Robust Parametric Tests of Constant Conditional Correlation in a MGARCH model: Full Monte Carlo and Detailed Proofs

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Abstract

This paper provides additional Monte Carlo information and detailed proofs of the results presented in Shadat and Orme (2015), "Robust Parametric Tests of Constant Conditional Correlation in a MGARCH model."

Introduction

In this paper, to accompany Shadat and Orme (2015), we provide (i) all the tables referred to in the Monte Carlo section of the main text; (ii) the main results and Propositions; and, (iii) detailed proofs of all results- which are dealt with only briefly in the main paper. All definitions are provided in Shadat and Orme (2015) are not reproduced here, except when done so within the context of stating or proving a Proposition.

Monte Carlo Results

Empirical Significance Levels

We employ AR(1)-CCC-GARCH (1,1) DGP for $N = 5$ as our null model; viz.,

$$\begin{aligned} y_{it} &= \varphi_{i0} + \varphi_{i1} y_{i,t-1} + \varepsilon_{it}, \quad i = 1, \cdots, 5 \\ Var(\varepsilon_t|\mathcal{F}_{t-1}) &= H_t \Rightarrow E\left[\varepsilon_{it}^2|\mathcal{F}_{t-1}\right] = h_{it},\ \varepsilon_t = H_t^{1/2}(\omega)\,\xi_t,\ \xi_t \sim N(0, I), \\ h_{i,t} &= \alpha_{i0} + \alpha_{i1}\varepsilon_{i,t-1}^2 + \beta_{i1} h_{i,t-1},\ \eta_i' = (\alpha_{i0}, \alpha_{i1}, \beta_{i1}) \\ H_t &= D_t \Gamma D_t,\ D_t = \text{diag}\left(\sqrt{h_{it}}\right) \text{ and} \\ \Gamma &= \left\{\rho_{ij}\right\}, i, j = 1, \cdots, 5 \text{ with } \rho_{ii} = 1. \end{aligned} \tag{1}$$

Three experiments are considered E1, E2 and E3 and the true parameter vectors employed are given in Table A1.

Table A2 reports the rejection frequencies when the null of the CCC is true under both Gaussian and non-Gaussian errors. Apart from investigating the robustness of these tests under non-normality, where the elements of ξ_{0t} are independently and identically distributed as $t(6)$ offers some evidence on the robustness of the procedure to violations of the underlying moment assumptions, since for the choice of test variables

$$r_{ij,t} = \frac{\varepsilon_{i,t-1}\varepsilon_{j,t-1}}{\sqrt{h_{i,t-1}}\sqrt{h_{j,t-1}}},$$

8^{th} order moments are required. The results are reported for a nominal significance level of 5%.

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Table A1: True parameter values for size simulation

	E1	E2	E3
$(\varphi_1, \eta_1)'$		$(1.00, 0.10, 0.01, 0.15, 0.70)$	
$(\varphi_2, \eta_2)'$		$(1.00, 0.50, 0.02, 0.10, 0.75)$	
$(\varphi_3, \eta_3)'$		$(1.00, 0.60, 0.15, 0.25, 0.60)$	
$(\varphi_4, \eta_4)'$		$(0.50, 0.70, 0.05, 0.16, 0.69)$	
$(\varphi_5, \eta_5)'$		$(1.00, 0.30, 0.05, 0.35, 0.50)$	
$\text{vecl}\,(\Gamma)'$	$\begin{pmatrix} 0.33 & 0.20 & 0.25 & 0.19 & 0.21 \\ 0.24 & 0.35 & 0.34 & 0.37 & 0.30 \end{pmatrix}$	$\begin{pmatrix} 0.30 & 0.45 & 0.75 & 0.60 & 0.55 \\ 0.80 & 0.40 & 0.70 & 0.65 & 0.35 \end{pmatrix}$	$\begin{pmatrix} 0.70 & 0.64 & 0.75 & 0.62 & 0.76 \\ 0.80 & 0.72 & 0.74 & 0.69 & 0.79 \end{pmatrix}$

Note: The parameter vectors refer to the AR(1)-CCC-GARCH(1,1) model in (1).

Robustness to Misspecified Univariate Volatility

We consider 12 experiments (M1a-M1c, M2a-M2c, M3a-M3c and M4a-M4c), each within the regression context to investigate, via Monte Carlo simulation, the impact of violations in the univariate GARCH specification, but when the true correlation structure for ζ_{0t} is constant with Gaussian error. The conditional mean parameters and the correlation structures remain the same as those previously employed, as detailed in Table A1. For M1, M2 and M3 the univariate volatility specifications of all five variables are governed by the GJR, higher order GARCH (i.e., GARCH(2,2)) and the EGARCH models, respectively whereas for M4 all 5 variables are subject to volatility spillover via an ECCC model. The suffix a, b or c associated with these experiments indicate low, mixed and high correlation structure, respectively, for Γ.

Table A3 and Table A4 report the rejection frequencies based on both the 5% empirical and nominal critical values (with the latter in the parenthesis) and with 2000 replications where the data are generated with normal errors; i.e., in the former case, and for each test procedure, "size-adjusted" rejection frequencies are reported, calculated using the empirical critical value that delivers a 5% significance level for the simulations reported in Table A1. Only Table A3 is reported in the main paper.

Table A2: CCC Models: Empirical Significance Levels against 5% nominal level

	AR(1)-CCC-GARCH(1,1) DGP					
	E1		E2		E3	
	T=500	T=1000	T=500	T=1000	T=500	T=1000
	Gaussian errors					
$\widehat{LM}_T^{*(o)}$	12.98	8.52	16.07	10.18	15.65	10.31
$\widehat{LM}_T^{*(r)}$	4.08	4.79	5.02	5.07	4.78	5.32
$\widetilde{LM}_T^{(o)}$	7.43	6.54	3.17	4.16	3.71	4.62
$\widetilde{LM}_T^{(r)}$	4.03	4.61	4.56	4.76	3.99	4.52
$\hat{S}_T^{*C(o)}$	15.33	10.01	16.75	11.61	17.81	11.77
$\hat{S}_T^{*C(r)}$	4.32	4.70	5.11	5.87	4.82	5.65
$\tilde{S}_T^{C(o)}$	11.84	8.71	6.57	6.82	6.81	7.08
$\tilde{S}_T^{C(r)}$	4.37	4.61	5.02	5.61	5.09	5.76
$\hat{S}_T^{*J(o)}$	24.74	15.04	24.93	16.10	26.15	16.77
$\hat{S}_T^{*J(r)}$	6.40	6.10	7.15	7.06	6.26	7.03
$\tilde{S}_T^{J(o)}$	16.92	12.04	9.78	9.40	10.54	9.99
$\tilde{S}_T^{J(r)}$	5.77	5.95	4.82	5.09	4.37	5.16
	$t(6)$ errors					
$\widehat{LM}_T^{*(o)}$	20.90	11.80	27.25	19.40	27.95	20.10
$\widehat{LM}_T^{*(r)}$	3.10	4.00	4.80	6.90	5.75	5.75
$\widetilde{LM}_T^{(o)}$	10.25	7.70	2.95	4.75	3.45	3.40
$\widetilde{LM}_T^{(r)}$	3.15	4.30	4.80	6.30	4.70	4.40
$\hat{S}_T^{*C(o)}$	26.30	16.85	32.10	24.70	30.80	23.10
$\hat{S}_T^{*C(r)}$	4.90	4.95	5.75	7.30	5.40	6.60
$\tilde{S}_T^{C(o)}$	15.50	10.80	8.90	9.45	7.70	8.40
$\tilde{S}_T^{C(r)}$	4.75	4.65	4.80	6.15	4.95	5.55
$\hat{S}_T^{*J(o)}$	46.15	32.45	46.50	34.65	46.55	33.70
$\hat{S}_T^{*J(r)}$	7.15	7.95	7.15	9.80	7.60	8.80
$\tilde{S}_T^{J(o)}$	31.85	22.60	16.10	17.10	15.25	14.95
$\tilde{S}_T^{J(r)}$	5.80	7.45	3.90	7.00	3.95	4.55

Notes:
1. Parameter values as detailed in Table 2.
2. The first block reports results for Tse's LM test, the second and third blocks those for CCM and FCM tests, respectively. Within each block the order is: OPG-FQMLE, ROBUST-FQMLE, OPG-PQMLE, ROBUST-PQMLE.
3. T is the sample size and results are based on 10,000 simulations for Gaussian errors and 2,000 for t(6) errors.

Table A3: Univariate volatility misspecification I: Empirical Rejection Rates against 5% empirical (nominal) critical value

	AR(1)-GJR-CCC DGP						AR(1)-EGARCH-CCC DGP					
	M1a		M1b		M1c		M3a		M3b		M3c	
	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000
$\widehat{LM}_T^{*(o)}$	5.70	5.25	6.05	5.20	5.90	4.70	19.45	12.05	77.35	86.30	78.30	86.00
	(15.25)	(9.45)	(16.70)	(11.20)	(15.10)	(10.90)	(34.50)	(16.70)	(88.90)	(90.90)	(88.50)	(90.50)
$\widehat{LM}_T^{*(r)}$	5.85	4.80	5.30	4.60	4.95	4.55	4.80	4.00	7.65	14.30	11.70	18.00
	(4.85)	(4.55)	(4.70)	(4.70)	(4.75)	(5.05)	(4.10)	(3.70)	(7.75)	(14.50)	(11.25)	(20.00)
$\widetilde{LM}_T^{(o)}$	5.50	5.00	5.80	5.20	5.25	4.65	14.65	9.30	27.15	45.50	30.30	56.00
	(8.15)	(6.60)	(4.60)	(4.15)	(4.10)	(4.50)	(18.80)	(11.30)	(22.20)	(42.15)	(26.30)	(51.00)
$\widetilde{LM}_T^{(r)}$	5.60	4.80	6.60	5.75	5.90	5.20	5.25	4.25	7.05	12.95	8.00	15.00
	(4.75)	(4.40)	(5.60)	(5.40)	(4.85)	(4.85)	(4.20)	(3.85)	(6.60)	(12.40)	(6.85)	(13.00)
$\hat{S}_T^{*C(o)}$	4.65	5.00	5.00	5.30	4.95	5.15	29.50	12.60	83.65	97.20	89.95	97.70
	(15.90)	(9.75)	(18.00)	(11.55)	(18.70)	(12.45)	(48.70)	(18.95)	(92.35)	(98.40)	(96.15)	(99.10)
$\hat{S}_T^{*C(r)}$	4.90	4.25	5.15	4.90	5.00	5.20	7.40	5.00	30.90	61.80	37.15	71.50
	(4.25)	(4.10)	(5.55)	(6.00)	(4.90)	(5.80)	(6.30)	(4.65)	(31.60)	(65.35)	(36.60)	(77.00)
$\tilde{S}_T^{C(o)}$	3.95	3.80	4.95	4.65	5.30	4.90	18.65	9.05	55.80	85.30	68.95	95.00
	(9.70)	(6.85)	(7.15)	(6.05)	(7.05)	(7.30)	(29.85)	(14.10)	(60.30)	(88.15)	(72.95)	(98.00)
$\tilde{S}_T^{C(r)}$	4.60	4.25	4.90	4.75	5.05	5.05	7.20	4.95	29.10	62.95	39.10	83.10
	(4.15)	(3.95)	(5.05)	(5.40)	(5.05)	(5.95)	(6.50)	(4.60)	(29.25)	(64.35)	(39.50)	(85.00)
$\hat{S}_T^{*J(o)}$	4.65	5.45	4.20	4.75	4.30	4.65	51.75	37.95	89.35	99.30	91.80	99.10
	(25.15)	(15.40)	(23.80)	(14.50)	(25.80)	(15.65)	(80.65)	(58.15)	(98.15)	(99.90)	(98.20)	(99.95)
$\hat{S}_T^{*J(r)}$	5.25	5.25	4.60	4.45	5.30	5.00	5.75	7.20	24.20	68.80	25.75	71.20
	(6.40)	(6.35)	(6.70)	(6.65)	(6.75)	(6.70)	(7.60)	(9.70)	(30.25)	(76.35)	(31.30)	(79.00)
$\tilde{S}_T^{J(o)}$	4.20	4.50	4.50	4.50	5.25	4.95	23.70	25.00	54.40	91.95	62.90	93.80
	(15.60)	(11.45)	(11.90)	(9.30)	(11.15)	(9.45)	(52.25)	(41.85)	(67.95)	(95.25)	(74.00)	(97.00)
$\tilde{S}_T^{J(r)}$	4.85	5.10	5.40	5.45	5.00	4.85	5.90	7.85	29.10	72.25	33.80	80.00
	(5.30)	(6.20)	(4.75)	(5.70)	(4.15)	(5.10)	(6.40)	(9.50)	(28.40)	(72.80)	(31.15)	(81.00)

Notes:
1. The suffix a,b,c denotes the low, mixed, high correlation structure, respectively.
2. T is the sample size and results are based on 2,000 simulations.
3. For each test the first (second) row report rejection rates using empirical (nominal) critical values; i.e., figures in the first row for ecah test statistic report size-adjusted rejection rates.

