Picard groups for blocks

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This thesis concerns the study of Picard groups for blocks, in the context of modular representation theory of finite groups.

The first chapter establishes the notation and provides an introduction to modular representation theory of finite groups, with a special focus on Morita equivalences.

In Chapter 2 we address the problem of calculating Picard groups. The results of this chapter rely on the existence of stable equivalence of Morita type and on methods that were successfully used by other authors to provide the first examples of Picard groups for blocks.

In Chapter 3 we investigate Picent for blocks, proving that it is trivial for a perhaps surprisingly large family of blocks. We also provide examples of blocks with non-trivial Picent, even with normal abelian defect group and abelian inertial quotient. The content of this chapter is joint work with Michael Livesey.

In Chapter 4 we prove that, for blocks with normal defect groups in odd characteristic, any bimodule inducing a Morita auto-equivalence must have endopermutation source, providing evidence to an existing open problem. We also have partial results for blocks with normal defect groups in even characteristic. The content of this chapter is joint work with Michael Livesey.

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Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Copyright Statement

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Thesis format

This thesis is submitted in Journal format, since we believe it to be the best format for including and presenting the results we obtained in these years and have been submitted or published.

The first chapter is a purely introductory chapter, and the first original results appear at the end of Section 1.6. Section 2.3 has appeared in a slightly different form in [64].

Chapters 3 and 4 have been presented as research papers, since they are respectively [60] and [61]. Results in these two chapters have been obtained in collaboration with Michael Livesey, but both authors agreed that our contribution are equal.

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"Who are we, who is each one of us, if not a combination of experiences, information, books we have read, things imagined? Each life is an encyclopaedia, a library, an inventory of objects, a series of styles, and everything can be constantly shuffled and reordered in every way conceivable."

(Italo Calvino-Six Memos For The Next Millennium)

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Introduction

In this thesis we study Picard groups of blocks, in the context of modular representation theory of finite groups. Modular representation theory is a fertile area of mathematics that involves the study of group algebras and their indecomposable factors (the *blocks*) by the investigation of invariants associated with these algebras. Some of the invariants most relevant to this thesis are introduced in Chapter 1, where we also describe the transformations these invariants are invariant under: Morita equivalences.

Picard groups are one of these invariants, and are defined as the sets of Morita auto-equivalences of blocks. The interest for Picard groups of blocks has just recently been fueled, and many problems connected with this invariant need to be addressed. We detected and dealt with three main problems, whose study is divided into Chapters 2, 3 and 4.

Picard groups have been successfully employed in the context of classifying blocks up to Morita equivalence, but a crucial problem arose: calculating Picard groups. At the time of writing, there are not effective ways to calculate the isomorphism type of the Picard group for a given block. Therefore one of the first problems we explored was the following: providing examples and techniques to calculate Picard groups for coherent classes of blocks. Chapter 2 collects all the results we obtained about this problem, mainly illustrating two ways of calculating Picard groups. In Section 2.1 we calculate Picard groups for some principal blocks with defect groups $C_3 \times C_3$; Section 2.2 explores the problem of calculating Picard groups for blocks that are built from smaller blocks whose Picard group is known; Section 2.3 provides the first examples of Picard groups of blocks with non-abelian and non-normal defect groups, namely principal 2-blocks of Suzuki groups; finally, Section 2.4 illustrates a brief application of Weiss' criterion to calculate Picard groups for blocks with defect groups Q_8.

Another interesting problem regarding Picard groups is the study of one of its normal subgroups: Picent. The Picent of a block can be regarded as the set of Morita auto-equivalences that act trivially on the irreducible characters of the block. Even if Picent is not trivial in general for blocks with non-abelian defect groups, it is not clear whether the same holds for blocks with abelian defect group. If Picent were trivial it would be possible to bound the size of the Picard groups, making calculations easier. In Chapter 3 we show that for some (perhaps surprisingly large) families of blocks with abelian defect groups Picent is trivial, but it is not in general. In fact, we provide counterexamples: even blocks with normal abelian defect group, abelian inertial quotient and non-trivial Picent. The content of Chapter 3 is joint work with Michael Livesey and has been published, [60].

The last problem we deal with is concerned with the structure of Picard groups. An open problem is whether all Morita auto-equivalences of blocks are induced by Morita equivalences with endopermutation source. This has been proved in the literature for group algebras of p-groups, nilpotent blocks and blocks with cyclic defect groups. Our contribution to this problem is given in Chapter 4, where we show that, for blocks with normal defect groups in odd characteristic, Picard groups are given by invertible bimodules with linear source. We also provide partial results for blocks with normal defect groups in even characteristic. The content of this chapter is joint work with Michael Livesey and has been submitted to a journal for publication; it is available on arXiv, [61].

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Chapter 1

Preliminaries

Throughout this work we try to avoid repeating ourselves too much, so some assumptions will be tacitly made for all the length of the document. Namely, modules and algebras are finitely generated and p is a prime number, unless differently stated. Most of the groups involved are going to be finite, but we will usually point this out.

1.1 Blocks

Let G be a finite group, R a commutative ring. We can then define the group algebra as the set of formal sums

$$RG = \left\{ \sum_{g\in G} r_g g | r_g \in R \right\}.$$

The structure of an R-module is naturally given and the element of RG with $r_g = 1$ and $r_h = 0$ for $h \neq g$ is identified with the group element g, for all group elements g. Through this identification we actually have an embedding of G in RG and, thus, formal sums can be seen as "real" sums, i.e. linear combinations of group elements. Multiplication on RG is then introduced in a standard way: first, multiplication of basis vectors (group elements) is defined via the aforementioned embedding of G in RG; then, multiplication of general elements is given by linear extension.

The complexity of RG as an algebra depends on the choice of the ring R and its interaction with the group G. Such complexity is reflected in RG-modules and their structure, and the simplest algebras we can deal with, in our framework, are called semi-simple. We say that an algebra A is semi-simple if every A-module is semi-simple, i.e. it is a direct sum of simple modules. Semi-simple group algebras over algebraically closed fields are characterized by the following well-known result.

Theorem 1.1.1 (Maschke). *Let G be a group and k an algebraically closed field. Then kG is semi-simple if and only if* $\mathrm{char}(k)$ *does not divide the order of G.*

In this setting, another incredibly powerful tool is provided: the Artin-Wedderburn theorem completely describes the structure of semi-simple algebras.

Theorem 1.1.2 (Artin-Wedderburn). *Let A be a finite dimensional semi-simple algebra over a field k. Then A is a direct product of matrix algebras over division rings. Specifically, if A, as a left A-module, decomposes as*

$$A \simeq S_1^{n_1} \oplus ... \oplus S_r^{n_r},$$

where the $S_1, ..., S_r$ are non-isomorphic simple modules occurring with multiplicities $n_1, ..., n_r$, then, as an algebra,

$$A \simeq M_{n_1}(D_1) \times ... \times M_{n_r}(D_r),$$

where $D_i = \mathrm{End}_A(S_i)^{\mathrm{op}}$. Furthermore, if k is algebraically closed, then $D_i = k$ for all i.

Note that, for a ring B, we denote by B^{op} the opposite ring, that is isomorphic as an abelian group to B, but multiplication is defined by

$$a \cdot b = ba.$$

Also the last statement of Artin-Wedderburn theorem is a direct consequence of Schur's Lemma, [86, Theorem 2.1.1].

However, not all group algebras are semi-simple. The most notable case is kG where k is an algebraically closed field with characteristic dividing the order of the group G. In fact, by Maschke's Theorem we know that kG is not semi-simple, so the Artin-Wedderburn Theorem can't be applied. Nevertheless, kG can be still decomposed as direct product of indecomposable algebras:

$$kG = B_1 \times ... \times B_r.$$

The indecomposable factors B_i can be defined in several equivalent ways, for example by considering kG as a kG-kG-bimodule.

Proposition 1.1.3 ([55], Theorem 1.7.1). *Let A be a k-algebra, B a k-submodule of A. The following are equivalent:*

- *B is a direct summand of A as an A-A-bimodule.*
- *B is a direct factor of A as a k-algebra.*

- *$B = Ab$ for some idempotent b in $Z(A)$*

Moreover A is indecomposable as a k-algebra and, equivalently, as a bimodule if and only if 1_A is primitive in $Z(A)$.

A block is a primitive idempotent b in $Z(A)$, and the algebra $B = Ab$ is called a block algebra. The next result lays the foundations for a sensible investigation of block algebras, since, for example, it states that the set of blocks of certain algebras is finite. In particular, the theorem applies to group algebras over Noetherian rings.

Theorem 1.1.4 ([55], Theorem 1.7.7)**.** *Suppose that R is a Noetherian ring, and A is an R-algebra. Then the following hold.*

- *The set of blocks $\mathcal{B}$ of A is the unique primitive decomposition of 1_A in $Z(A)$. In particular, $\mathcal{B}$ is finite and any two different blocks are orthogonal.*

- *$A = \prod_{b\in\mathcal{B}} Ab$ is the unique decomposition of A as a direct product of indecomposable k-algebras.*

- *$A = \oplus_{b\in\mathcal{B}} Ab$ is the unique decomposition of A as a direct sum of indecomposable A-A-bimodules.*

For an algebraically closed field K containing $|G|^{\text{th}}$-roots of unity and with characteristic zero, we can explicitly describe the blocks of KG. In fact, character theory yields the answer for such algebras (a common reference for a deep and splendid account of this theory is [41]). For each irreducible character χ we can define an associated central primitive idempotent by

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g,$$

and, by Artin-Wedderburn Theorem, we know that KGe_χ are matrix algebras. However, in general, block algebras have a richer structure.

We define the setting we are going to work in for the rest of this thesis.

Definition 1.1.5. A triple $(K, \mathcal{O}, k)$ is called a *p-modular system* if

- k is an algebraic closure of $\mathbb{F}_p$, the field with p elements.

- $\mathcal{O}$ is a complete discrete valuation ring with $\text{char}(\mathcal{O}) = 0$.

- K is the field of fractions of $\mathcal{O}$.

From now on the symbols $k, \mathcal{O}, K$ will uniquely refer to the object defined above. The assumption on the characteristic of $\mathcal{O}$ implies that we are not allowing $\mathcal{O} = k$, and that will be crucial for results on Picard groups, like Theorem 1.6.6. On the other hand, the completeness of $\mathcal{O}$ allows the lifting of primitive idempotents. In particular, the following holds.

Proposition 1.1.6 ([7],Proposition 4.9)**.** *The ring homomorphism $\mathcal{O}G \to kG$ induces a bijection between $\mathcal{O}G$-conjugacy classes of primitive idempotents of $\mathcal{O}G$ and kG-conjugacy classes of primitive idempotents of kG and also a bijection between blocks of $\mathcal{O}G$ and the blocks of kG.*

Moreover, we make an additional assumptions every time we use a p-modular system: for all groups G involved, K contains a primitive $|G|^{\text{th}}$-root of unity. Thanks to this assumption, KG is semisimple and then we can use ordinary representation theory to describe it. On the other side, kG is object of study of modular representation theory, therefore $\mathcal{O}G$ plays the role of a bridge between these two theories. Together with a p-modular system, we have a natural surjective map given by:

$$k \otimes_{\mathcal{O}} \text{-} : \mathcal{O}G \to kG.$$

This map *preserves projectivity and blocks.* In fact, take

$$\mathcal{O}G = B_1 \times ... \times B_r,$$

a decomposition of $\mathcal{O}G$ in blocks, then a decomposition in indecomposable factors of kG is given by

$$kG = \overline{B}_r \times ... \times \overline{B}_r,$$

where $\overline{B}_i = k \otimes_{\mathcal{O}} B_i$. The statement about projectivity is explained in the following proposition.

Proposition 1.1.7 ([55], Proposition 4.5.10)**.** *Let A be an $\mathcal{O}$-algebra, and set $\overline{A} = A/J(\mathcal{O})A$. For A-module U denote by $\overline{U}$ the $\overline{A}$-module $k \otimes_{\mathcal{O}} U$.*

(i) For any finitely generated projective $\overline{A}$-module Q there is, up to isomorphism, a unique finitely generated projective A-module P such that $\overline{P} \simeq Q$ as $\overline{A}$-modules.

(ii) Let U be an finitely generated A-module and P a finitely generated projective A-module. Then P is a projective cover of U if and only if $\overline{P}$ is a projective cover of $\overline{U}$.

(iii) Suppose that A is free of finite rank as an $\mathcal{O}$-module. Let U be a finitely generated $\mathcal{O}$-free A-module. Then U is a projective A-module if and only if $\overline{U}$ is a projective $\overline{A}$-module.

From now on the word *block* will denote either a block idempotent or a block algebra of $\mathcal{O}G$. Context, however, should allow effortless interpretation of the word used. Moreover capital letters will usually denote block algebras, while lower-case letters will refer to block idempotents. There are some local exceptions to this last rule, like the specific notations adopted in Section 2.1, but they will always be spelled out.

Linckelmann in [55] says that *the dominant feature of block theory is the dichotomy of invariants associated with block algebras.* In fact, block algebras have all the global invariants associated to algebras, like module categories, but also those invariants coming from the groups that underlie the blocks, like defect groups, and block theory revolves around the interplay of these two worlds. We are going to describe these invariants in the next sections.

1.2 Characters and blocks of finite groups

Character theory is the object of study of ordinary representation theory, but characters play an important role in block theory as well. Let G be a finite group and $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. For each block B of G, we can associate a set $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G)$ whose elements are called characters lying in B. There are many ways for defining these sets; we are going to present them by exclusively looking at summands of algebras, but a *more character theoretic* way of doing this is illustrated in [67]. Given a decomposition in block algebras of $\mathcal{O}G$,

$$\mathcal{O}G = B_1 \times ... \times B_n,$$

the injective map $K\otimes_{\mathcal{O}}$- yields a decomposition

$$KG = K \otimes_{\mathcal{O}} B_1 \times ... \times K \otimes_{\mathcal{O}} B_n.$$

However, in general, this is not a decomposition of KG as a product of indecomposable factors. Nevertheless, from character theory, a decomposition of KG in indecomposable factors is well known, and is provided by characters idempotents:

$$KG = \prod_{\chi \in \mathrm{Irr}(G)} KGe_{\chi}.$$

It then follows that, for each block B, there is a decomposition in indecomposable factors given by

$$K \otimes_{\mathcal{O}} B = \prod_{\chi \in \Gamma_B \subseteq \mathrm{Irr}(G)} KGe_\chi.$$

The characters in Γ_B are then called the characters of G that lie in B, and we write $\mathrm{Irr}(B) = \Gamma_B$. It is immediate seeing that these sets are disjoint and their union is the whole $\mathrm{Irr}(G)$.

There analogously is a notion of *Brauer characters lying in B*. First we recall the definition of Brauer characters. Note that $g \in G$ is called p-regular if $p \nmid |g|$.

Definition 1.2.1. Let M be a kG-module. Consider a matrix representation for M, and for each $x \in G^0$, the set of p-regular elements of G, denote by $\{\xi_1^x, ..., \xi_n^x\}$ the set of eigenvalues of x. For each of these eigenvalues, let $\tilde{\xi}_i^x$ be the unique lift of the said eigenvalue to $\mathcal{O}$. We define then the *Brauer character* of M, $\psi_M : G^0 \to \mathcal{O}$, by

$$\psi_M(x) = \sum_i \tilde{\xi}_i^x.$$

Remark 1.2.2. Note that the fact that each eigenvalue has a unique lift to $\mathcal{O}$ is a direct consequence of Hensel's Lemma [77, Proposition 2.4.7].

A Brauer character ψ_M is irreducible if the module M is simple, and we denote by $\mathrm{IBr}(G)$ the set of irreducible Brauer characters of G. Then, we say a Brauer character ψ_M lies in the block B if the module M lies in $\overline{B} = k \otimes_{\mathcal{O}} B$.

Definition 1.2.3. A kG-module M *lies in the block $\overline{B}$* of kG, where $B = \mathcal{O}Gb$, if $\bar{b}M = M$ and $\bar{c}M = 0$ for every block $c \neq b$ of $\mathcal{O}G$.

Obviously each simple kG-module lies in a unique block, and we denote by $\mathrm{IBr}(B)$ the set of Brauer characters lying in the block B. The two sets we have just defined allow us to define two important invariants of blocks: if B is a block, then we write $l(B) = |\,\mathrm{IBr}(B)|$ and $k(B) = |\,\mathrm{Irr}(B)|$. These two numbers will not play a central role in this thesis, however they are far from being understood. In fact, one of the oldest conjectures in modular representation theory is about one of these numbers:

Conjecture 1.2.4 (Brauer's $k(B)$-conjecture)**.** *Let B be a block of a finite group with defect group D. Then $k(B) \leq |D|$.*

Remark 1.2.5. For a recent study on the existence of counterexamples to this long-standing conjecture see [62].

We now turn our attention to the relationship between $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$. As with ordinary characters, we can define Brauer characters as non-negative integer linear combination of $\mathrm{IBr}(G)$. The following is very well-known.

Proposition 1.2.6 ([41], Theorem 15.6). *Let G be a group, χ an irreducible character of G. Then χ^0, the restriction of χ to the p-regular elements of G, is a Brauer character. In particular, if $\chi \in \mathrm{Irr}(B)$ then χ^0 is a non-negative integer linear combination of* $\mathrm{IBr}(B)$.

Definition 1.2.7. For each $\chi \in \mathrm{Irr}(B)$ we write

$$\chi^0 = \sum_{\psi \in \mathrm{IBr}(B)} \mathrm{d}_{\chi\psi}\psi,$$

and the $d_{\chi\psi}s$ are called *decomposition numbers* of χ. The matrix $D = (d_{\chi\psi})_{\chi,\psi}$ with $\chi \in \mathrm{Irr}(B), \psi \in \mathrm{IBr}(B)$ is called the *decomposition matrix* of B. We say that $\chi \in \mathrm{Irr}(B)$ *lifts* $\psi \in \mathrm{IBr}(B)$ if $d_{\chi\psi} = 1$ and $d_{\chi\eta} = 0$ for all Brauer characters $\eta \neq \psi$.

The decomposition matrix and the decomposition numbers can also be defined in an alternative way, as illustrated below.

Theorem 1.2.8 ([55], Theorem 4.16.2). *Let B be a block of $\mathcal{O}G$, X a simple $K \otimes_{\mathcal{O}} B$-module affording the character χ, S a simple $k \otimes_{\mathcal{O}} B$-module with Brauer character ψ. Let P be the projective cover of S, whose $\mathcal{O}$-lift $\hat{P}$ affords the character Φ_ψ. Then $d_{\chi\psi}$ is the number of direct summands isomorphic to X in a decomposition of $K \otimes_{\mathcal{O}} \hat{P}$ as a direct sum of KG-modules. In particular,*

$$\Phi_\psi = \sum_{\chi \in \mathrm{Irr}(B)} d_{\chi\psi}\chi.$$

Remark 1.2.9. In the last theorem we have mentioned projective covers without properly defining them, we do it now. It suffices to say that for each kG or $\mathcal{O}G$ module M we can uniquely define, up to isomorphism, a projective module P_M such that there is a surjective kG(or $\mathcal{O}G$)-morphism $\pi_M : P_M \to M$ which induces an isomorphism $P_M/\mathrm{rad}(P_M) \simeq M/\mathrm{rad}(M)$; P_M is called a projective cover of M, and we will sometimes denote it by $P(M)$. The projective covers of simple $k \otimes_{\mathcal{O}} B$-modules, or their $\mathcal{O}$-lifts, are called the projective indecomposable modules of $k \otimes_{\mathcal{O}} B$ or B.

A matrix closely related to the one we have just seen is the Cartan matrix, that we immediately define.

Definition 1.2.10. Let B be a block and $S_1, ..., S_{l(B)}$ representatives for the isomorphism classes of simple $k \otimes_{\mathcal{O}} B$ modules. The *Cartan matrix* of B is the square matrix of non negative integers $C = (c_{ij})_{i,j \in I}$, where c_{ij} is the number of composition factors isomorphic to S_i in a composition series of $P(S_j)$.

Proposition 1.2.11 ([86], Corollary 9.5.6)**.** *Let C, D respectively be the Cartan matrix and the decomposition matrix of B. Then $D^T D = C$. In particular, C is symmetric.*

1.3 Vertices, sources and Green correspondence

In the last section we saw how we can define for each block B, a set $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G)$. Together with this set, we can introduce an integer $d(B)$, that measures the *complexity* of $\mathrm{Irr}(B)$.

Definition 1.3.1. Let B be a block of G, $|G|_p = p^s$. Then we call the defect of B the following integer:

$$d(B) = \min \left\{ k \mid p^{s-k} | \chi(1), \forall \chi \in \mathrm{Irr}(B) \right\}$$

A much clearer viewpoint that describes *how* this integer is supposed to measure the complexity of B arises from the study of the algebra structure of the block.

We recall that an $\mathcal{O}G$-module V is projective if it is a direct summand of a free $\mathcal{O}G$-module. This notion can actually be generalized when we are talking about group algebras, so that subgroups appear in the picture. In fact, we say that an $\mathcal{O}G$ module V is relatively H-projective, for $H \leq G$, if there exists an $\mathcal{O}H$-module U such that $V | U \uparrow_H^G$, where the "|" notation means "is a direct summand of" and the arrow denotes the induction from H to G. Note that the usual definition of projectivity is retained taking $H = 1$. A trivial, but important, result is the following.

Proposition 1.3.2 ([1], Section 9, Theorem 2)**.** *If H is a subgroup of G containing a Sylow p-subgroup of G, then every $\mathcal{O}G$-module is relatively H-projective.*

This leads to the following definition.

Definition 1.3.3. Let V be an indecomposable $\mathcal{O}G$-module, then we say that (P, U) is a *vertex-source pair* for V if

(i) P is a minimal p-subgroup such that V is relatively P-projective.

(ii) U is an indecomposable $\mathcal{O}P$-module such that $V | U \uparrow_P^G$.

P is then called a vertex and U a source of V.

Obviously, such a pair always exists, since every $\mathcal{O}G$-module is relatively S-projective for S a Sylow p-subgroup. Moreover, it can be showed that all such pairs are conjugate.

Theorem 1.3.4 ([36]). *Let V be an indecomposable $\mathcal{O}G$-module.*

(i) If P and P' are vertices of V, then P and P' are conjugate in G.

(ii) If S and S' are both sources of V with the same vertex P, then $S' \simeq S^g$ for some $g \in N_G(P)$.

Vertices can be seen as a way of measuring how far an indecomposable module is from being projective. In fact, it can be proved that an indecomposable $\mathcal{O}G$-module is projective if and only if has vertex the trivial subgroup $\{1_G\}$. So, the *most complex* indecomposable modules are those who have a Sylow p-subgroup as a vertex. These concepts can obviously translate to blocks.

Blocks of $\mathcal{O}G$ are indecomposable factors of the algebra $\mathcal{O}G$, so they naturally inherit a structure of $\mathcal{O}G$-$\mathcal{O}G$-bimodules. Such bimodules can actually be seen as $\mathcal{O}(G \times G)$-modules, via a common identification: each element $(g, h) \in G \times G$ acts on m as gmh^{-1}. We can then talk about vertices for blocks as well, and the following completely describes their form.

Theorem 1.3.5 ([1], Section 13, Theorem 4). *If B is a block of $\mathcal{O}G$, then B has a vertex, as an $\mathcal{O}(G \times G)$-module, of the form $\Delta D = \{(g, g)|g \in D\} \leq G \times G$, where D is a p-subgroup of G. D is then called a defect group of G.*

Defect groups are central objects in modular representation theory, and, by Theorem 1.3.4, the set of defect groups of a block form a conjugacy class of G. We recall some standard properties of defect groups in the following proposition.

Proposition 1.3.6. *Let B be a block of $\mathcal{O}G$, D a defect group for B. Then*

(i) $O_p(G)$, the largest normal p-subgroup of G, is contained in D [56, Theorem 6.2.6].

(ii) $d(B) = |D|$.

(iii) Every indecomposable $\mathcal{O}G$-module lying in B has a vertex contained in D. Moreover, if D is abelian, then D is a vertex for every simple kG-module [44].

It is now definitely more clear how the defect of a block measures complexity. For example in blocks of defect zero all $\mathcal{O}$-free indecomposable modules are projective, while $\mathcal{O}$-free indecomposable modules in blocks whose defect groups are Sylow p-subgroups can have all kinds of vertices. One of these blocks with defect group a Sylow p-subgroup is distinguished, and it is the block $B_0(G)$ that contains the trivial character. This block is called the principal block of G.

We mentioned before that for blocks of defect zero all indecomposable modules are projective, but we can actually say more. In fact defect zero blocks are well understood, and we collect some characterizations below.

Theorem 1.3.7 ([56], Theorem 6.6.2)**.** *Let B be a block of $\mathcal{O}G$, $\chi \in \mathrm{Irr}(B)$, $\overline{B} = k \otimes_{\mathcal{O}} B$. Then the following are equivalent:*

(i) B has defect zero.

(ii) $\overline{B}$ has a projective simple module

(iii) χ is the character of a projective B-module

(iv) $\mathrm{Irr}(B) = \{\chi\}$

(v) The order of a Sylow p-subgroup of G divides $\chi(1)$.

The theory of vertices and sources, together with giving a tool for measuring complexity of blocks, yields one of the main tools for studying module categories: Green correspondence. Green's theory provides in fact a vertex and source preserving correspondence between indecomposable modules of a finite group G and normalisers of p-subgroups of G. We first state the main theorem, consequences of this correspondence will be described later.

Theorem 1.3.8 ([37])**.** *Let G be a finite group, Q a p-subgroup of G and $N_G(Q) \leq H \leq G$.*

(i) If U is an indecomposable $\mathcal{O}G$-module having vertex Q, then there is, up to isomorphism, a unique indecomposable direct summand $f(U)$ of $U \downarrow_H^G$, having Q as a vertex. Moreover, every source of $f(U)$, with respect to Q, is a source of U, and every other direct summand of $U \downarrow_H^G$ has a vertex that is contained in $Q^x \cap H$ for some $x \in G \setminus H$ and that is not H-conjugate to Q.

(ii) If V is an indecomposable $\mathcal{O}H$-module having vertex Q, then there is, up to isomorphism, a unique indecomposable direct summand $g(V)$ of $V \uparrow_H^G$, having Q as a vertex. Moreover, every source of V, with respect to Q, is a source of $g(V)$, and every other direct summand of $V \uparrow_H^G$ has a vertex that is contained in $Q \cap Q^x$ for some $x \in G \setminus H$.

Finally it holds $g(f(U)) \simeq U$ and $f(g(V)) \simeq V$ for all indecomposable $\mathcal{O}G$-module U and indecomposable $\mathcal{O}H$-module V.

One of the most interesting applications of Green correspondence lies in the so-called trivial intersection (TI) case. We say that a Sylow p-subgroup P is a trivial intersection subgroup for G, if, for every $g \in G$, $P^g \cap P$ is either P or trivial.

Corollary 1.3.9 ([1], Section 10, Theorem 1)**.** *Let G be a finite group, P a Sylow p-subgroup of G, and suppose that P is a TI subgroup for G. Then there is a one-to-one correspondence between isomorphism classes of non-projective indecomposable $\mathcal{O}G$ and $\mathcal{O}N_G(P)$-modules, given by f, and $g = f^{-1}$, satisfying the following:*

$$U \downarrow^G_{N_G(P)} = f(U) + Y$$
$$V \uparrow^G_{N_G(P)} = g(V) + X,$$

where U, V are, respectively, any indecomposable $\mathcal{O}G, \mathcal{O}N_G(P)$-modules, and Y, X are projective $\mathcal{O}N_G(P), \mathcal{O}G$-modules,.

The corollary for the TI case has actually deeper consequences, since the correspondence described above induces an equivalence of stable module categories. We will talk about these in the next section.

1.4 Equivalences of module categories

Module categories play an important role in modular representation theory: what is detected as bijections between irreducible characters of blocks, can sometimes be the shadow of an equivalence between the module categories of the blocks, giving much more insight on what is actually happening.

Moreover, Donovan's conjecture, one of the long-standing conjectures in modular representation theory, states that, once the defect group is fixed, there is just a finite number of blocks with that defect group, up to equivalence of the module categories. Therefore the study of the module categories of blocks should improve our comprehension of blocks and their properties.

We start by precisely defining the equivalences we will be interested in, and outlining their most interesting features. Recall that, for an algebra A, we denote by $\mathrm{Mod}(A)$ the category with objects all the A-modules, and morphisms the module homomorphisms. By $\mathrm{mod}(A)$ we will instead denote the category whose objects are just the finitely generated A-modules.

Definition 1.4.1. Two $\mathcal{O}$-algebras B and B' are *Morita equivalent* if the $\mathcal{O}$-linear categories $\mathrm{Mod}(B)$ and $\mathrm{Mod}(B')$ are equivalent. A *Morita equivalence* between B and B' is an equivalence between the two module categories.

Theorem 1.4.2 (Morita, [66])**.** *Let A, B be symmetric $\mathcal{O}$-algebras. The following are equivalent:*

1. *The algebras A and B are Morita equivalent.*

2. *The categories* $\mathrm{mod}(A)$ *and* $\mathrm{mod}(B)$ *are equivalent.*

3. *There is an A-B-bimodule M finitely generated projective as left and right module, such that $M \otimes_B M^* \simeq A$, as A-A-bimodules, and $M^* \otimes_A M \simeq B$, as B-B-bimodules. Moreover, $M \otimes_B - : \mathrm{mod}(B) \to \mathrm{mod}(A)$ is an equivalence of categories and $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ induces the inverse functor $M^* \otimes_A -$.*

Since blocks are symmetric algebras, all Morita equivalences of blocks are induced by bimodules, and the inverse of a Morita equivalence induced by a bimodule M is just induced by its $\mathcal{O}$-dual, M^*. We mentioned before Donovan's conjecture; we are now ready to formally state it.

Conjecture 1.4.3 (Donovan)**.** *Let D be a p-group. Then there is a finite number of Morita equivalence classes of blocks with defect group isomorphic to D.*

Even though Donovan's conjecture is not one of the main topics of this thesis, Picard groups have had a relevant role for its progress. A wiki page has been set up by Charles Eaton at [27], with the scope of recording advancement of the classification of blocks, up to Morita equivalence.

Donovan's conjecture emphasizes the need to determine whether two given blocks are Morita equivalent. This problem cannot be easily solved in general, but in some cases it can be showed in a straightforward way that two blocks are not Morita equivalent, relying on some features that need to be preserved by Morita equivalences. In the next omnibus theorem we are going to collect some well known facts, mostly available in [55] and [56].

Theorem 1.4.4. *Let B, B' be two Morita equivalent $\mathcal{O}$-blocks. Then*

(i) *The centers $Z(B)$ and $Z(B')$ are isomorphic.*

(ii) $k(B) = k(B')$.

(iii) $l(B) = l(B')$.

(iv) $d(B) = d(B')$.

(v) *The Loewy structure of the two blocks is the same.[50]*

(vi) *The Cartan matrices of B and B' are the same, up to reordering of rows and columns.*

(vii) *The decomposition matrices of B and B' are the same, up to reordering of rows and columns.*

Obviously two Morita equivalent blocks must have the same Morita invariants, but often this is not enough for ruling out all the candidates. The first four invariants in the above Theorem are actually preserved by a weaker relation between blocks that is induced by a Morita equivalence: a perfect isometry.

Definition 1.4.5. Let G, H be finite groups. A generalized K-valued *character* $\mu \in \mathbb{Z}\operatorname{Irr}_K(G\times H)$ is called *perfect* if the following two conditions hold for any $x \in G, y \in H$:

1. $\mu(x,y)$ is divisible in $\mathcal{O}$ by the orders of $|C_G(x)|$ and $|C_H(y)|$;
2. if exactly one of x, y has order prime to p (we say it is p-regular) but the other has order divisible by p (we say it is p-singular), then $\mu(x,y) = 0$.

Definition 1.4.6 ([15])**.** Let G, H be finite groups and $\mathcal{O}Gb$, $\mathcal{O}Hc$ be block algebras. An *isometry* (i.e. a group isomorphism preserving the scalar product) $\Phi : \mathbb{Z}\operatorname{Irr}(\mathcal{O}Gb) \to \mathbb{Z}\operatorname{Irr}(\mathcal{O}Hc)$ is called *perfect* if the associated generalized character $\mu \in \mathbb{Z}\operatorname{Irr}(G\times H)$, defined by

$$\mu(x,y) = \sum_{\eta\in\operatorname{Irr}(\mathcal{O}Hc)} \Phi(\eta)(x)\eta(y) \quad \forall (x,y) \in G\times H$$

is perfect.

The condition on centralisers can be partially replaced, as the next Proposition shows.

Proposition 1.4.7 ([76], Proposition 2.2)**.** *If the generalized character μ induced by an isometry satisfies the following:*

1. *for any p-singular elements $x \in G$ and $y \in H$ $\frac{1}{|C_G(x)|_p}\mu(x,y) \in \mathcal{O}$ and $\frac{1}{|C_H(y)|_p}\mu(x,y) \in \mathcal{O}$.*
2. *if exactly one of x, y is p-regular then $\mu(x,y) = 0$.*

then μ is perfect.

As mentioned before, every Morita equivalence induces a perfect isometry. Perfect isometries induce a signed bijection of the irreducible characters lying in the blocks, meaning that if $\chi \in \operatorname{Irr}(\mathcal{O}Gb)$, and Φ is a perfect isometry, then $\Phi(\chi) = \delta_\chi\eta$, where $\eta \in \operatorname{Irr}(\mathcal{O}Hc)$ and $\delta_\chi \in \{1, -1\}$. Morita equivalences always induce perfect isometries with *positive signs*, i.e. $\delta_\chi = 1$ for all irreducible characters χ if Φ is induced by a Morita equivalence. It can be also shown that perfect isometries preserve the height of characters, so, with the notation above, $\operatorname{ht}(\chi) = \operatorname{ht}(\eta)$.

Note that perfect isometries can be defined in an alternative way, that will be useful in Section 2.2. We will just give an alternative definition for perfect self-isometries of blocks, but it can obviously be adapted to arbitrary perfect isometries. Let B be a block of a group G, and denote by $\mathrm{CF}(G, B, K)$ the K-subspace of class functions on G spanned by $\mathrm{Irr}(B)$. Moreover, denote by $\mathrm{CF}(G, B, \mathcal{O})$ the $\mathcal{O}$-submodule

$$\{\phi \in \mathrm{CF}(G, \mathcal{O}P \otimes_{\mathcal{O}} B, K) : \phi(g) \in \mathcal{O} \text{ for all } g \in G\}$$

A perfect self-isometry of B is an isometry

$$\Phi : \mathbb{Z}\,\mathrm{Irr}(B) \to \mathbb{Z}\,\mathrm{Irr}(B)$$

such that

$$\Phi_K = \Phi \otimes_{\mathbb{Z}} K : K\,\mathrm{Irr}(B) \to K\,\mathrm{Irr}(B)$$

induces an $\mathcal{O}$-module automorphism of $\mathrm{CF}(G, B, \mathcal{O})$ and also an automorphism of $\mathrm{CF}_{p'}(G, B, \mathcal{O})$, where $\mathrm{CF}_{p'}(G, B, \mathcal{O})$ is the following $\mathcal{O}$-submodule of $\mathrm{CF}(G, B, \mathcal{O})$:

$$\left\{\phi \in \mathrm{CF}(G, \mathcal{O}P \otimes_{\mathcal{O}} B, K) : \phi(g) = 0 \text{ for all } g \in G \setminus G_{p'}\right\},$$

where $G_{p'}$ is the set of p-regular elements. This approach to perfect isometries can be found in [29] or [15].

Another important feature of perfect isometries is their good behaviour on two well known subsets of generalized characters. We briefly define in the following lines these subsets and recall some basic properties, available in [56, Section 9.2]. Let $\mathrm{Pr}_{\mathcal{O}}(G, b)$ be the subgroup of $\mathbb{Z}\,\mathrm{Irr}(\mathcal{O}Gb)$ generated by the characters of finitely generated projective $\mathcal{O}Gb$ modules. This subgroup coincides with all generalized characters of $\mathbb{Z}\,\mathrm{Irr}(\mathcal{O}Gb)$ vanishing on every p-singular element. Its orthogonal subgroup is $L^0(G, b)$, the subgroup of all generalized characters in $\mathbb{Z}\,\mathrm{Irr}(G, b)$ that vanish on all p'-elements, which coincides with the kernel of the decomposition map $d : \mathbb{Z}\,\mathrm{Irr}(\mathcal{O}Gb) \to \mathbb{Z}\,\mathrm{IBr}(\mathcal{O}Gb)$.

Proposition 1.4.8 ([56], Proposition 9.2.6)**.** *Let G, H be finite groups and $\mathcal{O}Gb$, $\mathcal{O}Hc$ be block algebras. If $\Phi : \mathbb{Z}\,\mathrm{Irr}(\mathcal{O}Gb) \to \mathbb{Z}\,\mathrm{Irr}(\mathcal{O}Hc)$ is a perfect isometry, then Φ maps $\mathrm{Pr}_{\mathcal{O}}(G, b)$ to $\mathrm{Pr}_{\mathcal{O}}(H, c)$ and $L^0(G, b)$ to $L^0(H, c)$.*

We now explore another kind of equivalence, stable equivalences of Morita type. Before doing it, we need to recall the definition of stable categories. The stable category of $\mathrm{Mod}(A)$, where A is a symmetric $\mathcal{O}$-algebra, is the category $\underline{\mathrm{Mod}}(A)$ whose objects are the A-modules and, for any two A-modules U, V, morphisms are given by the quotients space

$$\underline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\,\mathrm{Hom}_A^{pr}(U, V),$$

where $\operatorname{Hom}_A^{pr}(U,V)$ are the A-homomorphisms that factor through a projective A-module. Therefore modules in the stable category are isomorphic if they differ by a projective module. Analogously to Morita equivalences for module categories, we have a notion of equivalences of Morita type for stable categories.

Definition 1.4.9 ([16]). Let A,B be symmetric $\mathcal{O}$-algebras, let M be an A-B-bimodule and N a B-A-bimodule. We say that M and N *induce a stable equivalence of Morita type* between A and B if M,N are finitely generated projective as left and right modules and $M \otimes_B N \simeq A$ in $\underline{\mathrm{Mod}}\,(A \otimes_{\mathcal{O}} A^{\mathrm{op}})$ and $N \otimes_A M \simeq B$ in $\underline{\mathrm{Mod}}\,(B \otimes_{\mathcal{O}} B^{\mathrm{op}})$. In particular, the functors $M \otimes_B -$ and $N \otimes_A -$ induce inverse equivalences $\underline{\mathrm{Mod}}(A) \simeq \underline{\mathrm{Mod}}(B)$.

We have already seen an example of these equivalences in this thesis: Corollary 1.3.9 actually shows that Green correspondence for groups with TI Sylow subgroups induces a stable equivalence of Morita type between $\mathcal{O}G$ and $\mathcal{O}N_G(P)$.

Obviously every Morita equivalence induces a stable equivalence of Morita type, but the converse is false. Moreover, stable equivalences of Morita type don't necessarily induce perfect isometries, however they still induce a partial isometry between the L^0 groups.

Proposition 1.4.10 ([56], Proposition 9.3.5). *Let $\mathcal{O}Gb, \mathcal{O}Hc$ be two blocks and suppose there is a stable equivalence of Morita type between them, induced by a $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule M. Let μ be the character of M, as a $G \times H$-module, and define the group homomorphism $\Phi_M : \mathbb{Z}\operatorname{Irr}(\mathcal{O}Gb) \to \mathbb{Z}\operatorname{Irr}(\mathcal{O}Hc)$ by*

$$\Phi_M(\eta)(x) = \frac{1}{|H|} \sum_{y \in H} \mu(x, y^{-1})\eta(y).$$

Then

(i) Φ_M and Φ_{M^*} induce inverse isomorphisms between $\mathbb{Z}\operatorname{Irr}(\mathcal{O}Gb)/\operatorname{Pr}_{\mathcal{O}}(G,b)$ and $\mathbb{Z}\operatorname{Irr}(\mathcal{O}Hc)/\operatorname{Pr}_{\mathcal{O}}(H,c)$.

(ii) Φ_M and Φ_{M^*} induce inverse isometries between $L^0(G,b)$ and $L^0(H,c)$.

So far we have just worked over $\mathcal{O}$, however all these concepts can be translated to k without losing too much information. In particular, Morita equivalences and stable equivalences have a *descent* property that yields analogous equivalences for the reduction modulo k of blocks. This is crucial for a strong theorem of Linckelmann we will make use of, that gives sufficient properties for a stable equivalence of Morita type to actually be a Morita equivalence.

Proposition 1.4.11 ([55], Proposition 4.14.4)**.** *Let A, B be symmetric $\mathcal{O}$-algebras. Let M be an A-B-bimodule that is finitely generated projective as left and right module. Then M and its $\mathcal{O}$-dual M^* induce a stable equivalence of Morita type (resp. Morita equivalence) between A and B if and only if $k \otimes_{\mathcal{O}} M$ and its k-dual $(k \otimes_{\mathcal{O}} M)^*$ induce a stable equivalence of Morita type (resp. Morita equivalence) between $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$.*

Theorem 1.4.12 ([54], Theorem 2.1)**.** *Let kGb, kHc be k-blocks, M a kHc-kGb-bimodule such that $M \otimes_{kGb} -$ induces a stable equivalence of Morita type. If for all simple kGb-modules S, the kHc-module $M \otimes_{kGb} S$ is simple, then the functor $M \otimes_{kGb} -$ is a Morita equivalence.*

We conclude this section by introducing some special kinds of Morita equivalences. Recall that the identification between bimodules and left modules used for defining defect groups of blocks allows the definition of vertices and sources for bimodules. Also, permutations $\mathcal{O}G$-modules are $\mathcal{O}$-free modules that have an $\mathcal{O}$-basis that is G-stable. Equivalently, permutations modules are direct sums of modules of the form $1_H \uparrow_H^G$, where $H \leq G$. See for example [42, Chapter 11, Lemma 1.1].

Definition 1.4.13 ([24])**.** Let P be a finite p-group. An *endopermutation module $\mathcal{O}P$*-module is a finitely generated $\mathcal{O}$-free $\mathcal{O}P$-module V with the property that $V \otimes_{\mathcal{O}} V^*$ is a permutation $\mathcal{O}P$-module.

Definition 1.4.14. Let $\mathcal{O}Gb$, $\mathcal{O}Hc$ be blocks, M a $\mathcal{O}Hc$-$\mathcal{O}Gb$-bimodule inducing a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$. Let V be a source for M as a $\mathcal{O}(H \times G)$-module.

(i) If V is trivial, then we say *M induces a splendid Morita equivalence* between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

(ii) If V is an endopermutation module, then we say that *M induces a basic Morita equivalence* between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

Note that the meaning we reserved to the term *splendid* is the one that became customary since the publication of [73], as noticed in the introduction of Section 9.7 of [56]. These two equivalences preserve many properties of blocks, we list some of them in the next theorem.

Theorem 1.4.15 ([71], Remark 7.6; [43], Corollary 1.7)**.** *Let $\mathcal{O}Gb$, $\mathcal{O}Hc$ be blocks, and suppose that M is an indecomposable endopermutation source $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule inducing a Morita equivalence between the two blocks. Then:*

(i) $\mathcal{O}Gb$ and $\mathcal{O}Hc$ have isomorphic defect groups.

(ii) *The bijection of characters* $\operatorname{Irr}(\mathcal{O}Gb) \to \operatorname{Irr}(\mathcal{O}Hc)$ *induced by* M *sends* p*-rational characters to* p*-rational characters (*χ *is* p*-rational if* $\mathbb{Q}(\chi) = \mathbb{Q}(e^{2\pi i/n})$*, with* $(n,p) = 1$*).*

These are not all the invariants preserved by these two equivalences, in particular there are some important structures, source algebras and fusion systems, that need to be introduced and are invariant under one or both the equivalences defined above. The following brief introduction to Brauer map and source idempotents follows the approach used in Sections 6.4 and 6.7 of [56].

Let G be a finite group, Q a p-subgroup of G. We define the Brauer map as the projection onto $C_G(Q)$, i.e.:

$$\begin{aligned} \operatorname{Br}_Q : kG &\longrightarrow kC_G(Q) \\ \sum_{g\in G} a_g g &\mapsto \sum_{g\in C_G(Q)} a_g g \end{aligned}$$

The Brauer map can also be defined on elements of $\mathcal{O}G$, pre-composing it with the canonical map $\mathcal{O} \to k$. Moreover, if restricted to $(kG)^Q$, the Q-stable elements of kG, Br_Q becomes an algebra homomorphism, and can be used to define what a defect group is, among other things. In fact, if b is a block idempotent for kG, a defect group for kGb is a p-subgroup D of G that is maximal among the subgroups of G such that $\operatorname{Br}_D(b) \neq 0$. Note that, if D is a defect group, then $\operatorname{Br}_D(b)$ is non-zero and, since b is central and lies in $(\mathcal{O}G)^D$,

$$\operatorname{Br}(b) = \operatorname{Br}(b^2) = \operatorname{Br}(b)^2.$$

Therefore $\operatorname{Br}(b)$ is a central idempotent of $kC_G(D)$, in fact one can show that it is a central idempotent of $kN_G(D)$, paving the way for Brauer correspondence.

Theorem 1.4.16 (Brauer's correspondence, [12])**.** *Let* G *be a finite group and* P *a* p*-subgroup of* G*. Let* H *be a subgroup containing* $N_G(P)$*. For any block* b *of* $\mathcal{O}G$ *with* P *as a defect group there is a unique block* c *of* H *with* P *as a defect group such that* $\operatorname{Br}_P(b) = \operatorname{Br}_P(c)$*, and this correspondence defines a bijection between the blocks of* $\mathcal{O}G$ *and* $\mathcal{O}H$ *with* P *as a defect group.*

Definition 1.4.17. With the above notation, the unique block c of $\mathcal{O}N_G(P)$ with P as a defect group satisfying $\operatorname{Br}_P(b) = \operatorname{Br}_P(c)$ is called the *Brauer correspondent* of b. If e is a block of $\mathcal{O}C_G(P)$ satisfying $ec = e$, then $E = N_G(P,e)/PC_G(P)$ is called the *inertial quotient* of b.

Brauer correspondence can be interpreted as a special case of Green correspondence, but we don't explore this in detail. A proof can be however found in [56, Theorem 6.7.2].

The Brauer map also allows us to define the following structures.

Definition 1.4.18. Let G be a finite group, $\mathcal{O}Gb$ a block. A *Brauer pair* on $\mathcal{O}G$ is a pair (P, e) consisting of a p-subgroup P of G and a block e of $kC_G(P)$. A (G, b)-*Brauer pair* is a Brauer pair (P, e) on $\mathcal{O}G$ with the additional property $\mathrm{Br}_P(b)e = e$.

Definition 1.4.19. Let G be a finite group, $\mathcal{O}Gb$ a block of $\mathcal{O}G$ with defect group D.

(i) An *interior G-algebra* is an $\mathcal{O}$-algebra A together with a homomorphism $G \to A^\times$.

(ii) A primitive idempotent $i \in (\mathcal{O}Gb)^D$ satisfying $\mathrm{Br}_D(i) \neq 0$ is called a source idempotent of b.

If $i \in (\mathcal{O}Gb)^D$ is a source idempotent, the interior D-algebra $i\mathcal{O}Gi$, with structural homomorphism sending $u \in D$ to $ui = iui = iu$ is called a *source algebra* of b.

Source algebras are strictly related to splendid Morita equivalences, in fact, as the following results shows, every Morita equivalence that preserves source algebras is splendid, and the converse also holds.

Theorem 1.4.20 ([71]). *Let G, H be finite groups, $\mathcal{O}Gb$, $\mathcal{O}Hc$ blocks having a common defect group P, and $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be source idempotents. The following are equivalent.*

1. *$\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendid Morita equivalent via an indecomposable direct summand of $\mathcal{O}Gi \otimes_{\mathcal{O}P} j\mathcal{O}H$.*

2. *There is an isomorphism of interior P-algebras $\varphi : i\mathcal{O}Gi \to j\mathcal{O}Hj$*

The next section will be devoted to defining another important structure, that will be preserved under basic Morita equivalences (and thus under splendid equivalences): fusion systems. The theory of fusion systems is very rich, and many important results rely on these structures. However we will just give the basic definitions and properties that are relevant to our context.

1.5 Fusion systems

The presentation of fusion systems in this section follows the approach used in [56].

Definition 1.5.1. A *category on a finite p-group* P is a category $\mathcal{F}$ having as objects the subgroups of P, and such that for any two subgroups $Q, R \leq P$ the morphisms $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ satisfy the following:

(i) Every $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ is an injective group homomorphism.

(ii) If $Q \leq R$ then the inclusion morphism is a morphism in $\mathcal{F}$

(iii) If $\phi : Q \to R$ is a morphism in $\mathcal{F}$ then the induced isomorphism $\phi : Q \to \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.

(iv) Composition of group morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Definition 1.5.2. Let p be a prime, P a finite p-group, $\mathcal{F}$ a category on P. A subgroup Q of P is called *fully $\mathcal{F}$-normalized* if $|N_P(Q)| \geq |N_P(Q')|$ for all $Q' \leq P$ such that there is an $\mathcal{F}$-isomorphism $Q \simeq Q'$.

