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A Proof for 'Who is a J' Impossibility Theorem

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Abstract

In the analysis of group identification, Kasher and Rubinstein (1997), Logique Analyse 160, 385-395, have shown that any method to aggregate the opinions of a group of agents about the individuals in the group that posses a specific attribute, such as race, nationality, profession, etc., must be dictatorial or, otherwise, it must violate either consensus or independence. This result is known in the literature as 'Who is a J' impossibility theorem. This note enhances slightly the result by weakening the axiom consensus, and it offers a direct proof of the theorem based on the structure of the family of decisive coalitions.

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1 Problem

Consider a group of agents $N = \{1, \dots, n\}$, with $n > 2$. Denote by α a specific attribute that each member of N might possess. For instance, N could be a set of countries and α might denote the property of being a democracy, a EU's member, a free market economy, a world's trading nation, a nuclear nonproliferation country, etc. Alternatively, if we interpret the set of agents merely as individuals, then α could represent for instance the attribute of being a Jewish, hence the name ('Who is a J') given by Kasher and Rubinstein (1997) to the problem of collectively defining those who possess the given nationality.[1] For the sake of concreteness, here we stick with the interpretation that N denotes a set of countries and that α is the attribute of being a democracy. We believe this interpretation stresses the importance of the result for international organizations such as the United Nations, the European Union and the like.

Let $A_i \subset N$ be the set of countries that, according with country i's views, should be recognized as democratic nations in the international community N. We assume that neither $A_i = \emptyset$ nor $A_i = N$. Otherwise, the attribute as a criterium of group identification would be vacuous for country i. This research focuses instead on the case where each country thinks that some of the members of N have the corresponding attribute, but certainly not all of them. The question addressed in this work is how these (potentially) conflicting opinions about the countries in the world that are democracies can be aggregated into a single view valid for the whole international community.

An aggregation rule F assigns a proper and nonempty subset of N to each profile $A = (A_i)_{i \in N}$. The outcome $F(A)$ denotes the set of all members of N who are considered to be democracies according with the aggregation rule in place and the views of the group. As a passing remark, notice that the aggregation rule in this paper aggregates each profile of subsets of individuals into a subset of those individuals, whereas in the Arrovian framework the social welfare functional aggregates each profile of *ordered* sets (individual preferences) of *all* social alternatives into an *ordered* set (social preference) of those same alternatives. Thus, the two problems are related, but they are not the same.[2]

2 Axioms

Consider the following two properties we might wish F exhibits. The first one is ***independence*** (IN). An aggregation rule satisfies independence if each country is judged on its own merits, independently of how other countries are assessed. That is, country j is judged to be a democracy on the basis of how the community views j individually, and not on how the group assesses other countries different from j. Formally,

Independence: An aggregation rule F satisfies independence if for any two profiles $A = (A_i)_{i \in N}$ and $A' = (A'_i)_{i \in N}$ with the property that for any $j \in N$, and all $i \in N$, $j \in A_i \;\Leftrightarrow\; j \in A'_i$, it follows that $j \in F(A) \;\Leftrightarrow\; j \in F(A')$.

The second property we may be interesting in is ***consensus*** (CO). An aggregation rule satisfies consensus if any agreement among *all* countries that a certain state is a democracy (resp., a non-democracy) is respected by the rule. Formally,

[1]See Dimitrov (2011) for a recent and comprehensive review of this literature.

[2]See Samet and Schmeidler (2003) for a discussion about the differences between these two problems.

Consensus: An aggregation rule F satisfies consensus if for every profile $A = (A_i)_{i \in N}$ and any agent $j \in N$, (i) $j \in \bigcap_{i \in N} A_i \Rightarrow j \in F(A)$, and (ii) $j \in \bigcap_{i \in N}(N \setminus A_i) \Rightarrow j \notin F(A)$.

It's also possible to define a weaker version of consensus which suffices for our purposes later in Sections 3 and 4.[3] This version, called ***weak consensus*** (WCO), requires that the aggregation rule F respects collective agreements in *only one* of the two conceivable possibilities, namely, agreement in favor or agreement against one candidate. For concreteness, here we opt for the positive version.

Weak Consensus: An aggregation rule F satisfies weak consensus if for every profile $A = (A_i)_{i \in N}$ and any agent $j \in N$, we have that $j \in \bigcap_{i \in N} A_i \Rightarrow j \in F(A)$.