Table A4: Univariate volatility misspecification II: Empirical Rejection Rates against 5% empirical (nominal) critical value

	AR(1)-GARCH(2,2)-CCC DGP						AR(1)-GARCH-ECCC DGP					
	M2a		M2b		M2c		M4a		M4b		M4c	
	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000
$\widehat{LM}_T^{*(o)}$	4.95	4.50	5.50	5.30	5.70	5.80	4.70	4.45	5.20	4.90	5.05	5.65
	(12.60)	(8.10)	(15.50)	(10.85)	(15.10)	(11.95)	(12.85)	(8.45)	(16.95)	(10.05)	(15.70)	(12.15)
$\widehat{LM}_T^{*(r)}$	5.15	4.60	5.65	5.75	6.05	6.05	4.40	3.95	5.05	5.00	5.05	5.55
	(4.20)	(4.30)	(5.65)	(5.75)	(5.90)	(6.60)	(3.65)	(3.85)	(5.05)	(5.15)	(4.75)	(5.80)
$\widetilde{LM}_T^{(o)}$	5.25	4.35	5.05	5.25	5.20	5.90	4.75	4.40	4.10	4.75	4.75	6.15
	(7.30)	(6.15)	(3.30)	(4.45)	(4.15)	(5.35)	(7.30)	(5.80)	(2.75)	(3.95)	(3.45)	(5.70)
$\widetilde{LM}_T^{(r)}$	5.30	4.45	5.50	5.30	6.10	6.35	4.65	4.25	5.05	5.10	5.40	6.40
	(4.15)	(4.10)	(5.00)	(5.05)	(4.85)	(5.70)	(3.65)	(3.85)	(4.45)	(4.70)	(4.45)	(5.45)
$\hat{S}_T^{*C(o)}$	4.20	4.80	6.45	6.85	6.10	5.75	12.95	11.85	11.70	9.95	10.75	8.70
	(13.55)	(10.30)	(19.50)	(13.90)	(20.30)	(13.85)	(28.90)	(21.00)	(28.65)	(19.35)	(26.20)	(16.50)
$\hat{S}_T^{*C(r)}$	4.90	5.35	6.65	6.85	6.55	5.95	4.35	4.65	5.85	4.90	6.40	5.15
	(4.20)	(5.05)	(6.70)	(8.00)	(6.40)	(6.80)	(3.50)	(4.25)	(6.00)	(6.15)	(6.30)	(5.60)
$\tilde{S}_T^{C(o)}$	4.35	4.90	6.80	8.25	7.60	6.95	5.10	4.70	5.55	4.85	5.85	5.60
	(10.75)	(8.55)	(8.95)	(10.50)	(9.45)	(9.70)	(11.55)	(9.00)	(7.15)	(6.75)	(7.70)	(8.10)
$\tilde{S}_T^{C(r)}$	4.90	5.40	6.10	7.30	5.70	6.50	4.35	4.75	5.25	4.80	5.25	5.45
	(4.50)	(5.15)	(6.10)	(7.85)	(5.85)	(7.15)	(3.80)	(4.45)	(5.25)	(5.35)	(5.40)	(6.00)
$\hat{S}_T^{*J(o)}$	7.50	9.50	7.25	11.50	8.15	10.10	10.00	10.05	9.35	7.80	8.70	6.55
	(31.50)	(23.45)	(33.05)	(26.45)	(33.50)	(26.85)	(36.85)	(25.00)	(34.75)	(21.75)	(34.20)	(21.00)
$\hat{S}_T^{*J(r)}$	8.20	10.20	8.20	11.95	8.45	10.50	4.35	4.95	4.95	4.90	5.80	4.50
	(10.25)	(12.30)	(10.95)	(15.60)	(10.20)	(14.05)	(5.70)	(6.20)	(7.25)	(6.95)	(7.20)	(6.10)
$\tilde{S}_T^{J(o)}$	8.50	10.10	8.25	12.45	9.20	10.20	4.40	4.70	5.00	4.55	5.25	4.55
	(25.00)	(20.75)	(15.90)	(18.95)	(17.75)	(18.60)	(16.15)	(11.80)	(10.20)	(9.50)	(11.05)	(9.30)
$\tilde{S}_T^{J(r)}$	8.50	9.70	7.75	11.30	8.00	9.60	4.95	5.30	6.40	5.80	5.55	5.70
	(9.50)	(11.55)	(7.45)	(11.40)	(7.00)	(9.90)	(5.55)	(6.30)	(6.20)	(5.95)	(4.85)	(5.75)

Notes:
1. The suffix a,b,c denotes the low, mixed, high correlation structure, respectively.
2. T is the sample size and results are based on 2,000 simulations.
3.For each test the first (second) row report rejection rates using empirical (nominal) critical values; i.e., figures in the first row for ecah test statistic report size-adjusted rejection rates

Power Results

To examine power, we consider three types of MGARCH models with time varying correlations. The AR(1) conditional mean specification, and parameters, remain as in (1) but now we examine three alternative specifications for the conditional variance matrix $H_t = Var(\varepsilon_t|\mathcal{F}_{t-1})$. The first is the DCC-GARCH(1,1) model where the dynamic correlation matrix, Γ_t, is given as

$$\begin{aligned} \Gamma_t &= (I \odot \Psi_t)^{-1/2} \Psi_t (I \odot \Psi_t)^{-1/2} = \text{diag}(\Psi_t)^{-1/2} \Psi_t \,\text{diag}(\Psi_t)^{-1/2}, \\ \Psi_t &= (1 - \tilde{\alpha} - \tilde{\beta})\bar{\Gamma} + \tilde{\alpha}\zeta_{t-1}\zeta_{t-1}' + \tilde{\beta}\Psi_{t-1}, \end{aligned} \tag{2}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are nonnegative scalar parameters and $\alpha + \beta < 1$ and $\bar{\Gamma}$ is constant (time invariant) 5×5 symmetric positive definite matrix, with ones on the diagonal. Secondly, we consider the following Varying Correlation (VC) model of Tse and Tsui (2002)

$$\Gamma_t = (1 - a - b)\,\bar{\Gamma} + a\Gamma_{t-1} + b\Psi_{t-1}, \tag{3}$$

where a and b are nonnegative scalar parameters, satisfying $a + b \leq 1$, and Ψ_{t-1} is the 5×5 sample correlation matrix of $\{\zeta_{t-1}, \cdots, \zeta_{t-5}\}$ and its $(i, j)^{th}$ element is given by:

$$\psi_{ij,t-1} = \frac{\sum_{m=1}^{5} \zeta_{i,t-m}\zeta_{j,t-m}}{\left(\sum_{m=1}^{5} \zeta_{i,t-m}^2\right)^{1/2} \left(\sum_{m=1}^{5} \zeta_{j,t-m}^2\right)^{1/2}}.$$

Finally we consider the BEKK model,

$$H_t = C_B + A_B' \left(\varepsilon_{t-1}\varepsilon_{t-1}'\right) A_B + B_B' H_{t-1} B_B. \tag{4}$$

In the following experiments the diagonal BEKK (DBEKK) model is employed where the parameter matrices A_B and B_B are 5×5 *diagonal* matrices.

Seven experiments are considered: P1, P2 and P3 follow the DCC DGP (2), P4 and P5 follow VC DGP (3) and remaining two, P6 and P7, follow the DBEKK DGP (4). In all cases, the individual volatility specification for all variables is retained from earlier size experiment, whilst for the DCC and VC DGPs the constant $\bar{\Gamma}$ matrix is set to the previously defined mixed correlation structure (see Table A1). The remaining true parameter vectors are given in Table A5.

Table A6 and Table A7 present the size-adjusted power (and nominal) results with 2000 replications for the above seven experiments, based on a 5% empirical (respectively nominal) significance level and the data are generated assuming normality. Only Table A6 is reported in the main paper. As a measure of the variability of the conditional correlation coefficients, we calculate the range (*maximum* - *minimum*) of the conditional correlation coefficients in each replicated sample of $T = 1000$ observations. In Table A8 we report, here, the *average, maximum* and *minimum* ranges of the true conditional correlation coefficients across the 2000 Monte Carlo samples.

Table A5: True parameter values for power simulation

	P1	P2	P3	P4	P5	P6	P7
$\widetilde{\alpha}$	0.05	0.10	0.15	-	-	-	-
$\widetilde{\beta}$	0.90	0.85	0.80	-	-	-	-
a	-	-	-	0.50	0.60	-	-
b	-	-	-	0.20	0.30	-	-
C_B	-	-	-			$\begin{pmatrix} 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 \end{pmatrix}$	
A_B	-	-	-			diag $\{0.30\}$	diag $\{0.40\}$
B_B	-	-	-			diag $\{0.40\}$	diag $\{0.40\}$

Notes: .

1. P1-P3: parameters $\widetilde{\alpha}, \widetilde{\beta}$ refer to the DCC-GARCH(1,1) model in (2)

2. P3-P4: parameters a, b refer to the VC-GARCH(1,1) model in (3)

3. P6-P7: parameter matrices A_B, B_B and C_B refer to the DBEKK(1,1) model in (4).

Table A6: DCC Models I: Empirical Rejection Rates against 5% empirical (nominal) critical value

	AR(1)-DCC-GARCH(1,1) DGP					
	P1		P2		P3	
	T=500	T=1000	T=500	T=1000	T=500	T=1000
$\widehat{LM}_T^{*(o)}$	14.45	29.95	60.60	91.45	93.10	99.95
	(34.85)	(42.75)	(79.30)	(95.00)	(97.35)	(99.95)
$\widehat{LM}_T^{*(r)}$	14.95	27.60	51.00	88.00	85.45	99.70
	(15.00)	(27.90)	(51.00)	(88.10)	(85.50)	(99.70)
$\widetilde{LM}_T^{(o)}$	13.90	24.85	44.00	85.60	74.80	98.75
	(9.80)	(27.60)	(37.55)	(83.50)	(69.20)	(98.30)
$\widetilde{LM}_T^{(r)}$	13.65	27.70	50.40	88.15	84.60	99.60
	(12.85)	(26.60)	(49.10)	(87.60)	(83.45)	(99.60)
$\hat{S}_T^{*C(o)}$	14.80	24.00	65.60	91.50	95.20	99.90
	(34.00)	(38.90)	(82.45)	(96.40)	(98.60)	(100.00)
$\hat{S}_T^{*C(r)}$	12.85	21.55	57.15	88.65	91.55	99.70
	(13.10)	(24.00)	(57.50)	(90.20)	(91.60)	(99.75)
$\tilde{S}_T^{C(o)}$	14.70	27.50	58.80	92.05	91.10	99.70
	(18.45)	(33.10)	(63.65)	(94.10)	(92.45)	(99.80)
$\tilde{S}_T^{C(r)}$	13.40	23.75	56.65	89.80	92.25	99.80
	(13.45)	(25.15)	(56.65)	(90.55)	(92.30)	(99.80)
$\hat{S}_T^{*J(o)}$	17.45	29.15	73.45	96.25	98.15	100.00
	(50.95)	(52.45)	(92.00)	(98.90)	(99.85)	(100.00)
$\hat{S}_T^{*J(r)}$	15.65	27.10	63.80	94.75	95.40	100.00
	(19.25)	(34.00)	(69.80)	(96.05)	(96.15)	(100.00)
$\tilde{S}_T^{J(o)}$	17.25	32.30	65.35	95.55	94.70	100.00
	(27.50)	(43.90)	(76.45)	(97.45)	(97.30)	(100.00)
$\tilde{S}_T^{J(r)}$	15.05	25.70	61.55	94.05	94.45	100.00
	(14.70)	(26.10)	(60.95)	(94.05)	(94.15)	(100.00)

