is an odd prime. We also know by results of Uğurlu that the fixed point subgroup $C_G(\alpha)$ embeds in $\mathrm{Aut}(S) \cong \mathrm{Aut}(PSL_2(F))$ and that $\overline{S} = G$.

Structure of $\mathrm{Aut}(S)$

The structure of $\mathrm{Aut}(S) \cong \mathrm{Aut}(PSL_2(F))$ is well-known as F is a perfect field (see Fact 2.3.5)—$\mathrm{Aut}(S)$ is a product of inner, diagonal, field and graph automorphisms:

$$\mathrm{Aut}(S) = \mathrm{Inn}(S)\mathrm{Diag}(S) \rtimes \mathrm{Aut}(F)\mathrm{Grp}(S).$$

Since $S \cong PSL_2(F)$, we have $\mathrm{Grp}(S) = \mathrm{Id}$, $\mathrm{Inn}(S) \cong S$ and $\mathrm{Inn}(S)\mathrm{Diag}(S) \cong PGL_2(F)$. Therefore,

$$\mathrm{Aut}(S) \cong PGL_2(F) \rtimes \mathrm{Aut}(F).$$

We also have $\mathrm{Aut}(S)/\mathrm{Inn}(S) \cong \mathrm{OutDiag}(S) \rtimes \mathrm{Aut}(F)$, and therefore, since $C_G(\alpha)$ embeds in $\mathrm{Aut}(S)$, $C_G(\alpha)/S$ is isomorphic to a subgroup of $\mathrm{OutDiag}(S) \rtimes \mathrm{Aut}(F)$.

Structure of $S \cong PSL_2(F)$

Trichotomy for elements The following *Trichotomy for elements* is well-known for finite groups $PSL_2(F_{p^j})$ and, as these properties can be written down in a first-order way, it also hold in our pseudofinite group $S \cong PSL_2(F)$:

(1.) An element $u \in S$ is called *unipotent* if and only if there exists elements $g, h \in S$ such that $u^h \neq u \neq u^g \neq u^h$ and $\langle u, u^g, u^h \rangle$ is commutative.

(2.) Non-unipotent elements are called *semisimple*.

(2.1) A semisimple element $d \in S$ is called *diagonal* or *split* if for some non-trivial unipotent element $u \in S$, $u \neq u^d$ and $[u, u^d] = 1$.

(2.2) A semisimple element that is not a split element is called a *non-split* element.

(3.) If $d \in S$ is a split element then $C_S(d)^{1/2}$ is a *split torus*. Similarly, if $s \in S$ is a non-split element then $C_S(s)^{1/2}$ is a *non-split torus*.

(4.) All split (resp. non-split) tori are conjugate.

(5.) Only one of the classes of tori contains involutions.

Action of S on $\mathbf{P}^1$ Since $S \cong PSL_2(F)$ we know its structure well. In particular, it is well known that (for any field F), $PSL_2(F)$ is a split Zassenhaus group with its action on the projective line $\mathbf{P}^1$. This action can be explained precisely once we make the following identifications.

- We identify $\mathbf{P}^1$ with $F \cup \{\infty\}$.
- $x \in F$ is denoted by $\left\langle \begin{pmatrix} x \\ 1 \end{pmatrix} \right\rangle$.
- ∞ is denoted by $\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$.
- $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in F^*, b \in F \right\} / \pm I$ (here $\pm I$ stands for the center $Z(SL_2(F))$).

- $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in F^* \right\} / \pm I.$
- $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F \right\} / \pm I.$

The action of $PSL_2(F)$ on $\mathbf{P}^1$ can be described by matrix multiplications and the following observations can be made.

- B stabilises ∞ and its action corresponds to the transformation $x \mapsto a^2x + ab$ for $a \in F^*$ and for $b \in F$. We say that B is a *Borel subgroup* of S.
- U fixes ∞ and its action corresponds to the transformation $x \mapsto x + b$ for $b \in F$. As a result, $U \cong F^+$. We say that U is a *unipotent subgroup* of S.
- T stabilises pointwise both 0 and ∞ and its action corresponds to the transformation $x \mapsto a^2x$ for $a \in F^*$. As a result, T is isomorphic to a multiplicative subgroup $(F^*)^2$ of F^* which consists of squares. One observers that T contains a unique involution i if and only if -1 is a square in F, and, in this case, i inverts U. *Throughout this section we assume that T contains the unique involution i.* The group T is called a maximal split torus of S.
- The maximal split torus T is invariant under the action of $\mathrm{Diag}(S) \rtimes \mathrm{Aut}(F)$.
- U has no proper non-trivial subgroups which are invariant under the action of T.
- Consider the involution $\omega_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in S$. One observes that ω_0 interchanges 0 and ∞ and hence normalises T. Moreover, $N_S(T) = \langle \omega_0, T \rangle$ and ω_0 inverts T. Therefore, as we assume that T contains the unique involution i, ω_0 centralises i and $C_T(\omega_0) = \langle i \rangle$.
- $B = U \rtimes T$ is a split Frobenius group.
- $S = \langle U, T, \omega_0 \rangle$.

Note that the structures of the Borel subgroup B, the unipotent subgroup U and the maximal split torus T of the pseudofinite group $S \cong PSL_2(F)$ are closely related to the structures of the corresponding subgroups of finite groups $PSL_2(F_{p^j})$. More precisely, $B \cong \prod_{j\in J} B_j/\mathcal{U}$, $U \cong \prod_{j\in J} U_j/\mathcal{U}$, and $T \cong \prod_{j\in J} T_j/\mathcal{U}$, where B_j, U_j and T_j denote the Borel subgroups, unipotent subgroups and maximal split tori of $PSL_2(F_{p^j})$ respectively and $B_j = U_j \rtimes T_j$.

Delahan–Nesin identification theorem

To prove Theorem 7.2.1, we invoke the already presented Delahan–Nesin identification result—we recall it here:

Fact 7.2.2 (Delahan and Nesin [35]). *Let G be an infinite split Zassenhaus group of finite Morley rank. If the stabiliser of two distinct points contains an involution, then $G \cong PSL_2(K)$ for some algebraically closed field K of characteristic different from 2.*

One observes that, to prove Theorem 7.2.1, it suffices to prove that the definable closures in G of subgroups of S described above behave "as one would expect". That is, one needs to prove that $\overline{S} = G$ is a split Zassenhaus group with a one-point stabiliser $\overline{B}$ and a two-point stabiliser $\overline{T}$.

7.2.2 Proof of Theorem 7.2.1

Our set-up

From now on, we let G be an infinite simple group of finite Morley rank with a pure group structure admitting a tight automorphism α and with $\mathrm{pr}_2(G)$=1. We assume that the fixed point subgroup $C_G(\alpha)$ is pseudofinite. Further, P stands for $C_G(\alpha)$ and $S \cong PSL_2(F)$, where F is a pseudofinite field of characteristic different from 2. Subgroups and elements B, U, T, ω_0 and i of S are as described in Subsection 7.2.1; note that we assume that T contains the unique involution i.

By the structure of S (see our observations in Subsection 7.2.1), we know the following things:

- The unique involution i of T inverts U.
- $N_S(T) = \langle \omega_0, T\rangle$, where ω_0 is an involution which inverts T.
- The unique involution i of T is centralised by ω_0 and $C_T(\omega_0) = \langle i\rangle$.

- $C_S(i) = N_S(T) = \langle \omega_0, T \rangle$ and $C_S(t) = T$ for all $t \in T^* \backslash \{i\}$.
- U contains no involutions.
- $C_S(u) = U$ for all $u \in U^*$.

We start by observing that the definable closure in G of any subgroup of P is stabilised by α—we use this observation repeatedly throughout the rest of this subsection.

Lemma 7.2.3. *The definable closure in G of any subgroup X of P is stabilised by α.*

Proof. Since G has a pure group structure, α is a group automorphism of G and thus it maps definable sets to definable sets in the language of groups. Therefore, for any subgroup X of P, $\alpha(\overline{X})$ is a definable subgroup of G containing $\alpha(X) = X$, and hence, it contains $\overline{X}$ as well. The same argument for α^{-1} gives $\alpha(\overline{X}) \leqslant \overline{X}$ and so $\alpha(\overline{X}) = \overline{X}$. □

Structures of $\overline{B}$, $\overline{U}$, $\overline{T}$ and P

In what follows, we study the structures of $\overline{B}$, $\overline{U}$, $\overline{T}$ and P. In particular, we prove the following things:

- $\overline{B}$ interprets an algebraically closed field K,
- P/S is finite, and,
- $\overline{B} \cap \overline{U}^g = 1$ for all $g \in G \setminus \overline{B}$.

By the properties of the definable closures of subgroups of G (see Fact 5.1.3), we may immediately observe the following things:

- $\overline{U}$ and $\overline{T}$ are abelian groups.
- $\overline{U} \lhd \overline{B}$.
- $\overline{B}$ is a solvable group of solvability class 2.
- $\overline{B}' = \overline{U}$.
- $\overline{B} = \overline{U}\,\overline{T}$.

We start by proving that ω_0 inverts $\overline{T}$ and that $C_{\overline{T}}(\overline{U}) = 1$.