As the next proposition shows, IN and WCO implies CO.

Lemma 1 *If F satisfies* WCO *and* IN*, then it also verifies* CO.

Proof Part (i) of CO is implied by WCO. To see that (ii) holds as well, suppose, by way of contradiction, that there exists a profile $A = (A_i)_{i \in N}$ an individual $j \in N$ such that $j \in F(A)$ and $j \notin A_i$ for all $i \in N$. Consider the profile $A' = (N \setminus \{j\}, N \setminus \{j\}, \ldots, N \setminus \{j\})$. By WCO, $N \setminus \{j\} \in F(A')$. By IN, $j \in F(A')$. Hence, $F(A') = N$, a contradiction. ∎

3 Theorem

One of the main results of Kasher and Rubinstein (1997) is the following theorem.

Theorem 1 (Kasher and Rubinstein, 1997) *An aggregation rule F satisfies consensus and independence if and only if there exists an agent $i^* \in N$ such that $F(A) = A_{i^*}$ for each profile $A = (A_i)_{i \in N}$.*

Notice that given the result stated in Lemma 1, Theorem 1 also holds if CO is replaced by WCO. Additionally, it is important to stress that the negative message provided by this result, namely, the impossibility of finding aggregation rules rather than the dictatorial that satisfy two appealing properties such as consensus and independence, depends crucially on (i) the specific way in which individuals are allowed to express their opinions, and (ii) how the aggregation rule itself is defined in this framework. Indeed, when individuals are allowed to nominate any subset of the universal set N, including the empty set and the entire set N, and the range of the aggregation rule is amended to include these possibilities as well, more positive results emerge from the model.

To mention one, the family of aggregation rules characterized by Samet and Schmeidler (2003), called *consent rules*, not only satisfy CO and IN, but also a property called symmetry, which means roughly that the group identification does not change if individuals switch their names. That is, the rule does not depend on the names of the individuals. Consent rules are clearly nondictatorial, as they are made of a whole spectrum of rules in which one extreme is unanimity and the other the liberal rule.

[3]We thank one of the referees for suggesting this improvement of the axiomatic characterization.

4 Proof

In the rest of the paper, we offer a *direct* proof of Theorem 1, based on the structure of the family of decisive coalitions, that resembles Mas-Colell et al's (1995) proof of Arrow's impossibility theorem. We believe this simplifies the existing general proof given by Rubinstein and Fishburn (1986), which is based on algebraic aggregation theory.

The following concepts will be useful along the proof. A coalition $L \subseteq N$ is said to be ***semi-decisive for agent*** $i \in N$, denoted by $SD_L|\, i$, if for any profile $A = (A_j)_{j \in N}$ the following two conditions are satisfied:

$$[\forall j \in L,\ i \in A_j \ \text{ and } \ \forall j \notin L,\ i \notin A_j] \quad \Rightarrow \quad i \in F(A); \tag{1}$$

and

$$[\forall j \in L,\ i \notin A_j \ \text{ and } \ \forall j \notin L,\ i \in A_j] \quad \Rightarrow \quad i \notin F(A). \tag{2}$$

A coalition $L \subseteq N$ is said to be ***semi-decisive***, noted $SD_L|\, N$, if it is semi-decisive for all $i \in N$. Finally, a coalition $L \subseteq N$ is said to be ***decisive for agent*** $i \in N$, denoted by $D_L|\, i$, if for any profile $A = (A_j)_{j \in N}$ the following two conditions are satisfied:

$$[\forall j \in L,\ i \in A_j] \quad \Rightarrow \quad i \in F(A);$$

and

$$[\forall j \in L,\ i \notin A_j] \quad \Rightarrow \quad i \notin F(A).$$

For the sake of simplicity, in what follows we restrict attention to the three agents case $N = \{1, 2, 3\}$. The argument generalizes easily to $n > 3$. Additionally, for expositional convenience, we organize the proof of Theorem 1 in a series of lemmas. The first one shows that, under the hypotheses of the theorem, there exists a group of nations $L \subset N$ that is semi-decisive for some country $i \in N$.