Notes:
1. Parameter values as detailed in Table 5.
2. T is the sample size and results are based on 2,000 simulations.
3.For each test the first (second) row report rejection rates using empirical (nominal) critical value; i.e., figures in the first row for ecah test statistic report size-adjusted rejection rates

Table A7: DCC Models II: Empirical Rejection Rates against 5% nominal (empirical) critical value

	AR(1)-VC-GARCH(1,1) DGP				DBEKK DGP			
	P4		P5		P6		P7	
	T=500	T=1000	T=500	T=1000	T=500	T=1000	T=500	T=1000
$\widehat{LM}_T^{*(o)}$	17.80	30.35	85.25	99.05	74.20	79.00	89.50	99.90
	(38.85)	(43.30)	(93.40)	(99.55)	(81.70)	(87.10)	(97.05)	(99.90)
$\widehat{LM}_T^{*(r)}$	13.80	25.85	72.80	97.80	61.75	76.05	85.10	99.85
	(13.80)	(25.95)	(72.85)	(97.85)	(62.15)	(76.25)	(85.15)	(99.85)
$\widetilde{LM}_T^{(o)}$	13.55	25.40	62.20	94.00	66.10	85.50	94.00	99.85
	(10.20)	(22.55)	(55.10)	(92.90)	(64.70)	(83.75)	(91.25)	(99.90)
$\widetilde{LM}_T^{(r)}$	14.60	27.00	72.20	97.45	66.20	78.05	86.05	99.85
	(13.55)	(26.10)	(70.75)	(97.25)	(65.50)	(77.25)	(84.85)	(99.85)
$\hat{S}_T^{*C(o)}$	13.95	21.90	90.80	99.30	42.50	42.70	53.95	88.15
	(35.05)	(37.20)	(97.15)	(99.70)	(61.00)	(61.25)	(79.80)	(94.85)
$\hat{S}_T^{*C(r)}$	12.10	19.95	84.85	98.70	24.35	27.20	36.70	81.80
	(12.25)	(22.05)	(85.10)	(99.00)	(26.90)	(30.25)	(37.20)	(83.95)
$\tilde{S}_T^{C(o)}$	15.90	26.40	96.65	99.20	55.00	58.90	72.75	97.50
	(19.20)	(31.80)	(88.80)	(99.40)	(59.25)	(64.40)	(77.25)	(97.90)
$\tilde{S}_T^{C(r)}$	12.15	21.70	85.70	98.95	32.60	35.25	46.60	89.90
	(12.15)	(23.25)	(85.75)	(99.15)	(34.50)	(37.30)	(46.65)	(90.55)
$\hat{S}_T^{*J(o)}$	21.15	31.25	97.95	100.00	56.95	56.80	72.45	97.50
	(53.75)	(55.45)	(99.65)	(100.00)	(78.50)	(78.80)	(94.15)	(99.50)
$\hat{S}_T^{*J(r)}$	17.55	29.35	94.70	99.95	42.55	44.40	55.60	95.55
	(22.00)	(35.30)	(95.85)	(100.00)	(50.30)	(52.50)	(61.50)	(97.25)
$\tilde{S}_T^{J(o)}$	18.75	32.85	88.50	99.70	45.30	47.65	69.15	96.50
	(30.35)	(43.90)	(93.50)	(99.85)	(55.85)	(58.80)	(79.40)	(98.25)
$\tilde{S}_T^{J(r)}$	16.10	27.95	88.30	99.85	29.10	30.70	41.70	88.35
	(15.90)	(28.25)	(88.10)	(99.85)	(29.50)	(31.10)	(40.40)	(88.50)

Notes:
1. Parameter values as detailed in Table 5.
2. T is the sample size and results are based on 2,000 simulations.
3.For each test the first (second) row report rejection rates using empirical (nominal) critical values; i.e., figures in the first row for ecah test statistic report size-adjusted rejection rates

Table A8: Average, Maximum and minimum range of true correlation parameters in simulated sample for T=1000

	P1			P2			P3			P4		
	Avg	Max	Min	Avg	Max	Min	Avg	Max	Min	Avg	Max	Min
ρ_{12}	0.739	1.047	0.526	1.228	1.551	0.904	1.515	1.743	1.186	0.723	0.777	0.645
ρ_{13}	0.674	1.031	0.435	1.165	1.516	0.708	1.473	1.722	0.951	0.710	0.782	0.568
ρ_{14}	0.416	0.836	0.235	0.867	1.457	0.425	1.250	1.723	0.588	0.645	0.778	0.437
ρ_{15}	0.568	0.936	0.357	1.056	1.501	0.616	1.398	1.754	0.856	0.685	0.783	0.529
ρ_{23}	0.607	1.010	0.378	1.097	1.521	0.697	1.426	1.752	1.020	0.695	0.777	0.570
ρ_{24}	0.351	0.646	0.197	0.781	1.471	0.378	1.184	1.761	0.539	0.618	0.780	0.352
ρ_{25}	0.698	1.043	0.463	1.188	1.556	0.836	1.490	1.765	1.143	0.715	0.779	0.608
ρ_{34}	0.473	0.852	0.295	0.945	1.463	0.519	1.320	1.722	0.732	0.659	0.773	0.414
ρ_{35}	0.522	0.935	0.312	0.999	1.460	0.590	1.355	1.711	0.846	0.675	0.781	0.483
ρ_{45}	0.722	1.029	0.520	1.211	1.532	0.844	1.506	1.741	1.130	0.719	0.778	0.616
	P5			P6			P7					
	Avg	Max	Min	Avg	Max	Min	Avg	Max	Min			
ρ_{12}	1.403	1.472	1.205	0.618	1.222	0.427	0.921	1.470	0.640			
ρ_{13}	1.393	1.479	1.187	0.618	1.247	0.421	0.921	1.514	0.647			
ρ_{14}	1.333	1.484	0.634	0.615	1.230	0.416	0.919	1.514	0.676			
ρ_{15}	1.371	1.477	1.065	0.611	1.249	0.419	0.917	1.599	0.640			
ρ_{23}	1.378	1.477	1.093	0.614	1.209	0.418	0.917	1.475	0.651			
ρ_{24}	1.310	1.476	0.778	0.618	1.198	0.432	0.923	1.511	0.662			
ρ_{25}	1.397	1.481	1.215	0.617	1.146	0.419	0.924	1.546	0.687			
ρ_{34}	1.348	1.471	0.810	0.618	1.258	0.436	0.921	1.516	0.692			
ρ_{35}	1.361	1.473	0.986	0.615	1.216	0.430	0.919	1.502	0.661			
ρ_{45}	1.400	1.477	1.204	0.613	1.246	0.428	0.919	1.502	0.644			

Main Results

Tests based on FQMLE

Proposition 1 *Suppose Assumptions A and B hold. Then,* $\Sigma^* = E\left[u_t^{\infty *}(\omega_0)u_t^{\infty *}(\omega_0)'\right]$ *is finite, where* $u_t^{\infty *}(\omega)' = \left(m_t^{\infty}(\omega)', g_t^{\infty *}(\omega)'\right)$. *Furthermore:*

(i) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t^{\infty *}(\omega_0) \xrightarrow{d} N(0, \Sigma^*)$, *where* $\Sigma^* = E\left[u_t^{\infty *}(\omega_0)u_t^{\infty *}(\omega_0)'\right]$ *is finite; and,*

(ii) $\frac{1}{T}\sum_{t=1}^{T} u_t^*(\hat{\omega})u_t^*(\hat{\omega})' - \Sigma^* = o_p(1)$, *for any* $\hat{\omega} - \omega_0 = o_p(1)$, *where* $u_t^*(\omega)' = \left(m_t(\omega)', g_t^*(\omega)'\right)$.

Proposition 2 *Under the assumptions of Proposition 1,*

$$\sqrt{T}\bar{m}_T(\hat{\omega}) \xrightarrow{d} N\left(0, V^*\right),$$

where $V^* = A_0^*\Sigma^* A_0^{*\prime}$, *and* $A_0^* = \left[I_q : -B_0^* J_0^{*-1}\right]$, *with* $J_0^* = -E\left[\frac{\partial g_t^{\infty *}(\omega_0)}{\partial \omega'}\right]$, *positive definite,* $B_0^* = -E\left[\frac{\partial m_t^{\infty}(\omega_0)}{\partial \omega'}\right]$, *and* I_q *is the* $(q \times q)$ *identity matrix.*

Proposition 3 *Under the Assumptions of Proposition 6, in the Appendix*

(i) $J_T^*(\hat{\omega}) - J_0^* = o_p(1)$, *where*

$$J_T^*(\omega) = \frac{1}{4T}\left[\begin{array}{cc} Z'\left(\Gamma_A \otimes I_T\right)Z & Z'\left(E_N' P L_N \otimes \iota_T\right) \\ (L_N' P E_N \otimes \iota_T')Z & 2L_N'\left(\Gamma^{-1} \otimes \Gamma^{-1}\right)L_N \end{array}\right] + \frac{1}{T}\left[\begin{array}{cc} F'\left(\Gamma^{-1} \otimes I_T\right)F & 0 \\ 0 & 0 \end{array}\right] \tag{5}$$

and positive definite.

(ii) $B_T^*(\hat{\omega}) - B_0^* = o_p(1)$, *where, in the case of* $\bar{m}_T^J(\hat{\omega})$, $B_T^*(\omega)$ *can be expressed in (vertically-stacked) "block-row" form, as follows*

$$B_T^*(\omega) = \left[B_{11T}^*(\omega)', B_{21T}^*(\omega)', B_{22T}^*(\omega)', B_{31T}^*(\omega)', ..., B_{N,N-1T}^*(\omega)', B_{NNT}^*(\omega)'\right]' \tag{6}$$

and the $B_{ijT}^*(\omega)$ *are ordered by* (i,j) *according to* s_N^J *with*

$$B_{ijT}^*(\omega) = \frac{1}{T}\left[\frac{1}{2}\rho_{ij}R_{ij}'(e_j' \otimes Z_j + e_i' \otimes Z_i),\ R_{ij}'(e_{ij}' \otimes \iota_T)\right]. \tag{7}$$

Tests based on PQMLE

Proposition 4 *Suppose Assumptions A and B, with B1 and B2, appropriately strengthened for the particular choice of* $r_{ij,t}$, *hold. Then,* $\Sigma = E\left[u_t^{\infty}(\omega_0)u_t^{\infty}(\omega_0)'\right]$ *is finite, where* $u_t^{\infty}(\omega)' = \left(m_t^{\infty}(\omega)', g_t^{\infty}(\theta)'\right)$. *Furthermore:*

(i) $\sqrt{T}\bar{u}_T^{\infty}(\omega) \xrightarrow{d} N(0, \Sigma)$, *where* $\Sigma = E\left[u_t^{\infty}(\omega_0)u_t^{\infty}(\omega_0)'\right]$ *is finite; and,*

(ii) $\frac{1}{T}\sum_{t=1}^{T} u_t(\tilde{\omega})u_t(\tilde{\omega})' - \Sigma = o_p(1)$, *for any* $\tilde{\omega} - \omega_0 = o_p(1)$, *where* $u_t(\omega)' = \left(m_t(\omega)', g_t(\theta)'\right)$.