Let p be a prime, P a finite p-group, $\mathcal{F}$ a category on P. Let Q be an object of $\mathcal{F}$, $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. For a lighter notation, denote by N_ϕ the subgroup of $N_P(Q)$ consisting of all elements $y \in N_P(Q)$ for which there exists an elements $z \in N_P(\phi(Q))$ such that $\phi({}^y u) = {}^z\phi(u)$ for all $u \in Q$.

Definition 1.5.3. Let p be a prime, P a finite p-group. A *fusion system* on P is a category $\mathcal{F}$ on P such that $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R)$ for any two objects in $\mathcal{F}$, where

$$\mathrm{Hom}_P(Q, R) = \{\phi : Q \to R | \exists x \in P : \phi(u) = {}^x u \, \forall u \in Q\},$$

and the following hold:

(i) The group $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(ii) For every morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\phi(Q)$ is fully normalized there exists a morphism $\psi : N_\phi \to P$ in $\mathcal{F}$ such that $\psi_{|Q} = \phi$.

Before going on with definitions, let's see a concrete example of a fusion system. Let G be a finite group and P be a p-subgroup of G. Denote then by $\mathcal{F}_P(G)$ the category with objects the subgroups of P and with morphisms the group homomorphisms induced by conjugation by an element of G, i.e. $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R)$. If P is a Sylow p-subgroup of G then $\mathcal{F}_P(G)$ is a fusion system on P.

We are going to see how source algebras, and blocks, determine fusion systems. Recall that (G,b)-Brauer pairs form a partially ordered set, by

$$(Q,f) \leq (P,e) \text{ if } Q \leq P \text{ and } \mathrm{Br}_P(i)e \neq 0,\ \mathrm{Br}_Q(i)f \neq 0$$
$$\text{for a primitive idempotent } i \in (\mathcal{O}G)^P.$$

It can actually be shown that, if $Q \leq P$, there is a unique block $kC_G(Q)f$ of $kC_G(Q)$ such that $(Q,f) \leq (P,e)$, [4, Theorem 3.4].

Definition 1.5.4. Let G be a finite group, b a block of $\mathcal{O}G$ and let (P,e) be a maximal (G,b)-Brauer pair. For any subgroup Q of P denote by e_Q *the unique block of* $kC_G(Q)$ *such that* $(Q,e_Q) \leq (P,e)$. We define a *category* $\mathcal{F} = \mathcal{F}_{(P,e)}(G,b)$ as follows:

(i) the objects of $\mathcal{F}$ are the subgroups of P

(ii) for any two subgroups Q,R of P, the morphism set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ is the set of all group homomorphisms $\phi : Q \to R$ for which there exists an element $x \in G$ satisfying $\phi(u) = {}^x u$ for all $u \in Q$ and such that ${}^x(Q,e_Q) \leq (R,e_R)$.

Thus if we have a block $\mathcal{O}Gb$ and a source idempotent $i \in (\mathcal{O}Gb)^P$ we have a category $\mathcal{F} = \mathcal{F}_{(P,i)}(G,b)$, and this is the category determined by the source algebra $A = iOGi$.

If k is large enough, i.e. is a splitting field for any subgroup of G, then the category previously defined is a fusion system. In particular if b is the principal block of OG then the following holds.

Proposition 1.5.5 ([56], Proposition 8.5.5)**.** *Let G be a finite group, b a block (idempotent) of principal type of $\mathcal{O}G$, and (P,e) a maximal (G,b)-Brauer pair. Suppose that k is a splitting field for the block e of $kC_G(P)$. We have $\mathcal{F}_{(P,e)}(G,b) = \mathcal{F}_P(G)$.*

Recall that we chose k to be an algebraically closed field, so certainly k is a splitting field for any subgroup of G.

We mentioned before that fusion systems are preserved by basic Morita equivalences, we now state explicitly this important result.

Theorem 1.5.6 ([56], Theorem 9.11.9)**.** *Let G,H be finite groups, let $\mathcal{O}Gb, \mathcal{O}Hc$ be blocks, having a common defect group P, and let $i \in (\mathcal{O}Gb)^{\Delta P}$, $j \in (\mathcal{O}Hc)^{\Delta P}$ be source idempotents. Set $A = i\mathcal{O}Gi, B = j\mathcal{O}Hcj$, and suppose M is an indecomposable endopermutation source $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule inducing a (basic) Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$. Then A and B determine the same fusion system on P.*

We conclude this section with some facts on automorphisms of fusion system, that will be useful later.

Definition 1.5.7 ([5], Definition 1.13)**.** An automorphism $\beta \in \mathrm{Aut}(P)$ is *fusion preserving* if for each $R, Q \leq P$ and each $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$

$$\beta|_Q \left(\phi \left(\beta|_R \right)^{-1} \right) \in \mathrm{Hom}_{\mathcal{F}}(\beta(R), \beta(Q))$$

In particular, each such β normalizes $\mathrm{Aut}_{\mathcal{F}}(P)$. We denote by $\mathrm{Aut}(P, \mathcal{F})$ the set of fusion preserving automorphisms. Set $\mathrm{Out}(P, \mathcal{F}) = \mathrm{Aut}(P, \mathcal{F}) / \mathrm{Aut}_{\mathcal{F}}(P)$.

Soon, we will need an explicit description of $\mathrm{Out}(P, \mathcal{F})$. The motivation for this lies in the exact sequences defined by Theorem 1.6.11; in fact, $\mathrm{Out}(P, \mathcal{F})$ plays an important role for the computation of Picard groups, as will be explicitly seen in Section 2. Describing $\mathrm{Out}(P, \mathcal{F})$ is not always possible, however for fusion systems on abelian p-groups it can be done and the following result comes particularly handy.

Proposition 1.5.8 ([56], Proposition 8.1.14)**.** *Let P be an abelian p-group, $\mathcal{F}$ be a fusion system on P. Then $E = \mathrm{Aut}_{\mathcal{F}}(P)$ is a p'-subgroup of $\mathrm{Aut}(P)$ and we have $\mathcal{F} = \mathcal{F}_P(P \rtimes E)$. In particular, if P is a Sylow p-subgroup of a group G, then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$.*

Abelian p-groups are a special case of what are usually called *resistant* p-groups [79], i.e. p-groups for which every fusion system on P is just that of $P \rtimes A$, where A is a p'-subgroup of $\mathrm{Aut}(P)$. We are going to see that $\mathrm{Out}(P, \mathcal{F})$ can be easily described not only for fusion systems on abelian p-groups, but in a more general setting. We give all the relevant definitions and lemmas needed.

Definition 1.5.9. Let $\mathcal{F}$ be a category on a finite p-group P. A subgroup Q of P is called *$\mathcal{F}$-centric* if $C_P(R) = Z(Q)$ for any subgroup R of P such that there is an isomorphism $R \simeq Q$ in $\mathcal{F}$. If Q is $\mathcal{F}$-centric and $\mathrm{Aut}_{\mathcal{F}}(Q)$ has a strongly p-embedded subgroup, i.e. there is $H \leq \mathrm{Aut}_{\mathcal{F}}(Q)$ such that p divides $|H|$ but not $|H \cap H^x|$ for any $x \in \mathrm{Aut}_{\mathcal{F}}(Q) \setminus H$, then Q is called *$\mathcal{F}$-essential.*

Theorem 1.5.10 (Alperin's fusion theorem, [3])**.** *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Every isomorphism in $\mathcal{F}$ can be written as a composition of finitely many isomorphisms of the form $\varphi : R \to S$ in $\mathcal{F}$ for which there is a subgroup Q containing both R, S and an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\alpha \downarrow_R = \varphi$ and such that either $Q = P$ or Q is fully $\mathcal{F}$-normalised essential and α has p-power order.*

Note that the formulation above is the one given in [56, Theorem 8.2.8]. Detecting $\mathcal{F}$-essentials subgroups is not in general a trivial task, but the following Lemma is a fundamental tool for fusion systems on Sylow p-subgroups of a group G.

Lemma 1.5.11 ([23], Theorem 4.27 and Lemma 7.3)**.** *Let P be a Sylow p-subgroup of a group G and $\mathcal{F} = \mathcal{F}_P(G)$. There are no $\mathcal{F}$-essential subgroups of P if and only if $\mathcal{F} = \mathcal{F}_P\left(N_G(P)\right)$.*

We finally provide the description of $\mathrm{Out}(P, \mathcal{F})$ previously promised.

Proposition 1.5.12. *Let G be a finite group, $B = \mathcal{O}Gb$ the principal block of G and (P, e) a maximal (G, b)-Brauer pair. Suppose that P is normal in G or P is abelian, and let $\mathcal{F} = \mathcal{F}_{P,e}(G, b)$. Then $\mathrm{Out}(P, \mathcal{F}) \simeq N_{\mathrm{Out}(P)}(E)/E$, where E is the inertial quotient of B.*

Proof. By Proposition 1.5.5, $\mathcal{F} = \mathcal{F}_P(G)$. Let now $L = \mathrm{Aut}_{\mathcal{F}}(P)$. We start by giving a description of $\mathrm{Aut}(P, \mathcal{F})$. If $\beta \in \mathrm{Aut}(P, \mathcal{F})$, then, for each $R, Q \leq P$ and each $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$

$$\beta|_Q \left(\phi(\beta|_R)^{-1}\right) \in \mathrm{Hom}_{\mathcal{F}}(\beta(R), \beta(Q)).$$

However, P is either abelian or normal in G, therefore $\mathcal{F} = \mathcal{F}_P(N_G(P))$ by Proposition 1.5.8. By Lemma 1.5.11 there are no $\mathcal{F}$-essential subgroups of P, so Theorem 1.5.10 yields that $\hat{\phi} \in \mathrm{Hom}_{\mathcal{F}}(R, \phi(R))$ is the composition of isomorphisms of the form $\alpha|_R$, for $\alpha \in L$. We can assume that $\phi = \alpha|_R$. Now if $R = Q = P$, then β must satisfy

$$\beta\alpha\beta^{-1} \in L$$

This however means that $\beta \in N_{\mathrm{Aut}(P)}\left(L\right)$. Now, if γ is in $N_{\mathrm{Aut}(P)}\left(L\right)$ and $\alpha \in L$, then we have $\gamma\alpha\gamma^{-1} \in L$. Therefore, for any $Q, R \leq P$,

$$\left(\gamma\alpha\gamma^{-1}\right)|_{\gamma(R)} = \gamma|_Q \left(\alpha|_R(\gamma|_R)^{-1}\right) \in \mathrm{Hom}_{\mathcal{F}}(\gamma(R), \gamma(Q)).$$

We have thus shown that $\mathrm{Aut}(P, \mathcal{F}) = N_{\mathrm{Aut}(P)}(L)$. Now, since P is a Sylow p-subgroup of G, then $L \simeq N_G(P)/C_G(P)$. Note that the inertial quotient for B is $E = N_G(P, e)/PC_G(P)$, but, since B is principal, then $E \simeq N_G(P)/PC_G(P)$. If P is abelian, the proof is concluded.

Assume then that P is normal. It can be immediately noticed that $L \simeq (P \rtimes E)/Z(P)$. Certainly $N_{\mathrm{Aut}(P)}(L)$ contains a normal subgroup H isomorphic to $P/Z(P)$, and then

$$\begin{aligned}\mathrm{Out}(P, \mathcal{F}) = N_{\mathrm{Aut}(P)}(L)/L &\simeq \frac{N_{\mathrm{Aut}(P)}(L)}{(P \rtimes E)/Z(P)} \\ &\simeq \frac{N_{\mathrm{Aut}(P)}(L)/H}{\left((P \rtimes E)/Z(P)\right)/(P/Z(P))} \simeq N_{\mathrm{Out}(P)}(E)/E\end{aligned}$$

□

Remark 1.5.13. The proof of Proposition 1.5.12 holds for blocks of principal type with no difficulty. It should also hold for arbitrary blocks, provided that a defect group P has no $\mathcal{F}$-essential subgroups. Perhaps the proof could be modified to remove any restrictions on blocks and defect groups.

This last result will be crucial for calculating the isomorphism type of Picard groups for blocks with defect groups $C_3 \times C_3$.

1.6 Picard groups

The first appearance of Picard groups of non-commutative algebras in the literature can be traced quite far back in the past, but a systematic approach to Picard groups was first adopted by Albert Fröhlich in [33]. In fact, before the main interest of non-commutative algebraists was the special case of Azumaya algebras, central separable algebras, where the Picard group of the algebra coincides with the Picard group of the center of the algebra. In general, Picard groups have a much richer structure. We start writing down the definition and we then present the basic properties of Picard groups for blocks.

Definition 1.6.1. Let A be an $\mathcal{O}$-algebra. If M is an A-A-bimodule that is finitely generated projective as a left and right A-module such that the functor $M \otimes_A -$ is an equivalence on $\mathrm{mod}(A)$, then M is called an *invertible bimodule.*

Definition 1.6.2. Let A be an $\mathcal{O}$-algebra. The group of isomorphism classes of invertible A-A-bimodules, with group multiplication induced by $- \otimes_A -$, is called the *Picard group* of A, denoted by $\mathrm{Pic}(A)$.

Since blocks are the main object of our investigation, we restrict our attention to block algebras. Moreover, we will adopt a lighter notation in proofs and say that $M \in \mathrm{Pic}(B)$ to mean $[M]_\sim \in \mathrm{Pic}(B)$. Note that blocks are symmetric $\mathcal{O}$-algebras and therefore, in view of Theorem 1.4.2, the inverse of $M \in \mathrm{Pic}(B)$ is just the $\mathcal{O}$-dual M^*. Moreover $\mathrm{Pic}(B)$ can naturally be interpreted as the group of Morita auto-equivalences of B, and it follows immediately that Picard groups are invariant under Morita equivalences.

Proposition 1.6.3. *Let B, B' be blocks. If B and B' are Morita equivalent, then* $\mathrm{Pic}(B) \simeq \mathrm{Pic}(B')$.

Proof. Let M be a B-B'-bimodule that induces a Morita equivalence between B and B'.

Define then the following map:

$$\gamma : \mathrm{Pic}(B) \to \mathrm{Pic}(B')$$
$$N \mapsto M^* \otimes_B N \otimes_B M$$

γ is obviously an homomorphism, and an inverse map can be easily defined, namely:

$$\gamma^{-1} : \mathrm{Pic}(B') \to \mathrm{Pic}(B)$$
$$L \mapsto M \otimes_{B'} L \otimes_{B'} M^*$$

It follows that $\mathrm{Pic}(B) \simeq \mathrm{Pic}(B')$. □

Obviously, the definition of the Picard group can be adapted to k-algebras and a crucial question is how $\mathrm{Pic}(B)$ relates to $\mathrm{Pic}(k \otimes_{\mathcal{O}} B)$. These two groups are quite different, as we are going to see soon.

Proposition 1.6.4 ([55], Proposition 2.8.16)**.** *Let A be an $\mathcal{O}$-algebra (or a k-algebra). Then there is a natural map*

$$\mathrm{Aut}(A) \ni \alpha \mapsto A_\alpha \in \mathrm{Pic}(A),$$

where A_α is the A-A-bimodule whose right action is twisted by α. Moreover the kernel of the aforementioned map is $\mathrm{Inn}(A)$*.*

This last proposition allows us to find a large subgroup of $\mathrm{Pic}(k \otimes_{\mathcal{O}} B)$, and to show that Picard groups of k-blocks are not finite in general.

Example 1.6.5. *Let $B = \mathcal{O}C_p$; then, considering $k \otimes_{\mathcal{O}} B$, there is a well known isomorphism $kC_p \simeq k[x]/(x^p)$. In fact, if g is a generator of C_p, then the map defined by*

$$g - 1 \mapsto x + (x^p)$$

realizes the isomorphism. Moreover, k-algebra automorphisms of $k \otimes_{\mathcal{O}} B \simeq k[x]/(x^p)$ are completely determined by the image of x. Therefore

$$x \mapsto \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + ...$$

for $\lambda_1, \lambda_2, ... \in k$ and $\lambda_1 \neq 0$ defines such an automorphism. By the previous proposition, each of these automorphisms gives rise to a distinct element of $\mathrm{Pic}(k \otimes_{\mathcal{O}} B)$*, thus* $\mathrm{Pic}(kC_p)$ *is not finite.*

However for blocks of $\mathcal{O}G$ the situation is very different, since all Picard groups of $\mathcal{O}$-blocks are finite groups, as Florian Eisele proved.

Theorem 1.6.6 ([31])**.** *Let B be an $\mathcal{O}$-block. Then* $\mathrm{Pic}(B)$ *is finite.*

Eisele's result is one of the main reasons why working over a complete discrete valuation ring with characteristic zero is the natural setting for studying Picard groups.

We outline some additional motivations for studying Picard groups of blocks, even though the fact that they are Morita invariant already seems to be enough of one. The first reason why Picard groups were extensively studied in the past years was somehow related to Donovan's conjecture, which we have already seen as Conjecture 1.4.3. Specifically, for defect groups for which Donovan's conjecture is known to be true, describing a list of representatives of blocks for the Morita equivalence classes can be an interesting task that could provide counterexamples to other conjectures or suggest patterns. A complete classification for given defect groups was achieved in [26] and [6] for defect groups isomorphic to C_2^4 and C_2^5, with the use of Picard groups.

Donovan's conjecture is not a main theme of our research, so we are not going to dwell too much on this topic, but it should be noted that the procedure described in the aforementioned papers requires, among other things, the knowledge of isomorphism types of Picard groups for some blocks.

Another important reason for studying Picard groups is understanding the action of Morita equivalences on irreducible characters. To introduce this topic, we need to give the definition of a certain normal subgroup of $\mathrm{Pic}(B)$.

Definition 1.6.7. Let B be a block, then

$$\mathrm{Picent}(B) = \{[M]_{\sim} \in \mathrm{Pic}(B) |\, cm = mc \text{ for every } c \in Z(B), m \in M\}$$

Proposition 1.6.8. *Let B be a block, $M \in \mathrm{Pic}(B)$. Then $M \in \mathrm{Picent}(B)$ if and only if the perfect isometry induced by M fixes every $\chi \in \mathrm{Irr}(B)$.*

Proof. Let Φ be the perfect isometry induced by M, then, by [56, Theorem 9.2.3], Φ induces an automorphism of the center of B, as an $\mathcal{O}$-algebra. In particular, such an automorphism is induced by the permutation of characters idempotents given by:

$$Z(K \otimes_{\mathcal{O}} B) \ni e_\chi \mapsto e_{\Phi(\chi)} \in Z(K \otimes_{\mathcal{O}} B).$$

It is routine checking that if the above map is trivial, then the automorphism induced on $Z(B)$ is trivial as well. However, by [55, Corollary 2.8.6], the automorphism induced by M on the center can also be described by the map $\phi_M : Z(B) \to Z(B)$ by setting $\phi_M(z)$ as the unique element of $Z(B)$ such that

$$zm = m\phi_M(z) \text{ for every } m \in M.$$

Since M acts trivially on $Z(B)$, we conclude that $M \in \mathrm{Picent}(B)$.

Conversely, if $M \in \mathrm{Picent}(B)$, then M fixes $Z(B)$ pointwise and thus, from the definition of the map induced by Φ, it actually follows that Φ must act trivially on irreducible characters of B. □

By Proposition 1.6.8 we can immediately conclude that $\mathrm{Picent}(B)$ is a normal subgroup of $\mathrm{Pic}(B)$, since it can be seen as the kernel of the map $\mathrm{Pic}(B) \to \mathrm{Perf}(B)$, where $\mathrm{Perf}(B)$ is the group of perfect self-isometries of B (this map is naturally a group homomorphism). $\mathrm{Picent}(B)$, as $\mathrm{Pic}(B)$, is also a Morita invariant.

Proposition 1.6.9 ([72], Theorem 37.9). *If B and B' are Morita equivalent, then* $\mathrm{Picent}(B) \simeq \mathrm{Picent}(B')$.

An interesting problem regarding the Picent of blocks is concerned with when this group is trivial. In fact, if $\mathrm{Picent}(B)$ is trivial, then every Morita auto-equivalence of B can be recognized by its action on irreducible characters, and this would allow us to bound the size of $\mathrm{Pic}(B)$ in terms of the defect of the block. We will deal with this problem in a later section.

One last stimulating question stems naturally from the study of Picard groups and basic Morita equivalences, and is perhaps a more structural dilemma. We first need to introduce three notable subgroups of $\mathrm{Pic}(B)$.

Definition 1.6.10 ([10]). Let B be a block of $\mathcal{O}G$, and define the following subsets of $\mathrm{Pic}(B)$.

$$\mathcal{E}(B) = \{[M]_\sim \in \mathrm{Pic}(B) | M \text{ has endopermutation source as a } G \times G\text{-module}\}$$

$$\mathcal{L}(B) = \{[M]_\sim \in \mathrm{Pic}(B) | M \text{ has linear source as a } G \times G\text{-module}\}$$

$$\mathcal{T}(B) = \{[M]_\sim \in \mathrm{Pic}(B) | M \text{ has trivial source as a } G \times G\text{-module}\}$$

It can be shown that these subsets are actually subgroups of $\mathrm{Pic}(B)$.

In the paper where these three subgroups first appeared, the authors also described three exact sequences associated to them. Even if they are not short exact sequences, they are quite effective in the calculation of the isomorphism type of $\mathrm{Pic}(B)$, once we know that $\mathrm{Pic}(B) = \mathcal{E}(B)$. In the following theorem, $\mathrm{Out}_P(A)$, where A is a source algebra for a block with defect group P, denotes the set of algebra automorphisms of A that fix the image of P in A pointwise, modulo inner automorphisms. We won't define the groups $D_\mathcal{O}(P, \mathcal{F})$ and $\mathrm{foc}(P)$, since they won't be used in the rest of the thesis. The interested reader can however refer to [10].

Theorem 1.6.11 ([10], Theorem 1.1)**.** *Let G be a finite group and $B = \mathcal{O}Gb$ be a block of $\mathcal{O}G$ with defect group P and source idempotent i in $(\mathcal{O}Gb)^P$. Set $A = i\mathcal{O}Gi$ and denote by $\mathcal{F}$ the fusion system on P determined by A.*

1. *Let M be a (B, B)-bimodule which induces a Morita equivalence and which has an endopermutation module as a source. Then M is isomorphic to a direct summand of*

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta\phi}^{P\times P}(V) \otimes_{\mathcal{O}P} i\mathcal{O}G$$

for some $\phi \in \mathrm{Aut}(P, \mathcal{F})$, and some $\mathcal{F}$-stable indecomposable endopermutation $\mathcal{O}P$-module V, regarded as an $\mathcal{O}\Delta\phi$-module via the isomorphism $P \simeq \Delta\phi$ sending $x \in P$ to $(x, \phi(x))$.

2. *The correspondence sending M to the pair (V, ϕ) induces a group homomorphism Φ making the following diagram commutative with exact rows:*

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathrm{Out}_P(A) & \longrightarrow & \mathcal{E}(B) & \xrightarrow{\Phi} & \mathcal{D}_{\mathcal{O}}(P,\mathcal{F}) \rtimes \mathrm{Out}(P,\mathcal{F}) \\
 & & \| & & \uparrow & & \uparrow \\
1 & \longrightarrow & \mathrm{Out}_P(A) & \longrightarrow & \mathcal{L}(B) & \xrightarrow{\Phi} & \mathrm{Hom}(P/\mathrm{foc}(P), \mathcal{O}^\times) \rtimes \mathrm{Out}(P,\mathcal{F}) \\
 & & \| & & \uparrow & & \uparrow \\
1 & \longrightarrow & \mathrm{Out}_P(A) & \longrightarrow & \mathcal{T}(B) & \xrightarrow{\Phi} & \mathrm{Out}(P,\mathcal{F})
\end{array}$$

Some important remarks need to be made on these exact sequences, most of which were already made in [10]. The first one is the following result by Puig.

Proposition 1.6.12 ([69], Proposition 14.9)**.** *With the notation used in Theorem 1.6.11, $\mathrm{Out}_P(A)$ is canonically isomorphic to a subgroup of the finite abelian p'-group $\mathrm{Hom}(E, k^\times)$, where E is the inertial quotient of B.*

Another result by Puig in [71] describes the image of $\mathrm{Out}_P(A)$ in $\mathrm{Pic}(B)$; namely, it is the subgroup of $\mathcal{T}(B)$ given by trivial source bimodules with diagonal vertex ΔP. In general, endopermutation source bimodules inducing Morita auto-equivalences of B have vertices

$$\Delta\phi = \{(x, \phi(x)) | \, x \in P\},$$

for some $\phi \in \mathrm{Aut}(P, \mathcal{F})$, as it follows from (i) of Theorem 1.6.11. It should also be noted that the maps Φ in the sequences in Theorem 1.6.11 are not surjective in general. An example that involves nilpotent blocks can be found in [10, Example 7.2]. Finally recall that, by Proposition 1.5.12, if P is an abelian or normal Sylow p-subgroup of G and B is the principal block of G, then $\mathrm{Out}(P, \mathcal{F}) \simeq N_{\mathrm{Out}(P)}(E)/E$, where E is the inertial quotient of B. In particular, if P is abelian, $\mathrm{Out}(P, \mathcal{F}) \simeq N_{\mathrm{Aut}(P)}(E)/E$.

The subgroups $\mathcal{L}(B), \mathcal{T}(B)$ are not, in general, invariant under Morita equivalences. An important example arises with nilpotent blocks: in [10, Example 7.2] the authors provide an example of a nilpotent block B with a non-linear source Morita auto-equivalence. Since a nilpotent block is Morita equivalent to $\mathcal{O}P$ for P a p-group and, by [74], $\mathrm{Pic}(\mathcal{O}P) = \mathcal{L}(\mathcal{O}P)$, $\mathcal{L}(B)$ is then not Morita invariant. The same example provides a nilpotent block B for which the map Φ in Theorem 1.6.11 is not surjective and for which $\mathcal{T}(B)$ is not invariant under Morita equivalence. For $\mathcal{E}(B)$ the situation is quite different, since a crucial question regarding Picard groups is the following.

Conjecture 1.6.13. *Let G be a finite group and B a block of $\mathcal{O}G$. Then* $\mathrm{Pic}(B) = \mathcal{E}(B)$.

This conjecture was first aired in [31], and, at the time of the writing of this thesis, there are no known examples of blocks such that $\mathcal{E}(B)$ is properly contained in $\mathrm{Pic}(B)$. It should be noted that there are counterexamples to Conjecture 1.6.13, when formulated for blocks of kG. We postpone a detailed discussion of this matter to the end of this section.

Evidence for the validity of Conjecture 1.6.13 has been exhibited for certain classes of blocks; we are going to collect some results of the literature in the next theorem.

Theorem 1.6.14. *Let B be a block of a finite group G.*

(i) If G is a p-group, then $\mathrm{Pic}(B) = \mathcal{E}(B)$, *[74].*

(ii) If B is a nilpotent block, then $\mathrm{Pic}(B) = \mathcal{E}(B)$, *[70].*

(iii) If B has cyclic defect group, then $\mathrm{Pic}(B) = \mathcal{E}(B)$, *[54].*

(iv) If B has normal abelian defect group and abelian inertial quotient, then $\mathrm{Pic}(B) = \mathcal{E}(B)$, *[59].*

Except the last result, all the other facts presented above were proved for Morita equivalences between arbitrary blocks. In particular, an open question regarding Morita equivalences between blocks is still unanswered:

Conjecture 1.6.15. *All Morita equivalences of blocks are basic Morita equivalences.*

However, it can be shown that Conjectures 1.6.15 and 1.6.13 are actually equivalent; we are going to do this in the remaining part of this section.

Set up the following piece of notation, that is going to be valid just until the end of the section: if V, W are, respectively, $\mathcal{O}G$,$\mathcal{O}H$-modules then we consider $V \otimes_{\mathcal{O}} W$ as an $\mathcal{O}(G \times H)$-module via

$$(g,h)(v \otimes w) = (gv \otimes hw), \text{ for } v \in V, w \in W, g \in G, h \in H$$

Lemma 1.6.16. *Let V, W be finitely generated $\mathcal{O}$-free $\mathcal{O}G, \mathcal{O}H$-modules. Then*

$$\mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}}(W) \simeq \mathrm{End}_{\mathcal{O}}(V \otimes_{\mathcal{O}} W)$$

as left $\mathcal{O}(G \times H)$-modules.

Proof. Consider the canonical map

$$\mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}}(W) \ni \phi \otimes \psi \mapsto \alpha_{\phi\otimes\psi} \in \mathrm{End}_{\mathcal{O}}(V \otimes_{\mathcal{O}} W),$$

where $\alpha_{\phi\otimes\psi}(v \otimes w) = \phi(v) \otimes \psi(w)$, for $v, w \in V, W$. This map is an isomorphism of $\mathcal{O}$-modules, since the modules we deal with are finitely generated free $\mathcal{O}$-modules. Moreover, the action of $G \times H$ is preserved; in fact,

$$\begin{aligned}\alpha_{(g,h)\phi\otimes\psi}(v \otimes w) &= ((g, h)\,(\phi \otimes \psi))\,(v \otimes w) = (g, h)\,((\phi \otimes \psi)(v \otimes w)) \\ &= (g, h)\,(\alpha_{\phi\otimes\psi}(v \otimes w)) \ \text{ for } (g, h) \in G \times H \text{ and } v \otimes w \in V \otimes_{\mathcal{O}} W.\end{aligned}$$

Therefore, the isomorphism is actually a map of $\mathcal{O}(G \times H)$-modules. □

Lemma 1.6.17. *Let V, W be indecomposable $\mathcal{O}P, \mathcal{O}Q$-modules, where P, Q are p-groups; if $V \otimes_{\mathcal{O}} W$ is a permutation $\mathcal{O}(P \times Q)$-module, then V and W are permutation $\mathcal{O}P, \mathcal{O}Q$-modules.*

Proof. By [51, Proposition 1.1] $V \otimes_{\mathcal{O}} W$ is indecomposable; note that the quoted result is stated for modules over an algebraically closed field, but it holds over $\mathcal{O}$ without difficulty. Recall that an indecomposable $\mathcal{O}P$-module is a permutation $\mathcal{O}P$-module if and only if it is a trivial source $\mathcal{O}P$-module. By [51, Proposition 1.2], $V \otimes_{\mathcal{O}} W$ has vertex $H \times L$, where H is a vertex for V and L is a vertex for W. Moreover, a source for $V \otimes_{\mathcal{O}} W$ is given by the $\mathcal{O}(H \times L)$-module $S' \otimes_{\mathcal{O}} S''$, where S', S'' are, respectively, sources for V and W. Again, the quoted result is stated over k, but it holds over $\mathcal{O}$ as well. Therefore, if S is trivial, S' and S'' are both trivial, concluding the proof. □

Proposition 1.6.18. *Let $\mathcal{B}$ be a family of blocks closed under taking direct products, i.e. if $\mathcal{O}Gb, \mathcal{O}Hc \in \mathcal{B}$ then $\mathcal{O}Gb \otimes_{\mathcal{O}} \mathcal{O}Hc \in \mathcal{B}$. Then* $\mathrm{Pic}(B) = \mathcal{E}(B)$ *for all blocks $B \in \mathcal{B}$ if and only if for every $B_1, B_2 \in \mathcal{B}$, every Morita equivalence of blocks*

$$M \otimes_{B_1} - : \mathrm{mod}(B_1) \to \mathrm{mod}(B_2),$$

is basic.

Proof. One of the two implications is obvious. Let then G, H be finite groups and suppose by contradiction that two blocks $\mathcal{O}Gb, \mathcal{O}Hc \in \mathcal{B}$ are Morita equivalent via an $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule M and that M does not have endopermutation source. Consider then the group

$G \times H$, and the block $B = \mathcal{O}Gb \otimes_{\mathcal{O}} \mathcal{O}Hc$. Take the Morita equivalence between B and $C = \mathcal{O}Hc \otimes_{\mathcal{O}} \mathcal{O}Gb$ given by tensoring with the bimodule

$$\widehat{M} = M^* \otimes_{\mathcal{O}} M$$

Take now the B-C-bimodule N that is isomorphic as a left B-module to B, and where the right action of C is the natural one. Then N has trivial source and vertex $\Delta\varphi$, where, if D_1 is a defect group for $\mathcal{O}Gb$ and D_2 is a defect group for $\mathcal{O}Hc$, φ is described by

$$D_1 \times D_2 \to D_2 \times D_1$$
$$(g_1, g_2) \mapsto (g_2, g_1)$$

The Morita equivalence induced by N is then the *swap* of the two blocks $\mathcal{O}Gb$ and $\mathcal{O}Hc$. In particular, the composition $\widetilde{M} = N \otimes_C \widehat{M}$ is a Morita auto-equivalence of B, and thus, by assumption, $\widetilde{M}$ has endopermutation source. By [10, Lemma 2.6], $\widehat{M}$ must then have endopermutation source as well.

Now, a source S of $\widehat{M}$ is just the tensor product of sources S', S'' for M^*, M by a minor modification of [51, Proposition 1.2]. Therefore, Lemma 1.6.16 yields the isomorphism

$$\mathrm{End}_{\mathcal{O}}(S') \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{O}}(S'') \simeq \mathrm{End}_{\mathcal{O}}(S)$$

and, by Lemma 1.6.17, S' and S'' are both endopermutation modules, giving a contradiction. □

We need to remark that Conjecture 1.6.15 makes sense just for blocks defined over a complete discrete valuation ring with characteristic zero. In fact, an immediate consequence of the truth of the aforementioned conjecture for blocks defined over an algebraically closed field would be an affirmative answer to the modular isomorphism problem, whose statement we recall.

Conjecture 1.6.19 (Modular Isomorphism Problem)**.** *Let R be a field with characteristic p or $R = \mathcal{O}$. Take P, Q any two p-groups. Then $P \simeq Q$ if and only if $RP \simeq RQ$.*

Proposition 1.6.20. *Let R be either $\mathcal{O}$ or k. If all R-Morita equivalences of blocks defined over R are basic, then the Modular Isomorphism Problem has a positive answer.*

Proof. Let P, Q be p-groups for which $RP \simeq RQ$. Then RP and RQ are Morita equivalent, therefore they are actually basic Morita equivalent. Since basic Morita equivalent blocks have isomorphic defect groups, and RP, RQ are, respectively, blocks with defect groups P, Q, it follows that $P \simeq Q$. Therefore the Modular Isomorphism Problem has a positive answer. □

The Modular Isomorphism Problem obviously depends on the ring chosen. In particular, for complete discrete valuation rings with characteristic zero having k as its residue field, the Modular Isomorphism Problem holds by [74], therefore Conjecture 1.6.15 might still hold. On the other hand, it has been recently shown in [34] that the Modular Isomorphism Problem over an algebraically closed field does not always have a positive answer, thus there exist Morita equivalences over k that are not basic. Moreover, since $\mathcal{O}$-Morita equivalences of block algebras of p-groups are basic and so imply an isomorphism of the p-groups, the counterexample provided by Garcia, Margolis and del Rio also exhibits non-Morita equivalent $\mathcal{O}$-blocks of finite groups whose reductions modulo k are Morita equivalent. In any case, Conjecture 1.6.15 is still valid and it is worth trying to get more evidence.

1.7 Weiss' criterion

This last section briefly introduces Weiss' criterion, a classic result in integral representation theory that turned out to be really useful in the calculation of Picard groups. Note that Weiss' criterion was originally stated for the ring of p-adic integers but the formulation for group algebras over a discrete valuation ring was obtained just recently, in [63]. Since this last improvement of the criterion is the one we are interested in, we state the version for modules of group algebras defined over discrete valuation rings.

Theorem 1.7.1 (Weiss' criterion, [85])**.** *Let P be a finite p-group, M a finitely generated $\mathcal{O}P$-module and $Q \trianglelefteq P$ such that $M\downarrow_Q^P$ is free and ${}^QM$ is a permutation $\mathcal{O}(P/Q)$-module, where*

$$ {}^QM = \{m \in M | gm = m \text{ for all } g \in H\}. $$

Then M is a permutation $\mathcal{O}P$-module.

One of the applications of Weiss' criterion to the theory of Picard groups of blocks has been given in [29]. We first need to introduce the notation used by the authors. If B is a block of a group G, and $P \trianglelefteq G$ is a p-group, then B^P denotes the sum of blocks of $\mathcal{O}(G/P)$ dominated by B, i.e. those blocks not annihilated by the image of e_B under the natural $\mathcal{O}$-algebra homomorphism $\rho_P : \mathcal{O}G \to \mathcal{O}(G/P)$, where e_B is the block idempotent of B. Also, $\mathrm{Irr}(B)^P$ denotes the set of irreducible characters of B with P in the kernel.

Proposition 1.7.2 ([29], Proposition 4.3,4.4)**.** *Let $P \trianglelefteq G$ be a p-group and B a block of G.*

1. *The inflation map* $\mathrm{Inf} : \mathrm{Irr}(G/P) \to \mathrm{Irr}(G)$ *induces a bijection between* $\mathrm{Irr}(B^P)$ *and* $\mathrm{Irr}(B)^P$.

2. *Suppose M is a B-B-bimodule inducing a Morita auto-equivalence of B that permutes the elements of* $\mathrm{Irr}(B)^P$. *Then* ${}^P M = M^P$ *induces a Morita auto-equivalence of* B^P. *Furthermore, the permutation of* $\mathrm{Irr}(B^P)$ *induced by* ${}^P M$ *is equal to the permutation that M induces on* $\mathrm{Irr}(B)^P$*, once these two sets have been identified using 1. Moreover, if* ${}^P M \in \mathcal{T}(B^P)$*, then* $M \in \mathcal{T}(B)$.

So if every Morita auto-equivalence of a block permutes characters with P in their kernel, where P is a normal subgroup of G, then it can be shown that $\mathrm{Pic}(B) = \mathcal{T}(B)$, provided that ${}^P M \in \mathcal{T}(B^P)$. A useful criterion in this context is the following:

Proposition 1.7.3 ([29], Corollary 4.5)**.** *Let G be a finite group and B a block of $\mathcal{O}G$ with normal defect group $D \trianglelefteq G$. If*

$$\mathrm{Irr}(B)^D = \{\chi \in \mathrm{Irr}(B) | \chi \text{ is a lift of some } \phi \in \mathrm{IBr}(B)\},$$

then $\mathrm{Pic}(B) = \mathcal{T}(B)$. *In particular, if D and G/D are abelian and $Z(G) \cap D = \{1\}$, then* $\mathrm{Pic}(B) = \mathcal{T}(B)$.

Weiss' criterion and the results we presented in this section are crucial when B is a block with normal defect group, as it will be clear in Chapter 4. However, Weiss' criterion can be useful also for blocks with non-normal defect groups, as illustrated in Section 2.4.

Chapter 2

Calculation of Picard groups

Calculation of Picard groups is one of the first tasks we approached. As previously said, calculating Picard groups is crucial for classifying blocks in the context of Donovan's conjecture, so finding new methods for these calculations is desirable. In this chapter two main methods are used for calculating Picard groups: one uses a method based on a result by Carlson and Rouquier in [18], while the other relies on Weiss' criterion ([85]). This chapter is divided in four parts.

In the first part we calculate Picard groups for some blocks with defect group $C_3 \times C_3$ using the method based on Carlson and Rouquier's result, and, in particular, mimicking the successful applications of this method for the calculation of $\mathrm{Pic}(\mathcal{O}A_4), \mathrm{Pic}(B_0(\mathcal{O}A_5))$ for $p = 2$ in [10]. The motivation for these calculations mainly is the fact that non-principal blocks with that defect group are not completely classified up to Morita equivalence yet, so our work should be useful to complete this task. We remark that we could not perform such calculations for all principal blocks with defect group $C_3 \times C_3$, due to the complexity of Green correspondents of blocks with larger inertial quotient. However, for completeness, we performed those calculations adding an additional hypothesis. In the second part, the problem of calculating Picard groups for blocks with shape $\mathcal{O}P \otimes_{\mathcal{O}} B$, where B is a given block and P a p-group, is briefly addressed. The third section contains calculations of Picard groups for principal 2-blocks of Suzuki groups, the first examples of Picard groups for blocks with non-abelian and non-normal defect groups. In particular, the proof uses a minor extension of the method outlined by Carlson and Rouquier. Finally, in the last part of this chapter, we provide the Picard groups for blocks with defect groups Q_8, showing how Weiss' criterion can be used in this context.

2.1 Picard groups for blocks with defect group $C_3 \times C_3$

We outline the method that will be used for calculating the Picard groups for this family of blocks. From now on G is a finite group with Sylow 3-subgroup $D \simeq C_3 \times C_3$, $H = N_G(D)$. Denote by f the Green correspondence with respect to the subgroup $\Delta D \leq G \times G$, that sends an indecomposable $k[G \times G]$-module with vertex ΔD to an indecomposable $k[H \times G]$-module with the same vertex. Note that we will use the same symbol for denoting Green correspondence over $\mathcal{O}$.

Lemma 2.1.1 ([46], Lemma 3.8)**.** *$B_0(\mathcal{O}G)$ and $B_0(\mathcal{O}H)$ are stably equivalent of Morita type via $f(B_0(\mathcal{O}G))$.*

Proof. In [46] is actually shown that $B_0(kG)$ and $B_0(kH)$ are stably equivalent via $f(B_0(kG))$, but the result for $B_0(\mathcal{O}G)$ and $B_0(\mathcal{O}H)$ follows immediately. In fact, let $M = f(B_0(\mathcal{O}G))$, where here f is the Green correspondence over $\mathcal{O}$, then M is a trivial source $\mathcal{O}[H \times G]$-bimodule. By [58, Theorem 5.10.2] $k \otimes_{\mathcal{O}} M$ remains indecomposable and it is also a summand of $kG \downarrow^{G\times G}_{H\times G}$ with vertex ΔD. Thus $f(B_0(kG)) = k \otimes_{\mathcal{O}} M$ and M induces a stable equivalence of Morita type, by Proposition 1.4.11. □

Note that if $A = B_0(\mathcal{O}G)$, $B = B_0(\mathcal{O}H)$ and $\hat{A}$, $\hat{B}$ are the corresponding blocks over k, then M is a direct summand of $1_B\mathcal{O}G1_A$ and $k \otimes_{\mathcal{O}} M$ is a direct summand of $1_{\hat{B}}kG1_{\hat{A}}$.

Proposition 2.1.2 ([47], Lemma A.3)**.** *(Holds also over k) Let G be a finite group, $H \leq G$. Suppose that the block algebras $A = \mathcal{O}Gb$ and $B = \mathcal{O}Hc$ are stably equivalent of Morita type via a B-A-bimodule M such that $M | 1_B\mathcal{O}G1_A$.*

(i) Assume that X is a non-projective indecomposable $\mathcal{O}G$-module in A with vertex Q. Then there exists a non-projective indecomposable $\mathcal{O}H$-module Y in B, unique up to isomorphism, such that $(M \otimes_A X) = Y \oplus (proj)$ and Q^g is a vertex of Y for some element $g \in G$. In particular this means that X and Y have a common vertex since Q^g is a vertex of X.

(ii) Assume that Y is a non-projective indecomposable $\mathcal{O}H$-module in B with vertex Q. Then there exists a non-projective indecomposable $\mathcal{O}G$-module X in A, unique up to isomorphism, such that $(M^* \otimes_B Y) = X \oplus (proj)$ and Q is a vertex of X.

(iii) Let X, Y and $Q \leq H$ be as in (i). Then there is an indecomposable $\mathcal{O}Q$-module L such that L is a source of both X and Y.

(iv) *Let X, Y and $Q \leq H$ be as in (ii). Then there is an indecomposable $\mathcal{O}Q$-module L such that L is a source of both X and Y.*

(v) *Let X, Y, Q and L be the same as in (iii). Suppose that A and B have common defect group P and $H \geq N_G(P)$. Let f be the Green correspondent with respect to (G, D, H). If*

$$Q \in U = U(G, D, H) = \{K | K \leq D,\, K \neq (D^s \cap D)^g, \text{ with } s \in G \setminus H, g \in G\}$$

then we have $(M \otimes_A X) = f(X) + (proj)$.

(vi) *Let X, Y, Q and L be the same as in (iv). Suppose that A and B have common defect group D and $H \geq N_G(D)$. Let f and U be the same as in (v), then we have $(M^* \otimes_B Y) = f^{-1}(Y) + (proj)$.*

If $M = f\left(B_0(\mathcal{O}G)\right)$, then, by Lemma 2.1.1 and the last proposition, it induces a stable equivalence of Morita type between $B_0(\mathcal{O}G)$ and $B_0(\mathcal{O}H)$ such that

$$(k \otimes_{\mathcal{O}} M) \otimes_{B_0(kG)} S \simeq f(S)$$

for all simple $B_0(kG)$-modules S. Note in fact that if D is a defect group of $B_0(\mathcal{O}G)$, then by [44] all simple $B_0(kG)$-modules have vertex D and, since M is indecomposable, $k \otimes_{\mathcal{O}} S$ is indecomposable by [53, Theorem 2.1]. For calculating Picard groups of blocks $B = B_0(\mathcal{O}G)$ with (non-normal) defect group $C_3 \times C_3$ we will then proceed as follows:

- Determine the possible permutations of simple $B_0(kG)$-modules that an element of $\operatorname{Pic}(B)$ could give raise to, via examination of the decomposition matrix.
- Compute via Magma ([11]) the Green correspondence for simple modules of $B_0(kG)$ and $B_0(kH)$. Code for printing the Loewy structure of a module is available at [27], in the section *Guide to contributing:Code.* Green correspondents can be easily computed using well known Magma functions.
- Prove that all the possible elements of $\operatorname{Pic}(B)$ permute the Green correspondents of the simple $k \otimes_{\mathcal{O}} b$-modules, where $b = B_0(\mathcal{O}H)$.

Proving this last fact then give us an injective map

$$M \otimes_B - \otimes_B M^* : \operatorname{Pic}(B) \to \operatorname{Pic}(b), \tag{2.1}$$

by [53, Theorem 2.1]. Moreover, since M has trivial source, if $\operatorname{Pic}(b) = \mathcal{T}(b)$ then $\operatorname{Pic}(B) = \mathcal{T}(B)$, by [10, Lemma 2.6]. The techniques that we will use to calculate a given Picard group will depend on the case we are working on. All the blocks we will deal with will have $\mathrm{Pic}(B) = \mathcal{T}(B)$, therefore the exact sequence in Theorem 1.6.11 is going to be one of the main tools used. We remark now that, since D is abelian, it will always hold that $\mathrm{Out}(D, \mathcal{F}) \simeq N_{\mathrm{Aut}(D)}(E)/E$, so we will not repeat this multiple times.

The last thing that we want to point out is that, since M has trivial source and vertex ΔD, it will map elements of $\mathcal{T}(B)$ with vertex ΔD to elements of $\mathcal{T}(b)$ with the same vertex, again by [10, Lemma 2.6]. Thus there is also an injective map from $\mathrm{Out}_D(A)$ to $\mathrm{Out}_D(a)$, where A, a are, respectively, source algebras for B and b. Moreover bimodules with vertex $\Delta\alpha = \{(g, \alpha(g)) | g \in D\}$, where α is a non-trivial group automorphism of P, cannot be sent by the map in (2.1) to bimodules with vertex ΔD by [10, Lemma 2.6].

Finally, a list of Morita equivalence classes of principal blocks with defect group $C_3 \times C_3$ can be found in [45]. Note that $\mathrm{Pic}\left(\mathcal{O}(S_3 \times S_3)\right), \mathrm{Pic}\left(\mathcal{O}((C_3 \times C_3) : C_2)\right)$ can be calculated by [10, Proposition 4.3]. Also, the isomorphism types for $\mathrm{Pic}\left(\mathcal{O}(S_3 \times C_3)\right)$, $\mathrm{Pic}\left(\mathcal{O}(C_3 \times C_3) : C_4\right)$, $\mathrm{Pic}\left(\mathcal{O}(C_3 \times C_3)\right)$ follow from [59, Corollary 6.4] and the description of Picard groups for group algebras of p-groups. Thus we started our calculations from the principal block of $B_0(A_6)$. Note in addition that $\mathrm{Pic}(\mathcal{O}(C_3 \times C_3) : C_4)$ is given by trivial sources bimodules, by Proposition 1.7.3. We want to stress that the proof for $B_0(A_6)$ is the most detailed one. We put less details in the other blocks since the argument is extremely similar to the one for A_6 and therefore the reader can always go back to the proof for A_6 to have all the details.

2.1.1 Preliminary results

We fix the notation that will be used in the rest of the section. $B = B_0(\mathcal{O}G)$ is the 3-block we want to calculate the Picard group of, $D \simeq C_3 \times C_3$ is its defect group, E its inertial quotient, $H = N_G(D)$ and b is the principal block of $\mathcal{O}H$. By A we will usually denote a source algebra for B, while a is a source algebra for b. To have a cleaner notation we write $\overline{B}$ and $\bar{b}$ for denoting the correspondent k-blocks of, respectively, G and H. $F : \underline{\mathrm{mod}}(b) \to \underline{\mathrm{mod}}(B)$ and $F^* : \underline{\mathrm{mod}}(B) \to \underline{\mathrm{mod}}(b)$ are functors giving a stable equivalence of Morita type and induced by $f(B_0(\mathcal{O}G))$ and its $\mathcal{O}$-dual, as described in the previous section. We analogously denote by $\overline{F}$ and $\overline{F}^*$ the functors induced by $f(B_0(kG))$ and its k-dual, that we recall send simple modules to their Green correspondents. Finally, for $M \in \mathrm{Pic}(B)$ we will denote by M^F the stable equivalence of Morita type given by

$$M^F := F^* \circ (M \otimes_B \text{-}) \circ F : \underline{\mathrm{mod}}(b) \to \underline{\mathrm{mod}}(b)$$

We collect some useful lemmas that will often be used in the next section. Ext groups will be used, but we won't list all their elementary properties that are, for example, readily available in [8]. However we briefly recall their definition.