Lemma 2 *If F satisfies* WCO *and* IN*, then there exists a coalition $L \subset N$ and an agent $i \in N$ such that L is semi-decisive for i.*

Proof Consider the profile $A = (\{2\}, \{1\}, \{3\})$. By hypothesis, $F(A) \subset N$. First, suppose $F(A) = \{j, k\}$, for some $j \neq k$. Without loss of generality, let $F(A) = \{1, 2\}$. Take the profile $A^* = (\{2, 3\}, \{1, 3\}, \{3\})$. By IN, $\{1, 2\} \subseteq F(A^*)$. By CO, $3 \in F(A^*)$. Hence, $F(A^*) = N$, a contradiction. Thus, $F(A) = \{i\}$, for some $i \in N$. Without loss of generality, suppose $F(A) = \{2\}$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [2 \in \hat{A}_1,\ 2 \notin \hat{A}_2,\ \text{and } 2 \notin \hat{A}_3] \ \Rightarrow \ 2 \in F(\hat{A}). \tag{3}$$

That is, (1) holds for $L = \{1\}$ and $i = 2$. If (2) also holds, then $SD_{\{1\}}|\, 2$. Otherwise, consider the profile $A' = (\{1\}, \{2\}, \{2\})$, and suppose, by contradiction, $2 \in F(A')$. By Lemma 1, $3 \notin F(A')$. Moreover, $1 \notin F(A')$ either. Otherwise, if $1 \in F(A')$, the profile $A'' = (\{1, 3\}, \{2, 3\}, \{2, 3\})$ would lead to a contradiction, because by IN, $\{1, 2\} \subseteq F(A'')$; and by CO, $3 \in F(A'')$, implying that $F(A'')$ would be equal to N. Thus, $F(A') = \{2\}$; and, by IN, $\forall \hat{A} = (\hat{A}_j)_{j \in N}$ such that $2 \notin \hat{A}_1$, $2 \in \hat{A}_2$ and $2 \in \hat{A}_3$, we have that $2 \in F(\hat{A})$.

Consider next the profile $\widetilde{A} = (\{3\}, \{1\}, \{1\})$. By Lemma 1, $2 \notin F(\widetilde{A})$. If $3 \in F(\widetilde{A})$, then $F(\{3,1\}, \{1,2\}, \{1,2\}) = N$, a contradiction. Hence, $F(\widetilde{A}) = \{1\}$; and by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [1 \notin \hat{A}_1,\ 1 \in \hat{A}_2,\ \text{and } 1 \in \hat{A}_3] \ \Rightarrow\ 1 \in F(\hat{A}). \tag{4}$$

Finally, consider $\overline{A} = (\{2,3\}, \{3,1\}, \{3,1\})$. By (3) and (4), $\{2,1\} \subseteq F(\overline{A})$. By CO, $3 \in F(\overline{A})$. Hence, $F(\overline{A}) = N$, which provides the desired contradiction. Thus, (2) also holds for $L = \{1\}$ and $i = 2$, implying that $SD_{\{1\}}|\,2$. ∎

The next lemma shows that if there exists a group of nations that is semi-decisive in assessing the democratic status of a country, then the same group is semi-decisive for all countries.

Lemma 3 *If F satisfies* WCO *and* IN *and there is a coalition $L \subset N$ with the property that L is semi-decisive for some agent $i \in N$, then L is semi-decisive.*

Proof Without loss of generality, assume that $SD_{\{1\}}|\,2$. Let $A = (\{3\}, \{1\}, \{1\})$. By Lemma 1, $2 \notin F(A)$. Moreover, if $F(A) = \{1\}$, then by IN, CO and $SD_{\{1\}}|\,2$, we would have that $F(\{3,2\}, \{1,3\}, \{1,3\}) = N$, a contradiction. Therefore, $3 \in F(A)$ and, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [3 \in \hat{A}_1,\ 3 \notin \hat{A}_2,\ \text{and } 3 \notin \hat{A}_3] \ \Rightarrow\ 3 \in F(\hat{A}).$$

That is, (1) holds for $L = \{1\}$ and $i = 3$. If (2) also holds, then $SD_{\{1\}}|\,3$. Otherwise, consider the profile $A' = (\{1\}, \{3\}, \{3\})$. If $3 \in F(A')$, then IN, CO and $SD_{\{1\}}|\,2$ implies that $F(\{1,2\}, \{1,3\}, \{1,3\}) = N$, a contradiction. Thus, $3 \notin F(A)$ and, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [3 \notin \hat{A}_1,\ 3 \in \hat{A}_2,\ \text{and } 3 \in \hat{A}_3] \ \Rightarrow\ 3 \notin F(\hat{A}).$$

That is, $SD_{\{1\}}|\,3$. Repeating the argument once again, it follows that $SD_{\{1\}}|\,1$ as well. Therefore, by definition, $SD_{\{1\}}|\,N$. ∎

The next lemma shows that if there exist two semi-decisive groups, then they have some agents in common which are by themselves semi-decisive.

