STOCHASTIC DIFFERENTIAL EQUATIONS: OF JUMP TYPE AND WITH SINGULAR DRIFT

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Contents

Abstract

This thesis explores three different topics that relate the stochastic differential equations (SDE), including SDE with jumps, elliptic equations driven by Brownian motion with singular drift and reflected Brownian motion with singular drift.

For the SDE of jump types, we consider stochastic differential equations driven by compensated Poisson random measure. We showed that the solution of the SDE admits non-explosion for a class of super-linear coefficients, and moreover, we proved that the pathwise uniqueness holds under certain non-Lipschitzian conditions on the coefficients.

Moreover, we obtained the existence and uniqueness of solution u of elliptic equation associated with Brownian motion with singular drift. We then used the regularity of the weak solution u and the Zvonkin-type transformation to show that there is a unique weak solution to a stochastic differential equation when the drift is a measurable function.

Furthermore, we showed that there exists a unique weak solution to the reflected Brownian motion with singular drift μ, where μ is a vector-valued Kato class measure on $\mathbb{R}^d$. To serve the purpose of monitoring the weak solution, we also established some Gaussian type estimates of the transition density function of the solution.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

Stochastic differential equations have drawn lots of research in probability theory. Series of studies have been carried out about the existence of solution to an SDE. One of the most famous theorems, which is first introduced in [11, 14] by Friedman and Skorohod (readers can also find detailed proof in [17, 27]), demonstrates that a strong unique solution to an SDE:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt,$$

can be guaranteed by $\sigma(\cdot)$ and $b(\cdot)$ being Lipschitz-continuous functions. In this thesis, we will consider some exotic SDEs. By being exotic, we meant the SDE could be driven by stochastic processes of jump types, the drift of an SDE could no longer be a Lipschitz-continuous but merely a measurable function or even a vector-valued measure. These groups of SDEs can provide more accurate models when we try to describe industry objects, such as underlying assets price in financial market. We will monitor different SDEs and find out the criterion under which the SDE has a strong or weak solution. The thesis is organised as follow.

Fang and Zhang generalised the above theorem in [9] and proved the existence of a unique strong solution to SDEs with respect to Brownian motion under non-Lipschitz coefficients $\sigma(\cdot)$ and $b(\cdot)$. In Chapter 2, we can extend Fang and Zhang's work to SDEs driven by compensated Poisson random measures. The Section 2.1 of this chapter will introduce some fundamental definitions. In the Section 2.2, we show that the lifetime of the solution is infinite for a class of super-linear growth coefficients, so that the solution admits non-explosion. Our approach is similar to that in [9], however, we need to treat the jumps with great care. The general idea is to construct a proper stopping time τ with respect to the stochastic process that is defined by our SDE:

$$X_t = X_0 + \int_0^{t+} \int f(X_{s-}, u)\widetilde{N}_p(dsdu),$$

where $\widetilde{N}_p(dsdu)$ is the compensated Poisson random measure associated with the Poisson point process p. We then define a non-negative function $\Phi(\cdot)$ which satisfies some certain constrains so that we can apply Gronwall's inequality to obtain estimates on $E[\Phi(\cdot)]$. These estimates would lead us to a conclusion about the stopping time τ which implies the non-explosion of X_t. In the Section 2.3, we will prove that pathwise uniqueness holds if the coefficient satisfies certain non-Lipschitzian conditions. The method is similar to that in Section 2.2.

In Chapter 3, we consider the elliptic equation driven by Brownian motion with singular drift. For an elliptic equation:

$$\lambda u - Lu = \nu, \tag{1.0.1}$$

where $\lambda > 0$, L is the infinitesimal generator of the Brownian motion with drift $\mu = (\mu_1, \mu_2, \cdots, \mu_d)$. Formally:

$$L := \frac{1}{2}\Delta + \langle \mu, \nabla \rangle.$$

And we assume that $d \geq 3$ throughout this chapter. The reason for this will be given later in this chapter. If it is given that $\mu(dx) = b(x)dx$ and $\nu(dx) = f(x)dx$ for some bounded Borel measurable functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$, then the equation (1.0.1) has a unique weak solution on $\mathbb{R}^d$. In a more general case, when ν and μ_i for $1 \leq i \leq d$ are Borel measures on $\mathbb{R}^d$, on a bounded domain in $\mathbb{R}^d$, Yang and Zhang in [37] proved that there exists a unique weak solution of the elliptic equation (1.0.1) with Dirichlet boundary condition under appropriate conditions. In Chpater 3, we will prove that if $\lambda > 0$ is large enough, then the elliptic equation (1.0.1) on $\mathbb{R}^d$ has at least one weak solution given that ν and μ_i belong to the Kato class $K_{d,1}$. The definition of Kato class will be given in this chapter. Furthermore, if ν and μ_i are in $K_{d,1-\alpha_0}$, for some $0 < \alpha_0 < 1$, the solution is unique. In this chapter, we will also prove that there is a unique weak solution to the stochastic differential equation:

$$X_t = x + W_t + \int_0^t b(X_s)ds, \quad x \in \mathbb{R}^d, \tag{1.0.2}$$

where $W = \{W_t, t \geq 0\}$ is a d-dimensional standard Brownian motion, b is a measurable

$\mathbb{R}^d$-valued function with $|b| \in K_{d,1}$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. In Section 3.1 of this chapter, we will introduce some definitions and the ideas of proofs briefly. In Section 3.2, we present some potential theory with respect to Brownian motion with singular drift and establish the existence and uniqueness of the weak solution to the elliptic equation (1.0.1) on $\mathbb{R}^d$. Section 3.3 is devoted to obtaining the existence and uniqueness of the weak solution to SDE (1.0.2).

Bass and Chen in [1] extended (1.0.2) further so that b is a measure rather than a function. In other words, we have a bounded variation term A_t such that:

$$X_t = x + W_t + A_t, \tag{1.0.3}$$

where $W = \{W_t, t \geq 0\}$ is a d-dimensional standard Brownian motion and A_t is a measure of Kato class $K_{d,1}$. Bass and Chen showed that there exists a weak solution to (1.0.3) and it is unique in probability (see [1]). The same result was obtained for Lévy processes by Kim and Song in [22]. On the other hand, let us consider a domain $D \subseteq \mathbb{R}^d$ with smooth boundary ∂D, let $n(x)$ be an inward unit normal vector at the boundary $x \in \partial D$. Then one can define a process R_t:

$$R_t = R_0 + W_t - L_t,$$

where $W = \{W_t, t \geq 0\}$ is a d-dimensional standard Brownian motion and L is given by:

$$\begin{aligned} |L|_t &= \int_0^t \mathbf{1}(R_s \in \partial D) d|L|_s, \\ L_t &= \int_0^t n(R_s) d|L|_s. \end{aligned}$$

We can see this process R_t behaves as a d-dimensional Brownian motion inside the domain D. But it is reflected back to the domain D via the normal vector $n(R_t)$ when it hits the boundary ∂D. The existence and uniqueness of such a process R_t have been proved by Lions and Sznitman in [24]. In Chapter 4, we will combine the previous two SDEs and prove the existence and uniqueness of Brownian motion with measure-valued drift and reflection. The Section 4.1 will briefly summarise some previous works and introduce the SDE we are concerned with. We will outline some definitions as well. In Section 4.2, we will construct the transition density function of the reflected Brownian motion with singular drift and derive some useful estimates using the fact that the drift belongs to the Kato class $K_{d,1}$. We then prove the existence and uniqueness in Section

4.3 and 4.4.

Chapter 2

Pathwise Uniqueness and Non-Explosion of SDEs driven by Compensated Poisson Random Measures

2.1 Introduction

In this chapter, we focus on the stochastic differential equations driven by compensated Poisson random measures with non-Lipschitzian coefficients.

Let $(U, \mathcal{F})$ be a measurable space and the stochastic differential equation we are looking at is as follow:

$$X_t = X_0 + \int_0^{t+} \int_U f(X_{s-}, u) \widetilde{N}_p(dsdu), \tag{2.1.1}$$

where X_0 is a given initial value, p is a stationary Poisson point process defined on U. And the process $\widetilde{N}_p(t, du) = N_p(t, du) - \widehat{N}_p(t, du)$ is the compensated Poisson random measure associated with the Poisson point process p, where $\widehat{N}_p(dtdu) = dt\upsilon(du)$, $\upsilon(du)$ is the characteristic measure of p (see [17]).

Suppose $f(x, u)$ is a measurable function $R^d \times U \to R^d$ and assumed to be continuous with respect to x. Under this assumption, there exist weak solutions to (2.1.1) up to lifetime ([19, Chapter 6]). If the coefficient $f(x, u)$ is Lipschitz continuous and satisfies the linear growth, then it has been shown that the SDE (2.1.1) admits a unique strong solution (see [17]). The purpose of this chapter is to show that the non-explosion and pathwise uniqueness still hold for the SDE (2.1.1) under a class of non-Lipschitzian and super-linear growth coefficients. Concrete examples are also provided at the end of each of the following sections.

2.2 Non-Explosion for the stochastic differential equations

In this section, we will show that SDE (2.1.1) does not have explosion under a class of super-linear growth conditions on the coefficient. Consider the stochastic differential equation (2.1.1), we define a stochastic process ξ_t as follow:

$$\xi_t = |X_t|^2.$$

By Itô's formula, we can obtain:

$$\begin{aligned}\xi_t =& |X_0|^2 + \int_0^{t+}\int_U \left[|X_{s-} + f(X_{s-},u)|^2 - |X_{s-}|^2\right]\widetilde{N}_p(dsdu) \\ &+ \int_0^t \int_U \left[|X_s + f(X_s,u)|^2 - |X_s|^2 - 2 < X_s, f(X_s,u) > \right]\upsilon(du)ds. \qquad (2.2.1)\end{aligned}$$

For $M > 0$, define a stopping time τ_M:

$$\tau_M = \inf\{t \geq 0 : \xi_t \geq M\}.$$

Lemma 2.2.1. *Let ρ be a strictly positive C^1-function defined on $[0,+\infty)$. And let Φ be a strictly positive, C^2-function on $[0,+\infty)$ such that:*

$$0 \leq \Phi'(\xi) \leq \frac{C_1\Phi(\xi)}{\xi\rho(\xi)+1} \text{ and } \Phi''(\xi) \leq 0 \text{ where } \xi \in [0,\infty), \qquad (2.2.2)$$

where C_1 is a positive constant. If $f(x,u)$ in (2.1.1) satisfies:

(1). $F(x) := \int_U |f(x,u)|\upsilon(du)$ is bounded on bounded domains.

(2). $\int_U |f(x,u)|^2\upsilon(du) \leq C_2(|x|^2\rho(|x|^2)+1)$, where C_2 is a constant.

Then the following estimate is true:

$$E\left[\Phi(\xi_{t\wedge\tau_M})\right] \leq \Phi(\xi_0)e^{Kt},$$

where K is a constant.

Proof. By Itô's formula and (2.2.1), we have:

$$
\begin{aligned}
&\Phi(\xi_{t\wedge\tau_M})-\Phi(\xi_0)\\
=&\int_0^{t\wedge\tau_M}\Phi'(\xi_s)\int_U\left[|X_s+f(X_s,u)|^2-|X_s|^2-2<X_s,f(X_s,u)>\right]\upsilon(du)ds\\
&+\int_0^{t+\wedge\tau_M}\int_U\left[\Phi\left(|X_{s-}+f(X_{s-},u)|^2\right)-\Phi(\xi_{s-})\right]\widetilde{N}_p(dsdu)\\
&+\int_0^{t\wedge\tau_M}\int_U\left[\Phi\left(|X_s+f(X_s,u)|^2\right)-\Phi(\xi_s)-\Phi'(\xi_s)\left(|X_s+f(X_s,u)|^2-\xi_s\right)\right]\upsilon(du)ds\\
=&I_1(t)+I_2(t)+I_3(t).
\end{aligned}
$$

By the assumption (2.2.2):

$$
\begin{aligned}
&E\left[\int_0^{t\wedge\tau_M}\Phi'(\xi_s)\int_U\left[|X_s+f(X_s,u)|^2-|X_s|^2-2<X_s,f(X_s,u)>\right]\upsilon(du)ds\right]\\
=&E\left[\int_0^t\mathbf{1}(s<\tau_M)\Phi'(\xi_s)\int_U|f(X_s,u)|^2\upsilon(du)ds\right]\\
\le&E\left[\int_0^t\mathbf{1}(s<\tau_M)\Phi(\xi_s)\frac{C_1}{\xi_s\rho(\xi_s)+1}C_2(\xi_s\rho(\xi_s)+1)ds\right]\\
=&C_3E\left[\int_0^t\mathbf{1}(s<\tau_M)\Phi(\xi_s)ds\right].
\end{aligned}
$$

Hence we get:

$$
E[I_1(t)]\le C_3E\left[\int_0^t\mathbf{1}(s<\tau_M)\Phi(\xi_s)ds\right]\le C_3\int_0^tE[\Phi(\xi_{s\wedge\tau_M})]ds.
$$

By the mean-value theorem, we have:

$$
\begin{aligned}
&E\left[\int_0^t\mathbf{1}(s<\tau_M)\int_U\left|\Phi\left(|X_s+f(X_s,u)|^2\right)-\Phi(\xi_s)\right|\upsilon(du)ds\right]\\
\le&E\left[\int_0^t\mathbf{1}(s<\tau_M)\int_U\Phi'\left((1-\alpha_s)\xi_s+\alpha_s|X_s+f(X_s,u)|^2\right)\right.\\
&\left.\cdot\left||X_s+f(X_s,u)|^2-\xi_s\right|\upsilon(du)ds\right],
\end{aligned}
$$

where α_s is a stochastic process taking value on $[0,1]$. In view of (2.2.2), $\Phi'(\xi)$ is a decreasing function. Hence:

$$
\begin{aligned}
& E\left[\int_0^t \mathbf{1}(s<\tau_M)\int_U \left|\Phi\left(|X_s+f(X_s,u)|^2\right)-\Phi(\xi_s)\right|\upsilon(du)ds\right] \\
\leq & E\left[\int_0^t \mathbf{1}(s<\tau_M)\int_U \Phi'((1-\alpha_s)\xi_s)\cdot\left|\left(|X_s+f(X_s,u)|^2-\xi_s\right)\right|\upsilon(du)ds\right] \\
= & E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)\int_U \left||f(X_s,u)|^2+2<X_s,f(X_s,u)>\right|\upsilon(du)ds\right] \\
\leq & E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)\int_U |f(X_s,u)|^2\upsilon(du)ds\right] \\
& +2E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)\int_U |X_s|\cdot|f(X_s,u)|\upsilon(du)ds\right] \\
\leq & C_2E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)(\xi_s\rho(\xi_s)+1)ds\right] \\
& +2E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)\cdot|X_s|\cdot F(X_s)ds\right],
\end{aligned}
$$

since $s<\tau_M$, we have $|X_s|$ and ξ_s are bounded. Then by boundness of function $F(x)$ (property (1) above) together with continuity of function $\rho(x)$ and $\Phi(x)$, we deduce:

$$
E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)(\xi_s\rho(\xi_s)+1)ds\right]<\infty,
$$

$$
E\left[\int_0^t \mathbf{1}(s<\tau_M)\Phi'((1-\alpha_s)\xi_s)\cdot|X_s|\cdot F(X_s)ds\right]<\infty.
$$

Hence we have:

$$
E\left[\int_0^t \mathbf{1}(s<\tau_M)\int_U \left|\Phi\left(|X_s+f(X_s,u)|^2\right)-\Phi(\xi_s)\right|\upsilon(du)ds\right]<\infty,
$$

this implies that $I_2(t)$ is a martingale, so that $E[I_2(t)]=0$. For the term $I_3(t)$, we have:

$$
\begin{aligned}
& E\left[\int_0^t \mathbf{1}(s<\tau_M)\int_U \left[\Phi\left(|X_s+f(X_s,u)|^2\right)\right.\right. \\
& \qquad \left.\left.-\Phi(\xi_s)-\Phi'(\xi_s)\left(|X_s+f(X_s,u)|^2-\xi_s\right)\right]\upsilon(du)ds\right] \\
= & E\left[\int_0^t \mathbf{1}(s<\tau_M)\int_U \frac{1}{2}\Phi''\left(\alpha_s|X_s+f(X_s,u)|^2+(1-\alpha_s)\xi_s\right)\right. \\
& \qquad \left.\cdot\left(|X_s+f(X_s,u)|^2-\xi_s\right)^2\upsilon(du)ds\right],
\end{aligned}
$$

where α_s is a stochastic process that lies in the interval $[0,1]$. Due to $\Phi''(\xi) \leq 0$ on $[0,+\infty)$, we obtain:

$$E[I_3(t)] \leq 0.$$

Combining the above estimates together leads us to obtain:

$$E[\Phi(\xi_{t\wedge\tau_M})] = \Phi(\xi_0) + E[I_1(t)] + E[I_2(t)] + E[I_3(t)] \leq \Phi(\xi_0) + C_3 \int_0^t E[\Phi(\xi_{s\wedge\tau_M})]ds.$$

By Gronwall's inequality, we get:

$$E[\Phi(\xi_{t\wedge\tau_M})] \leq \Phi(\xi_0)e^{C_3 t},$$

as required.

□

If we assume $\upsilon(du) = du$ to be a Lebesgue measure, then $\upsilon(du)$ is σ-finite which makes the measure $\upsilon(du)$ the characteristic measure of a stationary Poisson point process on U (see [17]). Let us consider the function

$$f(x,u) = \sqrt{|x|^2\log(|x|^2)\mathbf{1}_{\{|x|^2\geq 1\}}(x)+1} \cdot (\mathbf{1}_{\{u\geq 0\}}e^{-u} + \mathbf{1}_{\{u<0\}}e^{u}),$$

we can see that condition (1) and (2) in the Lemma 2.2.1 are both satisfied.

With the Lemma 2.2.1, we are now ready to state the main result regarding the non-explosion of the solution to equation (2.1.1).

Theorem 2.2.2. *Let ρ be a strictly positive, C^1-function defined on $[K,+\infty)$ for some large positive constant K satisfying (i) $\int_K^{+\infty} \frac{1}{s\rho(s)+1} = +\infty$ and (ii) $1-\rho(s)-s\rho'(s) \leq 0$. Assume that:*

(1). $F(x) := \int_U |f(x,u)|\upsilon(du)$ is bounded on bounded domains.

(2). For $|x| \geq K$, $\int_U |f(x,u)|^2\upsilon(du) \leq C_2(|x|^2\rho(|x|^2)+1)$, where C_2 is a constant.

Then the solution, X, to the SDE (2.1.1) admits no explosion.

Proof. Let ζ denote the lifetime of the solution X. We will show $P(\zeta = +\infty) = 1$. With no loss of generality, we extend the function ρ to the whole half line $[0,+\infty)$. Define a function $\Phi(\xi)$ as follow:

$$\Phi(\xi) = e^{\int_0^\xi \frac{1}{s\rho(s)+1}ds}$$

It is then obvious that $\Phi(\xi)$ is a C^2-function and we have

$$\Phi'(\xi) = \Phi(\xi) \cdot \frac{1}{\xi\rho(\xi)+1} \quad \text{and} \quad \Phi''(\xi) = \frac{\Phi(\xi)(1-\rho(\xi)-\xi\rho'(\xi))}{(\xi\rho(\xi)+1)^2}.$$

Since $1-\rho(s)-s\rho'(s) \leq 0$, we have $\Phi''(s) \leq 0$. Thus both of the conditions in (2.2.2) are satisfied. Due to the hypothesis (1) and (2), we can now apply Lemma 2.2.1 to obtain:

$$E[\Phi(\xi_{t\wedge\tau_M})] \leq \Phi(\xi_0)e^{Ct},$$

where C is a constant and τ_M is the stopping time defined by:

$$\tau_M = \inf\{t \geq 0 : |\xi_t| \geq M\}.$$

Now we let $M \to +\infty$, by Fatou's lemma, we have:

$$E[\Phi(\xi_{t\wedge\zeta})] \leq \Phi(\xi_0)e^{Ct}. \tag{2.2.3}$$

This implies $P(\zeta = +\infty) = 1$. Indeed, suppose that $P(\zeta < +\infty) > 0$, then for some large enough $T > 0$, the set $P(\{\zeta \leq T\}) > 0$. If we take a fixed $t \geq T$, then by (2.2.3),

$$E[\mathbf{1}(\zeta \leq t)\Phi(\xi_\zeta)] \leq E[\Phi(\xi_{t\wedge\zeta})] \leq \Phi(\xi_0)e^{Ct}. \tag{2.2.4}$$

By the definition of function $\Phi(\xi)$ and the condition on ρ, $\Phi(\xi_\zeta) = \Phi(+\infty) = +\infty$. Then the left hand side of (2.2.4) is infinite while the right hand side is finite, which is impossible. Therefore we can conclude that $P(\zeta = +\infty) = 1$ as required.

□

The function ρ in Theorem 2.2.2 indeed exists.

Example 2.2.3. *Let $\rho(x) = \log(x)\mathbf{1}_{\{x\geq 1\}}(x)$, For $|x| \geq 1$, if $f(x,u)$ in (2.1.1) satisfies (1) and (2) in Theorem (2.2.2), then X_t admits non-explosion property.*

Indeed, for $\rho(s) = \log(s)\mathbf{1}_{\{s\geq 1\}}(s)$, we see that $\rho(s)$ is a C^1-function on $[1,+\infty)$, and $\int_1^{+\infty} \frac{1}{s\rho(s)+1} ds = +\infty$. On the interval $[1,+\infty)$, $1-\rho(s)-s\rho'(s) = -\rho(s) \leq 0$.

2.3 Pathwise Uniqueness

In this section, we are going to prove the pathwise uniqueness of solutions to (2.1.1) for a class of non-Lipschitz coefficients. We assume that the lifetime ζ of the SDE (2.1.1) is infinite, otherwise the uniqueness is up to the lifetime.

Let X_t and Y_t be two solutions to the SDE (2.1.1) with same initial data. Define stochastic processes η_t and ξ_t:

$$\eta_t = X_t - Y_t,$$
$$\xi_t = |\eta_t|^2.$$

Then:
$$\eta_t = X_t - Y_t = \int_0^{t+} \int_U [f(X_{s-},u) - f(Y_{s-},u)]\widetilde{N}_p(duds).$$
By Itô's formula, we obtain:

$$\begin{aligned}\xi_t = &\int_0^{t+} \int_U \left[|\eta_{s-} + f(X_{s-},u) - f(Y_{s-},u)|^2 - |\eta_{s-}|^2\right]\widetilde{N}_p(duds) \\ &+ \int_0^t \int_U \left[|\eta_s + f(X_s,u) - f(Y_s,u)|^2 - |\eta_s|^2 \right. \\ &\qquad \left. - 2 < \eta_s, f(X_s,u) - f(Y_s,u) > \right] \upsilon(du)ds. \end{aligned} \tag{2.3.1}$$

For $c > 0$, we define a stopping time:

$$\tau_c = \inf\{t > 0 : \xi_t \geq c\}.$$

Then we have the following lemma:

Lemma 2.3.1. *Let $r(x)$ be a C^1-function on $(0,+\infty)$ and be strictly positive for $0 < x \leq c_0$, where c_0 is some fixed constant. Suppose $\Phi(\xi)$ is a strictly positive C^2-function on $[0,+\infty)$ satisfying that there exists $C_4, \delta > 0$ such that:*

$$0 \leq \Phi'(\xi) \leq \frac{C_4\Phi(\xi)}{\xi r(\xi) + \delta} \text{ and } \Phi''(\xi) \leq 0 \text{ for } 0 \leq \xi \leq c_0. \tag{2.3.2}$$

Assume:

(1). $\int_U |f(x,u) - f(y,u)|^2 \upsilon(du) \leq C|x-y|^2 r(|x-y|^2)$ *for* $|x-y| \leq c_0$.

(2). $\int_U |f(x,u) - f(y,u)|\upsilon(du) = F(x,y)$ *where the function* $F(x,y)$ *is bounded when* $|x-y|$ *is bounded.*

(3). for every real number $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for $|x-y| \leq \delta_\varepsilon$, we have $|f(x,u) - f(y,u)| \leq \varepsilon$ for all u.

Then there exists a constant $\delta_0 > 0$ such that for any $0 < \delta_1 \leq \delta_0$,

$$E[\Phi(\xi_{t\wedge\tau_{\delta_1}})] \leq \Phi(0)e^{Kt},$$

where K is a constant which is independent of δ_1.