Let V, W be two kG-modules, and consider a projective resolution for V:

$$\mathbf{P}: \quad \cdots \to P_2 \to P_1 \to P_0.$$

Take then the complex

$$\mathrm{Hom}_{kG}(\mathbf{P}, W): \quad \mathrm{Hom}_{kG}(P_0, W) \to \mathrm{Hom}_{kG}(P_1, W) \to \mathrm{Hom}_{kG}(P_2, W) \to \cdots$$

Thus we define $\mathrm{Ext}^n_{kG}(V, W) = H^n(\mathrm{Hom}_{kG}(\mathbf{P}, W))$, where $H^n(-)$ are the cohomology groups of a given complex.

Lemma 2.1.3. *Let U, V be kG-modules such that*

$$\dim_k(\mathrm{Ext}^1_{kG}(V, U)) = 1.$$

If there is an indecomposable module W that is an extension of V by U, then there are just two isomorphism classes of modules realizing extensions of V by U: the isomorphism class of W and the one of $V \oplus U$.

Proof. This follows immediately from noticing that $\mathrm{Aut}_{kG}(V)$ and $\mathrm{Aut}_{kG}(U)$ act on $\mathrm{Ext}^1_{kG}(V, S)$ changing the extension but not the module in the middle. We show this for automorphisms induced by multiplication with a unit of k. Let then c_λ be the automorphism of U that acts by multiplication by $\lambda \in k^\times$, then there is no automorphism ϕ of W that makes the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{i} & W & \xrightarrow{\pi} & V & \longrightarrow & 0 \\
 & & \| & & \downarrow{\phi} & & \| & & \\
0 & \longrightarrow & U & \xrightarrow{c_\lambda} & W & \xrightarrow{\pi} & V & \longrightarrow & 0
\end{array}$$

Suppose for a moment such a ϕ existed. Then we could define a map

$$\begin{aligned} \varphi : W &\to V \oplus U \\ w &\mapsto (\pi(w), w - \phi(w)). \end{aligned}$$

For $\lambda \neq 1$ this map would actually be an injective morphism, therefore yielding an isomorphism $W \simeq V \oplus U$ that contradicts the assumptions on W. Since $k^\times$ is obviously contained in the automorphism groups of V and U, there are just two orbits of extensions: one that contains the split extension; and one where all the non-split extensions are stored. In particular, all non-split extensions of V by U are realized by W, or a module isomorphic to it. □

The next result allows us to describe Green correspondents and their Loewy structure computationally, working over a splitting field. A splitting field for G is a field for which every simple $\mathbb{F}G$-module S is absolutely simple, i.e. $\mathbb{E} \otimes_{\mathbb{F}} S$ is simple for every field extension $\mathbb{E}|\mathbb{F}$. From now on we will denote by $V^{\mathbb{E}}$ the $\mathbb{E}G$-module $\mathbb{E} \otimes_{\mathbb{F}} V$, where V is any indecomposable $\mathbb{F}G$-module and $\mathbb{E}|\mathbb{F}$ is a field extension.

Lemma 2.1.4 ([82], Theorem 1.8.8). *Let U be an indecomposable $\mathbb{F}G$-module with vertex Q and Green correspondent V an $\mathbb{F}N_G(Q)$-module U. If $\mathbb{E} \mid \mathbb{F}$ is a field extension, then*

$$U^{\mathbb{E}} = U_1 + ... + U_r$$
$$V^{\mathbb{E}} = V_1 + ... + V_r,$$

where all summands U_i have vertex Q and V_i is the Green correspondent of U_i for all $i = 1, ..., r$.

We will also make wide use of the next elementary lemma for calculating Ext groups. We denote by $\Omega^1(V)$, where V is a kG-module, the Heller translate of V, i.e. the kernel of the map $P(V) \to V$, where $P(V)$ is the projective cover of V. $\Omega^i(V)$ is iteratively defined by $\Omega^{i+1}(V) = \Omega^1\left(\Omega^i(V)\right)$.

Lemma 2.1.5 ([55], Corollary 4.13.8). *Let V be a finitely generated kG-module and S a non-projective simple kG-module. For any positive integer n we have*

$$\mathrm{Ext}^n_{kG}(V, S) \simeq \mathrm{Hom}_{kG}\left(\Omega^n(V), S\right) \simeq \mathrm{Hom}_{kG}\left(\Omega^n(V)/\operatorname{rad}(\Omega^n(V)), S\right).$$

Note that if S is not simple, it still holds that

$$\mathrm{Ext}^n_{kG}(V, S) \simeq \mathrm{Hom}_{kG}\left(\Omega^n(V), S\right).$$

Finally, the table below summarises all the results we are going to prove.

Block	**Inertial quotient**	**Picard group**
$\mathcal{O}\left((C_3 \times C_3) : C_4\right)$	C_4	$C_2 \times D_8$
$B_0(\mathcal{O}A_6)$	C_4	$C_2 \times C_2$
$B_0(\mathcal{O}A_7)$	C_4	$C_2 \times C_2$
$\mathcal{O}\left((C_3 \times C_3) : C_8\right)$	C_8	SD_{16}
$B_0(\mathcal{O}PGL(2,9))$	C_8	$C_2 \times C_2$
$\mathcal{O}\left((C_3 \times C_3) : D_8\right)$	D_8	D_8
$B_0(\mathcal{O}A_8)$	D_8	C_2
$B_0(\mathcal{O}S_6)$	D_8	$C_2 \times C_2$
$B_0(\mathcal{O}S_7)$	D_8	$C_2 \times C_2$

$\mathcal{O}\left((C_3 \times C_3) : Q_8\right)$	Q_8	S_4
$B_0(\mathcal{O}M_{10})$	Q_8	$C_2 \times C_2$
$B_0(\mathcal{O}L_3(4))^*$	Q_8	S_3
$\mathcal{O}\left((C_3 \times C_3) : SD_{16}\right)$	SD_{16}	$C_2 \times C_2$
$B_0(\mathcal{O}M_{11})^*$	SD_{16}	1
$B_0(\mathcal{O}M_{23})$	SD_{16}	1
$B_0(\mathcal{O}P\Sigma L(3,4))^*$	SD_{16}	C_2
$B_0(\mathcal{O}\,\mathrm{Aut}(S_6))$	SD_{16}	$C_2 \times C_2$
$B_0(\mathcal{O}HS)^*$	SD_{16}	1

Remark 2.1.6. Some blocks in the table have the * mark because to prove the result we had to add an additional hypothesis on the block, specifically on the Green correspondents with non-simple head. However, for completeness, we included also these partial proofs.

2.1.2 Principal block of A_6

We label the simple modules of $\overline{B}$ and $\bar{b}$ in the following way:

$$\mathrm{IBr}(B) = \{1, 3_1, 3_2, 4_1\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d\}.$$

We adopted the symbol used for Brauer characters to denote the sets of simple $\overline{B}, \bar{b}$ modules because of the well known correspondence between irreducible Brauer characters and simple modules. The notation n_i, for n, i positive integers, always denotes a simple module with dimension n. Note that B has inertial quotient C_4, and b is $\mathcal{O}\left((C_3 \times C_3) : C_4\right)$.

The Green correspondents of simple $\bar{b}$-modules are well known, but their Loewy structure can also be described using Magma. We explicitly write down their structure here. We remark that in the Loewy structure of a kG-module V, the i-th *horizontal layer* (where counting starts from the top) corresponds to the decomposition of $\mathrm{rad}^{i-1}(V)/\,\mathrm{rad}^i(V)$ as a sum of simple kG-modules, and this is everything we mean by the diagrams we are going to write.

$$\overline{F}(1_a) = 1; \;\; \overline{F}(1_b) = \begin{array}{ccc} & 4 & \\ 1 & & 1 \\ & 4 & \end{array}; \;\; \overline{F}(1_c) = \begin{array}{c} 3_1 \\ 4 \\ 3_2 \end{array}; \;\; \overline{F}(1_d) = \begin{array}{c} 3_2 \\ 4 \\ 3_1 \end{array}.$$

The decomposition matrix of B is the following:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

where the labelling of the columns is exactly the one we chose for simple $\overline{B}$-modules. We will use the same notation for all the other blocks.

Examination of the decomposition matrix yields that, if M is an element of $\operatorname{Pic}(B)$, then $\overline{M} := k \otimes_{\mathcal{O}} M$ can only permute 3_1 and 3_2 and must fix 1 and 4. Now we show that $\overline{F}(1_c)$ and $\overline{F}(1_d)$ are the only isotypes (i.e. isomorphism type) of $\overline{B}$-modules with their Loewy structure. First we compute the Loewy structure of their projective covers, with Magma.

$$\begin{array}{ccc} & 3_1 & \\ & 4 & \\ 1 & & 3_2 \\ & 4 & \\ & 3_1 & \end{array} \qquad \begin{array}{ccc} & 3_2 & \\ & 4 & \\ 1 & & 3_1 \\ & 4 & \\ & 3_2 & \end{array}$$

Since the structures of the two Green correspondents we are considering are very similar, and the same holds for the structures of their projective covers, we describe in detail just the procedure for dealing with one of them, $\overline{F}(1_c)$ for example. First note that there is just one isomorphism class of indecomposable modules with the Loewy structure of $V = \overline{F}(1_c)/\operatorname{soc}(\overline{F}(1_c))$; in fact, all such modules would have to be a quotient of $P(3_1)$ by a submodule containing $\operatorname{rad}^2(P(3_1))$, and, since $P(3_1)/\operatorname{rad}^2(P(3_1))$ has the desired Loewy structure, they must be isomorphic to it. Now, by Lemma 2.1.5,

$$\dim_k \left(\operatorname{Ext}^1_{kG}(V, 3_2)\right) = 1,$$

therefore we can apply Lemma 2.1.3 and show that $\overline{F}(1_c)$ is the unique isotype of indecomposable $\overline{B}$-modules with its Loewy structure. The same reasoning applies to $\overline{F}(1_d)$, thus these two modules are either swapped or fixed by a Morita auto-equivalence, since we recall that Morita auto-equivalences of B must permute 3_1 and 3_2, and fix 1 and 4.

We turn our attention to $\overline{F}(1_b)$. The Loewy structure of the projective cover of 4, $P(4)$, can be computed with Magma and is given by

$$\begin{array}{ccccc} & & 4 & & \\ 1\quad 1 & & & & 3_1\quad 3_2 \\ 4 & & 4 & & 4 \\ 3_1\quad 3_2 & & & & 1\quad 1 \\ & & 4 & & \end{array}$$

We first notice that there is just one isomorphism classes of $\overline{B}$-modules with the following Loewy structure,

$$\begin{array}{ccc} & 4 & \\ 1 & & 1 \end{array}$$

In fact, if W is a module with such a Loewy structure, then it is the quotient of $P(4)$ by the preimage of a submodule isomorphic to $3_1 \oplus 3_2$ under the map $\operatorname{rad}(P(4)) \to \operatorname{rad}(P(4))/\operatorname{rad}^2(P(4))$. Since there is just one submodule of $\operatorname{rad}(P(4))/\operatorname{rad}^2(P(4))$ isomorphic to $3_1 \oplus 3_2$, W is uniquely defined. Thus the only isotype of modules with the above Loewy structure is given by $W = \overline{F}(1_b)/\operatorname{soc}(\overline{F}(1_b))$.

Notice now that $\overline{F}\,(1_b) \simeq k \otimes_{\mathbb{F}} U$ and $\operatorname{soc}\left(\overline{F}(1_b)\right) \simeq k \otimes_{\mathbb{F}} \operatorname{soc}(U)$, where $\mathbb{F}$ is a splitting field for G and H, and U is the Green correspondent of the $\mathbb{F}H$-module ${}^{\mathbb{F}}1_b$, that is the absolutely simple $\mathbb{F}H$-module for which $k \otimes_{\mathbb{F}} {}^{\mathbb{F}}1_b = 1_b$. In particular, $k \otimes_{\mathbb{F}} \Omega^1(U/\operatorname{soc}(U)) \simeq \Omega^1(W)$. By Lemma 2.1.5, $\operatorname{Ext}^1_{kG}(W,4) \simeq \operatorname{Hom}\left(\Omega^1(W)/\operatorname{rad}(\Omega^1(W)),4\right)$ and its k-dimension is just the number of times 4 appears in the first Loewy layer of $\Omega^1(W)$. Thus we can calculate this just by computing the Loewy structure of $\Omega^1\left(U/\operatorname{soc}(U)\right)$ over $\mathbb{F}$ with Magma. It can be checked that

$$\Omega^1(U/\operatorname{soc}(U)) = \begin{array}{ccccc} & 3_1 & 4 & 3_2 & \\ & 4 & & 4 & \\ 3_1 & 3_2 & & 1 & 1 \\ & & 4 & & \end{array}$$

Note that for a more readable notation we used the symbol for simple $\overline{b}$-modules, in place of writing the correspondent absolutely simple $\mathbb{F}G$-module. Now the number of times 4 appears as a composition factor in the first Loewy layer of $\Omega^1(U/\operatorname{soc}(U))$ is just one, then $\operatorname{Ext}^1_{kG}(W,4) \simeq k$. By Lemma 2.1.3, there is just one isotype of modules with Loewy structure equal to $\overline{F}(1_b)$.

In particular $\overline{F}(1_b)$ must be fixed by any Morita auto-equivalence of B and therefore all Green correspondents of simple $\overline{b}$-modules are permuted by Morita auto-equivalences of B. Thus, for all $L \in \operatorname{Pic}(B)$, $L^F \in \operatorname{Pic}(b)$. Following the remarks just before Subsection 2.1.1,

$\mathrm{Pic}(b) = \mathcal{T}(b)$ and $\mathrm{Pic}(B) = \mathcal{T}(B)$. We now turn our attention to the exact sequence for $\mathcal{T}(B)$ described in Theorem 1.6.11.

Suppose that $M \in \mathrm{Out}_D(A)$ is non-trivial. Then, $M^F \in \mathrm{Out}_D(b)$ is also non-trivial and, since M preserves the trivial $\overline{B}$-module, M^F does preserve the trivial $\bar{b}$-module as well. However, b has normal defect group and non-trivial elements of $\mathrm{Out}_D(b)$ act on 1-dimensional simple modules by multiplication with a linear character of E by Proposition 1.6.12, therefore they don't fix the trivial b-module. The only chance is that M^F is trivial, but this happens if and only if M is trivial, so we have a contradiction. Thus $\mathrm{Out}_D(A)$ is trivial.

For what concerns the right side of the exact sequence for $\mathcal{T}(B)$, it can be checked that $N_{\mathrm{Aut}(D)}(E)/E \simeq C_2 \times C_2$ and we can realize these elements. In fact, $A_6 \trianglelefteq S_6$ and the unique conjugacy class of S_6 represented by an element of order 5 splits in two conjugacy classes of A_6 represented by elements of order 5 by a classical result. Therefore we can find an element of S_6 that swaps the conjugacy classes of order 5 of A_6. This element therefore induces a non-inner automorphism of B, since it swaps the irreducible characters of B with degree 8, that correspond to the fourth and fifth row of the decomposition matrix of B. Thus, by Proposition 1.6.4, this automorphism induces a non-trivial element α of $\mathrm{Pic}(B)$.

We also know that $A_6 \trianglelefteq PGL_2(\mathbb{F}_9)$, and it can be checked, computationally or studying the group structure, that the conjugacy class of $PGL_2(\mathbb{F}_9)$ represented by an element of order 3 splits in two conjugacy classes of A_6. Moreover there is an element of $PGL_2(\mathbb{F}_9)$ that swaps these two conjugacy classes of A_6 and fixes the others. In particular, it swaps the irreducible characters of B with degree 5, that correspond to the second and third row of the decomposition matrix, and fixes the others. Conjugation by this element induces again a non-inner algebra automorphism of B, and then again a non-trivial element of $\mathrm{Pic}(B)$. Note that this last element acts trivially on simple $\overline{B}$-modules, otherwise it would move other irreducible characters, by examination of the decomposition matrix. Therefore we can conclude that $\mathrm{Pic}(B) \simeq C_2 \times C_2$.

Remark 2.1.7. The argument for the principal block of A_6 is the blueprint for all the other blocks. We will put less detail in the next proofs, but the reader may refer to this case for the whole formal argument. The notations adopted for the other blocks will be identical to the ones introduced for the principal block of A_6.

2.1.3 Principal block of A_7

The labelling for simple modules is the following:

$$\mathrm{IBr}(B) = \{1, 10_1, 10_2, 13\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d\},$$

B has inertial quotient C_4 and b is $\mathcal{O}((C_3 \times C_3) : C_8)$. The Loewy structures of the Green correspondents for simple $\overline{b}$-modules are the following:

$$\overline{F}(1_a) = 1;\ \overline{F}(1_b) = \begin{matrix} & 13 & \\ 1 & & 1 \\ & 13 & \end{matrix};\ \overline{F}(1_c) = 10_2;\ \overline{F}(1_d) = 10_1,$$

and the decomposition matrix of B is

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}.$$

A Morita auto-equivalence of B can just permute the ten-dimensional simple modules, and act trivially on the others, so the Loewy structures of the Green correspondents are permuted. $\overline{F}(1_b)$ is the unique isotype of indecomposable $\overline{B}$-modules with its Loewy structure, by the same argument used with A_6. In fact, $P(13)$ has the following Loewy structure

$$\begin{matrix} & 13 & \\ 1 & & 1 \\ 10_1 & 13 & 10_2 \\ 1 & & 1 \\ & 13 & \end{matrix}$$

Therefore indecomposable modules with the Loewy structure of $\overline{F}(1_b)$ must be quotients of $P(13)$ by preimages of modules isomorphic to $10_1 \oplus 10_2$ under the map $\mathrm{rad}^2(P(13)) \to \mathrm{rad}^2(P(13))/\mathrm{rad}^3(P(13))$. Since there is just one such module, we can conclude that there is just one isomorphism class of indecomposable $\overline{B}$-modules with the same Loewy structure of $\overline{F}(1_b)$. Therefore, there is a map $(\text{-})^F : \mathrm{Pic}(B) \to \mathrm{Pic}(b)$ as described just before Subsection 2.1.1. In particular, since $\mathrm{Pic}(b) = \mathcal{T}(b)$, it follows again $\mathrm{Pic}(B) = \mathcal{T}(B)$.

We can immediately say that $\mathrm{Out}_D(A)$ is trivial, since the trivial B-module is fixed by any element of $\mathrm{Pic}(B)$. In fact, elements of $\mathrm{Out}_D(A)$ are sent to elements of $\mathrm{Out}_D(a)$ that fix the trivial b-module. Since the trivial element of $\mathrm{Pic}(b)$ is the unique Morita auto-equivalence of b in $\mathrm{Out}_D(a)$ that fixes the trivial b-module, we conclude by the injectivity of the map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$ that $\mathrm{Out}_D(A) = 1$.

From the paragraph above then $\mathrm{Pic}(B)$ injects in $C_2 \times C_2 \simeq N_{\mathrm{Aut}(D)}(E)/E$. In a similar fashion to A_6, the conjugacy class of S_7 represented by an element of order 7 splits in two conjugacy classes of A_7 represented by elements of order 7. Therefore, we can consider the outer automorphism of A_7 given by conjugation by an element of S_7 that permutes these two conjugacy classes and fixes the others. This induces an automorphism of G and it can be checked by looking at the character table that it swaps the irreducible characters of degree 10 that lift the two Brauer characters with degree 10, but fixes all the others.

The corresponding automorphism of B is then, in particular, an outer automorphism that swaps the 10-dimensional simple $\overline{B}$-modules and fixes the others. The same can be said about the Morita auto-equivalence of B corresponding to the automorphism. So we found $1 \neq L \in \mathrm{Pic}(B)$ whose action on simple $\overline{B}$-modules is described by the permutation $(10_1, 10_2)$. Moreover, the action of L on irreducible characters of B swaps the lifts of the Brauer characters with degree 10, but fixes all the others.

We now show that $\mathrm{Pic}(B) \simeq C_2 \times C_2$. Let T be a subgroup of A_7 isomorphic to A_6 containing H, and B_2 the principal block of T. Denote by $J : \mathrm{mod}(b) \to \mathrm{mod}(B_2)$ the stable equivalence of Morita type that "coincides" with Green correspondence; then, by what was shown for A_6, we can define a map $(\text{-})^J : \mathrm{Pic}(B_2) \to \mathrm{Pic}(b)$. Thus, for each $N \in \mathrm{Pic}(B_2)$ we have a stable equivalence of Morita type, given by

$$F \circ \left(N^J \otimes_b \text{-}\right) \circ F^* = F \circ J^* \circ (N \otimes_{B_2} \text{-}) \circ J \circ F^* : \underline{\mathrm{mod}}(B) \to \underline{\mathrm{mod}}(B).$$

For each $N \in \mathrm{Pic}(B_2)$ call N^{JF^*} the B-B-bimodule inducing the previously defined stable equivalence. We want to underline that not necessarily N^{JF^*} is an element of $\mathrm{Pic}(B)$.

Consider as N the B_2-B_2-bimodule that induces the Morita auto-equivalence that swaps the irreducible characters with degree 5 and fixes the other, as described at the end of the proof for A_6. $k \otimes_{\mathcal{O}} N$ acts trivially on simple $\overline{B}_2$-modules, thus $k \otimes_{\mathcal{O}} N^J$ acts trivially on simple $\overline{b}$-modules. (Note: we know that $N^J \in \mathrm{Pic}(b)$, by the argument for the principal block of A_6)

We can then see that $N^{JF^*} \in \mathrm{Pic}(B)$. In fact, the images under $\overline{F}^*$ of simple $\overline{B}$-modules are the following:

$$\overline{F}^*(1) = 1_a \quad \overline{F}^*(10_1) = 1_c \quad \overline{F}^*(10_2) = 1_d \quad \overline{F}^*(13) = \begin{array}{ccc} & 1_b & \\ 1_c & & 1_d \\ & 1_b & \end{array}$$

The non-simple Green correspondent is uniquely defined by its Loewy structure as an indecomposable $\bar{b}$-module, by an argument analogous to the one used at the beginning of this proof. Indeed, the projective cover of $\overline{F}^*(13)$ is

$$P(1_b) = \begin{matrix} & 1_b & \\ 1_c & & 1_d \\ 1_a & 1_b & 1_a \\ 1_d & & 1_c \\ & 1_b & \end{matrix}$$

Indecomposable $\bar{b}$-modules with the Loewy structure of M must be quotients of $P(1_b)$ by the preimage of a submodule isomorphic to $1_a \oplus 1_a$ under the map $P(1_b) \to P(1_b)/\operatorname{rad}^3(P(1_b))$, but there is just one of these submodules. Therefore $\overline{F}^*(13)$ is the unique isotype of indecomposable $\bar{b}$-modules with its Loewy structure. Thus N^J fixes the images of simple $\overline{B}$-modules via $\overline{F}^*$, and $N^{JF^*} \in \operatorname{Pic}(B)$.

We claim that N^{JF^*} is a non-trivial element of $\operatorname{Pic}(B)$ and different from the element L we described before. For showing this we will actually analyse the perfect isometry that it induces.

If F^* and F induced a perfect isometry then we would just need to compose these with the perfect isometry of N^J and we would be done, but this is not the case. However by Proposition 1.4.10 there are two maps

$$\Phi_F : \mathbb{Z}\operatorname{Irr}(b) \to \mathbb{Z}\operatorname{Irr}(B), \;\; \Phi_{F^*} : \mathbb{Z}\operatorname{Irr}(B) \to \mathbb{Z}\operatorname{Irr}(b),$$

defined as follows: Φ_F sends the character of $K \otimes_{\mathcal{O}} U$ to the character of $K \otimes_{\mathcal{O}} F(U)$, and Φ_{F^*} is defined analogously. Moreover Φ_F and Φ_{F^*} induce inverse isomorphisms

$$\overline{\Phi}_F : \mathbb{Z}\operatorname{Irr}(b)/\operatorname{Pr}_{\mathcal{O}}(b) \to \mathbb{Z}\operatorname{Irr}(B)/\operatorname{Pr}_{\mathcal{O}}(B), \;\; \overline{\Phi}_{F^*} : \mathbb{Z}\operatorname{Irr}(B)/\operatorname{Pr}_{\mathcal{O}}(B) \to \mathbb{Z}\operatorname{Irr}(b)/\operatorname{Pr}_{\mathcal{O}}(b)$$

Denote by $[\chi]$ the image of an irreducible character $\chi \in \operatorname{Irr}(B)$ in $\mathbb{Z}\operatorname{Irr}(B)/\operatorname{Pr}_{\mathcal{O}}(B)$. Each χ is the character of $K \otimes_{\mathcal{O}} V$ for some B-module V, and then $\overline{\Phi}_{F^*}([\chi]) = [\mu]$, where μ is the character of $K \otimes_{\mathcal{O}} U$, with $U = F^*(V)$ an indecomposable summand of $V \downarrow_H^G$. In particular, $\Phi_{F^*}([\chi]) = \mu + \varphi$, where φ is the character of a projective $\mathcal{O}H$-module.

Note that $\operatorname{Irr}(b) = \{1_a, 1_b, 1_c, 1_d, 4_a, 4_b\}$ and that the decomposition matrix of b is just given by

$$\mathbb{D}_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

In particular, the characters of projective indecomposable $\overline{b}$-modules are

$$\{1_a + 4_a + 4_b, 1_b + 4_a + 4_b, 1_c + 4_a + 4_b, 1_d + 4_a + 4_b\}$$

It immediately follows that a set of generators for $\mathbb{Z}\operatorname{Irr}(b)/\operatorname{Pr}_{\mathcal{O}}(b)$ is given by $\{[1_a], [4_a]\}$.

On the other hand, the irreducible characters of B are:

$$\operatorname{Irr}(B) = \{1, 10_1, 10_2, 14_1, 14_2, 35\}$$

and the group $\mathbb{Z}\operatorname{Irr}(B)/\operatorname{Pr}_{\mathcal{O}}(B)$ has generators $[1], [14_1]$. This can be checked by looking at the character table of G, since

$$10_1 - 1 \in \operatorname{Pr}_{\mathcal{O}}(B)$$
$$10_2 - 1 \in \operatorname{Pr}_{\mathcal{O}}(B)$$
$$14_2 - 1 + 14_1 \in \operatorname{Pr}_{\mathcal{O}}(B)$$
$$35 + 1 \in \operatorname{Pr}_{\mathcal{O}}(B)$$

We want to explicitly write down the map from $\mathbb{Z}\operatorname{Irr}(B)/\operatorname{Pr}_{\mathcal{O}}(B)$ to itself, given by N^{JF^*}.

We study $\overline{\Phi}_{F^*}$ first of all. Naturally $\overline{\Phi}_{F^*}([1]) = [1_a]$. For what concerns $[14_1]$, observe that

$$14_1 \downarrow_H^G = 1_a + 1_b + 4_a + 2 \cdot 4_b,$$

and $1_b + 4_a + 4_b$ is projective, thus $\overline{\Phi}_{F^*}([14_1]) = [1_a + 4_b]$. We can apply the same strategy to $\overline{\Phi}_F$. Obviously $\overline{\Phi}_F([1_a]) = 1_1$. Moreover, it can be checked that

$$4_a \uparrow_H^G = 6 + 10_1 + 10_2 + 14_1 + 2 \cdot (14_2 + 15_1 + 21_1) + 4 \cdot 35_1.$$

Note that some irreducible characters of G not in B appear when 4_a is induced up. However Φ_F is induced by F, that sends b-modules to B-modules, so we have to get rid of characters not in B. In particular, after taking out projectives, it follows that $\overline{\Phi}_F([4_a]) = [14_2 + 35] = [-14_1]$.

Now $\overline{\Phi}_{N^J}$ is described by the perfect isometry induced by N^J. Recall that N^J acts trivially on simple $\overline{b}$-modules, since N acts trivially on simple $\overline{B}$-modules. Moreover note that, by [60, Theorem 3.3.3], $\mathrm{Picent}(b) = 1$. Certainly N^J is not trivial, since N is not trivial, and therefore, by examination of the decomposition matrix of b, the action of N^J on $\mathrm{Irr}(b)$ swaps the characters of order 4 and fixes the linear ones. So we have that N^{JF^*} induces the following map:

$$[1] \xrightarrow{\overline{\Phi}_{F^*}} [1_a] \xrightarrow{\overline{\Phi}_{N^J}} [1_a] \xrightarrow{\overline{\Phi}_{F}} [1]$$

$$[14_1] \xrightarrow{\overline{\Phi}_{F^*}} [1_a + 4_b] \xrightarrow{\overline{\Phi}_{N^J}} [1_a + 4_a] \xrightarrow{\overline{\Phi}_{F}} [1 - 14_1] = [14_2]$$

Note that in particular $\overline{\Phi}_{N^{JF^*}}$ is not the trivial map, in fact $14_1 - 14_2$ is not projective. Thus N^{JF^*} is not the trivial element of $\mathrm{Pic}(B)$ and it is different from L, since L fixes 1_1 and 14_1, and therefore induces the trivial map on $\mathbb{Z}\,\mathrm{Irr}(B)/\mathrm{Pr}_{\mathcal{O}}(B)$. Then $\mathrm{Pic}(B) \simeq C_2 \times C_2$.

2.1.4 Principal block of $(C_3 \times C_3) : C_8$

B is the group algebra $\mathcal{O}\left((C_3 \times C_3) : C_8\right) = \mathcal{O}G$. From [10, Proposition 4.3] it follows

$$\mathrm{Pic}(B) = \mathrm{Pic}(\mathcal{O}G) = \mathcal{T}(\mathcal{O}G) \simeq \mathrm{Out}_D(\mathcal{O}G) \rtimes N_{\mathrm{Aut}(D)}(E)/E \simeq C_8 \rtimes C_2.$$

Obviously $\mathrm{IBr}(B)$ consists of the one-dimensional simple $\mathcal{O}G$-modules corresponding to the irreducible characters of E, and the action of $\mathrm{Out}_D(A)$ on $\mathrm{IBr}(B)$ is induced by multiplication with linear characters.

We can describe a non-trivial element of $N_{\mathrm{Aut}(D)}(E)/E$ by considering the non-trivial outer automorphism σ of G induced by the Frobenius automorphism of $\mathbb{F}_9$, the field with 9 elements. It can be checked that σ acts on $\mathrm{Irr}(B)$ by sending each linear character to its third power and therefore the action of the Morita auto-equivalence induced by σ on $\mathrm{IBr}(B)$ is given by sending each Brauer character to its third power.

Note that elements of $\mathrm{Pic}(B)$ are uniquely recognized by their action on $\mathrm{IBr}(B)$ and, if a is the generator of $\mathrm{Out}_D(A)$ and b the generator of $N_{\mathrm{Aut}(D)}(E)/E$, then $a^b = a^3$. Thus we immediately conclude that $\mathrm{Pic}(B) \simeq SD_{16}$.

2.1.5 Principal block of $PGL(2,9)$

We label simple modules as follows:

$$\mathrm{IBr}(B) = \{1_1, 1_2, 3_1, 3_2, 3_3, 3_4, 4_1, 4_2\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 1_e, 1_f, 1_g, 1_h\}.$$

B has inertial quotient C_8 and b is $\mathcal{O}\left((C_3 \times C_3) : C_8\right)$. The Loewy structure of Green correspondents of simple $\overline{b}$-modules is given by

$$\overline{F}(1_a) = 1_a;\ \overline{F}(1_b) = 1_b;\ \overline{F}(1_c) = \begin{matrix} 3_4 \\ 4_1 \\ 3_1 \end{matrix}\ ;\ \overline{F}(1_d) = \begin{matrix} 3_3 \\ 4_2 \\ 3_2 \end{matrix}\ ;$$

$$\overline{F}(1_e) = \begin{matrix} 3_1 \\ 4_1 \\ 3_4 \end{matrix}\ ;\ \overline{F}(1_f) = \begin{matrix} 3_2 \\ 4_2 \\ 3_3 \end{matrix}\ ;\ \overline{F}(1_g) = \begin{matrix} & 4_1 & \\ 1_1 & & 1_2 \\ & 4_2 & \end{matrix}\ ;\ \overline{F}(1_h) = \begin{matrix} & 4_2 & \\ 1_1 & & 1_2 \\ & 4_1 & \end{matrix}\ .$$

Now the decomposition matrix of B is the following:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Close examination of the decomposition matrix yields that the possible non-trivial permutations of simple $\overline{B}$-modules are:

$$(1_1, 1_2)(3_2, 3_3)(3_1, 3_4); (3_3, 3_4)(3_1, 3_2)(4_1, 4_2); (1_1, 1_2)(4_1, 4_2)(3_1, 3_3)(3_2, 3_4).$$

In fact, 1_1 and 1_2 must be permuted, since they are the only two lifts of Brauer characters. Suppose they are fixed. Then the irreducible character χ corresponding to the third row is either fixed or sent to the irreducible characters ρ corresponding to the fifth row. If χ is fixed, it means that 4_1 and 3_3 are fixed. However 4_1 and 4_2 must be permuted, since they are the only non-liftable Brauer characters that appear in 4 irreducible characters, so if 4_1 is fixed, 4_2 is fixed as well. Proceeding like this, it can be shown that if 1_1 and χ are fixed, then all simple modules are fixed.

Suppose then that 1_1 is fixed, but χ is not. Then χ is sent to ρ, 4_1 and 4_2 are (non-trivially) permuted and 3_3 and 3_4 are (non-trivially) permuted. Since the character corresponding to the fourth row must be sent to the one corresponding to the sixth, 3_2 and 3_1 must be permuted, so the action on $\mathrm{IBr}(B)$ is described by the permutation $(3_3, 3_4)(3_1, 3_2)(4_1, 4_2)$.

The other two permutations can be obtained with similar arguments, supposing that 1_1 is not fixed, and checking what needs to happen depending on the fact that 4_1 is fixed or not.

Analogously to the previous cases, the Green correspondents of simple $\bar{b}$-modules can be recognized as the unique indecomposable kG-modules with their Loewy structure. We quickly give a proof for two of them, since the others follow in a similar fashion. The Loewy structure of the projective cover of 3_4 is

$$\begin{array}{ccc} & 3_4 & \\ & 4_1 & \\ 1_1 & & 3_1 \\ & 4_1 & \\ & 3_4 & \end{array}$$

Therefore, a module with the same Loewy structure of $\overline{F}(1_c)$ must be isomorphic to the quotient of $P(3_4)$ by the pre-image of a submodule isomorphic to 1_1 under the map $\operatorname{rad}^2(P(3_4)) \to \operatorname{rad}^2(P(3_4))/\operatorname{rad}^3(P(3_4))$. Since there is just one such submodule, there is also just one such pre-image, and thus $\overline{F}(1_c)$ is uniquely determined by its Loewy structure.

Now we turn our attention to $\overline{F}(1_g)$. The projective cover of 4_1 has Loewy structure

$$\begin{array}{ccccc} & & 4_1 & & \\ 1_1 & 3_1 & & 3_4 & 1_2 \\ 4_1 & & 4_2 & & 4_1 \\ 1_1 & 3_1 & & 3_4 & 1_2 \\ & & 4_1 & & \end{array}$$

We have to proceed in two steps here: first we show that there is just one isomorphism class of submodules with Loewy structure

$$\begin{array}{ccc} & 4_1 & \\ 1_1 & & 1_2 \end{array}$$

in analogous fashion to what we did before. Then, if V is a module with such structure, we can compute its Heller translate with Magma

$$\Omega^1(V) = \begin{array}{ccccc} 3_1 & & 4_2 & & 3_4 \\ & 4_1 & & 4_1 & \\ 1_1 & 3_1 & & 3_4 & 1_2 \\ & & 4_1 & & \end{array}$$

and notice that $\dim_k(\operatorname{Ext}^1_{kG}(V, 4_2)) = 1$. This yields, by Lemma 2.1.5 and Lemma 2.1.3, that $\overline{F}(1_g)$ is uniquely recognized by its Loewy structure. Finally, since each permutation of simple modules given by a Morita auto-equivalence of B permutes the Green correspondents of simple $\overline{b}$-modules, we can define an injective map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$ as described just before Subsection 2.1.1. Moreover $\mathrm{Pic}(B) = \mathcal{T}(B)$, since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.4.

We now need to compute the Loewy structures of Green correspondents with respect to restriction:

$$\overline{F}^*(1_1) = 1_a;\ \overline{F}^*(1_2) = 1_b;\ \overline{F}^*(3_1) = \begin{array}{c} 1_c \\ 1_a \\ 1_e \end{array};\ \overline{F}^*(3_2) = \begin{array}{c} 1_d \\ 1_a \\ 1_f \end{array};\ \overline{F}^*(3_3) = \begin{array}{c} 1_f \\ 1_b \\ 1_d \end{array};$$

$$\overline{F}^*(3_4) = \begin{array}{c} 1_e \\ 1_b \\ 1_c \end{array};\ \overline{F}^*(4_1) = \begin{array}{ccc} & 1_h & \\ 1_d & & 1_f \\ & 1_g & \end{array};\ \overline{F}^*(4_2) = \begin{array}{ccc} & 1_g & \\ 1_c & & 1_e \\ & 1_h & \end{array};$$

Note that the notation we chose for simple $\overline{b}$-modules is according to the following Brauer character table for b.

	1A	**2A**	**4A**	**4B**	**8A**	**8B**	**8C**	**8D**
1_a	1	1	1	1	1	1	1	1
1_b	1	1	1	1	-1	-1	-1	-1
1_c	1	-1	$-\zeta^2$	ζ^2	ζ^3	ζ	$-\zeta^3$	$-\zeta$
1_d	1	-1	ζ^2	$-\zeta^2$	ζ	ζ^3	$-\zeta$	$-\zeta^3$
1_e	1	-1	ζ^2	$-\zeta^2$	$-\zeta$	$-\zeta^3$	ζ	ζ^3
1_f	1	-1	$-\zeta^2$	ζ^2	$-\zeta^3$	$-\zeta$	ζ^3	ζ
1_g	1	1	-1	-1	$-\zeta^2$	ζ^2	$-\zeta^2$	ζ^2
1_h	1	1	-1	-1	ζ^2	$-\zeta^2$	ζ^2	$-\zeta^2$

In the above character table ζ is a primitive root of unity of order 8. In particular, note that $1_c^3 = 1_d$, $1_e^3 = 1_f$ and $1_g^3 = 1_h$.

Now, the Green correspondents of simple $\overline{B}$-modules are uniquely defined as indecomposable kH-modules by their Loewy structure, and this can be shown analogously to before. We provide two projective covers for the two different structures of Green correspondents of simple $\overline{B}$-modules and the structure of one of the Heller translate to give proof; the other projective indecomposable modules have a very similar structure and can be easily computed in Magma. U is going to be $\overline{F}^*(3_1)/\operatorname{soc}\left(\overline{F}^*(3_1)\right)$.

$$P(1_c) = \begin{array}{ccc} & 1_c & \\ 1_h & & 1_a \\ 1_f & 1_e & 1_d \\ 1_g & & 1_b \\ & 1_c & \end{array} \qquad \Omega^1(U) = \begin{array}{ccc} 1_h & & 1_f \\ 1_d & & 1_e \\ 1_g & & 1_b \\ & 1_c & \end{array} \qquad P(1_g) = \begin{array}{ccc} & 1_g & \\ 1_c & & 1_e \\ 1_a & 1_h & 1_b \\ 1_d & & 1_f \\ & 1_g & \end{array}$$

It is then clear how applying the arguments used a few lines above it can be shown that each Green correspondent of a simple $\overline{B}$-module is uniquely determined, up to isomorphism, by its Loewy structure.

To describe $\mathcal{T}(B)$ we first consider $\mathrm{Out}_D(A)$. Note that multiplication by 1_b permutes the Green correspondents of simple $\overline{B}$-modules, therefore it can be lifted via F^* to an element of $\mathrm{Out}_D(A)$ that permutes 1_a and 1_b, and thus $\mathrm{Out}_D(A) \gtrsim C_2$. Let now N be an element of $\mathrm{Out}_D(A)$. N must either permute the simple $\overline{B}$-modules trivially or by one of the permutations listed before, right after the examination of the decomposition matrix of B. In particular N^F is an element of $\mathrm{Pic}(b)$ with diagonal vertex ΔD and trivial source that permutes 1_a and 1_b. But then N^F must either be the trivial element of $\mathrm{Pic}(b)$ or the Morita auto-equivalence of b that acts on simple $\bar{b}$-modules by multiplication with 1_b. By injectivity of the map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$ we conclude that $\mathrm{Out}_D(A) \simeq C_2$.

If we now consider the automorphism of H that sends each linear character to their third power, then we can build $L \in \mathrm{Pic}(b)$, as noted in Subsection 2.1.4. Now L permutes the Green correspondents of simple $\overline{B}$-modules. In fact, according to our notation, L acts on simple $\bar{b}$-modules via the permutation $(1_c, 1_d)(1_e, 1_f)(1_g, 1_h)$. Thus $L^{F^*} \in \mathrm{Pic}(B)$. Since $N_{\mathrm{Aut}(D)}(E)/E \simeq C_2$, we can conclude that $\mathrm{Pic}(B) \simeq C_2 \times C_2$.

2.1.6 Principal block of $(C_3 \times C_3) : D_8$

This block has normal defect group, and it can be easily verified that

$$\{\chi \in \mathrm{Irr}(B) | D \leq \mathrm{Ker}(\chi)\} = \{\chi \in \mathrm{Irr}(B) |\ \chi \text{ is a lift of some } \phi \text{ in } \mathrm{IBr}(B)\}.$$

In fact, all irreducible characters of B that don't lift Brauer characters have degree 4 and therefore have not D in their kernel.

So, by Proposition 1.7.3, $\mathrm{Pic}(B) = \mathcal{T}(B)$. In order to describe the isomorphism class of $\mathrm{Pic}(B)$ we need to write down the actions of its generators on irreducible characters. We describe the Brauer characters of B and also establish the notation that will be used for blocks with B as Brauer correspondent. Note that the order was chosen accordingly to how Magma lists simple $\overline{b}$-modules.

	1A	**2A**	**2B**	**2C**	**4**
1_a	1	1	1	1	1
1_b	1	-1	1	1	-1
1_c	1	-1	-1	1	1
1_d	1	1	-1	1	-1
2	2	0	0	-2	0

We have that $\mathrm{Out}_D(A) \simeq \mathrm{Hom}(E, k^\times) \simeq C_2 \times C_2$ and elements of $\mathrm{Pic}(B)$ corresponding to multiplication by linear characters of E act on the 1-dimensional simple $\overline{b}$-modules by one of the following permutations: $id, (1_a, 1_b)(1_c, 1_d), (1_a, 1_c)(1_b, 1_d), (1_a, 1_d)(1_b, 1_c)$.

Now it can be computed that $N_{\mathrm{Aut}(D)}(E) \simeq C_3 \times C_3 : SD_{16}$ and $N_{\mathrm{Aut}(D)}(E)/E \simeq C_2$. We can also describe a Morita auto-equivalence of B induced by a non-trivial outer automorphism of G. In fact look at $D \rtimes E$ inside $D \rtimes N_{\mathrm{Aut}(D)}(E)$, for example taking

$$SD_{16} = \left\langle \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \right\rangle \leq GL_2(3) \text{ and } D = \left\langle \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle = \mathbb{F}_3^2,$$

with SD_{16} acting naturally by conjugation on D, and

$$E = \left\langle \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \right\rangle \leq GL_2(3)$$

Consider the automorphism φ of D given by conjugation with the generating element of order 8 of SD_{16}. Then φ induces the automorphism $\tilde{\varphi}$ of $G \simeq D \rtimes E$. Now the number of elements of order 2 of G is the same as the number of elements of $T = D \rtimes N_{\mathrm{Aut}(D)}(E)$, but one of the conjugacy classes of T with elements of order two splits in two conjugacy classes, $2A$ and $2B$, for G. It can be checked that $\tilde{\varphi}$ permutes $2A$ and $2B$, inducing an outer automorphism of B. Moreover, the bimodule ${}_{\tilde{\varphi}}B$ is an element of $\mathrm{Pic}(B)$. In particular the action of ${}_{\tilde{\varphi}}B$ on $\mathrm{IBr}(B)$ is just the permutation $(1_b, 1_d)$.

Finally this permutation does not commute with the action given by the previously described $\mathrm{Out}_D(A)$, thus $\mathcal{T}(B) \simeq D_8$.

2.1.7 Principal block of A_8

We label simple modules as follows:

$$\mathrm{IBr}(B) = \{1, 7, 13, 28, 35\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2\}.$$

B has inertial quotient D_8 and b is $\mathcal{O}\left((C_3 \times C_3) : D_8\right)$. The Loewy structure of Green correspondents of simple $\overline{b}$-modules is given by

$$\overline{F}(1_a) = 1;\ \ \overline{F}(1_b) = 28;\ \ \overline{F}(1_c) = 7;\ \ \overline{F}(1_d) = \begin{matrix} & 13 & \\ 1 & & 7 \\ & 13 & \end{matrix}\ ;\ \ \overline{F}(2) = 35,$$

and the decomposition matrix of B is the following:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Note that a Morita auto-equivalence of B can just permute 1 and 7. Now the only non-simple Green correspondent is $\overline{F}(1_d)$, and its projective cover has the following structure:

$$\begin{matrix} & 13 & \\ 1 & & 7 \\ 13 & & 35 \\ 1 & & 7 \\ & 13 & \end{matrix}$$

We can immediately conclude that $F(1_d)$ is the unique isomorphism type of indecomposable kG-modules with such a Loewy structure, since it corresponds to the the unique quotient of $P(13)$ by the pre-image of a submodule isomorphic to 35 under the map $\mathrm{rad}^2\left(P(13)\right) \to \mathrm{rad}^2\left(P(13)\right) / \mathrm{rad}^3\left(P(13)\right)$. Therefore all Green correspondents of simple $\overline{b}$-modules are permuted and there is an injective map $(\text{-})^F : \mathrm{Pic}(B) \to \mathrm{Pic}(b)$, as described just before Subsection 2.1.1. Moreover, $\mathrm{Pic}(B) = \mathcal{T}(B)$, since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.6.

We claim that $\mathrm{Out}_D(A)$ is trivial. In fact take an element $V \in \mathrm{Pic}(B)$ with vertex ΔD. V necessarily permutes 1 and 7, and fixes 28, by examination of the decomposition matrix. Thus V^F is a Morita auto-equivalence of b with vertex ΔD and trivial source whose action on linear characters of b is either the transposition $(1_a 1_c)$ or the trivial one. The first case can't obviously happen, since elements of $\mathrm{Out}_D(a)$ act by multiplication with a linear character.

Therefore V^F needs to act on simple $\bar{b}$-modules trivially. The injectivity of $(\text{-})^F$ allows us to conclude that $\mathrm{Out}_D(A) = \{1\}$.

Finally, we can find a non trivial element of $\mathrm{Pic}(B)$. First we write down the Loewy structure of Green correspondents of simple $\overline{B}$-modules:

$$\overline{F}^*(1) = 1_a;\ \overline{F}^*(7) = 1_c;\ \overline{F}^*(13) = \begin{matrix} 1_d \\ 2 \\ 1_d \end{matrix};\ \overline{F}^*(28) = 1_b;\ \overline{F}^*(35) = 2;\ ,$$

Moreover, the projective cover of 1_d has Loewy structure

$$\begin{matrix} & 1_d & \\ & 2 & \\ 1_a & 1_d & 1_c \\ & 2 & \\ & 1_d & \end{matrix}$$

We can immediately conclude that $\overline{F}^*(13)$ is uniquely defined by its Loewy structure, up to isomorphism, as a $\bar{b}$-module just proceeding as above.

Consider $M_1, M_2 \in \mathrm{Pic}(b)$ inducing, respectively, the permutations of $\mathrm{IBr}(b)$ given by $(1_b, 1_d)$ and $(1_a, 1_c)(1_b, 1_d)$; these are respectively realized by an outer automorphism of H and multiplication by a linear character, as seen when describing $\mathrm{Pic}(b)$ in Subsection 2.1.6. Take $M = M_1 \otimes_b M_2$. Then M acts on simple $\bar{b}$-modules by the permutation $(1_a, 1_c)$ and thus permutes the Green correspondents of $\overline{B}$-modules. Therefore $M^{F^*} \in \mathrm{Pic}(B)$ and $C_2 \lesssim \mathrm{Pic}(B)$. Since $N_{\mathrm{Aut}(D)}(E)/E \simeq C_2$, then $\mathrm{Pic}(B) \simeq C_2$.