Lemma 7.2.4. *ω_0 inverts $\overline{T}$ and $C_{\overline{T}}(\overline{U}) = 1$.*

Proof. We first observe that ω_0 inverts $\overline{T}$. Let $X = \{x \in \overline{T} : \omega_0 x \omega_0 = x^{-1}\}$. We have, $T \subseteq X \subseteq \overline{T}$. Clearly, X is definable in G and one may easily observe that X is a subgroup of G. Since X contains T, we have $X = \overline{T}$ by the definition of the definable closure.

It is clear that $C_{\overline{T}}(\overline{U})$ is T and U invariant. Moreover, as ω_0 inverts $\overline{T}$ it leaves invariant $C_{\overline{T}}(\overline{U})$. Therefore, $\langle U, T, \omega_0 \rangle = S$ normalises the definable group $C_{\overline{T}}(\overline{U})$. But then $\overline{S} = G$ normalises $C_{\overline{T}}(\overline{U})$. This is possible if and only if $C_{\overline{T}}(\overline{U}) = 1$ as G is simple. □

At this point we know that $\overline{T} \cap \overline{U} = 1$ which implies that $\overline{B} = \overline{U} \rtimes \overline{T}$. We continue to prove that $\overline{U}$ is connected.

Lemma 7.2.5. $\overline{U} = \overline{U}^{\circ}$.

Proof. Let us consider $X = C_{\overline{U}^{\circ}}(\alpha)$ which has finite index in $Y = C_{\overline{U}}(\alpha)$. It can be easily observed that X is T-normal and therefore $X \cap U$ is T-normal as well. By the minimality of U under the action of T, either $X \cap U = 1$ or $X \cap U = U$. The former is not possible since in this case $U \cong UX/X \hookrightarrow Y/X < \infty$ but U is infinite as it is isomorphic to F^+, the additive group of the pseudofinite field F. Therefore, $X \cap U = U$, that is, $U \leqslant X$ and hence $\overline{U} \leqslant \overline{X}$. At the same time we know that $\overline{X} = \overline{U}^{\circ}$ by the definition of a tight automorphism α. It follows that $\overline{U} = \overline{U}^{\circ}$. □

Next we prove that i inverts $\overline{U}$ and $\overline{U}$ has no involutions.

Lemma 7.2.6. *The involution i inverts $\overline{U}$ and $\overline{U}$ has no involutions. In particular, $C_{\overline{U}}(\overline{T}) = C_{\overline{U}}(T) = 1$ and $C_{\overline{B}}(i) = \overline{T}$.*

Proof. By similar arguments as used in Lemma 7.2.4, it is clear that i inverts $\overline{U}$; $X = \{u \in \overline{U} : iui = u^{-1}\}$ is a definable subgroup of G containing U, that is, $X = \overline{U}$.

The rest of the claim can be observed as follows. If the connected group $\overline{B}^{\circ} = \overline{U} \rtimes \overline{T}^{\circ}$ is of degenerated type, then $\overline{B}^{\circ}$ contains no involutions by Fact 5.3.24. Therefore, we may assume that $\overline{B}^{\circ}$ is of odd type. Since $\overline{B}^{\circ}$ is connected, its Sylow 2-subgroups are connected by Fact 5.3.18. Therefore, a Sylow 2-subgroup of $\overline{B}^{\circ}$ contains a unique involution. One may now easily observe that if there is an involution in $\overline{U}$, then it is a unique involution, say j. But, since $\overline{U}$ is α-invariant, $\alpha(j) = j$, that is, $j \in U$—a contradiction. We haven proven that $\overline{U}$ has no involutions. Therefore, $C_{\overline{U}}(i) = 1$ which gives us $C_{\overline{U}}(\overline{T}) = 1$ and $C_{\overline{B}}(i) = \overline{T}$. Moreover, $C_{\overline{U}}(\overline{T}) = C_{\overline{U}}(T)$ by the properties of the definable closures. □

We have now enough information to observe that $C_{\overline{U}}(\alpha) = U$.

Lemma 7.2.7. $C_{\overline{U}}(\alpha) = U$.

Proof. Recall that P/S is isomorphic to a subgroup of $\mathrm{OutDiag}(S) \rtimes \mathrm{Aut}(F)$. Note also that $C_{\overline{U}}(\alpha) \leqslant C_P(U)$ as $\overline{U}$ is abelian.

Let us assume towards a contradiction that $C_{\overline{U}}(\alpha) > U$. Then, there exists a non-trivial element $d \in P$ which centralises U such that $d \in \mathrm{OutDiag}(S) \rtimes \mathrm{Aut}(F) \cap C_{\overline{U}}(\alpha)$. At the same time, as all elements of $\mathrm{OutDiag}(S) \rtimes \mathrm{Aut}(F)$ leave the maximal split torus T invariant, d fixes the unique involution i of T. Therefore $d \in C_P(i)$, that is, $d \in C_{\overline{U}}(i)$. However, by Lemma 7.2.6, $C_{\overline{U}}(i) = 1$—we have derived a contradiction. □

At this point we know that $\overline{B}$ is a centerless group since $C_{\overline{T}}(\overline{U}) = C_{\overline{U}}(\overline{T}) = 1$. We move on to prove that $\overline{B}^{\circ}$ is centerless as well.

Lemma 7.2.8. $C_{\overline{U}}(\overline{T}^{\circ}) = 1$.

Proof. Let $t \in T^* \cap \overline{T}^{\circ}$. We know that $C_{\overline{U}}(\alpha) = U$ by Lemma 7.2.7. Therefore,

$$C_{C^{\circ}_{\overline{U}}(t)}(\alpha) \leqslant_{f.i} C_{C_{\overline{U}}(t)}(\alpha) = C_{C_{\overline{U}}(\alpha)}(t) = C_U(t) = 1,$$

where $f.i$ stands for finite index. As a result, $\overline{C_{C^{\circ}_{\overline{U}}(t)}(\alpha)} = C^{\circ}_{\overline{U}}(t) = 1$. Therefore, $C_{\overline{U}}(\overline{T}^{\circ})$ is finite.

As $\overline{B}^{\circ} = \overline{U} \rtimes \overline{T}^{\circ}$, we have $N_{\overline{U}}(\overline{T}^{\circ}) = C_{\overline{U}}(\overline{T}^{\circ})$. Therefore, we have:

$$N_{\overline{B}}(\overline{T}^{\circ}) = (\overline{U} \cap N_{\overline{B}^{\circ}}(\overline{T}^{\circ})) \rtimes \overline{T}^{\circ} = N_{\overline{U}}(\overline{T}^{\circ}) \rtimes \overline{T}^{\circ} = C_{\overline{U}}(\overline{T}^{\circ}) \rtimes \overline{T}^{\circ}.$$

As $C_{\overline{U}}(\overline{T}^{\circ})$ is finite, we get $[N_{\overline{B}^{\circ}}(\overline{T}^{\circ}) : \overline{T}^{\circ}] < \infty$. Now, since $\overline{B}^{\circ}$ and $\overline{T}^{\circ}$ are connected, Fact 5.3.10 immediately gives us $N_{\overline{B}^{\circ}}(\overline{T}^{\circ}) = C_{\overline{B}^{\circ}}(\overline{T}^{\circ}) = \overline{T}^{\circ}$, that is, $C_{\overline{U}}(\overline{T}^{\circ}) = 1$. □

We have now enough information to prove that $\overline{U}$ is a direct sum of finitely many $\overline{B}$-minimal (and $\overline{B}^{\circ}$-minimal) subgroups.

Recall that the socle $S(\overline{B})$ of a group of finite Morley rank $\overline{B}$ is the subgroup generated by all minimal (finite or $\overline{B}$-minimal) normal subgroups of $\overline{B}$ and that $S(\overline{B})_{\circ}$ is the subgroup generated by all $\overline{B}$-minimal subgroups of $\overline{B}$.

Lemma 7.2.9. $\overline{U} = S(\overline{B}) = S(\overline{B}^{\circ})$. *Therefore,* $\overline{U} = \bigoplus_{j=1}^{m} X_j$, *where each* X_j *is* $\overline{B}$*-minimal and* $\overline{B}^{\circ}$*-minimal. In particular, each* X_j *is* $\overline{T}$*-minimal and* $\overline{T}^{\circ}$*-minimal.*

Proof. The group $\overline{B}$ can have two kinds of minimal normal subgroups; infinite ones that are $\overline{B}$-minimal and finite ones. Infinite ones being $\overline{B}$-minimal are connected and thus live in $\overline{B}^{\circ}$. Let X be a finite normal subgroup of $\overline{B}$. Then $X \leqslant C_{\overline{B}}(\overline{B}^{\circ})$ and since $Z(\overline{B}^{\circ}) = 1$ we get $X \cap \overline{U} = 1$. At the same time, as $X \leqslant C_{\overline{B}}(\overline{B}^{\circ})$, we have $X \leqslant C_{\overline{B}}(\overline{U}) = \overline{U}$. Therefore, $X = 1$, that is, $\overline{B}$ has no non-trivial finite normal subgroups, which implies that $S(\overline{B})_{\circ} = S(\overline{B})$.