Proposition 5 *Under the assumptions of Proposition 4, and provided* Σ *is positive definite,*

$$\sqrt{T}\bar{m}_T(\tilde{\omega}) \xrightarrow{d} N\left(0, V\right)$$

where $V = A_0 \Sigma A_0'$, $A_0 = \left[I_q, -B_0 J_0^{-1}\right]$, *with* $J_0 = -E\left[\frac{\partial g_t^{\infty}(\theta_0)}{\partial \theta'}\right]$, $B_0 = -E\left[\frac{\partial n_t^{\infty}(\omega_0)}{\partial \theta'}\right]$, *and* I_q *is the* $(q \times q)$ *identity matrix.*

Appendix:Assumptions and Proofs

Write $w'_{it}\varphi_i = \varphi_{i1}(L)y_{it} + d'_{it}\varphi_{i2}$ and $h_{it} = \alpha_{i0} + A_i(L)\varepsilon^2_{it} + B_i(L)h_{it} = a_{it} + B_i(L)h_{it}$, where $a_{it} = \alpha_{i0} + A_i(L)\varepsilon^2_{it} = \alpha_{i0} + \sum_{k=1}^{q}\alpha_{ik}\varepsilon^2_{i,t-k}$. As employed, for example, in Ling and McAleer (2003), Berkes, Horváth and Kokoszka (2003) and Halunga and Orme (2009), the following assumptions ensure the identifiability, stationarity and ergodicity of the above process. A3(i) is a stationarity assumption imposed over the whole parameter space. Notice that, with A3(ii), this implies that roots of $1-B_i(z)=0$ lie outside the unit circle. Thus, in addition to A3(ii) which restricts the parameter space so that zero values in η_i are ruled out, $\sum_{j=1}^{p}\beta_{ij} < 1$. These restrictions are also imposed on Θ by Berkes, Horváth and Kokoszka (2003) and are employed here because they afford uniform convergence of second derivatives of the log-likelihood over Θ, removing the need for third derivatives, thus greatly simplifying the algebra required to justify the substantive contribution.

Assumptions A

A1 The parameter space, Θ, is compact and ω_0 lies in the interior of Θ.

A2 The elements of d'_{it} are strictly stationary and ergodic and $1-\varphi_{i1}(L) = 1-\varphi_{i11}L - \phi_{i12}L^2 - ... - \phi_{i1p}L^p = 0$, $\phi_{i1p} \neq 0$, has all roots lying outside the unit circle, for all i, with p, the lag length, known.

A3 (i) All the roots of $1 - A_i(z) - B_i(z) = 0$ lie outside the unit circle.
(ii) The parameter space is constrained such that $0 < \lambda \leq \min_{i,l}\{\eta_{il}\} \leq \max_{i,l}\{\eta_{il}\} < \Lambda$, $l = 1, ..., p+q+1$, where λ and Λ are independent of ω.
(iii) The polynomials $A_i(z)$ and $1 - B_i(z)$ are coprimes.

Unless stated otherwise, all definitions are as in the main text; $\bar{K}$ will denote a generic positive finite constant, independent of ω, which is employed to bound certain expressions but whose value might change from line to line in the proofs as required; and, throughout, a superscript of ∞ signifies that h_{it} has been replaced by h^∞_{it} where necessary. The Euclidean norm of a matrix A is denoted $\|A\| = \sqrt{tr(A'A)}$.

The derivations below will exploit the properties of h_{it} and h^∞_{it}, as discussed in Halunga and Orme (2009, Appendix), and it will be useful to note here that:

Summary 1 *1.* $h^\infty_{it} = \sum_{l=0}^{\infty}\psi_{il}a_{i,t-l} \geq \lambda > 0$ *and* $h^\infty_{it} \geq \alpha_{i0} + \psi_{il}a_{i,t-l}$ *for all* $l \geq 1$; $h_{it} = \sum_{l=0}^{t-1}\psi_{il}a_{i,t-l} \geq \lambda > 0$ *and* $h_{it} \geq \alpha_{i0} + \psi_{il}a_{i,t-l}$ *for all* $l = 1, ..., t-1$.

2. $|h^\infty_{it} - h_{it}| = |\sum_{l=t}^{\infty}\psi_{il}a_{i,t-l}| \leq \bar{K}\sum_{l=t}^{\infty}\tau^l a_{t-l}$ *and* $\left|\frac{h^\infty_{it} - h_{it}}{h^\infty_{it}}\right| = \left|\sum_{l=t}^{\infty}\frac{\psi_{il}a_{i,t-l}}{\alpha_{i0} + \psi_{il}a_{t-l}}\right| \leq \bar{K}\sum_{l=t}^{\infty}\tau^l a^s_{i,t-l}$, *for some* $s \in (0,1)$, *using* $x/(1+x) \leq x^s$ *for some* $s \in (0,1)$ *and all* $x \geq 0$.

3. $0 < h_{it}/h^\infty_{it} < \sqrt{h_{it}/h^\infty_{it}} < 1$, *and* $0 < \dfrac{\sqrt{h^\infty_{it}} - \sqrt{h_{it}}}{\sqrt{h^\infty_{it}}} < \Delta h^\infty_{it} = \dfrac{h^\infty_{it} - h_{it}}{h^\infty_{it}} < 1$. *Similarly,* $0 < \frac{\sqrt{h^\infty_{it}h^\infty_{jt}} - \sqrt{h_{it}h_{jt}}}{\sqrt{h^\infty_{it}h^\infty_{jt}}} < \frac{h^\infty_{it}h^\infty_{jt} - h_{it}h_{jt}}{h^\infty_{it}h^\infty_{jt}} < 1$.

4. Since $h^\infty_{it} = h_{it} + (h^\infty_{it} - h_{it})$, $h^\infty_{it}h^\infty_{jt} = h_{it}h_{jt} + (h^\infty_{it} - h_{it})(h^\infty_{jt} - h_{jt}) + h_{it}(h^\infty_{jt} - h_{jt}) + h_{jt}(h^\infty_{it} - h_{it})$, *so that*

$$\frac{h^\infty_{it}h^\infty_{jt} - h_{it}h_{jt}}{h^\infty_{it}h^\infty_{jt}} = \Delta h^\infty_{it}\Delta h^\infty_{jt} + \frac{h_{it}}{h^\infty_{it}}\Delta h^\infty_{jt} + \frac{h_{jt}}{h^\infty_{jt}}\Delta h^\infty_{it} < \Delta h^\infty_{it} + \Delta h^\infty_{jt} + \Delta h^\infty_{it}$$

and thus

$$\left|\frac{\sqrt{h_{it}^{\infty}h_{jt}^{\infty}}-\sqrt{h_{it}h_{jt}}}{\sqrt{h_{it}^{\infty}h_{jt}^{\infty}}}\right| < \left|\frac{h_{it}^{\infty}h_{jt}^{\infty}-h_{it}h_{jt}}{h_{it}^{\infty}h_{jt}^{\infty}}\right| \leq 2\left|\Delta h_{it}^{\infty}\right|+\left|\Delta h_{jt}^{\infty}\right| \leq 2\left\{\left|\Delta h_{it}^{\infty}\right|+\left|\Delta h_{jt}^{\infty}\right|\right\}.$$

Assumptions B

B1 $E\left|\varepsilon_{0it}\right|^{6}<\infty$ for all i,t.

B2 $E\left[\left\|d_{it}\right\|^{6}\right]<\infty$, for all i,t.

B3 $\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^{l}\left\|r_{ij,t}^{\infty}-r_{ij,t}\right\|=O(1)$, at most, for all i,j,t and $l=0,1$.

B4 $E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^{l}\left\|r_{ij,t}^{\infty}\right\|^{2}<\infty$ for all i,j,t, and $l=0,1,2$.

B5 $E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^{l}\left\|\frac{\partial r_{ij,t}^{\infty}}{\partial\omega}\right\|<\infty$, at most, for all i,j,t and $l=0,1$.

Remark 1 *(i) A1, A2, B1 and B2 imply that* $E\sup_{\omega}\left|\varepsilon_{it}\right|^{6}<\infty$ *uniformly in* i,t*, where* $\varepsilon_{it}=\varepsilon_{0it}-w_{it}^{\prime}\left(\varphi_{i}-\varphi_{i0}\right)$, *and also that* $E\left|y_{it}\right|^{6}<\infty$ *for all* i,t*, so that* $E\left[\left\|d_{it}\right\|^{6}\right]<\infty$*, for all* i,t. ***(ii)*** *Extensions of Halunga and Orme (2009, Proposition 4) imply that B3-B5 also hold with* z_{it} *replacing* $r_{ij,t}$*. (iii) Assumptions A, B1 and B2 are sufficient to establish the consistency and asymptotic normality of both the FQMLE and PQMLE, and the consistency of variance estimators based on an OPG formulation. (iv) Depending on the choice of* $r_{ij,t}$*, B1 and B2 may need strengthening, in view of the demands of B4, in order to establish both the asymptotic normality of our test indicators and the consistency of the various asymptotic variance estimators employed in constructing the* χ^{2} *test statistics.*

Case 1 *For* $r_{ij,t}=\frac{\varepsilon_{i,t-1}\varepsilon_{j,t-1}}{\sqrt{h_{i,t-1}}\sqrt{h_{j,t-1}}}$*, B3-B5 hold provided B1 and B2 are replaced by*

*B1** $E\left|\varepsilon_{0it}\right|^{8}<\infty$ *for all* i,t.

*B2** $E\left[\left\|d_{it}\right\|^{8}\right]<\infty$ *for all* i,t.

Proof. Firstly, for B3, with $r_{ij,t}=\frac{\varepsilon_{i,t-1}\varepsilon_{j,t-1}}{\sqrt{h_{i,t-1}}\sqrt{h_{j,t-1}}}$, a scalar, we have

$$\begin{aligned}\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left|r_{ij,t}^{\infty}-r_{ij,t}\right| &= \sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right| \\ &\quad\times\left|\frac{1}{\sqrt{h_{i,t-1}^{\infty}}\sqrt{h_{j,t-1}^{\infty}}}-\frac{1}{\sqrt{h_{i,t-1}}\sqrt{h_{j,t-1}}}\right| \\ &= \sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|\left|\frac{1}{\sqrt{h_{i,t-1}}\sqrt{h_{j,t-1}}}\right| \\ &\quad\times\left|\frac{\sqrt{h_{i,t-1}}\sqrt{h_{j,t-1}}-\sqrt{h_{i,t-1}^{\infty}}\sqrt{h_{j,t-1}^{\infty}}}{\sqrt{h_{i,t-1}^{\infty}}\sqrt{h_{j,t-1}^{\infty}}}\right| \\ &\leq 2\lambda^{-1}\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|\left|\left|\Delta h_{i,t-1}^{\infty}\right|+\left|\Delta h_{j,t-1}^{\infty}\right|\right|.\end{aligned}$$

Now $\left|\Delta h_{i,t-1}^{\infty}\right| \leq \bar{K}\sum_{l=t-1}^{\infty}\tau^l a_{i,t-1-l}^s$, for any $s\in(0,1)$, so that

$$\begin{aligned}\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|\left|\Delta h_{i,t-1}^{\infty}\right| &\leq \bar{K}\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|\sum_{l=t-1}^{\infty}\tau^l a_{i,t-1-l}^s \\ &\leq \bar{K}\sum_{t=1}^{T}O(\tau^{t-1})=O(1)\end{aligned}$$

because by Holder's Inequality and, then, Cauchy-Schwartz (twice),

$$\begin{aligned}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-1}\varepsilon_{j,t-1}\varepsilon_{i,t-1-l}^{2s}\right| &\leq \left(E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|^{1+s}\right)^{1/(1+s)}\left(E\sup_{\omega}\left|\varepsilon_{i,t-1-l}\right|^{2(1+s)}\right)^{s/(1+s)} \\ &\leq K\sqrt{E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^{2(1+s)}E\sup_{\omega}\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|^{2(1+s)}}<\infty\end{aligned}$$

since $E\sup_{\omega}\left|\varepsilon_{it}\right|^{4(1+s)}<\infty$, for any $s\in(0,1)$, and this is satisfied by Assumptions B1 and B2. For B4, first, it is clear that $E\sup_{\omega}\left|r_{ij,t}^{\infty}\right|^2 \leq \lambda^{-2}E\sup_{\omega}\left|\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|^2<\infty$ by Cauchy-Schwartz and B1. Second,

$$E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}r_{ij,t}^{\infty}\right|^2 \leq \lambda^{-2}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-1}\varepsilon_{j,t-1}\right|^2<\infty$$

by Cauchy-Schwartz, B1* and B2*. For B5, note that, $\dfrac{\partial r_{ij,t}^{\infty}}{\partial\theta_i}=\zeta_{j,t-1}^{\infty}\left(f_{i,t-1}^{\infty}+\dfrac{1}{2}\zeta_{i,t-1}^{\infty}z_{i,t-1}^{\infty}\right)$, $i\neq j$; whilst $\dfrac{\partial r_{ij,t}^{\infty}}{\partial\theta_k}=0$, for $k\neq i$, $k\neq j$ and $\dfrac{\partial r_{ij,t}^{\infty}}{\partial\rho'}=0'$. Thus, for example,