2.1.8 Principal block of S_6

The labelling of simple modules is as follows:

$$\mathrm{IBr}(B) = \{1_1, 1_2, 4_1, 4_2, 6\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2\}.$$

S_6 has inertial quotient D_8 and b is $\mathcal{O}((C_3 \times C_3) : D_8)$. The Loewy structures of Green correspondents of simple $\bar{b}$-modules are given by

$$\overline{F}(1_a) = 1_1;\ \overline{F}(1_b) = \begin{matrix} & 4_1 & \\ 1_1 & & 1_2 \\ & 4_1 & \end{matrix};\ \overline{F}(1_c) = 1_2;\ \overline{F}(1_d) = \begin{matrix} & 4_2 & \\ 1_1 & & 1_2 \\ & 4_2 & \end{matrix};$$

$$\overline{F}(2) = \begin{matrix} & 6 & \\ 4_1 & & 4_2 \\ & 6 & \end{matrix},$$

and the decomposition matrix of B is

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Close examination of the decomposition matrix yields that a Morita auto-equivalence of B must preserve the dimensions of simple modules. There are three non-simple Green correspondents, but two of them can be treated in the same way; we begin analysing $\overline{F}(1_b)$. First, the Loewy structure of $P(4_1)$, computed via Magma, is

$$\begin{array}{ccc} & 4_1 & \\ 1_1 & 6 & 1_2 \\ 4_1 & 4_2 & 4_1 \\ 1_1 & 6 & 1_2 \\ & 4_1 & \end{array}$$

There is just one isotype of indecomposable $\overline{B}$-modules with Loewy structure

$$\begin{array}{ccc} & 4_1 & \\ 1_1 & & 1_2 \end{array}$$

since they correspond to quotients of $P(4_1)$ by a preimage of a submodule isomorphic to 6 under the map $\mathrm{rad}(P) \to \mathrm{rad}(P)/\mathrm{rad}^2(P)$, and the uniqueness of such submodules yields the uniqueness of such quotients. Therefore, $V = \overline{F}(1_b)/\operatorname{soc}(\overline{F}(1_b))$ is a representative for this isomorphism class. As already explained with the principal block of A_6, we can compute the Loewy structure of $\Omega^1(V)$ in Magma, that is

$$\begin{array}{ccc} 4_1 & & 6 \\ 4_1 & & 4_2 \\ 1_1 & 6 & 1_2 \\ & 4_1 & \end{array}$$

Therefore, by Lemma 2.1.3 and Lemma 2.1.5, we get that there is just one isomorphism class of indecomposable $\overline{B}$-modules with the Loewy structure of $\overline{F}(1_b)$. The same argument applies to $\overline{F}(1_d)$.

Now, turning our attention to $\overline{F}(2)$, the projective cover of 6 has structure

$$\begin{array}{ccc} & 6 & \\ 4_1 & & 4_2 \\ 1_1 & 6 & 1_2 \\ 4_1 & & 4_2 \\ & 6 & \end{array}$$

so it can be proven easily, as happened in previous cases, that there is just one isomorphism class of indecomposable $\overline{B}$-modules with the Loewy structure of $\overline{F}(2)$. Since any Morita auto-equivalence of B preserves the dimension of simple $\overline{B}$-modules, the Green correspondents of simple $\bar{b}$-modules are permuted by any element of $\mathrm{Pic}(B)$. It then follows, as described just before Subsection 2.1.1, that there is a map $(\text{-})^F : \mathrm{Pic}(B) \to \mathrm{Pic}(b)$. Moreover, since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.6, $\mathrm{Pic}(B) = \mathcal{T}(B)$.

We show that some elements of $\mathrm{Pic}(b)$ can be lifted. The Green correspondence with respect to restriction is given by

$$\overline{F}^*(1_1) = 1_a;\ \overline{F}^*(1_2) = 1_c;\ \overline{F}^*(4_1) = \begin{array}{c} 1_b \\ 2 \\ 1_b \end{array};\ \overline{F}^*(4_2) = \begin{array}{c} 1_d \\ 2 \\ 1_d \end{array};$$

$$\overline{F}^*(6) = \begin{array}{ccc} & 2 & \\ 1_c & & 1_a \\ & 2 & \end{array},$$

and, for all simple $\overline{B}$-modules S, there is just one isomorphism class of indecomposable modules with the Loewy structure of $\overline{F}^*(S)$. This can be shown in the usual way, we just write down the projective covers of the non simple Green correspondents.

$$\begin{array}{ccc} & 1_b & \\ & 2 & \\ 1_a & 1_b & 1_c \\ & 2 & \\ & 1_b & \end{array} \qquad \begin{array}{ccc} & 1_d & \\ & 2 & \\ 1_a & 1_d & 1_c \\ & 2 & \\ & 1_d & \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ 1_c & 1_a & & 1_b & 1_d \\ & 2 & 2 & 2 & \\ 1_c & 1_a & & 1_b & 1_d \\ & & 2 & & \end{array}$$

These situation have already been treated analogously a few lines above, just notice that

$\Omega^1\left(\overline{F}^*(6)/\operatorname{soc}(\overline{F}^*(6))\right)$ has structure

$$\begin{array}{ccccc} 1_b & & 2 & & 1_d \\ & 2 & & 2 & \\ 1_c & 1_a & & 1_b & 1_d \\ & & 2 & & \end{array}$$

We can then lift the Morita auto-equivalence of b induced by the automorphism of G that acts on $\operatorname{IBr}(b)$ by $(1_b, 1_d)$ described in Subsection 2.1.6. In fact this Morita auto-equivalence of b permutes the Green correspondents of simple $\overline{B}$-modules, and can then be lifted to $M \in \operatorname{Pic}(B)$ via F^*. Also the one induced by multiplication with the linear character 1_c (whose action on $\operatorname{IBr}(b)$ is $(1_a, 1_c)(1_b, 1_d)$) can be lifted to $N \in \operatorname{Pic}(B)$. However N has vertex ΔD, while M has not, so they are two different elements of $\operatorname{Pic}(B)$.

We already know by Subsection 2.1.6 that $N_{\operatorname{Aut}(D)}(E)/E \simeq C_2$, we claim that $\operatorname{Out}_D(A) \simeq C_2$. Let $N' \in \operatorname{Out}_D(A)$, then N' must preserve the dimension of simple $\overline{B}$-modules, as mentioned before. In particular, N' permutes 1_1 and 1_2, therefore $(N')^F$ is either the trivial element of $\operatorname{Out}_D(a)$ or the Morita auto-equivalence of b induced by multiplication with 1_c. By injectivity of the map between $\operatorname{Pic}(B)$ and $\operatorname{Pic}(b)$, it then follows $\operatorname{Pic}(B) \simeq C_2 \times C_2$.

2.1.9 Principal block of S_7

The labelling of simple modules is given by:

$$\operatorname{IBr}(B) = \{1_1, 1_2, 13_1, 13_2, 20\}, \qquad \operatorname{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2\}.$$

B has inertial quotient D_8 and b is $\mathcal{O}\left((C_3 \times C_3) : D_8\right)$. The Loewy structures of Green correspondents of simple b-modules are the following:

$$\overline{F}(1_a) = 1_1;\ \ \overline{F}(1_b) = \begin{array}{ccc} & 13_1 & \\ 1_1 & & 1_2 \\ & 13_1 & \end{array};\ \ \overline{F}(1_c) = 1_2;\ \ \overline{F}(1_d) = \begin{array}{ccc} & 13_2 & \\ 1_1 & & 1_2 \\ & 13_2 & \end{array};$$

$$\overline{F}(2) = 20,$$

The decomposition matrix of B is

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Looking at the decomposition matrix we note that a Morita auto-equivalence of B necessarily preserves dimensions of simple $\overline{B}$-modules. Below we write down the Loewy structure of $P(13_1)$, computed with Magma. The structure of $P(13_2)$ is very similar and the procedure for $\overline{F}(1_d)$ is analogous.

$$\begin{matrix} & 13_1 & \\ 1_1 & & 1_2 \\ 13_1 & & 20 \\ 1_1 & & 1_2 \\ & 13_1 & \end{matrix}$$

We can then conclude easily that all indecomposable $\overline{B}$-modules with the Loewy structure of $\overline{F}(1_b)$ must be isomorphic to it, since they correspond to quotients of $P(13_1)$ by pre-images of submodules isomorphic to 20, under the map $\operatorname{rad}^2(P(13_1)) \to \operatorname{rad}^2(P(13_1))/\operatorname{rad}^3(P(13_1))$. Since any Morita auto-equivalence of B preserves dimension of simple $\overline{B}$-modules, the Green correspondents of simple $\overline{b}$-modules are permuted by any element of $\operatorname{Pic}(B)$. Then there is a map $(\text{-})^F : \operatorname{Pic}(B) \to \operatorname{Pic}(b)$, as described just before Subsection 2.1.1, and $\operatorname{Pic}(B) = \mathcal{T}(B)$, since $\operatorname{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.6.

Now the Loewy structures for Green correspondents of simple $\overline{B}$-modules are given by

$$\overline{F}^*(1_1) = 1_a;\ \overline{F}^*(1_2) = 1_c;\ \overline{F}^*(13_1) = \begin{matrix} 1_b \\ 2 \\ 1_b \end{matrix};\ \overline{F}^*(13_2) = \begin{matrix} 1_d \\ 2 \\ 1_d \end{matrix};$$

$$\overline{F}^*(20) = 2,$$

We see that the Green correspondents for simple $\overline{B}$-modules are extremely similar to the ones for $B_0(kS_6)$, and the projective covers already described in Subsection 2.1.8 are enough to show that the Green correspondents of simple $\overline{B}$-modules are uniquely recognized by their Loewy structures as indecomposable $\bar{b}$-modules. Moreover, the same argument can be used to show that $\mathrm{Pic}(B) \simeq C_2 \times C_2$, so we won't repeat ourselves.

2.1.10 Principal block of $(C_3 \times C_3) : Q_8$

G has normal defect group, and it can be easily checked that

$$\{\chi \in \mathrm{Irr}(B) | D \leq \mathrm{Ker}(\chi)\} = \{\chi \in \mathrm{Irr}(B) |\ \chi \text{ is a lift of some } \varphi \text{ in } \mathrm{IBr}(B)\},$$

In fact the irreducible characters of B are either lifts of Brauer character, or the unique irreducible character of G with degree 8, that obviously has not D in its kernel. Therefore $\mathrm{Pic}(B) = \mathcal{T}(B)$.

$\mathrm{Out}_D(A)$ corresponds to Morita auto-equivalences induced by multiplication with linear characters as usual, and therefore $\mathrm{Out}_D(A) \simeq C_2 \times C_2$.

Now it can be checked computationally that $N_{\mathrm{Aut}(D)}(E)/E \simeq S_3$. Moreover, since G has no class-preserving automorphisms in $\mathrm{Out}(G) \simeq S_3$ by [13], i.e. all elements of $\mathrm{Out}(G)$ permute non-trivially the conjugacy classes of G, then each of the automorphisms in $\mathrm{Out}(G)$ induces a different permutation of $\mathrm{Irr}(B)$. We print the character table of G for making things clear.

	1A	**2A**	**3A**	**4A**	**4B**	**4C**
1_a	1	1	1	1	1	1
1_b	1	1	1	-1	-1	1
1_c	1	1	1	-1	1	-1
1_d	1	1	1	1	-1	-1
2_a	2	-2	2	0	0	0
8	0	-1	0	0	0	0

Since the conjugacy classes $1A$, $2A$ and $3A$ must be fixed by automorphisms of G, then $\mathrm{Out}(G)$ must permute the conjugacy classes $4A, 4B, 4C$. In particular, for each transposition σ of the three conjugacy classes represented by elements of order 4 there is an element α_σ in $\mathrm{Out}(G)$ inducing it. Moreover, for each α_σ we can define a Morita auto-equivalence of B that acts on Brauer characters of B as a transposition of non-trivial linear Brauer characters. In fact, each element of $\mathrm{Out}(G)$ induces an automorphism of D not in E, and therefore a non-trivial Morita auto-equivalence of the block B.

Note that every Morita auto-equivalences induced by elements of $\mathrm{Out}(G)$ is not in $\mathrm{Out}_D(A)$, since they fix the trivial character of B. Therefore $\mathrm{Pic}(B) \simeq (C_2 \times C_2) \rtimes S_3$, since $\mathrm{Out}_D(A)$ and the image of $\mathrm{Out}(G)$ in $\mathrm{Pic}(B)$ intersect trivially and $|\mathrm{Pic}(B)| \leq 24$ by the exact sequence described in Theorem 1.6.11.

Since multiplying by a linear character acts as a product of disjoint transpositions on all linear (Brauer) characters, we can actually find an element of $\mathrm{Pic}(B)$ for each one of the transpositions of the four linear characters of B. Therefore, since S_n is generated by its transposition, we conclude that $\mathrm{Pic}(B) \simeq S_4$.

2.1.11 Principal block of M_{10}

G is the Mathieu group on 10 points. The simple modules are labelled as follows:

$$\mathrm{IBr}(B) = \{1_1, 1_2, 4_1, 4_2, 6\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2\}.$$

B has inertial quotient Q_8 and b is $\mathcal{O}\left((C_3 \times C_3) : Q_8\right)$. The Loewy structures of Green correspondents are given by

$$\overline{F}(1_a) = 1_1;\ \overline{F}(1_b) = \begin{array}{ccc} & 4_2 & \\ 1_1 & & 1_2 \\ & 4_1 & \end{array};\ \overline{F}(1_c) = 1_2;\ \overline{F}(1_d) = \begin{array}{ccc} & 4_1 & \\ 1_1 & & 1_2 \\ & 4_2 & \end{array};$$

$$\overline{F}(2) = \begin{array}{ccc} & 6 & \\ 4_1 & & 4_2 \\ & 6 & \end{array},$$

We write down the structures of projective covers just for 4_1 and 6; $P(4_1)$ and $P(4_2)$ have very similar structures and can be treated analogously.

$$\begin{array}{ccc} & 4_1 & \\ 1_1 & 6 & 1_2 \\ 4_2 & 4_1 & 4_2 \\ 1_1 & 6 & 1_2 \\ & 4_1 & \end{array} \qquad \begin{array}{ccc} & 6 & \\ 4_1 & & 4_2 \\ 1_1 & 6 & 1_2 \\ 4_1 & & 4_2 \\ & 6 & \end{array}$$

There is nothing new at this point, and the procedure is completely analogous to previous cases. $\overline{F}(2)$ corresponds to quotients of $P(6)$ by the pre-image of $1_1 \oplus 1_2$ under the map $\mathrm{rad}^2(P) \to \mathrm{rad}^2(P)/\mathrm{rad}^3(P)$ and therefore is uniquely determined by its Loewy structure. It can be computed with Magma that $\Omega^1\left(\overline{F}(1_d)/\mathrm{soc}(\overline{F}(1_d))\right)$ has Loewy structure

$$\begin{array}{ccc} 4_2 & & 6 \\ 4_1 & & 4_2 \\ 1_1 & 6 & 1_2 \\ & 4_1 & \end{array}$$

and thus also $\overline{F}(2)$ is the unique iso-type of indecomposable modules with its Loewy structure, by Lemma 2.1.5 and Lemma 2.1.3. Therefore if a Green correspondent of a simple $\overline{b}$-module and another indecomposable $\overline{B}$-module have the same Loewy structure then they are actually isomorphic. Now the decomposition matrix of B is the following:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Examination of the decomposition matrix yields that any Morita auto-equivalence of B must preserve the dimension of simple $\overline{B}$-modules. Therefore the Green correspondents of simple $\overline{b}$-modules are permuted and thus, as described just before Subsection 2.1.1, we have a morphism $(\text{-})^F : \mathrm{Pic}(B) \to \mathrm{Pic}(b)$. Since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.10, $\mathrm{Pic}(B) = \mathcal{T}(B)$.

We then compute the Loewy structures of Green correspondents for simple $\overline{B}$-modules.

$$\overline{F}^*(1_1) = 1_a;\ \ \overline{F}^*(1_2) = 1_c;\ \ \overline{F}^*(4_1) = \begin{array}{c} 1_b \\ 2 \\ 1_d \end{array};\ \ \overline{F}^*(4_2) = \begin{array}{c} 1_d \\ 2 \\ 1_b \end{array};$$

$$\overline{F}^*(6) = \begin{array}{ccc} & 2 & \\ 1_c & & 1_a \\ & 2 & \end{array}$$

We give the structure of just two projective covers, since $F^*(4_1)$ and $F^*(4_2)$ can be treated analogously.

$$\begin{array}{ccc} & 1_b & \\ & 2 & \\ 1_a & 1_d & 1_c \\ & 2 & \\ & 1_b & \end{array} \qquad \begin{array}{ccccccc} & & & 2 & & & \\ 1_c & 1_a & & & & 1_b & 1_d \\ & & 2 & 2 & 2 & & \\ 1_c & 1_a & & & & 1_b & 1_d \\ & & & 2 & & & \end{array}$$

Note that the Heller translate of $F^*(6)/\operatorname{soc}(F^*(6))$ has the following Loewy structure:

$$\begin{array}{ccccc} 1_b & & 2 & & 1_d \\ & 2 & & 2 & \\ 1_c\ 1_a & & & & 1_b\ 1_d \\ & & 2 & & \end{array}$$

Therefore using the same arguments as a few lines above it can be shown that the Green correspondents of simple $\overline{B}$-modules are uniquely determined by their Loewy structure.

Note that the Morita auto-equivalence of b with vertex ΔD, whose action on $\mathrm{IBr}(b)$ is given by multiplication with 1_c, can be lifted to a non-trivial element of $\mathrm{Pic}(B)$. In fact, the Green correspondents of simple $\overline{B}$-modules are permuted by the above element of $\mathrm{Pic}(b)$. Since Morita auto-equivalences of B need to permute $\{1_1, 1_2\}$ we can conclude, by injectivity of the map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$ and the fact that elements of $\mathrm{Out}_D(A)$ are mapped to $\mathrm{Out}_D(a)$, that $\mathrm{Out}_D(A) \simeq C_2$.

If now we take the Morita auto-equivalence of b that swaps 1_b and 1_d then it can be lifted to a Morita auto-equivalence of B and it must have vertex of the form $\Delta\varphi$, with φ a non-trivial element of $N_{\mathrm{Aut}(D)}(E)/E$. Suppose that the map

$$\Phi : \mathcal{T}(B) \to N_{\mathrm{Aut}(D)}(E)/E \simeq S_3$$

defined in the exact sequence in Theorem 1.6.11 were surjective, then there would be $U \in \mathrm{Pic}(B)$ with vertex $\Delta\psi$ with $\psi \in N_{\mathrm{Aut}(D)}(E)/E$ of order 3. But then U would act trivially on simple $\overline{B}$-modules, since every Morita auto-equivalence of B must induce permutations of the following two subsets of $\mathrm{IBr}(B)$: $\{1_1, 1_2\}$ and $\{4_1, 4_2\}$. Therefore U would act trivially on $\mathrm{IBr}(B)$, and, U^F would act trivially on simple $\overline{b}$-modules. However the description of $\mathrm{Pic}(b)$ yields that U^F would necessarily be the trivial element of $\mathrm{Pic}(b)$ and, by injectivity of the map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$, we would have a contradiction. Since we showed that $\mathrm{Pic}(B)$ has order 4 and we exhibited two different elements of order 2, then $\mathrm{Pic}(B) \simeq C_2 \times C_2$.

2.1.12 Principal block of $L_3(4)$

The simple modules are labelled as follows:

$$\mathrm{IBr}(B) = \{1, 15_1, 15_2, 15_3, 19\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2\},$$

B has inertial quotient Q_8 and and b is $\mathcal{O}\left((C_3 \times C_3) : Q_8\right)$. The Green correspondents for simple $\overline{b}$-modules have the following Loewy structures

$$\overline{F}(1_a) = 1; \;\; \overline{F}(1_b) = \begin{matrix} & 19 & \\ 1 & 15_2 & 1 \\ & 19 & \end{matrix}; \;\; \overline{F}(1_c) = \begin{matrix} & 19 & \\ 1 & 15_3 & 1 \\ & 19 & \end{matrix}; \;\; \overline{F}(1_d) = \begin{matrix} & 19 & \\ 1 & 15_1 & 1 \\ & 19 & \end{matrix};$$

$$\overline{F}(2) = \begin{matrix} 15_1 & 15_2 & 15_3 \\ 19 & & 19 \\ 15_1 & 15_2 & 15_3 \end{matrix}.$$

For the first time a Green correspondent has non-simple first layer, making the argument hardly adaptable. This is the reason why we introduce an additional assumption for this block, and the same will be done for other similar blocks.

Assumption 2.1.8. *Every element of* $\mathrm{Pic}(B)$ *fixes* $\overline{F}(2)$*.*

Note that it is reasonable expecting this assumption to hold. In fact, as we will see later with the examination of the decomposition matrix, every element of $\mathrm{Pic}(B)$ sends $\overline{F}(2)$ to a module with the same Loewy diagram. A priori $\overline{F}(2)$ could be sent to a non-isomorphic module with the same Loewy diagram, but we do expect that, as for the other cases, $\overline{F}(2)$ is a representative of the unique isomorphism class of modules with its Loewy structure. We checked computationally that this last fact is true when we consider the corresponding module defined over a (finite) splitting field.

The same argument used before is still valid however for the Green correspondents with non-simple first layer. We proceed as usual, writing down the Loewy structure of the projective cover of the simple module with dimension 19.

$$\begin{array}{ccccc} & & 19 & & \\ 1 & 15_1 & 15_2 & 15_3 & 1 \\ & 19 & 19 & 19 & \\ 1 & 15_1 & 15_2 & 15_3 & 1 \\ & & 19 & & \end{array}$$

Obviously the procedure is analogous for all the three non-simple Green correspondents with simple socle. For instance, it can be shown that there is just one isomorphism class of modules, M, with Loewy structure

$$\begin{array}{ccc} & 19 & \\ 1 & 15_1 & 1 \end{array}$$

and it corresponds to the quotient of $P(19)$ by the preimage of the unique submodule isomorphic to $15_2 \oplus 15_3$ under the map $\mathrm{rad}(P(19)) \mapsto \mathrm{rad}(P(19))/\,\mathrm{rad}^2(P(19))$. Checking with Magma that $\Omega^1(M)$ has Loewy structure

$$\begin{array}{ccccc} & 15_2 & 15_3 & 19 & \\ & 19 & & 19 & \\ 1 & 15_1 & 15_2 & 15_3 & 1 \\ & & 19 & & \end{array}$$

shows that $\overline{F}(1_d)$ is a representative of the unique isomorphism class of indecomposable $\overline{B}$-modules with its Loewy structure. The same argument applies to $\overline{F}(1_b), \overline{F}(1_c)$. Since the decomposition matrix of B is:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

an element in $\mathrm{Pic}(B)$ can just permute the 15-dimensional modules, and therefore, under the assumption that $\overline{F}(2)$ is fixed by any element of $\mathrm{Pic}(B)$, all Green correspondents are fixed by a Morita auto-equivalence of B. Thus we have, as described just before Subsection 2.1.1, a morphism $(\text{-})^F : \mathrm{Pic}(B) \to \mathrm{Pic}(b)$. Moreover, since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.10, $\mathrm{Pic}(B) = \mathcal{T}(B)$.

The trivial character of B must be fixed by any element of $\mathrm{Pic}(B)$, so $\mathrm{Out}_D(A)$ is trivial. In fact an element of $\mathrm{Out}_D(A)$ is necessarily mapped to the unique Morita auto-equivalence of b with diagonal vertex ΔD that fixes the trivial b-module, i.e. the identity element of $\mathrm{Pic}(b)$.

For what concerns elements of $\mathrm{Pic}(B)$ with vertex different from ΔD, we compute the Green correspondence with respect to restriction:

$$\overline{F}^*(1) = 1_a; \quad \overline{F}^*(15_1) = \begin{array}{ccc} & 2 & \\ 1_a & & 1_d \\ & 2 & \end{array}; \quad \overline{F}^*(15_2) = \begin{array}{ccc} & 2 & \\ 1_a & & 1_b \\ & 2 & \end{array}; \quad \overline{F}^*(15_3) = \begin{array}{ccc} & 2 & \\ 1_a & & 1_c \\ & 2 & \end{array};$$

$$\overline{F}^*(19) = \begin{array}{ccc} 1_d & 1_c & 1_b \\ 2 & & 2 \\ 1_d & 1_c & 1_b \end{array}$$

Again, we need to work under an additional assumption.

Assumption 2.1.9. *There is a unique isomorphism class of modules with Loewy structure identical to the one of* $\overline{F}^*(19)$.

We checked with Magma that modules over a splitting field with Loewy structure identical to the one of $\overline{F}^*(19)$ lie in a unique isomorphism class. This doesn't yield that the same happens for modules over an arbitrary field, but at least gives a reason to expect the assumption to hold.

Showing that each of the other Green correspondents is a representative of the unique isomorphism class of modules with that Loewy structure can be done exactly like before. The structure of $P(2)$ is in fact

$$\begin{array}{ccccc} & & 2 & & \\ 1_a & 1_b & & 1_c & 1_d \\ & 2 & 2 & 2 & \\ 1_a & 1_b & & 1_c & 1_d \\ & & 2 & & \end{array}$$

and, for example, the structure of $\Omega^1(M)$, where M is isomorphic to $\overline{F}^*(15_1)/\operatorname{soc}\left(\overline{F}^*(15_1)\right)$, is the following:

$$\begin{array}{ccccc} 1_b & & 2 & & 1_c \\ & 2 & & 2 & \\ 1_a & 1_b & & 1_c & 1_d \\ & & 2 & & \end{array}.$$

Thus the same arguments can be carried on to show that the Green correspondents of simple $\overline{B}$-modules are indecomposable modules uniquely determined by their Loewy structure, up to isomorphism.

Now all the Morita auto-equivalences of b given by b-b-bimodules induced by elements of $\operatorname{Out}(H)$ permute the Green correspondents of the rank 15 simple $\overline{B}$-modules and fix the ones of 19 and 1_1. In fact, $\operatorname{Out}(H)$ acts on linear non-trivial characters of b as the symmetric group on three elements by Subsection 2.1.10, and therefore the Morita auto-equivalences induced by $\operatorname{Out}(H)$ permute the linear non-trivial Brauer characters of b. Since all such elements permute the Green correspondents of b, they can be lifted to Morita auto-equivalences of B, via F^*, and thus $\operatorname{Pic}(B) \simeq S_3$.

2.1.13 Principal block of $C_3 \times C_3 : SD_{16}$

G has normal defect group and

$$\{\chi \in \operatorname{Irr}(B) | D \leq \operatorname{Ker}(\chi)\} = \{\chi \in \operatorname{Irr}(B) |\ \chi \text{ is a lift of some } \varphi \text{ in } \operatorname{IBr}(B)\}.$$

In fact irreducible characters of G are either lifts of Brauer characters or characters with degree 8, which certainly don't have D in their kernel. Therefore $\operatorname{Pic}(B) = \mathcal{T}(B)$. Since $N_{\operatorname{Aut}(D)}(E)/E$ is trivial, we immediately conclude that $\operatorname{Pic}(B) \simeq C_2 \times C_2$. Note that $\overline{B}$ has three 2-dimensional simple modules and one of them is distinguished. For making this more clear, we attach the Brauer character table of B.

	1A	**2A**	**2B**	**4A**	**4B**	**8A**	**8B**
1_a	1	1	1	1	1	1	1
1_b	1	1	1	1	-1	-1	-1
1_c	1	1	-1	1	-1	1	1
1_d	1	1	-1	1	1	-1	-1
2_a	2	2	0	-2	0	0	0
2_b	2	-2	0	0	0	$\zeta^3 + \zeta$	$-\zeta^3 - \zeta$
2_c	2	-2	0	0	0	$-\zeta^3 - \zeta$	$+\zeta^3 + \zeta$

In the above character table ζ denotes a primitive root of unity of order 8. We will obviously use a coherent notation and 2_a will always denote the rational character with degree 2.

2.1.14 Principal block of M_{11}

G is the Mathieu group on 11 points. The labelling for simple modules is the following:

$$\mathrm{IBr}(B) = \{1, 5_1, 5_2, 10_1, 10_2, 10_3, 24\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2_a, 2_b, 2_c\}.$$

B has inertial quotient SD_{16} and b is $\mathcal{O}\left((C_3 \times C_3) : SD_{16}\right)$. The Green correspondents of simple $\overline{b}$-modules can be computed on Magma or found in [82]. In particular, they have the following Loewy structure:

$$\overline{F}(1_a) = 1;\ \overline{F}(1_b) = 10_1;\ \overline{F}(1_c) = \begin{array}{ccc} & 5_2 & \\ 5_1 & & 10_1 \\ 1 & & 24 \\ & 5_2 & \\ & 5_1 & \end{array};\ \overline{F}(1_d) = \begin{array}{ccc} & 10_2 & \\ & 5_1 & \\ 1 & & 24 \\ & 5_2 & \\ & 10_3 & \end{array}$$

$$\overline{F}(2_a) = \begin{array}{c} 5_1 \\ 1 \\ 5_2 \end{array};\ \overline{F}(2_b) = \begin{array}{ccc} 24 & & 10_2 \\ & 5_2 & \\ 5_1 & & 10_3 \\ & 1 & \\ & 10_2 & \end{array};\ \overline{F}(2_c) = \begin{array}{ccc} & 10_3 & \\ & 1 & \\ 5_2 & & 10_2 \\ 5_1 & & 10_3 \\ & 24 & \end{array}$$

There are several non-simple Green correspondents, therefore we need to apply the well-known arguments multiple times. We write down the projective covers involved:

$$
P(5_1) = \begin{array}{ccccc}
 & & 5_1 & & \\
 & 1 & & 24 & \\
5_2 & & 10_2 & & 5_2 \\
5_1 & 10_1 & & 10_3 & 5_1 \\
1 & & 24 & & 1 \\
 & 5_2 & & 10_2 & \\
 & & 5_2 & &
\end{array}
\quad
P(5_2) = \begin{array}{ccccc}
 & & 5_2 & & \\
10_1 & & 5_1 & & 10_3 \\
1 & & 24 & & 1 \\
5_2 & & 10_2 & & 5_2 \\
5_1 & & 10_3 & & 5_1 \\
 & 1 & & 24 & \\
 & & 5_2 & &
\end{array}
$$

$$
P(10_2) = \begin{array}{ccc}
 & 10_2 & \\
 & 5_1 & \\
1 & & 24 \\
5_2 & & 10_2 \\
5_1 & & 10_3 \\
 & 1 & \\
 & 10_2 &
\end{array}
\quad
P(10_3) = \begin{array}{ccc}
 & 10_3 & \\
 & 1 & \\
5_2 & & 10_2 \\
5_1 & & 10_3 \\
1 & & 24 \\
 & 5_2 & \\
 & 10_3 &
\end{array}
$$

The Green correspondents of simple $\overline{B}$-modules have greater Loewy length than the ones seen in previous cases, so more steps need to performed. Consider $Y = \overline{F}(1_c)$. It follows immediately, using well known arguments, that there is just one isomorphism class of modules with the same Loewy structure as $Y_1 = Y/\operatorname{rad}^2(Y)$. Now we need to show that $\dim(\operatorname{Ext}^1_{kG}(Y_1, 1 \oplus 24)) = 1$. The Loewy structure of the Heller translate, computed with Magma, is given by

$$
\Omega^1(Y_1) = \begin{array}{ccc}
1 & 24 & 10_3 \\
1 & & 5_2 \\
5_2 & 10_2 & 5_1 \\
5_1 & & 10_3 \\
1 & & 24 \\
 & 5_2 &
\end{array},
$$

Therefore $\dim(\operatorname{Ext}^1_{kG}(Y_1, 1 \oplus 24)) = 1$ and, by Lemma 2.1.3, there is just one isomorphism classes of modules with the same Loewy structure as $Y_2 = Y/\operatorname{rad}^3(Y)$. We have to iterate this argument, so we write the Heller translate of Y_2, computed once again with Magma.

$$\Omega^1(Y_2) = \begin{array}{ccc} 5_2 & & 10_3 \\ 1 & & 5_1 \\ 5_2 & & 10_2 \\ 5_1 & & 10_3 \\ 1 & & 24 \\ & 5_2 & \end{array}$$

Everything works here as well; finally, the last step asks for another Heller translate, namely the one of $Y_3 = Y/\operatorname{rad}^4(Y)$. Computations on Magma yield that the structure is the following:

$$\Omega^1(Y_3) = \begin{array}{ccc} 5_1 & & 10_3 \\ & 1 & \\ 5_2 & & 10_2 \\ 5_1 & & 10_3 \\ 1 & & 24 \\ & 5_2 & \end{array}$$

This way we showed that there is just one isomorphism class for modules with same Loewy structure as $\overline{F}(1_c)$. We now deal with $U = \overline{F}(1_d)$, $W = \overline{F}(2_a)$ and $V = \overline{F}(2_c)$. V corresponds to the pre-image of 1 under the quotient map $\operatorname{rad}^4(V) \to \operatorname{rad}^4(V)/\operatorname{rad}^5(V)$, therefore we don't need any additional arguments. W involves two steps: first, it is shown that there is a unique isomorphism class of modules with Loewy structure as $W/\operatorname{rad}^2(W)$ since they corresponds to pre-images of 24; then, we need to compute the Heller translate of $W_1 = W/\operatorname{rad}^2(W)$:

$$\Omega^1(W_1) = \begin{array}{ccccc} 5_2 & & 24 & & 10_2 \\ & 5_1 & & 5_2 & \\ 1 & 5_2 & 24 & 10_1 & 10_3 \\ & & 1 & & \\ & 5_2 & & 10_2 & \\ & & 5_1 & & \end{array}$$

Therefore it follows that W is determined as an indecomposable module by its Loewy structure, up to isomorphism. Finally we deal with U. First, $U_1 = U/\operatorname{rad}^4(U)$ corresponds to the pre-image of 10_2 under the quotient map $\operatorname{rad}^3(U) \to \operatorname{rad}^3(U)/\operatorname{rad}^4(U)$ and is therefore a representative of the unique isomorphism class of indecomposable modules with such a

Loewy structure. Now we compute the Heller translate with Magma:

$$\Omega^1(U_1) = \begin{array}{ccc} 10_2 & & 10_3 \\ & 5_1 & \\ & 1 & \\ & 10_2 & \end{array}$$

Thus, it follows that dim $\left(\mathrm{Ext}^1_{kG}(U_1, 10_3)\right) = 1$ and therefore U is determined as an indecomposable module by its Loewy structure, up to isomorphism. We need to make an additional assumption for the Green correspondent with non-simple head.

Assumption 2.1.10. *Every element of* $\mathrm{Pic}(B)$ *fixes* $\overline{F}(2_b)$.

As with the previous case, this assumption is reasonable since elements of $\mathrm{Pic}(B)$ send $\overline{F}(2_b)$ to modules with the same Loewy structure and we checked with Magma that modules over a splitting field with Loewy structure identical to the one of $\overline{F}(2_b)$ lie in a unique isomorphism class.

We are finally ready for a close examination of the decomposition matrix.

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The action of an element of $\mathrm{Pic}(B)$ on simple $\overline{B}$-modules must necessarily be trivial. In fact, 1 must be fixed, since it is the unique liftable Brauer character that appears five times as a Brauer constituents of irreducible characters. Also 24 needs to be fixed, since it is the unique non-liftable Brauer character that appears as a Brauer constituent of two irreducible characters. Analogously, 10_1 is the unique liftable Brauer characters that appears as a Brauer constituent of two irreducible characters, and the it must be fixed. Since 5_1 and 5_2 are, respectively, the unique non-liftable Brauer characters that appear four and five times as constituents of irreducible characters, they must be fixed as well. We just need to show that 10_2 and 10_3 are fixed, but this is immediate. In fact, the character corresponding to the sixth row must be fixed, and therefore 10_2 must be fixed.

This means that all Green correspondents are therefore fixed, and thus there is an injective map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$, as described just before Subsection 2.1.1. As with the other cases, since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.13, it actually holds that $\mathrm{Pic}(B) = \mathcal{T}(B)$.

Let $M \in \mathrm{Out}_D(A)$. Since M fixes the trivial $\overline{B}$-module, it must be mapped to the unique element of $\mathrm{Out}_D(a)$ that fixes the trivial module of $\bar{b}$, i.e. the trivial element of $\mathrm{Pic}(b)$. But, by injectivity of the map, we can immediately conclude that $\mathrm{Out}_D(A) = 1$. Since $N_{\mathrm{Aut}(D)}(E)/E$ is trivial, then $\mathrm{Pic}(B)$ is trivial.

2.1.15 Principal block of M_{23}

G is the simple Mathieu group on 23 points. The labelling for simple modules is the following:

$$\mathrm{IBr}(B) = \{1, 22, 104_1, 104_2, 253, 770_1, 770_2\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2_a, 2_b, 2_c\}.$$

B has inertial quotient SD_{16} and b is $\mathcal{O}\left((C_3 \times C_3) : SD_{16}\right)$. Computing Loewy structures of Green correspondents in Magma is still possible in this case, but they need more memory than the one available on the online Magma calculator and can require much time. However we were able to retrieve the relevant results from the literature. By [25] and [82] the Green correspondents of simple modules have the following Loewy structure:

$$\overline{F}(1_a) = 1; \;\; \overline{F}(1_b) = 253; \;\; \overline{F}(1_c) = 22; \;\; \overline{F}(1_d) = \begin{array}{ccc} & 104_1 & \\ 1 & 104_2 & 770_1 \\ 22 & 253 & 22 \\ 104_1 & & 770_2 \\ & 104_2 & \end{array}$$

$$\overline{F}(2_a) = \begin{array}{c} 104_2 \\ 22 \\ 104_1 \end{array}; \;\; \overline{F}(2_b) = 770_1; \;\; \overline{F}(2_c) = 770_2;$$

and the decomposition matrix of B is:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

An element of $\mathrm{Pic}(B)$ can either swap 1 and 253, act via $(104_1 104_2)(770_1 770_2)$ or is a product of these two permutations. We show however that none of these actions are actually realized by any Morita equivalence. The projective covers of 104_1 and 104_2 have the following structure, by [88]:

$$P(104_1) = \begin{array}{ccc} & 104_1 & \\ 1 & 104_2 & 770_1 \\ 22 & 253 & 22 \\ 104_1 & & 770_2 \\ 104_2 & & 770_1 \\ & 22 & \\ & 104_1 & \end{array} \qquad P(104_2) = \begin{array}{ccc} & 104_2 & \\ & 22 & \\ 104_1 & & 770_2 \\ 1 & 104_2 & 770_1 \\ 22 & 253 & 22 \\ 104_1 & & 770_2 \\ & 104_2 & \end{array} .$$

Note that we changed the labelling according to the one chosen by [82]. Suppose there were an element $M \in \mathrm{Pic}(B)$ acting on simple modules via $(770_1, 770_2)(104_1, 104_2)$, then $M \otimes_B P(104_2)$ would need to be isomorphic to $P(104_1)$, but this is not possible.

Analogously if $M \in \mathrm{Pic}(B)$ acted by $(1, 253)$, then M would act non-trivially on $P(104_2)$, but it would also fix 104_2 and this is a contradiction. Thus $\mathrm{Pic}(B) = \mathrm{Picent}(B)$ and every Morita auto-equivalence of B acts trivially on simple $\overline{B}$-modules..

We can now check that the Green correspondents are uniquely determined as indecomposable modules, up to isomorphism, by their Loewy structure. This is however immediate: $\overline{F}(1_d)$ is the pre-image of the unique submodule of $\mathrm{rad}^4\left(P(104_1)\right) / \mathrm{rad}^5\left(P(104_1)\right)$ isomorphic to 770_1, under the quotient map from $\mathrm{rad}^4\left(P(104_1)\right)$; analogously $\overline{F}(2_a)$ is the preimage of the unique submodule of $\mathrm{rad}^2\left(P(104_2)\right) / \mathrm{rad}^3\left(P(104_2)\right)$ isomorphic to 770_2, under the quotient map from $\mathrm{rad}^2\left(P(104_2)\right)$.

Thus the Green correspondents of simple $\overline{b}$-modules are permuted by any Morita auto-equivalence of B and, as described just before Subsection 2.1.1, there is an injective map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$. Since $\mathrm{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.13, we actually have that $\mathrm{Pic}(B) = \mathcal{T}(B)$.

Let M be a Morita auto-equivalence of B with diagonal vertex ΔD and trivial source. M must fix 22, by examination of the decomposition matrix, thus M^F must fix 1_c. But we know that $\mathrm{Out}_D(A)$ is mapped to $\mathrm{Out}_D(a)$ injectively, and that the unique element of $\mathrm{Out}_D(a)$ that a linear simple b-module is the trivial element of $\mathrm{Pic}(b)$. Therefore M must be trivial. Since $N_{\mathrm{Aut}(D)}(E)/E$ is trivial, it follows that $\mathrm{Pic}(B) = 1$.

2.1.16 Principal block of $P\Sigma L(3,4)$

G is the semidirect product of $L_3(4)$ with the field automorphisms of $L_3(4)$. The simple modules are labelled as follows:

$$\mathrm{IBr}(B) = \{1_1, 1_2, 15_1, 15_2, 19_1, 19_2, 30\}, \qquad \mathrm{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2_a, 2_b, 2_c\}.$$

B has inertial quotient SD_{16} and b is $\mathcal{O}\left((C_3 \times C_3) : SD_{16}\right)$. The Loewy structures of Green correspondents are:

$$\overline{F}(1_a) = 1_1;\; \overline{F}(1_b) = \begin{matrix} & 19_1 & \\ 1_1 & 15_1 & 1_2 \\ & 19_1 & \end{matrix};\; \overline{F}(1_c) = \begin{matrix} & 19_2 & \\ 1_1 & 15_2 & 1_2 \\ & 19_2 & \end{matrix};\; \overline{F}(1_d) = 1_2$$

$$\overline{F}(2_a) = \begin{matrix} & 19_1 & & 19_2 & \\ 1_1 & 1_2 & 30 & 1_2 & 1_1 \\ & 19_1 & & 19_2 & \end{matrix}\; \overline{F}(2_b) = \begin{matrix} 15_1 & 30 \\ 19_1 & 19_2 \\ 15_2 & 30 \end{matrix};\; \overline{F}(2_c) = \begin{matrix} 15_2 & 30 \\ 19_1 & 19_2 \\ 15_1 & 30 \end{matrix},$$

We do have again Green correspondents with non-simple head, thus an additional assumption has to be made.

Assumption 2.1.11. *Every element of* $\mathrm{Pic}(B)$ *fixes* $\overline{F}(2_a)$ *and permutes* $\overline{F}(2_b)$ *and* $\overline{F}(2_c)$.

We remark that this assumption is reasonable, since, as it will be clear from examination of the decomposition matrix, $\mathrm{Pic}(B)$ sends $\overline{F}(2_a)$ to a module with the same Loewy structure. Moreover $\overline{F}(2_b)$ is either sent to a module with the same Loewy structure or sent to a module with Loewy structure as $\overline{F}(2_c)$. The action of a Morita auto-equivalence on $\overline{F}(2_c)$ is analogous.

Consider now $\overline{F}(1_b)$ and $\overline{F}(1_c)$; because of their very similar structure, we will just analyze $\overline{F}(1_b)$. The Loewy structure of the projective cover of 19_1, computed with Magma, is the following

$$P(19_1) = \begin{array}{ccccccc} & & & 19_1 & & & \\ 1_1 & 15_1 & & & & 1_2 & 30 \\ 19_1 & & & 19_2 & & & 19_1 \\ 1_1 & 15_1 & & & & 1_2 & 30 \\ & & & 19_1 & & & \end{array}$$

As already seen, it can be first shown that exists just one isomorphism class of indecomposable $\overline{B}$-modules with Loewy structure

$$\begin{array}{ccc} & 19_2 & \\ 1_1 & 15_2 & 1_2 \end{array}$$

since these modules correspond to preimages of submodules of $\mathrm{rad}(P(19_1))/\,\mathrm{rad}^2(P(19_1))$ isomorphic to 30, and there is just one such submodule. If $M = F(1_b)/\,\mathrm{soc}(F(1_b))$, we can then compute the Loewy structure of its Heller translate with Magma and it is given by

$$\Omega^1(M) = \begin{array}{cccccc} & 19_1 & & & 30 & \\ & 19_1 & & & 19_2 & \\ 1_1 & 15_1 & & & 1_2 & 30 \\ & & & 19_1 & & \end{array}$$

In particular $\dim_k \left(\mathrm{Ext}^1_{kG}(M, 19_1)\right) = 1$ and thus there is just one isomorphism class of indecomposable modules with the same Loewy structure of $\overline{F}(1_b)$. Analogous steps can be performed with $\overline{F}(1_c)$. Now, the decomposition matrix of B is

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Thus the only possible non-trivial action of an element in $\mathrm{Pic}(B)$ on simple modules is given by the permutation $(1_1 1_2)(15_1 15_2)(19_1 19_2)$. In fact, 30 must be fixed, since it is the unique Brauer character that appears as a Brauer constituent of three irreducible characters. 1_1 and 1_2 must be permuted, since they are the unique liftable Brauer characters. Moreover, looking at the number of times they appear as Brauer constituents of irreducible characters, 15_1 and 15_2 must be permuted and the same holds for 19_1 and 19_2. Suppose now that 1_1 is fixed, then 1_2 is fixed as well, and by the third and forth row of the decomposition matrix, 19_1 and 19_2 are both fixed. Looking at the fifth and sixth row, 15_1 and 15_2 must be fixed as well. Suppose now that 1_1 and 1_2 are swapped. Then, by looking at the third and fourth row of $\mathbb{D}$, 19_1 and 19_2 must be swapped as well, and the same holds for 15_1 and 15_2 by the fifth and the sixth row. Therefore, a Morita auto-equivalence of B either acts trivially on simple modules or by the permutation $(1_1 1_2)(15_1 15_2)(19_1 19_2)$.

What has been shown for $\overline{F}(1_b)$ and $\overline{F}(1_c)$, together with the starting assumption, yields that the Green correspondents are permuted by elements of $\mathrm{Pic}(B)$ and therefore there is a map, as described just before Subsection 2.1.1, $(\text{-})^F : \mathrm{Pic}(B) \to \mathrm{Pic}(b)$. Since $\mathrm{Pic}(b) = \mathcal{T}(b)$ as proved in Subsection 2.1.13, we actually have $\mathrm{Pic}(B) = \mathcal{T}(B)$.

We compute also the Green correspondence for the restriction:

$$\overline{F}^*(1_1) = 1_a;\ \overline{F}^*(1_2) = 1_d;\ \overline{F}^*(15_1) = \begin{array}{ccc} & 2_b & \\ 1_a & & 1_c \\ & 2_c & \end{array};\ \overline{F}^*(15_2) = \begin{array}{ccc} & 2_c & \\ 1_b & & 1_d \\ & 2_b & \end{array}$$

$$\overline{F}^*(19_1) = \begin{array}{cc} 1_b & 2_a \\ 2_b & 2_c \\ 1_b & 2_a \end{array}\ \overline{F}^*(19_2) = \begin{array}{cc} 1_c & 2_a \\ 2_b & 2_c \\ 1_c & 2_a \end{array};\ \overline{F}^*(30) = \begin{array}{ccc} 2_b & & 2_c \\ 1_d & 2_a & 1_a \\ 2_c & & 2_b \end{array}.$$

Here again there are many Green correspondents with non-simple head, so we need to make another assumption.

Assumption 2.1.12. $\overline{F}^*(19_1), \overline{F}^*(19_2), \overline{F}^*(30)$ *are representatives of the unique isomorphism class of indecomposable $\bar{b}$-modules with their Loewy structure.*

We checked that the assumption holds for the corresponding modules defined over a splitting field with Magma, so it is reasonable expecting that this holds for the modules defined over k as well.

A similar procedure can be carried on for Green correspondents with simple head. For example, consider $\overline{F}^*(15_1)$; the Loewy structure of the projective cover of 2_b is given by

$$P(2_b) = \begin{array}{ccc} & 2_b & \\ 1_a & 2_a & 1_c \\ 2_c & 2_b & 2_c \\ 1_d & 2_a & 1_b \\ & 2_b & \end{array}$$

The same argument used for the Green correspondents of simple $\bar{b}$-modules can be used here. We provide the structure of the Heller translate of $V = \overline{F}^*(15_1)/\operatorname{soc}(\overline{F}^*(15_1))$ to show that even the final part of the proof can be carried on as usual.

$$\Omega^1(V) = \begin{array}{ccc} 2_a & & 2_c \\ 2_b & & 2_a \\ 1_d & 2_a & 1_b \\ & 2_b & \end{array}$$

The same can be shown for $\overline{F}^*(15_2)$, and thus, by the assumption, all Green correspondents are uniquely defined as indecomposable modules by their Loewy structure. We can then lift the element of $\operatorname{Pic}(b)$ acting on simple modules by multiplication by 1_d, since it permutes the Green correspondents of simple $\overline{B}$-modules. Then $\operatorname{Pic}(B) \geq C_2$.

Any element of $\operatorname{Pic}(B)$ must permute $\{1_1, 1_2\}$, therefore an element of $\operatorname{Out}_D(A)$ is either mapped to the element of $\operatorname{Out}_D(a)$ that acts by multiplication with 1_d or to the trivial Morita auto-equivalence of b. Thus, by the injectivity of $(\text{-})^F$, we conclude that $\operatorname{Pic}(B) \simeq C_2$.