Since $S(\overline{B})$ is a characteristic subgroup of $\overline{B}$, it is α-invariant. Further, since $S(\overline{B}) = S(\overline{B})_{\circ}$, the socle $S(\overline{B})$ is definable and connected by Zilber's Indecomposability Theorem (Fact 5.3.15). Clearly, $S(\overline{B}) \leqslant \overline{U}$. We have observed that $S(\overline{B})$ is an infinite α-invariant connected and definable subgroup of $\overline{U}$. Moreover, as $S(\overline{B})$ is $\overline{B}$-normal, and therefore $\overline{T}$-normal, one may easily observe that $C_{S(\overline{B})}(\alpha)$ is T-normal. We have, $C_{S(\overline{B})}(\alpha) \leqslant C_{\overline{U}}(\alpha) = U$, where $C_{S(\overline{B})}(\alpha)$ is T-normal. Now, by the minimality of U under the action of T, $C_{S(\overline{B})}(\alpha) = U$. Therefore, by the definition of a tight automorphism α, $S(\overline{B}) = \overline{C_{S(\overline{B})}(\alpha)} = \overline{U}$.

By Fact 5.3.13, $S(\overline{B}^{\circ}) \leqslant \overline{U}$ is a direct sum of m many $\overline{B}^{\circ}$-minimal subgroups X_j. We also know, by Fact 5.3.14, that $S(\overline{B}) \leqslant S(\overline{B}^{\circ})$. Therefore, $S(\overline{B}) = S(\overline{B}^{\circ}) = \overline{U}$. We have observed that $\overline{U} = S(\overline{B}) = S(\overline{B}^{\circ}) = \bigoplus_{j=1}^{m} X_j$, where each X_j is $\overline{B}$-minimal and $\overline{B}^{\circ}$-minimal. Further, it is easy to observe that the $\overline{B}$-minimal (resp. $\overline{B}^{\circ}$-minimal) subgroups of $\overline{U}$ are exactly the $\overline{T}$-minimal (resp. $\overline{T}^{\circ}$-minimal) subgroups of $\overline{U}$. □

At this point Zilber's Field Theorem (Fact 5.3.17) immediately gives us that $\overline{U} \cong \bigoplus_{j=1}^{m} K_j^{+}$ and $\overline{T}^{\circ} \leqslant \prod_{j=1}^{m} K^{*}$.

Lemma 7.2.10. $\overline{B}^{\circ} = \overline{U} \rtimes \overline{T}^{\circ} \cong \bigoplus_{j=1}^{m} K_j^{+} \rtimes \prod_{j=1}^{m} H_j$, *where each* K_j *is an algebraically closed field and* $H_j \leqslant K_j^{*}$. *Therefore,* $\overline{T}^{\circ}$ *is a good torus.*

Proof. We know, by Lemma 7.2.9, that $\overline{U} = \bigoplus_{j=1}^{m} X_j$, where each X_j is $\overline{T}^{\circ}$-minimal. Therefore, by Zilber's Field Theorem, we have:

$$X_j \rtimes \overline{T}^{\circ}/C_{\overline{T}^{\circ}}(X_j) \cong K_j^{+} \rtimes H_j,$$

where $H_j \leqslant K_j^{*}$ for some algebraically closed field K_j. It follows that

$$\overline{U} \cong \bigoplus_{j=1}^{m} K_j^{+}.$$

Let us then consider the natural map

$$\varphi : \overline{T}^{\circ} \longmapsto \prod_{j=1}^{m} \overline{T}^{\circ}/C_{\overline{T}^{\circ}}(X_j) \cong \prod_{j=1}^{m} H_j.$$

Kernel of this map is $\bigcap_{j=1}^{m} C_{\overline{T}^{\circ}}(X_j) = C_{\overline{T}^{\circ}}(\overline{U}) = 1$. Therefore, $\overline{T}^{\circ} \cong \prod_{j=1}^{m} H_j$. Since each H_j is a definable connected subgroup of a good torus K_j^{*} (Fact 5.3.5), $\overline{T}^{\circ}$ is a finite product of good tori and thus a good torus itself (Fact 5.3.6). □

Next we observe that $[C_P(T) : T] \leqslant 2$, which later on allows us to prove that P/S is finite.

Lemma 7.2.11. $[C_P(T) : T] \leqslant 2$. *In particular,* $[C_{\overline{T}}(\alpha) : T] \leqslant 2$. *Further,* $\overline{T}^{\circ} = C_G^{\circ}(\overline{T}) = N_G^{\circ}(\overline{T})$, *that is,* $\overline{T}^{\circ}$ *is a maximal good torus of* G.

Proof. We start by proving that $[C_P(T) : T] \leqslant 2$. Note that, since $\overline{T}$ is an abelian group, we have

$$T \leqslant C_{\overline{T}}(\alpha) \leqslant C_P(T).$$

Therefore, $[C_P(T) : T] \leqslant 2$ immediately implies that $[C_{\overline{T}}(\alpha) : T] \leqslant 2$. Let $x \in C_P(T)$. One observes that $x = yf$ where $y \in PGL_2(F)$ and $f \in \mathrm{Aut}(F)$ since P embeds in $\mathrm{Aut}(S) \cong PGL_2(F) \rtimes \mathrm{Aut}(F)$. Since x fixes T pointwise and the field automorphism f leaves $T = (F^{*})^2$ invariant, T is left invariant by y as well. Now, restricting x, y and f to T we get:

$$x|_T = y|_T f|_T,$$

where $x|_T = \mathrm{Id}$. So, we get $f^{-1}|_T = y|_T$. Since T is invariant under the action of y, the unique involution i of T is fixed by y. Therefore, $y \in C_{PGL_2(F^{alg})}(i)$, where F^{alg}

denotes the algebraic closure of F. One may now observe that y induces an algebraic automorphism of a maximal torus T_1 of $PGL_2(F^{alg})$ containing T—such automorphism acts on T_1 trivially or by inversion. Therefore, $y|_T = \pm\mathrm{Id}$, that is, $f^{-1}|_T = y|_T = \pm\mathrm{Id}$. However, as the field automorphism f can not act as $-\mathrm{Id}$ we get $f|_T = \mathrm{Id}$. One easily observes that if $f|_T = \mathrm{Id}$ then $f = \mathrm{Id}$. We have proven that $x \in PGL_2(F)$. Therefore, $C_P(T) \leqslant PGL_2(F)$. We know that $[PGL_2(F) : PSL_2(F)] = 2$ and hence

$$[PGL_2(F) \cap C_P(T) : PSL_2(F) \cap C_P(T)] \leqslant 2.$$

We have proven that $[C_P(T) : T] \leqslant 2$ since $C_S(T) = C_{PSL_2(F)}(T) = T$.

We move on to prove that $\overline{T}^\circ = C_G^\circ(\overline{T}) = N_G^\circ(\overline{T})$. By Lemma 7.2.10 and Fact 5.3.8, it is enough to prove that $\overline{T}^\circ = C_G^\circ(\overline{T})$. This equivalence can be observed as follows. Clearly we have:

$$C_{\overline{T}^\circ}(\alpha) \leqslant C_{C_G^\circ(\overline{T})}(\alpha) \leqslant C_P(T).$$

We also know that $\overline{T}^\circ = \overline{C_P(T)}^\circ$ as $[C_P(T) : T] \leqslant 2$. Therefore, by passing to definable closures and taking connected components, we get

$$\overline{T}^\circ \leqslant C_G^\circ(\overline{T}) = \overline{C_P(T)}^\circ = \overline{T}^\circ,$$

which proves the claim. □

We have now enough information to prove that P is a finite extension of S.

Theorem 7.2.12. *P/S is finite.*

Proof. We start by observing that $P = SN_P(T)$. Note that we identify here the elements of P with the corresponding automorphisms of S. Let $x \in P$. Then $x = sdf$ where $s \in \mathrm{Inn}(S) \cong S$, $d \in \mathrm{Diag}(S)$ and $f \in \mathrm{Aut}(F)$. We know that the maximal split torus T is invariant under the action of the group $\mathrm{Diag}(S) \rtimes \mathrm{Aut}(F)$. Therefore, $xs^{-1} = df \in N_P(T)$, that is, $x \in SN_P(T)$.

We have

$$P/S = SN_P(T)/S \cong N_P(T)/N_P(T) \cap S = N_P(T)/N_S(T).$$

Moreover, we know that $[N_S(T) : T] = 2$. Therefore, in order to show that P/S is finite it is enough to prove that $N_P(T)/T$ is finite.

We first observe that if $N_P(T)/T$ is infinite then also $\overline{N_P(T)}/\overline{T}$ is infinite: Towards a contradiction, assume that $N_P(T)/T$ is infinite but $\overline{N_P(T)}/\overline{T}$ is finite. We have

$$T \leqslant C_{\overline{T}}(\alpha) \leqslant C_P(T) \leqslant N_P(T) \leqslant C_{\overline{N_P(T)}}(\alpha).$$

By Lemma 7.2.11, we know that $[C_{\overline{T}}(\alpha) : T] < \infty$. Moreover, since $\overline{N_P(T)}/\overline{T}$ is assumed to be finite, we have $[C_{\overline{N_P(T)}}(\alpha) : C_{\overline{T}}(\alpha)] < \infty$. Therefore, $[C_{\overline{N_P(T)}}(\alpha) : T] < \infty$ which implies that $N_P(T)/T$ is finite—a contradiction.