Proof. In view of hypothesis (3), one can find a constant $\delta_0 > 0$ such that:

$$2|f(x,u) - f(y,u)| + 2\delta_0 \leq c_0 \tag{2.3.3}$$

for $|x-y| \leq \delta_0$. Let us consider the stopping time τ_{δ_1} with $0 < \delta_1 \leq \delta_0$. Then by (2.3.1) and Itô's formula, we have:

$$\begin{aligned}
\Phi(\xi_{t\wedge\tau_{\delta_1}}) =& \Phi(0) + \int_0^{t\wedge\tau_{\delta_1}} \int_U \Phi'(\xi_s)\big[|\eta_s + f(X_s,u) - f(Y_s,u)|^2 - |\eta_s|^2 \\
& - 2 < \eta_s, f(X_s,u) - f(Y_s,u) > \big]\upsilon(du)ds \\
& + \int_0^{t+\wedge\tau_{\delta_1}} \int_U \big[\Phi\left(|\eta_{s-} + f(X_{s-},u) - f(Y_{s-},u)|^2\right) - \Phi(\xi_{s-})\big]\widetilde{N}_p(duds) \\
& + \int_0^{t\wedge\tau_{\delta_1}} \int_U \big[\Phi\left(|\eta_s + f(X_s,u) - f(Y_s,u)|^2\right) - \Phi(\xi_s) \\
& - \Phi'(\xi_s)\left(|\eta_s + f(X_s,u) - f(Y_s,u)|^2 - \xi_s\right)\big]\upsilon(du)ds \\
=& \Phi(0) + I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{2.3.4}$$

For $I_1(t)$, we have:

$$\begin{aligned}
& E\left[\int_0^t \mathbf{1}(s < \tau_{\delta_1}) \int_U \Phi'(\xi_s)\big[|\eta_s + f(X_s,u) - f(Y_s,u)|^2 - |\eta_s|^2 \right. \\
& \quad \left. - 2 < \eta_s, f(X_s,u) - f(Y_s,u) > \big]\upsilon(du)ds\right] \\
\leq & E\left[\int_0^t \mathbf{1}(s < \tau_{\delta_1})\Phi(\xi_s) \cdot \frac{1}{\xi_s r(\xi_s) + \delta} \cdot C_4 \cdot C \cdot \xi_s r(\xi_s) ds\right] \leq C_5 E\left[\int_0^t \mathbf{1}(s < \tau_{\delta_1})\Phi(\xi_s) ds\right].
\end{aligned}$$

Hence,

$$E[I_1(t)] \leq C_5 E\left[\int_0^t \Phi(\xi_{s\wedge\tau_{\delta_1}}) ds\right] = C_5 \int_0^t E[\Phi(\xi_{s\wedge\tau_{\delta_1}})] ds.$$

To see that $I_2(t)$ is a martingale, we will need to show:

$$E\left[\int_0^{t+\wedge\tau_{\delta_1}}\int_U \left|\Phi\left(|\eta_s+f(X_s,u)-f(Y_s,u)|^2\right)-\Phi(\xi_s)\right|\upsilon(du)ds\right]<+\infty.$$

By the mean-value theorem,

$$\begin{aligned}
&E\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\int_U \left|\Phi\left(|\eta_s+f(X_s,u)-f(Y_s,u)|^2\right)-\Phi(\xi_s)\right|\upsilon(du)ds\right]\\
\leq &E\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\int_U \left|\Phi'\left(\alpha_s|\eta_s+f(X_s,u)-f(Y_s,u)|^2+(1-\alpha_s)\xi_s\right)\right|\right.\\
&\left.\cdot\left||\eta_s+f(X_s,u)-f(Y_s,u)|^2-\xi_s\right|\upsilon(du)ds\right],
\end{aligned}$$

where α_s is a stochastic process taking value in $[0,1]$. In view of (2.3.3), the inequality:

$$\alpha_s|\eta_s+f(X_s,u)-f(Y_s,u)|^2+(1-\alpha_s)\xi_s\leq 2\alpha_s|f(X_s,u)-f(Y_s,u)|^2+(1+\alpha_s)\xi_s, \tag{2.3.5}$$

and the fact that Φ' is a positive and decreasing function lead us to:

$$\begin{aligned}
&E\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\int_U \left|\Phi\left(|\eta_s+f(X_s,u)-f(Y_s,u)|^2\right)-\Phi(\xi_s)\right|\upsilon(du)ds\right]\\
\leq &E\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\Phi'((1-\alpha_s)\xi_s)\int_U |f(X_s,u)-f(Y_s,u)|^2\upsilon(du)ds\right]\\
&+2E\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\Phi'((1-\alpha_s)\xi_s)|\eta_s|\int_U |f(X_s,u)-f(Y_s,u)|\upsilon(du)ds\right]\\
\leq &CE\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\Phi'((1-\alpha_s)\xi_s)\xi_s r(\xi_s)ds\right]\\
&+2E\left[\int_0^t \mathbf{1}(s<\tau_{\delta_1})\Phi'((1-\alpha_s)\xi_s)|\eta_s|F(X_s,Y_s)ds\right]<\infty.
\end{aligned}$$

Hence $I_2(t)$ is a martingale. For $I_3(t)$, Taylor expansion gives

$$\begin{aligned}
&\Phi\left(|\eta_s+f(X_s,u)-f(Y_s,u)|^2\right)-\Phi(\xi_s)\\
&-\Phi'(\xi_s)\left(|\eta_s+f(X_s,u)-f(Y_s,u)|^2-\xi_s\right)\\
=&\Phi''\left(\alpha_s|\eta_s+f(X_s,u)-f(Y_s,u)|^2+(1-\alpha_s)\xi_s\right)\\
&\cdot\left(|\eta_s+f(X_s,u)-f(Y_s,u)|^2-\xi_s\right)^2,
\end{aligned}$$

where α_s is a stochastic process taking values in $[0,1]$. In view of (2.3.3) and the inequality (2.3.5), we conclude:

$$\Phi''(\alpha_s|\eta_s + f(X_s,u) - f(Y_s,u)|^2 + (1-\alpha_s)\xi_s) \le 0,$$

when $s < \tau_{\delta_1}$. Thus $E[I_3(t)] \le 0$. Hence, taking expectation in (2.3.4) implies:

$$E[\Phi(\xi_{t\wedge\tau_{\delta_1}})] \le \Phi(0) + C_5 \int_0^t E[\Phi(\xi_{s\wedge\tau_{\delta_1}})]ds.$$

By Gronwall's theorem, we are able to conclude

$$E[\Phi(\xi_{t\wedge\tau_{\delta_1}})] \le \Phi(0)e^{C_5 t}$$

as claimed.

□

Now we can state the main result of this section.

Theorem 2.3.2. *Let r be a C^1-function on $(0,+\infty)$ satisfying (i) $\lim_{s\to 0} r(s) = +\infty$, (ii) $\int_0^a \frac{1}{sr(s)}ds = +\infty$ for any $a > 0$ and (iii) $1 - r(\xi) - \xi r'(\xi) \le 0$ on $(0,c_0]$ for some small enough positive constant c_0. Assume:*

(1). For $|x-y| \le c_0$, $\int_U |f(x,u) - f(y,u)|^2 \upsilon(du) \le C|x-y|^2 r(|x-y|^2)$.

(2). $\int_U |f(x,u) - f(y,u)|\upsilon(du) = F(x,y)$ where the function $F(x,y)$ is bounded when $|x-y|$ is bounded.

(3). for every real number $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-y| \le \delta$, we have $|f(x,u) - f(y,u)| \le \varepsilon$ for all $u \in U$.

Then the pathwise uniqueness of the solutions to (2.1.1) holds.

Proof. We define a function $\Phi_\delta(\xi)$ with some positive constant δ as:

$$\Phi_\delta(\xi) = e^{\int_0^\xi \frac{1}{sr(s)+\delta}ds}.$$

Then:

$$\Phi'_\delta(\xi) = \Phi_\delta(\xi) \cdot \frac{1}{\xi r(\xi) + \delta} \text{ and } \Phi''_\delta(\xi) = \Phi_\delta(\xi)\frac{1 - r(\xi) - \xi r'(\xi)}{(\xi r(\xi) + \delta)^2}.$$

Since $\lim_{\xi\to 0} r(\xi) = +\infty$, one can assume that $r(\xi) > 0$ when $\xi \in (0, c_0]$. Hence conditions in (2.3.2) are both satisfied. Recall that X and Y are two solutions to equation (2.1.1) and $\xi_t = |X_t - Y_t|^2$. Due to hypothesis (1), (2) and (3) on function $f(x,u)$, we can now apply the Lemma 2.3.1 to conclude that there exists a small enough positive constant δ_0 such that for $0 < \delta_1 \le \delta_0$:

$$E[\Phi_\delta(\xi_{t\wedge\tau_{\delta_1}})] \le \Phi_\delta(0)e^{Ct} = e^{Ct},$$

where C is a constant which is independent of δ_1. Now let $\delta \to 0$, by monotone convergence:

$$E\left[e^{\int_0^{\xi_{t\wedge\tau_{\delta_1}}} \frac{1}{sr(s)} ds}\right] \le e^{Ct}. \tag{2.3.6}$$

In view of the condition (ii) on the function $r(s)$, (2.3.6) implies that:

$$\xi_{t\wedge\tau_{\delta_1}} = 0,$$

for a given t almost surely. Recall the definition of the stopping time τ_{δ_1}:

$$\tau_{\delta_1} = \inf\{t > 0 : \xi_t \ge \delta_1\}.$$

If $P(\tau_{\delta_1} < +\infty) > 0$, then there exists some large enough constants $T > 0$ such that $P(\tau_{\delta_1} \le T) > 0$. Then on subset $\{\omega : \tau_{\delta_1} \le T\}$ with positive measure, one can conclude $\xi_{\tau_{\delta_1}} = 0$, which is obviously a contradiction to the definition of stopping time τ_{δ_1}. Hence $\tau_{\delta_1} = +\infty$ almost surely. Therefore for any given t, $\xi_t = 0$ almost surely. By the right continuity, two solutions X_t and Y_t are indistinguishable as required.

□

Example 2.3.3. *Let $r(s) = \log(\frac{1}{s})$. If the function $f(x,u)$ in (2.1.1) satisfies (1), (2) and (3) in Theorem (2.3.2), then we can gurantee the pathwise uniqueness of solutions.*

Indeed, for the function $r(s) = \log(\frac{1}{s})$, we can see that $\lim_{s\to 0} r(s) = +\infty$ and $\int_0^a \frac{1}{sr(s)} ds = +\infty$ for any $a > 0$. Moreover, $1 - r(s) - sr'(s) = 2 - \log(\frac{1}{s})$, if c_0 takes the value of e^{-2}, then $1 - r(s) - sr'(s) \le 0$ on $(0, c_0]$.

Chapter 3

Elliptic equations associated with Brownian motion with singular drift

3.1 Introduction

Throughout this chapter, we assume $d \geq 3$. Notice that our main results also hold for $d = 2, 1$. We can show this by following the similar method when $d \geq 3$ but with different definition of Kato class and transition density function corresponding to different dimension. Let us consider the following elliptic equation with measure-valued coefficients on $\mathbb{R}^d$:

$$\lambda u - Lu = \nu, \tag{3.1.1}$$

where $\lambda > 0$, L is the infinitesimal generator of the Brownian motion with singular drift $\mu = (\mu_1, \mu_2, ..., \mu_d)$, and ν, μ_i, $1 \leq i \leq d$, are appropriate Borel measures on $\mathbb{R}^d$. Formally

$$L := \frac{1}{2}\Delta + \langle \mu, \nabla \rangle.$$

The first goal of this chapter is to show that if λ is a large enough positive constant, and ν, $\{\mu_i, 1 \leq i \leq d\}$ belong to the Kato class $K_{d,1}$ (See Definition 3.1.2 below for the precise definition of the Kato class), then the elliptic equation (3.1.1) on $\mathbb{R}^d$ has at least one weak solution, furthermore, if ν, $\{\mu_i, 1 \leq i \leq d\}$ belong to the Kato class $K_{d,1-\alpha_0}$ for some $0 < \alpha_0 < 1$, then the solution is unique.

Our second purpose is to show that there is a unique weak solution to the stochastic differential equation (SDE):

$$X_t = x + W_t + \int_0^t b(X_s)ds, \; x \in \mathbb{R}^d, \tag{3.1.2}$$

here $W = \{W_t, t \geq 0\}$ is a d-dimensional standard Brownian motion, b is a measurable $\mathbb{R}^d$-valued function with $|b| \in K_{d,1}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Recall that

Definition 3.1.1. *We say that the SDE (3.1.2) has a weak solution, if there exists a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ on which a d-dimensional standard Brownian motion W is defined so that (3.1.2) holds, where X_t is a continuous $\mathcal{F}_t$-adapted process such that*

$$\int_0^t |b(X_s)|ds < \infty,\ \forall t > 0,\ P\text{-a.e.}. \tag{3.1.3}$$

And the Kato class is defined as follows:

Definition 3.1.2. *A signed Radon measure π on $\mathbb{R}^d$ is said to be in the Kato class $K_{d,\alpha}$ $(0 < \alpha \leq 2)$ if*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|\leq r} \frac{|\pi|(dy)}{|x-y|^{d-\alpha}} = 0,$$

where $|\pi|$ is the total variation measure of π. A measurable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be in the Kato class $K_{d,\alpha}$ if $|f(x)|dx \in K_{d,\alpha}$.

To obtain the existence of the weak solutions of the elliptic equation (3.1.1) on $\mathbb{R}^d$, we first establish some new estimates on the regularity of the Kato measure. Then we approximate the coefficients $\{\mu_i, 1 \leq i \leq d\}$, ν by smooth functions through a sequence of mollifiers, and consider the elliptic equations on $\mathbb{R}^d$ corresponding to the smooth functions. We will show that the solutions and the gradients of the solutions of the approximating problems converge uniformly on compact subsets to the corresponding solution and its gradient using the new estimates established in this chapter.

In order to obtain the existence of weak solutions to SDE (3.1.2), we need the following definition and result from [1].

We fix a non-negative function $\varphi \in C_0^\infty(B(0,1))$ with $\int_{B(0,1)} \varphi(x)dx = 1$, where $B(0,1) := \{x \in \mathbb{R}^d : |x| < 1\}$. For any positive integer n, we put $\varphi_n(x) := 2^{nd}\varphi(2^n x)$. Given measures $\mu_i \in K_{d,1}$, for $1 \leq i \leq d$, and set $\mu := (\mu_1, \mu_2, ..., \mu_d)$, we introduce

$$G_i^n(x) := \int_{\mathbb{R}^d} \varphi_n(x-y)\mu_i(dy),$$

and $G^n := (G_1^n, G_2^n, \cdots, G_d^n)$. We recall the following definition from [1],

Definition 3.1.3. *By calling $\{X, \mathbf{P}_x, x \in \mathbb{R}^d\}$ a Brownian motion with singular drift μ, we mean that $\{\mathbf{P}_x, x \in \mathbb{R}^d\}$ is a family of probability measures on $C([0,\infty), \mathbb{R}^d)$ such* *that under each $\mathbf{P}_x$, the coordinator process X has the decomposition*

$$X_t = x + W_t + A_t,$$

where

(a) $A_t = \lim_{n\to\infty} \int_0^t G^n(X_s)ds$ uniformly in t over finite intervals, where the convergence is in probability;

(b) there exists a subsequence $\{n_k\}_{k\geq 1}$ such that for every $t > 0$,

$$\sup_{k\geq 1} \int_0^t |G^{n_k}(X_s)|ds < \infty, \quad P_x - a.e.$$

(c) W is a d-dimensional standard Brownian motion starting from the origin.

When the drift μ is in the Kato class $K_{d,1}$, the existence and uniqueness of the Brownian motion with drift μ were shown in [1]. In this chapter, we will use this result to prove the existence of weak solutions to (3.1.2). That is, when the drift $\mu(dx) = b(x)dx$ with $|b| \in K_{d,1}$, we can find a Brownian motion with drift μ, denoted by $\{X, \mathbf{P}_x, x \in \mathbb{R}^d\}$, such that $\int_0^t b(X_s)ds = A_t = \lim_{n\to\infty} \int_0^t G^n(X_s)ds$, and it is easy to see that X is a weak solution to (3.1.2). However, for any weak solution Y to (3.1.2), we do not know whether $\int_0^t b(Y_s)ds = \lim_{n\to\infty} \int_0^t G^n(Y_s)ds$. We can not obtain the uniqueness of the SDE (3.1.2) from the result in [1].

In order to prove the uniqueness of weak solutions to (3.1.2), we need some new ideas. Recently in [5], the authors proved the uniqueness of the weak solution to SDE driven by symmetric α-stable process, and their methods can also apply to the Brownian motion case. In this chapter, we will give another proof of the existence and uniqueness of the weak solution to (3.1.2) by using Zvonkin transformation (introduced in [44]), which has an independent interest and is simpler.

As we shall see, with the aid of the regularity of the weak solution u to the elliptic equation (3.1.1), we can easily establish the Zvonkin transformation of the SDE (3.1.2). On the other hand, Zvonkin transformation is a popular and effective method to solve the SDEs with singular drift by transforming the original SDEs into some new SDEs without drift, for instance, see [23, 42, 44]. Hence, we will adopt this approach to prove the uniqueness of weak solution to (3.1.2). Particularly, our idea is as follows. Let v be the weak solution to the elliptic equation (3.1.1) on $\mathbb{R}^d$ under the special case that $\mu(dx) = b(x)dx$ and set $F(x) := v(x) - x$. If X is a weak solution to the

SDE (3.1.2), we will establish the generalized Itô's formula of $F(X_t)$, which satisfies another SDE with regular drift. Since the new SDE has a unique weak solution and F is a homeomorphism on $\mathbb{R}^d$ for sufficiently large λ, we obtain the uniqueness of the weak solution to the SDE (3.1.2).

We close the introduction by mentioning some conventions used throughout the chapter. We will use $\{X, \mathbf{P}_x, x \in \mathbb{R}^d\}$ to denote the weak solution given in the Definition 3.1.3. Let $\langle \cdot, \cdot \rangle$ and $B(x,r)$ denote the inner product in $\mathbb{R}^d$ and the ball centered at x with radius r respectively. Let $d(\cdot,\cdot)$ stand for the distance in $\mathbb{R}^d$. For a $d \times d$-matrix $A := (a_{ij})_{1 \le i,j \le d}$, define $\|A\| := \sup_{1 \le i \le d} |a_{ij}|$. Let the letter C with or without subscripts stand for an unimportant positive constant, whose value may change in different places.

3.2 Elliptic equations associated with Brownian motion with drift

In this section, we will show that there exists a unique weak solution to the elliptic equation (3.1.1). Keep in mind that we assume $d \ge 3$ throughout this chapter.

Firstly we give the definition of the weak solution in the way below:

Definition 3.2.1. *We say that u is a weak solution of equation (3.1.1) if u belongs to $C^1(\mathbb{R}^d)$ and for every $\phi \in C_0^1(\mathbb{R}^d)$, where $C_0^1(\mathbb{R}^d)$ denotes a class of 1st-order differentiable functions that vanish at infinity,*

$$\begin{aligned} &\lambda \int_{\mathbb{R}^d} u(x)\phi(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla u(x), \nabla \phi(x)\rangle dx \\ = \;& \int_{\mathbb{R}^d} \phi(x)\langle \nabla u(x), \mu(dx)\rangle + \int_{\mathbb{R}^d} \phi(x)\nu(dx). \end{aligned}$$

Recall that $\mu = (\mu_1, \mu_2, ..., \mu_d)$, we have the following heat kernel estimates from Theorem 3.14 in [21] and Proposition 3.1 in [37].

Theorem 3.2.2. *Assume $\mu_i \in K_{d,1}$ for $1 \le i \le d$, then the Brownian motion with drift μ, $X_t, t \ge 0$ admits a transition density $p(t,x,y)$ which is jointly continuous on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. And there exist positive constants $M_i, i = 1, ..., 6$, such that*

$$M_1 e^{-M_2 t} t^{-d/2} e^{-M_3 |x-y|^2/t} \le p(t,x,y) \le M_4 e^{M_5 t} t^{-d/2} e^{-M_6 |x-y|^2/t}, \tag{3.2.1}$$

$$|\nabla_x p(t,x,y)| \le M_4 e^{M_5 t} t^{-(d+1)/2} e^{-M_6 |x-y|^2/t}. \tag{3.2.2}$$

Furthermore, if there exists a constant $0 < \alpha_0 < 1$ *such that* $\mu_i \in K_{d,1-\alpha_0}$ *for* $1 \le i \le d$, *then there exist positive constants* M_i, $i = 7,8,9$, *such that for all* $t > 0$, $x, x', y \in \mathbb{R}^d$ *and* $1 \le i \le d$,

$$|\frac{\partial p(t,x,y)}{\partial x_i} - \frac{\partial p(t,x',y)}{\partial x_i}| \le M_7|x-x'|^{\alpha_0} e^{M_8 t} t^{-\frac{d+1+\alpha_0}{2}} (e^{-\frac{M_9|x-y|^2}{t}} + e^{-\frac{M_9|x'-y|^2}{t}}). \quad (3.2.3)$$

Now we show the following lemma, which plays an important role in proving the results on the regularity of the weak solution of equation (3.1.1). In fact, in the other part of this chapter, we only use the case $\nu \in K_{d,1}$ and $\nu \in K_{d,1-\alpha_0}$, $0 < \alpha_0 < 1$. Since $K_{d,\alpha} \subset K_{d,\beta}$ for $0 \le \alpha < \beta$, indeed, if $\pi \in K_{d,\alpha}$, then we have $\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le r} \frac{|\pi|(dy)}{|x-y|^{d-\alpha}} = 0$, hence there exists a postive constant $c_0 := \beta - \alpha$ such that

$$\begin{aligned}
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le r} \frac{1}{|x-y|^{d-\beta}} \cdot |\pi|(dy) &= \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le r} \frac{1}{|x-y|^{d-\alpha-c_0}} \cdot |\pi|(dy) \\
&= \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le r} \frac{1}{|x-y|^{d-\alpha}} \cdot |x-y|^{c_0} \cdot |\pi|(dy) \\
&\le \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \le r} \frac{1}{|x-y|^{d-\alpha}} \cdot |\pi|(dy) \cdot r^{c_0} \\
&= 0,
\end{aligned}$$

the following lemma implies more information than what our main results need.

Lemma 3.2.3. *Assume* $\nu \in K_{d,2}$. *We have*
(i) if $r > 0$, *then*

$$\lim_{\delta \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\{|x-y| \ge r\}} e^{-\delta|x-y|^2} |\nu|(dy) = 0. \quad (3.2.4)$$

(ii) if $r \ge 3$, *then for any positive constant* δ,

$$\sup_{x \in \mathbb{R}^d} \int_{\{|x-y| \ge r\}} e^{-\delta|x-y|^2} |\nu|(dy) \le M(\nu) \sum_{k \in Z^d : |k| \ge 2r-2} e^{-\frac{\delta}{16}|k|^2}, \quad (3.2.5)$$

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\delta|x-y|^2} |\nu|(dy) \le M(\nu) e^{\delta} (\sum_{m=-\infty}^{\infty} e^{-\frac{\delta}{8}m^2})^d, \quad (3.2.6)$$

where $M(\nu) := \sup_{z \in \mathbb{R}^d} |\nu|(B(z,1))$.

Proof. For any $\delta, r > 0$, take an integer $N > 0$ such that $\frac{3}{N} \le r$. Then we have

$$\begin{aligned}
&\int_{\{|x-y|\ge r\}} e^{-\delta|x-y|^2}|\nu|(dy) \\
\le\ & \sum_{k\in Z^d}\int_{\{y-x\in B(\frac{k}{2N},\frac{1}{N})\}\cap\{|x-y|\ge r\}} e^{-\delta|x-y|^2}|\nu|(dy) \\
=\ & \sum_{k\in Z^d:|k|\ge 2Nr-2}\int_{\{y-x\in B(\frac{k}{2N},\frac{1}{N})\}\cap\{|x-y|\ge r\}} e^{-\delta|x-y|^2}|\nu|(dy) \\
\le\ & \sum_{k\in Z^d:|k|\ge 2Nr-2}\int_{\{|y-x-\frac{k}{2N}|\le\frac{1}{N}\}} e^{-\delta[\frac{1}{2}|\frac{k}{2N}|^2-|y-x-\frac{k}{2N}|^2]}|\nu|(dy) \\
\le\ & e^{\frac{\delta}{N^2}}\sum_{k\in Z^d:|k|\ge 2Nr-2} e^{-\delta\frac{1}{2}|\frac{k}{2N}|^2}\int_{\{|y-x-\frac{k}{2N}|\le\frac{1}{N}\}}|\nu|(dy) \\
\le\ & \sup_{z\in\mathbb{R}^d}|\nu|(B(z,\frac{1}{N}))\sum_{k\in Z^d:|k|\ge 2Nr-2} e^{-\delta\frac{1}{4}|\frac{k}{2N}|^2}e^{-\delta\frac{1}{4}|\frac{k}{2N}|^2+\frac{\delta}{N^2}} \\
\le\ & \sup_{z\in\mathbb{R}^d}|\nu|(B(z,\frac{1}{N}))\sum_{k\in Z^d:|k|\ge 2Nr-2} e^{-\delta\frac{1}{4}|\frac{k}{2N}|^2}.
\end{aligned} \tag{3.2.7}$$

Hence (3.2.5) follows from (3.2.7) with $N = 1$ and $M(\nu) := \sup_{z\in\mathbb{R}^d}|\nu|(B(z,1))$.

By (3.2.7), we have

$$\begin{aligned}
&\lim_{\delta\to\infty}\sup_{x\in\mathbb{R}^d}\int_{\{|x-y|\ge r\}} e^{-\delta|x-y|^2}|\nu|(dy) \\
\le\ & \sup_{z\in\mathbb{R}^d}|\nu|(B(z,\frac{1}{N}))\lim_{\delta\to\infty}\sum_{k\in Z^d} e^{-\delta\frac{1}{4}|\frac{k}{2N}|^2} \\
=\ & \sup_{z\in\mathbb{R}^d}|\nu|(B(z,\frac{1}{N}))\lim_{\delta\to\infty}(\sum_{m=-\infty}^{\infty} e^{-\frac{\delta}{16N^2}m^2})^d = 0.
\end{aligned}$$

This proves (3.2.4).

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{-\delta|x-y|^2}|\nu|(dy) \le\ & \sum_{k\in Z^d}\int_{\{y-x\in B(\frac{k}{2},1)\}} e^{-\delta|x-y|^2}|\nu|(dy) \\
\le\ & \sum_{k\in Z^d}\int_{\{|y-x-\frac{k}{2}|\le 1\}} e^{-\delta[\frac{1}{2}|\frac{k}{2}|^2-|y-x-\frac{k}{2}|^2]}|\nu|(dy) \\
\le\ & e^{\delta}\sum_{k\in Z^d} e^{-\delta\frac{1}{2}|\frac{k}{2}|^2}\int_{\{|y-x-\frac{k}{2}|\le 1\}}|\nu|(dy) \\
\le\ & M(\nu)e^{\delta}(\sum_{m=-\infty}^{\infty} e^{-\frac{\delta}{8}m^2})^d,
\end{aligned}$$

hence (3.2.6) holds.