$$E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|\left\|\frac{\partial r_{ij,t}^{\infty}}{\partial\theta_i}\right\| \leq \bar{K}\left\{E\sup_{\omega}\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{j,t-1}w_{i,t-1}\right\|+E\sup_{\omega}\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{j,t-1}\varepsilon_{i,t-1}z_{i,t-1}^{\infty}\right\|\right\}<\infty$$

since both $E\sup_{\omega}\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{j,t-1}w_{i,t-1}\right\|<\infty$ and $E\sup_{\omega}\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-1}\varepsilon_{j,t-1}z_{i,t-1}^{\infty}\right\|<\infty$, by Cauchy-Schwartz, B1 and B2, and Remark 1(ii). For example, note that

$$E\sup_{\omega}\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-1}\varepsilon_{j,t-1}z_{i,t-1}^{\infty}\right\| \leq \sqrt{E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^2 E\sup_{\omega}\left\|\varepsilon_{i,t-1}\varepsilon_{j,t-1}z_{i,t-1}^{\infty}\right\|^2}$$

and $E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^2<\infty$ by Cauchy-Schwartz and B1 and B2. Also, $E\sup_{\omega}\left\|\varepsilon_{i,t-1}\varepsilon_{j,t-1}z_{i,t-1}^{\infty}\right\|^2<\infty$ since the following moments are bounded by Cauchy-Schwartz and Assumptions B1, B2 (noting the expressions for $\|x_{it}^{\infty}\|$ and $\|c_{it}^{\infty}\|$ given by Halanga and Orme (2008, A5 and A8, respectively):

(i) $E\sup_{\omega}\left\|\varepsilon_{i,t-1}^2\varepsilon_{j,t-1}^2\varepsilon_{i,t-k}^{2s}\varepsilon_{i,t-l}^{2s}\right\| \leq \sqrt{E\sup_{\omega}\left\|\varepsilon_{i,t-1}^2\varepsilon_{i,t-k}^{2s}\right\|^2 E\sup_{\omega}\left\|\varepsilon_{j,t-1}^2\varepsilon_{i,t-l}^{2s}\right\|^2}$, and $E\sup_{\omega}\left\|\varepsilon_{i,t-1}^2\varepsilon_{i,t-k}^{2s}\right\|^2 \leq \left(E\sup_{\omega}\left|\varepsilon_{i,t-1}\right|^{4(1+s)}\right)^{1/(1+s)}\left(E\sup_{\omega}\left\|\varepsilon_{i,t-k}\right\|^{4(1+s)}\right)^{s/(1+s)}<\infty$, by Holder's inequality.

(ii) Similarly, $E\sup_{\omega}\left\|\varepsilon_{i,t-1}^2\varepsilon_{j,t-1}^2 w_{i,t-1-k}w_{i,t-1-l}'\right\| \leq \sqrt{E\sup_{\omega}\left\|\varepsilon_{i,t-1}^2 w_{i,t-1-k}\right\|^2 E\sup_{\omega}\left\|\varepsilon_{j,t-1}^2 w_{i,t-1-l}\right\|^2}$, and $E\sup_{\omega}\left\|\varepsilon_{i,t-1}^2 w_{i,t-1-k}\right\|^2 \leq \left(E\sup_{\omega}\left|\varepsilon_{i,t-1}\right|^6\right)^{2/3}\left(E\sup_{\omega}\left\|w_{i,t-1-k}\right\|^6\right)^{1/3}<\infty$, by Holder's inequality.

■

We first establish some preliminary results that will be of use later.

Lemma 1 *Let* $\{x_t\}_{t=1}^T$ *be a sample of stationary ergodic random variables, such that the random vector functions* $w_t(\omega)\equiv w(x_t;\omega)$ *and* $z_t(\omega)\equiv z(x_t;\omega)$, $t=1,...,T$, *satisfy* $\frac{1}{\sqrt{T}}\sum_{t=1}^T\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|=o_p(1)$.

(i) Then

$$\sup_{\omega}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}w_t(\omega)-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_t(\omega)\right\|=o_p(1).$$

(ii) If $E\sup_{\omega}\|w(x;\omega)\|^2<\infty$, *where* $\omega\in\Omega$ *a compact set, then (in addition)*

$$\sup_{\omega}\left\|\frac{1}{T}\sum_{t=1}^{T}w_t(\omega)w_t(\omega)'-\frac{1}{T}\sum_{t=1}^{T}z_t(\omega)z_t(\omega)'\right\|=o_p(1).$$

Proof. (i) First,

$$\begin{aligned}\sup_{\omega}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}w_t(\omega)-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}z_t(\omega)\right\| &= \sup_{\omega}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(w_t(\omega)-z_t(\omega))\right\| \\ &\leq \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\| \\ &= o_p(1).\end{aligned}$$

(ii) Second, and similarly,

$$\begin{aligned}\sup_{\omega}\left\|\frac{1}{T}\sum_{t=1}^{T}w_t(\omega)w_t(\omega)'-\frac{1}{T}\sum_{t=1}^{T}z_t(\omega)z_t(\omega)'\right\| &= \sup_{\omega}\left\|\frac{1}{T}\sum_{t=1}^{T}(w_t(\omega)w_t(\omega)'-z_t(\omega)z_t(\omega)')\right\| \\ &\leq \frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)w_t(\omega)'-z_t(\omega)z_t(\omega)'\|,\end{aligned}$$

then it is sufficient to show that $\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)w_t(\omega)'-z_t(\omega)z_t(\omega)'\|=o_p(1)$, as follows. Because $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|=o_p(1)$, it follows that $\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|^2=o_p(1)$, since

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|^2 &\leq \left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|\right)^2 \\ &= o_p(1).\end{aligned}$$

Finally, since $\|wz'\|=\|zw'\|=\|w\|\,\|z\|$ and

$$\begin{aligned}\|ww'-zz'\| &= \|(w-z)w'+w(w-z)'-(w-z)(w-z)'\| \\ &\leq 2\|w-z\|\,\|w\|+\|w-z\|^2,\end{aligned}$$

we can write, by the preceding result, that

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)w_t(\omega)'-z_t(\omega)z_t(\omega)'\| &\leq 2\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|\,\|w_t(\omega)\| \\ &\quad +\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|^2 \\ &\leq 2\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|\sup_{\omega}\|w_t(\omega)\|+o_p(1).\end{aligned}$$

Then by Cauchy-Schwartz

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|\sup_{\omega}\|w_t(\omega)\| &\le \sqrt{\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|^2\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)\|^2} \\ &= o_p(1).\end{aligned}$$

since $\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)-z_t(\omega)\|^2 = o_p(1)$ and $\frac{1}{T}\sum_{t=1}^{T}\sup_{\omega}\|w_t(\omega)\|^2 = O_p(1)$, by Markov's Inequality since $E\sup_{\omega}\|w_t(\omega)\|^2 < \infty$. ■

Remark 2 *Under the conditions of Lemma 1,* $E\sup_{\omega}\|w(x;\omega)\|^2 < \infty$ *so that* $E[w_t(\omega_0)w_t(\omega_0)']$ *is finite. Then, by a Uniform Law of Large Numbers and the triangle inequality, for any* $\hat{\omega}-\omega_0 = o_p(1)$, $\frac{1}{T}\sum_{t=1}^{T} z_t(\hat{\omega})z_t(\hat{\omega})' - E[w_t(\omega_0)w_t(\omega_0)'] = o_p(1)$. *In fact, the method proof also reveals that for any* $\hat{\omega}-\omega_0 = o_p(1)$, *and provided both* $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\|w_t(\hat{\omega})-z_t(\hat{\omega})\| = o_p(1)$ *and* $E\sup_{\omega}\|w(x;\omega)\|^2 < \infty$, *then* $\frac{1}{T}\sum_{t=1}^{T} z_t(\hat{\omega})z_t(\hat{\omega})' - E[w_t(\omega_0)w_t(\omega_0)'] = o_p(1)$.

Proposition 6 *Under Assumptions A and B1, B2:*

(i) $E\sup_{\omega}\|g_t^{\infty *}(\omega)\|^2 < \infty$;

(ii) $\dfrac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\|g_t^{\infty *}(\omega)-g_t^{*}(\omega)\| = o_p(1)$.

In addition, and adding B3 and B4:

(iii) $E\sup_{\omega}\|m_t^{\infty}(\omega)\|^2 < \infty$;

(iv) $\dfrac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\|m_t^{\infty}(\omega)-m_t(\omega)\| = o_p(1)$.

Proof.
(i) We can write the likelihood function, $l_t^{\infty *}(\omega)$ as:

$$l_t^{\infty *}(\omega) = -\frac{1}{2}\ln|\Gamma| - \frac{1}{2}\sum_{j=1}^{N}\ln h_{jt}^{\infty} - \frac{1}{2}\sum_{j=1}^{N}\zeta_{jt}^{\infty}\varepsilon_{jt}^{\infty *},$$

where $\varepsilon_{it}^{\infty *} = \sum_{m=1}^{N}\rho^{im}\zeta_{mt}^{\infty}$, $i = 1,...,N$ with $\rho^{im} = \rho^{mi}$. Then, exploiting $\dfrac{\partial\zeta_{it}^{\infty}}{\partial\theta_i} = -f_{it}^{\infty} - \dfrac{1}{2}\zeta_{it}^{\infty}z_{it}^{\infty}$, $\dfrac{\partial\varepsilon_{it}^{\infty *}}{\partial\theta_j} = \rho^{ij}\dfrac{\partial\zeta_{it}^{\infty}}{\partial\theta_j}$, $\dfrac{\partial\rho^{ij}}{\partial\rho_{km}} = -\rho^{ik}\rho^{jm} - \rho^{im}\rho^{jk}$ and $\dfrac{\partial\varepsilon_{it}^{\infty *}}{\partial\rho_{km}} = -\rho^{ik}\varepsilon_{mt}^{\infty *} - \rho^{im}\varepsilon_{kt}^{\infty *}$, we have the following scores:

$$\frac{\partial l_t^{\infty *}(\omega)}{\partial\theta_i} = f_{it}^{\infty}\varepsilon_{it}^{\infty *} + \frac{1}{2}\left(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty *} - 1\right)z_{it}^{\infty}, \tag{8}$$

$$\frac{\partial l_t^{\infty *}(\omega)}{\partial\rho_{ij}} = \varepsilon_{it}^{\infty *}\varepsilon_{jt}^{\infty *} - \rho^{ij}, \quad i > j, \tag{9}$$

and it is sufficient to show that the result holds for each of the above.
Firstly, and since $h_{it}^{\infty} \ge \lambda > 0$ we have $|\varepsilon_{it}^{\infty *}| \le \lambda^{-1}\sum_{m=1}^{N}\rho^{im}\varepsilon_{mt}$, and since $|\rho^{im}| < \infty$, we also have by Minkowski's Inequality for any $s > 0$,

$$E\sup_{\omega}|\varepsilon_{it}^{\infty *}|^s \le \bar{K}\left[\sum_{m=1}^{N}\left\{E\sup_{\omega}|\varepsilon_{mt}|^s\right\}^{1/s}\right]^s < \infty,$$

so that by Assumptions B1 and B2, $E\sup_\omega |\varepsilon_{it}^{\infty *}|^6 < \infty$.