Towards the final contradiction, assume that $N_P(T)/T$ is infinite. Then $\overline{N_P(T)}/\overline{T}$ is also infinite. But, if $\overline{N_P(T)}/\overline{T}$ is infinite then $N_G(\overline{T})/\overline{T}$ is infinite since $\overline{N_P(T)} \leqslant N_G(\overline{T})$. We have derived a contradiction since, by Lemma 7.2.11, $N_G^\circ(\overline{T}) = \overline{T}^\circ$. □

Theorem 7.2.12 gives us many useful corollaries:

Corollary 7.2.13. *$C_G^\circ(u) = \overline{U}$ for all $u \in U^*$ and $C_G^\circ(t) = \overline{T}^\circ$ for all $t \in T^*$.*

Proof. Let $u \in U^*$. Then,

$$C_{C_G^\circ(u)}(\alpha) \leqslant_{f.i} C_{C_G(u)}(\alpha) = C_{C_G(\alpha)}(u) \geqslant_{f.i} C_S(u) = U,$$

where $f.i$ stands for finite index. Passing to definable closures gives us $C_G^\circ(u) = \overline{U}$. Similar argument shows that $C_G^\circ(t) = \overline{T}^\circ$ for all $t \in T^*$. □

Corollary 7.2.14. $\overline{T} = \overline{T}^\circ = C_G(\overline{T})$. *Therefore, $\overline{B} = \overline{B}^\circ$. In particular, $\overline{T}$ is a maximal good torus of G and so any maximal good torus of G is of the form $\overline{T}^g$ for some $g \in G$.*

Proof. By Fact 5.3.7 and Lemma 7.2.10, we have $C_G(\overline{T}^\circ) = C_G^\circ(\overline{T}^\circ)$. Let $t \in T^* \cap \overline{T}^\circ$. We know that $\overline{T} \leqslant C_G(\overline{T}^\circ)$ and $C_G(\overline{T}^\circ) \leqslant C_G(t)$. At the same time, by Corollary 7.2.13, $\overline{T}^\circ = C_G^\circ(t)$. Therefore, $\overline{T}$ is a finite index subgroup of $C_G(\overline{T}^\circ)$ and thus $\overline{T} = C_G(\overline{T}^\circ)$ by the connectness of $C_G(\overline{T}^\circ)$. Clearly, $\overline{T}^\circ = \overline{T} = C_G(\overline{T}^\circ) = C_G(\overline{T})$.

The rest is clear; $\overline{T}$ is a maximal good torus by Lemma 7.2.11 and any two maximal good tori of G are conjugate by Fact 5.3.9. □

Corollary 7.2.15. *i is the unique involution of $\overline{T}$.*

Proof. Since $\overline{T}$ is a maximal (good) torus of G we have $S_{\overline{T}} = S_G^\circ$, where $S_{\overline{T}}$ and S_G° are the Sylow 2-subgroup of $\overline{T}$ containing i and the connected component of the Sylow 2-subgroup of G containing i, respectively. Therefore, $\overline{T}$ is a group of finite Morley rank of odd type and with $\mathrm{pr}_2(\overline{T}) = 1$. Now Fact 5.3.19 immediately implies that i is the unique involution of $\overline{T}$. □

We remind the reader that the Fitting subgroup $F(\overline{B})$ of $\overline{B}$ is the subgroup generated by all normal nilpotent subgroups of $\overline{B}$. We move on to prove that $\overline{U} = F^\circ(\overline{B})$.

Lemma 7.2.16. $\overline{U} = F^\circ(\overline{B})$.

Proof. Obviously, $\overline{U} \leqslant F^\circ(\overline{B})$. It follows that $F^\circ(\overline{B}) = \overline{U} \rtimes (\overline{T} \cap F^\circ(\overline{B}))$. Since $F^\circ(\overline{B})$ is a nilpotent characteristic subgroup of $\overline{B}$ (see Subsection 5.3.4), $H = (Z(F^\circ(\overline{B})) \cap \overline{U})^\circ$ is an infinite (see Fact 5.3.11) definable connected $\overline{T}$-normal and α-invariant subgroup of $\overline{U}$. Therefore, $C_H(\alpha) \leqslant C_{\overline{U}}(\alpha) = U$, where $C_H(\alpha)$ is a T-normal subgroup of U. By minimality of U under the action of T, we get $H = \overline{U}$. Hence, $\overline{U} \leqslant Z(F^\circ(\overline{B}))$ and therefore $\overline{T} \cap F^\circ(\overline{B}) \leqslant C_{\overline{T}}(\overline{U}) = 1$. □

Recall at this point that a subgroup M of a group of finite Morley rank H is a strongly embedded subgroup of H if M contains an involution and $M \cap M^h$ contains no involutions for any $h \in H \setminus M$. Using this notion, we move on to prove that $\overline{B}$ is a split Frobenius groups with a Frobelius complement $\overline{T}$ and a Frobenius kernel $\overline{U}$.

Lemma 7.2.17. $\overline{B} = \overline{U} \rtimes \overline{T}$ *is a split Frobenius group with a Frobenius complement* $\overline{T}$ *and a Frobenius kernel* $\overline{U}$. *In particular,* $C_{\overline{B}}(u) = \overline{U}$ *for all* $u \in \overline{U}^*$ *and* $C_{\overline{B}}(t) = \overline{T}$ *for all* $t \in \overline{T}^*$. *Further,* $\{\overline{T}^b : b \in \overline{B} \setminus \overline{T}\} = \{\overline{T}^u : u \in \overline{U}^*\}$.

Proof. To prove that $\overline{B}$ is a split Frobenius group, we need to prove that $\overline{T} \cap \overline{T}^b = 1$ for all $b \in \overline{B} \setminus \overline{T}$. We start by proving the further part of the claim, that is, by proving that it is enough to prove that $\overline{T} \cap \overline{T}^u = 1$ for all $u \in \overline{U}^*$. By Fact 5.3.22 and Lemma 7.2.16, the set of all involutions of $\overline{B}$ is $i^{\overline{U}}$. Consider the conjugate $\overline{T}^b$ of $\overline{T}$ for some $b \in \overline{B} \setminus \overline{T}$. By Corollary 7.2.15, $\overline{T}^b$ contains the unique involution i^b, which must be of the form $i^b = i^u$ for some $u \in \overline{U}^*$. By Lemma 7.2.6, $C_{\overline{B}}(i) = \overline{T}$. Therefore we have,

$$\overline{T}^b = C_{\overline{B}}(i)^b = C_{\overline{B}}(i^b) = C_{\overline{B}}(i^u) = C_{\overline{B}}(i)^u = \overline{T}^u,$$

that is, $\{\overline{T}^b : b \in \overline{B} \setminus \overline{T}\} = \{\overline{T}^u : u \in \overline{U}^*\}$.

We next observe that $\overline{T}$ is a strongly embedded subgroup of $\overline{B}$. Since $\overline{T}$ contains the unique involution i, it is enough to observe that $i \notin \overline{T} \cap \overline{T}^u$ for any $u \in \overline{U}^*$. Let $u \in \overline{U}^*$ and assume that $i \in \overline{T} \cap \overline{T}^u$. Then $i = i^u$ and so $u \in C_{\overline{B}}(i) = \overline{T}$—a contradiction.

Towards the final contradiction, assume that there exists a non-trivial element $t \in \overline{T} \cap \overline{T}^u$ for some $u \in \overline{U}^*$. Clearly, $\overline{T} \leqslant C_{\overline{B}}(t)$ and therefore $C_{\overline{B}}(t)$ is a strongly embedded subgroup of $\overline{B}$ by Fact 5.3.25 ($C_{\overline{B}}(t) \neq \overline{B}$ as $\overline{B}$ is centerless). Note also that $\overline{T}^u \leqslant C_{\overline{B}}(t)$, that is, $i^u \in C_{\overline{B}}(t)$. Since $C_{\overline{B}}(t)$ is strongly embedded in $\overline{B}$, the intersection $R = C_{\overline{B}}(t) \cap C_{\overline{B}}(t)^b$ contains no involutions for any $b \in \overline{B} \setminus C_{\overline{B}}(t)$. We may observe that $\overline{U} \cap (\overline{B} \setminus C_{\overline{B}}(t)) \neq \emptyset$ (otherwise $C_{\overline{U}}(t) = \overline{U}$ which would imply that $C_{\overline{T}}(\overline{U}) \neq 1$). Choose an element $b \in \overline{B} \setminus C_{\overline{B}}(t)$ in such way that $b \in \overline{U} \cap (\overline{B} \setminus C_{\overline{B}}(t))$. Then R contains the involution $i^u = i^{u^b}$—a contradiction.

The in particular part follows immediately from Fact 5.3.26. □

Recall that $C^*_G(x) = \{g \in G : x^g = x \text{ or } x^g = x^{-1}\}$ is a generalised centraliser of an element $x \in G$ and that $[C^*_G(x) : C_G(x)] = 1$ or 2. We have now enough information to prove that $(C^*_G(u))^\circ = C^\circ_G(u) = \overline{U}$ for all $u \in \overline{U}^*$.