□

Furthermore, we are able to see:

Corollary 3.2.4. *Let* $\nu \in K_{d,\alpha}$ *for some* $0 < \alpha \leq 2$. *For any fixed constant* $\delta > 0$, *we have*

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{|y-x|^{d-\alpha}}e^{-\delta|x-y|^2}|\nu|(dy) < \infty. \tag{3.2.8}$$

Proof. Splitting the integral over $\mathbb{R}^d$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d}\frac{1}{|y-x|^{d-\alpha}}e^{-\delta|x-y|^2}|\nu|(dy) \\ \leq\ & \int_{\{|y-x|\leq 1\}}\frac{1}{|y-x|^{d-\alpha}}|\nu|(dy) + \int_{\{|y-x|>1\}}e^{-\delta|x-y|^2}|\nu|(dy). \end{aligned}$$

Equation (3.2.8) now follows Lemma 3.2.3 and the definition of $K_{d,\alpha}$.

□

For $\lambda > 0$, define

$$u_\nu(x) := \int_0^\infty e^{-\lambda t}dt\int_{\mathbb{R}^d}p(t,x,y)\nu(dy).$$

In the following, we will show that u_ν is a weak solution to the elliptic equation (3.1.1). For this purpose, we will establish some results on the regularity of the function u_ν.

Firstly, we will bound $|\nabla u_\nu|_\infty$ in terms of the parameter λ. We have:

Proposition 3.2.5. *Assume* $\nu, \mu_i \in K_{d,1}$ *for* $1 \leq i \leq d$ *and* $\lambda > M_5$. *Then we have* $u_\nu \in C_b^1(\mathbb{R}^d)$ *and*

$$\lim_{\lambda\to\infty}\sup_{x\in\mathbb{R}^d}|\nabla u_\nu(x)| = 0, \tag{3.2.9}$$

Proof. Without lose of generality, we assume ν is a positive measure. By Theorem 3.2.2, it is easy to see that $u_\nu \in C_b^1(\mathbb{R}^d)$. Now we show (3.2.9). For any $r > 0$ and

$\lambda > M_5$, in view of (3.2.2) we have

$$\begin{aligned}
|\nabla u_\nu(x)| &= |\int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} \nabla_x p(t,x,y)\nu(dy)| \\
&\le M_4 \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} e^{M_5 t} t^{-(d+1)/2} e^{-M_6|x-y|^2/t} \nu(dy) \\
&= M_4 \int_{\mathbb{R}^d} \nu(dy) \int_0^\infty e^{-(\lambda - M_5)t} t^{-(d+1)/2} e^{-M_6|x-y|^2/t} dt \\
&= M_4 \int_{\mathbb{R}^d} \frac{\nu(dy)}{|x-y|^{d-1}} \int_0^\infty e^{-(\lambda-M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
&= M_4 \int_{|x-y|\le r} \frac{\nu(dy)}{|x-y|^{d-1}} \int_0^\infty e^{-(\lambda-M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
&\quad + M_4 \int_{|x-y|> r} \frac{\nu(dy)}{|x-y|^{d-1}} \int_0^\infty e^{-(\lambda-M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
&:= J_1(x) + J_2(x). \qquad (3.2.10)
\end{aligned}$$

For the term J_1, we have

$$\begin{aligned}
J_1(x) &= M_4 \int_{|x-y|\le r} \frac{\nu(dy)}{|x-y|^{d-1}} \int_0^\infty e^{-(\lambda-M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
&\le M_4 \int_{|x-y|\le r} \frac{\nu(dy)}{|x-y|^{d-1}} \int_0^\infty u^{(d+1)/2-2} e^{-M_6 u} du \\
&\le C \int_{|x-y|\le r} \frac{\nu(dy)}{|x-y|^{d-1}}. \qquad (3.2.11)
\end{aligned}$$

For the term J_2,

$$\begin{aligned}
J_2(x) &= M_4 \int_{|x-y|> r} \frac{\nu(dy)}{|x-y|^{d-1}} \int_0^1 e^{-(\lambda-M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
&\quad + M_4 \int_{|x-y|> r} \frac{\nu(dy)}{|x-y|^{d-1}} \int_1^\infty e^{-(\lambda-M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
&:= J_3(x) + J_4(x). \qquad (3.2.12)
\end{aligned}$$

Take some constant $\delta > 0$ such that $\lambda > M_5 + \delta$, in view of (3.2.6),

$$\begin{aligned}
J_3(x) &\le C_r \int_{|x-y|>r} e^{-(\lambda-M_5)|x-y|^2} \nu(dy) \int_0^1 u^{(d+1)/2-2} e^{-M_6 u} du \\
&\le C_r e^{-(\lambda-M_5-\delta)r^2} \int_{\mathbb{R}^d} e^{-\delta|x-y|^2} \nu(dy) \\
&\le C_r e^{-(\lambda-M_5-\delta)r^2}. \qquad (3.2.13)
\end{aligned}$$

For the term J_4, by Lemma 3.2.3 and monotone convergence theorem, we have

$$\begin{aligned} & \lim_{\lambda\to\infty} \sup_{x\in\mathbb{R}^d} J_4(x) \\ \le\ & M_4 \frac{1}{r^{d-1}} \int_1^\infty u^{(d+1)/2-2} e^{-M_6 u} du \lim_{\lambda\to\infty} \sup_{x\in\mathbb{R}^d} \int_{|x-y|>r} e^{-(\lambda-M_5)|x-y|^2/u} \nu(dy) = 0. \end{aligned} \tag{3.2.14}$$

Putting (3.2.10)–(3.2.14) together, we obtain that

$$\lim_{\lambda\to\infty} \sup_{x\in\mathbb{R}^d} |\nabla u_\nu(x)| \le C \int_{|x-y|\le r} \frac{\nu(dy)}{|x-y|^{d-1}}.$$

Since r is arbitrary and $\nu \in K_{d,1}$, hence (3.2.9) is proved.

□

Next we will construct a sequence of smooth approximations for the function u_ν. Let $p_n(t,x,y)$ denote the transition density of the diffusion process determined by the SDE:

$$dX_t^n = dW_t + G^n(X_t^n)dt,\ X_0^n = x \in \mathbb{R}^d.$$

It was proved in [21] that $p_n(t,x,y) \to p(t,x,y)$, $\nabla_x p_n(t,x,y) \to \nabla_x p(t,x,y)$ uniformly on compact subsets of $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Let φ_n be given in Section 3.1. Define

$$\nu^n(x) := \int_{\mathbb{R}^d} \varphi_n(x-y)\nu(dy), \quad u^n(x) := \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} p_n(t,x,y)\nu^n(y)dy.$$

Due to the smoothness of the drift G^n, u^n is smooth. We will show that $u^n, \nabla u^n$ converge respectively to u_ν and ∇u_ν. In order to do so, the following lemmas are needed.

Lemma 3.2.6. *Assume $\nu, \mu_i \in K_{d,1}$ for $1 \le i \le d$ and $\lambda > M_5$. Then the following holds.* *(i)*

$$\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^d} \sup_{n\ge 1} \Big| \int_0^\varepsilon e^{-\lambda t} dt \int_{\mathbb{R}^d} \nabla_x p_n(t,x,y)\nu^n(y)dy \Big| = 0,$$

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{n\ge 1} \Big| \int_T^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} \nabla_x p_n(t,x,y)\nu^n(y)dy \Big| = 0.$$

(ii)

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^d}\Big|\int_0^{\varepsilon} e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)\nu(dy)\Big| = 0, \tag{3.2.15}$$

$$\lim_{T\to\infty}\sup_{x\in\mathbb{R}^d}\Big|\int_T^{\infty} e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla p(t,x,y)\nu(dy)\Big| = 0. \tag{3.2.16}$$

Proof. Let us prove (ii). (i) can be proved similarly using the fact

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\sup_{n\geq 1}\int_{\{|y-x|\leq r\}}\frac{1}{|y-x|^{d-1}}|\nu^n(y)|dy = 0.$$

We first prove that (3.2.15) holds. Without lose of generality, we assume ν is a positive measure. In view of (3.2.2), for any $\varepsilon \leq 1$ and $r > 0$, we have

$$\begin{aligned}
&\Big|\int_0^{\varepsilon} e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)\nu(dy)\Big| \\
\leq\ & M_4\int_{\mathbb{R}^d}\nu(dy)\int_0^{\varepsilon} e^{-(\lambda-M_5)t}t^{-(d+1)/2}e^{-M_6|x-y|^2/t}dt \\
\leq\ & M_4\int_{\mathbb{R}^d}e^{-M_6|x-y|^2/2}\nu(dy)\int_0^{\varepsilon} e^{-(\lambda-M_5)t}t^{-(d+1)/2}e^{-M_6|x-y|^2/2t}dt \\
\leq\ & M_4\int_{\mathbb{R}^d}e^{-M_6|x-y|^2/2}\frac{1}{|y-x|^{d-1}}\nu(dy)\int_{\frac{|y-x|^2}{\varepsilon}}^{\infty} u^{\frac{d+1}{2}-2}e^{-\frac{M_6}{2}u}du \\
\leq\ & M_4\int_{\{|y-x|\leq r\}}\frac{1}{|y-x|^{d-1}}\nu(dy)\int_0^{\infty} u^{\frac{d+1}{2}-2}e^{-\frac{M_6}{2}u}du \\
&+M_4\int_{\{|y-x|>r\}}e^{-M_6|x-y|^2/2}\frac{1}{|y-x|^{d-1}}\nu(dy)\int_{\frac{r^2}{\varepsilon}}^{\infty} u^{\frac{d+1}{2}-2}e^{-\frac{M_6}{2}u}du.
\end{aligned} \tag{3.2.17}$$

For any positive constant $\delta > 0$, we first choose r sufficiently small so that

$$M_4\sup_{x\in\mathbb{R}^d}\int_{\{|y-x|\leq r\}}\frac{1}{|y-x|^{d-1}}\nu(dy)\int_0^{\infty} u^{\frac{d+1}{2}-2}e^{-\frac{M_6}{2}u}du \leq \delta. \tag{3.2.18}$$

For the fixed number r, in view of Corollary 3.2.4, we see that

$$\lim_{\varepsilon\to 0}M_4\sup_{x\in\mathbb{R}^d}\int_{\{|y-x|>r\}}e^{-M_6|x-y|^2/2}\frac{1}{|y-x|^{d-1}}\nu(dy)\int_{\frac{r^2}{\varepsilon}}^{\infty} u^{\frac{d+1}{2}-2}e^{-\frac{M_6}{2}u}du = 0. \tag{3.2.19}$$

Since δ is arbitrary, putting together (3.2.17)–(3.2.19) we obtain (3.2.15).

Next we show (3.2.16). By (3.2.2) and (3.2.6), we have

$$\begin{aligned}
& |\int_T^\infty e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)\nu(dy)| \\
\leq\ & M_4\int_T^\infty e^{-(\lambda-M_5)t}t^{-(d+1)/2}(\int_{\mathbb{R}^d}e^{-M_6|x-y|^2/t}\nu(dy))dt \\
\leq\ & M_4M(\nu)\int_T^\infty e^{-(\lambda-M_5)t}t^{-(d+1)/2}e^{\frac{M_6}{t}}(\sum_{m=-\infty}^{\infty}e^{-\frac{M_6}{8t}m^2})^d dt \\
\leq\ & C\int_T^\infty e^{-(\lambda-M_5)t}t^{-(d+1)/2}e^{\frac{M_6}{t}}t^d(\sum_{m=-\infty}^{\infty}\frac{1}{m^2})^d dt \\
\leq\ & C\int_T^\infty e^{-(\lambda-M_5)t}t^{-(d+1)/2}e^{\frac{M_6}{t}}t^d dt.
\end{aligned}$$

This implies (3.2.16).

□

The convergence of u^n and ∇u^n can now be shown.

Lemma 3.2.7. *Assume $\nu,\mu_i\in K_{d,1}$ for $1\leq i\leq d$ and $\lambda>M_5$. Then*
(i) $u^n(x)$ converges to $u_\nu(x)$ uniformly on every compact subset of $\mathbb{R}^d$.
(ii) $\nabla u^n(x)$ converges to $\nabla u_\nu(x)$ uniformly on every compact subset of $\mathbb{R}^d$.

Proof. Let us prove (ii). The proof of (i) is similar. We have

$$\begin{aligned}
& \nabla u^n(x)-\nabla u_\nu(x) \\
=\ & \int_0^\infty e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p_n(t,x,y)\nu^n(y)dy-\int_0^\infty e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)\nu(dy).
\end{aligned}$$

In view of Lemma 3.2.6, we only need to prove that for any compact subset $K\subset\mathbb{R}^d$, and positive constants $\varepsilon<T$,

$$\begin{aligned}
\lim_{n\to\infty}\sup_{x\in K}|\int_\varepsilon^T e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p_n(t,x,y)\nu^n(y)dy \\
-\int_\varepsilon^T e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)\nu(dy)|=0.
\end{aligned}$$

Notice that

$$\begin{aligned}
& \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} \nabla_x p_n(t,x,y)\nu^n(y)dy - \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} \nabla_x p(t,x,y)\nu(dy) \\
= \; & \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} [\nabla_x p_n(t,x,y) - \nabla_x p(t,x,y)]\nu^n(y)dy \\
& + \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} \nabla_x p(t,x,y)[\nu^n(y)dy - \nu(dy)].
\end{aligned}$$

Thus, it sufficient to show

$$\lim_{n\to\infty} \sup_{x\in K} \Big| \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} [\nabla_x p_n(t,x,y) - \nabla_x p(t,x,y)]\nu^n(y)dy \Big| = 0, \tag{3.2.20}$$

and

$$\lim_{n\to\infty} \sup_{x\in K} \Big| \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} \nabla_x p(t,x,y)[\nu^n(y)dy - \nu(dy)] \Big| = 0. \tag{3.2.21}$$

For any $r > 0$, we pick a jointly continuous function $\psi_r^x(y)$ such that $\psi_r^x(\cdot) \in C_0^\infty(B(x,r+1))$ and that $\psi_r^x(y) = 1$ for $y \in B(x,r)$. Let us now prove (3.2.20). We deduce

$$\begin{aligned}
& \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} [\nabla_x p_n(t,x,y) - \nabla_x p(t,x,y)]\nu^n(y)dy \\
= \; & \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} [\nabla_x p_n(t,x,y) - \nabla_x p(t,x,y)]\psi_r^x(y)\nu^n(y)dy \\
& + \int_\varepsilon^T e^{-\lambda t}dt \int_{\mathbb{R}^d} [\nabla_x p_n(t,x,y) - \nabla_x p(t,x,y)](1-\psi_r^x(y))\nu^n(y)dy \\
:= \; & L_1^n(x) + L_2^n(x).
\end{aligned} \tag{3.2.22}$$

Note that $\nabla_x p_n(t,x,y)$ has the same upper estimates as those in (3.2.2) by Theorem 3.14 in [21]. Thus, by (3.2.5), we have for $r \geq 3$ and a positive constant c_0,

$$\begin{aligned}
& \sup_{n\geq 1} \sup_{x\in\mathbb{R}^d} |L_2^n(x)| \\
\leq \; & c_0 \sup_{n\geq 1} \sup_{x\in\mathbb{R}^d} \int_\varepsilon^T e^{-(\lambda-M_5)t} t^{-(d+1)/2} dt \int_{|y-x|\geq r} e^{-M_6|x-y|^2/t} |\nu^n(y)|dy \\
\leq \; & c_0 \sup_{n\geq 1} \sup_{x\in\mathbb{R}^d} \int_{B(x,1)} |\nu^n(y)|dy \int_\varepsilon^T e^{-(\lambda-M_5)t} t^{-(d+1)/2} \sum_{|k|\geq 2r-2} e^{-\frac{M_6}{16t}|k|^2} dt \\
\leq \; & C \sum_{|k|\geq 2r-2} e^{-\frac{M_6}{16T}|k|^2},
\end{aligned}$$

which yields that

$$\lim_{r\to\infty}\sup_{n\geq 1}\sup_{x\in\mathbb{R}^d}|L_2^n(x)|=0. \tag{3.2.23}$$

Since $\nabla_x p_n(t,x,y)$ converges to $\nabla_x p(t,x,y)$ uniformly on

$$\{(t,x,y): t\in[\varepsilon,T], x\in K, y\in B(x,r+1)\},$$

for any given $\delta>0$ there exists an integer $N>0$ such that for $n\geq N$, $|\nabla_x p_n(t,x,y)-\nabla_x p(t,x,y)|\leq\delta$ for $(t,x)\in[\varepsilon,T]\times K$ and $y\in B(x,r+1)$. Consequently for $n\geq N$,

$$\begin{aligned}
|L_1^n(x)| &\leq \delta\int_\varepsilon^T e^{-\lambda t}dt\int_{B(x,r+1)}\psi_r^x(y)|\nu^n(y)|dy\\
&\leq C\delta\int_\varepsilon^T e^{-\lambda t}dt\sup_{n\geq 1}\sup_{x\in\mathbb{R}^d}\int_{B(x,r+1)}|\nu^n(y)|dy\\
&\leq C_r\delta\int_\varepsilon^T e^{-\lambda t}dt.
\end{aligned}$$

As δ is arbitrary, we deduce that

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}|L_1^n(x)|=0. \tag{3.2.24}$$

Combining (3.2.22)-(3.2.24) we can prove (3.2.20).

Let us now start the proof of (3.2.21). We have

$$\begin{aligned}
&\int_\varepsilon^T e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)[\nu^n(y)dy-\nu(dy)]\\
=\;&\int_\varepsilon^T e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)\psi_r^x(y)[\nu^n(y)dy-\nu(dy)]\\
&+\int_\varepsilon^T e^{-\lambda t}dt\int_{\mathbb{R}^d}\nabla_x p(t,x,y)(1-\psi_r^x(y))[\nu^n(y)dy-\nu(dy)]\\
:=\;&M_1^n(x)+M_2^n(x).
\end{aligned} \tag{3.2.25}$$

As the proof of (3.2.23), we can show that

$$\lim_{r\to\infty}\sup_{n\geq 1}\sup_{x\in\mathbb{R}^d}|M_2^n(x)|=0. \tag{3.2.26}$$

From the definition of $\nu^n(y)$ we have

$$\begin{aligned}
& M_1^n(x) \\
= & \int_\varepsilon^T e^{-\lambda t} dt \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} \nabla_x p(t,x,y) \psi_r^x(y) \varphi_n(y-z) dy] \nu(dz) \\
& - \int_\varepsilon^T e^{-\lambda t} dt \int_{\mathbb{R}^d} \nabla_x p(t,x,z) \psi_r^x(z) \nu(dz) \\
= & \int_\varepsilon^T e^{-\lambda t} dt \int_{\mathbb{R}^d} \nu(dz) \int_{\mathbb{R}^d} [\nabla_x p(t,x,y) \psi_r^x(y) \\
& - \nabla_x p(t,x,z) \psi_r^x(z)] \varphi_n(y-z) dy].
\end{aligned}$$

Considering the supports of $\psi_r^x(\cdot)$ and $\varphi_n(\cdot)$, we see that

$$f(t,x,z) := \int_{\mathbb{R}^d} [\nabla_x p(t,x,y) \psi_r^x(y) - \nabla_x p(t,x,z) \psi_r^x(z)] \varphi_n(y-z) dy] = 0,$$

for z belongs to outside of the ball $B(x,r+2)$. Hence, we obtain

$$\begin{aligned}
& |M_1^n(x)| \\
\leq & \int_\varepsilon^T e^{-\lambda t} dt \int_{B(x,r+2)} |\nu|(dz) \int_{\mathbb{R}^d} |\nabla_x p(t,x,y) \psi_r^x(y) \\
& - \nabla_x p(t,x,z) \psi_r^x(z)| \varphi_n(y-z) dy.
\end{aligned} \tag{3.2.27}$$

Note that $\nabla_x p(t,x,y)\psi_r^x(y)$ is uniformly continuous on $[\varepsilon,T] \times K \times K^{r+2}$, where $K^{r+2} := \{y : d(y,K) \leq r+2\}$. Hence, for any given $\delta > 0$ there exists an integer N such that for $n \geq N$ and $x \in K$, we have

$$\int_{\mathbb{R}^d} |\nabla_x p(t,x,y) \psi_r^x(y) - \nabla_x p(t,x,z) \psi_r^x(z)| \varphi_n(y-z) dy \leq \delta \int_{\mathbb{R}^d} \varphi_n(y-z) dy = \delta.$$

It follows from (3.2.27) that

$$\begin{aligned}
& |M_1^n(x)| \\
\leq & \delta \int_\varepsilon^T e^{-\lambda t} dt \cdot |\nu|(B(x,r+2)) \\
\leq & \delta \int_\varepsilon^T e^{-\lambda t} dt \cdot \sup_{x \in \mathbb{R}^d} |\nu|(B(x,r+2)).
\end{aligned}$$

Since δ is arbitrary, we conclude that

$$\lim_{n\to\infty} \sup_{x \in K} |M_1^n(x)| = 0. \tag{3.2.28}$$

Combining (3.2.25), (3.2.26) and (3.2.28), we obtain (3.2.21).

□

Consequently, we are able to show the following result, which concerns the existence of the solution to equation (3.1.1).

Theorem 3.2.8. *Assume $\nu, \mu_i \in K_{d,1}$ for $1 \le i \le d$ and $\lambda > M_5$. Then u_ν defined by*

$$u_\nu(x) := \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} p(t,x,y)\nu(dy) \tag{3.2.29}$$

is a weak solution to equation (3.1.1).

Proof. For any $n \ge 1$ and $r > 0$, by the strong Markov property of X^n, we have

$$\begin{aligned} u^n(x) &= E_x\left[\int_0^\infty e^{-\lambda t}\nu^n(X_t^n)dt\right] \\ &= E_x\left[\int_0^{\tau_{B(0,r)}} e^{-\lambda t}\nu^n(X_t^n)dt\right] + E_x\left[\int_{\tau_{B(0,r)}}^\infty e^{-\lambda t}\nu^n(X_t^n)dt\right] \\ &= E_x\left[\int_0^{\tau_{B(0,r)}} e^{-\lambda t}\nu^n(X_t^n)dt\right] + E_x\left[e^{-\lambda\tau_{B(0,r)}} u^n(X^n_{\tau_{B(0,r)}})\right], \end{aligned}$$

where $\tau_{B(0,r)}$ is the first exiting time of X from $B(0,r)$. Due to the smoothness of the drift G^n and ν^n, by Theorem 4.1 in [40], we know that u^n is the solution of the following equation on $B(0,r)$:

$$\begin{cases} \lambda u^n(x) - \frac{1}{2}\Delta u^n(x) - \langle G^n(x), \nabla u^n(x)\rangle = \nu^n(x), & \forall x \in B(0,r), \\ u^n(x)|_{\partial B(0,r)} = u^n(x), & \forall x \in \partial B(0,r). \end{cases}$$

Hence for every $\phi \in C_0^1(B(0,r))$, we have

$$\begin{aligned} &\lambda \int_{\mathbb{R}^d} u^n(x)\phi(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla u^n(x), \nabla\phi(x)\rangle dx \\ &= \int_{\mathbb{R}^d} \phi(x)\langle \nabla u^n(x), G^n(x)\rangle dx + \int_{\mathbb{R}^d} \phi(x)\nu^n(x)dx. \end{aligned} \tag{3.2.30}$$

Since $\phi(x)\nabla u^n(x)$ converges uniformly to $\phi(x)\nabla u_\nu(x)$ on the support of ϕ by Lemma

3.2.7, take $n \to \infty$ in (3.2.30) to get

$$\lambda \int_{\mathbb{R}^d} u_\nu(x)\phi(x)dx + \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla u_\nu(x), \nabla\phi(x)\rangle dx$$
$$= \int_{\mathbb{R}^d} \phi(x)\langle \nabla u_\nu(x), \mu(dx)\rangle + \int_{\mathbb{R}^d} \phi(x)\nu(dx).$$

Since r is arbitrary, we see that u_ν is a weak solution to equation (3.1.1).

□

Lemma 3.2.9. *Assume* $\lambda > M_5 \vee M_8$ *and there exists a constant* $0 < \alpha_0 < 1$ *such that* $\nu, \mu_i \in K_{d,1-\alpha_0}$ *for* $1 \le i \le d$. *Then*

$$\nabla u_\nu \in C^{\alpha_0}(\mathbb{R}^d).$$

Proof. Recall the definition of $u_\nu(x)$:

$$u_\nu(x) = \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} p(t,x,y)\nu(dy).$$

For a particular element of $\nabla u_\nu = (\frac{\partial u_\nu}{\partial x_1}, \frac{\partial u_\nu}{\partial x_2}, \cdots, \frac{\partial u_\nu}{\partial x_d})$,

$$\frac{\partial u_\nu}{\partial x_i} = \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} \frac{\partial p(t,x,y)}{\partial x_i}\nu(dy),\ i = 1, 2, \cdots, d.$$

Then:

$$\left|\frac{\partial u_\nu}{\partial x_i}(x) - \frac{\partial u_\nu}{\partial x_i}(x')\right| \le \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} \left|\frac{\partial p(t,x,y)}{\partial x_i} - \frac{\partial p(t,x',y)}{\partial x_i}\right| \cdot |\nu|(dy).$$

By (3.2.3), we have

$$\left|\frac{\partial u_\nu}{\partial x_i}(x) - \frac{\partial u_\nu}{\partial x_i}(x')\right| \le M_7|x-x'|^{\alpha_0} \int_0^\infty e^{-\lambda t} e^{M_8 t} dt \int_{\mathbb{R}^d} t^{-\frac{d+1+\alpha_0}{2}} (e^{-\frac{M_9|x-y|^2}{t}} + e^{-\frac{M_9|x'-y|^2}{t}})|\nu|(dy),$$

similar to the proof of Proposition 3.2.5, using Corollary 3.2.4 we can obtain:

$$\sup_{x',x\in\mathbb{R}^d} \int_0^\infty e^{-\lambda t} e^{M_8 t} dt \int_{\mathbb{R}^d} t^{-\frac{d+1+\alpha_0}{2}} (e^{-\frac{M_9|x-y|^2}{t}} + e^{-\frac{M_9|x'-y|^2}{t}})|\nu|(dy) < \infty.$$

This completes the proof.

□

Now we can state the main result in this section.

