

ON CONTACT BETWEEN SUBMANIFOLDS

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The principal aim of this article is to clarify the relationship between the contact of submanifolds and the singularity type (more precisely, the \mathcal{K} -class) of maps. In [3], Golubitsky and Guillemin consider the equidimensional case, but the general case has not been treated previously.

We define the local notion of contact type in the obvious way: Given two pairs of submanifold-germs at the origin in \mathbf{R}^n , then the pairs have the same contact type if there is a diffeomorphism-germ of $(\mathbf{R}^n, 0)$ taking one pair to the other. (Clearly, the dimensions of one pair must be the same as the dimensions of the other.) We denote the contact type of X and Y by $K(X, Y)$.

The main result is the following.

THEOREM. *For $i = 1, 2$, let $g_i: (X_i, x_i) \hookrightarrow (\mathbf{R}^n, 0)$ be immersion-germs and $f_i: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be submersion-germs, with $Y_i = f_i^{-1}(0)$. Then the pairs (X_1, Y_1) and (X_2, Y_2) have the same contact type if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are \mathcal{K} -equivalent.*

This relationship also serves to elucidate the work of Porteous and others [6, 7] on the distance-squared function and the geometry of submanifolds of Euclidean space. The singularities of the distance-squared functions that occur for a given submanifold can be viewed as types of contact of the submanifolds with hyperspheres in the Euclidean space. In subsequent articles we will take this idea further and consider the contact with spheres of higher codimension.

This article formed part of my Ph.D. thesis [5] at the University of Liverpool. I would like to thank I. R. Porteous — my supervisor — for his advice and encouragement, and also C. T. C. Wall and C. G. Gibson for helpful discussions.

Since completing this work I have learned that J. W. Bruce has thought about similar questions. In particular, the Symmetry Lemma is proved in an article of his entitled “Wavefronts and Parallels in Euclidean Space,” to appear in the *Mathematical Proceedings of the Cambridge Philosophical Society*.

Preliminaries. Recall that two map-germs $\phi, \psi: (\mathbf{R}^k, 0) \rightarrow (\mathbf{R}^p, 0)$ are \mathcal{K} -equivalent if there are diffeomorphism-germs h of $(\mathbf{R}^k, 0)$ and H of $(\mathbf{R}^k \times \mathbf{R}^p, (0, 0))$ such that $H(x, 0) = (h(x), 0)$ and $H(x, \phi(x)) = (h(x), \psi \circ h(x))$. Equivalently, we require H to map the graph of the zero-map to itself and the graph of ϕ to the graph of ψ . Also useful is the fact that H can be written in the form $H(x, y) = (h(x), \theta(x, y))$, where $\theta: (\mathbf{R}^k \times \mathbf{R}^p, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ and the differential of θ with respect to its second argument $d_y \theta$ at 0 is surjective.

One other fact we use is the following: Let $\phi, \psi: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be map-germs, and $\bar{\phi}, \bar{\psi}: (\mathbf{R}^{n+k}, 0) \rightarrow (\mathbf{R}^{p+k}, 0)$ be suspensions of ϕ and ψ respectively.

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Then ϕ and ψ are \mathcal{K} -equivalent if and only if $\bar{\phi}$ and $\bar{\psi}$ are. It therefore makes sense to extend the notion of \mathcal{K} -equivalence to maps involving different dimensions. Suppose now $\phi: (\mathbf{R}^{n_1}, 0) \rightarrow (\mathbf{R}^{p_1}, 0)$ and $\psi: (\mathbf{R}^{n_2}, 0) \rightarrow (\mathbf{R}^{p_2}, 0)$ ($n_1 - n_2 = p_1 - p_2$); then we can say that ϕ and ψ are \mathcal{K} -equivalent if one is \mathcal{K} -equivalent to a suspension of the other (in the traditional sense). Note that with this extended definition, it is still true that two map-germs are \mathcal{K} -equivalent if and only if their local algebras are induced isomorphic.

For further details on \mathcal{K} -equivalence see [2] or [4].