2.1.17 Principal block of $\operatorname{Aut}(S_6)$

Simple modules are labelled as follows:

$$\operatorname{IBr}(B) = \{1_1, 1_2, 1_3, 1_4, 6_1, 6_2, 8\}, \qquad \operatorname{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2_a, 2_b, 2_c\}.$$

B has inertial quotient SD_{16} and b is $\mathcal{O}\left((C_3 \times C_3) : SD_{16}\right)$. The Green correspondents for simple $\bar{b}$-modules have the following Loewy structures:

$$\overline{F}(1_a) = 1_1;\ \ \overline{F}(1_b) = 1_2;\ \ \overline{F}(1_c) = 1_3;\ \ \overline{F}(1_d) = 1_4;$$

$$\overline{F}(2_a) = \begin{array}{ccccc} & & 8 & & \\ 1_1 & 1_2 & & 1_3 & 1_4 \\ & & 8 & & \end{array};\ \ \overline{F}(2_b) = \begin{array}{c} 6_1 \\ 8 \\ 6_2 \end{array};\ \ \overline{F}(2_c) = \begin{array}{c} 6_2 \\ 8 \\ 6_1 \end{array}$$

The Green correspondents in this case have simple heads, therefore usual arguments can be carried on. Take $\overline{F}(2_a)$; the projective cover of 8 has the following Loewy structure:

$$P(8) = \begin{array}{ccccccc} & & & 8 & & & \\ 6_1 & 1_1 & 1_2 & & 1_3 & 1_4 & 6_2 \\ 8 & & & 8 & & & 8 \\ 6_1 & 1_1 & 1_2 & & 1_3 & 1_4 & 6_2 \\ & & & 8 & & & \end{array}$$

We can therefore show the existence of just one isomorphism class of modules with Loewy structure as $M = \overline{F}(2_a)/\operatorname{soc}(\overline{F}(2_a))$. The Loewy structure of $\Omega^1(M)$ can be computed with Magma and is given by

$$\Omega^1(M) = \begin{array}{ccccccc} 6_1 & & & 8 & & & 6_2 \\ & & 8 & & 8 & & \\ 6_1 & 1_1 & 1_2 & & 1_3 & 1_4 & 6_2 \\ & & & 8 & & & \end{array}$$

Such structure implies the uniqueness, up to isomorphism, of $\overline{B}$-modules with Loewy structure as $\overline{F}(2_a)$. A more immediate argument can be used for the other non-simple Green correspondents; for example the projective cover of 6_1 has the following Loewy structure:

$$P(6_1) = \begin{array}{ccc} & 6_1 & \\ & 8 & \\ 1_1 & 6_2 & 1_3 \\ & 8 & \\ & 6_1 & \end{array}$$

We now turn our attention to the decomposition matrix of B.

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

An element of $\operatorname{Pic}(B)$ must then preserve degrees of simple modules, so, since the Green correspondents are uniquely determined as indecomposable modules by their Loewy structure, they are permuted by Morita auto-equivalence. Thus there is a map, as described just before Subsection 2.1.1, $(\text{-})^F : \operatorname{Pic}(B) \to \operatorname{Pic}(b)$. Moreover, $\operatorname{Pic}(B) = \mathcal{T}(B)$, since $\operatorname{Pic}(b) = \mathcal{T}(b)$ by Subsection 2.1.13.

Examining the restriction, the Loewy structures of Green correspondents are:

$$\overline{F}^*(1_1) = 1_a;\ \overline{F}^*(1_2) = 1_b;\ \overline{F}^*(1_3) = 1_c;\ \overline{F}^*(1_4) = 1_d$$

$$\overline{F}^*(6_1) = \begin{array}{ccc} & 2_c & \\ 1_a & & 1_c \\ & 2_b & \end{array}\ \overline{F}^*(6_2) = \begin{array}{ccc} & 2_b & \\ 1_b & & 1_d \\ & 2_c & \end{array};\ \overline{F}^*(8) = \begin{array}{ccc} & 2_a & \\ 2_b & & 2_c \\ & 2_a & \end{array};$$

The usual procedure can be applied to the non-simple Green correspondents; their projective covers have structure:

$$P(2_a) = \begin{array}{ccccc} & & 2_a & & \\ & 2_b & & 2_c & \\ 1_a & 1_b & 2_a & 1_c & 1_d \\ & 2_c & & 2_b & \\ & & 2_a & & \end{array};\quad P(2_b) = \begin{array}{ccc} & 2_b & \\ 1_b & 2_a & 1_d \\ 2_b & 2_c & 2_b \\ 1_a & 2_a & 1_c \\ & 2_b & \end{array};\quad P(2_c) = \begin{array}{ccc} & 2_c & \\ 1_a & 2_a & 1_c \\ 2_b & 2_c & 2_b \\ 1_b & 2_a & 1_d \\ & 2_c & \end{array};$$

The arguments are the usual ones; the Heller translate for $M = \overline{F}^*(6_1)/\operatorname{soc}(\overline{F}^*(6_1))$ and $N = \overline{F}^*(6_2)/\operatorname{soc}(\overline{F}^*(6_2))$ are, for instance, the following:

$$\Omega^1(M) = \begin{array}{ccc} 2_a & & 2_b \\ 2_b & & 2_c \\ 1_d & 2_a & 1_b \\ & 2_c & \end{array};\quad \Omega^1(N) = \begin{array}{ccc} 2_a & & 2_c \\ 2_c & & 2_b \\ 1_a & 2_a & 1_c \\ & 2_b & \end{array}$$

After showing that the Green correspondents for restriction are uniquely determined by their Loewy structure, it can be easily checked, using the Brauer character table provided in Subsection 2.1.13, that all elements of $\operatorname{Pic}(b)$ permute the Green correspondents of simple $\overline{B}$-modules. Therefore they can be lifted via F^* and $\operatorname{Pic}(B) \simeq C_2 \times C_2$.

2.1.18 Principal block of HS

G is the Higman-Sims group. The simple modules are labelled by

$$\operatorname{IBr}(B) = \{1_1, 22, 154, 321, 748, 1176, 1253\}, \qquad \operatorname{IBr}(b) = \{1_a, 1_b, 1_c, 1_d, 2_a, 2_b, 2_c\}.$$

B has inertial quotient SD_{16}. Note that in this case H has non-trivial centre, and therefore b is not the whole group algebra of $\mathcal{O}H$, even if it is isomorphic to $\mathcal{O}\left((C_3 \times C_3) : SD_{16}\right)$. However it can still be checked that $\operatorname{Pic}(b) = \mathcal{T}(b)$, using Proposition 1.7.3, so we don't need to make any additional steps. The Green correspondents have the following structures, by

[82]:

$$\overline{F}(1_a) = 1_1; \ \overline{F}(1_b) = 154; \ \overline{F}(1_c) = 22; \ \overline{F}(1_d) = \begin{array}{ccc} & 748 & \\ 1_1 & 22 & 1176 \\ & 748 & \end{array}$$

$$\overline{F}(2_a) = \begin{array}{cccccc} & & 1253 & 748 & & \\ 1_1 & 22 & 154 & 1176 & 22 & 1_1 \\ & & 1253 & 748 & & \end{array} \ \overline{F}(2_b) = \begin{array}{c} 321 \\ 1253 \\ 1176 \end{array}; \ \overline{F}(2_c) = \begin{array}{c} 1176 \\ 1253 \\ 321 \end{array};$$

Now the decomposition matrix of B is:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

From examination of the decomposition matrix, an element of $\mathrm{Pic}(B)$ must then act trivially on simple $\overline{B}$-modules.

In fact, let $M \in \mathrm{Pic}(B)$. The permutation of simple $\overline{B}$-modules induced by M needs to fix 154, 321 and 1253. This follows just by counting the number of irreducible characters with these Brauer characters as constituents, and noticing that 154 lifts to an irreduible character. In fact, 1253 can either be fixed or sent to 1_1 or 22, since it appears as a constituent of 4 irreducible characters. But 1253 doesn't lift to an irreducible character, while 1_1 and 22 do, so it must be fixed. Moreover, if χ is the irreducible character corresponding to the 8$^{\text{th}}$ row, M must fix χ, since 321 and 1253 are fixed. Thus 1176 must be fixed.

If now we turn our attention to η, the irreducible character corresponding to the last row, then η must be fixed, since it has five Brauer constituents. Moreover, looking at the Brauer constituents of η, we can also say that 748 must be fixed. In fact 1253 and 1176 are fixed, and 748 can't be sent to 1_1 and 22, since these last two Brauer characters are liftable to an irreducible character. Now the character corresponding to the 7$^{\text{th}}$ row must be fixed, since 1176 and 748 are fixed, and therefore 1_1 is fixed. It then follows that all simple $\overline{B}$-modules are fixed.

We need to show that the Green correspondents are uniquely determined as indecomposable modules by their Loewy structure. The structure of projective indecomposable modules can be recovered also from [89] and is the following, for the modules we are concerned with:

$$P(321) = \begin{array}{ccc} & 321 & \\ & 1253 & \\ 22 & 154 & 1176 \\ & 1253 & \\ & 321 & \end{array} ; \quad P(748) = \begin{array}{ccc} & 748 & \\ 1 & 22 & 1176 \\ 748 & & 1253 \\ 1 & 22 & 1176 \\ & 748 & \end{array} ;$$

$$P(1176) = \begin{array}{ccccc} & & 1176 & & \\ & 748 & & 1253 & \\ 1 & 321 & 1176 & 22 & 1 \\ & 1253 & & 748 & \\ & & 1176 & & \end{array}$$

For easing the notation used, we call $\overline{F}(1_d), \overline{F}(2_b), \overline{F}(2_c)$, respectively, U, V, W. U corresponds to the pre-image of 1253 under the quotient map from $\mathrm{rad}^2(P(U))$ to $\mathrm{rad}^2(P(U))/\mathrm{rad}^3(P(U))$, and therefore is uniquely determined by its Loewy structure, as an indecomposable module. Similarly, V corresponds to the pre-image of $22 \oplus 1254$ under the map from $\mathrm{rad}^2(P(V))$ to $\mathrm{rad}^2(P(V))/\mathrm{rad}^3(P(V))$.

W needs two steps: in fact, $W_1 = W/\mathrm{soc}(W)$ is the pre-image of 748 under the map $\mathrm{rad}(P(W)) \to \mathrm{rad}(P(W))/\mathrm{rad}^2(P(W)$. We then notice that $\mathrm{Ext}^1_{kG}(W_1, 321)$ is not zero, since W exists. Certainly $\mathrm{Hom}_{kG}\left(\Omega^1(W_1), 321\right)$ has dimension 0 or 1, since 321 appears once in the Loewy structure of $\Omega^1(W_1)$. Therefore, by Lemma 2.1.5, $\mathrm{Ext}^1_{kG}(W_1, 321)$ has dimension one and by Lemma 2.1.3 we can then conclude that W is uniquely determined as an indecomposable module by its Loewy structure. For the non-simple head Green correspondent, we need again an additional hypothesis:

Assumption 2.1.13. *Every element of* $\mathrm{Pic}(B)$ *fixes* $\overline{F}(2_a)$.

This assumption is reasonable, since elements of $\mathrm{Pic}(B)$ act trivially on simple modules and then send $\overline{F}(2_a)$ to a module with the same Loewy structure.

Together with the assumption, we have then shown that Morita auto-equivalences of B permute the Green correspondents of simple $\bar{b}$-modules. Then, as described just before Subsection 2.1.1, there is an injective map $\mathrm{Pic}(B) \to \mathrm{Pic}(b)$. Since $\mathrm{Pic}(b) = \mathcal{T}(b)$ as remarked at the beginning of the proof, then $\mathrm{Pic}(B) = \mathcal{T}(B)$.

Moreover we know that $\mathrm{Out}_D(A)$ is mapped (injectively) to $\mathrm{Out}_D(a)$ and Morita auto-equivalences of B fix the trivial $\overline{B}$-module, thus their images in $\mathrm{Pic}(b)$ need to fix the trivial module as well. However, Morita auto-equivalences of b are given by multiplication with a linear character and therefore only the trivial element of $\mathrm{Pic}(B)$ fixes the trivial character of b. It then follows that $\mathrm{Pic}(B) = 1$.

2.2 Picard groups for some families of principal 2-blocks

The objective of this section is performing calculations of Picard groups for principal blocks of groups of the form $P \times G$, where P is a p-group and G a group whose principal block's Picard group has already been calculated. Finding a general result for this class of blocks is at the moment out of reach, but it is something that needs to be addressed to make calculations of Picard groups easier: many blocks are in fact built from *atomic* blocks with this procedure, i.e. taking the principal block of a direct product of a group with certain Sylow p-subgroups and an appropriate p-group. We will work with 2-blocks, since the aim of this section is to calculate Picard groups for the principal blocks of groups $P \times G$, for groups G whose principal blocks have non-normal defect group C_2^3 and their Picard groups are known. To achieve these results we are going to heavily rely on properties of perfect self-isometries groups and apply Weiss' criterion, in the same fashion as in [29]. Perfect isometries are a crucial tool for the next set of results, therefore we refer the reader to Section 1.4, where they were introduced.

We have already underlined that Morita equivalences induce a perfect isometry with positive signs. In certain cases, perfect isometries induced by Morita auto-equivalences of a block of $P \times G$ can be *partially factorized*, as proved in the next proposition. Note that we say that a matrix has *distinct rows* if the rows are distinct as n-tuples.

Proposition 2.2.1. *Let H be a group, B a block of H such that its decomposition matrix has distinct rows. Let P be an abelian p-group and establish a labelling*

$$\mathrm{Irr}(P) = \left\{\theta_1, ..., \theta_{|P|}\right\} \quad \mathrm{Irr}(B) = \left\{\chi_1, ..., \chi_{k(B)}\right\}.$$

Let $M \in \mathrm{Pic}(\mathcal{O}P \otimes_{\mathcal{O}} B)$ and let $T \in \mathrm{Perf}(\mathcal{O}P \otimes_{\mathcal{O}} B)$ denote the perfect isometry induced by M. Then there exist a perfect self-isometry τ of $\mathbb{Z}\,\mathrm{Irr}(B)$, and permutations $\sigma_j \in \mathrm{S}_{|P|}$, for all $j \in \{1, ..., k(B)\}$, such that

$$T(\theta_i \otimes \chi_j) = \theta_{\sigma_j(i)} \otimes \tau(\chi_j), \quad \forall \theta_i \in \mathrm{Irr}(P), \chi_j \in \mathrm{Irr}(B).$$

Proof. We construct τ. Suppose that, for some $i, h \in \{1, ..., |P|\}$ and $j, s, r \in \{1, ..., k(B)\}$, $T(\theta_i \otimes \chi_s) \downarrow_H = \chi_j$ and $T(\theta_h \otimes \chi_s) \downarrow_H = \chi_r$, with $r \neq j$. Now $\theta_h \otimes \chi_s - \theta_i \otimes \chi_k$ is zero on all the p-regular elements of $P \times H$, i.e. the elements of the form $(1, g)$ with g a p-regular element of H. Thus $T(\theta_i \otimes \chi_s) - T(\theta_h \otimes \chi_s)$ must be zero on p-regular elements of $P \times H$ as well, by Proposition 1.4.8. But this means that $\chi_r(g) - \chi_j(g) = 0$ for all p-regular elements $g \in H$, i.e. $\chi_r - \chi_j$ is in the kernel of the decomposition map. This is a contradiction, considered that the decomposition matrix of B has distinct rows. So we can define a map

$$\begin{aligned} \tau : \mathrm{Irr}(B) &\to \mathrm{Irr}(B) \\ \chi &\mapsto T(1_P \otimes \chi) \downarrow_H \end{aligned}$$

Obviously τ is a bijection, since $\tau(\chi_1) = \tau(\chi_2)$ means that $T(1_P \otimes \chi_1 - 1_P \otimes \chi_2)$ is zero on p-regular elements and this implies that $1_P \otimes \chi_1 - 1_P \otimes \chi_2$ is zero on p-regular elements, giving that $\chi_1 = \chi_2$, as above.

We now show the existence of the permutations σ_j as described in the statement. What we need to prove is that, for $\theta_i \neq \theta_s$, $T(\theta_i \otimes \chi_j) \downarrow_P \neq T(\theta_s \otimes \chi_j) \downarrow_P$. Suppose by contradiction that this doesn't happen; so $T(\theta_i \otimes \chi_j) - T(\theta_s \otimes \chi_j)$ is zero on p-singular elements. But T is a perfect isometry and therefore, by Proposition 1.4.8, the same must happen for $\theta_i \otimes \chi_j - \theta_s \otimes \chi_j$. This is obviously a contradiction, since θ_i and θ_s were chosen to be different. So it means that we can define, for each s, a map

$$\begin{aligned} \Sigma_s : \mathrm{Irr}(P) &\to \mathrm{Irr}(P) \\ \theta &\mapsto \frac{1}{\chi_s(1)} T(\theta \otimes \chi_s) \downarrow_P \end{aligned}$$

This map is a bijection by the paragraph above. In particular, we can find σ_s a permutation of $\{1, ..., |P|\}$ such that $\Sigma_s(\theta_i) = \theta_{\sigma_s(i)}$. Then, we have shown that

$$T(\theta_i \otimes \chi_s) = \theta_{\sigma_s(i)} \otimes \chi_{\tau(s)}$$

for all i, s. We just need to prove that τ actually defines a perfect self-isometry. Abusing the notation, we will write $\tau(\chi_s) = \chi_{\tau(s)}$. Also note that

$$\tau(\chi)(-) = T(\theta \otimes \chi)(1, -),$$

where θ is any irreducible character of P. The character associated to τ is defined by

$$\eta(g, h) = \sum_{\chi \in \mathrm{Irr}(B)} \tau(\chi(g))\chi(h)$$

We prove that this character is perfect. Suppose that μ is the character of T, and take g and h in H such that exactly one of them is p-regular. Thus

$$0=\mu((1,g),(1,h))=\sum_{i,\chi}T(\theta_i\otimes\chi)(1,g)\left((\theta_i\otimes\chi)(1,h)\right)=|P|\sum_{\chi}\tau(\chi)(g)\chi(h)=|P|\eta(g,h).$$

Moreover we know that for all g,h in H

$$|C_{P\times H}((1,g))|=|P||C_H(g)| \text{ divides } \mu((1,g),(1,h))=|P|\eta(g,h)$$

and so η is perfect. □

The above proposition is now applied to some special cases to obtain a *full factorization* of perfect isometries induced by elements of the Picard group. We first include a Lemma that will be repeatedly used, for benefit of the reader.

Lemma 2.2.2 ([28], Lemma 2.9)**.** *Let m be a positive integer and suppose $\sum_{i=0}^{2^m-1}\zeta^{l_i}\in 2^m\mathcal{O}$, where $l_i\in\mathbb{Z}$ for $0\le i<2^m$. Then either $\zeta^{l_o}=\cdots=\zeta^{l_{2^m-1}}$ or $\sum_{i=0}^{2^m-1}l^i=0$.*

Proposition 2.2.3. *Let T be a perfect self-isometry of the principal 2-block of $P\times SL_2(8)$ induced by a Morita auto-equivalence, where P is an abelian 2-group. Then $T=\sigma\otimes\gamma$ where $\sigma\in\mathrm{Sym}(\mathrm{Irr}(P))$ and γ is a perfect self-isometry of the principal block of $P\times SL_2(8)$.*

Proof. Let $G=P\times SL_2(8)$, B the principal block of $SL_2(8)$. Write

$$\mathrm{Irr}(B)=\{\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_7,\chi_8,\chi_9\};$$
$$\mathrm{Irr}(\mathcal{O}P\otimes_{\mathcal{O}}B)=\{\theta\otimes\chi_i|\theta\in\mathrm{Irr}(P),\chi_i\in\mathrm{Irr}(B)\},$$

with the Atlas labelling ([22]). In particular: χ_1 is the trivial character; $\chi_2,\chi_3,\chi_4,\chi_5$ are the irreducible characters of B with degree 7; the others are characters with degree 9. Also note that $\chi_6\notin\mathrm{Irr}(B)$, since usually χ_6 denotes the projective irreducible character of $SL_2(8)$. Now the characters of the projective indecomposable modules of $\mathcal{O}P\otimes_{\mathcal{O}}B$ are the following:

$$\chi_{pr_1}=\sum_{\theta\in\mathrm{Irr}(P),i}\theta\otimes\chi_i \qquad \chi_{pr_2}=\sum_{\theta\in\mathrm{Irr}(P)}\theta\otimes(\chi_2+\chi_3+\chi_7+\chi_8)$$
$$\chi_{pr_3}=\sum_{\theta\in\mathrm{Irr}(P)}\theta\otimes(\chi_2+\chi_5+\chi_8+\chi_9) \qquad \chi_{pr_4}=\sum_{\theta\in\mathrm{Irr}(P)}\theta\otimes(\chi_2+\chi_4+\chi_7+\chi_9)$$
$$\chi_{pr_5}=\sum_{\theta\in\mathrm{Irr}(P)}\theta\otimes(\chi_5+\chi_9) \qquad \chi_{pr_6}=\sum_{\theta\in\mathrm{Irr}(P)}\theta\otimes(\chi_4+\chi_7)$$
$$\chi_{pr_7}=\sum_{\theta\in\mathrm{Irr}(P)}\theta\otimes(\chi_3+\chi_8)$$

Now a Morita auto-equivalence of $\mathcal{O}P \otimes_{\mathcal{O}} B$ must obviously permute them. The decomposition matrix of $SL_2(8)$ is the following:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Since $\mathbb{D}$ has distinct rows, if T is the perfect isometry induced by a Morita auto-equivalence of $\mathcal{O}P \otimes_{\mathcal{O}} B$, we know by Proposition 2.2.1 that T acts by

$$T(\theta_i \otimes \chi_s) = \theta_{\sigma_s(i)} \otimes \tau(\chi_s).$$

Just by inspecting the number of times irreducible characters appear as constituents of characters of projective indecomposable modules and looking at the number of irreducible constituents of the χ_{pr_i}, we can describe the action of τ. For example, we immediately see that χ_1 is fixed, since it is the unique irreducible character of B that appears just in χ_{pr_1}. Also, χ_2 is the unique irreducible character of B that appears as a constituent of the characters of projective indecomposable modules of $\mathcal{O}P \otimes_{\mathcal{O}} B$ that have $4|P|$ irreducible constituents, i.e. χ_{pr_2}, χ_{pr_3}, and χ_{pr_4}. Now, if τ acts non trivially on χ_8 then it must send χ_8 to χ_7 or χ_9. In particular, if τ is non-trivial, the permutation induced by τ on $\mathrm{Irr}(B)$ is one of the following:

$$(789)(345) \text{ or } (798)(354).$$

Now, for all $\theta \in \mathrm{Irr}(P)$

$$\theta \otimes \sum_{\chi_i \in \mathrm{Irr}(B)} \chi_i(1)\chi_i \in |SL_2(8)|\,\mathrm{CF}(G, \mathcal{O}P \otimes_{\mathcal{O}} B, \mathcal{O}), \tag{2.2}$$

where we are using the notation introduced in Section 1.4. Note that (2.2) follows from the fact that the identity is a perfect self-isometry of B, and thus

$$|C_{SL_2(8)}(1)| = |SL_2(8)| \mid \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi(g) \; \forall g \in SL_2(8).$$

Applying T to 2.2 yields again an element of $|SL_2(8)|\,\mathrm{CF}(G, \mathcal{O}P \otimes_{\mathcal{O}} B, \mathcal{O})$. Thus, fixing θ and writing $T(\theta \otimes \chi_i) = \theta_i \otimes \chi_i$ for $i = 1, 2$ and $T(\theta \otimes \chi_i) = \theta_i \otimes \chi_{\tau(i)}$ for the remaining indices, it follows that

$$\theta_1\chi_1 + 7\left(\theta_2\chi_2 + \theta_3\chi_{\tau(3)} + \theta_4\chi_{\tau(4)} + \theta_5\chi_{\tau(5)}\right) + 9\left(\theta_7\chi_{\tau(7)} + \theta_8\chi_{\tau(8)} + \theta_9\chi_{\tau(9)}\right) \in 8\,\mathrm{CF}(G, \mathcal{O}P\otimes_{\mathcal{O}} B, \mathcal{O}). \tag{2.3}$$

We remark that $\chi_2, \chi_3, \chi_4, \chi_5$ have degree 7, while χ_7, χ_8, χ_9 have degree 9. We now evaluate the class function in (2.3) on various elements $(x, g) \in P\times SL_2(8)$. Assume for a moment that τ acts trivially; we will check in a moment that it is a natural assumption. The elements $g \in SL_2(8)$ we choose are respectively in the conjugacy classes: $1A, 3A, 7A, 7B, 7C, 9A, 9B, 9C$, where the Atlas notation is adopted; for example, $7A, 7B, 7C$ are the three conjugacy classes of $SL_2(8)$ represented by elements of order 7. The class function defined above is then evaluated on all the pairs (x, g) where x is an element of P and g is an element lying in one of the conjugacy classes above.

$$\textbf{(x,1A)}\quad \theta_1(x)+\theta_2(x)+\theta_3(x)+\theta_4(x)+\theta_5(x)+\theta_7(x)+\theta_8(x)+\theta_9(x) \in 8\mathcal{O} \tag{2.4}$$

$$\textbf{(x,3A)}\quad \theta_1(x)+2\theta_2(x)+(-\theta_3(x))+(-\theta_4(x))+(-\theta_5(x)) \in 8\mathcal{O} \tag{2.5}$$

$$\textbf{(x,7A)}\quad \theta_1(x)+9Z_1\theta_7(x)+9Z_1^{\sigma_2}\theta_8(x)+9Z_1^{\sigma_3}\theta_9(x) \in 8\mathcal{O} \tag{2.6}$$

$$\textbf{(x,7B)}\quad \theta_1(x)+9Z_1^{\sigma_3}\theta_7(x)+9Z_1\theta_8(x)+9Z_1^{\sigma_2}\theta_9(x) \in 8\mathcal{O} \tag{2.7}$$

$$\textbf{(x,7C)}\quad \theta_1(x)+9Z_1^{\sigma_2}\theta_7(x)+9Z_1^{\sigma_3}\theta_8(x)+9Z_1\theta_9(x) \in 8\mathcal{O} \tag{2.8}$$

$$\textbf{(x,9A)}\quad \theta_1(x)+(-\theta_2(x))+7Z_2\theta_3(x)+7Z_2^{\sigma_2}\theta_4(x)+7Z_2^{\sigma_4}\theta_5(x) \in 8\mathcal{O} \tag{2.9}$$

$$\textbf{(x,9B)}\quad \theta_1(x)+(-\theta_2(x))+7Z_2^{\sigma_4}\theta_3(x)+7Z_2\theta_4(x)+7Z_2^{\sigma_2}\theta_5(x) \in 8\mathcal{O} \tag{2.10}$$

$$\textbf{(x,9C)}\quad \theta_1(x)+(-\theta_2(x))+7Z_2^{\sigma_2}\theta_3(x)+7Z_2^{\sigma_4}\theta_4(x)+7Z_2\theta_5(x) \in 8\mathcal{O} \tag{2.11}$$

where ζ_1, ζ_2 are, respectively, primitive roots of unity of order $7, 9$, and $Z_1 = \zeta_1^2 + \zeta_1^5$, $Z_2 = -\zeta_2^4 - \zeta_2^5$. Moreover, σ denotes algebraic conjugation, that is σ_i indicates replacing the root of unity ζ_j by ζ_j^i. Note that if τ was not trivial, but one of the non-trivial permutations listed a few lines above, then the equations we just wrote would be permuted. Therefore, assuming that τ is trivial or not doesn't really make a difference. Now, the following equalities hold:

$$Z_1 + Z_1^{\sigma_2} + Z_1^{\sigma_3} = -1;$$
$$Z_2 + Z_2^{\sigma_2} + Z_2^{\sigma_4} = 0.$$

The sums $(2.6)+(2.7)+(2.8)$ and $(2.9)+(2.10)+(2.11)$ yield the following new expressions:

$$3\theta_1(x)+(-\theta_7(x))+(-\theta_8(x))+(-\theta_9(x)) \in 8\mathcal{O} \tag{2.12}$$

$$3\theta_1(x)+3(-\theta_2(x)) \in 8\mathcal{O} \tag{2.13}$$

The expression (2.13) implies that $3\theta_1(x) + 5\theta_2(x) \in 8\mathcal{O}$, so by Lemma 2.2.2 we have that $\theta_1(x) = \theta_2(x)$, since it can't be zero. But then (2.12) is equivalent to

$$4\theta_1(x) + (-\theta_1(x)) + (-\theta_7(x)) + (-\theta_8(x)) + (-\theta_9(x)) \in 8\mathcal{O}$$

and either the roots of unity are all equal or their sum is zero, again using Lemma 2.2.2. Certainly they are not all equal, since $\theta_1(x) \neq -\theta_1(x)$. Then

$$3\theta_1(x) + (-\theta_7(x)) + (-\theta_8(x)) + (-\theta_9(x)) = 0.$$

In particular, looking at the complex norm, this would mean that

$$|3\theta_1(x)| = 3 = |\theta_7(x) + \theta_8(x) + \theta_9(x)|$$

Therefore

$$\theta_7(x) = \theta_8(x) = \theta_9(x) = \theta_1(x) = \theta_2(x).$$

Then, (2.4) becomes

$$5\theta_1(x) + \theta_3(x) + \theta_4(x) + \theta_5(x) \in 8\mathcal{O}.$$

This sum being zero would give a contradiction. In fact, as above, it would mean that

$$|5\theta_1(x)| = 5 = |-(\theta_3(x) + \theta_4(x) + \theta_5(x))|$$

Therefore $\theta_i(x) = \theta_j(x)$ for all i, j and $x \in P$. Thus $\theta_i = \theta_j$ for all i, j, and there is r such that $T(\theta \otimes \chi_s) = \theta_r \otimes \chi_s$, for all s. Since this holds for all θ (recall that θ was a fixed character), then T has the desired form. □

Proposition 2.2.4. *Let T be a perfect self-isometry of the principal 2-block of $P \times \mathrm{Aut}(SL_2(8))$ induced by a Morita auto-equivalence, where P is an abelian 2-group. Then $T = \sigma \otimes \gamma$ where $\sigma \in \mathrm{Sym}(\mathrm{Irr}(P))$ and γ is a perfect self-isometry of the principal block of $\mathrm{Aut}(SL_2(8))$.*

Proof. Let $G = P \times \mathrm{Aut}(SL_2(8))$, B the principal block of $\mathrm{Aut}(SL_2(8))$, and, with the labelling given by Magma,

$$\mathrm{Irr}(B) = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_{10}, \chi_{11}\};$$
$$\mathrm{Irr}(\mathcal{O}P \otimes_{\mathcal{O}} B) = \{\theta \otimes \chi_i | \theta \in \mathrm{Irr}(P), \chi_i \in \mathrm{Irr}(B)\}.$$

In particular: χ_1 is the trivial character; χ_2, χ_3 are linear characters; χ_4, χ_5, χ_6 are characters with degree 7; χ_{10} has degree 21 and χ_{11} has degree 27. Note that χ_7, χ_8, χ_9 all lie in different blocks. The characters of the projective indecomposable modules of $\mathcal{O}P \otimes_{\mathcal{O}} B$ are the following:

$$\chi_{pr_1} = \sum_{\theta \in \mathrm{Irr}(P)} \theta \otimes (\chi_1 + \chi_4 + \chi_{10} + \chi_{11}) \qquad \chi_{pr_2} = \sum_{\theta \in \mathrm{Irr}(P)} \theta \otimes (\chi_2 + \chi_6 + \chi_{10} + \chi_{11})$$

$$\chi_{pr_3} = \sum_{\theta \in \mathrm{Irr}(P)} \theta \otimes (\chi_3 + \chi_5 + \chi_{10} + \chi_{11}) \qquad \chi_{pr_4} = \sum_{\theta \in \mathrm{Irr}(P)} \theta \otimes (\chi_4 + \chi_5 + \chi_6 + \chi_{10} + 2\chi_{11})$$

$$\chi_{pr_5} = \sum_{\theta \in \mathrm{Irr}(P)} \theta \otimes (\chi_{10} + \chi_{11})$$

We proceed as before, so we check the decomposition matrix of B.

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

$\mathbb{D}$ has distinct rows and, by Proposition 2.2.1, we know that, if T is the perfect isometry induced by a Morita auto-equivalence, T acts by

$$T(\theta_i \otimes \chi_s) = \theta_{\sigma_s(i)} \otimes \chi_{\tau(s)}.$$

We clearly see then that χ_n must be fixed for $n \in \{10, 11\}$. If τ acts non trivially then it must act via an element of the group generated by the permutations

$$\{(1,2)(4,6), (1,3)(4,5), (2,3)(6,5)\}.$$

Doing exactly as in the previous propositions we fix $\theta \in \mathrm{Irr}(P)$ and we have

$$\begin{aligned} \theta_1\chi_{\tau(1)} + \theta_2\chi_{\tau(2)} + \theta_3\chi_{\tau(3)} + 7\left(\theta_4\chi_{\tau(4)} + \theta_5\chi_{\tau(5)} + \theta_6\chi_{\tau(6)}\right) \\ + 21\theta_{10}\chi_{10} + 27\theta_{11}\chi_{11} \in \mathrm{CF}(G, \mathcal{O}P \otimes_{\mathcal{O}} B, \mathcal{O}). \end{aligned} \tag{2.14}$$

We evaluate the previous expression to pairs (x, g), where x is an arbitrary element of P and g is an element that lies in one of the following conjugacy classes: $1A, 7A, 3A$. Note that the notation adopted here for conjugacy classes is the one adopted by Magma and, as usual, the number denotes the order of the elements in that conjugacy class. The action of τ can again be assumed to be trivial, since the following expressions would not be different.

$$\textbf{(x,1A)} \quad \theta_1(x)+\theta_2(x)+\theta_3(x)+\theta_4(x)+\theta_5(x)+\theta_6(x)+\theta_{10}(x)+\theta_{11}(x) \in 8\mathcal{O} \qquad (2.15)$$

$$\textbf{(x,7A)} \quad \theta_1(x)+\theta_2(x)+\theta_3(x)+5\theta_{11}(x) \in 8\mathcal{O} \qquad (2.16)$$

$$\textbf{(x,3A)} \quad \theta_1(x)+\theta_2(x)+\theta_3(x)+2\theta_4(x)+2\theta_5(x)+2\theta_6(x)-\theta_{10}(x) \in 8\mathcal{O} \qquad (2.17)$$

By expression (2.16) and Lemma 2.2.2 clearly follows that $\theta_1(x)=\theta_2(x)=\theta_3(x)=\theta_{11}(x)$. If the $\theta_i(x)$ were not all equal, then (2.15) would yield

$$\theta_4(x)+\theta_5(x)+\theta_6(x)+\theta_{10}(x)=-4\theta_{11}(x)$$

that means, looking at the complex norm, $\theta_4(x)=\theta_5(x)=\theta_6(x)=\theta_{10}(x)=-\theta_{11}(x)$. Therefore, by expression (2.17), we have $-2\theta_{11}(x)\in 8\mathcal{O}$, that is a contradiction. Thus the $\theta_i(x)$ are all the same and, by arbitrariness of x, we can conclude the proof as in the previous Proposition. □

Now we can finally state the result about Picard groups, through the application of Weiss' criterion.

Proposition 2.2.5. *Let P be an abelian 2-group. Then*

1. *If $B_1 = B_0\left(\mathcal{O}(P\times SL_2(8))\right)$ then $\mathrm{Pic}(B_1)=\mathcal{L}(B_1)=(P\rtimes \mathrm{Out}(P))\times C_3$*
2. *If $B_2 = B_0\left(\mathcal{O}(P\times \mathrm{Aut}(SL_2(8)))\right)$ then $\mathrm{Pic}(B_2)=\mathcal{L}(B_2)=(P\rtimes \mathrm{Out}(P))\times C_3$*

Proof. The proof for the two cases is completely analogous and follows the proof of [29, Theorem 4.7(b)]. In particular we will adopt the notation used there, that was introduced in Section 1.7 of this thesis. We start considering $SL_2(8)$.

Let $M\in \mathrm{Pic}(B)$, by [29, Lemma 3.5(b)] and Proposition 2.2.3 we can compose M with a Morita auto-equivalence induced by some element of $\mathrm{Aut}(\mathcal{O}P)$ such that the induced permutation Ψ of $\mathrm{Irr}(P\times SL_2(8))$ satisfies $\Psi(\theta\otimes\chi)=\theta\otimes\gamma(\chi)$, for all $\theta\in\mathrm{Irr}(P)$ and $\chi\in\mathrm{Irr}(SL_2(8))$; in particular this composition permutes $\mathrm{Irr}(B)^P$.

Suppose now that M does permute $\mathrm{Irr}(B)^P$. By Proposition 1.7.2, ${}^PM$ induces a Morita auto-equivalence of $B^P=B_0\left(SL_2(8)\right)$ and thus M^P has trivial source, by [29, Proposition 5.3]. But then by Proposition 1.7.2 again, M has trivial source as well, and we have shown that $\mathrm{Pic}(B_1)$ is generated by $\mathrm{Aut}(\mathcal{O}P)$ and $\mathcal{T}(B_1)$. Therefore, [29, Lemma 3.5(a)], implies $\mathrm{Pic}(B_1)=\mathcal{L}(B_1)$.

We now calculate $\mathcal{T}(B_1)$. By the proof of [29, Lemma 5.3] and [29, Lemma 2.3], $\mathrm{Out}_D(A)=1$, for A a source algebra for B_1 and D the defect group of B_1. Thus $\mathcal{T}(B_1)\leq$

$\mathrm{Out}(D,\mathcal{F}) \simeq \mathrm{Aut}(P) \times C_n$. All the elements of $\mathrm{Out}(D,\mathcal{F}) \simeq \mathrm{Aut}(P) \times C_n$ can however be realized and, noting that $\mathrm{Aut}(\mathcal{O}P) \cap \mathcal{T}(B_1) = \mathrm{Aut}(P)$, the result follows.

We now consider B_2. It can be analogously shown that $\mathrm{Pic}(B_2) = \mathcal{L}(B_2)$, so we won't repeat ourselves. We just calculate $\mathcal{T}(B_2)$. By [29, Proposition 5.4] and [29, Lemma 2.3] $\mathrm{Out}_D(A) \simeq C_3$, for A a source algebra for B_2 and D the defect group of B_2, and this is exactly the set of Morita auto-equivalences given by tensoring with the linear characters of $P \times \mathrm{Aut}\,(SL_2(8))$, with $P \times SL_2(8)$ in the kernel. By the proof of [29, Proposition 5.4], $\mathrm{Out}(D,\mathcal{F}) \simeq \mathrm{Aut}(P)$ and therefore, noting again that $\mathrm{Aut}(\mathcal{O}P) \cap \mathcal{T}(B_2) = \mathrm{Aut}(P)$, the result follows. □

We have tried to find a generalization of the factorization results, the following is a partial one.

Proposition 2.2.6. *Let $p = 2$ and G a finite group with $|G|_2 = 2^s$, for some $s \geq 2$, and abelian Sylow 2-subgroups. Let B be the principal block of G and suppose that the decomposition matrix of B has distinct rows. Suppose there is not a proper, non-empty, set $J \subseteq \mathrm{Irr}(B)$ such that*

$$2^{s-1} | \sum_{\chi \in J} \chi(g)\chi(h) \text{ in } \mathcal{O} \;\; \forall g \in G^0, \forall h \text{ non trivial 2-element.}$$

Then, if $M \in \mathrm{Pic}\,(B_0(\mathcal{O}C_2 \times G))$ and $\Phi \in \mathrm{Perf}\,(B_0(\mathcal{O}C_2 \times G))$ is induced by M, we have $\Phi = \sigma \otimes \Phi_B$ where σ is a permutation of $\mathrm{Irr}(C_2)$ and Φ_B is an element of $\mathrm{Perf}(B)$.

Proof. Since the decomposition matrix of B has distinct rows, Proposition 2.2.1 gives a partial factorization of Φ. In particular, we can define a map

$$\mathrm{Pic}((B_0(\mathcal{O}C_2 \times G)) \ni M \mapsto \Phi_G \in \mathrm{Perf}(B)$$

by $\Phi_G(\chi)(-) = \Phi(1_{C_2} \otimes \chi)(1,-)$.

We can now assume that Φ_G is the identity, up to composing Φ with $id \otimes \Phi_G^{-1}$. Note that this composition may not be induced by an element of the Picard group, but it would still be a perfect isometry with positive signs and therefore Proposition 2.2.1 applies. Define now the following sets

$$\mathrm{Fix}(\Phi) = \{\chi \in \mathrm{Irr}(B) | \Phi(1_{C_2} \otimes \chi) = 1_{C_2} \otimes \chi\}$$
$$\mathrm{Swap}(\Phi) = \{\chi \in \mathrm{Irr}(B) | \Phi(1_{C_2} \otimes \chi) = \delta \otimes \chi\},$$

where δ is the non-trivial irreducible character of C_2. These sets are a partition of $\mathrm{Irr}(B)$ and if $\chi \in \mathrm{Fix}(\Phi)$ then $\Phi(\delta \otimes \chi) = \delta \otimes \chi$. Now, if μ is the character of Φ and $x \in C_2$,

$g, h \in G$, we have

$$\begin{aligned}\mu((x,g)(x,h)) &= 2\sum_{\chi\in\mathrm{Fix}(\Phi)}\chi(g)\chi(h) - 2\sum_{\chi\in\mathrm{Swap}(\Phi)}\chi(g)\chi(h)\\ &= 2\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\chi(h) - 4\sum_{\chi\in\mathrm{Swap}(\Phi)}\chi(g)\chi(h)\\ &= 2\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\chi(h) - 4\sum_{\chi\in\mathrm{Fix}(\Phi)}\chi(g)\chi(h)\end{aligned}$$

In particular if g is 2-regular and h is a (non-trivial) p-element of G we have that

$$\mu((x,g)(x,h)) = -4\sum_{\chi\in\mathrm{Swap}(\Phi)}\chi(g)\chi(h) = 4\sum_{\chi\in\mathrm{Fix}(\Phi)}\chi(g)\chi(h).$$

Recall that, by the definition of perfect isometries, $C_{C_2\times G}((x,h))$ must divide $\mu((x,g)(x,h))$. Moreover, since $C_{C_2\times G}((x,h)) \geq C_2 \times P$, where P is the defect group of B containing h, then, recalling that $|P| = 2^s$, we have that

$$2^{s-1}|\sum_{\chi\in\mathrm{Fix}(\Phi)}\chi(g)\chi(h) = -\sum_{\chi\in\mathrm{Swap}(\Phi)}\chi(g)\chi(h). \tag{2.18}$$

Thus we must have that one among $\mathrm{Fix}(\Phi)$ and $\mathrm{Swap}(\Phi)$ is the entire block $\mathrm{Irr}(B)$, and then $\Phi = \sigma\otimes\Phi_B$ with $\sigma \in S_2$. Finally note that if $\mathrm{Fix}(\Phi) = \mathrm{Irr}(B)$ then σ is the trivial permutation, otherwise it is the non-trivial element of S_2. □

Showing that the hypotheses of the last proposition are satisfied obviously requires a detailed knowledge of the character table of the group G, but it can be shown that the condition on subsets of $\mathrm{Irr}(B)$ are less unnatural than they appear. We are going to address this matter shortly, but first we illustrate two examples of the application of the previous Proposition.

Proposition 2.2.7. *Let $G = A_4\times A_5$ and B its principal 2-block. If $J \subseteq \mathrm{Irr}(B)$ is non-trivial and such that*

$$2^3|\sum_{\chi\in J}\chi(g)\chi(h) \text{ in } \mathcal{O}\ \ \forall g\in G^0, \forall h \text{ non-trivial 2-element,} \tag{2.19}$$

then it must be $J = \mathrm{Irr}(B)$.

Proof. The proof can be performed computationally on Magma, checking that for each subset J with $|J| \leq \lfloor\frac{|\mathrm{Irr}(B)|}{2}\rfloor$ the statement holds. In fact, if $|J|$ is greater than $\lfloor\frac{|\mathrm{Irr}(B)|}{2}\rfloor$, we can consider $\mathrm{Irr}(B)\setminus J$, since from

$$\sum_{\chi\in\mathrm{Irr}(B)}\chi(g)\chi(h) = 0$$

it follows that the property holds for J if and only if it holds for $\mathrm{Irr}(B)\setminus J$. □

Proposition 2.2.8. *Let $n \geq 2$, $G = SL_2(2^n)$, and B its principal block. If $J \subseteq \mathrm{Irr}(B)$ is non-trivial and such that*

$$2^{n-1} | \sum_{\chi \in J} \chi(g)\chi(h) \text{ in } \mathcal{O} \ \forall g \in G^0, \forall h \text{ 2-element,} \tag{2.20}$$

then it must be $J = \mathrm{Irr}(B)$.

Proof. The character table of $SL_2(2^n)$ is well known and can be for example found in [91]. It is the following:

	$\mathbf{1_G}$	$\mathbf{c}$	$\mathbf{a}^l$	$\mathbf{b}^m$
1_G	1	1	1	1
ψ	2^n	0	1	-1
χ_i	$2^n + 1$	1	$\rho^{il} + \rho^{-il}$	0
θ_j	$2^n - 1$	-1	0	$-\sigma^{jm} - \sigma^{-jm}$

where $1 \leq i, l \leq 2^{n-1} - 1$; $1 \leq j, m \leq 2^{n-1}$; and ρ, σ are primitive $2^n - 1, 2^n + 1$-roots of unity. Note that $\psi \notin \mathrm{Irr}(B)$ and the conjugacy classes corresponding to 2-elements are just the trivial one and the one represented by c, while all the other conjugacy classes are represented by p-regular elements. Suppose without loss of generality that 1_G is in J, otherwise just take its complement in $\mathrm{Irr}(B)$, and define the following sets: $S_\theta := \{\theta_j\}_j$, $S_\chi := \{\chi_i\}_i$.

Taking $g = 1$ and $h = c$ in (2.20) yields that $|J \cap S_\theta| + |J \cap S_\chi|$ is equal to either $2^{n-1} - 1$ or $2^n - 1$. In fact, $|J \cap S_\theta| \leq 2^{n-1}$, $|J \cap S_\chi| \leq 2^{n-1} - 1$, and

$$\begin{aligned}
\sum_{\chi \in J} \chi(1)\chi(c) &= 1 + \sum_{\chi_i \in J} (2^n + 1) - \sum_{\theta_j \in J} (2^n - 1) \\
&= 1 + (2^n + 1)\,|J \cap S_\chi| - (2^n - 1)\,|J \cap S_\theta| \\
&\equiv 1 + |J \cap S_\chi| + |J \cap S_\theta| \pmod{2^{n-1}}
\end{aligned}$$

Suppose that J is a proper subset of $\mathrm{Irr}(B)$, so that the second eventuality can be ruled out. Thus

$$|J \cap S_\theta| + |J \cap S_\chi| = 2^{n-1} - 1.$$

Take this time $g = b^1$ and $h = c$ in (2.20). We have that

$$2^{n-1} | \ 1 + \sum_{\theta_j \in J} (2^n - 1)\left(\sigma^j + \sigma^{-j}\right) \equiv 1 + \sum_{\theta_j \in J} \left(\sigma^j + \sigma^{-j}\right) \pmod{2^{n-1}}.$$

Looking at the norm, it follows that either $|J \cap S_\theta| \geq 2^{n-2}$ or the sum is zero. If the sum is zero, then $J \cap S_\theta = S_\theta$, since σ is a primitive root of unity of order 2^{n+1}, and there would be a contradiction. In fact, we are working under the assumption that

$$|J \cap S_\theta| + |J \cap S_\chi| = 2^{n-1} - 1$$

and therefore $J \cap S_\theta = S_\theta$ would mean that the complement of J would be S_χ. Now, recall that, as remarked in the proof Proposition 2.2.7, property (2.20) holds for the set $\mathrm{Irr}(B) \setminus J$ as well. Thus taking $g = a^1$ and $h = c$ in (2.20) with $\mathrm{Irr}(B) \setminus J$ in place of J would yield

$$2^{n-1} | \sum_{i=1}^{2^{n-1}} \rho^i + \rho^{-i} = -1$$

Thus we must have $|J \bigcap S_\theta| \geq 2^{n-2}$. We can now use a very similar argument for $J \cap S_\chi$. Take $g = a^1$ and $h = c$ for J in (2.20). Then

$$2^{n-1} | \ 1 + \sum_{\chi_i \in J} (2^n + 1) \left(\rho^i + \rho^{-i}\right) \equiv 1 + \sum_{\chi^i \in J} \left(\rho^i + \rho^{-i}\right) \pmod{2^{n-1}}.$$

We can once again look at the complex norm and then it follows that either $|J \cap S_\chi| \geq 2^{n-2}$ or the sum is zero. If the sum is zero, then $J \cap S_\chi = S_\chi$, since ρ is a primitive root of unity of order $2^n - 1$, and we have another contradiction. In fact, analogously to what we shown for $J \cap S_\theta$, we would need to have that the complement of J is S_θ. But then, taking $g = b^1$, $h = c$ and $\mathrm{Irr}(B) \setminus J$ in (2.20) would follow

$$2^{n-1} | \sum_{i=1}^{2^{n-1}} \sigma^i + \sigma^{-i} = -1$$

So we must have $|J \cap S_\chi| \geq 2^{n-2}$. Recall that $|J \cap S_\theta| \geq 2^{n-2}$ as well. We supposed a few lines back that J is a proper subset of $\mathrm{Irr}(B)$, and it lead to the following equality:

$$|J \cap S_\theta| + |J \cap S_\chi| = 2^{n-1} - 1.$$

Thus we have a contradiction and J must be $\mathrm{Irr}(B)$. □

Corollary 2.2.9. *Let* $n \geq 2$, $B = B_0\left(\mathcal{O}(C_2 \times SL_2(2^n))\right)$, *then*

$$\mathrm{Pic}(B) = \mathcal{L}(B) = (C_2 \times C_n).$$

Proof. Since the Sylow 2-subgroup of $SL_2(2^n)$ has order 2^n, we can use Proposition 2.2.8 and Proposition 2.2.6 to get a factorization of perfect isometries induced by a Morita auto-equivalence of B. We can then use the same argument of Proposition 2.2.5(1.) to conclude the proof. □

Corollary 2.2.10. *Let $B = B_0(\mathcal{O}(C_2 \times A_4 \times A_5))$, then*

$$\mathrm{Pic}(B) = \mathcal{L}(B).$$

Proof. Since the Sylow 2-subgroups of $A_4 \times A_5$ has order 2^3, Proposition 2.2.7 and Proposition 2.2.6 yield a factorization of perfect isometries induced by Morita auto-equivalences. Note that it can be easily checked $\mathrm{Pic}(B_0(\mathcal{O}(A_4 \times A_5))) = \mathcal{T}(B)$ using Proposition 1.7.3. Therefore the argument is analogous to Proposition 2.2.5. □

We now want to briefly address the divisibility condition on subsets of irreducible characters introduced in the statement of Proposition 2.2.6. Recall that the condition we are interested in is the following.