Theorem 3.2.10. *Assume $\lambda > M_5 \vee M_8$ and $\nu, \mu_i \in K_{d,1}$ for $1 \le i \le d$, then u_ν defined in (3.2.29) is a weak solution to equation (3.1.1). Furthermore, if there exists a constant $0 < \alpha_0 < 1$ such that $\nu, \mu_i \in K_{d,1-\alpha_0}$ for $1 \le i \le d$, then u_ν is the unique bounded weak solution to equation (3.1.1) in $C^{1,\alpha_0}(\mathbb{R}^d)$.*

Proof. Let u_ν be defined in (3.2.29) and $\lambda > M_5 \vee M_8$. If $\nu, \mu_i \in K_{d,1}$ for $1 \le i \le d$, it is clear that the bounded function u_ν is a weak solution to equation (3.1.1) by Proposition 3.2.5 and Theorem 3.2.8. If $\nu, \mu_i \in K_{d,1-\alpha_0}$ for some constant $0 < \alpha_0 < 1$ and each $1 \le i \le d$, then $u_\nu \in C^{1,\alpha_0}(\mathbb{R}^d)$ by Lemma 3.2.9.

Now we show the uniqueness. Suppose the bounded function $u \in C^{1,\alpha_0}(\mathbb{R}^d)$ is another weak solution to equation (3.1.1). Set $v := u_\nu - u$. Then for any $r > 0$, v is a weak solution to the following equation on $B(0,r)$:

$$\begin{cases} \frac{1}{2}\Delta v + \nabla v \cdot \mu - \lambda v = 0, & \forall x \in B(0,r), \\ v(x)|_{\partial B(0,r)} = v(x), & \forall x \in \partial B(0,r). \end{cases}$$

Then by Theorem 5.3 in [37], $v(x)$ is given by:

$$v(x) = E_x \left[e^{-\lambda \tau_{B(0,r)}} v(X_{\tau_{B(0,r)}}) \right],$$

where $\tau_{B(0,r)}$ is the first exiting time of X from $B(0,r)$. Since v is bounded and $\lim_{r\to\infty} \tau_{B(0,r)} = \infty$, we have

$$|v(x)| \le \lim_{r\to\infty} E_x \left[e^{-\lambda \tau_{B(0,r)}} |v(X_{\tau_{B(0,r)}})| \right] \le \|v\|_{L^\infty} \lim_{r\to\infty} E_x \left[e^{-\lambda \tau_{B(0,r)}} \right] = 0, \ \forall x \in \mathbb{R}^d.$$

The uniqueness is proved.

□

3.3 A Zvonkin-type transformation and solution to SDE with singular drift

In this section, we study a Zvonkin-type transformation for the Brownian motion with drift term μ. We then obtain the existence and uniqueness of the weak solution to the SDE (3.1.2).

Recall that in this chapter we assume $d \geq 3$. Given a measurable function $b(x) = (b_1(x), b_2(x), ..., b_d(x))$ on $\mathbb{R}^d$. Set $b_i^n(x) := (-n) \vee b_i(x) \wedge n$ and $b^n(x) := (b_1^n(x), b_2^n(x), ..., b_d^n(x))$ for each $n \geq 1$ and $1 \leq i \leq d$.

Let $\{\bar{X}, \bar{\mathbf{P}}_x, x \in \mathbb{R}^d\}$ denote the Brownian motion with drift $b(x)dx$ in the sense of Definition 3.1.3 and let $\bar{p}(t,x,y)$ denote the transition density of $\bar{X}$. Without loss of generality, if $b_i \in K_{d,1}$ for $1 \leq i \leq d$, we assume $\bar{p}(t,x,y)$ admits the same estimates as those in Theorem 3.2.2. We consider in particular

$$v(x) := -\int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} \bar{p}(t,x,y) b(y) dy, \tag{3.3.1}$$

$$v^n(x) := -\int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} \bar{p}(t,x,y) b^n(y) dy. \tag{3.3.2}$$

Lemma 3.3.1. *Assume $b_i \in K_{d,1}$ for $1 \leq i \leq d$ and $\lambda > M_5$. Then we have*

$$\sup_{n \geq 1} \sup_{x \in \mathbb{R}^d} \|\nabla v^n(x)\| < \infty, \tag{3.3.3}$$

$$\lim_{n \to \infty} \|\nabla v^n(x) - \nabla v(x)\| = 0, \ \forall x \in \mathbb{R}^d. \tag{3.3.4}$$

Proof. By proposition 3.2.5, it is easy to see that (3.3.3) holds and

$$\begin{aligned}
& \|\nabla v^n(x) - \nabla v(x)\| \\
\leq\ & \sum_{1 \leq i \leq d} M_4 \int_{\{|x-y| \leq r\} \bigcap \{|b_i(y)| > n\}} \frac{|b_i(y)| dy}{|x-y|^{d-1}} \int_0^\infty e^{-(\lambda - M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
& + \sum_{1 \leq i \leq d} M_4 \int_{\{|x-y| > r\} \bigcap \{|b_i(y)| > n\}} \frac{|b_i(y)| dy}{|x-y|^{d-1}} \int_0^1 e^{-(\lambda - M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
& + \sum_{1 \leq i \leq d} M_4 \int_{\{|x-y| > r\} \bigcap \{|b_i(y)| > n\}} \frac{|b_i(y)| dy}{|x-y|^{d-1}} \int_1^\infty e^{-(\lambda - M_5)|x-y|^2/u} u^{(d+1)/2-2} e^{-M_6 u} du \\
\leq\ & C_r \sum_{1 \leq i \leq d} [\int_{\{|x-y| \leq r\} \bigcap \{|b_i(y)| > n\}} \frac{|b_i(y)| dy}{|x-y|^{d-1}} + \int_{\{|b_i(y)| > n\}} e^{-(\lambda - M_5)|x-y|^2} |b_i(y)| dy] \\
& + C_r \sum_{1 \leq i \leq d} \int_1^\infty u^{(d+1)/2-2} e^{-M_6 u} du \int_{\{|x-y| > r\} \bigcap \{|b_i(y)| > n\}} e^{-(\lambda - M_5)|x-y|^2/u} |b_i(y)| dy.
\end{aligned}$$

Hence by Lemma 3.2.3 and the dominated convergence theorem, the left hand side above converges to 0 as $n \to \infty$ for any $x \in \mathbb{R}^d$.

□

Recall that $p(t,x,y)$ is the transition density of the process X, *i.e.* Brownian motion with drift μ. Define

$$u(x) := -\int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} p(t,x,y)\mu(dy).$$

Then we have the following Zvonkin-type transformation.

Theorem 3.3.2. *Assume $\mu_i, b_i \in K_{d,1}$ for $1 \le i \le d$, $\lambda > M_5$ and Y is a solution to the SDE (3.1.2). Then we have*

$$u(X_t) = u(x) + \int_0^t \langle \nabla u(X_s), dW_s\rangle + \lambda \int_0^t u(X_s) ds + A_t, \tag{3.3.5}$$

$$v(Y_t) = v(x) + \int_0^t \langle \nabla v(Y_s), dW_s\rangle + \lambda \int_0^t v(Y_s) ds + \int_0^t b(Y_s) ds. \tag{3.3.6}$$

Proof. First we prove (3.3.5). Let us define:

$$u^n(x) = -\int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^d} p^n(t,x,y) G^n(y) dy,$$

where $p^n(t,x,y)$ is the transition density determined by the SDE:

$$X_t^n = X_0^n + W_t + \int_0^t G^n(X_s^n) ds,$$

and $G^n(x)$ is given in Section 3.1. By the smoothness of $G^n(x)$, we know that $u^n(x)$ is smooth. Moreover, $u^n(x)$ is the solution to:

$$\frac{1}{2}\triangle u^n(x) + \langle \nabla u^n(x), G^n(x)\rangle = \lambda u^n(x) + G^n(x).$$

Consider the SDE:

$$X_t = X_0 + W_t + A_t,$$

which is introduced in Section 3.1. By Itô's formula, we have:

$$\begin{aligned}
&u^n(X_t)\\
=&u^n(X_0) + \int_0^t \langle \nabla u^n(X_s), dW_s\rangle + \int_0^t \langle \nabla u^n(X_s), dA_s\rangle + \frac{1}{2}\int_0^t \triangle u^n(X_s) ds\\
=&u^n(X_0) + \int_0^t \langle \nabla u^n(X_s), dW_s\rangle + \int_0^t \langle \nabla u^n(X_s), dA_s\rangle + \lambda \int_0^t u^n(X_s) ds\\
&+ \int_0^t G^n(X_s) ds - \int_0^t \langle \nabla u^n(X_s), G^n(X_s)\rangle ds\\
=&u^n(X_0) + I_1^n + I_2^n + I_3^n + I_4^n - I_5^n.
\end{aligned}$$

We take the limit $n \to \infty$ on both sides of the above equation. By Lemma 3.2.7 with ν replaced by μ, we see that $u^n(x)$ and $\nabla u^n(x)$ converge respectively to $u(x)$ and $\nabla u(x)$ uniformly on every compact subset of $\mathbb{R}^d$. Hence for the left hand side, we have:

$$u^n(X_t) \to u(X_t) \text{ as } n \to \infty,$$

immediately. For the right hand side, $u^n(X_0) \to u(X_0)$ as $n \to \infty$ is again an immediate result. For I_1^n, let $I_1 := \int_0^t \langle \nabla u(X_s), dW_s \rangle$. Now for any fixed positive constant m, we define τ_m:

$$\tau_m = \inf\{s \geq 0 : |X_s| > m\}.$$

Then for any $\delta > 0$, we can obtain:

$$\begin{aligned} P(|I_1^n - I_1| > \delta) &= P(|I_1^n - I_1| > \delta, t \leq \tau_m) + P(|I_1^n - I_1| > \delta, \tau_m < t) \\ &\leq P\left(\left| \int_0^{t \wedge \tau_m} \langle \nabla u^n(X_s) - \nabla u(X_s), dW_s \rangle \right| > \delta \right) + P(\tau_m < t) \\ &\leq E\left[\left| \int_0^{t \wedge \tau_m} \langle \nabla u^n(X_s) - \nabla u(X_s), dW_s \rangle \right|^2 \right] / \delta^2 + P(\tau_m < t). \end{aligned}$$

Notice that:

$$\begin{aligned} & E\left[\left| \int_0^{t \wedge \tau_m} \langle \nabla u^n(X_s) - \nabla u(X_s), dW_s \rangle \right|^2 \right] \\ = \;& E\left[\int_0^t 1(s \leq \tau_m) \|\nabla u^n(X_s) - \nabla u(X_s)\|^2 ds \right] \\ = \;& \int_0^t E\left[\|\nabla u^n(X_{s \wedge \tau_m}) - \nabla u(X_{s \wedge \tau_m})\|^2 \right] ds \\ \leq \;& \sup_{x \in B(0,m)} \|\nabla u^n(x) - \nabla u(x)\|^2 \cdot t. \end{aligned}$$

Thus $E\left[\left| \int_0^{t \wedge \tau_m} \langle \nabla u^n(X_s) - \nabla u(X_s), dW_s \rangle \right|^2 \right] \to 0$ as $n \to \infty$ by the uniform convergence of $\nabla u^n(x)$. Hence:

$$\lim_{n \to \infty} P(|I_1^n - I_1| > \delta) \leq P(\tau_m < t).$$

Since $\tau_m \to \infty$ almost sure as $m \to \infty$, we let $m \to \infty$ on the both sides of the above inequality to get:

$$\lim_{n \to \infty} P(|I_1^n - I_1| > \delta) = 0,$$

for arbitrary positive constant δ. Therefore we can obtain that $I_1^n \to I_1$ as $n \to \infty$.

For I_3^n, we introduce:

$$I_3 := \lambda \int_0^t u(X_s)ds,$$

then

$$\begin{aligned} |I_3^n - I_3| &= \lambda \left| \int_0^t u^n(X_s) - u(X_s)ds \right| \\ &\leq \lambda \int_0^t |u^n(X_s) - u(X_s)|ds. \end{aligned}$$

If we fix $\omega \in \Omega$, we have:

$$|I_3^n - I_3| \leq \sup_{x \in \{X_s, 0 \leq s \leq t\}} |u^n(x) - u(x)| \cdot \lambda t.$$

By the uniform convergence of $u^n(x)$, we can therefore conclude the convergence of I_3^n for fixed $\omega \in \Omega$, $I_3^n \to I_3$ as $n \to \infty$. Moreover, for I_4^n, it is an immediate consequence from the definition of A_t that $I_4^n \to A_t$ as $n \to \infty$. It is now only $I_2^n - I_5^n$ left.

$$\begin{aligned} &|I_2^n - I_5^n| \\ =&|\int_0^t \langle \nabla u^n(X_s), dA_s \rangle - \int_0^t \langle \nabla u^n(X_s), G^n(X_s) \rangle ds| \\ =&|\int_0^t \langle \nabla u^n(X_s) - \nabla u(X_s), dA_s \rangle + \int_0^t \langle \nabla u(X_s), dA_s \rangle \\ &- \int_0^t \langle \nabla u^n(X_s) - \nabla u(X_s), G^n(X_s) \rangle ds - \int_0^t \langle \nabla u(X_s), G^n(X_s) \rangle ds| \\ \leq& \int_0^t \|\nabla u^n(X_s) - \nabla u(X_s)\| d|A|_s + \int_0^t \|\nabla u^n(X_s) - \nabla u(X_s)\| |G^n(X_s)| ds \\ &+ |\int_0^t \langle \nabla u(X_s), dA_s \rangle - \int_0^t \langle \nabla u(X_s), G^n(X_s) \rangle ds|. \end{aligned}$$

If we fix $\omega \in \Omega$, for the first term:

$$\int_0^t \|\nabla u^n(X_s) - \nabla u(X_s)\| d|A|_s \leq \sup_{x \in \{X_s, 0 \leq s \leq t\}} \|\nabla u^n(x) - \nabla u(x)\| |A|_t.$$

Then by the uniform convergence of $\nabla u^n(x)$, $\int_0^t \|\nabla u^n(X_s) - \nabla u(X_s)\| d|A|_s$ converges to 0 as $n \to \infty$. For the second term,

$$\begin{aligned}&\int_0^t \|\nabla u^n(X_s) - \nabla u(X_s)\||G^n(X_s)|ds \\ \leq &\sup_{x\in\{X_s, 0\leq s\leq t\}} \|\nabla u^n(x) - \nabla u(x)\| \int_0^t |G^n(X_s)|ds.\end{aligned}$$

By the definition and property regarding $G^n(x)$ given in Section 3.1, we can assume $\sup_n \int_0^t |G^n(X_s)|ds < \infty$. Hence $\int_0^t \|\nabla u^n(X_s) - \nabla u(X_s)\||G^n(X_s)|ds \to 0$ as $n \to \infty$. For the last term, denote $G_t^n := \int_0^t G^n(X_s)ds$. Then there exists a subsequence $\{G_t^{n_k}\}_{n_k\geq 0}$, which is a sequence of bounded variation process such that $\lim_{n_k\to\infty} G_t^{n_k} = A_t$, then the corresponding signed measure defined by $G_t^{n_k}$ weakly converges to the one defined by A_t. Note that $\nabla u(x)$ is continous, hence we have:

$$\int_0^t \langle \nabla u(X_s), dG_s^{n_k}\rangle \to \int_0^t \langle \nabla u(X_s), dA_s\rangle$$

$$\Rightarrow \left|\int_0^t \langle \nabla u(X_s), dA_s\rangle - \int_0^t \langle \nabla u(X_s), G^{n_k}(X_s)\rangle ds\right| \to 0$$

as $n_k \to \infty$. Hence $|I_2^n - I_5^n| \to 0$ as $n \to \infty$. Therefore:

$$u(X_t) = u(X_0) + \int_0^t \langle \nabla u(X_s), dW_s\rangle + \lambda \int_0^t u(X_s)ds + A_t$$

as required.

Now we prove (3.3.6). Recall that $|b^n|$ is bounded on $\mathbb{R}^d$ and v^n is defined in (3.3.2). Following the proof of (3.3.5), we also have

$$v^n(Y_t) = v^n(x) + \int_0^t \langle \nabla v^n(Y_s), dW_s\rangle + \lambda \int_0^t v^n(Y_s)ds + \int_0^t b^n(Y_s)ds.$$

Note that $\int_0^t |b(Y_s)|ds < \infty$, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^t b^n(Y_s)ds = \int_0^t b(Y_s)ds.$$

Combining this with (3.3.3) and (3.3.4), we have (3.3.6).

□

Now we show the main result in this section.

Theorem 3.3.3. *Assume $b_i \in K_{d,1}$ for $1 \le i \le d$. Then for any given $x \in \mathbb{R}^d$, there exists a unique weak solution to the following SDE:*

$$Y_t = x + W_t + \int_0^t b(Y_s)ds. \tag{3.3.7}$$

Proof. We first prove the existence. Assume $\{\bar{X}, \bar{\mathbf{P}}_x, x \in \mathbb{R}^d\}$ is the Brownian motion with drift $\mu(dx) = b(x)dx$ in the sense of Definition 3.1.3, which is given by:

$$d\bar{X}_t = d\bar{W}_t + d\bar{A}_t,\ X_0 = x.$$

and $\bar{p}(t,x,y)$ is the transition density of $\bar{X}$. Recall that

$$G^n(x) = \int_{\mathbb{R}^d} \varphi_n(x-y)\mu(dy) = \int_{\mathbb{R}^d} \varphi_n(x-y)b(y)dy.$$

Then by using (3.2.1) and the proof of Lemma 3.2.7, we have

$$\begin{aligned} &\lim_{n\to\infty} \bar{E}_x \left[\int_0^t |b(\bar{X}_s) - G^n(\bar{X}_s)| ds \right] \\ \le\ &\lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^d} M_4 e^{M_5 s} s^{-d/2} e^{-M_6 |x-y|^2/s} |b(y) - G^n(y)| dy = 0, \end{aligned}$$

where $\bar{E}_x$ stands for the expectation under $\bar{P}_x$. Hence by the definition of $\bar{X}_t$, we have $\bar{A}_t = \lim_{n\to\infty} \int_0^t G^n(\bar{X}_s)ds = \int_0^t b(\bar{X}_s)ds$. On the other hand, since $|b| \in K_{d,1}$, by Theorem 3.2.2 we have

$$\bar{E}_x \left[\int_0^t |b(\bar{X}_s)| ds \right] = \int_0^t \int_{\mathbb{R}^d} \bar{p}(s,x,y)|b(y)| dyds < \infty,$$

which implies $\int_0^t |b(\bar{X}_s)|ds < \infty$, *P-a.e.*. Hence $\bar{X}$ is a weak solution to SDE (3.3.7).

Next we prove the uniqueness. Assume Y is a weak solutions to SDE (3.3.7). By (3.2.9), we can take λ large enough such that $\sup_{x\in\mathbb{R}^d} \|\nabla v(x)\| < \frac{1}{2d}$, where $v(x)$ is defined in (3.3.1). Let $F(x) := v(x) - x$, we have that

$$\frac{1}{2}|x-y| \le |F(x) - F(y)| \le 2|x-y|.$$

Indeed, by mean value theorem, we have

$$|F(x) - F(y)| \le \sup_{x\in\mathbb{R}^d} ||\nabla v(x)|| \cdot |x-y| + |x-y|,$$

by the estimates $\sup_{x\in\mathbb{R}^d}||\nabla v(x)|| < \frac{1}{2d} < 1$, we have $|F(x)-F(y)| \leq 2|x-y|$. On the other hand, by the mean value theorem again, we have the lower bound

$$|F(x)-F(y)| \geq |x-y|\cdot(1-\sup_{x\in\mathbb{R}^d}||\nabla v(x)||) > |x-y|\cdot(1-\frac{1}{2d}) \geq \frac{1}{2}|x-y|.$$

Denote F^{-1} as the inverse of F. Then by the Zvonkin-type transformation (3.3.6), we have

$$F(Y_t) = F(x) + \int_0^t \langle \nabla F(Y_s), dW_s\rangle + \lambda\int_0^t v(Y_s)ds.$$

Set $\widetilde{Y}_t := F(Y_t)$, then

$$\widetilde{Y}_t = \widetilde{Y}_0 + \int_0^t \langle \nabla F\circ F^{-1}(\widetilde{Y}_s), dW_s\rangle + \lambda\int_0^t v\circ F^{-1}(\widetilde{Y}_s)ds. \tag{3.3.8}$$

Since $\nabla F\circ F^{-1}$ is bounded continuous and uniformly non-degenerate, and $v\circ F^{-1}$ is bounded, $\widetilde{Y}_t$ is the unique weak solution to SDE (3.3.8) by [17]. Thus $Y_t := F^{-1}(\widetilde{Y}_t)$ is the unique weak solution to SDE (3.3.7).

□

Chapter 4

Reflected Brownian Motion with Singular Drift

4.1 Introduction

In this chapter, we are going to show that there exists a unique weak solution to the reflected Brownian motion with singular drift μ, where μ is a vector-valued Kato class measure on $\mathbb{R}^d$ with $d \geq 3$, which usually is not absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}^d$. Notice that our main results also hold for $d = 2, 1$. We can show this by following the similar method when $d \geq 3$ but with different definition of Kato class and transition density function corresponding to different dimension. For simplicity, through out this chapter, we assume $d \geq 3$. One of the motivation is to construct a reflected Brownian motion which drifts upwards when it hits fractal-like sets (see [1, p. 792]).

We now introduce the Kato class $K_{d,\alpha}$ for $\alpha \in (0, 2]$. Recall a measure π on $\mathbb{R}^d$ is called Radon measure, if $\pi(K) < \infty$ for each compact set $K \subset \mathbb{R}^d$ and for any Borel measurable set B,

$$\pi(B) = \sup\{\pi(K) : K \text{ is a compact set with } K \subset B\}.$$

Let π be a general signed Radon measure on $\mathbb{R}^d$ and we say that $\pi \in K_{d,\alpha}$, if

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} |x-y|^{\alpha-d} |\pi|(dy) = 0,$$

where $|\pi|$ is the total variation measure of π.

In this chapter, we only consider the Kato class $K_{d,1}$. There are many examples of measures belonging to $K_{d,1}$. By [1, Proposition 2.1], we know that for a signed measure π on $\mathbb{R}^d$, if there exist constants $c_1, c_2 > 0$ such that for any $x \in \mathbb{R}^d$ and $0 < r \leq 1$, $|\pi|(B(x,r)) \leq c_1 r^{d-1+c_2}$, where $B(x,r)$ denotes the ball in $\mathbb{R}^d$ centered at x with radius r, then $\pi \in K_{d,1}$. Therefore, if $f \in L^p$ for some $p > d$, then $\pi(dx) = f(x)dx \in K_{d,1}$. Let $\mathbb{H}^\lambda$ be the λ-dimensional Hausdorff measure for some $\lambda \in (0,d]$. We say a Borel measurable set $\Gamma \subset \mathbb{R}^d$ is a λ-set (see [18]), if there exist positive constants c_3 and c_4 such that, for all $x \in \Gamma$ and $r \in (0,1]$,

$$c_3 r^\lambda \leq \mathbb{H}^\lambda(\Gamma \cap B(x,r)) \leq c_4 r^\lambda.$$

Then $\mathbb{H}^\lambda$ restricted to a λ-set belongs to the Kato class $K_{d,1}$ if $\lambda \in (d-1,d]$. For more examples of measures of Kato class $K_{d,1}$, we refer to [1].

Throughout this chapter, we assume that D is a bounded smooth domain in $\mathbb{R}^d$ with $d \geq 3$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_d)$ with $\mu_i(dx) \in K_{d,1}$ for each $1 \leq i \leq d$. Let $n(x) = (n_1(x), n_2(x), \cdots, n_d(x))$ be the inward normal vector field on the boundary of the domain D, ∂D. We denote by $\psi(x)$ a non-negative C^∞ function on $\mathbb{R}^d$ with compact support and

$$\int_{\mathbb{R}^d} \psi(x)dx = 1.$$

Now we define $\psi_n(x) := 2^{dn}\psi(2^n x)$. We can construct a sequence of smooth functions $(G_{i,n})_{n\geq 1}$ for $i = 1, \cdots, d$, by

$$G_{i,n}(x) := \int_{\mathbb{R}^d} \psi_n(x-y)\mu_i(dy).$$

and set $G_n := (G_{1,n}, \cdots, G_{d,n})$.

Let us first introduce the following notations, which will be used throughout this chapter. Let $\Omega := C([0,\infty), \bar{D})$ be the set of continuous functions mapping $[0,\infty)$ to $\bar{D}$ and $X(t,\omega) := \omega(t)$ be the the coordinate map on Ω. We will use $\mathcal{F}_t$ and $\mathcal{F}_\infty$ to denote the σ-field $\sigma\{X_s : s \leq t\}$ and $\sigma\{X_s : s < \infty\}$ respectively. Recall the shift operator $\theta_t : \Omega \to \Omega$ is defined as $\theta_t(\omega)(\cdot) := \omega(t+\cdot)$. Given a strong Markov process $\{X, P_x, x \in \bar{D}\}$, we call a continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted process $C = (C_t)_{t\geq 0}$ a continuous additive functional (CAF) of X, if for each $x \in \bar{D}$, $C_0 = 0$, P_x-*a.e.*, and

$$C_{s+t} = C_s + C_t \circ \theta_s, \ \forall s,t \geq 0, \ P_x - a.e..$$

We say that the Markov process $\{\Omega, X, P_x, x \in \bar{D}\}$ is strong Feller if for any bounded

Borel measurable function f and $t>0$, $P_t f(x) := \mathbb{E}_x[f(X_t)]$ is continuous on $\bar{D}$, where $\mathbb{E}_x$ denotes the expectation with respect to P_x. We now define a weak solution to the reflected Brownian motion with singular drift μ.

Definition 4.1.1. *Given $x \in \bar{D}$, and we say that a probability measure P on $(\Omega, \mathcal{F}_\infty)$ is a weak solution to the reflected Brownian motion with drift μ, if there exist a d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $W=(W_t)_{t\geq 0}$ with $W_0 = 0$ under P, and continuous $(\mathcal{F}_t)_{t\geq 0}$-addapted processes $A=(A_t)_{t\geq 0}$ and $L=(L_t)_{t\geq 0}$, having bounded variations, such that*

(a)

$$\begin{cases} X_t = x + A_t + W_t + L_t, \\ L_t = \displaystyle\int_0^t n(X_s) I_{\partial D}(X_s)\, d|L|_s, \end{cases} \tag{4.1.1}$$

where $|L|_t$ denotes the total variation of L_t on $[0,t]$.