For $E\sup_\omega \left\| \dfrac{\partial l_t^{\infty *}(\omega)}{\partial \theta_i} \right\|^2 < \infty$, we need both $E\sup_\omega \|f_{it}^{\infty}\varepsilon_{it}^{\infty *}\|^2 < \infty$ and $E\sup_\omega \|(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty *} - 1)\, z_{it}^{\infty}\|^2 < \infty$. For the former

$$E\sup_\omega \|f_{it}^{\infty}\varepsilon_{it}^{\infty *}\|^2 \leq \frac{1}{\lambda}\sqrt{E\sup_\omega \|w_{it}\|^2\, E\sup_\omega |\varepsilon_{it}^{\infty *}|^2} < \infty,$$

by Cauchy-Schwartz and Assumptions B1 and B2. For the latter, $E\sup_\omega \|z_{it}^{\infty}\|^2 < \infty$, as shown by Halunga and Orme (2009, Proposition 4a), and from Remark 1(ii)

$$E\sup_\omega \|\zeta_{it}^{\infty}\varepsilon_{it}^{\infty *} z_{it}^{\infty}\|^2 \leq \bar{K}\sum_{m=1}^{N} E\sup_\omega |\varepsilon_{it}\varepsilon_{mt}|^2\, \|z_{it}^{\infty}\|^2 < \infty,$$

since the following moments are bounded by Cauchy-Schwartz and Assumptions B1, B2 (similar to (i) and (ii) in the proof of Case 1):

(a) $E\sup_\omega \left|\varepsilon_{it}^2\varepsilon_{mt}^2\varepsilon_{i,t-k}^{2s}\varepsilon_{i,t-l}^{2s}\right| \leq \left(E\sup_\omega |\varepsilon_{it}\varepsilon_{mt}|^{2(1+s)}\right)^{1/(1+s)} \left(E\sup_\omega |\varepsilon_{i,t-k}\varepsilon_{i,t-l}|^{2(1+s)}\right)^{s/(1+s)} < \infty$, by Holder's Inequality, since $E\sup_\omega |\varepsilon_{it}|^{4(1+s)} < \infty$, for any $s \in (0,1)$;

(b) $E\sup_\omega \left\|\varepsilon_{it}^2\varepsilon_{mt}^2 w_{i,t-k}w_{i,t-l}'\right\| \leq \sqrt{E\sup_\omega \|\varepsilon_{it}^2 w_{i,t-k}\|^2\, E\sup_\omega \|\varepsilon_{mt}^2 w_{i,t-l}\|^2}$; $E\sup_\omega \left\|\varepsilon_{it}^2 w_{i,t-k}\right\|^2 \leq \left(E\sup_\omega |\varepsilon_{it}|^6\right)^{2/3}\left(E\sup_\omega \|w_{i,t-k}\|^6\right)^{1/3} < \infty$.

Finally, and trivially, $E\sup_\omega \left|\varepsilon_{it}^{\infty *}\varepsilon_{jt}^{\infty *} - \rho^{ij}\right|^2 < \infty$, since $E\sup_\omega \left|\varepsilon_{it}^{\infty *}\varepsilon_{jt}^{\infty *}\right|^2 < \infty$, by Cauchy-Schwartz. Thus, $E\sup_\omega \|g_t^{\infty *}(\omega)\|^2 < \infty$.

(ii) From (8),

$$\begin{aligned}
\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_\omega \left\|\frac{\partial l_t^{\infty *}(\omega)}{\partial \theta_i} - \frac{\partial l_t^{*}(\omega)}{\partial \theta_i}\right\| &\leq \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_\omega \|f_{it}^{\infty}\varepsilon_{it}^{\infty *} - f_{it}\varepsilon_{it}^{*}\| \\
&\quad + \frac{1}{2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_\omega \|(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty *} - 1)\, z_{it}^{\infty} - (\zeta_{it}\varepsilon_{it}^{*} - 1)\, z_{it}\| \\
&= R_{1T} + R_{2T},
\end{aligned}$$

and, by Markov's Inequality, it is sufficient to show that $E\left[R_{jT}\right] = o(1)$, $j = 1, 2$. Note that

$$\begin{aligned}
\frac{w_{it}}{\sqrt{h_{it}^{\infty}}}\varepsilon_{it}^{\infty *} - \frac{w_{it}}{\sqrt{h_{it}}}\varepsilon_{it}^{*} &= \frac{w_{it}}{\sqrt{h_{it}^{\infty}}}\sum_{m=1}^{N}\rho^{im}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}^{\infty}}} - \frac{w_{it}}{\sqrt{h_{it}}}\sum_{m=1}^{N}\rho^{im}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}}} \\
&= \sum_{m=1}^{N}\rho^{im}\frac{w_{it}\varepsilon_{mt}}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}} - \sum_{m=1}^{N}\rho^{im}\frac{w_{it}\varepsilon_{mt}}{\sqrt{h_{it}}\sqrt{h_{mt}}} \\
&= \sum_{m=1}^{N}\rho^{im}w_{it}\varepsilon_{mt}\left\{\frac{1}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}} - \frac{1}{\sqrt{h_{it}}\sqrt{h_{mt}}}\right\},
\end{aligned}$$

so that, and since $h_{it} \geq \lambda > 0$, and $\left|\rho^{im}\right| < \infty$,

$$\begin{aligned}
E\left[R_{1T}\right] &\leq \bar{K}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{m=1}^{N} E\sup_\omega \left\|\rho^{im}w_{it}\varepsilon_{mt}\left\{|\Delta h_{it}^{\infty}| + |\Delta h_{mt}^{\infty}|\right\}\right\| \\
&\leq \bar{K}\sum_{m=1}^{N}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{E\sup_\omega \|w_{it}\varepsilon_{mt}\Delta h_{it}^{\infty}\| + E\sup_\omega \|w_{it}\varepsilon_{mt}\Delta h_{mt}^{\infty}\|\right\}.
\end{aligned}$$

Now, because we can write $\Delta h_{it}^{\infty} \leq \bar{K}\sum_{l=t}^{\infty}\tau^l a_{i,t-l}^s$, for some $s \in (0,1)$, and $0 < \tau < 1$, independent of

ω, we have (for example)

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\sup_{\omega}\|w_{it}\varepsilon_{mt}\Delta h_{it}^{\infty}\|\leq\bar{K}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\sup_{\omega}\left\|w_{it}\varepsilon_{mt}\sum_{l=t}^{\infty}\tau^{l}a_{i,t-l}^{s}\right\|.$$

But, by Cauchy-Schwartz, $E\sup_{\omega}\left\|w_{it}\varepsilon_{mt}\varepsilon_{i,t-k}^{2s}\right\|\leq\sqrt{E\sup_{\omega}\|w_{it}\varepsilon_{mt}\|^{2}E\sup_{\omega}|\varepsilon_{i,t-k}|^{4s}}<\infty$, since $E\sup_{\omega}\|w_{it}\varepsilon_{mt}\|^{2}<\infty$, by another application of Cauchy-Schwartz and B1, B2. Therefore,

$$E\left[R_{1T}\right]\leq\bar{K}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{l=t}^{\infty}\tau^{l}=\bar{K}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}O(\tau^{t})=o(1).$$

Turning to R_{2T}, now, we have

$$\begin{aligned}\left(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty*}-1\right)z_{it}^{\infty}-\left(\zeta_{it}\varepsilon_{it}^{*}-1\right)z_{it} &= \left(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty*}-1\right)\left(z_{it}^{\infty}-z_{it}\right)-\left(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty*}-\zeta_{it}\varepsilon_{it}^{*}\right)\left(z_{it}^{\infty}-c_{it}\right)\\ &\quad+\left(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty*}-\zeta_{it}\varepsilon_{it}^{*}\right)z_{it}^{\infty},\end{aligned}$$

so that

$$\begin{aligned}\left\|\left(\zeta_{it}^{\infty}\varepsilon_{it}^{\infty*}-1\right)z_{it}^{\infty}-\left(\zeta_{it}\varepsilon_{it}^{*}-1\right)z_{it}\right\| &\leq \left\|\left(\sum_{m=1}^{N}\rho^{im}\frac{\varepsilon_{it}}{\sqrt{h_{it}^{\infty}}}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}^{\infty}}}-1\right)\left(z_{it}^{\infty}-z_{it}\right)\right\|\\ &\quad+\left\|\sum_{m=1}^{N}\rho^{im}\varepsilon_{it}\varepsilon_{mt}\left\{\frac{\sqrt{h_{it}}\sqrt{h_{mt}}-\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}\sqrt{h_{it}}\sqrt{h_{mt}}}\right\}\left(z_{it}^{\infty}-z_{it}\right)\right\|\\ &\quad+\left\|\sum_{m=1}^{N}\rho^{im}\varepsilon_{it}\varepsilon_{mt}\left\{\frac{\sqrt{h_{it}}\sqrt{h_{mt}}-\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}\sqrt{h_{it}}\sqrt{h_{mt}}}\right\}z_{it}^{\infty}\right\|.\end{aligned}$$

Taking each of the above in turn, $\sum_{m=1}^{N}\rho^{im}\frac{\varepsilon_{it}}{\sqrt{h_{it}^{\infty}}}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}^{\infty}}}-1=\sum_{m=1}^{N}\rho^{im}\left(\frac{\varepsilon_{it}}{\sqrt{h_{it}^{\infty}}}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}^{\infty}}}-\rho_{mi}\right)$, and $\left|\rho^{im}\left(\frac{\varepsilon_{it}}{\sqrt{h_{it}^{\infty}}}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}^{\infty}}}-\rho_{mi}\right)\right|\leq\bar{K}\left|\varepsilon_{it}\varepsilon_{mt}+1\right|$, $\bar{K}=\max\left(1,\lambda^{-1}\right)$, so that

$$\left\|\left(\sum_{m=1}^{N}\rho^{im}\frac{\varepsilon_{it}}{\sqrt{h_{it}^{\infty}}}\frac{\varepsilon_{mt}}{\sqrt{h_{mt}^{\infty}}}-1\right)\left(z_{it}^{\infty}-z_{it}\right)\right\|\leq\bar{K}\sum_{m=1}^{N}\left|\varepsilon_{it}\varepsilon_{mt}+1\right|\left\|z_{it}^{\infty}-z_{it}\right\|=O(1),$$

as noted in Remark 1(ii), above. The same is true of the second term since

$$\left\|\varepsilon_{it}\varepsilon_{mt}\left\{\frac{\sqrt{h_{it}}\sqrt{h_{mt}}-\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}\sqrt{h_{it}}\sqrt{h_{mt}}}\right\}\left(z_{it}^{\infty}-z_{it}\right)\right\|\leq\bar{K}\left|\varepsilon_{it}\varepsilon_{mt}\right|\left\|z_{it}^{\infty}-z_{it}\right\|.$$

For the third term, we can write

$$\left\|\varepsilon_{it}\varepsilon_{mt}\left\{\frac{\sqrt{h_{it}}\sqrt{h_{mt}}-\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}\sqrt{h_{it}}\sqrt{h_{mt}}}\right\}z_{it}^{\infty}\right\|\leq\bar{K}\left|\varepsilon_{it}\varepsilon_{mt}\left(|\Delta h_{it}^{\infty}|+|\Delta h_{mt}^{\infty}|\right)\right|\left\|z_{it}^{\infty}\right\|.$$

By Cauchy-Schwartz, $E\sup_{\omega}|\varepsilon_{it}\varepsilon_{mt}\Delta h_{it}^{\infty}|\,\|z_{it}^{\infty}\|\leq\sqrt{E\sup_{\omega}|\Delta h_{it}^{\infty}|^{2}E\sup_{\omega}|\varepsilon_{it}\varepsilon_{mt}|^{2}\|z_{it}^{\infty}\|^{2}}$ and, from the part (i), $E\sup_{\omega}|\varepsilon_{it}\varepsilon_{mt}|^{2}\|z_{it}^{\infty}\|^{2}<\infty$. Also, since $|\Delta h_{it}^{\infty}|^{2}\leq\bar{K}\left|\sum_{l=t}^{\infty}\tau^{l}a_{i,t-l}^{s}\right|^{2}$, it also follows that $E\sup_{\omega}|\Delta h_{it}^{\infty}|^{2}=O(\tau^{2l})$. Thus, $E\left[R_{2T}\right]\leq\bar{K}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sum_{l=t}^{\infty}\tau^{2l}=\bar{K}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}O(\tau^{2t})=o(1)$.