Condition 2.2.11. *Let G be a finite group and B its principal block. Suppose that B has defect group P, and $|P| = p^s$, with s greater than 1. If $J \subseteq \mathrm{Irr}(B)$ is non-trivial and such that*

$$p^{s-1} | \sum_{\chi \in J} \chi(g)\chi(h) \text{ in } \mathcal{O} \ \ \forall g \in G^0, \forall h \text{ non-trivial } p\text{-elements}$$

then $J = \mathrm{Irr}(B)$.

This condition seems to be happening for every principal block we have considered, no matter if $p = 2$, G has abelian Sylow p-subgroup or if the block has other restrictions. However it doesn't hold for every non-principal block, in fact, as Gabriel Navarro pointed out in private correspondence, a non-principal block of $2.S_5$ provides a counterexample.

A natural question arises in this setting, yielding a condition that could be satisfied by a larger family of blocks.

Condition 2.2.12. *Let B be a block of G, and $|G|_p = p^s$. If $J \subseteq \mathrm{Irr}(B)$ is non-trivial and such that*

$$p^{s-1} | \sum_{\chi \in J} \chi(g)\chi(h) \text{ in } \mathcal{O}, \ \forall g \in G^0, \forall h \in G \setminus G^0,$$

then $J = \mathrm{Irr}(B)$.

The main reason that pushed us to this generalization of Condition 2.2.11 is its loose connection to Osima's theorem, of which we provide here a slight improvement that we weren't able to find in the literature.

Proposition 2.2.13. *Let B be a block of G, $|G|_p = p^s$. Suppose that $J \subseteq \mathrm{Irr}(B)$ is non-trivial and such that*

$$p^{s} | \sum_{\chi \in J} \chi(g)\chi(h) \text{ in } \mathcal{O} \ \ \forall g \in G^0, \forall h \in G \setminus G^0.$$

Then $J = \mathrm{Irr}(B)$. *In particular Osima's theorem ([68, Theorem 3]) holds, i.e. if*

$$\sum_{\chi \in J} \chi(g)\chi(h) = 0 \; \forall g \in G^0, \, \forall h \in G \setminus G^0,$$

then $J = \mathrm{Irr}(B)$.

Proof. From the hypothesis follows immediately that

$$\frac{1}{|G|} \sum_{\chi \in J} \chi(1)\chi(h) \in \mathcal{O}, \; \forall\, h \in G \setminus G^0.$$

If g is p-regular and P is a Sylow p-subgroup of G we have that

$$\begin{aligned} \mathcal{O} \ni \sum_{\chi \in J} \chi(g) \left[\chi \downarrow_P^G, 1_P\right]_P &= \frac{1}{|P|} \sum_{x \in P} \sum_{\chi \in J} \chi(g)\chi(x) \\ &= \frac{1}{|P|} \sum_{\chi \in J} \chi(g)\chi(1) + \frac{1}{|P|} \sum_{x \in P \setminus \{1\}} \sum_{\chi \in J} \chi(g)\chi(x). \end{aligned}$$

By the hypothesis on J, for all $x \in P \setminus \{1\}$ we have

$$\frac{1}{|P|} \sum_{\chi \in J} \chi(g)\chi(x) \in \mathcal{O}$$

Therefore $\frac{1}{|P|} \sum_{\chi \in J} \chi(g)\chi(1)$ is in $\mathcal{O}$ as well. Then

$$\frac{1}{|G|} \sum_{\chi \in J} \chi(1)\chi(h) \in \mathcal{O} \; \forall h \in G.$$

So

$$\sum_{\chi \in J} e_\chi = \sum_{\chi \in J} \frac{1}{|G|} \sum_{g \in G} \chi(1)\chi(g) g \in Z(\mathcal{O}G)$$

and thus, by [67, Theorem 3.9], we conclude that $J = \mathrm{Irr}(B)$. □

Obviously Condition 2.2.12 requires a stronger divisibility condition than the hypothesis in the previous theorem, but maybe it could still be satisfied by a reasonably large class of blocks. Finally, we remark that the result we just proved suggests that a stronger version of Harada's conjecture ([38]) may hold in general, namely:

Conjecture 2.2.14 (Strong Harada)**.** *Let* G *be a finite group and* B *a block of* G. *If* $J \subseteq \mathrm{Irr}(B)$ *is non-trivial and*

$$|G|_p \mid \sum_{\chi \in J} \chi(1)\chi(h) \text{ in } \mathcal{O}, \; \forall\, h \in G \setminus G^0$$

then $J = \mathrm{Irr}(B)$. *In particular, Harada's conjecture holds for* B, *i.e, if*

$$\sum_{\chi \in J} \chi(1)\chi(h) = 0, \; \forall h \in G \setminus G^0$$

then $J = \mathrm{Irr}(B)$.

Remark 2.2.15. In Conjecture 2.2.14 $|G|_p$ cannot be replaced by the order of a defect group for B. In fact, as Gabriel Navarro pointed out in private correspondence, the non-principal 2-block of A_7 would be a counterexample.

2.3 Picard groups for principal 2-blocks of Suzuki groups

This section is devoted to calculating Picard groups for principal 2-blocks of Suzuki groups. Since we are dealing with an infinite family of blocks, we can't employ a computational approach. However these blocks still admit a stable Morita equivalence induced by the Green correspondence. Therefore it is possible to deduce information on the Picard group by looking at the stable module category, and the stable Picard group. Some of the results included in this section appeared in a preprint available on arXiv, [64].

2.3.1 Stable auto-equivalences of Morita type

Recall that the stable category of $\mathrm{Mod}(A)$, where A is a k-algebra, is the category $\underline{\mathrm{Mod}}(A)$ whose objects are the A modules and, for any two A-modules U, V, morphisms are given by the quotients space

$$\underline{\mathrm{Hom}}_A(U,V) = \mathrm{Hom}_A(U,V)/\,\mathrm{Hom}_A^{pr}(U,V),$$

where $\mathrm{Hom}_A^{pr}(U,V)$ are the A-homomorphisms that factor through a projective A-module. In this subsection we are going to completely describe the stable Picard group for some group algebras, $\underline{\mathrm{Pic}}(kG)$, whose elements are the isomorphism classes in stable category of kG-kG-bimodules that induce a stable auto-equivalence of Morita type of kG. In general it is not known how this group is related to the Picard group of kG, but we will show that, in some cases, $\underline{\mathrm{Pic}}(kG)$ is generated by $\mathrm{Pic}(kG)$ and a cyclic, central, subgroup.

The aim of this section is to calculate the Picard group for principal 2-blocks of Suzuki groups. Such blocks have TI Sylow 2-subgroups P by [81] and, by [55, Proposition 5.2.5], there is a stable equivalence of Morita type given by induction and restriction, that we denote

$$\Psi : \underline{\mathrm{Mod}}(B) \xrightarrow{\sim} \underline{\mathrm{Mod}}\left(k(N_G(P))\right).$$

The stable Picard groups of B and $kN_G(P)$ are then isomorphic, so this is the ultimate reason for being interested in the stable Picard group of $N_G(P)$. The following result is an adaptation of [18, Lemma 3.1] to a slightly different setting. We are going to make use of endo-trivial modules and $T(G)$ will always denote the group of endo-trivial modules of kG.

We recall that a kG-module V is endo-trivial if $V^* \otimes_k V \simeq k$ in $\underline{\mathrm{Mod}}(kG)$. The set of endo-trivial kG-modules has a natural group structure induced by the tensor product $\otimes_k$.

A complete account of the topic can be found in [65]. Before stating the result we need a couple more definitions and basic properties.

Proposition 2.3.1 ([78], Theorem 3.10)**.** *Let $\mathbb{F}_p$ be the field with p-elements. A finite group G has periodic cohomology if for all primes $p \mid |G|$ one of the following equivalent statements holds:*

(i) Every Sylow p-subgroup of G is either cyclic or generalized quaternion 2-group.

(ii) $\Omega^d_{\mathbb{F}_pG}(\mathbb{F}_p) \simeq \mathbb{F}_p$ for some $d \geq 1$.

(iii) $H^{i+d}(G, \mathbb{F}_p) \simeq H^i(G, \mathbb{F}_p)$ for some $d \geq 1$, all $i \geq 1$.

Definition 2.3.2 ([18])**.** Let G be a finite group. We denote by $\mathcal{T}$ the *thick subcategory of* $\underline{\text{mod}}(kG)$ *generated by the trivial module k.* The objects of $\mathcal{T}$ are the same of the smallest full subcategory of $\underline{\text{mod}}(kG)$ containing the Heller translates of k and closed under taking extensions and direct summands.

Proposition 2.3.3. *Let $G = P \rtimes E$ with E a cyclic p'-group, P a non-abelian p-group with non-periodic cohomology. Denote by $T(P)$ the group of endo-trivial kP-modules and suppose that the following hold:*

- *$T(P) \simeq \mathbb{Z}$ corresponds to the Heller translates of the trivial kP-module*
- *$\mathcal{T}$ is the full stable category of kG.*

If $F : \underline{\text{Mod}}(kG) \to \underline{\text{Mod}}(kG)$ is a stable equivalence of Morita type, then there is $\sigma \in \text{Sym}(\text{Hom}(E, k^\times))$ and an integer n such that $\forall\, V \in \text{Hom}(E, k^\times)$ (identifying the characters of E with the underlying modules) we have $\Omega^{-n}F(V) \simeq \sigma(V)$ in $\underline{\text{Mod}}(kG)$.

Proof. Let $V \in \text{Hom}(E, k^\times)$, then V is an endo-trivial kG-module, thus $F(V)$ is endo-trivial as well, by [18, Proposition 2.2]. But then its restriction to P is endo-trivial by [17, Proposition 2.6], thus there is an integer n such that $F(V)$ is a direct summand of

$$\text{Ind}_P^G(\Omega^n_{kP}(k) + \text{proj}) \simeq \bigoplus_{U \in \text{Hom}(E,k^\times)} \Omega^n_{kG} U + \text{proj},$$

In particular $F(V) \simeq \Omega^n_{kG}U$ for some $U \in \text{Hom}\,(E, k^\times)$. So there are two functions, $n : \hat{E} \to \mathbb{Z}$, $\sigma : \hat{E} \to \hat{E}$, such that

$$F(V) \simeq \Omega^{n(V)}_{kG}\sigma(V).$$

We want to show that σ is a permutation. Suppose $\sigma(V) = \sigma(V')$, then

$$F(V') = \Omega^{n(V')}_{kG}\sigma(V') = \Omega^{n(V')-n(V)}_{kG}\Omega^{n(V)}_{kG}\sigma(V) = \Omega^{n(V')-n(V)}_{kG}F(V)$$

Since Heller translates commute with stable equivalences of Morita type, by [55, Proposition 2.14.6], then $V' = \Omega_{kG}^{n(V')-n(V)}V$. Suppose that $V \neq V'$, then $n(V') \neq n(V)$. Thus, after restriction to P, we have $k = \Omega_{kP}^{n(V)-n(V')}(k)$. But this is a contradiction by Proposition 2.3.1, since P has not periodic cohomology. Therefore $V = V'$ and σ is a permutation.

Take now $W \in \mathrm{Hom}\,(E, k^\times)$ with $n(W)$ minimal. Define $h : \mathrm{Hom}\,(E, k^\times) \to \mathbb{Z}[[t, t^{-1}]]$ by the following power series:

$$h(V) = \sum_{n\in\mathbb{Z}} \dim\left(\mathrm{Ext}_{kG}^n(k, V)\right) t^n.$$

Proceeding exactly as in [18, Lemma 3.1] we get the desired result. In fact,

$$\begin{aligned}\mathrm{Ext}_{kG}^i\left(k, VW^{-1}\right) &\simeq \mathrm{Ext}_{kG}^i\left(W, V\right) \simeq \mathrm{Ext}_{kG}^i\left(F(W), F(V)\right)\\ &\simeq \mathrm{Ext}_{kG}^{i+n(W)-n(V)}\left(k, \sigma(V)\sigma(W)^{-1}\right),\end{aligned}$$

hence, if we define a bijection $\tau : \mathrm{Hom}\,(E, k^\times) \to \mathrm{Hom}\,(E, k^\times)$ by $\tau(V) = \sigma(VW)\sigma(W)^{-1}$,

$$h\left(VW^{-1}\right) = t^{n(V)-n(W)} h(\sigma(V)\sigma(W)^{-1}).$$

Moreover, using VW in place of V, we have

$$h\left(V\right) = t^{n(VW)-n(W)} h(\sigma(VW)\sigma(W)^{-1}) = t^{n(VW)-n(W)} h(\tau(V)). \tag{2.21}$$

Fix $V \in \mathrm{Hom}\,(E, k^\times)$, then we can find an integer s such that $\tau^s(V) = V$. Thus, by repeated use of (2.21),

$$h(V) = t^{(n(VW)-n(W))+(n(\tau(V)W)-n(W))+\cdots+\left(n(\tau^{s-1}(V)W)-n(W)\right)} h(V). \tag{2.22}$$

We now claim that $h(V)$ is not periodic as a power series, i.e. there is not a positive integer d for which

$$h(V) = t^d h(V).$$

Suppose there was such a d, then $\dim\left(\mathrm{Ext}_{kG}^i(k, V)\right) = \dim\left(\mathrm{Ext}_{kG}^{i+d}(k, V)\right)$. In particular, $\dim\left(\mathrm{Ext}_{kG}^i(k, V)\right)$ are uniformly bounded. Now V is in $\mathcal{T}$, but then k in is the thick subcategory generated by V as well, since we can consider the equivalence given by tensoring with V. Therefore k can be obtained as a direct summand of a module U obtained by taking a sequence of extensions of $\Omega_{kG}^n(V)$ for various n. Any such module U would necessarily have the dimensions of $\mathrm{Ext}_{kG}^i(k, U)$ uniformly bounded. Then, the dimensions of $\mathrm{Ext}_{kG}^i(k, k)$ would be bounded as well. Since $\mathcal{T}$ is actually the whole stable module category, we can analogously show that the dimensions of $\mathrm{Ext}_{kG}^i(k, W)$ are bounded for each finitely generated module W. However, by [2, Theorem 1], this means that k is a periodic module, and this is a contradiction. Therefore $h(V)$ is not periodic. Note that $n(W)$ is minimal, thus $n\left(\tau^i(V)W\right) - n(W) \geq 0$ for all i. Then (2.22) implies

$$(n(VW) - n(W)) + (n(\tau(V)W) - n(W)) + \cdots + \left(n(\tau^{s-1}(V)W) - n(W)\right) = 0$$

and $n(VW) = n(W)$. This happens for all $V \in \operatorname{Hom}(E, k^\times)$ and then n is constant. □

Corollary 2.3.4. *Let P a non-abelian p-group, $H \simeq P \rtimes E$ such that:*

(i) P has non-periodic cohomology.

(ii) $T(P) \simeq \mathbb{Z}$.

(iii) E is a cyclic p'-group.

(iv) $\mathcal{T}$ is the full stable category of kH

Then

$$\underline{\operatorname{Pic}}(\mathcal{O}H) = \operatorname{Pic}(\mathcal{O}H) \cdot \langle \Omega_{\mathcal{O}(H \times H^0)} \mathcal{O}H \rangle.$$

Proof. We obviously have a map $\underline{\operatorname{Pic}}(\mathcal{O}H) \to \underline{\operatorname{Pic}}(kH)$, and the Kernel of this map is contained in $\operatorname{Pic}(\mathcal{O}H)$. In fact, let M be an indecomposable $\mathcal{O}H$-$\mathcal{O}H$-bimodule in $\underline{\operatorname{Pic}}(\mathcal{O}H)$. Then $k \otimes_{\mathcal{O}} M$ trivial in $\underline{\operatorname{Pic}}(kH)$ means that $k \otimes_{\mathcal{O}} M \simeq kH \oplus P$, where P is projective. However, since M is indecomposable, $P = 0$ by [54, Lemma 2.6]. So $k \otimes_{\mathcal{O}} M \simeq kH$, hence

$$k \otimes_{\mathcal{O}} (M \otimes_{\mathcal{O}H} M^*) \simeq kG$$

Since $M \otimes_{\mathcal{O}H} M^* \simeq \mathcal{O}G \oplus X$, for some projective module X, we get $X = 0$ and then $M \in \operatorname{Pic}(\mathcal{O}H)$. In particular, we need to prove the result just over k.

We want to prove then that

$$\underline{\operatorname{Pic}}(kH) = \operatorname{Pic}(kH) \cdot \langle \Omega_{k(H \times H^{\mathcal{O}})} kH \rangle.$$

Consider M a kH-kH-bimodule in $\underline{\operatorname{Pic}}(\mathcal{O}H)$. By Proposition 2.3.3 there is an n such that if $M_1 = \Omega_{kH}^{-n} M = \Omega_{k(H \times H^{\mathrm{op}})}^{-n} kH \otimes_{kH} M$, then $M_1 \otimes_{kH} -$ permutes the simple kH-modules in the stable category, i.e. up to projectives. But then by [53, Proposition 2.4] we can consider (uniquely) an indecomposable summand M_2 of M_1 that actually permutes the simple kH-modules in the module category. This means, by [53, Proposition 2.5], that $M_2 \in \operatorname{Pic}(kH)$. □

2.3.2 Picard groups

For the next results we use properties of Suzuki groups and their Sylow 2-subgroups that are mainly available in Suzuki's original paper [80]. We will denote Suzuki groups with the Atlas notation, i.e. ${}^2B_2(q)$, where $n \geq 1, m = 2n+1, q = 2^m$.

Proposition 2.3.5. *Let*

$$L = {}^2B_2(q),\ P \in \mathrm{Syl}_2(L).$$

Then $G = N_L(P)$ satisfies the hypotheses of Proposition 2.3.3.

Proof. P is a non-abelian 2-group of order q^2, and is usually called in the literature a Suzuki 2-group of type A. Detailed information on these groups can be found in [39] and obviously [80]. However they are not generalized quaternion 2-groups, therefore P has non-periodic cohomology. It is also well known that P is a special group, i.e

$$Z(P) = \Phi(P) = P',$$

and $Z(P) \simeq C_2^{2n+1}$. Moreover all involutions of P are central, so we conclude by [19, Corollary 1.3] that the group of endo-trivial kP-modules is $T(P) \simeq \mathbb{Z}$, generated by $\Omega^1(k)$.

For what concerns $G = N_L(P)$, we have that $G \simeq P \rtimes C_{2^{2n+1}-1} = P \rtimes \langle \xi \rangle$. By [80, Lemma 5], conjugation by ξ^s permutes transitively the involutions of G and acts with no non-trivial fixed points on P for $s \geq 1$. Thus for any involution $x \in G$ it holds that $C_G(x) = P$ is (2)-nilpotent. It then follows by [9, Proposition 4.1] that $\mathcal{T}$ is the full stable category of kG. □

We are now ready to compute the Picard groups. Note that we will actually show that $\mathrm{Pic}(B) = \mathcal{T}(B)$, which obviously implies $\mathrm{Pic}(B) = \mathcal{E}(B)$, since $\mathcal{T}(B) \leq \mathcal{E}(B) \leq \mathrm{Pic}(B)$.

Theorem 2.3.6. *Let $q = 2^m$, with m an odd integer greater than 2, $G = {}^2B_2(q)$, and B be the principal 2-block of $\mathcal{O}G$. Then*

$$\mathrm{Pic}(B) = \mathcal{T}(B) \simeq C_m$$

Proof. Let P be a Sylow p-subgroup of G and $H = N_G(P)$. We claim that $\underline{\mathrm{Pic}}(\mathcal{O}H) = \underline{\mathcal{E}}(\mathcal{O}H)$, where

$$\underline{\mathcal{E}}(\mathcal{O}H) = \{M \in \underline{\mathrm{Pic}}(\mathcal{O}H) |\, M \text{ has endo-permutation source}\}$$

First we prove that $\mathrm{Pic}(\mathcal{O}H) = \mathcal{T}(\mathcal{O}H)$, i.e. all Morita auto-equivalences of $\mathcal{O}H$ are given by trivial source bimodules. Recall that

$$H \simeq P \rtimes C_{2^q-1} =: P \rtimes E,$$

and by [80, Lemma 5] the action of E on P is free. Then the irreducible characters of H are either irreducible characters of H/P or induced from irreducible characters of P. Moreover, if $\chi \in \mathrm{Irr}(H)$ is induced from a character of P, then χ is zero outside P. Thus

$$\{\chi \in \mathrm{Irr}(H) \mid P \leq \mathrm{Ker}(\chi)\} = \{\chi \in \mathrm{Irr}(H) \mid \chi \text{ is a lift of } \psi \in \mathrm{IBr}(H)\}.$$

So, by Proposition 1.7.3, it follows that $\mathrm{Pic}(\mathcal{O}H) = \mathcal{T}(\mathcal{O}H)$.

Now by Corollary 2.3.4 and Proposition 2.3.5, we have that elements of $\underline{\mathrm{Pic}}(\mathcal{O}H)$ are either bimodules inducing a Morita auto-equivalence of $\mathcal{O}H$ or Heller translate of those. Since Heller translates of bimodules with trivial source have endo-trivial source, we can immediately conclude that $\underline{\mathrm{Pic}}(\mathcal{O}H) = \underline{\mathcal{E}}(\mathcal{O}H)$.

It is well known that the Sylow 2-subgroups of G are TI-groups, thus induction and restriction give a stable equivalence of Morita type between $\mathcal{O}H$ and B. In particular there is an $\mathcal{O}H$-B-bimodule M, with trivial source and diagonal vertex ΔP, such that

$$\Psi : M \otimes_B - \otimes_B M^* : \underline{\mathrm{Pic}}(B) \to \underline{\mathrm{Pic}}(\mathcal{O}H)$$

is an injective group homomorphism. We know that H has non-periodic cohomology, thus $\langle \Omega_{\mathcal{O}(H\times H^0)}\mathcal{O}H\rangle$ is an infinite central cyclic subgroup of $\underline{\mathrm{Pic}}(\mathcal{O}H)$, by [55, Proposition 2.17.7]. But then all finite subgroups of $\underline{\mathrm{Pic}}(\mathcal{O}H)$ are contained in $\mathrm{Pic}(\mathcal{O}H)$, and in particular this holds for the image of the map Ψ as well. Thus, since $\mathrm{Pic}(B)$ is finite by Theorem 1.6.6, we have an injective map $\mathrm{Pic}(B) \hookrightarrow \mathrm{Pic}(\mathcal{O}H)$, that maps $\mathrm{Out}_P(A)$ to $\mathrm{Out}_P(\mathcal{O}H)$, that is isomorphic to $\mathrm{Hom}(E, k^\times)$.

We already observed that $\mathrm{Pic}(\mathcal{O}H) = \mathcal{T}(\mathcal{O}H)$, and that Ψ is induced by tensoring with trivial source bimodules, so $\mathrm{Pic}(B) = \mathcal{T}(B)$.

Using the exact sequence given in Theorem 1.6.11 we have

$$1 \longrightarrow \mathrm{Out}_P(A) \longrightarrow \mathcal{T}(B) \longrightarrow \mathrm{Out}(P, \mathcal{F})$$

Now, the trivial character is the only lift of a Brauer character. In fact, the set of character degrees of G is, by [80, Theorem 5]

$$cd(G) = \left\{1, q^2, q^2+1, (q-r+1)(q-1), (q+r+1)(q-1), r(q-1)/2\right\}$$

where $r^2 = 2q$. While, by [21, Theorem 3.2] the simple kG-modules have dimension

$$cd_{\mathrm{Br}}(G) = \left\{1, 4, 4^2, ..., 4^m = q^2\right\}$$

It is then immediate seeing that all elements in $\mathrm{cd}(G)$ are not zero modulo 4, with exception of the degrees $r(q-1)/2$ and q^2. However, $r(q-1)/2$ is not a power of 4 and there is just one irreducible character ψ with degree q^2, the Steinberg character, and $\psi \notin \mathrm{Irr}(B)$. Since there is just one linear simple kG-module (the trivial one) and one linear character, we then conclude that the trivial character of B is the only lift of a Brauer character of B.

Thus we must have that $\mathrm{Out}_P(A) = 1$, otherwise a non-trivial element M of $\mathrm{Out}_P(A)$ would correspond via Ψ to a non-trivial element of $\mathrm{Hom}(E, k^\times)$ that fixes the character 1_H. Since multiplication by a non-trivial linear character does not fix the trivial character, M would be sent to the trivial element of $\mathrm{Pic}(\mathcal{O}H)$, and, by injectivity of Ψ, we would have a contradiction.

All we need to understand is then the "fusion part" of $\mathcal{T}(B)$. Since B is the principal block, $\mathcal{F} = \mathcal{F}_P(G)$. Moreover, by [14, Theorem A], $\mathrm{Out}(P, \mathcal{F}) \simeq \mathrm{Out}(G)$. Automorphisms of Suzuki groups are well known and they are just the *field automorphisms* of the field with q elements, by [87] for example. In particular, $\mathrm{Out}(G) \simeq C_m = \langle\sigma\rangle$. Finally, since elements of $\mathrm{Out}(G)$ induce an outer automorphism of D, it follows that ${}_\sigma B$ is isomorphic to a direct summand of $\mathcal{O}_{\Delta\sigma} \uparrow_{\Delta\sigma}^{G\times G}$ and therefore it has vertex $\Delta\sigma$. Moreover, since σ is not induced by an element of H, $\Delta\sigma$ is not conjugated to ΔD in G and then ${}_\sigma B$ is a non-trivial element of $\mathrm{Pic}(B)$. Thus $\mathrm{Pic}(B) = \mathcal{T}(B) \simeq C_m$. □

2.4 Picard groups for blocks with defect group Q_8

This last section outlines the calculation of Picard groups for blocks with defect group Q_8, thanks to the application of Weiss' criterion, one of the most powerful tools in the study of Picard groups that has been introduced in Section 1.7. Blocks with defect group Q_8 were first classified over k in [32] and then over $\mathcal{O}$ by [40] and [30]. In particular, a list of representatives of Morita equivalence classes of blocks with defect group Q_8 is given by $\mathcal{O}Q_8$, $B_0(\mathcal{O}\mathrm{SL}_2(3))$ and $B_0(\mathcal{O}\mathrm{SL}_2(5))$.

Weiss' criterion and its application have been crucial for obtaining results for blocks with normal defect groups, as will be clear from Chapter 4, but they have revealed themselves to be useful also when dealing with blocks of groups G and defect group D for which $O_p(G) < D$. The following two results clearly give a concrete example of this.

Proposition 2.4.1. *Let $G = \mathrm{SL}_2(3) \simeq Q_8 \rtimes C_3$, $B = B_0(\mathcal{O}G)$. Then*

$$\mathrm{Pic}(B) = \mathcal{T}(B) \simeq S_3.$$

Proof. Let $P = Z(G)$. Note that all elements of $\mathrm{Pic}(B)$ permute the elements of $\mathrm{Irr}(B)^P$, in fact $\mathrm{Irr}(B) = \{1_1, 1_2, 1_3, 2_1, 2_2, 2_3, 3\}$ and the only characters with P in their kernel are the three lifts of Brauer characters and 3, whose restriction to p-regular elements is the sum of the three Brauer characters. Thus, by Proposition 1.7.2, if $M \in \mathrm{Pic}(B)$ then $M^P \in \mathrm{Pic}(B^P)$. But B^P is just the principal block of A_4 and, by [10, Theorem 1.5], $\mathrm{Pic}(B_0(A_4)) = \mathcal{T}(B_0(A_4))$, so $M^P \in \mathcal{T}(B^P)$ and $M \in \mathcal{T}(B)$.

We have just proved $\mathrm{Pic}(B) = \mathcal{T}(B)$, now we need to determine the isomorphism type of $\mathcal{T}(B)$, using the exact sequence in Theorem 1.6.11. Since B has normal defect group D, we can immediately compute this. In fact, by [75, Theorem 3.19] $\mathcal{F}$ has no $\mathcal{F}$-essential p-subgroups and then, by Alperin's fusion theorem (Theorem 1.5.10), all morphisms in $\mathcal{F}$ are restrictions of automorphisms of D. It can be then shown, as in Proposition 1.5.12, that $\mathrm{Out}(D, \mathcal{F}) = N_{\mathrm{Out}(D)}(E)/E$, where $E \simeq C_3$ is the inertial quotient of B. Therefore it can be finally checked that $\mathrm{Out}(D, \mathcal{F}) \simeq C_2$.

The non-trivial automorphism can be realized in the following way. Use the common i, j, k notation for Q_8, and then E is generated by the automorphism

$$\begin{aligned} \sigma : & i \mapsto j \\ & j \mapsto k \\ & k \mapsto i \end{aligned}$$

Now the automorphism

$$\begin{aligned} \tau : & i \mapsto -j \\ & j \mapsto -i \\ & k \mapsto -k \end{aligned}$$

satisfies ${}^{\tau}\sigma = \sigma^2$, and we have just defined a non-trivial element of $N_{\mathrm{Out}(D)}(E)$ not in E. Obviously this last automorphism induces a Morita auto-equivalence of B.

Since $\mathrm{Out}_D(A) \simeq \mathrm{Hom}\,(E, k^\times) \simeq C_3$ is as always given by multiplication with a linear character, then $\mathrm{Pic}(B) \simeq S_3$. In fact, τ swaps the non-trivial linear characters of G and therefore doesn't commute with $\mathrm{Out}_D(A)$. □

Proposition 2.4.2. *Let $G = \mathrm{SL}_2(5) \simeq C_2.A_5$, $B = B_0(\mathcal{O}G)$. Then*

$$\mathrm{Pic}(B) = \mathcal{T}(B) \simeq C_2.$$

Proof. Let $P = Z(G)$. All elements of $\mathrm{Pic}(B)$ permute $\mathrm{Irr}(B)^P$ again. To see this, we write down the decomposition matrix of B:

$$\mathbb{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

We recall that irreducible characters and Brauer characters of B are given by

$$\mathrm{Irr}(B) = \{1, 2_1, 2_2, 3_1, 3_2, 5, 6\}, \quad \mathrm{IBr}(B) = \{1, 2_1, 2_2\}$$

and that the rows and columns of $\mathbb{D}$ are ordered according to how we listed irreducible characters and Brauer characters of B. It can be checked with the character table of G that characters in $\mathrm{Irr}(B)^P$ correspond to the rows $1, 4, 5, 6$ and we immediately see that they must be permuted by a Morita auto-equivalence of B. Let M be an element of $\mathrm{Pic}(B)$, then, by Proposition 1.7.2, $M^P \in \mathrm{Pic}(B^P)$. However, the only block dominated by B is the principal block of A_5 and, by [10, Theorem 1.5], $\mathrm{Pic}(B_0(A_5)) = \mathcal{T}(B_0(A_5))$, thus $M \in \mathcal{T}(B)$.

We have again to describe the isomorphism type of $\mathrm{Pic}(B)$. Note that by [56, Proposition 6.4.17], B is its own source algebra. In fact, simple $\overline{B}$-modules are either the trivial module or they have dimension two. Since D acts non-trivially on the 2-dimensional simple $\overline{B}$-modules, their restriction to D must be indecomposable and then $\overline{B}$ is its own source algebra.

Since G is a central extension of A_5 by C_2, we are going to show that any φ in $\mathrm{Out}_D(B)$ induces an outer automorphism of $B_0(\mathcal{O}A_5)$, that fixes pointwise a Sylow subgroup of A_5.

We first have to show that elements with the same reduction via the map $\pi : \mathcal{O}G \to \mathcal{O}(G/P)$ are sent to the same element by $\pi \circ \varphi$, where for the moment we assume that $\varphi \in \mathrm{Out}_D(\mathcal{O}G)$. Take a transversal T for $P = \langle x \rangle$, and consider two elements $m = \sum_{g \in T} \alpha_g g + \sum_{g \in T} \alpha_{gx} gx$, $n = \sum_{g \in T} \beta_g g + \sum_{g \in T} \beta_{gx} gx$ such that

$$\alpha_g + \alpha_{gx} = \beta_g + \beta_{gx}$$

for all $g \in T$, that is equivalent to $\pi(m) = \pi(n)$. Since $\varphi(gx) = \varphi(g)x$, it follows

$$\varphi\left(\sum \alpha_g g + \sum \alpha_{gx} gx\right) = \sum \alpha_g \varphi(g) + \sum \alpha_{gx} \varphi(g) x,$$

and then $\pi(\varphi(m)) = \pi(\varphi(n))$. So we have just defined a map $\overline{\varphi} : \mathcal{O}A_5 \to \mathcal{O}A_5$, where we identify $\mathcal{O}A_5$ with $\mathcal{O}(G/P)$. This is an algebra isomorphism (we can obviously define an inverse) and fixes Q, a Sylow subgroup of A_5, pointwise. We turn now our attention to blocks. Let b, c be the block idempotents corresponding, respectively, to B, and to the principal block of A_5. Since we can naturally extend π to a map from B to $B_0(A_5)$ by defining $\pi(b) = c$, the previous construction can be immediately adapted to blocks. Thus $\varphi \in \mathrm{Out}_D(B)$ induces an element $\overline{\varphi}$ of $\mathrm{Out}_Q(B_0(A_5))$.

However we already know that $\mathrm{Out}_Q(B_0(A_5)) = 1$, so the induced automorphism $\overline{\varphi}$ of $B_0(A_5)$ is inner. We can obviously lift $\overline{\varphi}$ to an inner automorphism of B and compose it with φ to get an automorphism $\hat{\varphi}$ of $\mathrm{Out}_D(B)$ for which the induced automorphism on $B_0(A_5)$ is the trivial one. With no loss of generality we can then assume that $\hat{\varphi} = \varphi$ and $\overline{\varphi}$ is the trivial automorphism of $B_0(A_5)$.

Now, for all $g \in G$,

$$gPc = \overline{\varphi}\,(gPc) = \pi\,(\varphi(gb)) = \tilde{\varphi}(g)Pc,$$

where $\varphi(gb) = \tilde{\varphi}(g)b$ for some automorphism $\tilde{\varphi}$ of $\mathcal{O}G$. Therefore there are $\alpha, \beta \in \mathcal{O}$ (that depend on g), for which

$$\varphi(gb) = \alpha gb + \beta gxb$$

In particular, $\alpha + \beta = 1$. Take now g an element of order 4 of G, then, since all elements of order 2 are central and φ fixes P, $\varphi(g^2b) = g^2b$. Thus

$$(\alpha gb + \beta gxb)^2 = g^2b.$$

Obviously this means that $(\alpha b + \beta xb)^2 = b$. Recall that $\alpha + \beta = 1$, and therefore

$$0 = (\alpha b + \beta xb)^2 - (\alpha b + \beta b)^2 = \alpha^2 b + \beta^2 b + 2\alpha\beta xb - \alpha^2 b - \beta^2 b - 2\alpha\beta b.$$

So $\alpha\beta = 0$ and then gb is either mapped to gxb or to itself. This happens for all elements of G with order 4, but note that it could happen that for some $g_1, g_2 \in G$ with $|g_1| = |g_2| = 4$, g_1b is mapped to itself and g_2b is mapped to g_2xb.

Now, it can be checked that all non-central elements whose order is a power of 2 have order 4, and therefore the subgroup L of G generated by elements of order 4 is exactly the subgroup generated by the Sylow 2-subgroups of G. In particular, L is a normal subgroup of G properly containing P, and it therefore is the whole group G. Thus, for every $g \in G$, it holds $\varphi(gb) = gb$ or $\varphi(gb) = gxb$. We can then define a morphism

$$\eta : G \to C_2 \text{ where } \eta(g) = \begin{cases} x & \text{if } \varphi(gb) = gxb \\ 1 & \text{if } \varphi(gb) = gb \end{cases}$$

If η is surjective then G has a normal subgroup of index 2 and this is clearly a contradiction. Therefore $\varphi(gb) = gb$ for all $g \in G$ and thus $\mathrm{Out}_D(B) = 1$.

Now, it is well known that the fusion systems on Q_8 are just two: one nilpotent and the one on the Sylow 2-subgroup of $SL_2(3)$. See for example [75, Theorem 3.19]. A fusion system $\mathcal{F}_S(R)$ for a finite group R and a Sylow p-subgroup S of R is called nilpotent if $\mathcal{F}_S(R) = \mathcal{F}_S(S)$ and, by [23, Theorem 1.12], this is equivalent to R having a normal p-complement.

In particular, if H is a copy of $SL_2(3)$ in G, then $\mathcal{F} = \mathcal{F}_D(G) = \mathcal{F}_D(H)$, since G has not a normal p-complement. Therefore $\mathrm{Out}(D, \mathcal{F}) \simeq C_2$ again. Since $\mathrm{Out}(G) \simeq C_2$, then $\mathrm{Pic}(B) = \mathcal{T}(B) \simeq C_2$. $\square$

Chapter 3

On Picent for blocks with normal defect group

We include a copy of a published paper [60]. The version presented here is the one freely available on arXiv, with the exceptions of the correction of a few typos and the rephrasing of the main theorem in the introduction. In this paper we show that Picent is trivial for some classes of blocks with normal defect groups, and examples of blocks with non-trivial Picent are also provided.

In accordance with the guidelines for the submission of journal format, the page numbering has been modified and the bibliography absorbed in the global bibliography of the thesis. The paper is joint work with Dr. Michael Livesey, and I hereby declare that all results were obtained in collaboration, with Dr. Livesey and I contributing equally to the proofs.

Abstract

We prove that if b is a block of a finite group with normal abelian defect group and inertial quotient a direct product of elementary abelian groups, then $\mathrm{Picent}(b)$ is trivial. We also provide examples of blocks b of finite groups with non-trivial $\mathrm{Picent}(b)$. We even have examples with normal abelian defect group and abelian inertial quotient.

3.1 Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with $k := \mathcal{O}/J(\mathcal{O})$ an algebraically closed field of prime characteristic p. Let K, the field of fractions of $\mathcal{O}$, have characteristic zero. We take K large enough, meaning that it contains all $|H|^{\text{th}}$ roots of unity, for all the groups H involved in the rest of the paper. By a block we always mean a block b of $\mathcal{O}H$ for some finite group H. We use $\mathrm{Irr}(H)$ to denote the set of irreducible characters of H and $\mathrm{Irr}(b)$ the subset of irreducible characters lying in the block b.

Let b be a block of a finite group H. The Picard group $\mathrm{Pic}(b)$ of b consists of isomorphism classes of b-b-bimodules which induce $\mathcal{O}$-linear Morita auto-equivalences of b. For $M, N \in \mathrm{Pic}(b)$, the group multiplication is given by $M \otimes_b N$. We will often view M as an $\mathcal{O}(H \times H)$-module via $(g,h).m = gmh^{-1}$, for all $g, h \in H$ and $m \in M$. This paper is concerned with $\mathrm{Picent}(b)$, the subgroup of $\mathrm{Pic}(b)$ consisting of Morita auto-equivalences that induce the trivial permutation of $\mathrm{Irr}(b)$.

In [29] it is proved that all 2-blocks with abelian defect group of rank at most three have trivial Picent. Our main result is that in certain situations Picent is always trivial (see Theorems 3.3.2 and 3.3.3).

Theorem. *Let b be a block with normal abelian defect group. Then* $\mathrm{Picent}(b)$ *is trivial in each of the following cases:*

(i) b has cyclic inertial quotient;

(ii) b has inertial quotient a product of elementary abelian groups;

(iii) b is principal and has abelian inertial quotient.

Notwithstanding the above theorem, in this paper we show that blocks with non-trivial Picent do exist. Our focus is entirely on blocks with normal defect group. We construct three families of examples. The first, see Example 3.4.1, is simply any p-group with a non-inner, class-preserving automorphism. The second example, see Proposition 3.4.2, is given by a non-inner, class-preserving automorphism of a group H, with normal abelian Sylow p-subgroup, that induces a non-trivial element of Picent for the principal block of $\mathcal{O}H$. Our final example, see Proposition 3.4.3, is concerned with a non-principal block with normal abelian defect group and abelian inertial quotient. An interesting point to note in this final case is that the relevant bimodule has vertex ΔD, where D is the defect group. This means the Morita auto-equivalence is simply given by tensoring with a linear character of the inertial quotient.

The following notation will hold for the remainder of the article. If $N \lhd H$, for a finite group H, and $\chi \in \mathrm{Irr}(N)$, then we denote by $\mathrm{Irr}(H|\chi)$ the set of irreducible characters of H appearing as constituents of $\chi \uparrow_N^H$. Similarly, for a block b of H, we define $\mathrm{Irr}(b|\chi) := \mathrm{Irr}(b) \cap \mathrm{Irr}(H|\chi)$. If $h \in H$, then we denote by $c_h \in \mathrm{Aut}(H)$ the automorphism given by $g \mapsto {}^h g := hgh^{-1}$. For any $\mathcal{O}N$-module M, ${}^h M$ will denote the $\mathcal{O}N$-module equal to M as an $\mathcal{O}$-module but with the action of N defined via $g.m = c_h^{-1}(g)m$, for all $g \in N$ and $m \in {}^h M$. The character ${}^h\chi \in \mathrm{Irr}(N)$ is defined analogously. If $X \leq H$, then we set $I_X(M) := \{h \in X | {}^h M \cong M\}$ and define $I_X(\chi)$ analogously. We write $\mathcal{O}_H$ for the $\mathcal{O}H$-module $\mathcal{O}$ with the trivial action of H and $1_H \in \mathrm{Irr}(H)$ will denote the trivial character of H. We use $e_b \in \mathcal{O}H$ to signify the block idempotent of b and $e_\eta \in KH$ the character idempotent associated to $\eta \in \mathrm{Irr}(H)$.

If $\psi \in \mathrm{Aut}(H)$, then we denote by $\Delta\psi := \{(h, \psi(h)) | h \in H\} \leq H \times H$. If $\psi = \mathrm{Id}_H$, then we just denote $\Delta\psi$ by ΔH. We will also view ψ as an $\mathcal{O}$-linear automorphism of $\mathcal{O}H$, where appropriate. For an $\mathcal{O}$-algebra A, $\mathrm{Aut}_{\mathcal{O}}(A)$ will stand for the set of $\mathcal{O}$-algebra automorphisms of A. If $\alpha \in \mathrm{Aut}_{\mathcal{O}}(b)$, we denote by ${}_\alpha b \in \mathrm{Pic}(b)$ the equivalence induced by α. In other words, ${}_\alpha b = b$ as sets but with $x.m.y := \alpha(x)my$, for all $x, y \in b$ and $m \in {}_\alpha b$.

The article is organised as follows. §2 contains an assortment of lemmas that will be needed in §3 and §4. §3 concerns Morita auto-equivalences induced by bimodules with trivial source and then goes on to prove our main theorems regarding families of blocks with trivial Picent. §4 gives our three classes of examples of blocks with non-trivial Picent.

3.2 Preliminaries

In this section we gather together various lemmas that will be used throughout the rest of the article.

Lemma 3.2.1. *Let V be an n-dimensional vector space over a field $\mathbb{F}$, for some $n \in \mathbb{N}$. In addition let $(V_i)_{i\in I}$ be a set of subspaces of V of codimension one such that $\bigcap_{i\in I} V_i = \{0\}$. Then there exists a subset $\{i_1, \dots, i_n\} \subseteq I$ such that $\bigcap_{j=1}^n V_{i_j} = \{0\}$. Furthermore, setting $U_l := \bigcap_{\substack{j=1 \\ j\neq l}}^n V_{i_j}$, for each $1 \leq l \leq n$, we have that each U_l has dimension one and $V = \bigoplus_{l=1}^n U_l$.*

Proof. Choose any $i_1 \in I$. Then choose the remaining i_l's iteratively by demanding that $\bigcap_{j=1}^l V_{i_j}$ is strictly contained in $\bigcap_{j=1}^{l-1} V_{i_j}$ for $2 \leq l \leq n$. Note that the next V_{i_l} always exists, until $\bigcap_{j=1}^l V_{i_j} = \{0\}$, since $\bigcap_{i\in I} V_i = \{0\}$. In fact $\bigcap_{j=1}^l V_{i_j} = \{0\}$ precisely when $l = n$, since $\bigcap_{j=1}^{l-1} V_{i_j}$ has codimension one in $\bigcap_{j=1}^l V_{i_j}$, for $1 \leq l \leq n$. In particular, each U_l has dimension one. Therefore, the final claim follows since

$$\left(\sum_{\substack{l=1 \\ l\neq m}}^n U_l\right) \cap U_m \subseteq V_{i_m} \cap U_m = \bigcap_{j=1}^n V_{i_j} = \{0\},$$

for all $1 \leq m \leq n$. □

Lemma 3.2.2. *Let H be a finite group, N a normal subgroup such that H/N is abelian, ℓ a prime and $N \leq L \leq H$ such that L/N is a cyclic ℓ-group and $\ell \nmid [H : L]$. If $\chi \in \mathrm{Irr}(N)$ is H-stable, then χ extends to L and every extension is H-stable.*

Proof. Since L/N is cyclic and χ is L-stable, χ certainly extends to L. Now let $h \in H$. Since H/N is abelian, L/N is a cyclic ℓ-group and $\ell \nmid [H : L]$, $L\langle h\rangle/N$ is cyclic. Therefore, χ extends to $L\langle h\rangle$ in $[L\langle h\rangle : N]$ different ways. In particular, every extension of χ to L extends to $L\langle h\rangle$ and so h stabilises χ. □

For a finite p-group P, we denote by $\Phi(P)$ the Frattini subgroup of P. For the following lemma we have in mind the semi-direct product $P \rtimes H$. We will borrow notation from this setup, for example $C_P(H)$ will denote the set of fixed points in P under the action of H.

Lemma 3.2.3. *Let P be a finite abelian p-group and H an abelian p'-subgroup of $\mathrm{Aut}(P)$.*

1. *$P = C_P(H) \times [H, P]$.*
2. *The natural homomorphism $H \to \mathrm{Aut}(P/\Phi(P))$ is injective. Furthermore, H acts indecomposably on P, that is there does not exist a non-trivial, H-invariant decomposition $P = P_1 \times P_2$, if and only if H acts indecomposably on $P/\Phi(P)$.*

3. *The homomorphism* $H \to \mathrm{Aut}(\mathrm{Irr}(P))$*,* $h(\lambda)(x) = \lambda(h^{-1}(x))$*, for all* $h \in H$*,* $x \in P$ *and* $\lambda \in \mathrm{Irr}(P)$*, is injective and this action of* H *on* $\mathrm{Irr}(P)$ *is indecomposable if and only if the action of* H *on* P *is.*

4. *If* H *does act indecomposably on* P*, then* H *is cyclic and* $C_P(h)$ *is trivial for all* $h \in H\backslash\{1\}$*.*

5. *Let* $\psi \in C_{\mathrm{Aut}(P)}(H)$ *such that, for all* $x \in P$*,* $\psi(x) = h(x)$*, for some* $h \in H$*. Then* $\psi \in H$*.*

Proof.

1. This is [35, §5, Theorem 2.3].

2. The fact that the homomorphism is injective follows from [35, §5, Theorem 1.4]. The indecomposability statement is a direct consequence of [35, §5, Theorem 2.2] and [58, Lemma 2.5].