(b) $A_t = \lim_{n\to\infty} \int_0^t G_n(X_s)ds$ uniformly over t in finite intervals, where the convergence is in probability.

(c) there exists a subsequence $\{n_k\}$ such that $\sup_{k\geq 1}\int_0^t |G_{n_k}(X_s)|ds < \infty$ a.e. for each $t>0$.

A weak solution to the reflected Brownian motion with drift μ is also referred as a weak solution to the reflected stochastic differential equation (SDE) (4.1.1). We say that the weak solution to the reflected SDE (4.1.1) is unique, if probability measures P and $\tilde{P}$ on $(\Omega, \mathcal{F}_\infty)$ are two weak solutions to the reflected SDE (4.1.1) satisfying that $P(X_0 = x) = \tilde{P}(X_0 = x)$, then $P = \tilde{P}$ on $(\Omega, \mathcal{F}_\infty)$.

We emphasise our main result below:

Theorem 4.1.1. *(a) For any $x \in \bar{D}$, there exists a unique weak solution to the reflected SDE (4.1.1).*

(b) Let (X, P_x) be the weak solution to the reflected SDE (4.1.1) with $X_0 = x$. Then the collection $(X, P_x, x \in \bar{D})$ forms a Feller process with strong Feller property, which has a jointly continuous transition density $p(t,x,y)$ on $(0,\infty)\times\bar{D}\times\bar{D}$ satisfying the following equation: for any $(t,x,y) \in (0,\infty)\times\bar{D}\times\bar{D}$,

$$p(t,x,y) = q(t,x,y) + \int_0^t \int_D q(t-s,x,z)\langle \nabla_z p(s,z,y), \mu(dz)\rangle ds, \tag{4.1.2}$$

where $q(t,x,y)$ is the transition density of the reflected Brownian motion on D.

(c) *There exist positive constants C_i, $1 \leq i \leq 3$, such that*

$$p(t,x,y) \leq C_1 e^{C_2 t} t^{-\frac{d}{2}} e^{-\frac{C_3|x-y|^2}{2t}}, \ \forall (t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D},$$

$$|\nabla_x p(t,x,y)| \leq C_1 e^{C_2 t} t^{-\frac{d+1}{2}} e^{-\frac{C_3|x-y|^2}{2t}}, \ \forall (t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D}.$$

Furthermore, each component A_i of A and $|L|$ are CAFs of X, whose Revuz measures are μ_i and $\frac{1}{2}\sigma$ respectively, where σ is the $(d-1)$-dimensional volume element on ∂D, i.e. for every $T > 0$ and continuous function f on $[0,T] \times \bar{D}$,

$$\begin{aligned} E_x\left[\int_0^T f(s,X(s))dA_i(s)\right] &= \int_0^T \int_D p(s,x,y) f(s,y) \mu_i(dy) ds, \ \forall x \in \bar{D}, \\ E_x\left[\int_0^T f(s,X(s))d|L|(s)\right] &= \frac{1}{2}\int_0^T \int_{\partial D} p(s,x,y) f(s,y) \sigma(dy) ds, \ \forall x \in \bar{D}, \end{aligned} \tag{4.1.3}$$

where E_x is the expectation with respect to P_x.

Remark 4.1.1. *By (4.1.2), we see that the solution to the reflected SDE (4.1.1) is independent on the choice of the smooth function $\psi(x)$. Moreover, following from the uniqueness of CAF of the Revuz measure (see the proof of [12, Theorem 5.1.2]), we also see that the processes A and L in (4.1.1) are independent on the choice of the smooth function $\psi(x)$.*

There is a great deal of literature concerning SDEs with singular coefficients. In the celebrated work [44], Zvonkin introduced a transformation of the phase space to remove the drifts and obtain the existence and uniqueness of strong solutions to SDEs with singular coefficients. Since then, there are many works devoted to extending the Zvonkin transformation in various ways to obtain the strong solutions to SDEs with singular coefficients, see [15, 23, 35, 36, 43]. Bass and Chen established the existence and uniqueness of weak solutions of the Brownian motion with measure-valued drift in [1]. Stable processes with measure-valued drift were later constructed in [22] by Kim and Song. Flandoli, Issoglio and Russo in [10] showed the existence and uniqueness of weak solutions to SDE with distributional drift. However, the distribution-valued drift does not include the measure-valued drift.

On the other hand, the study of reflected diffusion processes have been investigated.

The pioneering work goes back to A.V. Skorokhod [31] and [32], who considered a reflected diffusion process on $[0,\infty)$. H. Tanaka in [34] obtained the strong solutions of the reflected SDEs in a convex domain based on solving the corresponding Skorokhod problem. P.L. Lions and A.S. Sznitman in [24] studied the reflected SDEs by a penalized method in a C^3-domain. C. Costantini in [7] proved the existence of the weak solutions to SDEs with oblique reflection boundary condition when the coefficients are bounded continuous and the boundary of the domain piecewise belongs to the space C_b^1. P. Dupuis and H. Ishii in [8] obtained the existence and uniqueness of the strong solution to reflected SDEs in a nonsmooth domain, which only requires the directions of reflection to be C^2. Z.Q. Chen in [6] constructed a reflected diffusion process on a general domain using the Dirichlet form theory, and obtained the Skorokhod decomposition of the reflected diffusion process. For further papers regarding reflected SDEs, we refer to [25, 26, 29, 30, 33].

In this chapter, we are going to construct a unique weak solution of the reflected Brownian motion with measure-valued drift. We will adopt a similar approach as that in [1] and [22]. However, we will need to deal with extra difficulties caused by the reflection. We stress that none of the methods in the above mentioned articles on reflected SDEs applies here because of the generality of the measure-valued drift.

We close this section by mentioning some conventions used throughout this chapter. We will use $\langle\cdot,\cdot\rangle$ to denote the inner product of the Euclidean space $\mathbb{R}^d$. We let $C(\bar{D})$ and $C^2(\bar{D})$ stand for the functional spaces of continuous functions and twice continuously differentiable functions respectively on $\bar{D}$. Meanwhile, let us denote with $C_c(D)$ the functional space of continuous functions with compact support on D. The letter c is a generic constant whose value might be different from one to another.

4.2 Construction of the Transition Density Function

In this section, we construct the transition density function for the reflected Brownian motion with singular measure-valued drift. We firstly establish the uniform Gaussian type estimates of the transition density functions $p_n(t,x,y)$ of the reflected SDEs with the approximating smooth drifts G_n, and show that $p_n(t,x,y)$ satisfies the equation (4.1.2) with $\mu(dz)$ replaced by $G_n(z)dz$. Then we derive that $p_n(t,x,y)$ and $\nabla_x p(t,x,y)$ converge uniformly to a transition density function $p(t,x,y)$ and respectively its gradient $\nabla_x p(t,x,y)$ on compact subsets in $(0,\infty)\times\bar{D}\times\bar{D}$. $p(t,x,y)$ is the desired transition density function of the reflected Brownian motion with drift μ.

Let us start with the transition density of the reflected Brownian motion on the domain D. Let $q(t,x,y)$ defined on $(0,\infty)\times\bar{D}\times\bar{D}$ be the transition density of the reflected Brownian motion on $\bar{D}$. By [3, theorem 3.1] and [13, Theorem VI.3.1], $q(t,x,y)$ admits the following estimates: there exist positive constants M_1 and M_2 such that for any $t>0$ and $x,y\in\bar{D}$,

$$q(t,x,y) \le M_1 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{2t}}, \tag{4.2.1}$$

$$|\nabla_x q(t,x,y)| \le M_1 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{2t}}. \tag{4.2.2}$$

Let $\widetilde{W}=(\widetilde{W}_t)_{t\ge 0}$ be a d-dimensional standard Brownian motion defined on a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}}_\infty,(\widetilde{\mathcal{F}})_{t\ge 0},\widetilde{P})$. By [24, theorem 3.1], there exists a unique strong solution $\widetilde{X}_n=(\widetilde{X}_n(t))_{t\ge 0}$ to the following reflected SDE:

$$\begin{cases} \widetilde{X}_n(t)=x+\int_0^t G_n(\widetilde{X}_n(s))ds+\widetilde{W}_t+\widetilde{L}_n(t), \\ \widetilde{X}_n(t)\in\bar{D}, \\ \widetilde{L}_n(t)=\int_0^t n(\widetilde{X}_n(s))1_{\partial D}(\widetilde{X}_n(s))d|\widetilde{L}_n|(s), \end{cases} \tag{4.2.3}$$

where each component of $\widetilde{L}_n$ is an adapted continuous process of finite variation, $|\widetilde{L}_n|(t)$ denotes the total variation of $\widetilde{L}_n$ on $[0,t]$.

Let $p_n(t,x,y)$ be the transition density of $\widetilde{X}_n$. By the standard parametrix method (see the proof of [38, Theorem 3.2]), p_n admits the following expression:

$$p_n(t,x,y)=q(t,x,y)+\int_0^t\int_D q(t-s,x,z)\phi_n(s,z,y)dzds, \tag{4.2.4}$$

where function ϕ satisfies

$$\phi_n(t,x,y)=\langle G_n(x),\nabla_x q(t,x,y)\rangle+\int_0^t\int_D\langle G_n(x),\nabla_x q(t-s,x,z)\rangle\phi_n(s,z,y)dzds. \tag{4.2.5}$$

Define a sequence of functions $\{f_{n,k}\}_{k\ge 0}$:

$$f_{n,0}(t,x,y):=\langle G_n(x),\nabla_x q(t,x,y)\rangle,$$

$$f_{n,k+1}(t,x,y):=\int_0^t\int_D\langle G_n(x),\nabla_x q(t-s,x,z)\rangle f_{n,k}(s,z,y)dzds. \tag{4.2.6}$$

Then iterating (4.2.5) gives

$$\phi_n(t,x,y) = \sum_{k\geq 0} f_{n,k}(t,x,y). \tag{4.2.7}$$

Next we will show that the transition density function $p_n(t,x,y)$ admits uniform Gaussian type estimates. For $\alpha > 0$, define

$$N_{\mu,\alpha}(t) := \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{\alpha|x-y|^2}{2s}} |\mu|(dy)ds,$$

$$N_{G_n,\alpha}(t) := \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-\frac{d+1}{2}} e^{-\frac{\alpha|x-y|^2}{2s}} |G_n(y)|dyds.$$

Then we have the following Lemma.

Lemma 4.2.1. *There exists a $M_3 > 0$ such that for any $(t,x,y) \in (0,1)\times\bar{D}\times\bar{D}$ and $n \geq 1$, we have*

$$\int_0^t \int_D (t-s)^{-\frac{d}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)|dzds \leq M_3 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{G_n,\frac{M_2}{2}}(t),$$
$$\int_0^t \int_D (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)|dzds \leq M_3 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{G_n,\frac{M_2}{2}}(t),$$

and

$$\int_0^t \int_D (t-s)^{-\frac{d}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |\mu(dz)|ds \leq M_3 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{\mu,\frac{M_2}{2}}(t),$$
$$\int_0^t \int_D (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |\mu(dz)|ds \leq M_3 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{\mu,\frac{M_2}{2}}(t).$$

Furthermore, we have

$$\lim_{t\to 0} \sup_{n\geq 1} N_{G_n,\frac{M_2}{2}}(t) = \lim_{t\to 0} N_{\mu,\frac{M_2}{2}}(t) = 0. \tag{4.2.8}$$

Proof. Following the arguments of [39, Lemma 3.1], one can see that there exists a $M_3 > 0$ such that for any $(t,x,y) \in (0,1)\times\bar{D}\times\bar{D}$ and $n \geq 1$,

$$\int_0^t \int_D (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)|dzds \leq M_3 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{G_n,\frac{M_2}{2}}(t),$$

$$\int_0^t \int_D (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |\mu|(dz)ds \leq M_3 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{\mu,\frac{M_2}{2}}(t).$$

Hence we have

$$\begin{aligned}
&\int_0^t\int_D (t-s)^{-\frac{d}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)| dz ds \\
\leq\; & \int_0^t\int_D t^{\frac{1}{2}}(t-s)^{-\frac{d+1}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)| dz ds \\
\leq\; & M_3 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{G_n,\frac{M_2}{2}}(t).
\end{aligned}$$

Similarly, we also have

$$\int_0^t\int_D (t-s)^{-\frac{d}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} \cdot s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |\mu|(dz) ds \leq M_3 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{\mu,\frac{M_2}{2}}(t).$$

Since $\mu \in K_{d,1}$, (4.2.8) follows from [21, Proposition 2.2 and Proposition 2.3]. □

Now we define a sequence of functions $\{\Lambda_{n,k}\}_{k\geq 0}$ by

$$\begin{aligned}
&\Lambda_{n,0}(t,x,y) := q(t,x,y), \\
&\Lambda_{n,k+1}(t,x,y) := \int_0^t\int_D q(t-s,x,z)\langle \nabla_z \Lambda_{n,k}(s,z,y), G_n(z)\rangle dz ds. \quad (4.2.9)
\end{aligned}$$

Then we have the following result.

Lemma 4.2.2. *There exists a positive constant T_1 such that for each integer $k \geq 0$, $n \geq 1$ and $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$, $\Lambda_{n,k}(t,x,y)$ is differentiable with respect to x and the following hold:*

$$\Lambda_{n,k}(t,x,y) \leq M_1 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} \cdot 2^{-k}, \quad (4.2.10)$$

$$|\nabla_x \Lambda_{n,k}(t,x,y)| \leq M_1 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} \cdot 2^{-k}, \quad (4.2.11)$$

$$\Lambda_{n,k+1}(t,x,y) = \int_0^t\int_D q(t-s,x,z) f_{n,k}(s,z,y) dz ds. \quad (4.2.12)$$

Proof. By (4.2.8), one can find a constant $T_1 \in (0,1)$ such that

$$\sup_{n\geq 1} M_1 M_3 N_{G_n,\frac{M_2}{2}}(T_1) \leq \frac{1}{2}. \quad (4.2.13)$$

Now we prove this Lemma by induction. For $k=0$, by (4.2.1) and (4.2.2) we know that for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$, $\Lambda_{n,0}(t,x,y)$ is differentiable with respect to x and (4.2.10)-(4.2.12) hold.

Assume the results hold for the case $k = m$. That is, we have

$$|\nabla_x \Lambda_{n,m}(t,x,y)| \le M_1 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} \cdot 2^{-m}, \tag{4.2.14}$$

$$\Lambda_{n,m+1}(t,x,y) = \int_0^t \int_D q(t-s,x,z) f_{n,m}(s,z,y)dzds. \tag{4.2.15}$$

Then by (4.2.1), (4.2.9), (4.2.13), (4.2.14) and Lemma 4.2.1, for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$ we have

$$\begin{aligned} &|\Lambda_{n,m+1}(t,x,y)| \\ &\le \int_0^t \int_D q(t-s,x,z)|\nabla_z \Lambda_{n,m}(s,z,y)||G_n(z)|dzds \\ &\le \int_0^t \int_D M_1^2 (t-s)^{-\frac{d}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)|dzds \cdot 2^{-m} \\ &\le M_1^2 M_3 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{G_n,\frac{M_2}{2}}(t) \cdot 2^{-m} \le M_1 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} \cdot 2^{-m-1}. \end{aligned} \tag{4.2.16}$$

For the gradient, using (4.2.2), (4.2.14) and Lemma 4.2.1, one can see that for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$, $\nabla_x q(t-s,x,z)\langle \nabla_z \Lambda_{n,m}(s,z,y), G_n(z)\rangle$ is integrable with respect to the measure $dsdz$ on $(0,t) \times D$. Hence $\Lambda_{n,m+1}(t,x,y)$ is differentiable with respect to x and

$$\begin{aligned} &|\nabla_x \Lambda_{n,m+1}(t,x,y)| \\ &\le \int_0^t \int_D |\nabla_x q(t-s,x,z)||\nabla_z \Lambda_{n,m}(s,z,y)||G_n(z)|dzds \\ &\le \int_0^t \int_D M_1^2 (t-s)^{-\frac{d+1}{2}} e^{-\frac{M_2|x-z|^2}{2(t-s)}} s^{-\frac{d+1}{2}} e^{-\frac{M_2|z-y|^2}{4s}} |G_n(z)|dzds \cdot 2^{-m} \\ &\le M_1^2 M_3 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} N_{G_n,\frac{M_2}{2}}(t) \cdot 2^{-m} \le M_1 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}} \cdot 2^{-m-1}. \end{aligned} \tag{4.2.17}$$

By (4.2.6), (4.2.9) and (4.2.15), we have for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$,

$$\begin{aligned} &\Lambda_{n,m+2}(t,x,y) \\ &= \int_0^t \int_D q(t-s,x,z)\langle \nabla_z \Lambda_{n,m+1}(s,z,y), G_n(z)\rangle dzds \\ &= \int_0^t \int_D q(t-s,x,z)dzds \int_0^s \int_D \langle \nabla_z q(s-\tau,z,w), G_n(z)\rangle f_{n,m}(\tau,w,y)dwd\tau \\ &= \int_0^t \int_D q(t-s,x,z) f_{n,m+1}(s,z,y)dzds. \end{aligned}$$

Together with (4.2.16) and (4.2.17), we see that the statements of this Lemma hold for

$k = m+1$. This completes the proof. □

By (4.2.4), (4.2.7) and (4.2.12), for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$ we have

$$\begin{aligned} p_n(t,x,y) &= q(t,x,y) + \sum_{k\geq 0} \int_0^t \int_D q(t-s,x,z) f_{n,k}(s,z,y)dzds \\ &= \sum_{k\geq 0} \Lambda_{n,k}(t,x,y). \end{aligned} \tag{4.2.18}$$

Then together with (4.2.11), we also have for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$,

$$\nabla_x p_n(t,x,y) = \sum_{k\geq 0} \nabla_x \Lambda_{n,k}(t,x,y). \tag{4.2.19}$$

Now we can derive the estimates of p_n and $\nabla_x p_n$.

Proposition 4.2.3. *There exist positive constants M_4, M_5 and M_6 such that for $n \geq 1$ and $(t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D}$,*

$$p_n(t,x,y) \leq M_4 e^{M_5 t} t^{-\frac{d}{2}} e^{-\frac{M_6|x-y|^2}{2t}}, \tag{4.2.20}$$

$$|\nabla_x p_n(t,x,y)| \leq M_4 e^{M_5 t} t^{-\frac{d+1}{2}} e^{-\frac{M_6|x-y|^2}{2t}}, \tag{4.2.21}$$

and

$$p_n(t,x,y) = q(t,x,y) + \int_0^t \int_D q(t-s,x,z) \langle \nabla_z p_n(s,z,y), G_n(z) \rangle dzds. \tag{4.2.22}$$

Proof. By (4.2.18), (4.2.19) and Lemma 4.2.2, we have for $(t,x,y) \in (0,T_1] \times \bar{D} \times \bar{D}$ and $n \geq 1$,

$$p_n(t,x,y) \leq 2M_1 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}}, \tag{4.2.23}$$

$$|\nabla_x p_n(t,x,y)| \leq 2M_1 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}}. \tag{4.2.24}$$

By (4.2.23), there exists a $c_1 > 0$ such that for $(t,x,y) \in (T_1,\infty) \times \bar{D} \times \bar{D}$ and $n \geq 1$,

$$\begin{aligned} p_n(t,x,y) &= \int_D p_n(t-T_1,x,z) p_n(T_1,z,y)dz \\ &\leq \sup_{z,y\in\bar{D}} p_n(T_1,z,y) \int_D p_n(t-T_1,x,z)dz \leq c_1. \end{aligned} \tag{4.2.25}$$

On the other hand, since D is bounded, for any $c_2 > 0$, there exists a $c_3 > 0$ such that

$$\inf_{(t,x,y)\in(T_1,\infty)\times\bar{D}\times\bar{D}} c_3 e^{c_2 t} t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}} \geq c_1. \tag{4.2.26}$$

Therefore by (4.2.25) and (4.2.26), we have for $(t,x,y) \in (T_1,\infty)\times\bar{D}\times\bar{D}$ and $n \geq 1$,

$$p_n(t,x,y) \leq c_3 e^{c_2 t} t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}}.$$

Combining this with (4.2.23), we obtain (4.2.20).

By (4.2.20) and (4.2.24), there exists a $c_4 > 0$ such that for $(t,x,y) \in (T_1,\infty)\times\bar{D}\times\bar{D}$ and $n \geq 1$,

$$\begin{aligned} |\nabla_x p_n(t,x,y)| &\leq \int_D |\nabla_x p_n(T_1,x,z)| p_n(t-T_1,z,y) dz \\ &\leq \sup_{x,z\in\bar{D}} |\nabla_x p_n(t,x,z)| \int_D M_4 e^{M_5(t-T_1)} (t-T_1)^{-\frac{d}{2}} e^{-\frac{M_6|z-y|^2}{2(t-T_1)}} dz \leq c_4 e^{M_5 t}. \end{aligned}$$

Hence by (4.2.26), we can two positive constants c_5 and c_6 such that for $n \geq 1$ and $(t,x,y) \in (T_1,\infty)\times\bar{D}\times\bar{D}$,

$$|\nabla_x p_n(t,x,y)| \leq c_5 e^{c_6 t} t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}}.$$

Combining this with (4.2.24), we obtain (4.2.21).

Now we show (4.2.22). Since $\Lambda_{n,k}$ satisfies (4.2.9), by (4.2.18) and (4.2.19), we see that for $(t,x,y) \in (0,T_1]\times\bar{D}\times\bar{D}$ and $n \geq 1$,

$$\begin{aligned} p_n(t,x,y) &= \Lambda_{n,0}(t,x,y) + \sum_{k\geq 1} \int_0^t \int_D q(t-s,x,z) \langle \nabla_z \Lambda_{n,k-1}(s,z,y), G_n(z) \rangle dz ds \\ &= q(t,x,y) + \int_0^t \int_D q(t-s,x,z) \langle \nabla_z p_n(s,z,y), G_n(z) \rangle dz ds. \end{aligned}$$

For each integer $k \geq 1$, if (4.2.22) holds for $(t,x,y) \in (0,kT_1]\times\bar{D}\times\bar{D}$, then for $(t,x,y) \in$

$(kT_1,(k+1)T_1]\times\bar{D}\times\bar{D}$,

$$
\begin{aligned}
& p_n(t,x,y) \\
=& \int_D p_n(kT_1,x,w)p_n(t-kT_1,w,y)dw \\
=& \int_D q(kT_1,x,w)p_n(t-kT_1,w,y)dw \\
& + \int_0^{kT_1}\int_D\int_D q(kT_1-s,x,z)\langle\nabla_z p_n(s,z,w),G_n(z)\rangle p_n(t-kT_1,w,y)dwdzds \\
=& \int_D q(kT_1,x,w)q(t-kT_1,w,y)dw \\
& + \int_0^{t-kT_1}\int_D\int_D q(kT_1,x,w)q(t-kT_1-s,w,z)\langle\nabla_z p_n(s,z,y),G_n(z)\rangle dwdzds \\
& + \int_0^{kT_1}\int_D\int_D q(kT_1-s,x,z)\langle\nabla_z p_n(s,z,w),G_n(z)\rangle p_n(t-kT_1,w,y)dwdzds \\
=& q(t,x,y)+\int_0^{t-kT_1}\int_D q(t-s,x,z)\langle\nabla_z p_n(s,z,y),G_n(z)\rangle dzds \\
& + \int_0^{kT_1}\int_D q(kT_1-s,x,z)\langle\nabla_z p_n(s+t-kT_1,z,y),G_n(z)\rangle dzds \\
=& q(t,x,y)+\int_0^{t}\int_D q(t-s,x,z)\langle\nabla_z p_n(s,z,y),G_n(z)\rangle \mathrm{d}z\mathrm{d}s.
\end{aligned}
$$

By induction, (4.2.22) holds for $(t,x,y)\in(0,\infty)\times\bar{D}\times\bar{D}$ and $n\geq 1$. □

We now can show that $p_n(t,x,y)$ and $\nabla_x p_n(t,x,y)$ converge uniformly to some transition density function $p(t,x,y)$ and $\nabla_x p(t,x,y)$ respectively on compact subsets of $(0,\infty)\times\bar{D}\times\bar{D}$. We now define a sequence of functions $\{\Lambda_k\}_{k\geq 0}$ by

$$
\begin{aligned}
\Lambda_0(t,x,y) &:= q(t,x,y), \\
\Lambda_{k+1}(t,x,y) &:= \int_0^t\int_D q(t-s,x,z)\langle\nabla_z\Lambda_k(s,z,y),\mu(dz)\rangle ds.
\end{aligned}
$$

As in the proof of Lemma 4.2.2, we can show that there exists a constant $T_2\in(0,T_1)$ such that for $k\geq 0$ and $(t,x,y)\in(0,T_2]\times\bar{D}\times\bar{D}$,

$$
\begin{aligned}
\Lambda_k(t,x,y) &\leq M_1 t^{-\frac{d}{2}} e^{-\frac{M_2|x-y|^2}{4t}}\cdot 2^{-k}, \\
|\nabla_x\Lambda_k(t,x,y)| &\leq M_1 t^{-\frac{d+1}{2}} e^{-\frac{M_2|x-y|^2}{4t}}\cdot 2^{-k}.
\end{aligned}
\tag{4.2.27}
$$

Since ∂D is smooth, by [21, Lemma 3.2], we see that $\mu(\partial D)=0$. Then the same proof of [21, Lemma 3.7 and Lemma 3.10] gives the following Lemma:

Lemma 4.2.4. *For each $k \geq 1$ and $0 < \delta < T \leq T_2$,*

$$\lim_{n\to\infty} \sup_{(t,x,y)\in[\delta,T]\times\bar{D}\times\bar{D}} |\Lambda_{n,k}(t,x,y) - \Lambda_k(t,x,y)| = 0,$$
$$\lim_{n\to\infty} \sup_{(t,x,y)\in[\delta,T]\times\bar{D}\times\bar{D}} |\nabla_x\Lambda_{n,k}(t,x,y) - \nabla_x\Lambda_k(t,x,y)| = 0.$$

Now we define a function p on $(0,\infty)\times\bar{D}\times\bar{D}$ as follows. For $x, y \in \bar{D}$, if $t \in (0, T_2]$, set

$$p(t,x,y) := \sum_{k\geq 0} \Lambda_k(t,x,y), \tag{4.2.28}$$

and if $t \in (\frac{mT_2}{2}, \frac{(m+1)T_2}{2}]$ for $m \geq 2$, set

$$p(t,x,y) := \int_D p(t - \frac{(m-1)T_2}{2}, x, z)p(\frac{(m-1)T_2}{2}, z, y)dz. \tag{4.2.29}$$

Theorem 4.2.5. *The function $p(t,x,y)$ above is a well defined, jointly continuous transition density function. For any $0 < \delta < T$, we have*

$$\lim_{n\to\infty} \sup_{(t,x,y)\in[\delta,T]\times\bar{D}\times\bar{D}} |p_n(t,x,y) - p(t,x,y)| = 0, \tag{4.2.30}$$
$$\lim_{n\to\infty} \sup_{(t,x,y)\in[\delta,T]\times\bar{D}\times\bar{D}} |\nabla_x p_n(t,x,y) - \nabla_x p(t,x,y)| = 0. \tag{4.2.31}$$

Moreover for any $(t,x,y) \in (0,\infty)\times\bar{D}\times\bar{D}$,

$$p(t,x,y) \leq M_4 e^{M_5 t} t^{-\frac{d}{2}} e^{-\frac{M_6|x-y|^2}{2t}}, \tag{4.2.32}$$
$$|\nabla_x p(t,x,y)| \leq M_4 e^{M_5 t} t^{-\frac{d+1}{2}} e^{-\frac{M_6|x-y|^2}{2t}}, \tag{4.2.33}$$
$$p(t,x,y) = q(t,x,y) + \int_0^t \int_{\bar{D}} q(t-s,x,z)\langle\nabla_z p(s,z,y), \mu(dz)\rangle ds. \tag{4.2.34}$$

Proof. By (4.2.27), we see that for any $\delta \in (0,T_2)$, $\sum_{k\geq 0} \Lambda_k(t,x,y)$ and $\sum_{k\geq 0} \nabla_x\Lambda_k(t,x,y)$ converge uniformly to $p(t,x,y)$ and $\nabla_x p(t,x,y)$ on $[\delta, T_2]\times\bar{D}\times\bar{D}$ respectively.