Lemma 4 *If F satisfies* WCO *and* IN *and there exist two semi-decisive coalitions $L \subset N$ and $L' \subset N$, then $L \cap L'$ is semi-decisive.*

Proof First we show that $L \cap L' \neq \emptyset$. Suppose not. Without loss of generality, let $L = \{1\}$ and $L' = \{2,3\}$. Consider the profile $A = (\{1\}, \{2\}, \{2\})$. Then, $\{1,2\} \subseteq F(A)$ and, by IN and CO, $F(\{1,3\}, \{2,3\}, \{2,3\}) = N$, a contradiction. Hence, $L \cap L' \neq \emptyset$.

Second, we prove $L \cap L'$ is semi-decisive. Without loss of generality, let $L = \{1,3\}$ and $L' = \{1,2\}$. We wish to show that $L \cap L' = \{1\}$ is semi-decisive. Consider the profile $A = (\{1\}, \{3\}, \{2\})$, and assume, by contradiction, $1 \notin F(A)$. If $2 \in F(A)$, then by IN, CO and $SD_{L'}|\,N$, $F(\{1,3\}, \{3,1\}, \{2,3\}) = N$, a contradiction. Alternatively, if $3 \in F(A)$, then consider the profile $A' = (\{1,2\}, \{3,1\}, \{2\})$. By IN, $3 \in F(A')$. By $SD_L|\,N$, $2 \in F(A')$. By $SD_{L'}|\,N$, $1 \in F(A')$. Thus, $F(A') = N$, a contradiction. Therefore, $1 \in F(A)$, and IN implies that

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [1 \in \hat{A}_1,\ 1 \notin \hat{A}_2,\ \text{and } 1 \notin \hat{A}_3] \ \Rightarrow\ 1 \in F(\hat{A}). \tag{5}$$

Next, consider the profile $A = (\{3\}, \{1\}, \{1\})$. If $1 \in F(A)$, then it follows from IN, $SD_L| N$ and $SD_{L'}| N$ that $F(\{3, 2\}, \{1, 2\}, \{1, 3\}) = N$, a contradiction. Hence, $1 \notin F(A)$ and, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j\in N},\ [1 \notin \hat{A}_1,\ 1 \in \hat{A}_2,\ \text{and } 1 \in \hat{A}_3] \Rightarrow 1 \notin F(\hat{A}). \tag{6}$$

Thus, by (5) and (6), $SD_{\{1\}}| 1$; and, by Lemma 3, $SD_{\{1\}}| N$. ∎

The next statement shows that for any coalition of countries, either the coalition is semi-decisive or otherwise those in the complement constitute a semi-decisive group.

Lemma 5 *If F satisfies* WCO *and* IN*, then for any coalition $L \subseteq N$, either L is semi-decisive or $N \setminus L$ is semi-decisive.*

Proof Note that by CO, $SD_N| N$. Without loss of generality, fix $L = \{1, 2\}$ and suppose, by way of contradiction, that $\{1, 2\}$ is not semi-decisive. Then, there must exist a profile, say $A = (A_j)_{j\in N}$, and an individual, say $i \in N$, such that either,

$$i \in A_1,\ i \in A_2,\ i \notin A_3,\ \text{and } i \notin F(A);$$

or

$$i \notin A_1,\ i \notin A_2,\ i \in A_3,\ \text{and } i \in F(A). \tag{7}$$

Without loss of generality, suppose (7) holds, and let $i = 1$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j\in N},\ [1 \in \hat{A}_3,\ 1 \notin \hat{A}_1,\ \text{and } 1 \notin \hat{A}_2] \Rightarrow 1 \in F(\hat{A}). \tag{8}$$