3. We can identify P with $\mathrm{Irr}(\mathrm{Irr}(P))$ as H-sets via $x \mapsto (\lambda \mapsto \lambda(x))$. Injectivity follows since, if some $h \in H$ fixes $\mathrm{Irr}(P)$ pointwise, then h also fixes $\mathrm{Irr}(\mathrm{Irr}(P))$ pointwise and hence also P. If $P = P_1 \times P_2$ is an H-invariant decomposition of P, then $\mathrm{Irr}(P) = \mathrm{Irr}(P, 1_{P_1}) \times \mathrm{Irr}(P, 1_{P_2})$ is an H-invariant decomposition of $\mathrm{Irr}(P)$. The reverse implication follows, once again, by identifying P with $\mathrm{Irr}(\mathrm{Irr}(P))$.

4. [35, §5, Theorem 2.2] and [58, Lemma 2.6] gives that H is cyclic. Next let $h \in H$. If $C_P(h)$ is non-trivial then, by part (1), $P = C_P(h) \times [\langle h \rangle, P]$ is an H-invariant decomposition. Therefore, since H acts indecomposably on P, $C_P(h) = P$ and $h = 1$.

5. Let's first assume that P is elementary abelian and H acts indecomposably. By postcomposing with a suitable element of H, we may assume that ψ has a non-trivial fixed point $x \in P$. Now ψ must fix all of P, since otherwise $C_P(\psi)$ is a non-trivial, proper H-invariant subgroup of P which, by [35, §3, Theorem 3.2], contradicts the indecomposability of the action of H on P.

   We next drop the assumption that H acts indecomposably. Decompose $P = P_1 \times \cdots \times P_n$ into indecomposable components. First note that the hypotheses of the lemma ensure that ψ respects this decomposition. Now choose $x_i \in P_i\backslash\{1\}$, for $1 \leq i \leq n$. As above, we may assume that ψ fixes $(x_1, \ldots, x_n) \in P$ and hence each x_i. Also as above, we must now have ψ acting as the identity on each P_i and therefore all of P.

   For the general case, by the previous paragraph, we may assume that ψ induces the trivial automorphism of $P/\Phi(P)$. Decompose $P = P_1 \times \cdots \times P_n$ into indecomposable components. Let $x_i \in P_i \backslash \Phi(P_i)$, for $1 \leq i \leq n$ and set $x := (x_1, \ldots, x_n) \in P$. Then

$$C_H(x\Phi(P)) = \bigcap_{i=1}^{n} C_H(x_i\Phi(P_i)) = \bigcap_{i=1}^{n} C_H(P_i/\Phi(P_i)) = \bigcap_{i=1}^{n} C_H(P_i) = \{1\},$$

where the second equality follows from part (4) and the third follows from part (2). Therefore, any $h \in H$ such that $\psi(x) = h(x)$ must be trivial. So ψ is trivial on $(P_1\backslash\Phi(P_1)) \times \cdots \times (P_n\backslash\Phi(P_n))$ and hence on all of P.

□

3.3 Blocks with trivial Picent

We set the following notation that will hold for the remainder of this section. Let D be a finite p-group, E a finite p'-group and $Z \leq E$ a central, cyclic subgroup such that we can identify $L := E/Z$ with a subgroup of $\mathrm{Aut}(D)$. Through this identification we define $G := D \rtimes E$ and $B := \mathcal{O}Ge_\varphi$ for some fixed $\varphi \in \mathrm{Irr}(Z)$. Since $D \lhd G$, any block idempotent of $\mathcal{O}G$ is supported on $C_G(D) = Z(D) \times Z$. Therefore, B is a block of $\mathcal{O}G$ with defect group D. We set $C := D \times Z$. $\mathcal{T}(B)$ will denote the subgroup of $\mathrm{Pic}(B)$ consisting of bimodules that have trivial source when viewed as $\mathcal{O}(G \times G)$-modules.

Proposition 3.3.1. *Let $M \in \mathcal{T}(B)$.*

1. *M has vertex $\Delta\psi$ for some $\psi \in N_{\mathrm{Aut}(D)}(L)$.*

We denote by $\psi_L \in \mathrm{Aut}(L)$ the automorphism of L induced by conjugation by ψ.

2. *$M \cong {}_\alpha B$, where $\alpha \in \mathrm{Aut}_\mathcal{O}(B)$ satisfies $\alpha(xe_\varphi) = \psi(x)e_\varphi$, for all $x \in D$ and $\alpha(\mathcal{O}gZe_\varphi) = \mathcal{O}\psi_L(gZ)e_\varphi$, for all $g \in E$.*
3. *If M has vertex ΔD, then M is just given by tensoring with some linear character $\lambda \in \mathrm{Irr}(G|1_C)$. Moreover, different λ's give non-isomorphic M's.*

Proof.

1. By [10, Theorem 1.1(i)], M has vertex $\Delta\psi$ for some $\psi \in \mathrm{Aut}(D, \mathcal{F})$, where $\mathcal{F}$ is the fusion system on D determined by B and $\mathrm{Aut}(D, \mathcal{F})$ is the subgroup of $\mathrm{Aut}(D)$ which stabilises $\mathcal{F}$. In particular, $\psi \in N_{\mathrm{Aut}(D)}(L)$.
2. Since M has vertex $\Delta\psi$ and $\mathcal{O}_{\Delta\psi} \uparrow_{\Delta\psi}^{D\times D} \cong {}_\psi(\mathcal{O}D)$ is indecomposable, M is a direct summand of ${}_\psi(\mathcal{O}D) \uparrow_{D\times D}^{G\times G}$. Now let $(g, h) \in E \times E$. Then ${}^{(g,h)}({}_\psi(\mathcal{O}D))$ has vertex

$$\begin{aligned} {}^{(g,h)}(\Delta\psi) =&\{({}^g x, {}^h\psi(x)|x \in D\} = \{(x, {}^h\psi({}^{g^{-1}}x))|x \in D\} \\ =&\Delta(hZ \circ \psi \circ g^{-1}Z), \end{aligned} \tag{3.1}$$

where we are viewing $gZ, hZ \in \mathrm{Aut}(D)$. Now if $hZ = \psi_L(gZ)$, then

$$hZ \circ \psi \circ g^{-1}Z = \psi_L(gZ) \circ \psi \circ g^{-1}Z = \psi \circ gZ \circ \psi^{-1} \circ \psi \circ g^{-1}Z = \psi.$$

Therefore, ${}^{(g,h)}(\Delta\psi) = \Delta\psi$ and $S_\psi \leq I_{G\times G}({}_\psi(\mathcal{O}D))$, where $S_\psi \leq G \times G$ is generated by $D \times D$ and

$$E_\psi := \{(g,h) \in E \times E | \psi_L(gZ) = hZ\}.$$

If $I_{G\times G}({}_\psi(\mathcal{O}D))$ properly contains S_ψ then there exists some $(g,1) \in I_{G\times G}({}_\psi(\mathcal{O}D))$, with $g \in E\backslash Z$. Since $p \nmid |L|$, gZ has order prime to p and so cannot be an inner automorphism of D. Suppose ${}^{(x,y)}(\Delta\psi) = \Delta(\psi \circ g^{-1}Z)$, for some $(x,y) \in D \times D$. Then, as in (3.1), $c_y \circ \psi \circ c_{x^{-1}} = \psi \circ g^{-1}Z$. Therefore,

$$c_{\psi^{-1}(y)} = \psi^{-1} \circ c_y \circ \psi = g^{-1}Z \circ c_x, \tag{3.2}$$

a contradiction as gZ is not inner, and so $S_\psi = I_{G\times G}({}_\psi(\mathcal{O}D))$. Furthermore, ${}_\psi(\mathcal{O}D)$ must extend to an $\mathcal{O}S_\psi$-module since if it didn't, then

$$\mathrm{rk}_\mathcal{O}(M) > \mathrm{rk}_\mathcal{O}({}_\psi(\mathcal{O}D)).[G \times G : S_\psi] = |D|.|L| = \mathrm{rk}_\mathcal{O}(B),$$

contradicting [58, Lemma 2.1]. Therefore $M \cong N \uparrow_{S_\psi}^{G\times G}$, where $N \downarrow_{D\times D}^{S_\psi} \cong {}_\psi(\mathcal{O}D)$. Moreover, since $e_\varphi M e_\varphi = M$, $N \downarrow_{C\times C}^{S_\psi} \cong {}_\psi(\mathcal{O}D) \otimes_\mathcal{O} \mathcal{O}Ze_\varphi$. Now

$$\begin{aligned}({}_\psi(KD) \otimes_K KZe_\varphi) \uparrow_{C\times C}^{S_\psi} e_{1_D} &\cong ({}_\psi(KD)e_{1_D} \otimes_K KZe_\varphi) \uparrow_{C\times C}^{S_\psi} \\ &\cong \bigoplus_{\chi \in \mathrm{Irr}(E_\psi|\varphi\otimes\varphi^{-1})} KS_\psi e_{\chi'},\end{aligned} \tag{3.3}$$

where $\chi' := \mathrm{Inf}_{E_\psi}^{S_\psi}(\chi)$, the inflation of χ to S_ψ. Since N is a direct summand of $({}_\psi(\mathcal{O}D) \otimes_\mathcal{O} \mathcal{O}Ze_\varphi) \uparrow_{C\times C}^{S_\psi}$ and $\dim_K((K \otimes_\mathcal{O} N)e_{1_D}) = 1$, we have that $(K \otimes_\mathcal{O} N)e_{1_D} \cong KS_\psi e_{\chi'}$ as KS_ψ-modules, for some linear character $\chi \in \mathrm{Irr}(E_\psi|\varphi \otimes \varphi^{-1})$ and, for each such χ, up to isomorphism, there is at most one extension N_χ of ${}_\psi(\mathcal{O}D) \otimes_\mathcal{O} \mathcal{O}Ze_\varphi$ to an $\mathcal{O}S_\psi$-module with $(K \otimes_\mathcal{O} N_\chi)e_{1_D} \cong KS_\psi e_{\chi'}$. Until further notice we fix the appropriate linear $\chi \in \mathrm{Irr}(E_\psi|\varphi \otimes \varphi^{-1})$ such that $N \cong N_\chi$ as $\mathcal{O}S_\psi$-modules.

Since $p \nmid |E_\psi|$, $\chi(g,h) \in \mathcal{O}$, for all $(g,h) \in E_\psi$. We define $\alpha \in \mathrm{Aut}_\mathcal{O}(B)$ via $\alpha(xe_\varphi) = \psi(x)e_\varphi$, for all $x \in D$ and $\alpha(ge_\varphi) = \chi(g,h)he_\varphi$, for all $(g,h) \in E_\psi$. One can readily check that α is a well-defined $\mathcal{O}$-algebra automorphism of B. We demonstrate the most difficult condition:

$$\begin{aligned}&\alpha(ge_\varphi)\alpha(xe_\varphi)\alpha(g^{-1}e_\varphi) = h\psi(x)h^{-1}e_\varphi = hZ(\psi(x))e_\varphi = \psi_L(gZ)(\psi(x))e_\varphi \\ &= (\psi \circ gZ \circ \psi^{-1})(\psi(x))e_\varphi = \psi(gZ(x))e_\varphi = \alpha(gxg^{-1}e_\varphi),\end{aligned}$$

where we view $gZ, hZ \in \mathrm{Aut}(D)$ and the third equality follows from the fact that $(g,h) \in E_\psi$. We claim that $N \uparrow_{S_\psi}^{G\times G} \cong {}_\alpha B$.

Set $G_\psi := \Delta\psi \rtimes E_\psi$ and $\mathcal{O}_\alpha$ the $\mathcal{O}G_\psi$-module $\mathcal{O}$ affording $\mathrm{Inf}_{E_\psi}^{G_\psi}(\chi)$. Since $\langle e_\varphi\rangle_{\mathcal{O}} \subseteq {}_\alpha B$ affords $\mathrm{Inf}_{E_\psi}^{G_\psi}(\chi)$, ${}_\alpha B \cong \mathcal{O}_\alpha \uparrow_{G_\psi}^{G\times G}$. Next we consider the $\mathcal{O}S_\psi$-module $\mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi}$. Now

$$\mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi}\downarrow_{D\times D}^{S_\psi} \cong \mathcal{O}_\alpha \downarrow_{\Delta\psi}^{G_\psi}\uparrow_{\Delta\psi}^{D\times D} \cong \mathcal{O}_{\Delta\psi} \uparrow_{\Delta\psi}^{D\times D} \cong {}_\psi(\mathcal{O}D).$$

Therefore, $\mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi}$ is an extension of ${}_\psi(\mathcal{O}D)\otimes_{\mathcal{O}}\mathcal{O}Ze_\varphi$ to $\mathcal{O}S_\psi$. In particular, $\dim_K((K\otimes_{\mathcal{O}} \mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi})e_{1_D}) = 1$. Hence, since

$$\begin{aligned}\langle \mathrm{Inf}_{E_\psi}^{G_\psi}(\chi)\uparrow_{G_\psi}^{S_\psi}, \mathrm{Inf}_{E_\psi}^{S_\psi}(\chi)\rangle_{S_\psi} &= \langle \mathrm{Inf}_{E_\psi}^{G_\psi}(\chi), \mathrm{Inf}_{E_\psi}^{S_\psi}(\chi)\downarrow_{G_\psi}^{S_\psi}\rangle_{G_\psi}\\ &= \langle \mathrm{Inf}_{E_\psi}^{G_\psi}(\chi), \mathrm{Inf}_{E_\psi}^{G_\psi}(\chi)\rangle_{G_\psi} = 1,\end{aligned}$$

$(K\otimes_{\mathcal{O}} \mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi})e_{1_D}$ affords $\mathrm{Inf}_{E_\psi}^{S_\psi}(\chi)$. Therefore, by the comments following (3.3) and the fact that $(K\otimes_{\mathcal{O}} N)e_{1_D}$ also affords $\mathrm{Inf}_{E_\psi}^{S_\psi}(\chi)$, $N \cong \mathcal{O}_\alpha \uparrow_{G_\psi}^{S_\psi}$ and so $M \cong {}_\alpha B$.

3. What was proved in part (2) was essentially that any $M \in \mathcal{T}(B)$ is uniquely determined by its vertex, say $\Delta\psi$, and the linear character $\chi \in \mathrm{Irr}(E_\psi|\varphi\otimes\varphi^{-1})$ afforded by $(K\otimes N)e_{1_D}$, where N is an extension of ${}_\psi(\mathcal{O}D)\otimes_{\mathcal{O}}\mathcal{O}Ze_\varphi$ to an $\mathcal{O}S_\psi$-module and $M \cong N\uparrow_{S_\psi}^{G\times G}$. If $M \in \mathcal{T}(B)$ induces the equivalence given by tensoring with λ, a linear character in $\mathrm{Irr}(G|1_C)$, then ψ can be taken to be Id_D and α to be given by $\alpha(ge_\varphi) = \lambda(g)ge_\varphi$, for all $g \in G$. The corresponding χ is then given by $\chi(g,h) = \lambda(g)\varphi(gh^{-1})$, for all $(g,h) \in E_{\mathrm{Id}_D} = (Z\times Z).(\Delta E)$. Since χ is a linear character in $\mathrm{Irr}(E_{\mathrm{Id}_D}|\varphi\otimes\varphi^{-1})$, in fact every χ is of this form for some $\lambda \in \mathrm{Irr}(G|1_C)$. Therefore, every $M \in \mathcal{T}(B)$, with vertex ΔD, is given by tensoring with some linear character $\lambda \in \mathrm{Irr}(G|1_C)$.

   Next suppose M is given by tensoring with some linear character $\lambda \in \mathrm{Irr}(G|1_C)$. As shown above M has vertex ΔD. Let N be the extension of $\mathcal{O}D\otimes_{\mathcal{O}}\mathcal{O}Ze_\varphi$ to $\mathcal{O}S_{\mathrm{Id}_D}$ from the proof of part (2). Since $S_{\mathrm{Id}_D} = I_{G\times G}(\mathcal{O}D)$ and $M \cong N\uparrow_{S_{\mathrm{Id}_D}}^{G\times G}$, N is in fact the unique summand of $M\downarrow_{S_{\mathrm{Id}_D}}^{G\times G}$ that extends $\mathcal{O}D\otimes_{\mathcal{O}}\mathcal{O}Ze_\varphi$. Therefore, once we've fixed ΔD as a vertex, χ is uniquely determined by M. Finally, by the previous paragraph, λ is uniquely determined by χ.

□

Theorem 3.3.2. *Let b be a block with normal abelian defect group and inertial quotient a product of elementary abelian groups. Then* Picent(b) *is trivial.*

Proof. By [48, Theorem A] (see also [56, Theorem 6.14.1] for a more detailed description), we may assume b is of the form of B as described just before Proposition 3.3.1. Note that in [56, Theorem 6.14.1], the isomorphism classes of the defect group and inertial quotient do not change when we move from b to B. In other words D is abelian and $L = E/Z$ is a product of elementary abelian groups. Let $M \in \mathrm{Picent}(B)$. By [29, Propositions 4.3,4.4], $M \in \mathcal{T}(B)$ and so we let ψ and α be as in Proposition 3.3.1.

We will use ψ^* to denote the self-bijection of $\mathrm{Irr}(D)$ given by $\psi^*(\chi)(x) = \chi(\psi^{-1}(x))$, for all $x \in D$ and $\chi \in \mathrm{Irr}(D)$. For any $C \leq H \leq G$, we denote by ${}^{\alpha}H$ the unique subgroup $C \leq {}^{\alpha}H \leq G$ such that $\mathcal{O}({}^{\alpha}H)e_{\varphi} = \alpha(\mathcal{O}He_{\varphi})$ or equivalently ${}^{\alpha}H/C = \psi_L(H/C)$. For any such H, we will use α^* to denote the bijection $\mathrm{Irr}(H|\varphi) \to \mathrm{Irr}({}^{\alpha}H|\varphi)$ given by $\alpha^*(\chi)(x) = \chi(\alpha^{-1}(x))$, for all $x \in {}^{\alpha}H$ and $\chi \in \mathrm{Irr}(H|\varphi)$. Since $M \in \mathrm{Picent}(B)$, when $H = G$, α^* is just the identity on $\mathrm{Irr}(B)$. Note that, since α permutes the left cosets of C in G, $\alpha^*(\chi)\uparrow_{{}^{\alpha}H}^{G} = \alpha^*(\chi\uparrow_H^G)$, for any $C \leq H \leq G$ and $\chi \in \mathrm{Irr}(H|\varphi)$.

Decompose $D = D_1 \times \cdots \times D_n$ into indecomposable components with respect to the action of L. Let $\theta_i \in \mathrm{Irr}(D_i)\backslash\{1_{D_i}\}$, for each $1 \leq i \leq n$ and set $\theta := \theta_1 \otimes \cdots \otimes \theta_n \in \mathrm{Irr}(D)$. Since $\alpha^*(\theta \otimes \varphi)\uparrow_C^G = \alpha^*((\theta \otimes \varphi)\uparrow_C^G) = (\theta \otimes \varphi)\uparrow_C^G$, $\alpha^*(\theta \otimes \varphi) = \psi^*(\theta) \otimes \varphi$ must be conjugate to $\theta \otimes \varphi$ via an element of E. Therefore, by composing ψ and α with an appropriately chosen c_g, we may assume that $\psi^*(\theta) = \theta$.

By an abuse of notation, we view each $\theta_i \in \mathrm{Irr}(D)$ by letting it act trivially on all the other D_j's. An identical argument to the previous paragraph applied to each $\theta_i \otimes \varphi \in \mathrm{Irr}(C)$, for $1 \leq i \leq n$, and the fact that ψ^* already fixes θ, gives that each θ_i is also fixed by ψ^*. In addition , by parts (3) and (4) of Lemma 3.2.3, $I_L(\theta_i) = C_L(D_i)$ and $I_E(\theta_i) = C_E(D_i)$.

Until further notice we fix a prime $\ell \mid |L|$. Since L is a product of elementary abelian groups, $O_\ell(L) \cong (C_\ell)^t$, for some $t \in \mathbb{N}$. Therefore, by part (4) of Proposition 3.2.3, for each $1 \leq i \leq n$,

$$O_\ell(L/I_L(\theta_i)) = O_\ell(L/C_L(D_i)) \cong \{1\} \text{ or } C_\ell.$$

Also, since we're identifying L with a subgroup of $\mathrm{Aut}(D)$,

$$\bigcap_{i=1}^{n} O_\ell(I_L(\theta_i)) = \bigcap_{i=1}^{n} O_\ell(C_L(D_i)) = O_\ell(C_L(D)) = \{1\}.$$

Therefore, viewing $O_\ell(L)$ as an $\mathbb{F}_\ell$-vector space, by Lemma 3.2.1 there exist $1 \leq i_1, \dots i_t \leq n$ such that $\bigcap_{j=1}^{t} O_\ell(I_L(\theta_{i_j})) = \{1\}$. In addition, setting

$$\vartheta_m := \left(\bigotimes_{\substack{j=1 \\ j \neq m}}^{t} \theta_{i_j} \right) \otimes \left(\bigotimes_{\substack{j=1 \\ j \notin \{i_1,\dots,i_t\}}}^{n} 1_{D_j} \right) \otimes 1_{D_{i_m}} \in \mathrm{Irr}(D),$$

for each $1 \leq m \leq t$, Lemma 3.2.1 also gives that

$$\mathcal{C}_m := O_\ell((I_L(\vartheta_m)) = \bigcap_{\substack{j=1 \\ j \neq m}}^{t} O_\ell(I_L(\theta_{i_j})) \cong C_\ell,$$

and that the $\mathcal{C}_m$'s generate $O_\ell(L)$.

For any subgroup H of L we denote by $\tilde{H}$ its preimage in E. Until further notice we fix some $1 \leq m \leq t$ and set $I_m := I_L(\vartheta_m \otimes \varphi)$. Since $\psi^*(\theta_i) = \theta_i$, for all $1 \leq i \leq n$, $\psi^*(\vartheta_m) = \vartheta_m$. Therefore, $\psi_L(I_m) = I_m$ and $\psi_L(\mathcal{C}_m) = \mathcal{C}_m$ implying ${}^{\alpha}(D \rtimes \tilde{I}_m) = D \rtimes \tilde{I}_m$ and ${}^{\alpha}(D \rtimes \tilde{\mathcal{C}}_m) = D \rtimes \tilde{\mathcal{C}}_m$.

By [52, Theorem 6.11(b)], $\mathrm{Irr}(B|\vartheta_m \otimes \varphi) = \mathrm{Irr}(G|\vartheta_m \otimes \varphi)$ is in one-to-one correspondence, via induction, with $\mathrm{Irr}(I_G(\vartheta_m \otimes \varphi)|\vartheta_m \otimes \varphi) = \mathrm{Irr}(D \rtimes \tilde{I}_m|\vartheta_m \otimes \varphi)$. Therefore, since ${}^{\alpha}(D \rtimes \tilde{I}_m) = D \rtimes \tilde{I}_m$ and α^* respects induction, α^* must be the identity on $\mathrm{Irr}(D \rtimes \tilde{I}_m|\vartheta_m \otimes \varphi)$. By Lemma 3.2.2, $\vartheta_m \otimes \varphi$ extends to $D \rtimes \tilde{\mathcal{C}}_m$ and every extension to $D \rtimes \tilde{\mathcal{C}}_m$ is stable in $D \rtimes \tilde{I}_m$. In other words, every character in $\mathrm{Irr}(D \rtimes \tilde{I}_m|\vartheta_m \otimes \varphi)$ lies above a unique character in $\mathrm{Irr}(D \rtimes \tilde{\mathcal{C}}_m|\vartheta_m \otimes \varphi)$. Therefore, again since α^* respects induction, α^* is also the identity on $\mathrm{Irr}(D \rtimes \tilde{\mathcal{C}}_m|\vartheta_m \otimes \varphi)$. Now every character in $\mathrm{Irr}(D \rtimes \tilde{\mathcal{C}}_m|\vartheta_m \otimes \varphi)$ is linear and is therefore determined by its restriction to $\tilde{\mathcal{C}}_m$. Hence, α is the identity on $Z(\mathcal{O}\tilde{\mathcal{C}}_m e_\varphi) = \mathcal{O}\tilde{\mathcal{C}}_m e_\varphi$.

Since the $\mathcal{C}_m$'s generate $O_\ell(L)$ and the $O_\ell(L)$'s generate L as ℓ runs over all primes dividing $|L|$, α is the identity on $\mathcal{O}Ee_\varphi$. In particular, ψ_L is the identity. As noted earlier in this proof, for every $\chi \in \mathrm{Irr}(D)$, $\psi^*(\chi)$ is conjugate to χ via an element of E. Therefore, by part (5) of Lemma 3.2.3, ψ^* is induced by an element of E and so, by composing ψ and α with an appropriately chosen c_g, we may assume that ψ is the identity.

We now repeat the entire proof with ψ being the identity until we've reproved that α is the identity on $\mathcal{O}Ee_\varphi$. Therefore, α is the identity and M is the trivial Morita autoequivalence. □

Theorem 3.3.3. *Let b be a block with normal abelian defect group and either cyclic inertial quotient or a principal block with abelian inertial quotient. Then* $\mathrm{Picent}(b)$ *is trivial.*

Proof. The proofs for both situations proceed similarly to that of Theorem 3.3.2. We first note that if b is a principal block of a group H with normal defect group P, then P must be a Sylow p-subgroup of H. Therefore, by the Schur-Zassenhaus theorem, $H = P \rtimes F$, for some p'-subgroup $F \leq H$. Then, $C_F(P)$ must be in the kernel of b and so, by factoring out by $C_F(P)$ and considering the relevant Morita equivalent block, we may assume that b is already of the form of B with Z trivial. In other words, the block B we reduce to is also principal. In particular, in both instances of the theorem, E is abelian.

Now replace $\vartheta_m \in \mathrm{Irr}(D)$, in the proof of Theorem 3.3.2, with 1_D. Proceeding as in that proof and noting that, since E is abelian, every character in $\mathrm{Irr}(E|\varphi)$ is linear, we prove that α is the identity on $\mathcal{O}Ee_\varphi$. We then continue to prove that M is the trivial Morita auto-equivalence exactly as before. □

3.4 Examples of non-trivial Picent

Picard groups have not been calculated for any Morita equivalence class of blocks with non-abelian defect group, excluding nilpotent blocks but, as our first example shows, just taking the principal p-block of a suitably chosen p-group already yields an example of a block with non-trivial Picent.

We first set up some notation. A class-preserving automorphism of a group H is an automorphism that leaves invariant every conjugacy class of H. Equivalently it leaves invariant every irreducible character of H. We define

$$\mathrm{Aut}_c(H) = \{\alpha \in \mathrm{Aut}(H) | \alpha \text{ is class-preserving}\},$$

$$\mathrm{Inn}(H) = \{c_h \in \mathrm{Aut}(H) | h \in G\},$$

$\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ and $\mathrm{Out}_c(H) = \mathrm{Aut}_c(H)/\mathrm{Inn}(H)$.

Example 3.4.1. *Let P be a finite p-group. We know, by [74], that* $\mathrm{Pic}(\mathcal{O}P) \cong \mathrm{Hom}(P, \mathcal{O}^\times) \rtimes \mathrm{Out}(P)$. *Since any element of* $\mathrm{Picent}(\mathcal{O}P)$ *must fix the trivial character,* $\mathrm{Picent}(\mathcal{O}P) \cong \mathrm{Out}_c(P)$ *and finding a non-inner, class-preserving automorphism of P immediately yields a non-trivial element of* $\mathrm{Picent}(\mathcal{O}P)$. *Many p-groups with such an automorphism have been* *constructed, [90] lists some of these. The smallest p-group P with* $\mathrm{Out}_c(P) \neq \{1\}$ *has order* 32 *and is described in [83].*

For blocks with abelian defect group the situation is different. As $\mathrm{Out}_c(P)$ is trivial, for any abelian p-group P, p-groups will no longer provide examples with non-trivial Picent. In addition, in all the cases the Picard group of a block with abelian defect group has been computed (namely [10] and [29]) Picent has been trivial. However, the families of blocks that we exhibit show explicitly that Picent of a block with abelian defect group is not trivial, in general. The first family we are going to describe yields a counterexample in the same spirit as Example 3.4.1, the non-trivial element of Picent(B) is given by an outer automorphism of the relevant group.

Before stating the result we set up some notation. Let $t > 1$ be an integer coprime to p. For any $i \in \mathbb{N}$, we denote by ω_i a primitive i^{th} root of unity and take n to be the smallest positive integer such that $\omega_{t^2} \in \mathbb{F}_{p^n}$. In other words, n is the multiplicative order of p mod t^2. Set $s := (p^{nt}-1)/(t(p^n-1))$ and $D := (C_p)^{nt}$ that we identify with $\mathbb{F}_{p^{nt}}$. Certainly s is coprime to p and

$$(p^{nt}-1)/(p^n-1) = p^{n(t-1)} + ... + p^n + 1.$$

Therefore, since each $p^{ni} \equiv 1 \mod t^2$, $s \equiv 1 \mod t$. In particular, s and t are coprime. For any $\lambda \in \mathbb{F}_{p^{nt}}^{\times}$, we denote by m_λ the group automorphism of $\mathbb{F}_{p^{nt}}$ given by multiplication by λ and we denote by Φ_{p^n} the automorphism of $\mathbb{F}_{p^{nt}}$ given by $(x \mapsto x^{p^n})$.

We set $G := D \rtimes E$, where $E := (\langle g \rangle \rtimes \langle h \rangle)$, $\langle g \rangle = \langle m_{\omega_s} \rangle \cong C_s$ and $\langle h \rangle = \langle \Phi_{p^n} \circ m_{\omega_{t^2}} \rangle \cong C_{t^2}$. Since ${}^h g = g^{p^n}$, E is well-defined. What's more, since s and t^2 are coprime, $E/C_E(D)$ has order st^2. In other words, E acts faithfully on D and $\mathcal{O}G$ is a block.

Proposition 3.4.2. *With the above notation,* $\mathrm{Out}_c(G)$ *is non-trivial. Moreover, the image of* $\mathrm{Out}_c(G)$ *in* Picent$(\mathcal{O}G)$ *is non-trivial.*

Proof. We define $\psi = m_{\omega_{t^2}} \in \mathrm{Aut}(D)$. This extends to an automorphism $\psi_G \in \mathrm{Aut}(G)$ that acts trivially on E. We claim that ψ_G gives a non-trivial element of $\mathrm{Out}_c(G)$.

Note that $\psi \in \mathrm{Aut}(D)$ is not already in E. Suppose it were, then $\psi^{-1} \circ h = \Phi_{p^n} \in E$ is an element of order t. By considering their images in $E/\langle g \rangle \cong C_{t^2}$, we see that all the elements of order t in E are in $\langle g \rangle \times \langle h^t \rangle$ and hence powers of $h^t = m_{\omega_{t^2}}^t$. However, Φ_{p^n} is not of this form as it has non-trivial fixed points on D.

We now show that ψ preserves the E-conjugacy classes of D, that is for each $x \in D$, there exists $y \in E$ such that $\psi(x) = {}^y x$.

For $x \in \mathbb{F}_{p^{nt}}^\times$, $0 \leq i < s$ and $0 \leq j < t$,

$$c_g^i \circ c_h^{1+jt}(x) = \psi(x) \Leftrightarrow \omega_s^i \omega_{t^2}^{1+jt} x^{p^{n(1+jt)}} = \omega_{t^2} x \Leftrightarrow x^{p^n - 1} = \omega_{t^2}^{-jt} w_s^{-i}.$$

Thus, since x^{p^n-1} is a (not necessarily primitive) $(st)^{\text{th}}$ root of unity, there exist unique i, j such that $c_g^i \circ c_h^{1+jt}(x) = \psi(x)$.

Next we prove that for all $x \in D\backslash\{1\}$, $C_E(x) = \{1\}$. This is crucial, since it will give us a description of the irreducible characters of the group G.

We first note that all the non-trivial elements of $\langle g \rangle \times \langle h^t \rangle = \langle m_{\omega_{st}} \rangle$ have no non-trivial fixed points on D. If $y \in E\backslash\langle g \rangle$, then, by considering its image in $E/\langle g \rangle \cong C_{t^2}$, there exists an integer l such that $y^l \in (\langle g \rangle \times \langle h^t \rangle)\backslash\{1\}$. If y has non-trivial fixed points on D then y^l would have non-trivial fixed points as well, thus y can't have any non-trivial fixed points. In other words, for all $x \in D\backslash\{1\}$, $C_E(x) = \{1\}$.

We now show that the analogous property holds, when we replace D with $\mathrm{Irr}(D)$. Let $y \in E\backslash\{1\}$ and decompose $D = D_1 \times \cdots \times D_m$ with respect to the action of $\langle y \rangle$. Let $\theta_i \in \mathrm{Irr}(D_i)$, for each $1 \leq i \leq m$, and set $\theta := \theta_1 \otimes \cdots \otimes \theta_m \in \mathrm{Irr}(D)$. If $\theta \neq 1_D$, say $\theta_j \neq 1_{D_j}$, for some $1 \leq j \leq m$, then

$$I_{\langle y \rangle}(\theta) \subseteq I_{\langle y \rangle}(\theta_j) = C_{\langle y \rangle}(D_j) = \{1\},$$

where the first equality follows from parts (3) and (4) of Lemma 3.2.3 and the final one from the previous paragraph. In other words, the action of E on $\mathrm{Irr}(D)$ is fixed-point-free as well. Thus all irreducible character of G are either irreducible characters with D in their kernel or irreducible characters induced from non-trivial characters of D. Characters of G with D in their kernel correspond to characters of G/D, that are obviously fixed by ψ_G, while the characters of G induced from D are characterised by their values on G-conjugacy classes of elements of D but ψ preserves the E-conjugacy classes of elements of D. Therefore, every character of G is ψ_G-stable and so $\psi_G \in \mathrm{Aut}_c(G)$.

Note that ψ_G is a non-inner group automorphism of G, since we have shown that ψ is not induced by an element of E. Finally, we note that, since ${}_{\psi_G}(\mathcal{O}G) \cong \mathcal{O}_{\Delta\psi_G} \uparrow_{\Delta\psi_G}^{G\times G}$, ${}_{\psi_G}(\mathcal{O}G)$ has vertex $\Delta\psi$. Similarly $\mathcal{O}G$ has vertex ΔD. Since ψ is not induced by an element of E, $\Delta\psi$ is not conjugate to ΔD in G. Therefore, ${}_{\psi_G}(\mathcal{O}G)$ is a non-trivial element of $\mathrm{Picent}(\mathcal{O}G)$. □

We now exhibit another family of blocks with non-trivial Picent. The main difference with the previous family is that, as well as having abelian defect group, these blocks also have abelian inertial quotient and diagonal vertex. In fact, the non-trivial element of $\mathrm{Picent}(B)$ we exhibit is given by multiplication by a linear character of the inertial quotient. Note that, by Theorem 3.3.3, these blocks are necessarily non-principal.

Let ℓ be a prime number different from p. Take

$$E := (\langle z\rangle \times \langle g\rangle) \rtimes \langle h\rangle \cong (C_\ell \times C_{\ell^2}) \rtimes C_\ell,$$

defined by $hz = zh$, ${}^h g = gz$ and set $F := \langle z\rangle \times \langle g^\ell\rangle \times \langle h\rangle \leq E$. Furthermore, let n be the multiplicative order of $p \mod \ell^2$ and set $D := (C_p)^n \times (C_p)^n =: D_1 \times D_2$ that we identify with $\mathbb{F}_{p^n} \oplus \mathbb{F}_{p^n}$.

We define $G := D \rtimes E$, where the action of E on D_1 has kernel $\langle z\rangle \times \langle h\rangle$, while on D_2 the kernel is $\langle z\rangle \times \langle g^\ell h\rangle$ and g acts on both components by multiplication by $\omega \in \mathbb{F}_{p^n}^\times$, a primitive ℓ^2-th root of of unity. Since $\omega \notin \mathbb{F}_{p^m}^\times$ for any $m < n$, no proper $\mathbb{F}_p$-subspace of $\mathbb{F}_{p^n}$ has an $\mathbb{F}_p$-linear map with eigenvalue ω. In particular, the actions of E on both D_1 and D_2 are indecomposable. Of course, $C_E(D) = Z := \langle z\rangle$. Let $\varphi \in \mathrm{Irr}(Z)\backslash\{1_Z\}$ and set $B := \mathcal{O}Ge_\varphi$.

Proposition 3.4.3. *With the notation above, tensoring by any non-trivial* $\lambda \in \mathrm{Irr}(G|1_{D\rtimes F})$ *yields a non-trivial element of* $\mathrm{Picent}(B)$.

Proof. Let $\lambda \in \mathrm{Irr}(G|1_{D\rtimes F})$. Every character in $\mathrm{Irr}(B|1_D)$ is of the form $\mathrm{Inf}_E^G(\chi)$, for some $\chi \in \mathrm{Irr}(E|\varphi)$. Then, by [58, Lemma 3.1(2)], we have $\chi^{\oplus m} = \eta \uparrow_{Z(E)}^E$, for some $m \in \mathbb{N}$ and $\eta \in \mathrm{Irr}(Z(E)|\varphi)$. Note that $Z(E) = Z \times \langle g^\ell\rangle \leq F$. Therefore,

$$(\lambda \downarrow_E^G).(\eta \uparrow_{Z(E)}^E) = (\lambda \downarrow_{Z(E)}^G .\eta) \uparrow_{Z(E)}^E = \eta \uparrow_{Z(E)}^E,$$

giving that $(\lambda \downarrow_E^G).\chi = \chi$ and finally that $\lambda.\mathrm{Inf}_E^G(\chi) = \mathrm{Inf}_E^G(\chi)$.

Now take $\chi \in \mathrm{Irr}(B|\theta)$, for some $\theta = \theta_1 \otimes \theta_2 \in \mathrm{Irr}(D)$, where $\theta_i \in \mathrm{Irr}(D_i)\backslash\{1_{D_i}\}$, for

$i = 1, 2$. By part (3) and (4) of Lemma 3.2.3, $I_E(\theta_i) = C_E(D_i)$, for $i = 1, 2$. Therefore, $I_E(\theta_1 \otimes \theta_2) = C_E(D_1) \cap C_E(D_2) = Z$ and so $\chi = (\theta \otimes \varphi) \uparrow_{D \times Z}^{G}$. Hence,

$$\lambda.\chi = \lambda.((\theta \otimes \varphi) \uparrow_{D\times Z}^{G}) = ((\lambda \downarrow_{D\times Z}^{G}).(\theta \otimes \varphi)) \uparrow_{D\times Z}^{G} = (\theta \otimes \varphi) \uparrow_{D\times Z}^{G} = \chi.$$

Finally, let $\chi \in \mathrm{Irr}(B|\theta)$, where $\theta = \theta_1 \otimes 1_{D_2} \in \mathrm{Irr}(D)$, for some $\theta_1 \in \mathrm{Irr}(D_1)\backslash\{1_{D_1}\}$. This time $I_E(\theta) = \langle z \rangle \times \langle h \rangle \leq F$ and the argument concludes similarly to the previous paragraph. Of course, the case of $\theta = 1_{D_1} \otimes \theta_2$, for some $\theta_2 \in \mathrm{Irr}(D_2)\backslash\{1_{D_2}\}$ is dealt with in an identical fashion.

We have proved that tensoring with λ fixes $\mathrm{Irr}(B)$ pointwise and thus defines an element of $\mathrm{Picent}(B)$. The fact that non-trivial λ induce non-trivial elements of $\mathrm{Picent}(B)$ follows from part (3) of Proposition 3.3.1. $\square$

Chapter 4

Picard groups for blocks with normal defect groups and linear source bimodules

We include a copy of a submitted paper, available on arXiv [61]. In this paper we show that $\mathrm{Pic}(B) = \mathcal{E}(B)$ for blocks with normal defect group in odd characteristic.

In accordance with the guidelines for the submission of journal format, the page numbering has been modified and the bibliography absorbed in the global bibliography of the thesis. The paper is joint work with Dr. Michael Livesey, and I hereby declare that all results were obtained in collaboration, with Dr. Livesey and I contributing equally to the proofs.

Abstract

It is an open problem as to whether any bimodule inducing a Morita auto-equivalence of a block must have endopermutation source. We prove that, for blocks b with normal defect groups in odd characteristic, a stronger result holds, namely that all such bimodules have linear source. We also prove the analogous result in characteristic 2, provided that the defect group is of a specific, slightly restrictive, form.

4.1 Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring, with $k := \mathcal{O}/J(\mathcal{O})$ an algebraically closed field of characteristic $p > 0$ and K, a field of characteristic zero, the field of fractions of $\mathcal{O}$. Let H be a finite group. In this setting K will always be large enough, meaning that it contains all $|H|^{\text{th}}$ roots of unity. For the remainder of the introduction let b be a block of H, by which we mean a block of $\mathcal{O}H$.

The Picard group $\text{Pic}(b)$ of b consists of isomorphism classes of invertible b-b-bimodules. If $M \in \text{Pic}(b)$, then M induces an $\mathcal{O}$-linear Morita auto-equivalence of b given by $M \otimes_b -$. There are three important subgroups of $\text{Pic}(b)$ that will form the main area of interest for this article. (For more details on $\mathcal{T}(b)$, $\mathcal{L}(b)$ and $\mathcal{E}(b)$ see [10].)

$$\mathcal{T}(b) = \{[M]_\sim \in \text{Pic}(b) | M \text{ has trivial source as an } \mathcal{O}(H \times H)\text{-module}\}$$
$$\mathcal{L}(b) = \{[M]_\sim \in \text{Pic}(b) | M \text{ has linear source as an } \mathcal{O}(H \times H)\text{-module}\}$$
$$\mathcal{E}(b) = \{[M]_\sim \in \text{Pic}(b) | M \text{ has endopermutation source as an } \mathcal{O}(H \times H)\text{-module}\}.$$

Morita equivalences given by endopermutation source bimodules seem to be very common in practice, in fact there are no known examples of Morita equivalences of blocks given by a bimodule that does not have endopermutation source. It is, therefore, a very natural question to ask if all elements of $\text{Pic}(b)$ have endopermutation source. If it were always the case that $\text{Pic}(b) = \mathcal{E}(b)$, then it would be known that $\text{Pic}(b)$ is bounded in terms of a function of the order of the defect group (see [57, Theorem 1.3]). In fact, it is already known that $\text{Pic}(b)$ is always finite (see [31, Corollary 1.2]).

For blocks with normal defect group, it is already known that $\mathcal{E}(b) = \mathcal{L}(b)$ (see [57, Theorem 1.5]). Therefore, in this case we ask whether $\operatorname{Pic}(b) = \mathcal{L}(b)$. In [58, Theorem 6.3] this was shown to hold for blocks with normal abelian defect and abelian inertial quotient. We improve this result, removing any hypothesis on the structure of the defect group and the inertial quotient, in our main theorem:

Theorem (4.3.2)**.** *Let $p > 2$ and b a block with normal defect group. Then* $\operatorname{Pic}(b) = \mathcal{L}(b)$*.*

The corresponding theorem for $p = 2$ is currently out of reach of the authors using the methods outlined in this article. However, if the abelianisation of the defect group is sufficiently "tall" the theorem still holds:

Theorem (4.3.3)**.** *Let $p = 2$ and b a block with normal defect group D such that $D/[D, D]$ has no direct factor isomorphic to C_2. Then* $\operatorname{Pic}(b) = \mathcal{L}(b)$*.*

The following notation will hold throughout the article. If $N \lhd H$ and $\chi \in \operatorname{Irr}(N)$, then we denote by $\operatorname{Irr}(H|\chi)$ the set of irreducible characters of H appearing as constituents of $\chi\uparrow_N^H$. Similarly, we set $\operatorname{Irr}(b|\chi) := \operatorname{Irr}(b) \cap \operatorname{Irr}(H|\chi)$. Let $F \leq H$. For any $h \in H$ and $\chi \in \operatorname{Irr}(F)$, we define ${}^hF = hFh^{-1}$ and ${}^h\chi \in \operatorname{Irr}({}^hF)$ by ${}^h\chi(g) = \chi(h^{-1}gh)$, for all $g \in {}^hF$. If now $E \leq H$ normalises F, then E_χ will symbolise the stabiliser of χ in E and if $\eta = {}^h\chi$, for some $h \in E$ and $\eta \in \operatorname{Irr}(F)$, then we write $\chi \sim_E \eta$. We also adopt all the analogous above notation for Brauer characters, where we replace Irr with IBr.

$\langle , \rangle_H$ will denote the usual inner product on $\mathbb{Z}\operatorname{Irr}(H)$. If $\chi \in \operatorname{Irr}(H)$ is a linear character then $\mathcal{O}_\chi$ will be the $\mathcal{O}H$-module $\mathcal{O}$ with action of H defined through χ. We use $1_H \in \operatorname{Irr}(H)$ to signify the trivial character of H, $e_b \in \mathcal{O}H$ the block idempotent of b and $e_\chi \in KH$ the character idempotent associated to any $\chi \in \operatorname{Irr}(H)$. We define $\overline{} : \mathcal{O} \to k$ to be the natural quotient map, $\overline{} : \mathbb{Z}\operatorname{Irr}(H) \to \mathbb{Z}\operatorname{IBr}(H)$ the corresponding reduction modulo p map and $\overline{M}$ to be $k \otimes_{\mathcal{O}} M$, for any $\mathcal{O}H$-module M. We adopt the convention that all $\mathcal{O}H$-modules are finitely generated and free as $\mathcal{O}$-modules.

The article is organised as follows. §4.2 is concerned with Ext groups and how they can be used to distinguish certain subsets of characters of a particular block. In §4.3 we prove our main theorems. The proofs heavily rely on an application of Weiss' criterion.

4.2 Ext groups

This section is concerned with using Ext groups to distinguish certain subsets of irreducible characters of a block with normal defect group. This will allow us to apply Weiss' criterion in §4.3 via Proposition 4.3.1. We first gather together various well known results on Ext groups with respect to group rings. In what follows all cohomology groups will be calculated over $\mathcal{O}$, not $\mathbb{Z}$. Also all isomorphisms of Ext and cohomology groups will assumed to be isomorphisms as $\mathcal{O}$ or k-modules and not just as abelian groups.

Lemma 4.2.1. *Let H be a finite group.*

(i) For any $i \in \mathbb{N}_0$ and $\mathcal{O}H$-modules M_1, M_2,

$$\mathrm{Ext}^i_{\mathcal{O}H}(M_1, M_2) \simeq H^i(H, M_1^* \otimes_{\mathcal{O}} M_2).$$

(ii) For any $N \leq H$, $\mathcal{O}N$-module M_1 and $\mathcal{O}H$-module M_2,

$$\mathrm{Ext}^i_{\mathcal{O}H}(M_1 \uparrow_N^H, M_2) \simeq \mathrm{Ext}^i_{\mathcal{O}N}(M_1, M_2 \downarrow_N^H),$$
$$\mathrm{Ext}^i_{\mathcal{O}H}(M_2, M_1 \uparrow_N^H) \simeq \mathrm{Ext}^i_{\mathcal{O}N}(M_2 \downarrow_N^H, M_1).$$

(iii) If $H = H_1 \times H_2$, then for any $n \in \mathbb{N}$, $\mathcal{O}H_1$-modules U_1, V_1 and $\mathcal{O}H_2$-module U_2, V_2, there exists a split, short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} \mathrm{Ext}^i_{\mathcal{O}H_1}(U_1, V_1) \otimes_{\mathcal{O}} \mathrm{Ext}^j_{\mathcal{O}H_2}(U_2, V_2) \longrightarrow \mathrm{Ext}^n_{\mathcal{O}H}(U_1 \otimes_{\mathcal{O}} U_2, U_1 \otimes_{\mathcal{O}} U_2)$$
$$\longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathcal{O}}(\mathrm{Ext}^i_{\mathcal{O}H_1}(U_1, V_1), \mathrm{Ext}^j_{\mathcal{O}H_2}(U_2, V_2)) \longrightarrow 0.$$

(iv) If H is an abelian p-group, say $H \simeq C_{p^{n_1}} \times \cdots \times C_{p^{n_t}}$, and $\lambda_1, \lambda_2 \in \mathrm{Irr}(H)$, then

$$\mathrm{Ext}^0_{\mathcal{O}H}(\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}) = \begin{cases} \mathcal{O} & \text{if } \lambda_1 = \lambda_2 \\ \{0\} & \text{otherwise} \end{cases},$$

$$\mathrm{Ext}^1_{\mathcal{O}H}(\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}) = \begin{cases} \{0\} & \text{if } \lambda_1 = \lambda_2 \\ \mathcal{O}/(1-\zeta)\mathcal{O} & \text{otherwise} \end{cases},$$

$$\mathrm{Ext}^2_{\mathcal{O}H}(\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}) = \begin{cases} \bigoplus_{i=1}^{t} \mathcal{O}/p^{n_i}\mathcal{O} & \text{if } \lambda \text{ is trivial} \\ [\mathcal{O}/(1-\zeta)\mathcal{O}]^{\oplus(t-1)} & \text{otherwise} \end{cases},$$

where it is assumed, if $\lambda_1 \neq \lambda_2$, that $\lambda_1.\lambda_2^{-1}$ has image $\{\zeta^i\}_{i \in \mathbb{Z}}$, for some p^{th}-power root of unity ζ.

(v) *For any $\mathcal{O}$-module M,*

$$\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}, M) = \mathrm{Tor}_1^{\mathcal{O}}(M, \mathcal{O}) = \{0\}.$$

Furthermore,

$$\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}/a\mathcal{O}, \mathcal{O}/b\mathcal{O}) \simeq \mathcal{O}/a\mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}/b\mathcal{O},$$

for all non-zero $a, b \in \mathcal{O}$.