Lemma 7.2.18. *$C^\circ_G(u) = \overline{U}$ for all $u \in \overline{U}^*$.*

Proof. Let $u \in \overline{U}^*$ and consider $C^*_G(u)$. By Lemma 7.2.6, i inverts $\overline{U}$, that is, $i \in C^*_G(u)$. Since $i \in T$, we know that $C^\circ_G(i) = \overline{T}$ by Corollary 7.2.13. Further, by Lemma 7.2.17, $C_{\overline{T}}(u) = 1$. Therefore, $C_G(i) \cap C^*_G(u)$ is finite. We have observed that $C_{C^*_G(u)}(i)$ is finite. Now Fact 5.3.20 implies that $(C^*_G(u))^\circ = C^\circ_G(u)$ is abelian. Let then $u_1 \in U^*$. By Corollary 7.2.13, $C^\circ_G(u_1) = \overline{U}$. Since $C^\circ_G(u)$ contains $C^\circ_G(u_1)$ and $C^\circ_G(u)$ is abelian, we have $C^\circ_G(u) = C^\circ_G(u_1) = \overline{U}$. □

We move on to study the structures of normalisers of $\overline{T}$, $\overline{U}$ and $\overline{B}$ in G. We start by proving that $N_G(\overline{T}) = C_G(i) = \langle \overline{T}, \omega_0 \rangle$.

Lemma 7.2.19. *$N_G(\overline{T}) = C_G(i) = \langle \overline{T}, \omega_0 \rangle$.*

Proof. By Fact 5.3.19, we know that the Sylow 2-subgroup S_G of G containing i is of the form $S^\circ_G \rtimes \langle \omega \rangle \cong Z(2^\infty) \rtimes \langle \omega \rangle$, where ω is either an involution or element of order 4 which acts on S°_G by inversion or $\omega = 1$. We also know that $S^\circ_G \leqslant C^\circ_G(i) = \overline{T}$ as $\overline{T}$ is a maximal (good) torus of G containing i. Therefore, we have $S^\circ_{C_G(i)} = S^\circ_G$, where

$S^\circ_{C_G(i)}$ denotes the connected component of the Sylow 2-subgroup of $C_G(i)$ containing i. We also know that $\omega_0 \in C_G(i)$ but $\omega_0 \notin C^\circ_G(i) = \overline{T}$ and that Sylow 2-subgroups of $C_G(i)$ are conjugate (Fact 5.3.23). These observations ensure that ω_0 lives outside of the connected component of any Sylow 2-subgroup of $C_G(i)$—note that such "outer involutions" are all conjugate. Therefore, $S_{C_G(i)} = S^\circ_{C_G(i)} \rtimes \langle\omega\rangle$, where ω is an involution which acts on S°_G by inversion and which is $C_G(i)$-conjugate to ω_0.

By Fact 5.3.21, $C_G(i)/C^\circ_G(i)$ has exponent 2. Therefore, $C_G(i)/C^\circ_G(i)$ is an elementary abelian 2-group. It follows that

$$C_G(i) = \langle C^\circ_G(i), \omega_0, \omega_1, \ldots, \omega_n\rangle = \langle \overline{T}, \omega_0, \omega_1, \ldots, \omega_n\rangle,$$

where each ω_i is an involution which is $C_G(i)$-conjugate to ω_0. Moreover, we have

$$\overline{T}^{\omega_i} = C^\circ_G(i)^{\omega_i} = C^\circ_G(i^{\omega_i}) = C^\circ_G(i) = \overline{T},$$

that is, each ω_i normalises $\overline{T}$. Since ω_i's are $C_G(i)$-conjugate to ω_0, one may observe that each ω_i inverts $\overline{T}$.

Recall that, by Corollary 7.2.15, i is the unique involution of $\overline{T}$ and that, by Lemma 7.2.4, ω_0 inverts $\overline{T}$. Let $t_1 \in \overline{T}^* \setminus \{i\}$. We know that $[C^*_G(t_1) : C_G(t_1)] = 2$ as $\omega_0 \in C^*_G(t_1)$ but $\omega_0 \notin C_G(t_1)$ (note also that $\omega_0 \in C^*_G(i) = C_G(i)$). Let then $t \in \overline{T}$ and conside $C^*_G(\overline{T}) = \bigcap_{t\in\overline{T}} C^*_G(t)$. We have $[C^*_G(\overline{T}) : C_G(\overline{T})] = 2$ and $\omega_0 \in C^*_G(\overline{T})$. Recall that, by Corollary 7.2.14, $C_G(\overline{T}) = \overline{T}$. We have observed that $C^*_G(\overline{T}) = \langle C_G(\overline{T}), \omega_0\rangle = \langle \overline{T}, \omega_0\rangle$. But $\omega_i \in C^*_G(\overline{T})$ for all i, that is, $\omega_i \in \langle \overline{T}, \omega_0\rangle$. We have proven that $C_G(i) = \langle \overline{T}, \omega_0\rangle$.

We move on to prove that $N_G(\overline{T}) \leqslant C_G(i) = \langle \overline{T}, \omega_0\rangle$, that is, $N_G(\overline{T}) = C_G(i)$. Let $n \in N_G(\overline{T})$. Then $\overline{T}^n = \overline{T}$ and so $i^n = t$ for some $t \in \overline{T}^*$. Now i^n is an involution in $\overline{T}$. But i is the unique involution of $\overline{T}$, that is, $i^n = i$ which proves that $n \in C_G(i)$. □

Next we prove that $N_G(\overline{U}) = \overline{B}$. To do so, we prove that, like $\overline{B}$, $N_G(\overline{U})$ is a split Frobenius groups with a Frobelius complement $\overline{T}$ and a Frobenius kernel $\overline{U}$.

Recall that the phrase "$H < K$ is a Frobenius group" will mean that K is a Frobenius group with a Frobenius complement H.

Lemma 7.2.20. *$N_G(\overline{U}) = \overline{B}$.*

Proof. Clearly, $\overline{B} \leqslant N_G(\overline{U})$ by the properties of the definable closure.

We start by proving that $N_G^\circ(\overline{U}) \leqslant \overline{B}$, which implies that $N_G^\circ(\overline{U}) = \overline{B}$ as $\overline{B}$ is connected.

Since $N_G^\circ(\overline{U})$ normalises $\overline{U}$, $X = C_{N_G^\circ(\overline{U})}(\alpha)$ normalises $C_{\overline{U}}(\alpha) = U$. Therefore, $X \leqslant N_P(U)$. By Theorem 7.2.12, we have $[N_P(U) : N_S(U)] < \infty$. Therefore, since $N_S(U) = B$, the normaliser $N_P(U)$ is a finite extension of B. Now passing to definable closures gives us $\overline{X} = N_G^\circ(\overline{U}) \leqslant \overline{B}$.

We move on to prove that $\overline{B} = N_G^\circ(\overline{U}) = N_G(\overline{U})$. We start by observing that $\overline{T} < N_G(\overline{U})$ is a split Frobenius group. Since, by Lemma 7.2.17, $C_{\overline{B}}(t) = \overline{T}$ for all $t \in \overline{T}^*$ and $[N_G(\overline{U}) : \overline{B}] < \infty$ we have $C^\circ_{N_G(\overline{U})}(t) = \overline{T}$ for all $t \in \overline{T}^*$. Let then $n \in N_G(\overline{U})$ and assume that there is a non-trivial element $x \in \overline{T}^n \cap \overline{T}$. We have:

$$\overline{T}^n = C^\circ_{N_G(\overline{U})}(t)^n = C^\circ_{N_G(\overline{U})}(t^n) = C^\circ_{N_G(\overline{U})}(x) = C^\circ_{N_G(\overline{U})}(t) = \overline{T}.$$

Therefore, $n \in N_G(\overline{T}) = \langle \overline{T}, \omega_0 \rangle$ (by Lemma 7.2.19). Clearly, $\omega_0 \notin N_G(\overline{U})$. Thus, $N_G(\overline{U}) \cap N_G(\overline{T}) = \overline{T}$ and so $n \in \overline{T}$. Therefore, $\overline{T} < N_G(\overline{U})$ is a Frobenius group. By Fact 5.3.27, the Frobenius group $N_G(\overline{U})$ splits, that is, $N_G(\overline{U}) = U_1 \rtimes \overline{T}$ for some $U_1 \lhd N_G(\overline{U})$. Moreover, by Fact 5.3.28, U_1 is connected which implies that $\overline{B} = \overline{U} \rtimes \overline{T} = N_G^\circ(\overline{U}) = U_1^\circ \rtimes \overline{T}^\circ = U_1 \rtimes \overline{T} = N_G(\overline{U})$. □

We remind the reader that a Borel subgroup K of a group of finite Morley rank H is a maximal definable connected and solvable subgroup of H. One should also recall a Borel subgroup K of a group of finite Morley rank H is almost self-normalising, that is, $K = N_H^\circ(K)$ (see Subsection 5.3.4).

At this point we have enough information to prove that $\overline{B}$ is a Borel subgroup of G and that $N_G(\overline{B}) = \overline{B}$.

Lemma 7.2.21. *$N_G(\overline{B}) = \overline{B}$. Moreover, $\overline{B}$ is a Borel subgroup of G.*

Proof. Clearly, $\overline{B} \leqslant N_G(\overline{B})$ by the properties of the definable closure.