We wish to prove $N \setminus L = \{3\}$ is semi-decisive for agent 1. To do that, it remains to be shown that

$$\forall \hat{A} = (\hat{A}_j)_{j\in N},\ [1 \notin \hat{A}_3,\ 1 \in \hat{A}_1,\ \text{and } 1 \in \hat{A}_2] \Rightarrow 1 \notin F(\hat{A}). \tag{9}$$

Consider the profile $A' = (\{1\}, \{1\}, \{2\})$. If $1 \notin F(A')$, then (9) follows from IN. Instead, if $1 \in F(A')$, then we proceed as follows. First, notice that $3 \notin F(A')$ by Lemma 1. Second, if $2 \in F(A')$, then $F(\{1, 3\}, \{1, 3\}, \{2, 3\}) = N$. Therefore, it must be that $F(A') = \{1\}$, and by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j\in N},\ [1 \notin \hat{A}_3,\ 1 \in \hat{A}_1,\ \text{and } 1 \in \hat{A}_2] \Rightarrow 1 \in F(\hat{A}). \tag{10}$$

Next, consider the profile $\widetilde{A} = (\{3\}, \{3\}, \{2\})$. By Lemma 1, $1 \notin F(\widetilde{A})$. If $2 \in F(\widetilde{A})$, then by (10), CO and IN, $F(\{3, 1\}, \{3, 1\}, \{2, 3\}) = N$. Thus, it has to be that $F(\widetilde{A}) = \{3\}$; and by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j\in N},\ [3 \notin \hat{A}_3,\ 3 \in \hat{A}_1,\ \text{and } 3 \in \hat{A}_2] \Rightarrow 3 \in F(\hat{A}). \tag{11}$$

But then, CO, (8) and (11) imply that $F(\{2, 3\}, \{2, 3\}, \{1, 2\}) = N$, a contradiction.

Hence, (9) holds and, together with (8), imply that $N \setminus L = \{3\}$ is semi-decisive for agent 1. Finally, by Lemma 3, we get the desired result, i.e., $SD_{\{3\}}| N$. ∎

Next we show that adding countries to a semi-decisive coalition does not erode its power to influence the social outcome.

Lemma 6 *If F satisfies* WCO *and* IN *and there is a semi-decisive coalition $L \subset N$, then the supra-coalition $L' \supset L$ is also semi-decisive.*

Proof Fix any $L \subset L' \subseteq N$, and suppose L is semi-decisive. Assume, by way of contradiction, that L' is not semi-decisive. By Lemma 5, $N \setminus L'$ is semi-decisive. Since $L \subset L'$, $(N \setminus L') \cap L = \emptyset$, which stands in contradiction with Lemma 4. Hence, $SD_{L'}| N$. ∎

Next we show the family of semi-decisive coalitions has a 'nested property,' in the sense that smaller nonempty subsets of a semi-decisive coalition are themselves semi-decisive.

Lemma 7 *If F satisfies* WCO *and* IN *and there is a semi-decisive coalition $L \subseteq N$, with $|L| > 1$, then there exists a sub-coalition $L' \subset L$ such that L' is semi-decisive.*

Proof Take any $h \in L$. If $L \setminus \{h\}$ is semi-decisive, we have proved the desired result. Otherwise, Lemma 5 implies that $N \setminus (L \setminus \{h\}) \equiv N \setminus L \cup \{h\}$ is semi-decisive. By Lemma 4, $N \setminus L \cup \{h\} \cap L = \{h\}$ is semi-decisive; and since $\{h\} \subset L$, this proves the lemma. ∎

The next lemma exploits the nested property alluded above and it shows that, under the conditions of Theorem 1, one country in the international community has semi-decisive power.