Proof. (i) This is very well known.

(ii) This is Shapiro's Lemma.

(iii) Set $M_1 = U_1^* \otimes_{\mathcal{O}} V_1$ and $M_2 = U_2^* \otimes_{\mathcal{O}} V_2$. The Künneth formula for group cohomology gives a split, short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(H_1, M_1) \otimes_{\mathcal{O}} H^j(H_2, M_2) \longrightarrow H^n(H, M_1 \otimes_{\mathcal{O}} M_2)$$
$$\longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathcal{O}}(H^i(H_1, M_1), H^j(H_2, M_2)) \longrightarrow 0.$$

For an explicit reference see [84, Exercise 6.1.8]. Also note that in [8, Theorem 3.5.6] it states that the underlying ring need not be $\mathbb{Z}$ but merely a hereditary ring. The result now follows from part (i).

(iv) This follows easily from the well known description of cohomology groups for cyclic groups (e.g. see [8, Corollary 3.5.2]) and parts (i) and (iii).

(v) The first claim follows simply from the fact that $\mathcal{O}$ is respectively projective/flat over itself. For the second claim we recall that

$$\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}/a\mathcal{O}, M) \simeq \mathrm{Ann}_M(a) := \{x \in M | ax = 0\},$$

for any non-zero (divisor) $a \in \mathcal{O}$ and $\mathcal{O}$-module M. However, if $b \in \mathcal{O}$ is also non-zero then

$$\mathrm{Ann}_{\mathcal{O}/b\mathcal{O}}(a) \simeq \mathcal{O}/c\mathcal{O} \simeq \mathcal{O}/a\mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}/b\mathcal{O},$$

where c is either a or b depending on which has the smaller valuation with respect to $J(\mathcal{O})$.

□

We now introduce a specific setup that will hold for the remainder of this section. Set $G = D \rtimes E$, where D is a finite p-group and E a finite p'-group. $Z := C_E(D)$ is a cyclic, central subgroup of E with quotient $L := E/Z$. We set $B := \mathcal{O}(D \rtimes E)e_\varphi$, for some faithful $\varphi \in \mathrm{Irr}(Z)$. Since $D \lhd G$, any block idempotent of $\mathcal{O}G$ must be supported on $C_G(D) = Z(D) \times Z$ and hence B is indeed a block.

We will require this general setup, where D is allowed to be non-abelian, in §4.3. However, for the remainder of this section all the above will hold with the added assumption that D is abelian. In this case, we set $D_1 = [D, E]$ and $D_2 = C_D(E)$. By [35, Theorem 2.3], we have a decomposition $D = D_1 \times D_2$. Say $D_1 \simeq C_{p^{n_1}} \times \cdots \times C_{p^{n_u}}$ and $D_2 \simeq C_{p^{n_{u+1}}} \times \cdots \times C_{p^{n_t}}$, for some $t \geq u \in \mathbb{N}_0$ and $n_i \in \mathbb{N}$. We also demand that all the $n_i > 1$, when $p = 2$. We record all these assumptions below.

Assumption 4.2.2. *(i) $B = \mathcal{O}(D \rtimes E)e_\varphi$ but D is not assumed to abelian. This setup will be used in §4.3.*

(ii) In addition to (i), D is assumed to be abelian and if $p = 2$ we assume that $n_i > 1$, for all $1 \leq i \leq t$, i.e. D has no direct factor isomorphic to C_2. This setup will hold for the remainder of this section.

We can describe $\mathrm{Irr}(B)$ very precisely. For any $\lambda \in \mathrm{Irr}(D)$ and $\chi \in \mathrm{Irr}(E_\lambda|\varphi)$ we define $(\lambda, \chi) \in \mathrm{Irr}(D \rtimes E_\lambda|\varphi)$ by

$$(\lambda, \chi)(xg) = \lambda(x)\chi(g) \text{ for all } x \in D, g \in E_\lambda.$$

Then

$$\mathrm{Irr}(B) = \left\{(\lambda, \chi)\uparrow_{D \rtimes E_\lambda}^{G} |\lambda \in \mathrm{Irr}(D), \chi \in \mathrm{Irr}(E_\lambda|\varphi)\right\}. \tag{4.1}$$

This was proved in [58, Lemma 3.2] for L abelian but the proof is identical in this more general setting.

Remark 4.2.3. Since $D \lhd G$, D is in the kernel of every Brauer character of B and so we can identify $\mathrm{IBr}(B)$ with $\mathrm{Irr}(E|\varphi)$. Furthermore, through this identification, we can identify the reduction map $\mathbb{Z}\,\mathrm{Irr}(B) \to \mathbb{Z}\,\mathrm{IBr}(B)$ with the restriction map $\mathbb{Z}\,\mathrm{Irr}(B) \to \mathbb{Z}\,\mathrm{Irr}(E|\varphi)$.

Next we need a lemma concerning $\mathcal{O}G$-modules with linear source that realise irreducible characters of B.

Lemma 4.2.4. *Let $\eta \in \mathrm{Irr}(B)$.*

(i) There exists a unique, up to isomorphism, linear source $\mathcal{O}G$-module M_η such that $K \otimes_{\mathcal{O}} M_\eta$ affords η.

(ii) If η lifts a Brauer character then M_η is the unique, up to isomorphism, $\mathcal{O}G$-module such that $K \otimes_{\mathcal{O}} M_\eta$ affords η.

Proof. (i) Suppose $\eta = (\lambda, \chi)\uparrow_{D\rtimes E_\lambda}^G$, for some $\lambda \in \mathrm{Irr}(D)$ and $\chi \in \mathrm{Irr}(E_\lambda|\varphi)$. Define V_χ to be an $\mathcal{O}E_\lambda$-module such that $K \otimes_{\mathcal{O}} V_\chi$ affords χ. We can extend V_χ to $\mathcal{O}(D \rtimes E_\lambda)$ by letting each $x \in D$ act via scalar multiplication by $\lambda(x)$. (Since E_λ is a p'-group, V_χ is uniquely defined by the above.) Now set $M_\eta = V_\chi \uparrow_{D\rtimes E_\lambda}^G$. Certainly V_χ and therefore M_η has linear source.

Next let M'_η be another linear source $\mathcal{O}G$-module such that $K \otimes_{\mathcal{O}} M'_\eta$ affords η. Since any source of M'_η must be of the form $\mathcal{O}_{\lambda'}$ for some $\lambda' \sim_E \lambda$, M'_η must be a direct summand of $\mathcal{O}_\lambda \uparrow_D^G$. Now each summand of $\mathcal{O}_\lambda \uparrow_D^{D\rtimes E_\lambda}$ is of the form $V_{\chi'}$ from the above paragraph, for some $\chi' \in \mathrm{Irr}(E_\lambda)$. Since each $V_{\chi'} \uparrow_{D\rtimes E_\lambda}^G$ is indecomposable, $M'_\eta \simeq V_{\chi'} \uparrow_{D\rtimes E_\lambda}^G$ for some $\chi' \in \mathrm{Irr}(E_\lambda)$. Fixing said χ', we must have

$$\begin{aligned} 1 &= \langle (\lambda,\chi')\uparrow_{D\rtimes E_\lambda}^G, (\lambda,\chi)\uparrow_{D\rtimes E_\lambda}^G\rangle_G = \langle (\lambda,\chi'), (\lambda,\chi)\uparrow_{D\rtimes E_\lambda}^G\downarrow_{D\rtimes E_\lambda}^G\rangle_{D\rtimes E_\lambda} \\ &= \langle (\lambda,\chi'), (\lambda,\chi)\rangle_{D\rtimes E_\lambda}, \end{aligned}$$

where the final equality follows since $(\lambda,\chi)\downarrow_D^{D\rtimes E_\lambda} = |E_\lambda|.\lambda$ and the stabiliser of λ in G is $D \rtimes E_\lambda$. In particular,

$$1 \leq \langle (\lambda,\chi')\downarrow_{E_\lambda}^{D\rtimes E_\lambda}, (\lambda,\chi)\downarrow_{E_\lambda}^{D\rtimes E_\lambda}\rangle_{E_\lambda} = \langle \chi', \chi\rangle_{E_\lambda}.$$

Therefore, $\chi' = \chi$ and $M'_\eta \simeq V_\chi \uparrow_{D\rtimes E_\lambda}^G = M_\eta$, as desired.

(ii) Suppose η lifts $\mu \in \mathrm{IBr}(G)$ and U is an $\mathcal{O}G$-module such that $K \otimes_{\mathcal{O}} U$ affords η. Let P_μ be the projective indecomposable $\mathcal{O}G$-module corresponding to μ. So $\overline{P}_\mu$ is the projective indecomposable kG-module corresponding to μ. In particular, we have a surjective $\mathcal{O}G$-module homomorphism $f : P_\mu \to \overline{U}$. Therefore, since P_μ is projective, there exists an $\mathcal{O}G$-module homomorphism $h : P_\mu \to U$ such that $\overline{} \circ h = f$ and, by Nakayama's lemma, h must surject onto U. The proof is completed by observing that, since η lifts μ, η appears with multiplicity one in the character of $K \otimes_{\mathcal{O}} P_\mu$ and so P_μ has a unique $\mathcal{O}$-free quotient whose K-span affords η.

□

We adopt the notation of M_η, from the above lemma, for the remainder of this section. In particular, for $n \in \mathbb{N}_0$ and $\chi, \eta \in \mathrm{Irr}(G)$, $\mathrm{Ext}^n_{\mathcal{O}G}(\chi, \eta)$ will denote $\mathrm{Ext}^n_{\mathcal{O}G}(M_\chi, M_\eta)$. We use an analogous convention for $\mathrm{Ext}^n_{kG}(\mu, \psi)$, where $\mu, \psi \in \mathrm{IBr}(B)$.

Lemma 4.2.5. *Let $\chi = \alpha \otimes \theta_\chi$, $\eta = \beta \otimes \theta_\eta \in \mathrm{Irr}(B)$, for some $\alpha, \beta \in \mathrm{Irr}(D_1 \rtimes E|\varphi)$ and $\theta_\chi, \theta_\eta \in \mathrm{Irr}(D_2)$.*

(i) For $0 \leq i \leq 2$, each $\mathrm{Ext}^i_{\mathcal{O}G}(\chi,\eta)$ is a finite direct sum of $\mathcal{O}$-modules of the form $\mathcal{O}/a\mathcal{O}$, for some $a \in \mathcal{O}$. Moreover, if $i = 0$, all the a's can be chosen to be zero, if $i = 1$, then all the a's can be chosen to be of the form $1-\zeta$, for some non-trivial p^{th}-power roots of unity ζ and if $i = 2$, then all the a's can be chosen to be non-zero. In particular, $\mathrm{Ext}^i_{\mathcal{O}G}(\chi,\eta)$ has a direct summand isomorphic to $\mathcal{O}$ only if $i = 0$ and in this case

$$\mathrm{Ext}^0_{\mathcal{O}G}(\chi,\eta) \simeq \mathcal{O}^{\oplus\delta_{\chi,\eta}}.$$

(ii) If $\theta_\chi = \theta_\eta$, then

$$\begin{aligned}\mathrm{Ext}^2_{\mathcal{O}G}(\chi,\eta) \simeq & \left[\bigoplus_{i=u+1}^{t} \mathcal{O}/p^{n_i}\mathcal{O} \otimes_{\mathcal{O}} \mathrm{Ext}^0_{\mathcal{O}(D_1\rtimes E)}(\alpha,\beta)\right] \\ & \oplus \mathrm{Ext}^2_{\mathcal{O}(D_1\rtimes E)}(\alpha,\beta) \oplus \left[\bigoplus_{i=u+1}^{t} \mathcal{O}/p^{n_i}\mathcal{O} \otimes_{\mathcal{O}} \mathrm{Ext}^1_{\mathcal{O}(D_1\rtimes E)}(\alpha,\beta)\right].\end{aligned}$$

(iii) If $\theta_\chi \neq \theta_\eta$ and $\theta_\chi^{-1}.\theta_\eta$ has image $\{\zeta^i\}_{i\in\mathbb{Z}}$, for some p^{th}-power root of unity ζ, then

$$\begin{aligned}\mathrm{Ext}^2_{\mathcal{O}G}(\chi,\eta) \simeq & \left[\mathcal{O}/(1-\zeta)\mathcal{O} \otimes_{\mathcal{O}} \mathrm{Ext}^0_{\mathcal{O}(D_1\rtimes E)}(\alpha,\beta)\right]^{\oplus(t-u-1)} \\ \oplus & \left[\mathcal{O}/(1-\zeta)\mathcal{O} \otimes_{\mathcal{O}} \mathrm{Ext}^1_{\mathcal{O}(D_1\rtimes E)}(\alpha,\beta)\right]^{\oplus(t-u)} \oplus \left[\mathcal{O}/(1-\zeta)\mathcal{O} \otimes_{\mathcal{O}} \mathrm{Ext}^2_{\mathcal{O}(D_1\rtimes E)}(\alpha,\beta)\right].\end{aligned}$$

Proof. (i) Note that for any $0 \leq i \leq 2$ and $\lambda_1, \lambda_2 \in \mathrm{Irr}(D)$, by Lemma 4.2.1(iii),

$$\mathrm{Ext}^i_{\mathcal{O}G}(\mathcal{O}_{\lambda_1}\uparrow_D^G, \mathcal{O}_{\lambda_2}\uparrow_D^G) \simeq \mathrm{Ext}^i_{\mathcal{O}D}(\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}\uparrow_D^G\downarrow_D^G).$$

The claim now follows, by Lemma 4.2.1(iv), since M_χ and M_η both have linear source. That $\mathrm{Ext}^0_{\mathcal{O}G}$ is of the desired form, can be seen by noting that $\mathrm{Ext}^0_{\mathcal{O}G} = \mathrm{Hom}_{\mathcal{O}G}$.

(ii),(iii) Applying Lemma 4.2.1(iii) we get the split, short exact sequence

$$\begin{aligned}0 \longrightarrow & \bigoplus_{i+j=2} \mathrm{Ext}^i_{\mathcal{O}(D\rtimes E)}(\alpha,\beta) \otimes_{\mathcal{O}} \mathrm{Ext}^j_{\mathcal{O}D_2}(\theta_\chi,\theta_\eta) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}G}(\alpha\otimes\theta_\chi, \beta\otimes\theta_\eta) \\ \longrightarrow & \bigoplus_{i+j=3} \mathrm{Tor}_1^{\mathcal{O}}(\mathrm{Ext}^i_{\mathcal{O}(D\rtimes E)}(\alpha,\beta), \mathrm{Ext}^j_{\mathcal{O}D_2}(\theta_\chi,\theta_\eta)) \longrightarrow 0.\end{aligned} \tag{4.2}$$

We now need only apply Lemma 4.2.1(iv) and (v) and part (i) of the current lemma to (4.2) to obtain the result.

□

Lemma 4.2.6. *(i) Let $\chi, \eta \in \mathrm{Irr}(B)$ be such that $\overline{\chi}$ and $\overline{\eta}$ have no irreducible constituents in common. Then*

$$k \otimes_{\mathcal{O}} \mathrm{Ext}^2_{\mathcal{O}G}(\chi,\eta) \simeq \mathrm{Ext}^1_{kG}(\overline{\chi},\overline{\eta}).$$

(ii) The graph with vertices labelled by $\mathrm{IBr}(B)$ and an edge between $\mu, \psi \in \mathrm{IBr}(B)$ if $\mathrm{Ext}^1_{kG}(\mu,\psi) \neq \{0\}$, is connected.

Proof. (i) Consider a projective resolution for M_χ

$$\cdots \to P_1 \to P_0 \to M_\chi \to 0,$$

and let X be the corresponding Hom chain complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{O}G}(P_1, M_\eta) \leftarrow \mathrm{Hom}_{\mathcal{O}G}(P_0, M_\eta) \leftarrow 0,$$

where we consider $\mathrm{Hom}_{\mathcal{O}G}(P_i, M_\eta)$ to be in position $-i$. By [20, Theorem 2.7.4], we have an exact sequence of k-modules,

$$0 \to H_{-1}(X) \otimes_{\mathcal{O}} k \to H_{-1}(X \otimes_{\mathcal{O}} k) \to \mathrm{Tor}_1^{\mathcal{O}}(H_{-2}(X), k) \to 0. \tag{4.3}$$

Of course, $H_{-1}(X) \simeq \mathrm{Ext}^1_{\mathcal{O}G}(\chi, \eta)$ and $H_{-2}(X) \simeq \mathrm{Ext}^2(\chi, \eta)$. By Lemma 4.2.1(v) and Lemma 4.2.5(i),

$$\mathrm{Tor}_1^{\mathcal{O}}(\mathrm{Ext}^2(\chi, \eta), k) \simeq \mathrm{Ext}^2(\chi, \eta) \otimes_{\mathcal{O}} k.$$

It remains to show that the middle term is $\mathrm{Ext}^1_{kG}(\overline{\chi}, \overline{\eta})$ and $\mathrm{Ext}^1_{\mathcal{O}G}(\chi, \eta) = \{0\}$. For the first claim we just need to prove that $X \otimes_{\mathcal{O}} k$ is the chain complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{O}G}(P_1 \otimes_{\mathcal{O}} k, M_\eta \otimes_{\mathcal{O}} k) \leftarrow \mathrm{Hom}_{\mathcal{O}G}(P_0 \otimes_{\mathcal{O}} k, M_\eta \otimes_{\mathcal{O}} k) \leftarrow 0.$$

In other words we need to show that every element of

$$\mathrm{Hom}_{kG}(P_i \otimes_{\mathcal{O}} k, M_\eta \otimes_{\mathcal{O}} k)$$

lifts to an element of $\mathrm{Hom}_{\mathcal{O}G}(P_i, M_\eta)$. If $G = D$, then the claim follows, since each P_i is a direct sum of $\mathcal{O}D$'s,

$$\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}D, \mathcal{O}_\lambda) = \langle g \mapsto \lambda(g)\rangle_{\mathcal{O}},$$

for any $\lambda \in \mathrm{Irr}(D)$, and

$$\mathrm{Hom}_{kG}(kD, k) = \langle g \mapsto 1\rangle_k.$$

The general case follows from tracing through the series of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}G}\left(\mathcal{O}D\uparrow_D^G, \mathcal{O}_\lambda\uparrow_D^G\right) \otimes_{\mathcal{O}} k &\simeq \mathrm{Hom}_{\mathcal{O}G}\left(\mathcal{O}D, \mathcal{O}_\lambda\uparrow_D^G\downarrow_D^G\right) \otimes_{\mathcal{O}} k \\ \simeq \mathrm{Hom}_{kG}\left(kD, k_\lambda\uparrow_D^G\downarrow_D^G\right) &\simeq \mathrm{Hom}_{kG}\left(\mathcal{O}D\uparrow_D^G \otimes_{\mathcal{O}} k, \mathcal{O}_\lambda\uparrow_D^G \otimes_{\mathcal{O}} k\right)\end{aligned}$$

and noting that every projective $\mathcal{O}G$-module is a direct summand of $\mathcal{O}D\uparrow_D^G$ and every M_η is a direct summand of some $\mathcal{O}_\lambda\uparrow_D^G$. (Note the first and third isomorphisms follow from Frobenius reciprocity and the second since we already know our claim holds for $G = D$.)

We now show that $\mathrm{Ext}^1_{\mathcal{O}G}(\chi, \eta) = \{0\}$ and this will conclude the proof. Take P_χ a projective cover of χ, then we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathcal{O}G}(M_\chi, M_\eta) \longrightarrow \mathrm{Hom}_{\mathcal{O}G}(P_\chi, M_\eta) \longrightarrow \mathrm{Hom}_{\mathcal{O}G}(\Omega^1(\chi), M_\eta) \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{O}G}(M_\chi, M_\eta) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}G}(P_\chi, M_\eta) \longrightarrow \dots \end{aligned}$$

where $\Omega^1(\chi)$ is the kernel of the natural map $P_\chi \to M_\chi$. We know that $\mathrm{Hom}_{\mathcal{O}G}(P_\chi, M_\eta)$ and $\mathrm{Hom}_{\mathcal{O}G}(\Omega^1(\chi), M_\eta)$ are both zero, since the hypotheses of the lemma ensure that η has multiplicity zero in the characters of $K \otimes_{\mathcal{O}} P_\chi$ and $K \otimes_{\mathcal{O}} \Omega^1(\chi)$. Moreover, as P_χ is projective, $\mathrm{Ext}^1_{\mathcal{O}G}(P_\chi, M_\eta)$ is zero and thus $\mathrm{Ext}^1_{\mathcal{O}G}(\chi, \eta) = \mathrm{Ext}^1_{\mathcal{O}G}(M_\chi, M_\eta) = \{0\}$ as required.

(ii) This is [1, Proposition 4.13.3].

□

Before proceeding we need to note the following. If $\zeta \in \mathcal{O}$ is a primitive $(p^n)^{\text{th}}$-root of unity, for some $n \in \mathbb{N}$, then

$$\prod_{\substack{i=1,\\ p\nmid i}}^{p^n-1} (X - \zeta^i) = (X^{p^n} - 1)/(X^{p^{n-1}} - 1) = \sum_{i=0}^{p-1} X^{ip^{n-1}} \in \mathbb{Z}[X].$$

In particular,

$$\prod_{\substack{i=1,\\ p\nmid i}}^{p^n-1} (1 - \zeta^i) = p$$

and so $1 - \zeta \in p\mathcal{O}$ if and only if $p = 2$ and $n = 2$. This is ultimately the reason that we do not have as strong a theorem in the $p = 2$ case.

We need one small lemma before continuing.

Lemma 4.2.7. *If $\lambda \in \mathrm{Irr}(D_1)$ is E-stable, then $\lambda = 1_{D_1}$.*

Proof. This was proved in [58, Lemma 2.4] under the assumption that L is abelian. However, this fact was totally unused in the proof. □

Lemma 4.2.8. *Let $\lambda_1, \lambda_2 \in \mathrm{Irr}(D)$, $\chi_1 \in \mathrm{Irr}(E_{\lambda_1}|\varphi)$ and $\chi_2 \in \mathrm{Irr}(E_{\lambda_2}|\varphi)$.*

(i) If $p > 2$ and

$$\mathrm{Ext}^2_{\mathcal{O}G}((\lambda_1,\chi_1)\uparrow^G_{D\rtimes E_{\lambda_1}},(\lambda_2,\chi_2)\uparrow^G_{D\rtimes E_{\lambda_2}})\simeq\bigoplus_{i=1}^s\mathcal{O}/p^{m_i}\mathcal{O},$$

for some $s\in\mathbb{N}$ and $m_1,\dots,m_s\in\mathbb{N}$, then $\lambda_1\sim_E\lambda_2$.

(ii) If $p = 2$ and

$$\mathrm{Ext}^2_{\mathcal{O}G}((\lambda_1,\chi_1)\uparrow^G_{D\rtimes E_{\lambda_1}},(\lambda_2,\chi_2)\uparrow^G_{D\rtimes E_{\lambda_2}})\simeq\bigoplus_{i=1}^s\mathcal{O}/p^{m_i}\mathcal{O},$$

for some $s\in\mathbb{N}$ and $m_1,\dots,m_s\in\mathbb{N}_{>1}$, then $\lambda_1\sim_E\lambda_2$.

Proof. We prove only the $p>2$ case. The $p=2$ case follows in a similar fashion. If $\zeta\in\mathcal{O}$ is a primitive $(p^n)^{\text{th}}$-root of unity, for some $n\in\mathbb{N}$, the only difference between the two cases is that, due to the comments preceding the lemma, $1-\zeta\notin p\mathcal{O}$ if $p>2$ but when $p=2$ we can have $1-\zeta\in 2\mathcal{O}$ but not $1-\zeta\in 4\mathcal{O}$.

The claim follows immediately if $(\lambda_1,\chi_1)\uparrow^G_{D\rtimes E_{\lambda_1}}=(\lambda_2,\chi_2)\uparrow^G_{D\rtimes E_{\lambda_2}}$ so let's assume from now on that $(\lambda_1,\chi_1)\uparrow^G_{D\rtimes E_{\lambda_1}}\neq(\lambda_2,\chi_2)\uparrow^G_{D\rtimes E_{\lambda_2}}$.

Suppose $\lambda_i=\alpha_i\otimes\beta_i$, for $i=1,2$, $\alpha_i\in\mathrm{Irr}(D_1)$ and $\beta_i\in\mathrm{Irr}(D_2)$. Then

$$\begin{aligned}&\mathrm{Ext}^2_{\mathcal{O}G}((\lambda_1,\chi_1)\uparrow^G_{D\rtimes E_{\lambda_1}},(\lambda_2,\chi_2)\uparrow^G_{D\rtimes E_{\lambda_2}})\\ \simeq\,&\mathrm{Ext}^2_{\mathcal{O}G}((\alpha_1,\chi_1)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_1}}\otimes\beta_1,(\alpha_2,\chi_2)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_2}}\otimes\beta_2).\end{aligned}$$

Therefore, by the hypotheses of the lemma and the comments preceding it, Lemma 4.2.5(iii) gives that $\beta_1=\beta_2$. Furthermore, by Lemma 4.2.5(ii),

$$\mathrm{Ext}^2_{\mathcal{O}(D_1\rtimes E)}((\alpha_1,\chi_1)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_1}},(\alpha_2,\chi_2)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_2}})\simeq\bigoplus_{i=1}^s\mathcal{O}/p^{m_i}\mathcal{O}.$$

Note that, since we are assuming the two characters are different, the Ext^0 term in Lemma 4.2.5(ii) is just $\{0\}$. Also, by Lemma 4.2.5(i), the hypotheses of the lemma and the comments preceding it, the Ext^1 term in Lemma 4.2.5(ii) is also $\{0\}$. We now need to show that $\alpha_1\sim_E\alpha_2$. Lemma 4.2.1(ii) gives

$$\begin{aligned}&\mathrm{Ext}^2_{\mathcal{O}(D_1\rtimes E)}((\alpha_1,\chi_1)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_1}},(\alpha_2,\chi_2)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_2}})\\ \simeq\,&\mathrm{Ext}^2_{\mathcal{O}(D_1\rtimes E_{\alpha_1})}((\alpha_1,\chi_1),(\alpha_2,\chi_2)\uparrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_2}}\downarrow^{D_1\rtimes E}_{D_1\rtimes E_{\alpha_1}})\end{aligned}\tag{4.4}$$

and, by the Mackey formula,

$$(\alpha_2,\chi_2)\uparrow^{D_1\rtimes E}_{D\rtimes E_{\alpha_2}}\downarrow^{D_1\rtimes E}_{D\rtimes E_{\alpha_1}}=\sum_i({}^{g_i}\alpha_2,\chi_{g_i})\uparrow^{D_1\rtimes E_{\alpha_1}}_{D_1\rtimes({}^{g_i}E_{\alpha_2}\cap E_{\alpha_1})},$$

for some g_i's in E and $\chi_{g_i} \in \mathrm{Irr}({}^{g_i}E_{\alpha_2} \cap E_{\alpha_1}, \varphi)$. Therefore, by the hypotheses of the lemma, there must exist some i with

$$\mathrm{Ext}^2_{\mathcal{O}(D_1 \rtimes E)}((\alpha_1, \chi_1), ({}^{g_i}\alpha_2, \chi_{g_i}) \uparrow^{D_1 \rtimes E_{\alpha_1}}_{D_1 \rtimes ({}^{g_i}E_{\alpha_2} \cap E_{\alpha_1})}) \simeq \bigoplus_{i=1}^{s'} \mathcal{O}/p^{m'_i}\mathcal{O}, \tag{4.5}$$

for some $s' \in \mathbb{N}$ and $m'_1, \dots, m'_s \in \mathbb{N}$. We now proceed by induction on $|E|$. We first note that the result holds in the base case $E = Z$ by Lemma 4.2.1(iii) and (iv). Next, we see that we can perform the inductive step, using (4.5), unless $E_{\alpha_1} = E$. In this case, by Lemma 4.2.7, $\alpha_1 = 1_{D_1}$. We now reverse the roles of $(\alpha_1, \chi_1) \uparrow^{D_1 \rtimes E}_{D_1 \rtimes E_{\alpha_1}}$ and $(\alpha_2, \chi_2) \uparrow^{D_1 \rtimes E}_{D_1 \rtimes E_{\alpha_2}}$ in (4.4), i.e.

$$\begin{aligned} &\mathrm{Ext}^2_{\mathcal{O}(D_1 \rtimes E)}((\alpha_1, \chi_1) \uparrow^{D_1 \rtimes E}_{D_1 \rtimes E_{\alpha_1}}, (\alpha_2, \chi_2) \uparrow^{D_1 \rtimes E}_{D_1 \rtimes E_{\alpha_2}}) \\ \simeq\ &\mathrm{Ext}^2_{\mathcal{O}(D_1 \rtimes E_{\alpha_2})}((\alpha_1, \chi_1) \uparrow^{D_1 \rtimes E}_{D_1 \rtimes E_{\alpha_1}} \downarrow^{D_1 \rtimes E}_{D_1 \rtimes E_{\alpha_2}}, (\alpha_2, \chi_2)), \end{aligned}$$

and proceed as above. This time the claim follows by induction unless $E_{\alpha_2} = E$. In this case we have $\alpha_1 = \alpha_2 = 1_{D_1}$ and the result is proved. □

Property 4.2.9. *We say a subset $X \subseteq \mathrm{Irr}(B)$ is good if it satisfies the following properties:*

(i) Each $\chi \in X$ is a lift of a Brauer character and each $\psi \in \mathrm{IBr}(B)$ has exactly one lift in X.

(ii) If $p > 2$, then for every $\chi_1, \chi_2 \in X$, there exist $s \in \mathbb{N}_0$ and $m_1, \dots, m_s \in \mathbb{N}$ such that

$$\mathrm{Ext}^2_{\mathcal{O}G}(\chi_1, \chi_2) \simeq \bigoplus_{i=1}^{s} \mathcal{O}/p^{m_i}\mathcal{O}.$$

If $p = 2$, then for every $\chi_1, \chi_2 \in X$, there exist $s \in \mathbb{N}_0$ and $m_1, \dots, m_s \in \mathbb{N}_{>1}$ such that

$$\mathrm{Ext}^2_{\mathcal{O}G}(\chi_1, \chi_2) \simeq \bigoplus_{i=1}^{s} \mathcal{O}/p^{m_i}\mathcal{O}.$$

Proposition 4.2.10. *A subset $X \subseteq \mathrm{Irr}(B)$ is good if and only if there exists some $\theta \in \mathrm{Irr}(D_2)$ such that*

$$X = \{(1_{D_1}, \chi) \otimes \theta | \chi \in \mathrm{Irr}(E|\varphi)\} = \mathrm{Irr}(B|1_{D_1} \otimes \theta).$$

Proof. As with Lemma 4.2.8, we only prove the $p > 2$ case. The proposition is proved for $p = 2$ in an identical way, once we have taken into account Assumption 4.2.2 and the differences in Lemma 4.2.8 and Property 4.2.9, in the $p = 2$ case.

We first assume X is of the desired form and let θ be the relevant irreducible character of D_2. By Remark 4.2.3, X satisfies Property 4.2.9(i). Now, by Lemma 4.2.1(ii) and (iv),

$$\begin{aligned} &\mathrm{Ext}^2_{\mathcal{O}G}((1_{D_1} \otimes \theta) \uparrow^G_D, (1_{D_1} \otimes \theta) \uparrow^G_D) \simeq \mathrm{Ext}^2_{\mathcal{O}D}((1_{D_1} \otimes \theta), (1_{D_1} \otimes \theta) \uparrow^G_D \downarrow^G_D) \\ \simeq\ &\mathrm{Ext}^2_{\mathcal{O}D}(1_{D_1} \otimes \theta, 1_{D_1} \otimes \theta)^{\oplus [G:D]} \simeq \bigoplus_{i=1}^{t} (\mathcal{O}/p^{n_i}\mathcal{O})^{\oplus [G:D]}. \end{aligned}$$

Since every $\chi \in X$ is an irreducible constituent of $(1_{D_1} \otimes \theta) \uparrow_D^G$, X satisfies Property 4.2.9(ii) and X is good.

For the remainder of the proof we identify $\mathrm{IBr}(B)$ with $\mathrm{Irr}(E|\varphi)$ (and do other analogous identifications), using Remark 4.2.3.

For the converse, suppose X is good. For any $\mu \in \mathrm{IBr}(B) = \mathrm{Irr}(E|\varphi)$, let $(\lambda_\mu, \chi_\mu) \uparrow_{G_{\lambda_\mu}}^G \in X$ be the unique lift in X of μ, for some $\lambda_\mu \in \mathrm{Irr}(D)$ and $\chi_\mu \in \mathrm{Irr}(E_{\lambda_\mu}|\varphi)$. In particular,

$$\mu = (\lambda_\mu, \chi_\mu) \uparrow_{D \rtimes E_{\lambda_\mu}}^G \downarrow_E^G = \chi_\mu \uparrow_{E_{\lambda_\mu}}^E, \tag{4.6}$$

for each $\mu \in \mathrm{Irr}(E|\varphi)$. Next, let $\mu \neq \psi \in \mathrm{Irr}(E|\varphi)$ satisfy $\mathrm{Ext}^1_{kG}(\mu, \psi) \neq \{0\}$. Since X satisfies Property 4.2.9(ii),

$$\mathrm{Ext}^2_{\mathcal{O}G}((\lambda_\mu, \chi_\mu) \uparrow_{G_{\lambda_\mu}}^G, (\lambda_\psi, \chi_\psi) \uparrow_{G_{\lambda_\psi}}^G) \simeq \bigoplus_{i=1}^s \mathcal{O}/p^{m_i}\mathcal{O},$$

for some $s \in \mathbb{N}$ and $m_1, \dots, m_s \in \mathbb{N}$. (Lemma 4.2.6(i) ensures that $s > 0$.) Therefore, by Lemma 4.2.8, $\lambda_\mu \sim_E \lambda_\psi$. Now, by Lemma 4.2.6(ii) and the fact that X satisfies Property 4.2.9(i), we can say the same thing for any pair of characters in X. In other words, we may assume that all the λ_μ's are equal to some fixed $\lambda \in \mathrm{Irr}(D)$. Now, for not necessarily distinct $\mu, \psi \in \mathrm{Irr}(E|\varphi)$,

$$\mathrm{Ext}^2_{\mathcal{O}G}((\lambda, \chi_\mu) \uparrow_{G_\lambda}^G, (\lambda, \chi_\psi) \uparrow_{G_\lambda}^G) \simeq \mathrm{Ext}^2_{\mathcal{O}G_\lambda}((\lambda, \chi_\mu), (\lambda, \chi_\psi) \uparrow_{G_\lambda}^G \downarrow_{G_\lambda}^G)$$
$$\simeq \mathrm{Ext}^2_{\mathcal{O}G_\lambda}\left((\lambda, \chi_\mu), \sum_{g \in E_\lambda \backslash E / E_\lambda} {}^g(\lambda, \chi_\psi) \downarrow^{{}^gG_\lambda}_{{}^gG_\lambda \cap G_\lambda} \uparrow^{G_\lambda}_{{}^gG_\lambda \cap G_\lambda}\right).$$

Of course, the only $\lambda' \in \mathrm{Irr}(D)$ E_λ-conjugate to λ is λ itself. Therefore, the only irreducible constituent of $(\lambda, \chi_\psi) \uparrow_{G_\lambda}^G \downarrow_{G_\lambda}^G$ of the form $(\lambda', \chi') \uparrow_{G_{\lambda'}}^G$, for some $\lambda' \in \mathrm{Irr}(D)$ and $\chi' \in \mathrm{Irr}(E_{\lambda'}|\varphi)$, with λ' E_λ-conjugate to λ is (λ, χ_ψ), with multiplicity one. Therefore, since X satisfies Property 4.2.9(ii), Lemma 4.2.8 applied to $\mathcal{O}G_\lambda e_\varphi$ gives that

$$\mathrm{Ext}^2_{\mathcal{O}G_\lambda}((\lambda, \chi_\mu) \uparrow_{G_\lambda}^G, (\lambda, \chi_\psi) \uparrow_{G_\lambda}^G) \simeq \mathrm{Ext}^2_{\mathcal{O}G_\lambda}((\lambda, \chi_\mu), (\lambda, \chi_\psi))$$

and

$$\mathrm{Ext}^2_{\mathcal{O}G_\lambda}((\lambda, \chi_\mu), {}^g(\lambda, \chi_\psi) \downarrow^{{}^gG_\lambda}_{{}^gG_\lambda \cap G_\lambda} \uparrow^{G_\lambda}_{{}^gG_\lambda \cap G_\lambda}) = \{0\}, \tag{4.7}$$

for any $g \in E \setminus E_\lambda$.

Now, since X satisfies Property 4.2.9(ii), (4.6) gives that

$$\begin{aligned}\delta_{\mu,\psi} = \langle \mu, \psi\rangle_E = \langle \chi_\mu \uparrow_{E_\lambda}^{E}, \chi_\psi \uparrow_{E_\lambda}^{E}\rangle_E = \langle \chi_\mu, \chi_\psi \uparrow_{E_\lambda}^{E}\downarrow_{E_\lambda}^{E}\rangle_{E_\lambda} \\ = \left\langle \chi_\mu, \sum_{g\in E_\lambda\backslash E/E_\lambda} {}^{g}\chi_\psi \downarrow^{{}^{g}E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\uparrow^{E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\right\rangle_{E_\lambda},\end{aligned}$$

for all $\mu, \psi \in \operatorname{Irr}(E|\varphi)$. In other words, for any $g \in E$,

$$\langle \chi_\mu, {}^{g}\chi_\psi \downarrow^{{}^{g}E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\uparrow^{E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\rangle_{E_\lambda} = \begin{cases} 1 & \text{if } \mu=\psi \text{ and } g\in E_\lambda, \\ 0 & \text{otherwise.}\end{cases}$$

Therefore, since

$${}^{g}(\lambda, \chi_\psi) \downarrow^{{}^{g}G_\lambda}_{{}^{g}G_\lambda\cap G_\lambda}\uparrow^{G_\lambda}_{{}^{g}G_\lambda\cap G_\lambda}\downarrow^{G_\lambda}_{E_\lambda} = {}^{g}\chi_\psi \downarrow^{{}^{g}E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\uparrow^{E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda},$$

we can apply Lemma 4.2.6(i) to (4.7) to give

$$\operatorname{Ext}^1_{kG_\lambda}(\chi_\mu, {}^{g}\chi_\psi \downarrow^{{}^{g}E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\uparrow^{E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}) = \{0\}, \tag{4.8}$$

for any $\mu, \psi \in \operatorname{Irr}(E|\varphi)$ and $g \in E \setminus E_\lambda$. Certainly every $\eta \in \operatorname{Irr}(E_\lambda|\varphi)$ is an irreducible constituent of

$$\psi \downarrow^{E}_{E_\lambda} = \chi_\psi \uparrow^{E}_{E_\lambda}\downarrow^{E}_{E_\lambda} = \sum_{g\in E_\lambda\backslash E/E_\lambda} {}^{g}\chi_\psi \downarrow^{{}^{g}E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda}\uparrow^{E_\lambda}_{{}^{g}E_\lambda\cap E_\lambda},$$

for some $\psi \in \operatorname{Irr}(E|\varphi)$, where the first equality follows from (4.6). Therefore, (4.8) implies that $\operatorname{Ext}^1_{kG_\lambda}(\chi_\mu, \eta) \neq \{0\}$, for some $\eta \in \operatorname{Irr}(E_\lambda|\varphi)$, only if $\eta = \chi_\psi$, for some $\psi \in \operatorname{Irr}(E|\varphi)$. Lemma 4.2.6(ii) applied to the block $\mathcal{O}(D \rtimes E_\lambda)e_\varphi$ now gives that

$$\operatorname{Irr}(E_\lambda|\varphi) = \{\chi_\mu | \mu \in \operatorname{Irr}(E|\varphi)\}.$$

So, again by (4.6), we have a one-to-one bijection between $\operatorname{Irr}(E_\lambda|\varphi)$ and $\operatorname{Irr}(E|\varphi)$ given by induction. This implies $E_\lambda = E$, since

$$\begin{aligned}[E : E_\lambda].\dim_K(KE_\lambda e_\varphi) = \dim_K(KEe_\varphi) = \sum_{\chi\in\operatorname{Irr}(E|\varphi)} \chi(1)^2 \\ = \sum_{\eta\in\operatorname{Irr}(E_\lambda|\varphi)} \eta\uparrow^{E}_{E_\lambda}(1)^2 = [E : E_\lambda]^2. \sum_{\eta\in\operatorname{Irr}(E_\lambda|\varphi)} \eta(1)^2 = [E : E_\lambda]^2.\dim_K(KE_\lambda e_\varphi).\end{aligned}$$

Therefore, by Lemma 4.2.7, $\lambda = 1_{D_1} \otimes \theta$, for some $\theta \in \operatorname{Irr}(D_2)$, as required. □

4.3 Weiss' criterion and the main theorems

Before proceeding we need to set up some notation. Let b be a block of $\mathcal{O}H$, for some finite group H and Q a normal p-subgroup of H. We denote by b^Q the direct sum of blocks of $\mathcal{O}(H/Q)$ dominated by b, that is those blocks not annihilated by the image of e_b under the natural $\mathcal{O}$-algebra homomorphism $\mathcal{O}H \to \mathcal{O}(H/Q)$. It will also be necessary to define $\mathcal{T}(b)$ for b a sum of blocks of H. Note our definition of $\mathrm{Pic}(b)$ already makes sense for b a sum of blocks.

$$\mathcal{T}(b) = \{[M]_\sim \in \mathrm{Pic}(b) | M \text{ is a direct sum of modules with trivial source} \\ \text{as } \mathcal{O}(H \times H)\text{-modules}\}.$$

The following Proposition was shown to be a consequence of Weiss' criterion in [29, Propositions 4.3,4.4]. Weiss' criterion is a statement about permutation modules originally stated in [85, Theorem 2] but proved in its most general form in [63, Theorem 1.2]. It has, in recently years, been a very important tool for calculating Picard groups.

Proposition 4.3.1.

(i) The inflation map $\mathrm{Inf}_{H/Q}^H : \mathrm{Irr}(H/Q) \to \mathrm{Irr}(H)$ *induces a bijection between* $\mathrm{Irr}(b^Q)$ *and* $\mathrm{Irr}(b|1_Q)$.

(ii) Suppose M is a b-b-bimodule inducing a Morita auto-equivalence of b that permutes the elements of $\mathrm{Irr}(b|1_Q)$.

(a) Then ${}^QM$, the set of fixed points of M under the left action of Q, induces a Morita auto-equivalence of b^Q. Furthermore, the permutation of $\mathrm{Irr}(b^Q)$ *induced by ${}^QM$ is identical to the permutation that M induces on* $\mathrm{Irr}(b|1_Q)$*, once these two sets have been identified using part (i).*

(b) If ${}^QM \in \mathcal{T}(b^Q)$, then $M \in \mathcal{T}(b)$.

We immediately drop all the assumptions on b made just before Proposition 4.3.1. We are now ready to prove our main theorem.

Theorem 4.3.2. *Let $p > 2$ and b a block with normal defect group. Then* $\mathrm{Pic}(b) = \mathcal{L}(b)$.

Proof. We first reduce to the situation that $b = B$ as in Assumption 4.2.2(i), where the defect group of b is isomorphic to D and its inertial quotient is isomorphic to L. Indeed it follows from [48, Theorem A] that such a B can be chosen to be Morita equivalent to b and even source algebra equivalent by [56, Theorem 6.14.1]. Note that, by [56, Theorem 6.7.6(v)], L must be prime to p and Z must be as well, since otherwise B has defect group larger than that of b. In other words E is indeed a p'-group as assumed in Assumption 4.2.2(i).

By [10, Lemma 2.8], the above equivalence yields isomorphisms $\mathrm{Pic}(b) \simeq \mathrm{Pic}(B)$ and $\mathcal{L}(b) \simeq \mathcal{L}(B)$. Therefore, since we are concerned only with $\mathrm{Pic}(b)$ and $\mathcal{L}(b)$, we may assume that $b = B$.

Let $M \in \mathrm{Pic}(B)$. We first show that the elements of $\mathrm{Irr}(B|1_{[D,D]})$ get permuted by the permutation of $\mathrm{Irr}(B)$ induced by M. This claim will follow from [15, Lemme 1.6] once we have shown that $\mathrm{Irr}(B|1_{[D,D]})$ is precisely the subset of irreducible characters of B of height zero, i.e. $\mathrm{Irr}(B|1_{[D,D]})$ is precisely the subset of irreducible characters of B of p'-degree.

If $\eta \in \mathrm{Irr}(B|1_{[D,D]})$ then we can view $\eta \in \mathrm{Irr}(B^{[D,D]})$ via Proposition 4.3.1(i) and hence as the character of a block with normal abelian defect group. After possibly applying the reduction from the beginning of the proof and [15, Lemme 1.6] again, it follows from the description around (4.1) that η has p'-degree.

Conversely if $\eta \in \mathrm{Irr}(B|\lambda)$, for some non-linear $\lambda \in \mathrm{Irr}(D)$, then $\eta \downarrow^G_D$ is a sum of conjugates of λ. In particular, $\lambda(1) \mid \eta(1)$ and η does not have p'-degree.

We can now apply Proposition 4.3.1(ii)(a) with respect to M and $[D, D]$ to obtain ${}^{[D,D]}M \in \mathrm{Pic}(B^{[D,D]})$. In this case [49, Corollary 4] tells us that $B^{[D,D]}$ is just a single block of $\mathcal{O}(G/[D, D])$. Actually, $B^{[D,D]}$ is of the form described in Assumption 4.2.2(ii). Indeed, if $h \in E$ acts trivially on $D/[D, D]$, then h must also act trivially on each factor group $\gamma_i(D)/\gamma_{i+1}(D)$ in the lower central series of D, so, by [35, Corollary 5.3.3], the subgroup generated by h in $\mathrm{Aut}(D)$ is a p-group, and, since h has p'-order, $h \in Z$. In conclusion we have $C_E(D/[D, D]) = Z$ and we adopt the slight abuse of notation $D/[D, D] = D_1 \times D_2$ from §4.2.

Note that by Lemma 4.2.4(ii), the notion of being good from Property 4.2.9 is a Morita invariant. Therefore, since $B^{[D,D]}$ is now of the desired form, we can apply Proposition 4.2.10 to obtain that there exists some $\theta \in \mathrm{Irr}(D_2)$ such that $\mathrm{Irr}(B^{[D,D]}|1_{D/[D,D]})$ gets sent to $\mathrm{Irr}(B^{[D,D]}|1_{D_1} \otimes \theta)$ under the permutation of $\mathrm{Irr}(B^{[D,D]})$ induced by ${}^{[D,D]}M$. The last sentence in Proposition 4.3.1(ii)(a) now implies that $\mathrm{Irr}(B|1_D)$ gets sent to $\mathrm{Irr}(B|\,\mathrm{Inf}^D_{D/[D,D]}(1_{D_1} \otimes \theta))$ under the permutation of $\mathrm{Irr}(B)$ induced by M.

Set $\mathbf{D}_1$ to be the preimage of D_1 in D. Since $D_2 \simeq G/(\mathbf{D}_1 \rtimes E)$, we can set $\omega \in \mathrm{Irr}(G)$ to be the inflation of θ to G and $M_{\omega^{-1}} \in \mathrm{Pic}(B)$ to be the B-B-bimodule inducing the Morita auto-equivalence given by tensoring with ω^{-1}. Now $\mathrm{Irr}(B|1_D)$ gets permuted under the permutation of $\mathrm{Irr}(B)$ induced by $M_{\omega^{-1}} \otimes_B M$. We can therefore apply Proposition 4.3.1(ii) with respect to $M_{\omega^{-1}} \otimes_B M$ and D to obtain that $M_{\omega^{-1}} \otimes_B M \in \mathcal{T}(B)$. ($G/D$ is a p'-group and so certainly ${}^{D}(M_{\omega^{-1}} \otimes_B M) \in \mathcal{T}(B^D)$.) Since $\mathcal{T}(B)$ is a subgroup of $\mathcal{L}(B)$, it remains only to show that $M_{\omega^{-1}} \in \mathcal{L}(B)$. However,

$$M_{\omega^{-1}} = \mathcal{O}_{\Delta\omega^{-1}} \uparrow_{\Delta G}^{G \times G},$$

where $\mathcal{O}_{\Delta\omega^{-1}}$ is the $\mathcal{O}(\Delta G)$-module $\mathcal{O}$ with the action of ΔG given by $(g,g).m = \omega(g)^{-1}m$, for all $m \in \mathcal{O}$, so $M_{\omega^{-1}}$ certainly has linear source. □

Theorem 4.3.3. *Let $p = 2$ and b a block with normal defect group D such that $D/[D,D]$ has no direct factor isomorphic to C_2. Then* $\mathrm{Pic}(b) = \mathcal{L}(b)$.

Proof. The proof is identical to that of Theorem 4.3.2. Assumption 4.2.2(ii) ensured that we were only considering blocks with abelian defect group that had no direct factor isomorphic to C_2 throughout §4.2. We can therefore apply Proposition 4.2.10 in exactly the same way we did in the proof of Theorem 4.3.2. □

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