By Lemma 4.2.2 and (4.2.27), for any $\delta \in (0,T_2)$ and $\varepsilon > 0$, there exists a large enough integer K_0 such that for $n \geq 1$ and $(t,x,y) \in [\delta,T_2]\times\bar{D}\times\bar{D}$,

$$\begin{aligned}(\sum_{k>K_0} \Lambda_{n,k}(t,x,y)) \vee \sum_{k>K_0} |\nabla_x\Lambda_{n,k}(t,x,y)| \leq \varepsilon,\\ (\sum_{k>K_0} \Lambda_k(t,x,y)) \vee \sum_{k>K_0} |\nabla_x\Lambda_k(t,x,y)| \leq \varepsilon.\end{aligned} \tag{4.2.35}$$

Hence it follows from (4.2.18), (4.2.28), (4.2.35) and Lemma 4.2.4 that

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{(t,x,y)\in[\delta,T_2]\times\bar{D}\times\bar{D}} |p_n(t,x,y)-p(t,x,y)| \\
\le &\lim_{n\to\infty} \sup_{(t,x,y)\in[\delta,T_2]\times\bar{D}\times\bar{D}} \sum_{0\le k\le K_0} |\Lambda_{n,k}(t,x,y)-\Lambda_k(t,x,y)| \\
&+\sup_{n\ge 1} \sup_{(t,x,y)\in[\delta,T_2]\times\bar{D}\times\bar{D}} \Big(\sum_{k>K_0} \Lambda_{n,k}(t,x,y)+\sum_{k>K_0}\Lambda_k(t,x,y)\Big) \le 2\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we have (4.2.30) for $T \le T_2$, which implies that $p(t,x,y)$ is continuous on $(0,T_2]\times\bar{D}\times\bar{D}$. Similarly, we also have (4.2.31) for $T\le T_2$. By Proposition 4.2.3, (4.2.30) and (4.2.31), (4.2.32) and (4.2.33) hold on $(0,T_2]\times\bar{D}\times\bar{D}$.

For each $m\ge 2$, if (4.2.32) holds for $t\in(0,\frac{mT_2}{2}]$, then the right hand side of (4.2.29) is integrable for $t\in(\frac{mT_2}{2},\frac{(m+1)T_2}{2}]$. Thus, $p(t,x,y)$ is well defined on $(0,\frac{(m+1)T_2}{2}]\times\bar{D}\times\bar{D}$. By induction, we see that $p(t,x,y)$ is well defined on $(0,\infty)\times\bar{D}\times\bar{D}$. Similarly, we have that (4.2.30) and (4.2.31) hold for any $0<\delta<T<\infty$, $p(t,x,y)$ is continuous on $(0,\infty)\times\bar{D}\times\bar{D}$, and (4.2.32) and (4.2.33) hold for any $(t,x,y)\in(0,\infty)\times\bar{D}\times\bar{D}$.

By Proposition 4.2.3, (4.2.30), (4.2.31), (4.2.33) and the dominated convergence theorem, we have (4.2.34) for any $(t,x,y)\in(0,\infty)\times\bar{D}\times\bar{D}$.

Since $p_n(t,x,y)$ is the transition density of $\widetilde{X}_n$, for any $s,t>0$ and $x,y\in\bar{D}$ we have

$$\begin{aligned}
&\int_D p_n(t,x,y)dy=1,\\
&\int_D p_n(t,x,z)p_n(s,z,y)dz=p_n(t+s,x,y).
\end{aligned}\tag{4.2.36}$$

Then by Proposition 4.2.3, (4.2.30), (4.2.32), (4.2.36) and the dominated convergence theorem, we see that

$$\begin{aligned}
&\int_D p(t,x,y)dy=1,\\
&\int_D p(t,x,z)p(s,z,y)dz=p(t+s,x,y),
\end{aligned}\tag{4.2.37}$$

Hence $p(t,x,y)$ is a transition density function. □

4.3 Existence of Weak Solution

In this section, we will prove that there exists a weak solution to the reflected Brownian motion with drift μ. We firstly show that there exists a continuous Feller process $(X, P_x, x \in \bar{D})$, which has the transition density $p(t,x,y)$ constructed in Section 4.2. Then by applying the potential theory, we construct CAFs A and L of X which admit the property (4.1.3). Finally, by the property (4.1.3) and the fact that $p_n(t,x,y)$ converges to $p(t,x,y)$, we show that $W(t) = X(t) - X(0) - A(t) - L(t)$, $t \geq 0$, is a Brownian motion. This implies that P_x is a weak solution to the reflected Brownian motion with drift μ for any given $x \in \bar{D}$.

Recall that $\Omega = C([0,\infty), \bar{D})$ is a set of continuous functions mapping $[0,\infty)$ to $\bar{D}$ and $X(t,\omega) = \omega(t)$ is the the coordinate map on Ω. $\mathcal{F}_\infty$ is the σ-field generated by $\{X(s) : s \geq 0\}$. Firstly, we have the following result.

Theorem 4.3.1. *There exists a family of probability measures $\{P_x\}_{x\in\bar{D}}$ on $(\Omega, \mathcal{F}_\infty)$ such that $(\Omega, X, P_x, x \in \bar{D})$ is a Feller process with strong Feller property and $p(t,x,y)$ is the transition density function of X.*

Proof. Let $\bar{\Omega}$ denote the set of all functions mapping $[0,\infty)$ to $\bar{D}$ and $\mathcal{C}$ denote the σ-field generated by the cylinder sets of finite dimensions in $\bar{\Omega}$. Set the coordinate map $\bar{X}(t,\omega) := \omega(t)$ for $\omega \in \bar{\Omega}$.

Since $p(t,x,y)$, $(t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D}$, is a family of continuous transition density function satisfying (4.2.37), by the Kolmogorov consistency theorem (see [20, Theorem II.2.2]), there exists a family of probability measures $\{\bar{P}_x\}_{x\in\bar{D}}$ on $(\bar{\Omega}, \mathcal{C})$ such that $(\bar{\Omega}, \bar{X}, \bar{P}_x, x \in \bar{D})$ is a Markov process admitting the transition density $p(t,x,y)$ and $\bar{P}_x(\bar{X}(0) = x) = 1$.

By Theorem 4.2.5 and the Markov property of $\bar{X}$, it is easy to see that for any $T > 0$, there exists a $C > 0$ such that for any $0 \leq s < t \leq T$ and $x \in \bar{D}$,

$$\begin{aligned}
\bar{E}_x\left[|\bar{X}(t) - \bar{X}(s)|^{2d+2}\right] &= \bar{E}_x\left[\bar{E}_{\bar{X}(s)}\left[|\bar{X}(t-s) - \bar{X}(0)|^{2d+2}\right]\right] \\
&\leq \sup_{x\in\bar{D}} \int_D M_4 e^{M_5(t-s)}(t-s)^{-\frac{d}{2}} e^{-\frac{M_6|z-y|^2}{2(t-s)}} |z-y|^{2d+2} dz \\
&\leq C|t-s|^{d+1},
\end{aligned}$$

where $\bar{E}_x$ is the expectation with respect to $\bar{P}_x$. This implies that the sample path of $\bar{X}$ admits a continuous version $\widetilde{X} = (\widetilde{X}(t))_{t\geq 0}$ under $\bar{P}_x$ for each $x \in \bar{D}$. Now let P_x be the law of $\widetilde{X}$ under $\bar{P}_x$. Then $(\Omega, X, P_x, x \in \bar{D})$ forms a continuous Markov process. Since

$p(t,x,y)$ is continuous on $(0,\infty)\times\bar{D}\times\bar{D}$ and yields the Gaussian type upper estimate by Theorem 4.2.5, we can see that $(\Omega,X,P_x,x\in\bar{D})$ is a Feller process with strong Feller property.

□

Now we show the following Lemma which will be helping to prove the uniqueness of weak solution to the reflected Brownian motion with drift μ.

Lemma 4.3.2. *Let $\beta>0$. Then exists $\delta<1$ such that if $\tau:=\inf\{t:|X(t)-X(0)|>\beta\}$, then*

$$\sup_{x\in\bar{D}} E_x[e^{-\tau}]\le\delta,$$

where E_x is the expectation with respect to P_x.

Proof. By (4.2.32) and the proof of [3, Theorem 2.7], we see that there exist constants $c_1,c_2>0$ such that for any $t<1$ and $\beta>0$,

$$\sup_{x\in\bar{D}} P_x(\sup_{s\le t}|X(s)-x|>\beta)\le c_1 t^{-\frac{d}{2}}e^{-\frac{c_2\beta^2}{t}},$$

which implies $\lim_{t\to 0}\sup_{x\in\bar{D}} P_x(\sup_{s\le t}|X(s)-x|>\beta)=0$. Similar to the proof of [1, Corollary 4.4], we can show that $\sup_{x\in\bar{D}} E_x[e^{-\tau}]\le\delta$ for some $\delta<1$. □

Next, we are going to show that the Feller process constructed above is a solution to the reflected Brownian motion with singular drift μ. To serve this purpose, we first construct the drift CAF A and the local time L of X by the potential theory.

We fix a constant $\lambda>M_5$. We say that a bounded positive function f is a λ-potential function if for any $t\ge 0$ and $x\in\bar{D}$, $P_{\lambda,t}f(x):=E_x[e^{-\lambda t}f(X(t))]$ satisfies

$$P_{\lambda,t}f(x)\le f(x),$$

and

$$\lim_{t\to 0}\sup_{x\in\bar{D}}|P_{\lambda,t}f(x)-f(x)|=0.$$

Let μ_+ and μ_- denote the positive and negative part of the signed measure μ. Then we define

$$G_{+,n}(x) := \int_{\mathbb{R}^d} \psi_n(x-y)\mu_+(dy), \;\; G_{-,n}(x) := \int_{\mathbb{R}^d} \psi_n(x-y)\mu_-(dy).$$
$$U_{+,n} := E_x\left[\int_0^\infty e^{-\lambda t} G_{+,n}(X(t))dt\right], \;\; U_{-,n}(x) := E_x\left[\int_0^\infty e^{-\lambda t} G_{-,n}(X(t))dt\right].$$
$$U_+(x) := \int_0^\infty e^{-\lambda t}\int_D p(t,x,y)\mu_+(dy)dt, \;\; U_-(x) := \int_0^\infty e^{-\lambda t}\int_D p(t,x,y)\mu_-(dy)dt.$$

By Lemma 4.2.1 and Theorem 4.2.5, we can see that $U_{+,n}(x)$ and $U_{-,n}(x)$ converge uniformly to $U_+(x)$ and $U_-(x)$ on $\bar{D}$ respectively. Hence U_+ and U_- are continuous λ-potential function. By [4, Theorem IV.3.16], we know that if f is a λ-potential function, then there exists a positive CAF $C = (C(t))_{t\geq 0}$ of X such that for any $x \in \bar{D}$, $f(x) = E_x\left[\int_0^\infty e^{-\lambda t} dC(t)\right]$. Hence there exist positive CAFs $A_+ = (A_+(t))_{t\geq 0}$ and $A_- = (A_-(t))_{t\geq 0}$ of X such that

$$U_+(x) = E_x\left[\int_0^\infty e^{-\lambda t} dA_+(t)\right], \;\; U_-(x) = E_x\left[\int_0^\infty e^{-\lambda t} dA_-(t)\right].$$

We have the following result.

Theorem 4.3.3. *Let*

$$A(t) := A_+(t) - A_-(t).$$

Then

$$\sup_{x\in\bar{D}} E_x\left[\int_0^\infty e^{-\lambda s} d|A|(s)\right] < \infty, \tag{4.3.1}$$

and $\int_0^t G_n(X(s))ds$ *converges uniformly to* $A(t)$ *over finite intervals in probability* P_x *for each* $x \in \bar{D}$ *as* $n \to \infty$. *Moreover, there exists a subsequence* $\{n_k\}_{k\geq 1}$ *such that*

$$\sup_{x\in\bar{D}} E_x\left[\sup_{k\geq 1}\int_0^\infty e^{-\lambda s}|G_{n_k}(X(s))|ds\right] < \infty. \tag{4.3.2}$$

Proof. Let $r_0 := \sup\{|x| : x \in D\}$. Recall that $B(x,r)$ is the ball in $\mathbb{R}^d$ centered at x with radius r. Set

$$M_{G_n}(r) := \sup_{x\in\mathbb{R}^d}\int_{B(x,r)} |x-y|^{2-d}|G_n(y)|dy,$$
$$M_\mu(r) := \sup_{x\in\mathbb{R}^d}\int_{B(x,r)} |x-y|^{2-d}|\mu|(dy).$$

Then following the proof of [1, Proposition 3.6] we can see that

$$M_{G_n}(r_0) \leq M_\mu(r_0), \ \forall\, n \geq 1.$$

Therefore,

$$\begin{aligned}
E_x\left[\int_0^\infty e^{-\lambda s}|G_n(X(s))|ds\right] &= \int_0^\infty e^{-\lambda s}\int_D p(s,x,y)|G_n(y)|dyds \\
&\leq \int_0^\infty e^{-\lambda s}\int_D M_4 s^{-\frac{d}{2}}e^{M_5 s}e^{-\frac{M_6|x-y|^2}{2s}}|G_n(y)|dyds \\
&\leq \int_D M_4\int_0^\infty s^{-\frac{d}{2}}e^{-\frac{M_6|x-y|^2}{2s}}ds|G_n(y)|dy \\
&\leq c\int_D \frac{1}{|x-y|^{d-2}}|G_n(y)|dy \\
&\leq c\int_{B(0,r_0)} \frac{1}{|x-y|^{d-2}}|G_n(y)|dy \\
&\leq cM_{G_n}(r_0) \leq cM_\mu(r_0).
\end{aligned}$$

Hence we obtain

$$\sup_{n\geq 1, x\in\bar{D}} E_x\left[\int_0^\infty e^{-\lambda s}|G_n(X(s))|ds\right] < \infty. \tag{4.3.3}$$

On the other hand, by the definition of $A(t)$ and the boundness of $|U_+(x)|$ and $|U_-(x)|$, we have

$$\sup_{x\in\bar{D}} E_x\left[\int_0^\infty e^{-\lambda s}d|A|(s)\right] \leq \sup_{x\in\bar{D}}(U_+(x)+U_-(x)) < \infty. \tag{4.3.4}$$

In view of Lemma 4.2.1 and Theorem 4.2.5, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{x\in\bar{D}}\left|E_x\left(\int_0^\infty e^{-\lambda t}G_n(X(t))dt\right) - E_x\left(\int_0^\infty e^{-\lambda t}dA(t)\right)\right| \\
&= \lim_{n\to\infty}\sup_{x\in\bar{D}}\left|\int_0^\infty\int_D e^{-\lambda t}p(t,x,y)G_n(y)dydt - \int_0^\infty\int_D e^{-\lambda t}p(t,x,y)\mu(dy)dt\right| = 0.
\end{aligned} \tag{4.3.5}$$

Combining (4.3.3)-(4.3.5) and [1, Lemma 3.10], we get that for any $t > 0$,

$$\lim_{n\to\infty}\sup_{x\in\bar{D}} E_x\left[\sup_{s\leq t}\left|\int_0^s G_n(X(r))dr - A(s)\right|^2\right] = 0.$$

In particular,

$$\int_0^t G_n(X(s))ds \to A(t)$$

uniformly as $n \to \infty$ with respect to t over finite intervals in probability P_x for each $x \in \bar{D}$.

Similarly, using [2, Proposition I.6.14] we also have

$$\lim_{n\to\infty} \sup_{x\in\bar{D}} E_x \left[\left| \int_0^\infty e^{-\lambda s}(G_{+,n}(X(s)) + G_{-,n}(X(s)))ds - \int_0^\infty e^{-\lambda s}(dA_+(s) + dA_-(s)) \right|^2 \right] = 0.$$

Hence there exists a subsequence $\{n_k\}_{k\geq 1}$ such that for any $x \in \bar{D}$,

$$E_x \left[\left| \int_0^\infty e^{-\lambda s}(G^+_{n_k}(X_s) + G^-_{n_k}(X_s))ds - \int_0^\infty e^{-\lambda s}(dA^+_s + dA^-_s) \right|^2 \right] \leq 2^{-k}.$$

It follows that

$$\begin{aligned}
&\sup_{x\in\bar{D}} E_x \left[\sup_{k\geq 1} \left| \int_0^\infty e^{-\lambda s}(G_{+,n_k}(X(s)) + G_{-,n_k}(X(s)))ds \right| \right] \\
\leq &\sup_{x\in\bar{D}} E_x \left[\sup_{k\geq 1} \left| \int_0^\infty e^{-\lambda s}(G_{+,n_k}(X(s)) + G_{-,n_k}(X(s)))ds - \int_0^\infty e^{-\lambda s}(dA_+(s) + dA_-(s)) \right| \right] \\
&+ \sup_{x\in\bar{D}} E_x \left[\left| \int_0^\infty e^{-\lambda s}(dA_+(s) + dA_-(s)) \right| \right] \\
\leq &\sup_{x\in\bar{D}} E_x \left[\sup_{k\geq 1} \left| \int_0^\infty e^{-\lambda s}(G_{+,n_k}(X(s)) + G_{-,n_k}(X(s)))ds - \int_0^\infty e^{-\lambda s}(dA_+(s) + dA_-(s)) \right|^2 \right]^{\frac{1}{2}} \\
&+ \sup_{x\in\bar{D}} E_x \left[\left| \int_0^\infty e^{-\lambda s}(dA_+(s) + dA_-(s)) \right| \right] \\
\leq &(\sum_{k\geq 1} 2^{-k})^{\frac{1}{2}} + \sup_{x\in\bar{D}} E_x \left[\left| \int_0^\infty e^{-\lambda s}(dA_+(s) + dA_-(s)) \right| \right] < \infty.
\end{aligned}$$

Note that $|G_n(x)| \leq |G_{+,n}(x) + G_{-,n}(x)|$, hence we obtain (4.3.2). □

Recall that $P_{\lambda,t} f(x) = E_x \left[e^{-\lambda t} f(X(t)) \right]$ and σ is the $(d-1)$-dimensional volume element on ∂D. Let $V(x)$ be defined by

$$V(x) := \int_0^\infty e^{-\lambda t} \int_{\partial D} p(t,x,y)\sigma(dy)dt.$$

Then we have

Lemma 4.3.4. *There exists a positive CAF $K = (K(t))_{t\geq 0}$ of X such that*

$$V(x) = E_x\left[\int_0^\infty e^{-\lambda t} dK(t)\right].$$

Proof. By [4, Theorem IV.3.16], it is sufficient to proved that $V(x)$ is a λ-potential function. This will be proved by showing

$$P_{\lambda,t}V(x) \leq V(x),\ \forall x \in \bar{D},\ \ t > 0, \tag{4.3.6}$$

$$\lim_{t\to\infty} P_{\lambda,t}V(x) = 0,\ \ \forall x \in \bar{D}, \tag{4.3.7}$$

$$\lim_{t\to 0}\sup_{x\in\bar{D}} \left|P_{\lambda,t}V(x) - V(x)\right| = 0. \tag{4.3.8}$$

We start with (4.3.6),

$$\begin{aligned}
P_{\lambda,t}V(x) &= E_x\left[e^{-\lambda t}V(X_t)\right] \\
&= e^{-\lambda t}\int_D p(t,x,y)\int_0^\infty e^{-\lambda s}\int_{\partial D} p(s,y,z)\sigma(dz)dsdy \\
&= e^{-\lambda t}\int_0^\infty e^{-\lambda s}\int_{\partial D}\sigma(dz)\cdot\int_D p(t,x,y)p(s,y,z)dyds \\
&= e^{-\lambda t}\int_0^\infty e^{-\lambda s}\int_{\partial D} p(t+s,x,z)\sigma(dz)ds \\
&= \int_t^\infty e^{-\lambda s}\int_{\partial D} p(s,x,y)\sigma(dy)ds \\
&\leq V(x)
\end{aligned} \tag{4.3.9}$$

(4.3.7) is a consequence of (4.3.9). To prove (4.3.8), note that

$$\left|P_{\lambda,t}V(x) - V(x)\right| = \int_0^t e^{-\lambda s}\int_{\partial D} p(s,x,y)\sigma(dy)ds.$$

Using a similar argument as in the proof of [16, Theorem I.2.1], we can show that there exists a $c_1 > 0$ such that

$$\sup_{x\in\bar{D}}\int_{\partial D} s^{-\frac{d}{2}} e^{\frac{-|x-y|^2}{2s}}\sigma(dy) \leq c_1 s^{-\frac{1}{2}},\ \forall s > 0. \tag{4.3.10}$$

Hence together with (4.2.32), we obtain $\left|P_{\lambda,t}V(x) - V(x)\right| \leq c_2\sqrt{t}$, which implies (4.3.8). □

Next we will provide a number of properties for the CAF K, which will be used later.

Lemma 4.3.5. *For $T > 0$, we have*

$$E_x[K(T)] = \int_0^T \int_{\partial D} p(t,x,y)\sigma(dy)dt. \tag{4.3.11}$$

Proof. For $\beta \geq \lambda$, define $S_\beta f(x) := E_x\left[\int_0^{+\infty} e^{-\beta t} f(X(t))dt\right]$ and

$$V_\beta(x) := E_x\left[\int_0^{+\infty} e^{-\beta t} dK(t)\right].$$

Since

$$+\infty > \sup_{x\in\bar{D}} V_\lambda(x) \geq \sup_{x\in\bar{D}} E_x\left[\int_0^T e^{-\lambda t} dK(t)\right] \geq e^{-\lambda T} \sup_{x\in\bar{D}} E_x[K(T)],$$

there exists a $c_1 = c_1(T) < \infty$ such that $\sup_{x\in\bar{D}} E_x[K_T] \leq c_1$. Thus,

$$\begin{aligned} \sup_{x\in\bar{D}} E_x[K(2T)] &= \sup_{x\in\bar{D}} E_x[K(T)] + \sup_{x\in\bar{D}} E_x\left[E_{X(T)}[K(T)]\right] \\ &\leq 2\sup_{x\in\bar{D}} E_x[K(T)] \leq 2c_1. \end{aligned}$$

Similarly we have $E_x[K(nT)] \leq nc_1$ for each $n \geq 1$. Hence, we obtain

$$\begin{aligned} V_\beta(x) &= \lim_{n\to\infty} E_x\left[\int_0^{nT} e^{-\beta t} dK(t)\right] \\ &= \lim_{n\to\infty}\left(e^{-\beta nT} E_x[K(nT)] + E_x\left[\int_0^{nT} \beta e^{-\beta t} K(t)dt\right]\right) \\ &= E_x\left[\int_0^\infty \beta e^{-\beta t} K(t)dt\right]. \end{aligned} \tag{4.3.12}$$

On the other hand, since K is a CAF of X, we have

$$\begin{aligned} V_\beta(x) - V_\lambda(x) &= (\lambda-\beta) S_\beta V_\lambda(x), \\ \Rightarrow V_\beta(x) &= V_\lambda(x) + (\lambda-\beta) S_\beta V_\lambda(x) \\ &= \int_0^\infty e^{-\beta t} \int_{\partial D} p(t,x,y)\sigma(dy)dt. \end{aligned} \tag{4.3.13}$$

Set $F(t) := \int_0^t \int_{\partial D} p(s,x,y)\sigma(dy)ds$, then $e^{-\lambda t}F(t) \to 0$ as $t \to \infty$ by (4.2.32) and (4.3.10).