Finally, from (9),

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left|\frac{\partial l_t^{\infty*}(\omega)}{\partial\rho_{ij}}-\frac{\partial l_t^{*}(\omega)}{\partial\rho_{ij}}\right| &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left|\varepsilon_{it}^{\infty*}\varepsilon_{jt}^{\infty*}-\varepsilon_{it}^{*}\varepsilon_{jt}^{*}\right| \\ &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left|\pi_{ij}'\operatorname{vech}\left(\zeta_t^{\infty}\zeta_t^{\infty\prime}-\zeta_t\zeta_t'\right)\right|,\end{aligned}$$

and we consider, simply,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left|\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\zeta_{it}\zeta_{jt}\right| = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{mt}\left\{\frac{\sqrt{h_{it}}\sqrt{h_{mt}}-\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}}{\sqrt{h_{it}^{\infty}}\sqrt{h_{mt}^{\infty}}\sqrt{h_{it}}\sqrt{h_{mt}}}\right\}\right|,$$

which is $o_p(1)$, by previous arguments.
(iii) $E\sup_{\omega}\left\|m_t^{\infty}(\omega)\right\|^2<\infty$ provided $E\sup_{\omega}\left\|(\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\rho_{ij})r_{ij,t}^{\infty}\right\|^2<\infty$, for i,j, which it is by B4.
(iv) First define $\Delta m_{ij,t}^{\infty}(\omega)=m_{ij,t}^{\infty}(\omega)-m_{ij,t}(\omega)$. We show that $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}E\sup_{\omega}\left\|\Delta m_{ij,t}^{\infty}(\omega)\right\|=o(1)$, and the result follows by Markov's inequality. Now,

$$\begin{aligned}\Delta m_{ij,t}^{\infty}(\omega) &= (\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\rho_{ij})r_{ij,t}^{\infty}-(\zeta_{it}\zeta_{jt}-\rho_{ij})r_{ij,t} \\ &= (\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\rho_{ij})\left(r_{ij,t}^{\infty}-r_{ij,t}\right)-\left(\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\zeta_{it}\zeta_{jt}\right)\left(r_{ij,t}^{\infty}-r_{ij,t}\right)+\left(\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\zeta_{it}\zeta_{jt}\right)r_{ij,t}^{\infty} \\ &= (\zeta_{it}^{\infty}\zeta_{jt}^{\infty}-\rho_{ij})\left(r_{ij,t}^{\infty}-r_{ij,t}\right)-\varepsilon_{it}\varepsilon_{jt}\left(\frac{1}{\sqrt{h_{it}^{\infty}h_{jt}^{\infty}}}-\frac{1}{\sqrt{h_{it}h_{jt}}}\right)\left(r_{ij,t}^{\infty}-r_{ij,t}\right) \\ &\quad +\varepsilon_{it}\varepsilon_{jt}\left(\frac{1}{\sqrt{h_{it}^{\infty}h_{jt}^{\infty}}}-\frac{1}{\sqrt{h_{it}h_{jt}}}\right)r_{ij,t}^{\infty},\end{aligned}$$

so that

$$\begin{aligned}\left\|\Delta m_{ij,t}^{\infty}(\omega)\right\| &\leq K\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left\|r_{ij,t}^{\infty}-r_{ij,t}\right\|+\left|\varepsilon_{it}\varepsilon_{jt}\right|\left\|r_{ij,t}^{\infty}-r_{ij,t}\right\| \\ &\quad +2\left|\varepsilon_{it}\varepsilon_{jt}\right|\left\{\left|\Delta h_{it}^{\infty}\right|+\left|\Delta h_{jt}^{\infty}\right|\right\}\left\|r_{ijt}^{\infty}\right\| \\ &\leq K\left\{\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left\|r_{ij,t}^{\infty}-r_{ij,t}\right\|+\left|\varepsilon_{it}\varepsilon_{jt}\right|\left\{\left|\Delta h_{it}^{\infty}\right|+\left|\Delta h_{jt}^{\infty}\right|\right\}\left\|r_{ij,t}^{\infty}\right\|\right\}.\end{aligned}$$

By B3, $\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}+1\right|\left\|r_{ij,t}^{\infty}-r_{ij,t}\right\|=O(1)$. For the second term involving $\left|\varepsilon_{it}\varepsilon_{jt}\right|\left|\Delta h_{it}^{\infty}\right|\left\|r_{ij,t}^{\infty}\right\|$, and since $\left|\Delta h_{it}^{\infty}\right|\leq\bar{K}\sum_{l=t}^{\infty}\tau^{l}a_{i,t-l}^{s}$, for some $s\in(0,1)$, with $a_{i,t-l}=\alpha_{i0}+A_i(L)\varepsilon_{i,t-l}^2$, we need to consider terms like $\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-l}^{2s}r_{ij,t}^{\infty}\right\|$, for any $s\in(0,1)$. In particular, by Cauchy-Schwartz,

$$E\sup_{\omega}\left\|\varepsilon_{it}\varepsilon_{jt}\varepsilon_{i,t-l}^{2s}r_{ij,t}^{\infty}\right\|\leq\sqrt{E\sup_{\omega}\left|\varepsilon_{it}^2\varepsilon_{jt}^2\varepsilon_{i,t-l}^{4s}\right|E\sup_{\omega}\left\|r_{ij,t}^{\infty}\right\|^2}<\infty,$$

because $E\sup_{\omega}\left\|r_{ij,t}^{\infty}\right\|^2<\infty$, by B4, and by Holder's Inequality and then Cauchy-Schwartz

$$E\sup_{\omega}\left|\varepsilon_{it}^2\varepsilon_{jt}^2\varepsilon_{i,t-l}^{4s}\right|\leq\left(E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|^{2(1+s)}\right)^{1/(1+s)}\left(E\sup_{\omega}\left|\varepsilon_{i,t-l}\right|^{4(1+s)}\right)^{s/(1+s)}.$$

By B1 and B2, $E\sup_{\omega}\left|\varepsilon_{it}\right|^{4(1+s)}<\infty$ for any $s>0$ and all i,t, so that $E\sup_{\omega}\left|\varepsilon_{it}^2\varepsilon_{jt}^2\varepsilon_{i,t-l}^{4s}\right|<\infty$. This implies that

$$\sum_{t=1}^{T}E\sup_{\omega}\left|\varepsilon_{it}\varepsilon_{jt}\right|\left|\Delta h_{it}^{\infty}\right|\left\|r_{ij,t}^{\infty}\right\|\leq\bar{K}\sum_{t=1}^{T}O(\tau^{t})=O(1),$$

since $0 < \tau < 1$. Thus, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} E\sup_{\omega}\left\|\Delta m_{ij,t}^{\infty}(\omega)\right\| = o(1)$. ■

Proof of Proposition 1: Σ^* is finite by Proposition 6(i) and (iii). As in Ling and McAleer (2003, Lemma 5.2), (i) follows from a Martingale Central Limit Theorem. Part (ii), follows from Proposition 6 and Remark 2.■

Proof of Proposition 2: The test indicator under consideration is $\bar{m}_T(\hat{\omega}) \equiv T^{-1}\sum_{t=1}^{T} m_t(\hat{\omega})$. Firstly, by Proposition 1, $\sqrt{T}\bar{m}_T(\hat{\omega}) = \sqrt{T}\bar{m}_T^{\infty}(\hat{\omega}) + o_p(1)$, so we work with $\sqrt{T}\bar{m}_T^{\infty}(\hat{\omega})$ whose limit distribution can be established more easily. Following Ling and McAleer (2003), as adapted by Halunga and Orme (2009), it is straightforward to show that firstly, $\hat{\omega} - \omega_0 = o_p(1)$ and, secondly, that $E\sup_{\omega}\left\|\frac{\partial^2 l_t^{\infty *}(\omega)}{\partial\omega\partial\omega'}\right\| < \infty$. Thus a Uniform of Large Numbers yields $T^{-1}\sum_{t=1}^{T}\frac{\partial^2 l_t^{\infty *}(\tilde{\omega})}{\partial\omega\partial\omega'} + J_0^* = o_p(1)$, for all $\tilde{\omega} - \omega_0 = o_p(1)$, and $J_0^* = -E\left[\frac{\partial^2 l_t^{\infty *}(\omega)}{\partial\omega\partial\omega'}\right]_{\omega=\omega_0}$ is finite. Therefore, by Proposition 6, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_t^{\infty *}(\hat{\omega}) = o_p(1)$, which yields by a mean value expansion,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_t^{\infty *}(\omega_0) + T^{-1}\sum_{t=1}^{T}\frac{\partial^2 l_t^{\infty *}(\bar{\omega})}{\partial\omega\partial\omega'}\sqrt{T}(\hat{\omega} - \omega_0) = o_p(1),$$

where $\bar{\omega} = \omega_0 + o_p(1)$ signifies the "usual" mean value. Because $E\sup_{\omega}\left\|\frac{\partial^2 l_t^{\infty *}(\omega)}{\partial\omega\partial\omega'}\right\| < \infty$, a the Uniform Law of Large Numbers on $T^{-1}\sum_{t=1}^{T}\frac{\partial^2 l_t^{\infty *}(\bar{\omega})}{\partial\omega\partial\omega'}$ yields $\sqrt{T}(\hat{\omega} - \omega_0) = J_0^{*-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_t^{\infty *}(\omega_0) + o_p(1)$, which is $O_p(1)$. Note that J_0^* is shown to be positive definite in the proof of Proposition 3 below and, since we exploit $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_t^{\infty *}(\hat{\omega}) = o_p(1)$ here, rather than $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_t^{*}(\hat{\omega}) = 0$, we do not explicitly require $\sup_{\omega}\left\|T^{-1}\sum_{t=1}^{T}\frac{\partial^2 l_t^{\infty *}(\omega)}{\partial\omega\partial\omega'} - T^{-1}\sum_{t=1}^{T}\frac{\partial^2 l_t^{*}(\omega)}{\partial\omega\partial\omega'}\right\| = o_p(1)$; c.f., Ling and McAleer (2003, Lemma 5.4b, Theorem 5.1) or Halunga and Orme (2009, Proposition 7b, Theorem 1). Next, if $E\sup_{\omega}\left\|\frac{\partial m_t^{\infty}(\omega)}{\partial\omega}\right\| < \infty$ then $\frac{\partial\bar{m}_T^{\infty}(\omega_T)}{\partial\omega'} \xrightarrow{p} -B_0^* = E\left[\frac{\partial m_t^{\infty}(\omega_0)}{\partial\omega'}\right]$, for any sequence $\omega_T = \omega_0 + o_p(1)$. Thus taking a mean value expansion of $\bar{m}_T^{\infty}(\hat{\omega})$ about ω_0, and ignoring asymptotically negligible terms, yields

$$\begin{aligned}\sqrt{T}\bar{m}_T^{\infty}(\hat{\omega}) &= \sqrt{T}\bar{m}_T^{\infty}(\omega_0) - B_0^*\sqrt{T}(\hat{\omega} - \omega_0) + o_p(1)\\ &= A_0^*\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t^{\infty *}(\omega_0) + o_p(1),\end{aligned}$$

and, from Proposition 1, $\sqrt{T}\bar{m}_T(\hat{\omega}) \xrightarrow{d} N(0, V^*)$. It just remains to show that $E\sup_{\omega}\left\|\frac{\partial m_t^{\infty}(\omega)}{\partial\omega}\right\| < \infty$, for which it is sufficient to consider $E\sup_{\omega}\left\|\frac{\partial m_{ij,t}^{\infty}(\omega)}{\partial\theta_k}\right\|$, $k = 1, ..., N$, and $E\sup_{\omega}\left\|\frac{\partial m_{ij,t}^{\infty}(\omega)}{\partial\rho}\right\|$. Since $\rho_{ii} \equiv 1$ and $\frac{\partial\zeta_{it}^{\infty}}{\partial\theta_i} = -f_{it}^{\infty} - \frac{1}{2}\zeta_{it}^{\infty}z_{it}^{\infty}$, $\frac{\partial(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{ij})}{\partial\theta_i} = -\zeta_{jt}^{\infty}\left(f_{it}^{\infty} + \frac{1}{2}\zeta_{it}^{\infty}z_{it}^{\infty}\right)$, $i \neq j$. Furthermore, $\frac{\partial(\zeta_{it}^{\infty 2} - 1)}{\partial\theta_i} = -2\zeta_{it}^{\infty}\left(f_{it}^{\infty} + \frac{1}{2}\zeta_{it}^{\infty}z_{it}^{\infty}\right)$ and $\frac{\partial(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{ij})}{\partial\rho_{ij}} = -1$, $i \neq j$, whilst $\frac{\partial(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{ij})}{\partial\theta_k} = 0$, for $k \neq i$, $k \neq j$, and $\frac{\partial(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{ij})}{\partial\rho_{km}} = 0$, for any other $(k, m) \neq (i, j)$. Hence (for general choice of $r_{ij,t}^{\infty}$), we obtain

$$\begin{aligned}\frac{\partial m_{ij,t}^{\infty}(\omega)}{\partial\theta_k'} &= -\zeta_{jt}^{\infty}r_{ij,t}^{\infty}\delta_{ki}\left(f_{it}^{\infty\prime} + \frac{1}{2}\zeta_{it}^{\infty}z_{it}^{\infty\prime}\right) - \zeta_{it}^{\infty}r_{ij,t}^{\infty}\delta_{kj}\left(f_{jt}^{\infty\prime} + \frac{1}{2}\zeta_{jt}^{\infty}z_{jt}^{\infty\prime}\right)\\ &\quad + \left(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{ij}\right)\frac{\partial r_{ij,t}^{\infty}}{\partial\theta_k'}, \qquad (10)\\ \frac{\partial m_{ij,t}^{\infty}(\omega)}{\partial\rho'} &= -r_{ij,t}^{\infty}e_{ij}' + \left(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{ij}\right)\frac{\partial r_{ij,t}^{\infty}}{\partial\rho'}. \qquad (11)\end{aligned}$$