We prove that $N_G^\circ(\overline{B}) \leqslant \overline{B}$, that is, $N_G^\circ(\overline{B}) = N_G(\overline{U}) = \overline{B}$. To do so, we prove that $\overline{B}$ is a Borel subgroup of G. Towards a contradiction, assume that $\overline{B}$ is not a Borel subgroup of G and let B_1 be a Borel subgroup of G containing $\overline{B}$. By Fact 5.3.12, we have $\overline{U} = \overline{B}' \leqslant B_1' \leqslant F^\circ(B_1)$, where $F^\circ(B_1)$ denotes the connected component of the Fitting subgroup of B_1. Moreover, by Lemma 7.2.20, we have $\overline{U} \leqslant N_{F^\circ(B_1)}(\overline{U}) \leqslant N_G(\overline{U}) = \overline{B}$. At the same time we know that $\overline{B}' = \overline{U} = F^\circ(\overline{B})$ and that $F^\circ(B_1)$ is a nilpotent characteristic subgroup of B_1. Therefore, $N_{F^\circ(B_1)}(\overline{U}) = F^\circ(\overline{B}) = \overline{U}$. However, as the connected component of the Fitting subgroup $F^\circ(B_1)$ of B_1 is nilpotent, it satisfies the normaliser condition, that is, $F^\circ(B_1) = \overline{U}$. Therefore, B_1 normalises $\overline{U}$. But $\overline{B} = N_G(\overline{U})$ by Lemma 7.2.20—this contradiction shows that $\overline{B} = B_1$. Since $\overline{B}$ is a Borel subgroup of G we have $N_G^\circ(\overline{B}) = \overline{B}$.

The proof of $\overline{B} = N_G^\circ(\overline{B}) = N_G(\overline{B})$ works exactly similarly as the proof of $\overline{B} = N_G^\circ(\overline{U}) = N_G(\overline{U})$ in Lemma 7.2.20. □

Finally, we have enough information to prove that $\overline{B} \cap \overline{U}^g = 1$ for all $g \in G \setminus \overline{B}$.

Lemma 7.2.22. *$\overline{B} \cap \overline{U}^g = 1$ for all $g \in G \setminus \overline{B}$.*

Proof. We first observe that $\overline{U} \cap \overline{U}^g = 1$ for all $g \in G \setminus \overline{B}$. Let $g \in G \setminus \overline{B}$ and assume towards a contradiction that there exists a non-trivial element $x \in \overline{U} \cap \overline{U}^g$. Then, by Lemma 7.2.18,

$$\overline{U} = C_G^\circ(u) = C_G^\circ(x) = C_G^\circ(u^g) = C_G^\circ(u)^g = \overline{U}^g,$$

and so, $g \in N_G(\overline{U})$. But $N_G(\overline{U}) = \overline{B}$ by Lemma 7.2.20—we have contradict the choice of g.

Let then $V = \overline{B} \cap \overline{U}^g$ and $g \in G \setminus \overline{B}$. We know that $V \cap \overline{U} = 1$ and thus V is conjugate to a subgroup of $\overline{T}$ in $\overline{B}$ by Fact 5.3.29. We have, $V = X^b$ where $X \leqslant \overline{T}$ for some $b \in \overline{B}$. Assume then that there exists a non-trivial element $v \in V$. Then,

$$\overline{T}^b \leqslant C_G^\circ(t)^b = C_G^\circ(t^b) = C_G^\circ(v) = C_G^\circ(u^g) = C_G^\circ(u)^g = \overline{U}^g,$$

that is, $\overline{T}^b \leqslant \overline{U}^g$ (the last equivalence above follows from Lemma 7.2.18). But $\overline{U}$ contains no involutions by Lemma 7.2.6—a contradiction. □

Final identification of G

Recall that our aim is to invoke the Delahan–Nesin identification result; Fact 7.2.2. Therefore, we need to prove that G is a split Zassenhaus group. In order to do so, we must prove that $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$.

Several people have studied odd type infinite simple groups of finite Morley rank with $\mathrm{pr}_2(G) = 1$, under different further assumptions—the aim is (of course) always to prove that such G is isomorphic to $PSL_2(K)$ for some algebraically closed field K of characteristic different from 2. We are aware of the following papers of the topic (in a chronological order) the reader might want to have a look at: [54], [32], [37], [36], [40] and [39]. In all of these earlier studies, the authors were eventually facing the same task as us—for subgroups and elements of their G corresponding to our $\overline{B}$, $\overline{U}$ and ω_0, it was necessary to prove that $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$.

From now on, we follow the strategy of Jaligot in [54] (at least to our best knowledge this kind of a strategy was first used in [54] but the reader should also see all the rest of the papers cited in the list above). Note that, Jaligot's preprint [54] was never published but a slightly modified version of this preprint can be found from the paper by Cherlin and Jaligot [32, Section 4].

Following terminology of Jaligot in [54], a *FT-group* is a non-solvable connected group of finite Morley rank whose proper definable and connected subgroups are solvable (in [32] and in many other places in literature such groups are called *minimal*). In [54], Jaligot proved the following result.

Fact 7.2.23 (Jaligot [54]; see also Cherlin and Jaligot [32, Theorem 4.1])**.** *Let G be an odd type simple FT-group with $\mathrm{pr}_2(G) = 1$. Let S be a Sylow 2-subgroup of G and i be the unique involution of S°. Assume that B is a Borel subgroup of G that contains $C_G^\circ(i)$. Assume further that U is a B-minimal subgroup of B such that $C_{C_G^\circ(i)}(U)$ is finite. Then $G \cong PSL_2(K)$ where K is an algebraically closed field of characteristic different from* 2*.*

To prove Fact 7.2.23, Jaligot used the Delahan–Nesin identification result; Fact 7.2.2. As mentioned above, he needed to prove that, for subgroups and elements of his G corresponding to our $\overline{B}$, $\overline{U}$ and ω_0, $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$. In [54], Jaligot used the FT-group assumption to prove that, for subgroups corresponding to our $\overline{U}$ and $\overline{B}$, the following hold:

- $C_G^\circ(u) = C_G^\circ(\overline{U})$ for all $u \in \overline{U}^*$, and,
- $\overline{U} \cap \overline{U}^g = 1$ and $\overline{B} \cap \overline{U}^g$ is finite for all $g \in G \setminus \overline{B}$. (Note that in our set-up we proved something slightly stronger, namely, we proved that $\overline{B} \cap \overline{U}^g = 1$ for all

$g \in G \setminus \overline{B}$.)

Since the presense of a tight automorphism α allows us to prove the bulletpoints above (Lemmas 7.2.18 and 7.2.22), for the rest of the proof of Theorem 7.2.1, we may mimic arguments of Jaligot without the FT-group assumption. To keep the text self-contained, we write down the arguments of the rest of our proof—all of which one may also find, for example, from [54] or [32, Section 4]. We wish to highlight that Lemmas 7.2.25, 7.2.26, 7.2.28 and 7.2.29 are all proven by Jaligot in [54] and, in our set-up, we may mimic his proofs almost word to word.

Fact 7.2.24 (Jaligot [54, Lemme 2.13], or see [32, Fact 2.36])**.** *Let G be an infinite simple group of finite Morley rank and M be a proper definable subgroup of G. Then, $rk(x^G \cap M) < rk(x^G)$ for every non-trivial element x of G.*

At this point we know the following things:

- $rk(G) = rk(i^G) + rk(C_G(i)) = rk(i^G) + rk(\overline{T})$.
- $rk(i^G) = rk((i^G \setminus \overline{B}))$ (by Fact 7.2.24), that is, $rk(G) = rk((i^G \setminus \overline{B})) + rk(\overline{T})$.

Mimicking Jaligot in [54] (see also [32, Section 4]), we define the following for an involution $\omega \in (i^G \setminus \overline{B})$:

$$T(\omega) = \{b \in \overline{B} : \omega b \omega = b^{-1}\}.$$

We may immediately observe that $T(\omega) \cap \overline{U} = 1$ for all $\omega \in (i^G \setminus \overline{B})$—this follows from Lemma 7.2.22 as $\omega \notin \overline{B}$. Therefore, by Fact 5.3.29, $T(\omega)$ is conjugate to a subgroup of $\overline{T}$. Still mimicking Jaligot in [54], we define the following for an involution $\omega \in (i^G \setminus \overline{B})$:

$$X_1 = \{\omega \in (i^G \setminus \overline{B}) : rk(T(\omega)) < rk(\overline{T})\},$$

$$X_2 = \{\omega \in (i^G \setminus \overline{B}) : rk(T(\omega)) = rk(\overline{T})\}.$$

We next prove that $rk(X_2) = rk(i^G)$, using exactly the same arguments as Jaligot in [54, Lemme 3.7.].

Lemma 7.2.25. $rk(X_2) = rk(i^G)$ (cf. Jaligot [54, Lemme 3.7.] or Cherlin and Jaligot [32, page 35]).

Proof. Since $rk(i^G) = rk((i^G \setminus \overline{B}))$, it is enough to prove that $rk(X_1) < rk(i^G)$.