Let $g: (X, x) \hookrightarrow (\mathbf{R}^n, 0)$ be an immersion-germ and $f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a submersion-germ, with $Y = f^{-1}(0)$. We call $f \circ g$ a *contact map* (-germ) for X and Y . Of course, there are many different contact map-germs for a given pair of submanifold germs. That they are all \mathcal{K} -equivalent follows as a corollary of the Symmetry Lemma below. The Symmetry Lemma itself ensures that the same information is obtained from the \mathcal{K} -class whether we consider X as immersed and Y as a zero-set of a map or vice versa.

SYMMETRY LEMMA. *Let X and Y be submanifold-germs of \mathbf{R}^n at 0. Let*

$$g: (X, x) \hookrightarrow (\mathbf{R}^n, 0) \quad \text{and} \quad \bar{g}: (Y, y) \hookrightarrow (\mathbf{R}^n, 0)$$

be immersion-germs, and let

$$\bar{f}: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0) \quad \text{and} \quad f: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^q, 0)$$

be submersion-germs, with $X = \bar{f}^{-1}(0)$ and $Y = f^{-1}(0)$. Then $f \circ g$ and $\bar{f} \circ \bar{g}$ are \mathcal{K} -equivalent (with the appropriate interpretation, described above, if $\dim X \neq \dim Y$).

Proof. [I am indebted to C. T. C. Wall for the idea behind this proof.] Consider the commutative diagram:

$$\begin{array}{ccc}
 (X, x) & \xrightarrow{\phi} & (\mathbf{R}^q, 0) \\
 \searrow g & & \nearrow \\
 (\mathbf{R}^n, 0) & \xrightarrow{\chi = (f, \bar{f})} & (\mathbf{R}^q \times \mathbf{R}^p, (0, 0)) \\
 \nearrow \bar{g} & & \searrow \\
 (Y, y) & \xrightarrow{\psi} & (\mathbf{R}^p, 0)
 \end{array}$$

where $\chi(x) = (f(x), \bar{f}(x))$, the maps from $(\mathbf{R}^q \times \mathbf{R}^p, (0, 0))$ are the obvious projections, $\phi = f \circ g$ and $\psi = \bar{f} \circ \bar{g}$. Now, if we express \mathbf{R}^n as $X \times \mathbf{R}^p$ so that $g(x) = (x, 0)$ and $\bar{f}(x, z) = z$ (which is possible, since $\bar{f} \circ g = 0$), then

$$\chi(x, z) = (f(x, z), z) \quad \text{and} \quad \chi(x, 0) = (f \circ g(x), 0),$$

and so χ is seen to be an unfolding of ϕ . It follows that χ is \mathcal{K} -equivalent to a suspension of ϕ . By a similar argument χ is \mathcal{K} -equivalent to a suspension of ψ , and the result follows. □

COROLLARY. *For any pair of submanifold-germs of \mathbf{R}^n , the \mathcal{K} -class of the contact map is independent of the choice of contact map, and so depends only on the submanifold-germs themselves.*

The following example shows that it is necessary to consider \mathcal{K} -equivalence, and that \mathcal{Q} - (left-right) equivalence is too restrictive: Let $f, f': \mathbf{R}^3 \rightarrow \mathbf{R}^2$ be given by (y, z) and $(y, z - xy)$ respectively, both of which cut out the x -axis. Now consider their contact with the curve $t \rightarrow (t, t^2, t^3)$; the contact maps become $t \rightarrow (t^2, t^3)$ and $t \rightarrow (t^2, 0)$, respectively. These two maps are certainly \mathcal{K} -equivalent, but that they are not \mathcal{Q} -equivalent follows from the fact that their images are not diffeomorphic.

From now on, X_i and Y_i will be submanifold-germs of \mathbf{R}^n at 0 ($i = 1, 2$, with $\dim X_1 = \dim X_2 = k$ and $\dim Y_1 = \dim Y_2 = n - p$). Also, g_i will be an immersion-germ of X_i , and $f_i: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ will be a submersion-germ with $f_i^{-1}(0) = Y_i$.

To prove the ‘if’ part of the theorem we need two lemmas, which we present before proving the theorem.

LEMMA A (invariance of contact under suspension). *Let a be any positive integer and let $X'_i = X_i \times \mathbf{R}^a$, $Y'_i = Y_i \times \{0\}$ (all in $\mathbf{R}^n \times \mathbf{R}^a$); then*

$$K(X_1, Y_1) = K(X_2, Y_2) \Leftrightarrow K(X'_1, Y'_1) = K(X'_2, Y'_2).$$

Proof of Lemma A.