Hence by (4.3.13),

$$\begin{aligned} V_\beta(x) &= \lim_{T\to\infty}\int_0^T e^{-\beta t}dF(t), \\ &= \lim_{T\to\infty}[e^{-\beta T}F(T)+\int_0^T \beta e^{-\beta t}F(t)dt] \\ &= \int_0^\infty \beta e^{-\beta t}F(t)dt. \end{aligned}$$

Combining this with (4.3.12), we get

$$\int_0^\infty e^{-\beta t}E_x[K(t)]dt = \int_0^\infty e^{-\beta t}F(t)dt,\ \forall \beta \geq \lambda,$$

which implies (4.3.11) by the uniqueness of the Laplace transform. □

Lemma 4.3.6. *For any $f \in C([0,T]\times\bar{D})$, it holds that*

$$E_x\left[\int_0^T f(s,X(s))dK(s)\right] = \int_0^T \int_{\partial D} p(s,x,y)f(s,y)\sigma(dy)ds, \tag{4.3.14}$$

$$E_x\left[\int_0^T f(s,X(s))dA_i(s)\right] = \int_0^T \int_D p(s,x,y)f(s,y)\mu_i(dy)ds. \tag{4.3.15}$$

where $A_i(s)$ is the i-th component of $A(t)$.

Proof. We only prove (4.3.14), (4.3.15) can be proved similarly. It is sufficient to prove that for any $t \in (0,T)$

$$E_x\left[\int_t^T f(s,X(s))dK(s)\right] = \int_t^T \int_{\partial D} p(s,x,y)f(s,y)\sigma(dy)ds. \tag{4.3.16}$$

Taking a sequence of $\{s_{n,j} : 0 \leq j \leq n_j\}$ of partitions of $[t,T]$ satisfying that $\max_j \Delta s_{n,j} :=$

$\max_j(s_{n,j+1} - s_{n,j}) \to 0$ as $n \to \infty$, then by the Markov property and (4.3.11), we have

$$\begin{aligned}
&E_x\left[\int_t^T f(s,X(s))dK(s)\right] \\
&= \lim_{n\to\infty}\sum_j E_x\left[f(s_{n,j},X(s_{n,j}))(K(s_{n,j+1})-K(s_{n,j}))\right] \\
&= \lim_{n\to\infty}\sum_j E_x\left[f(s_{n,j},X(s_{n,j}))E_{X(s_{n,j})}\left[K(\Delta s_{n,j})\right]\right] \\
&= \lim_{n\to\infty}\sum_j \int_D p(s_{n,j},x,y)f(s_{n,j},y)\int_0^{\Delta s_{n,j}}\int_{\partial D} p(u,y,z)\sigma(dz)dudy \\
&= \lim_{n\to\infty}\sum_j \int_{\partial D}\sigma(dz)\int_0^{\Delta s_{n,j}}\int_D p(s_{n,j},x,y)f(s_{n,j},y)p(u,y,z)dydu.
\end{aligned} \tag{4.3.17}$$

Note that the right hand side of (4.3.16) can be rewritten as

$$\begin{aligned}
&\int_t^T\int_{\partial D} f(s,z)p(s,x,z)\sigma(dz)ds \\
&= \lim_{n\to\infty}\sum_j\int_{\partial D}\sigma(dz)\cdot f(s_{n,j},z)\cdot p(s_{n,j},x,z)\Delta s_{n,j}.
\end{aligned} \tag{4.3.18}$$

On the other hand, by the Komogorov-Chapmann equation,

$$\begin{aligned}
&\left|f(s_{n,j},z)\cdot p(s_{n,j},x,z)\Delta s_{n,j} - \int_0^{\Delta s_{n,j}}\int_D p(s_{n,j},x,y)f(s_{n,j},y)p(u,y,z)dydu\right| \\
=&\left|\int_0^{\Delta s_{n,j}}\int_D p(s_{n,j}-u,x,y)p(u,y,z)f(s_{n,j},z)dydu - \int_0^{\Delta s_{n,j}}\int_D p(s_{n,j},x,y)f(s_{n,j},y)p(u,y,z)dydu\right| \\
\le&\left|\int_0^{\Delta s_{n,j}}\int_D \left(p(s_{n,j}-u,x,y)-p(s_{n,j},x,y)\right)p(u,y,z)dydu\cdot f(s_{n,j},z)\right| \\
&+\int_D\int_0^{\Delta s_{n,j}} p(s_{n,j},x,y)\left|f(s_{n,j},z)-f(s_{n,j},y)\right|p(u,y,z)dudy. \\
=&\left|\int_0^{\Delta s_{n,j}}\left(p(s_{n,j},x,z)-p(s_{n,j}+u,x,z)\right)f(s_{n,j},z)du\right| \\
&+\int_D\int_0^{\Delta s_{n,j}} p(s_{n,j},x,y)\left|f(s_{n,j},z)-f(s_{n,j},y)\right|p(u,y,z)dudy. \\
=&I_{n,j}+II_{n,j}.
\end{aligned}$$

Notice that

$$I_{n,j} \le \int_0^{\Delta s_{n,j}}\left|p(s_{n,j},x,z)-p(s_{n,j}+u,x,z)\right|du\cdot\|f\|_\infty.$$

where $||f||_\infty := \sup_{(t,x)\in[0,T]\times\bar{D}} |f(t,x)|$. By the continuity of $p(s,x,z)$, for any $\varepsilon > 0$, there exists a $N > 0$ such that any $n \geq N$ and $0 < u \leq \Delta s_{n,j}$,

$$|p(s,x,z) - p(s+u,x,z)| \leq \varepsilon,\ \forall (s,x,z) \in [t,T]\times\bar{D}\times\bar{D}.$$

Hence for $n \geq N$ and $0 \leq j \leq n_j$, we have

$$I_{n,j} \leq c_1 \varepsilon \Delta s_{n,j}.$$

Since f is uniformly continuous on $[t,T]\times\bar{D}$, there exits a constant $\delta > 0$ such that for $(s,y,z) \in [t,T]\times\bar{D}\times\bar{D}$ with $|y-z| \leq \delta$, we have $|f(s,z) - f(s,y)| < \varepsilon$. Hence, in view of (4.2.32),

$$\begin{aligned}
II_{n,j} \leq & \int_{|y-z|>\delta}\int_0^{\Delta s_{n,j}} p(s_{n,j},x,y)p(u,y,z)du dy \cdot 2\|f\|_\infty \\
& + \varepsilon \int_{|y-z|\leq\delta}\int_0^{\Delta s_{n,j}} p(s_{n,j},x,y)p(u,y,z)du dy \\
\leq & \int_{|y-z|>\delta}\int_0^{\Delta s_{n,j}} p(s_{n,j},x,y)c_1 u^{-\frac{d}{2}} e^{-\frac{M_6\delta^2}{2u}} du dy + \varepsilon \int_0^{\Delta s_{n,j}} p(s_{n,j}+u,x,z)du \\
\leq & \int_0^{\Delta s_{n,j}} c_1 u^{-\frac{d}{2}} e^{-\frac{M_6\delta^2}{2u}} du + \varepsilon \sup_{(s,x,y)\in[t,T]\times\bar{D}\times\bar{D}} |p(s,x,y)|\Delta s_{n,j} \\
\leq & c_2(\delta)(\Delta s_{n,j})^2 + \varepsilon \sup_{(s,x,y)\in[t,T]\times\bar{D}\times\bar{D}} |p(s,x,y)|\Delta s_{n,j},
\end{aligned}$$

where $c_2(\delta)$ is some positive constant dependent on δ. Hence we have

$$\begin{aligned}
& \lim_{n\to\infty}\sum_j \int_{\partial D} \sigma(dy)(I_{n,j} + II_{n,j}) \\
\leq & \lim_{n\to\infty}\sum_j \int_{\partial D} \sigma(dy)\left[\varepsilon\left(c_1 + \sup_{(s,x,y)\in[t,T]\times\bar{D}\times\bar{D}} |p(s,x,y)|\right)\Delta s_{n,j} + c_2(\delta)(\Delta s_{n,j})^2\right] \\
= & \lim_{n\to\infty}\sum_j \sigma(\partial D)\left[\varepsilon\left(c_1 + \sup_{(s,x,y)\in[t,T]\times\bar{D}\times\bar{D}} |p(s,x,y)|\right)\Delta s_{n,j} + c_2(\delta)(\Delta s_{n,j})^2\right] \\
\leq & \sigma(\partial D)\left[\varepsilon\left(c_1 + \sup_{(s,x,y)\in[t,T]\times\bar{D}\times\bar{D}} |p(s,x,y)|\right)\right] T + c_2(\delta)\lim_{n\to\infty}(\max_j \Delta s_{n,j})\sum_j(\Delta s_{n,j}) \\
\leq & \sigma(\partial D)\left(c_1 + \sup_{(s,x,y)\in[t,T]\times\bar{D}\times\bar{D}} |p(s,x,y)|\right)\cdot\varepsilon\cdot T.
\end{aligned}$$

Since ε is arbitrary, we obtain

$$\lim_{n\to\infty}\sum_j \int_{\partial D}\sigma(dy)(I_{n,j}+II_{n,j})=0.$$

Combining this with (4.3.17) and (4.3.18), we see that (4.3.16) holds. □

The Lemma 4.3.6 particularly implies that $K(t)$, $t \geq 0$, only increases when X hits the boundary of D, i.e.

$$\int_0^t \mathbf{1}_{\partial D}(X(s))dK(s)=K(t),\ t\geq 0.$$

Indeed, for a nonnegative function $g \in C_c(D)$, it follows from Lemma 4.3.6 that

$$E_x\left[\int_0^t g(X(s))dK(s)\right]=0.$$

This indicates that

$$\int_0^t g(X(s))dK(s)=0.$$

Therefore, choosing a sequence of functions $\{g_n\}_{n\geq 1}\subset C_c(D)$ such that g_n converges to $\mathbf{1}_D$ as $n\to\infty$, one can derive

$$\int_0^t \mathbf{1}_D(X(s))dK(s)=0 \ \Rightarrow\ \int_0^t \mathbf{1}_{\partial D}(X(s))dK(s)=K(t).$$

Set $L_i(t) := \frac{1}{2}\int_0^t n_i(X(s))\mathbf{1}_{\partial D}(X(s))dK(s)$ and $L(t) := (L_1(t), L_2(t), \cdots, L_d(t))$. We have

$$|L|(t)=\frac{1}{2}\int_0^t |\mathbf{1}_{\partial D}(X(s))n(X(s))|\,dK(s)=\frac{1}{2}\int_0^t \mathbf{1}_{\partial D}(X(s))dK(s)=\frac{1}{2}K(t),$$

Hence

$$L(t)=\int_0^t n(X(s))\mathbf{1}_{\partial D}(X(s))d|L|(s).$$

The existence of the reflected Brownian motion with the singular drift μ is contained in the following theorem.

Theorem 4.3.7. *(a) For any $x\in\bar{D}$, (X,P_x) is a weak solution to the reflected Brownian motion with drift μ satisfying that*

$$X(t)=x+W(t)+A(t)+L(t).$$

(b) *The collection $(X, P_x, x \in \bar{D})$ forms a Feller process with strong Feller property, which has a jointly continuous transition density $p(t,x,y)$ on $(0,\infty)\times\bar{D}\times\bar{D}$. Furthermore, each component A_i of A and $|L|$ are CAFs of X, whose Revuz measures are μ_i and $\frac{1}{2}\sigma$ respectively. i.e. for every $T>0$ and continuous function f on $[0,T]\times\bar{D}$,*

$$E_x\left[\int_0^T f(s,X(s))dA_i(s)\right] = \int_0^T\int_D p(s,x,y)f(s,y)\mu_i(dy)ds,\ \forall x\in\bar{D},$$

$$E_x\left[\int_0^T f(s,X(s))d|L|(s)\right] = \frac{1}{2}\int_0^T\int_{\partial D} p(s,x,y)f(s,y)\sigma(dy)ds,\ \forall x\in\bar{D}.$$

Proof. We first prove the part (a). For each $1\le i\le d$, define a stochastic process $W_i = (W_i(t))_{t\ge 0}$ as:

$$W_i(t) := X_i(t) - X_i(0) - A_i(t) - L_i(t),$$

where $X_i(t)$ is the i-th component of $X(t)$. By Theorem 4.3.3 and the construction of $L(t)$ above, it only remains to show that $W=(W_1,\cdots,W_d)$ is a d-dimensional Brownian motion. To do this, it is sufficient to prove that W is a martingale and $\langle W_i(t), W_j(t)\rangle = t\cdot\delta_{ij}$.

Recall that θ_t is the shift operator. Since A and L are CAFs of X, we have

$$W_i(t+s) = W_i(t) + W_i(s)\circ\theta_t.$$

For the martingale property, note that

$$E_x\left[W_i(t+s)|\mathcal{F}_t\right] = E_x\left[W_i(t)+W_i(s)\circ\theta_t|\mathcal{F}_t\right] = W_i(t) + E_{X(t)}\left[W_i(s)\right].$$

Therefore, if we can prove $E_x[W_i(t)]=0$ for $x\in D$, then W is a martingale. Recall the reflected SDEs (4.2.3), let $X_{i,n}(t)$, $L_{i,n}(t)$ and $\widetilde{W}_i(t)$ be the i-th components of $\widetilde{X}_n(t)$, $\widetilde{L}_n(t)$ and $\widetilde{W}(t)$ respectively, and let $\widetilde{P}_x$ be the conditional probability of $\widetilde{P}$ given $\widetilde{X}_n(0)=x$. Then

$$\widetilde{E}_x\left[X_{i,n}(t) - X_{i,n}(0) - \int_0^t G_{i,n}(\widetilde{X}_n(s))ds - L_{i,n}(t)\right] = \widetilde{E}_x\left[\widetilde{W}_i(t)\right] = 0,$$

where $\widetilde{E}_x$ is the expectation with respect to $\widetilde{P}_x$. Thus, to show $E_x[W_i(t)]=0$, it is sufficient to prove that

$$\lim_{n\to\infty} \widetilde{E}_x \left[X_{i,n}(t) - X_{i,n}(0) - \int_0^t G_{i,n}(\widetilde{X}_n(s))ds - L_{i,n}(t) \right] = E_x \left[X_i(t) - X_i(0) - A_i(t) - L_i(t) \right].$$

For the local time $|L_n|(t)$, similar to the proof of [16, Theorem 4.2], we can show that

$$\widetilde{E}_x \left[|L_n|(t) \right] = \frac{1}{2} \int_0^t \int_{\partial D} p_n(s,x,y) \sigma(dy) ds.$$

Hence

$$\begin{aligned} &\widetilde{E}_x \left[X_{i,n}(t) - X_{i,n}(0) - \int_0^t G_{i,n}(\widetilde{X}_n(s))ds - L_{i,n}(t) \right] \\ =& \int_D p_n(t,x,y) y_i dy - x_i - \int_0^t \int_D p_n(s,x,y) G_{i,n}(y) dy ds \\ &- \frac{1}{2} \int_0^t \int_{\partial D} p_n(s,x,y) n_i(y) \sigma(dy) ds, \end{aligned}$$

where y_i is the i-th component of $y \in \mathbb{R}^d$. If we take $n \to \infty$, by the convergence of p_n, we have

$$\int_D p_n(t,x,y) y_i dy \to \int_D p(t,x,y) y_i dy, \tag{4.3.19}$$

$$\int_0^t \int_D p_n(s,x,y) G_{i,n}(y) dy ds \to \int_0^t \int_D p(s,x,y) \mu_i(dy) ds. \tag{4.3.20}$$

For the term

$$I_n := \int_0^t \int_{\partial D} p_n(s,x,y) n_i(y) \sigma(dy) ds,$$

by Proposition 4.2.3, Theorem 4.2.5 and (4.3.10), for any $\varepsilon > 0$, we can choose a constant $\delta \in (0,t)$ sufficiently small, such that

$$\begin{aligned} &\left| I_n - \int_0^t \int_{\partial D} p(s,x,y) n_i(y) \sigma(dy) ds \right| \\ \le& \int_0^\delta \int_{\partial D} p(s,x,y) |n_i(y)| \sigma(dy) ds + \sup_{n\ge 1} \int_0^\delta \int_{\partial D} p_n(s,x,y) |n_i(y)| \sigma(dy) ds \\ &+ \int_\delta^t \int_{\partial D} |p(s,x,y) - p_n(s,x,y)| \, |n_i(y)| \sigma(dy) ds, \\ \le& \varepsilon + \int_\delta^t \int_{\partial D} |p(s,x,y) - p_n(s,x,y)| \, |n_i(y)| \sigma(dy) ds. \end{aligned}$$

In view of (4.2.30) , letting $n \to \infty$ on both sides of the above inequality, one obtains the following estimate:

$$\lim_{n\to\infty}\left|I_n-\int_0^t\int_{\partial D}p(s,x,y)n_i(y)\sigma(dy)ds\right|\le\varepsilon.$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty}I_n=\int_0^t\int_{\partial D}p(s,x,y)n_i(y)\sigma(dy)ds. \tag{4.3.21}$$

We can therefore conclude, in view of (4.3.19)-(4.3.21), that as $n\to\infty$,

$$\tilde{E}_x\left[X_{i,n}(t)-X_{i,n}(0)-\int_0^t G_{i,n}(\widetilde{X}_n(s))ds-L_{i,n}(t)\right]\to E_x\left[X_i(t)-X_i(0)-A_i(t)-L_i(t)\right].$$

In particular,

$$E_x\left[W_i(t)\right]=E_x\left[X_i(t)-X_i(0)-A_i(t)-L_i(t)\right]=0.$$

Since W_i and W_j are CAFs of X, to prove $\left\langle W_i,W_j\right\rangle(t)=t\delta_{ij}$, it suffices to show that

$$E_x\left[W_i(t)W_j(t)\right]=t\cdot\delta_{ij}. \tag{4.3.22}$$

Recall that

$$\widetilde{W}(t)=\widetilde{X}_n(t)-\widetilde{X}_n(0)-\int_0^t G_n(\widetilde{X}_n(s))ds-\widetilde{L}_n(t),$$

is a Brownian motion under $\widetilde{P}_x$. Therefore, (4.3.22) will follow if we show that

$$\begin{aligned}&\lim_{n\to\infty}\widetilde{E}_x\Bigg[\left(X_{i,n}(t)-X_{i,n}(0)-\int_0^t G_{i,n}(\widetilde{X}_n(s))ds-L_{i,n}(t)\right)\\&\qquad\cdot\left(X_{j,n}(t)-X_{j,n}(0)-\int_0^t G_{j,n}(\widetilde{X}_n(s))ds-L_{j,n}(t)\right)\Bigg]\\&=\ E_x\left[\left(X_i(t)-X_i(0)-A_i(t)-L_i(t)\right)\left(X_j(t)-X_j(0)-A_j(t)-L_j(t)\right)\right].\end{aligned}$$

To establish the above equality, we only need to show the following convergence:

$$\widetilde{E}_x\left[X_{i,n}(t)X_{j,n}(t)\right] \to E_x\left[X_i(t)X_j(t)\right], \tag{4.3.23}$$

$$\widetilde{E}_x\left[X_{i,n}(t)\int_0^t G_{j,n}(\widetilde{X}_n(s))ds\right] \to E_x\left[X_i(t)A_j(t)\right], \tag{4.3.24}$$

$$\widetilde{E}_x\left[X_{i,n}(t)L_{j,n}(t)\right] \to E_x\left[X_i(t)L_j(t)\right], \tag{4.3.25}$$

$$\widetilde{E}_x\left[\int_0^t G_{i,n}(\widetilde{X}_n(s))ds\int_0^t G_{j,n}(\widetilde{X}_n(s))ds\right] \to E_x\left[A_i(t)A_j(t)\right], \tag{4.3.26}$$

$$\widetilde{E}_x\left[\int_0^t G_{i,n}(\widetilde{X}_n(s))ds\cdot L_{j,n}(t)\right] \to E_x\left[A_i(t)L_j(t)\right], \tag{4.3.27}$$

$$\widetilde{E}_x\left[L_{i,n}(t)L_{j,n}(t)\right] \to E_x\left[L_i(t)L_j(t)\right]. \tag{4.3.28}$$

(4.3.23) is a consequence of convergence:

$$\int_D y_iy_jp_n(t,x,y)dy \to \int_D y_iy_jp(t,x,y)dy.$$

For (4.3.24), by Lemma (4.2.1) and the boundness of $|X_{i,n}(t)|$ and $|X_i(t)|$, for any $\varepsilon > 0$, there exists a $\delta_1 \in (0,t)$ such that

$$\sup_{n\geq 1}\int_0^{\delta_1}\int_D \widetilde{E}_y\left[|X_{i,n}(t-s)|\right]\cdot|G_{j,n}(y)|\cdot p_n(s,x,y)dyds \\ +\int_0^{\delta_1}\int_D E_y[|X_i(t-s)|]\cdot p(s,x,y)|\mu_j|(dy)ds \ < \ \varepsilon.$$

Together with Theorem 4.2.5 and Lemma 4.3.6, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\widetilde{E}_x\left[X_{i,n}(t)\int_0^t G_{j,n}(\widetilde{X}_n(s))ds\right]-E_x\left[X_i(t)A_j(t)\right]\right| \\
&=\lim_{n\to\infty}\left|\widetilde{E}_x\left[\int_0^t X_{i,n}(t)G_{j,n}(\widetilde{X}_n(s))ds\right]-E_x\left[\int_0^t X_i(t)dA_j(s)\right]\right| \\
&=\lim_{n\to\infty}\left|\widetilde{E}_x\left[\int_0^t \widetilde{E}_{\widetilde{X}_n(s)}[X_{i,n}(t-s)]G_{j,n}(\widetilde{X}_n(s))ds\right]-E_x\left[\int_0^t E_{X(s)}[X_i(t-s)]dA_j(s)\right]\right| \\
&=\lim_{n\to\infty}\left|\int_0^t\int_D \widetilde{E}_y[X_{i,n}(t-s)]G_{j,n}(y)p_n(s,x,y)dyds-\int_0^t\int_D E_y[X_i(t-s)]p(s,x,y)\mu_j(dy)ds\right| \\
&\leq \sup_{n\geq 1}\int_0^{\delta_1}\int_D \widetilde{E}_y[|X_{i,n}(t-s)|]|G_{j,n}(y)|p_n(s,x,y)dyds+\int_0^{\delta_1}\int_D E_y[|X_i(t-s)|]p(s,x,y)|\mu_j|(dy)ds \\
&\quad+\lim_{n\to\infty}\left|\int_{\delta_1}^t\int_D \widetilde{E}_y[X_{i,n}(t-s)]G_{j,n}(y)p_n(s,x,y)dyds-\int_{\delta_1}^t\int_D E_y[X_i(t-s)]p(s,x,y)\mu_j(dy)ds\right|<\varepsilon.
\end{aligned}$$

Since ε is arbitrary, (4.3.24) holds.

For the limit (4.3.25), for any $\varepsilon > 0$, there exists a $\delta_2 \in (0,t)$ such that

$$\sup_{n\geq 1}\int_0^{\delta_2}\int_{\partial D}\widetilde{E}_y[|X_{i,n}(t-s)|]p_n(s,x,y)\sigma(dy)ds+\int_0^{\delta_2}\int_{\partial D}E_y[|X_i(t-s)|]p(s,x,y)\sigma(dy)ds<\varepsilon.$$

Together with Theorem 4.2.5 and Lemma 4.3.6, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left|\widetilde{E}_x\left[X_{i,n}(t)L_{j,n}(t)\right]-E_x\left[X_i(t)L_j(t)\right]\right|\\
&=\lim_{n\to\infty}\left|\widetilde{E}_x\left[\int_0^t X_{i,n}(t)dL_{j,n}(s)\right]-E_x\left[\int_0^t X_i(t)dL_j(s)\right]\right|\\
&=\lim_{n\to\infty}\left|\widetilde{E}_x\left[\int_0^t \widetilde{E}_{\widetilde{X}_n(s)}[X_{i,n}(t-s)]dL_{j,n}(s)\right]-E_x\left[\int_0^t E_{X(s)}[X_i(t-s)]dL_j(s)\right]\right|\\
&=\lim_{n\to\infty}\frac{1}{2}\left|\int_0^t\int_{\partial D}\widetilde{E}_y[X_{i,n}(t-s)]p_n(s,x,y)n_j(y)\sigma(dy)ds-\int_0^t\int_{\partial D}E_y[X_i(t-s)]p(s,x,y)n_j(y)\sigma(dy)ds\right|\\
&\leq\sup_{n\geq 1}\frac{1}{2}\int_0^{\delta_2}\int_{\partial D}\widetilde{E}_y[|X_{i,n}(t-s)|]p_n(s,x,y)|n_j(y)|\sigma(dy)ds\\
&\quad+\frac{1}{2}\int_0^{\delta_2}\int_{\partial D}E_y[|X_i(t-s)|]p(s,x,y)|n_j(y)|\sigma(dy)ds\\
&\quad+\lim_{n\to\infty}\frac{1}{2}\int_{\delta_2}^t\int_{\partial D}\left|\widetilde{E}_y[X_{i,n}(t-s)]p_n(s,x,y)-E_y[X_i(t-s)]p(s,x,y)\right||n_j(y)|\sigma(dy)ds\\
&<\frac{\varepsilon}{2}.
\end{aligned}$$

Since ε is arbitrary, this implies (4.3.25).