Let $\sim$ be an equivalence relation on X_1 defined as follows. For $\omega_1, \omega_2 \in X_1$, $\omega_1 \sim \omega_2$ if and only if ω_1 and ω_2 are in the same coset of $\overline{B}$ (equivalently, $\omega_1 \sim \omega_2$ if and only if $\omega_1\omega_2 \in \overline{B}$, that is, $\omega_1\omega_2 \in T(\omega_1)$). Let us consider the natural definable projection

$$p : X_1 \longmapsto X_1/\sim .$$

Let

$$(X_1)_k = \{\omega_1 \in X_1 : rk(p^{-1}(p(\omega_1))) = k\}.$$

We have, $0 \leqslant k \leqslant rk(\overline{T}) - 1$ by the definition of X_1. It is clear that X_1 can be written as a disjoint union of finitely many $(X_1)_k$'s. Therefore, for some k_0, $(X_1)_{k_0}$ is generic in X_1, that is,

$$rk(X_1) = rk((X_1)_{k_0}) = rk(p((X_1)_{k_0})) + k_0 \leqslant rk(X_1/\sim) + k_0.$$

At the same time, $rk(\overline{B}) + rk(X_1/\sim) = rk(X_1\overline{B}) \leqslant rk(G) = rk(i^G) + rk(\overline{T})$, which re-writes to $rk(X_1/\sim) \leqslant rk(i^G) + rk(\overline{T}) - rk(\overline{B}) = rk(i^G) - rk(\overline{U})$. Therefore we have

$$rk(X_1) \leqslant rk(i^G) - rk(\overline{U}) + k_0.$$

But $k_0 < rk(\overline{T}) \leqslant rk(\overline{U})$, and therefore, $rk(X_1) < rk(i^G)$. □

We move on to prove that $rk(X_2) \leqslant rk(\overline{B})$ which implies that $rk(G) \leqslant rk(\overline{B}) + rk(\overline{U})$. Again, we wish to mention that we use exactly the same arguments as Jaligot in [54, Lemme 3.9.].

Lemma 7.2.26. $rk(X_2) \leqslant rk(\overline{B})$ (cf. Jaligot [54, Lemme 3.9.] or Cherlin and Jaligot [32, page 35]).

Proof. Recall that we have observed above that, for all $\omega \in (i^G \setminus \overline{B})$, $T(\omega)$ is $\overline{B}$-conjugate to a subgroup of $\overline{T}$. Therefore, by Lemma 7.2.17, given $\omega_1 \in X_2$, we may assume that $T(\omega_1) = \overline{T}^u$ for some unique element $u \in \overline{U}$.

Similarly as in Lemma 7.2.25, let $\sim$ be an equivalence relation on X_2 defined as follows. For $\omega_1, \omega_2 \in X_2$, $\omega_1 \sim \omega_2$ if and only if ω_1 and ω_2 are in the same coset of $\overline{B}$.

Consider the following map:

$$\phi : X_2/\sim \longrightarrow \overline{U},\ \ \omega_1/\sim \mapsto u,$$

where u is the unique element of $\overline{U}$ such that $T(\omega_1) = \overline{T}^u$. In what follows we observe that ϕ has finite fibers. By conjugation, it is enough to show this for $u = 1$. Recall that, by Lemma 7.2.11, $\overline{T} = N^\circ_G(\overline{T})$. Each element of $\phi^{-1}(1)$ is a coset of $N_G(\overline{T})/\overline{T}$, distinct of $\overline{T}$, and thus, $\phi^{-1}(1)$ is finite. Therefore $rk((X_2/\sim)) = rk(\overline{U})$. Since $rk(X_2) \leqslant rk((X_2/\sim)) + rk(\overline{T})$ we have proven that $rk(X_2) \leqslant rk(\overline{U}) + rk(\overline{T}) = rk(\overline{B})$. □

The following is an immediate corollary of Lemma 7.2.26.

Corollary 7.2.27. *$rk(G) \leqslant rk(\overline{B}) + rk(\overline{U})$.*

Proof. $rk(G) = rk(i^G) + rk(\overline{T}) = rk(X_2) + rk(\overline{T}) \leqslant rk(\overline{B}) + rk(\overline{T}) \leqslant rk(\overline{B}) + rk(\overline{U})$. □

Finally, we have enough information to prove that $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$—using exactly the same arguments as Jaligot in [54, Corollaire 3.14] and Cherlin and Jaligot in [32, Lemma 4.11].

Lemma 7.2.28. *$G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$ (cf. Jaligot [54, Corollaire 3.14] or Cherlin and Jaligot [32, Lemma 4.11]).*

Proof. Let us consider the following map:

$$\varphi_{\omega_0} : \overline{U} \times \overline{B} \longmapsto \overline{U}\omega_0\overline{B},\ (u, b) \mapsto u\omega_0 b.$$

This map has finite fibers—if $u_1\omega_0 b_1 = u_2\omega_0 b_2$ then $(u_2^{-1}u_1)^{\omega_0} = b_2 b_1^{-1} \in \overline{U}^{\omega_0} \cap \overline{B}$ which is trivial by Lemma 7.2.22. Thus, the rank of the image $\overline{U}\omega_0\overline{B}$ is $rk(\overline{B}) + rk(\overline{U})$. By Lemma 7.2.26, $rk(\overline{B}) + rk(\overline{U}) \geqslant rk(G)$. Of course, we also have $rk(\overline{B}) + rk(\overline{U}) \leqslant rk(G)$, and therefore, $rk(\overline{B}) + rk(\overline{U}) = rk(G)$. We have observed that the image $\overline{U}\omega_0\overline{B}$ is generic in G.

Let then $g \in G\backslash\overline{B}$ and consider the map φ_g defined as above. Similarly as above, we may observe that the image $\overline{U}g\overline{B}$ is generic in G. Since G is connected and both $\overline{U}\omega_0\overline{B}$ and $\overline{U}g\overline{B}$ are generic in G, they intersect non-trivially and the claim follows. □

At this point, it is routine to check that G is a split Zassenhaus group with an involution in a two-point stabiliser. Therefore, we finally have enough information to identify G with $PSL_2(K)$, where K is an algebraically closed field of characteristic different from 2.

Lemma 7.2.29. *G is a split Zassenhaus group with an involution in a two-point stabiliser.* (cf. Jaligot [54, page 9] or Cherlin-Jaligot [32, page 36])

Proof. We start by proving that the action of G by left multiplication on $G/\overline{B}$ is 2-transitive. We have proven that $N_G(\overline{B}) = \overline{B}$ (Lemma 7.2.21) and $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$ (Lemma 7.2.28). Let $G_{\overline{B}}$ denote the stabiliser of the coset $\overline{B}$. $G_{\overline{B}} = \overline{B}$ since $N_G(\overline{B}) = \overline{B}$. It is enough to prove that $G_{\overline{B}} = \overline{B}$ acts transitively on the cosets of the form $\overline{U}\omega_0\overline{B}$. Let $x_1 = u_1\omega_0\overline{B}$ and $x_2 = u_2\omega_0\overline{B}$ be two such cosets with $u_1 \neq u_2$. Then $u_2u_1^{-1}x_1 = x_2$ and since $1 \neq u_2u_1^{-1} \in \overline{B}$ we get the transitivity of $\overline{B}$.

We move on to prove that a stabiliser of two distinct points contains an involution. Let $G_{\{x,y\}}$ denote the two-point stabiliser of the points $x = \overline{B}$ and $y = \omega_0\overline{B}$. One observes that $G_{\{x,y\}} = \overline{B} \cap \overline{B}^{\omega_0}$. Clearly $\overline{T} \leqslant \overline{B}^{\omega_0} \cap \overline{B}$ and therefore the unigue involution i of $\overline{T}$ is contained in $G_{\{x,y\}}$. Further, by Lemma 7.2.22, $(\overline{B}^{\omega_0} \cap \overline{B}) \cap \overline{U} = 1$ and therefore, by the properties of solvable Frobenius groups, $\overline{B}^{\omega_0} \cap \overline{B}$ is conjugate to a subgroup of $\overline{T}$. By Lemma 7.2.17, conjugates of $\overline{T}$ (which are disjoint from $\overline{T}$) intersect $\overline{T}$ trivially in $\overline{B}$, and so, $\overline{B}^{\omega_0} \cap \overline{B} = \overline{T}$.

Finally, we observe that a stabiliser of three distinct points is trivial. Let $g \in G$ be the stabiliser of the points $\overline{B}, \omega_0\overline{B}, u_1\omega_0\overline{B}$ where $u_1 \in \overline{U}^*$. We have, $g \in \overline{B} \cap \overline{B}^{\omega_0} = \overline{T}$ and, moreover, $gu_1\omega_0\overline{B} = u_1\omega_0\overline{B}$. Therefore, $g^{u_1} \in \overline{B}^{\omega_0}$. We have observed that $g^{u_1} \in \overline{T}^{u_1} \cap \overline{B}^{\omega_0} \cap \overline{B} = \overline{T}^{u_1} \cap \overline{T}$. However, $\overline{T}^{u_1} \cap \overline{T} = 1$ by Lemma 7.2.17. Therefore, $g = 1$. □

Now the Delahan–Nesin identificatio result, Fact 7.2.2, proves Theorem 7.2.1. □

7.3 Landscape for the future work

In what follows, we briefly explain a possible strategy for tackling Conjecture 7.1.5 in the case in which G has a pure group structure. That is, we outline a possible strategy towards proving that if an infinite simple odd type group of finite Morley rank G with a pure group structure admits a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite, then G is isomorphic to a Chevalley group over an algebraically closed field of characteristic different from 2. Notice that the assumption that G has a pure group structure allows one to observe that the definable closure $\overline{X}$ in G of any subgroup X of $C_G(\alpha)$ is stabilised by α—like is done in Lemma 7.2.3. Below we explain our strategy case by case, depending on the $\mathrm{pr}_2(G)$.