(i) “ \Rightarrow ”: This part is immediate, as the suspension of the diffeomorphism taking X_1 and Y_1 to X_2 and Y_2 (respectively) itself takes X'_1 and Y'_1 to X'_2 and Y'_2 (respectively).

(ii) “ \Leftarrow ”: Let H' be the diffeomorphism required by the second statement, so $H'(X'_1) = X'_2$ and $H'(Y'_1) = Y'_2$. Write $H' = (H_1, H_2)$, where $H_1: (\mathbf{R}^{n+a}, 0) \rightarrow (\mathbf{R}^n, 0)$ and H_2 has image $(\mathbf{R}^a, 0)$. It follows that each of H_1 and H_2 are submersions.

Suppose that we can find a map-germ $u: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^a, 0)$ satisfying both (a) $u|_{Y_1} = 0$ and (b) the map $H: x \rightarrow H_1(x, u(x))$ is a diffeomorphism. Then the map H will be the required map, for:

$$\begin{aligned} x \in X_1 &\Rightarrow (x, u(x)) \in X'_1 \Rightarrow H(x, u(x)) \in X'_2 \Rightarrow H(x) = H_1(x, u(x)) \in X_2, \\ y \in Y_1 &\Rightarrow (y, u(y)) = (y, 0) \in Y'_1 \Rightarrow H'(y, 0) \in Y'_2 \Rightarrow H(y) \in Y_2. \end{aligned}$$

There remains to show that the map u does indeed exist. For (b) it is enough that the differential $dH: \hat{x} \rightarrow (\hat{x}, du(\hat{x}))$ be injective. Let $dH_1 = (A, B)$, with $A \in L(\mathbf{R}^n, \mathbf{R}^n)$, $B \in L(\mathbf{R}^a, \mathbf{R}^n)$, and $U = du$. Then (A, B) has rank n , and we require U so that $A + BU$ has rank n , and for condition (a) we require U to be zero on TY_1 . It is straightforward to show that such a U exists (since A restricted to TY_1 is injective). \square

LEMMA B (invariance of the \mathcal{K} -class of contact maps under suspension). *Let X_i, Y_i, g_i, f_i be as above, $i = 1, 2$. Let a, b be non-negative integers. Consider the submanifold-germs of $\mathbf{R}^n \times \mathbf{R}^a \times \mathbf{R}^b$ at 0:*

$$\begin{aligned} X'_i &= X_i \times \mathbf{R}^a \times \{0\} \\ Y'_i &= Y_i \times \{0\} \times \mathbf{R}^b; \end{aligned}$$

let g'_i be an immersion-germ with image X'_i and let f'_i be a submersion-germ with zero-set Y'_i . Then $f'_1 \circ g'_1$ and $f'_2 \circ g'_2$ are \mathcal{K} -equivalent if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are.

Proof of Lemma B. By the corollary to the Symmetry Lemma we can choose g'_i and f'_i without affecting the \mathcal{K} -classes of the composite maps. We therefore make the obvious choices:

$$g'_i(x, u) = (g_i(x), u, 0),$$

$$f'_i(y, u, v) = (f_i(y), u).$$

Thus $f'_i \circ g'_i(x, u) = (f_i \circ g_i(x), u)$ and the assertion follows immediately. □

Proof of the Theorem. Recall the theorem:

$$K(X_1, Y_1) = K(X_2, Y_2) \Leftrightarrow f_1 \circ g_1 \text{ and } f_2 \circ g_2 \text{ are } \mathcal{K}\text{-equivalent.}$$

(i) “ \Rightarrow ”: This part is elementary and does not require the above lemmas. Let H be the diffeomorphism of \mathbf{R}^n taking X_1 and Y_1 to X_2 and Y_2 respectively. Now H restricted to X_1 is a diffeomorphism onto X_2 , and thus there exists a diffeomorphism $h: X_1 \rightarrow X_2$ such that $H \circ g_1 = g_2 \circ h$. We also have that $(f_2 \circ H)^{-1}(0) = f_1^{-1}(0)$, and so by Hadamard’s lemma we can write

$$f_2 \circ H(y) = \sum_{i=1}^p f_{1i}(y) \cdot a_i(y),$$

where $f_1 = (f_{11}, \dots, f_{1p})$ and for each i , $a_i(y) \in \mathbf{R}^p$. Now $f_2 \circ H$ is a submersion, and therefore so is the right-hand side, which implies (since $f_1(0) = 0$) that the $p \times p$ matrix $[a_1(y), \dots, a_p(y)]$ is invertible.