For (4.3.26), we can observe that

$$E_x\left[A_i(t)A_j(t)\right]=E_x\left[\int_0^t A_i(t)dA_j(s)\right]=E_x\left[\int_0^t A_j(t)dA_i(s)\right],$$

and, due to integration by part,

$$E_x\left[A_i(t)A_j(t)\right]=E_x\left[\int_0^t A_i(s)dA_j(s)\right]+E_x\left[\int_0^t A_j(s)dA_i(s)\right],$$

hence we have

$$\begin{aligned} E_x\left[A_i(t)A_j(t)\right] =& E_x\left[\int_0^t A_i(t)dA_j(s)\right]+E_x\left[\int_0^t A_j(t)dA_i(s)\right] \\ &- E_x\left[\int_0^t A_i(s)dA_j(s)\right]-E_x\left[\int_0^t A_j(s)dA_i(s)\right] \\ =& E_x\left[\int_0^t (A_i(t)-A_i(s))dA_j(s)\right]+E_x\left[\int_0^t (A_j(t)-A_j(s))dA_i(s)\right] \\ =& E_x\left[\int_0^t A_i(t-s)\circ\theta_s dA_j(s)\right]+E_x\left[\int_0^t A_j(t-s)\circ\theta_s dA_i(s)\right] \\ =& E_x\left[\int_0^t E_x\left[A_i(t-s)\circ\theta_s|\mathcal{F}_s\right]dA_j(s)\right]+E_x\left[\int_0^t E_x\left[A_j(t-s)\circ\theta_s|\mathcal{F}_s\right]dA_i(s)\right] \\ =& E_x\left[\int_0^t E_{X(s)}\left[A_i(t-s)\right]dA_j(s)\right]+E_x\left[\int_0^t E_{X(s)}\left[A_j(t-s)\right]dA_i(s)\right]. \end{aligned}$$

Let

$$\begin{aligned} h_i(s,x) &:= E_x\left[A_i(s)\right]=\int_0^t\int_D p(s,x,y)\mu_i(dy)ds, \\ h_{i,n}(s,x) &:= \widetilde{E}_x\left[\int_0^s G_{i,n}(\widetilde{X}_n(u))du\right]=\int_0^t\int_D p_n(s,x,y)G_{i,n}(y)dyds. \end{aligned}$$

By Theorem 4.2.5, we see that

$$\begin{aligned} &\sup_{n\geq 1,(s,x)\in[0,t]\times\bar{D}}\left(|h_{i,n}(s,x)|+|h_i(s,x)|\right)<\infty, \\ &\lim_{n\to\infty}\sup_{(s,x)\in[0,t]\times\bar{D}}|h_{i,n}(s,x)-h_i(s,x)|=0. \end{aligned} \tag{4.3.29}$$

Note that

$$\begin{aligned} \widetilde{E}_x\left[\int_0^t G_{i,n}(\widetilde{X}_n(s))ds\int_0^t G_{j,n}(\widetilde{X}_n(s))ds\right] =& \widetilde{E}_x\left[\int_0^t h_{i,n}(t-s,\widetilde{X}_n(s))G_{j,n}(\widetilde{X}_n(s))ds\right] \\ &+\widetilde{E}_x\left[\int_0^t h_{j,n}(t-s,\widetilde{X}_n(s))G_{i,n}(\widetilde{X}_n(s))ds\right]. \end{aligned}$$

(4.3.26) will follow if

$$\int_0^t\int_D p_n(s,x,y)h_{i,n}(t-s,y)G_{j,n}(y)dyds\to\int_0^t\int_D p(s,x,y)h_i(t-s,y)\mu_j(dy)ds,$$

and this is true due to the convergence of p_n, G_n and (4.3.29). (4.3.27) and (4.3.28) can be proved similarly.

Hence, (X, P_x) is a weak solution to the reflected Brownian motion with drift μ. The part (b) follows from Theorem 4.3.1 and Lemma 4.3.6. □

4.4 Uniqueness of Weak Solution

In this section, we are going to prove the uniqueness of the weak solution to the reflected SDE (4.1.1). In order to do so, for $\lambda > M_5$ and $f \in C^2(\bar{D})$, we introduce the following notations:

$$\begin{aligned} R_\lambda f(x) &:= \int_0^\infty \int_D e^{-\lambda t} q(t,x,y) f(y) dy dt, \\ Bf &:= \langle \nabla f, \mu \rangle. \end{aligned}$$

We outline the result,

Theorem 4.4.1. *(a) For any $x \in \bar{D}$, there exists a unique weak solution to the reflected SDE (4.1.1).*

(b) Let (X, P_x) be the weak solution to the reflected SDE (4.1.1) with $X(0) = x$. Then the collection $(X, P_x, x \in \bar{D})$ forms a Feller process with strong Feller property, which has a jointly continuous transition density $p(t,x,y)$ on $(0,\infty) \times \bar{D} \times \bar{D}$ satisfying the following equation: for any $(t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D}$,

$$p(t,x,y) = q(t,x,y) + \int_0^t \int_D q(t-s,x,z) \langle \nabla_z p(s,z,y), \mu(dz) \rangle ds,$$

where $q(t,x,y)$ is the transition density of the reflected Brownian motion on D.

(c) There exist positive constants C_i, $1 \le i \le 3$, such that

$$p(t,x,y) \le C_1 e^{C_2 t} t^{-\frac{d}{2}} e^{-\frac{C_3 |x-y|^2}{2t}}, \ \forall (t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D},$$

$$|\nabla_x p(t,x,y)| \le C_1 e^{C_2 t} t^{-\frac{d+1}{2}} e^{-\frac{C_3 |x-y|^2}{2t}}, \ \forall (t,x,y) \in (0,\infty) \times \bar{D} \times \bar{D}.$$

Furthermore, each component A_i of A and $|L|$ are CAFs of X, whose Revuz measures are μ_i and $\frac{1}{2}\sigma$ respectively, where σ is the $(d-1)$-dimensional volume element on ∂D, i.e. for every $T > 0$ and continuous function f on $[0,T] \times \bar{D}$,

$$E_x \left[\int_0^T f(s, X(s)) dA_i(s) \right] = \int_0^T \int_D p(s,x,y) f(s,y) \mu_i(dy) ds, \ \forall x \in \bar{D},$$

$$E_x\left[\int_0^T f(s,X(s))d|L|(s)\right]=\frac{1}{2}\int_0^T\int_{\partial D}p(s,x,y)f(s,y)\sigma(dy)ds,\ \forall x\in\bar{D},$$

where E_x is the expectation with respect to P_x.

Proof. The part (b) and (c) follow from the uniqueness of the weak solution to the reflected SDE (4.1.1), Theorem 4.2.5 and Theorem 4.3.7. It only remains to prove the uniqueness. We first show that if the uniqueness of the weak solution to the reflected SDE (4.1.1) holds under the extra assumption that there exist a ball $B(x_0,r)\subset\mathbb{R}^d$ for some $r>0$ small enough, a positive constant λ and a subsequence of $\{n_k\}_{k\geq 1}$, such that

$$\begin{aligned}&\operatorname{supp}\mu\subseteq B(x_0,r),\\ &E_x\left[\sup_{k\geq 1}\int_0^\infty e^{-\lambda s}|G_{n_k}(X(s))|ds\right]<\infty,\ E_x\left[\int_0^\infty e^{-\lambda s}d|A|(s)\right]<\infty,\end{aligned}\tag{4.4.1}$$

then the uniqueness of the weak solution to the reflected SDE (4.1.1) follows.

This can be proved essentially by the same method as in Section 5 and Section 6 of [1]. For the readers convenience, we give a sketch.

Suppose that we have the uniqueness under the above extra assumption. Let $\mu|_{B(x_0,r)}$ denote the measure μ restricted to $B(x_0,r)$, that is,

$$\mu|_{B(x_0,r)}=(\mu_1|_{B(x_0,r)},\mu_2|_{B(x_0,r)},\cdots,\mu_d|_{B(x_0,r)}),$$

with $\mu_i|_{B(x_0,r)}(C)=\mu_i(C\cap B(x_0,r))$ for any $1\leq i\leq d$ and Borel set $C\subset\mathbb{R}^d$. And let $(X,P_x,x\in\bar{D})$ be the weak solution to the reflected Brownian motion with drift $\mu|_{B(x_0,r)}$ constructed as in Theorem 4.3.7 satisfying that

$$X(t)=X(0)+A(t)+W(t)+L(t).$$

Let $(X,\widehat{P}_x)$ be any weak solution to the reflected Brownian motion with drift μ satisfying

$$\begin{cases}X(t)=x+\widehat{A}(t)+\widehat{W}(t)+\widehat{L}(t),\\ \widehat{L}(t)=\int_0^t n(X(s))\mathbf{1}_{\partial D}(X(s))d\left|\widehat{L}\right|(s).\end{cases}\tag{4.4.2}$$

Define the stopping times $S_1:=\inf\{t>0:|X(t)-X(0)|\geq r\}$ and

$$S_{n+1}:=S_n+S_1\circ\theta_{S_n},$$

$$T_N := S_1 \wedge \inf\left\{ t > 0 : \sup_{k\geq 1} \int_0^t |G_{n_k}(X(s))|ds + |\widehat{A}|(t) \geq N \right\}.$$

Then we define a new probability measure $\widetilde{P}_x$ on $(\Omega, \mathcal{F}_\infty)$, such that for $B \in \mathcal{F}_{T_N}$ and $C \in \mathcal{F}_\infty$,

$$\widetilde{P}_x(B \cap (C \circ \theta_{T_N})) := \widehat{E}_x\left[P_{X(T_N)}(C); B\right],$$

where $\widehat{E}_x$ is the expectation with respect to $\widehat{P}_x$. Let

$$\widetilde{W}(t) := \widehat{W}(t)\mathbf{1}_{t\leq T_N} + (W(t) - W(T_N))\mathbf{1}_{t>T_N},$$

$$\widetilde{A}(t) := \widehat{A}(t)\mathbf{1}_{t\leq T_N} + (A(t) - A(T_N))\mathbf{1}_{t>T_N},$$

$$\widetilde{L}(t) := \widehat{W}(t)\mathbf{1}_{t\leq T_N} + (L(t) - L(T_N))\mathbf{1}_{t>T_N}.$$

Since $(X, P_x, x \in \bar{D})$ constructed as in Theorem 4.3.7 is a strong Markov process and A, L are CAFs of X, one can see that $\widetilde{P}_x$ is a weak solution to the reflected Brownian motion with drift $\mu|_{B(x_0,r)}$ satisfying that

$$X(t) = x + \widetilde{A}(t) + \widetilde{W}(t) + \widetilde{L}(t).$$

Moreover, by (4.3.1) and (4.3.2), we see that

$$\widetilde{E}_x\left[\int_0^\infty e^{-\lambda s} d|\widetilde{A}|(s)\right] = \widehat{E}_x\left[\int_0^{T_N} e^{-\lambda s} d|\widehat{A}|(s)\right] + \widehat{E}_x\left[e^{-\lambda T_N} E_{X(T_N)}\left[\int_0^\infty e^{-\lambda s} d|A|(s)\right]\right] < \infty,$$

$$\begin{aligned}\widetilde{E}_x\left[\sup_{k\geq 1}\int_0^\infty e^{-\lambda s}|G_{n_k}(X(s))|ds\right] &\leq \widehat{E}_x\left[\sup_{k\geq 1}\int_0^{T_N} e^{-\lambda s}|G_{n_k}(X(s))|ds\right] \\ &\quad + \widehat{E}_x\left[e^{-\lambda T_N} E_{X(T_N)}\left[\sup_{k\geq 1}\int_0^\infty e^{-\lambda s}|G_{n_k}(X(s))|ds\right]\right] < \infty,\end{aligned}$$

This shows that $\widetilde{P}_x$ is a solution to the reflected Brownian motion with drift $\mu|_{B(x_0,r)}$ satisfying (4.4.1). Therefore, $\widetilde{P}_x$ is uniquely determined. Hence from the definition of $\widetilde{P}_x$, we see that $\widehat{P}_x$ is uniquely determined on $\mathcal{F}_{T_N}$ for each $N \geq 1$. Combining this with the fact that $\lim_{N\to\infty} T_N = S_1$, we see that $\widehat{P}_x$ is uniquely determined on $\mathcal{F}_{S_1}$. Then by the standard argument (see [22, Theorem A.3]), it follows that $\widehat{P}_x$ is uniquely determined on $\mathcal{F}_{S_n}$ for $n \geq 1$. Since $\lim_{n\to\infty} S_n = +\infty$ by Lemma 4.3.2, we deduce that $\widehat{P}_x$ is uniquely determined on $\mathcal{F}_\infty$ and hence the uniqueness holds.

Hence to prove the part (a), by Theorem 4.3.7 and the proceeding argument, it is sufficient to prove that if $(X,\widehat{P}_x)$ is another weak solution to the reflected Brownian motion with drift μ satisfying (4.4.2) and the condition (4.4.1), then for any $s>0$ and $f\in C(\bar{D})$, we have $\widehat{E}_x[f(X(s))]=E_x[f(X(s))]$.

For $g\in C^2(\bar{D})$, by Itô's formula, we have

$$\begin{aligned}&g(X(t))-g(x)\\&=\int_0^t\langle\nabla g(X(s)),d\widehat{A}(s)\rangle+\int_0^t\langle\nabla g(X(s)),d\widehat{L}(s)\rangle+\frac{1}{2}\int_0^t\Delta g(X(s))ds+\int_0^t\langle\nabla g(X(s)),d\widehat{W}(s)\rangle.\end{aligned}$$

By integral by parts, we take expectation with respect to $\widehat{P}_x$ to obtain

$$\begin{aligned}\widehat{E}_x\left[e^{-\lambda t}g(X(t))\right]-g(x)=&\widehat{E}_x\left[\int_0^t e^{-\lambda s}\langle\nabla g(X(s)),d\widehat{A}(s)\rangle\right]+\frac{1}{2}\widehat{E}_x\left[\int_0^t e^{-\lambda s}\Delta g(X(s))ds\right]\\&+\widehat{E}_x\left[\int_0^t e^{-\lambda s}\langle\nabla g(X(s)),d\widehat{L}(s)\rangle\right]-\lambda\widehat{E}_x\left[\int_0^t e^{-\lambda s}g(X(s))ds\right].\end{aligned}$$

Since g is bounded, by letting $t\to\infty$ on both sides of the above equation, we get

$$\begin{aligned}\lambda\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}g(X(s))ds\right]-g(x)=&\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle\nabla g(X(s)),d\widehat{A}(s)\rangle\right]+\frac{1}{2}\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\Delta g(X(s))ds\right]\\&+\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle\nabla g(X(s)),d\widehat{L}(s)\rangle\right].\end{aligned}\tag{4.4.3}$$

For any given $f\in C^2(\bar{D})$, define $g(x):=R_\lambda f(x)=\int_0^\infty\int_D e^{-\lambda s}q(s,x,y)f(y)dyds$. Then $g\in C^2(\bar{D})$ is the solution to the following Neumann problem:

$$\begin{cases}\frac{1}{2}\Delta g(x)=\lambda g(x)-f(x), & \forall x\in D,\\ \frac{\partial g}{\partial n}(x)=0, & \forall x\in\partial D.\end{cases}\tag{4.4.4}$$

Substituting the function g into (4.4.3) gives

$$\begin{aligned}\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}f(X(s))ds\right]=&R_\lambda f(x)+\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle\nabla R_\lambda f(X(s)),d\widehat{A}(s)\rangle\right]\\&+\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle\nabla g(X(s)),d\widehat{L}(s)\rangle\right].\end{aligned}$$

Note that

$$
\begin{aligned}
\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla g(X(s)), d\widehat{L}(s)\rangle\right] &= \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla g(X(s)), n(X(s))\rangle \mathbf{1}_{\partial D}(X(s)) d|\widehat{L}|(s)\right] \\
&= \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\frac{\partial g}{\partial n}(X(s))\mathbf{1}_{\partial D}(X(s)) d|\widehat{L}|(s)\right],
\end{aligned}
$$

in view of (4.4.4), we obtain $\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla g(X(s)), d\widehat{L}(s)\rangle\right] = 0$ which implies that

$$
\widehat{E}_x\left[\int_0^\infty e^{-\lambda s} f(X(s))ds\right] = R_\lambda f(x) + \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda f(X(s)), d\widehat{A}(s)\rangle\right]. \quad (4.4.5)
$$

Now, we fix $f \in C(\bar{D})$. Choose a sequence of functions $\{f_n\}_{n\geq 1} \subset C^2(\bar{D})$ converging uniformly to f on $\bar{D}$. Since (4.4.5) holds for f_n, we see that it also holds for f. Define an operator S_λ by

$$
S_\lambda h(x) := \widehat{E}_x\left[\int_0^\infty e^{-\lambda s} h(X(s))ds\right].
$$

Then (4.4.5) can be written as

$$
S_\lambda f = R_\lambda f(x) + \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda f(X(s)), d\widehat{A}(s)\rangle\right]. \quad (4.4.6)
$$

On the other hand, considering (4.4.1), we have

$$
\begin{aligned}
\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda f(X(s)), d\widehat{A}(s)\rangle\right] &= \lim_{k\to\infty} \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda f(X(s)), G_{n_k}(X(s))\rangle ds\right] \\
&= \lim_{k\to\infty} S_\lambda((\nabla R_\lambda f) G_{n_k}).
\end{aligned} \quad (4.4.7)
$$

By taking $f = (\nabla R_\lambda f)\, G_{n_k}$ in (4.4.6), we again have

$$
S_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(x) = R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(x) + \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(X(s)), d\widehat{A}(s)\rangle\right]. \quad (4.4.8)
$$

By (4.2.1), (4.2.2) and Lemma 4.2.1, we have

$$
\lim_{k\to\infty} R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(x) = R_\lambda(BR_\lambda f)(x),
$$

$$
\lim_{k\to\infty} \nabla R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(x) = \nabla R_\lambda(BR_\lambda f)(x),
$$

$$
\sup_{k\geq 1, x\in\bar{D}} |\nabla R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(x)| < \infty.
$$

Combining this with (4.4.1), (4.4.6)-(4.4.8), using dominated convergence theorem, we get

$$\begin{aligned} &S_\lambda f(x) \\ &= R_\lambda f(x) + \lim_{k\to\infty}\left(R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(x) + \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda(\langle \nabla R_\lambda f, G_{n_k}\rangle)(X(s)), d\widehat{A}(s)\rangle\right]\right) \\ &= R_\lambda f(x) + R_\lambda(BR_\lambda f)(x) + \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda(BR_\lambda f)(X(s)), d\widehat{A}(s)\rangle\right]. \end{aligned}$$

If we repeat the steps of rewriting $\widehat{E}_x[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda(BR_\lambda f)(X(s)), d\widehat{A}(s)\rangle]$ as the limit of $S_\lambda\left(\langle \nabla R_\lambda(BR_\lambda f), G_{n_k}\rangle\right)$, we derive that for any $k \geq 1$,

$$S_\lambda f = \sum_{j=0}^{k} R_\lambda\left((BR_\lambda)^j f\right)(x) + \widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda(BR_\lambda)^k f(X(s)), d\widehat{A}(s)\rangle\right].$$

Since $\mu_i \in K_{d,1}$ for each $1 \leq i \leq d$, and supp $\mu \subset B(x_0, r)$ for some $r > 0$ sufficiently small, by (4.2.1), (4.2.2) and [1, Proposition 3.3 and Proposition 3.6], we have

$$\lim_{k\to\infty}\sup_{x\in\bar{D}}|\nabla R_\lambda(BR_\lambda)^k f(x)| = 0.$$

Hence

$$\lim_{k\to\infty}\widehat{E}_x\left[\int_0^\infty e^{-\lambda s}\langle \nabla R_\lambda(BR_\lambda)^k f(X(s)), d\widehat{A}(s)\rangle\right] = 0.$$

Therefore, we can obtain that

$$S_\lambda f(x) = \sum_{j=0}^{\infty} R_\lambda\left((BR_\lambda)^j f\right)(x).$$

In the same way, we can show that $E_x\left[\int_0^\infty e^{-\lambda s} f(X(s))ds\right] = \sum_{j=0}^{\infty} R_\lambda\left((BR_\lambda)^j f\right)(x)$. This implies that $S_\lambda f(x) = E_x\left[\int_0^\infty e^{-\lambda s} f(X(s))ds\right]$. By the uniqueness of the Laplace transform, we conclude that $\widehat{E}_x[f(X(s))] = E_x[f(X(s))]$ for any $s > 0$. Hence the uniqueness of the weak solution to the reflected SDE (4.1.1) follows.

□

Bibliography

[1] R.F. Bass and Z.Q. Chen, Brownian motion with singular drift. *The Annals of Probability,* 2003, 31(2): 791-817.

[2] R.F. Bass, Probabilistic techniques in analysis. Springer Science & Business Media, 1994.

[3] R.F. Bass and P. Hsu, Some potential theory for reflected Brownian motion in Holder and Lipschitz domains. *The Annals of Probability*, 1991: 486-508.

[4] R.M. Blumenthal and R.K. Getoor, Markov processes and potential theory. Courier Corporation, 2007.

[5] Z.Q. Chen, and L.M. Wang, Uniqueness of stable processes with drift. *Proceedings of the American Mathematical Society*, 2016, 144(6): 2661-2675.

[6] Z.Q. Chen, On reflected diffusion processes and Skorokhod decompositions. *Probability Theory and Related Fields*, 1993, 94(3): 281-315.

[7] C. Costantini, The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations. *Probability Theory and Related Fields*, 1992, 91(1): 43-70.

[8] P. Dupuis and H. Ishii, SDEs with oblique reflection on nonsmooth domains. *The Annals of Probability*, 1993: 554-580.

[9] S.Z. Fang and T.S. Zhang, A study of a class of stochastic differential equations with non-Lipschitzian coefficients. *Probability Theory and Related Fields.* **132** (2005), 356-390.

[10] F. Flandoli, E. Issoglio and F. Russo, Multidimensional stochastic differential equations with distributional drift. *Transactions of the American Mathematical Society*, 2017, 369(3): 1665-1688.

[11] A. Friedman, Stochastic Differential Equations and Applications. Vol. 1, Academic Press, New York, 1975.

[12] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes. Walter de Gruyter, 2010.

[13] M.G. Garroni and J.L. Menaldi, Green functions for second order parabolic integro-differential problems. Chapman & Hall/CRC, 1992.

[14] I.I. Gihman and A.V. Skorohod, Stochastic Differential Equations. Springer, New York, 1972.

[15] I. Gyöngy and T. Martnez, On stochastic differential equations with locally unbounded drift. *Czechoslovak Mathematical Journal*, 2001, 51(4): 763-783.

[16] P. Hsu, reflected Brownian motion, boundary local time and the Neumann problem. Stanford University, 1984.

[17] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes. Vol. 24. Elsevier Science, 2014.

[18] A. Jonsson and H. Wallin, Function spaces on subsets of $\mathbb{R}^n$. *Mathematical Reports 2*. Harwood, Reading, UK. 1984.

[19] G. Kallianpur and J. Xiong, Stochastic differential equations in infinite dimensional spaces. *IMS.* (1995).

[20] I. Karatzas and S.E. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed. Springer, New York, 1994.

[21] P. Kim and R. Song, Two-sided estimates on the density of Brownian motion with singular drift. *Illinois Journal of Mathematics*, 2006, 50(1-4): 635-688.

[22] P. Kim and R. Song, Stable process with singular drift. *Stochastic Processes and Their Applications*, 2014, 124(7): 2479-2516.

[23] N.V. Krylov and M. Roeckner, Strong solutions of stochastic equations with singular time dependent drift. *Probability Theory and Related Fields*, 2005, 131(2): 154-196.

[24] P.L. Lions and A.S. Sznitman, Stochastic differential equations with reflected boundary conditions. *Communications on Pure and Applied Mathematics*, 1984, 37(4): 511-537.

[25] N.L.P Lundström and T. Önskog, Stochastic and partial differential equations on non-smooth time-dependent domains. *Stochastic Processes and Their Applications*, 2019, 129(4): 1097-1131.

[26] K. Nyström and T. Önskog, The Skorohod oblique reflection problem in time-dependent domains. *The Annals of Probability*, 2010, 38(6): 2170-2223.

[27] B. Øksendal, Stochastic Differential Equations: An Introduction with Applications. 5th ed. Springer-Verlag Berlin Heidelberg, 1998.

[28] L. Overbeck, M. Rockner and Z.M. Ma, Markov processes associated with semi-Dirichlet forms. *Osaka Journal of Mathematics*, 1995, 32(1): 97-119.

[29] Y. Saisho, Stochastic differential equations for multi-dimensional domain with reflected boundary. *Probability Theory and Related Fields*, 1987, 74(3): 455-477.

[30] K. Sato and T. Ueno, Multi-dimensional diffusion and the Markov process on the boundary. *Journal of Mathematics of Kyoto University*, 1965, 4(3): 529-605.

[31] A.V. Skorokhod, Stochastic equations for diffusion processes in a bounded region. *Theory of Probability & Its Applications*, 1961, 6(3): 264-274.

[32] A.V. Skorokhod, Stochastic equations for diffusion processes in a bounded region. II. *Theory of Probability & Its Applications*, 1962, 7(1): 3-23.

[33] D.W. Stroock and S.R.S. Varadhan, Diffusion processes with boundary conditions. *Communications on Pure and Applied Mathematics*, 1971, 24(2): 147-225.

[34] H. Tanaka, Stochastic differential equations with reflected boundary condition in convex regions. *Stochastic Processes: Selected Papers of Hiroshi Tanaka*. 2002: 157-171.

[35] A.J. Veretennikov, On strong solutions and explicit formulas for solutions of stochastic integral equations. *Sbornik: Mathematics*, 1981, 39: 387-403.

[36] L. Xie and X. Zhang, Sobolev differentiable flows of SDEs with local Sobolev and super-linear growth coefficients. *The Annals of Probability*, 2016, 44(6): 3661-3687.

[37] S. Yang and T.S. Zhang, Dirichlet boundary value problems for elliptic operators with measure data: a probabilistic approach. arXiv preprint arXiv:1804.01819 (2018).

[38] X. Yang and T.S. Zhang, Estimates of Heat Kernels with Neumann Boundary Conditions. *Potential Analysis*, 2013, 38(2): 549-572.

[39] Q.S. Zhang, Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u)+B\nabla u-u_t=0$. *Manuscripta mathematica*, 1997, 93(1): 381-390.

[40] T.S. Zhang, A probabilistic approach to Dirichlet problems of semilinear elliptic PDEs with singular coefficients. *The Annals of Probability*, 2011, 39(4): 1502-1527.

[41] T.S. Zhang, On the small time asymptotics of diffusion processes on Hilbert spaces. *The Annals of Probability.* **28** (2000), 537-557.

[42] X. Zhang, Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients. *Stochastic Processes and Their Applications*, 2005, 115(11): 1805-1818.

[43] X. Zhang and G. Zhao, Singular Brownian diffusion processes. *Communications in Mathematics and Statistics*, 2018, 6(4): 533-581.

[44] A.K. Zvonkin, A transformation of the phase space of a diffusion process that removes the drift. *Mathematics of the USSR-Sbornik*, 1974, 22(1): 129.