Lemma 8 *If F satisfies* WCO *and* IN, *then there exists an agent $h \in N$ such that $\{h\}$ is semi-decisive.*

Proof By Lemma 1 CO holds, and N is semi-decisive. By Lemma 7, there exists $L' \subset N$ such that $N \setminus L'$ is semi-decisive. Using Lemma 7 once again, there must exist L'' such that $(N \setminus L') \setminus L''$ is semi-decisive; and since N is finite, repeated applications of Lemma 7 yield that there exists $h \in N$ such that $SD_{\{h\}}| N$. ∎

Lemma 9 *If F satisfies* WCO *and* IN *and there is a semi-decisive coalition $L \subseteq N$, then L is decisive for all $i \in N$.*

Proof Fix any semi-decisive coalition $L \subset N$. By Lemma 8, there exists $h \in L$ such that $SD_{\{h\}}| N$. Without loss of generality, assume $h = 1$. Take any $i \in N$ and suppose, by way of contradiction, that $\{1\}$ is not decisive for agent i. To simplify, let $i = 2$. Then, there must exist a profile $A = (A_j)_{j \in N}$ such that either (a) $2 \in A_1$ and $2 \notin F(A)$; or (b) $2 \notin A_1$ and $2 \in F(A)$. Suppose the former. The other case is similar. Since by hypothesis $\{1\}$ is semi-decisive for agent 2, there has to be a $j \neq 1$ such that $2 \in A_j$. Moreover, there must also exist an agent $k \in N \setminus \{1, j\}$ such that $2 \notin A_k$. Otherwise, by WCO, we would get $2 \in F(A)$. Without loss of generality, consider the case where $A = (\{2\}, \{3\}, \{2\})$. (Bear in mind that we have assumed $2 \notin F(A)$.)

By Lemma 1, $1 \notin F(A)$. Thus, $3 \in F(A)$. By IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [3 \in \hat{A}_2,\ 3 \notin \hat{A}_1,\ \text{and}\ 3 \notin \hat{A}_3] \ \Rightarrow\ 3 \in F(\hat{A}). \tag{12}$$

Consider next the profile $A' = (\{3\}, \{1\}, \{2, 3\})$. If $3 \notin F(A')$, then by IN

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [3 \notin \hat{A}_2,\ 3 \in \hat{A}_1,\ \text{and } 3 \in \hat{A}_3] \ \Rightarrow\ 3 \notin F(\hat{A}). \tag{13}$$

By (12) and (13), $SD_{\{2\}}|\,3$ and, by Lemma 3, it follows that $SD_{\{2\}}|\,N$. However, $\{1\} \cap \{2\} = \emptyset$, which contradicts Lemma 4. Therefore, $3 \in F(A')$. Moreover, by IN,

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [3 \in \hat{A}_1,\ 3 \notin \hat{A}_2,\ \text{and } 3 \in \hat{A}_3] \ \Rightarrow\ 3 \in F(\hat{A}).$$

Since $SD_{\{1\}}|\,3$, we also know that

$$\forall \hat{A} = (\hat{A}_j)_{j \in N},\ [3 \in \hat{A}_1,\ 3 \notin \hat{A}_2,\ \text{and } 3 \notin \hat{A}_3] \ \Rightarrow\ 3 \in F(\hat{A}).$$

Therefore, if for all $\hat{A} = (\hat{A}_j)_{j \in N}$, $[3 \in \hat{A}_1,\ 3 \in \hat{A}_2$ and $3 \notin \hat{A}_3] \Rightarrow 3 \in F(\hat{A})$, we would get the desired result: i.e., $\{1\}$ would be decisive for agent 3. Otherwise, we can repeat the previous argument and show that $SD_{\{3\}}|\,N$, which again would contradict Lemma 4 because $\{1\} \cap \{3\} = \emptyset$. Hence, $D_{\{1\}}|\,3$; and, since $1 \in L$, $\forall \hat{A} = (\hat{A}_j)_{j \in N}$, $[3 \in \hat{A}_j, \forall j \in L]$ $\Rightarrow 3 \in F(\hat{A})$. That is, $D_L|\,3$. Finally, a reasoning similar to the above shows that $D_L|\,i$ for all $i \in N$. ∎

We are now ready to complete the proof of Theorem 1. We do that by showing the 'only if' (necessity) part. The 'if' (sufficiency) part follows immediately. Under the hypotheses of the theorem, namely CO and IN, we know from Lemma 8 that there exists an agent $h \in N$ such that $\{h\}$ is semi-decisive. By Lemma 9, $\{h\}$ is also decisive for all $i \in N$. Hence, by definition, for all $A = (A_i)_{i \in N}$, $i \in A_h \ \Rightarrow\ i \in F(A)$; and $i \notin A_h \ \Rightarrow\ i \notin F(A)$. Therefore, $F(A) = A_h$ for all $A = (A_i)_{i \in N}$.

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