Define $\theta: \mathbf{R}^n \times \mathbf{R}^p \rightarrow \mathbf{R}^p$ by

$$\theta(y, z) = \sum_{i=1}^p z_i a_i(y),$$

and

$$\theta': X \times \mathbf{R}^p \rightarrow \mathbf{R}^p: (x, z) \rightarrow \theta(g_1(x), z).$$

Then $\theta'(x, f_1 \circ g_1(x)) = f_2 \circ g_2 \circ h(x)$, and it is easy to show that θ' has the required properties to ensure that $f_1 \circ g_1$ and $f_2 \circ g_2$ are \mathcal{K} -equivalent.

(ii) “ \Leftarrow ”: It is this part of the proof that requires Lemmas A and B, for it does not lend itself to a direct proof unless $\dim X_i = \dim Y_i$. First we treat this equidimensional case, and then use the lemmas to extend this to the general result.

Suppose $\dim X_i = \dim Y_i = k$. We wish to express each X_i as the graph of some map $\phi_i: \mathbf{R}^k \rightarrow \mathbf{R}^p$, and the Y_i as the graph of the zero map from \mathbf{R}^k to \mathbf{R}^p . To do this we first choose coordinates on \mathbf{R}^n so that $f_1(x_1, \dots, x_n) = (x_{k+1}, \dots, x_n)$; thus $Y_1 = \mathbf{R}^k \times \{0\}$. Secondly, choose a p -dimensional subspace V_1 transverse to both X_1 and Y_1 , and write $\mathbf{R}^n = Y_1 \times V_1$. Let $\pi: \mathbf{R}^n \rightarrow Y_1$ be the projection on to the first factor; then $\pi|_{X_1}: X_1 \rightarrow Y_1$ is a diffeomorphism which induces a coordinate system on X_1 . With respect to these coordinates Y_1 is the graph of the zero map, while X_1 is the graph of the map $f_1 \circ g_1$ (thinking of f_1 as the projection: $Y_1 \times V_1 \rightarrow V_1$). A similar construction can be done for X_2 and Y_2 . Then any diffeomorphism $H: \mathbf{R}^k \times \mathbf{R}^p \rightarrow \mathbf{R}^k \times \mathbf{R}^p$ preserving the graph of the zero map and taking the graph of $f_1 \circ g_1$ to the graph of $f_2 \circ g_2$ is then a diffeomorphism taking X_1 to X_2 and Y_1 to Y_2 , so concluding the equidimensional case.

In the case where $\dim X_i \neq \dim Y_i$, we can suspend whichever is of the lower dimension with \mathbf{R}^d , d being the difference in the dimensions, to give X'_i and Y'_i in $\mathbf{R}^n \times \mathbf{R}^d$, and define the appropriate maps g'_i and f'_i . We then have the following correspondences:

$$\begin{array}{ccc} K(X, Y) & \xrightarrow{(a)} & \mathcal{K}(f \circ g) \\ (d) \uparrow & & \downarrow (c) \\ K(X', Y') & \xleftarrow{(b)} & \mathcal{K}(f' \circ g'), \end{array}$$

with (a) from part (i) of this proof, (b) from part (ii), (c) from Lemma A, and (d) from Lemma B ($a = d$, $b = 0$ for $\dim Y > \dim X$, and $b = d$, $a = 0$ for $\dim Y < \dim X$). \square

Final remark. The isomorphism class of the local algebra $Q(\phi)$ of a map-germ ϕ is an invariant of the \mathcal{K} -class. It follows that the isomorphism class of the local algebra of the contact map-germ of a pair of submanifold-germs is an invariant of the contact type. Moreover, this isomorphism class determines the contact type (at least if the contact map is of finite \mathcal{K} -codimension). It follows from a paper of Damon and Galligo [1] that in many cases the multiplicity of contact of a pair of submanifolds is equal to the dimension of the local algebra of the contact map. In particular this is the case when the contact map has finite \mathcal{K} -codimension, and either has kernel rank at most 2 or is of discrete algebra type.

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