Note that, to prove Conjecture 7.1.5, one certainly needs to start by fully proving the case in which $\mathrm{pr}_2(G) = 1$ (part (i)) so that inductive arguments can be used in the cases where $\mathrm{pr}_2(G) = 2$ (part (ii)) and $\mathrm{pr}_2(G) \geqslant 3$ (part (iii)).

A possible strategy towards the proof of the part (i) of Conjecture 7.1.5 when the maximal split torus T is not assumed to contain an involution. Throughout this paragraph, G is an infinite simple group of finite Morley rank with a pure group structure and with $\mathrm{pr}_2(G)$=1 admitting a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite. We retain most of the notation from the previous section, that is, S, B, U and ω_0 are as described in Subsection 7.2.1. Note however that we do not assume that the maximal split torus T contains an involutions since that case is proven in Subsection 7.2.2.

The task is of course to identify G with $PSL_2(K)$ for some algebraically closed field K of characteristic different from 2. Therefore, similarly as in the proof of Theorem 7.2.1, one wishes to prove the following two things:

- $\overline{B} \cong \overline{U} \rtimes \overline{T} \cong K^+ \rtimes K^*$, for some algebraically closed field K, and;
- $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$.

Clearly, in view of Theorem 7.2.1, to prove this case it suffices to prove that $C_{\overline{T}}(\alpha)$ must contain an involution.

One possible strategy towards the identification of G is to concentrate on the field K (which one can certainly interpret using Zilber's Field Theorem). The following fact is proven in [5, Lemma 4.16] using results of Ludomir Newelski [66] and Wagner [87].

Fact 7.3.1 ([5, Lemma 4.16])**.** *Let K be a field of finite Morley rank and F an infinite (not necessarily definable) subfield of K. Let X be a definable multiplicative subgroup* *of K which contains F. Then $X = K^*$.*

It is natural to ask whether the following generalisation of Fact 7.3.1 is true.

Question 7.3.2. *Let K be a field of finite Morley rank and F be an infinite (not necessarily definable) subfield of K. Let X be a definable multiplicative subgroup of K which contains $(F^*)^2$. Is it true that in this case $X = K^*$?*

We believe that a positive answer to Question 7.3.2 would allow us to prove that $\overline{U} \cong K^+$ and $\overline{T} \cong K^*$, where K is an algebraically closed field. Then, $\overline{T}$ must contain a unique involution, say i. Moreover, as $\overline{T}$ is α-invariant, $\alpha(i) = i$, that, is $i \in C_{\overline{T}}(\alpha)$. Therefore, a positive answer to Question 7.3.2 would allow us apply Theorem 7.2.1 for the identification of G.

A possible strategy towards the proof of the part (ii) of Conjecture 7.1.5. Throughout this paragraph, G is an infinite simple group of finite Morley rank with a pure group structure and with $\mathrm{pr}_2(G)$=2 admitting a tight automorphism α whose fixed point subgroup $C_G(\alpha)$ is pseudofinite. We also assume that the part (i) of Conjecture 7.1.5 holds.

Before introducing identification results one wishes to use in this case, we need to introduce some definitions.

- A definable connected nilpotent subgroup $U \leqslant G$ is called *quasiunipotent* if U does not contain a p-torus for any prime p.
- $O(G)$ is the unique maximal connected definable normal $2^{\perp}$-subgroup of G.
- A section H/K of G is called a definable section if H and K are definable subgroups of G.
- G is a *K-group* if every infinite simple definable and connected section of G is an algebraic group over an algebraically closed field. Further, G is a *K^*-group* if every proper definable subgroup of G is a K-group.

In [8, 7], Christine Altseimer proved the following.

Fact 7.3.3 (Altseimer [8, Theorem 4.3])**.** *Let G be a simple K^*-group of finite Morley rank that does not interpret a bad field. Assume that the centralisers of involutions of G are isomorphic to $GL_2(K)$ for some algebraically closed fields K of characteristic different from 2. Then $G \cong PSL_3(K)$, if there is an involution i and a quasiunipotent subgroup $U \leqslant C_G(i)$ which is maximal in $C_G(i)$ but not in G.*

Fact 7.3.4 (Altseimer [7, Theorem 1]; see also [8, Theorem 4.44])**.** *Let G be a simple K^*-group of finite Morley rank that does not interpret a bad field. Assume that G contains an involution i such that either*

(1.) $C_G(i)^\circ/O(C_G(i)) \cong SL_2(K) * SL_2(K)$, where the two copies of $SL_2(K)$ intersect non-trivially, or

(2.) $C_G(i)^\circ/O(C_G(i)) \cong PSL_2(K) \times K^*$,

for an algebraically closed field K of characteristic neither 2 nor 3. Then, $G \cong PSp_4(K)$ or $G \cong G_2(K)$.

We believe that to prove that G is isomorphic to one of the Chevalley groups $PSL_3(K)$, $PSp_4(K)$ or $G_2(K)$, where K is an algebraically closed field of characteristic different from 2, results of Altseimer above are usable due to the presence of a tight automorphism α. To invoke these identification results, one needs to check that, in our set-up, the assumptions of Facts 7.3.3 and 7.3.4 are satisfied. Given that we assume that the part (i) of Conjecture 7.1.5 holds, we believe that the assumptions on centralisers of involutions should not be too hard to check. However, checking that G is a K^*-group which does not interpret a bad field might turn out to be quite challenging.

A possible strategy towards the proof the of part (iii) of Conjecture 7.1.5. We start this final paragraph by giving some final definitions.

Let β be an automorphism of a group H. We say that γ is a *weakening* of β if $\gamma = \beta^k$ for some $k \in \mathbb{N}$. If H is invariant under the action of some weakening γ of β then we say that H is *weakly invariant* under the action of β.

The connected component $L^\circ(G)$ of the layer $L(G)$ (product of all components, that is, quasisimple subnormal subgroups of G) of a group of finite Morley rank G

is denoted by $E(G)$. Note that a group of finite Morley rank G has finitely many components all of which are definable and normal in $L^\circ(G) = E(G)$.

At the moment, we only have a strategy towards the proof of the part (iii) of Conjecture 7.1.5 under the assumption that an infinite simple group of finite Morley rank G with $\mathrm{pr}_2(G) \geqslant 3$ admits a *supertight* automorphism instead of just a tight one. The definition of a supertight automorphism α was also given by Uğurlu in her thesis [85]: An automorphism α of an infinite simple group of finite Morley rank G is a supertight automorphism if, for each positive integer n, each power α^n is a tight automorphism. This tightening does not seem very strong to us; actually, we expect that a tight automorphism α is supertight, that is, we expect that the following questions has a positive answer.

Question 7.3.5. *Is a tight automorphism α supertight?*

Let now G be an infinite simple group of finite Morley rank with a pure group structure admitting a supertight automorphism α and with $\mathrm{pr}_2(G) \geqslant 3$. Assume that, for each power α^n of α, the fixed point subgroup $C_G(\alpha^n)$ is pseudofinite.

To identify G with a Chevalley group over an algebraically closed field K of characteristic different from 2, we wish to follow ideas of Ayşe Berkman and Borovik in [12] and [13]. We believe that their Generic Identification Theorem [12, Theorem 1.2] (see also [13, Theorem 1.1] for an unpublished but a slightly stronger result) can be generalised without much difficulties to the following form which seems suitable for our purposes.

Conjecture 7.3.6 (Strong version of the Berkman–Borovik Generic Identification Theorem; compare to [12, Theorem 1.2] and [13, Theorem 1.1])**.** *Let G be an infinite simple group of finite Morley rank and α an automorphism of G. Let D be a maximal p-torus in G; we assume that D is α-invariant and has Prüfer rank at least* 3*. Assume further that*

(i) Every proper connected definable subgroup of G which contains D and is weakly invariant under the action of α is a K-group.

(ii) For every element x of order p in D, the group $C^\circ_G(x)$ is of p'-type and $C^\circ_G(x) = F^\circ(C^\circ_G(x))E(C^\circ_G(x))$.

(iii) $\langle C_G^\circ(x) \mid x \in D,\ |x| = p \rangle = G$.

Then G is a Chevalley group over an algebraically closed field of characteristic distinct from p.

The known cases of Conjecture 7.3.6, Theorem 1.2 of [12] and Theorem 1.1 of [13], are based on stronger assumptions than assumption (i) in Conjecture 7.3.6, namely, that:

- *every* proper definable subgroup of G is a K-group [12], or that
- *every* proper definable subgroup of G containing D is a K-group [13].

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