Thus

$$\begin{aligned}
\left\| \frac{\partial m_{ij,t}^{\infty}(\omega)}{\partial \theta_k} \right\| &\leq \bar{K} \left\{ \| \varepsilon_{jt} w_{it} \| \left\| r_{ij,t}^{\infty} \right\| + |\varepsilon_{jt}\varepsilon_{it} + 1| \left(\left\| r_{ij,t}^{\infty} \right\| \left(\| z_{it}^{\infty} \| + \left\| z_{jt}^{\infty} \right\| \right) + \left\| \frac{\partial r_{ij,t}^{\infty}}{\partial \theta_k} \right\| \right) \right\}, \\
\left\| \frac{\partial m_{ij,t}^{\infty}(\omega)}{\partial \rho} \right\| &\leq \bar{K} \left\{ \left\| r_{ij,t}^{\infty} \right\| + |\varepsilon_{it}\varepsilon_{jt} + 1| \left(\left\| \frac{\partial r_{ij,t}^{\infty}}{\partial \rho} \right\| \right) \right\}.
\end{aligned}$$

The result follows from previous results and B4 and B5. For example, $E \sup_{\omega} \| \varepsilon_{jt} w_{it} \| \left\| r_{ij,t}^{\infty} \right\|$ is bounded by Cauchy-Schwartz and B1, B2 and B4; $E \sup_{\omega} |\varepsilon_{jt}\varepsilon_{it}|^{l} \| z_{it}^{\infty} \| \left\| r_{ij,t}^{\infty} \right\|$, $l = 0, 1$, is bounded by Cauchy-Schwartz, B4 and Remark 1(ii).■

Proof of Proposition 3

(i) Define $z_t^{\infty} = \operatorname{diag}(z_{it}^{\infty\prime})$, the $(N \times N(K + K^*))$ block diagonal matrix with $z_{it}^{\infty\prime}$, $(1 \times K + K^*)$, forming the diagonal blocks, and $f_t^{\infty} = \operatorname{diag}(f_{it}^{\infty\prime})$, $(N \times N(K + K^*))$, constructed in the same way. Note that if $X = \operatorname{diag}(x_i)$, $(N \times N)$, is a diagonal matrix with $x = \{x_i\}$, $(N \times 1)$, then $\operatorname{vec}(X) = E_N x$ and $E_N X = (X \otimes I_N) E_N = (I_N \otimes X) E_N$, whilst $\nabla \Gamma' \equiv \partial \operatorname{vec}(\Gamma)/\partial \rho' = L_N$. From the definitions of E_N and L_N it is easily verified that further general properties of E_N and L_N are: $E_N' E_N = I_N$, $L_N' L_N = 2 I_{\frac{1}{2}N(N-1)}$ and $E_N' L_N = 0$, so that (E_N, L_N) has full rank of $\frac{1}{2}N(N+1)$.Furthermore, $E_N' (A \otimes B) E_N = E_N' (B \otimes A) E_N = A \odot B$, and $E_N' (A \otimes B) L_N = E_N' (B \otimes A) L_N$ for any $(N \times N)$ matrices A and B, whilst $E_N' (a \otimes b) = E_N' (b \otimes a) = a \odot b$, for any $(N \times 1)$ vectors a and b, and $(a' \otimes I_N) E_N = (I_N \otimes a') E_N$. Using these results, $J_0^* = -E \left\{ E \left[\frac{\partial^2 l_t^{\infty *}(\omega_0)}{\partial \omega \partial \omega'} \middle| \mathcal{F}_{t-1} \right] \right\}$ can also obtained by direct differentiation of the scores (8) and (9) which can themselves can be expressed as

$$\begin{aligned}
\frac{\partial l_t^{\infty *}(\omega)}{\partial \theta} &= \frac{1}{2} z_t^{\infty\prime} \left(\varepsilon_t^{\infty *} \odot \zeta_t^{\infty} - \iota_N \right) + f_t^{\infty\prime} \varepsilon_t^{\infty *} \\
&= \frac{1}{2} z_t^{\infty\prime} \left(\left(\Gamma^{-1} \zeta_t^{\infty} \odot \zeta_t^{\infty} \right) - \iota_N \right) + f_t^{\infty\prime} \Gamma^{-1} \zeta_t^{\infty} \\
&= \frac{1}{4} z_t^{\infty\prime} (E_N' (\Gamma^{-1} \otimes I_N + I_N \otimes \Gamma^{-1})(\zeta_t^{\infty} \otimes \zeta_t^{\infty}) - 2\iota_N) + f_t^{\infty\prime} \Gamma^{-1} \zeta_t^{\infty} \\
&= \frac{1}{4} z_t^{\infty\prime} (E_N' P (\zeta_t^{\infty} \otimes \zeta_t^{\infty}) - 2\iota_N) + f_t^{\infty\prime} \Gamma^{-1} \zeta_t^{\infty},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial l_t^{\infty *}(\omega)}{\partial \rho} &= \operatorname{vecl} \left(\varepsilon_t^{\infty *} \varepsilon_t^{\infty * \prime} - \Gamma^{-1} \right) \\
&= \frac{1}{2} L_N' \operatorname{vec} \left(\varepsilon_t^{\infty *} \varepsilon_t^{\infty * \prime} - \Gamma^{-1} \right) \\
&= \frac{1}{2} L_N' \left(\Gamma^{-1} \otimes \Gamma^{-1} \right) \operatorname{vec} \left(\zeta_t^{\infty} \zeta_t^{\infty\prime} - \Gamma \right).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\partial \operatorname{vec} \left(\zeta_t^{\infty} \zeta_t^{\infty\prime} \right)}{\partial \theta'} &= \frac{\partial (\zeta_t^{\infty} \otimes \zeta_t^{\infty})}{\partial \theta'} = \left(\zeta_t^{\infty} \otimes I_N + I_N \otimes \zeta_t^{\infty} \right) \frac{\partial \zeta_t^{\infty}}{\partial \theta'}, \\
\frac{\partial \zeta_t^{\infty}}{\partial \theta'} &= D_t^{\infty -1} \frac{\partial \varepsilon_t}{\partial \theta'} + (\varepsilon_t' \otimes I_N) \frac{\partial \operatorname{vec} D_t^{\infty -1}}{\partial \theta'} \\
&= -f_t^{\infty} - \frac{1}{2} \left(\zeta_t^{\infty\prime} \otimes I_N \right) E_N z_t^{\infty},
\end{aligned}$$

exploiting $\zeta_t^{\infty} = D_t^{\infty -1} \varepsilon_t = \operatorname{vec} \left(D_t^{\infty -1} \varepsilon_t \right) = (\varepsilon_t' \otimes I_N) \operatorname{vec} D_t^{\infty -1}$. But, $\frac{\partial \operatorname{vec} D_t^{\infty -1}}{\partial \theta'} = -(D_t^{\infty -1} \otimes$

$D_t^{\infty-1})\frac{\partial \operatorname{vec} D_t^{\infty}}{\partial\theta'} = -\frac{1}{2}(D_t^{\infty-1}\otimes I_N)E_N z_t^{\infty}$. Thus, since $E\left[I_N\otimes\zeta_t^{\infty}|\mathcal{F}_{t-1}\right]_{\omega=\omega_0}=0$,

$$\begin{aligned}
E\left[\frac{\partial \operatorname{vec}\left(\zeta_t^{\infty}\zeta_t^{\infty\prime}\right)}{\partial\theta'}|\mathcal{F}_{t-1}\right]_{\omega=\omega_0} &= -\frac{1}{2}E\left[\left(\zeta_t^{\infty}\otimes I_N + I_N\otimes\zeta_t^{\infty}\right)\left(\zeta_t^{\infty\prime}\otimes I_N\right)E_N z_t^{\infty}|\mathcal{F}_{t-1}\right]_{\omega=\omega_0}\\
&= -\frac{1}{2}E\left[\left(\zeta_t^{\infty}\zeta_t^{\infty\prime}\otimes I_N + I_N\otimes\zeta_t^{\infty}\zeta_t^{\infty\prime}\right)E_N z_t^{\infty}|\mathcal{F}_{t-1}\right]_{\omega=\omega_0}\\
&= -\frac{1}{2}\left[\left(\Gamma\otimes I_N + I_N\otimes\Gamma\right)E_N z_t^{\infty}\right]_{\omega=\omega_0}.
\end{aligned}$$

Exploiting this and since $E\left[\zeta_t^{\infty}|\mathcal{F}_{t-1}\right]_{\omega=\omega_0}=0$, $E\left[\left(\Gamma^{-1}\zeta_t^{\infty}\odot\zeta_t^{\infty}\right)-\iota_N|\mathcal{F}_{t-1}\right]_{\omega=\omega_0}=0$ we have

$$\begin{aligned}
E\left[\frac{\partial^2 l_t^{\infty *}(\omega_0)}{\partial\theta\partial\theta'}|\mathcal{F}_{t-1}\right] &= -\frac{1}{8}\left[z_t^{\infty\prime}E_N' P(\Gamma\otimes I_N + I_N\otimes\Gamma)E_N z_t^{\infty}\right]_{\omega=\omega_0} - \left[f_t^{\infty\prime}\Gamma^{-1}f_t^{\infty}\right]_{\omega=\omega_0}\\
&= -\frac{1}{8}\left[z_t^{\infty\prime}E_N'(\Gamma^{-1}\otimes I_N + I_N\otimes\Gamma^{-1})(\Gamma\otimes I_N + I_N\otimes\Gamma)E_N z_t^{\infty}\right]_{\omega=\omega_0} - \left[f_t^{\infty\prime}\Gamma^{-1}f_t^{\infty}\right]_{\omega=\omega_0}\\
&= -\frac{1}{4}\left[z_t^{\infty\prime}(E_N'E_N + E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N)z_t^{\infty}\right]_{\omega=\omega_0} - \left[f_t^{\infty\prime}\Gamma^{-1}f_t^{\infty}\right]_{\omega=\omega_0}\\
&= -\frac{1}{4}\left[z_t^{\infty\prime}\Gamma_A z_t^{\infty}\right]_{\omega=\omega_0}' - \left[f_t^{\infty\prime}\Gamma^{-1}f_t^{\infty}\right]_{\omega=\omega_0},
\end{aligned}$$

where the last line follows because $E_N'E_N + E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N = \Gamma_A$.

Similarly, differentiating $\dfrac{\partial l_t^{\infty *}(\omega)}{\partial\rho}$

$$\begin{aligned}
E\left[\frac{\partial^2 l_t^{\infty *}(\omega_0)}{\partial\rho\partial\theta'}|\mathcal{F}_{t-1}\right] &= \frac{1}{2}L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)E\left[\frac{\partial \operatorname{vec}\left(\zeta_t^{\infty}\zeta_t^{\infty\prime}\right)}{\partial\theta'}|\mathcal{F}_{t-1}\right]_{\omega=\omega_0}\\
&= -\frac{1}{4}\left[L_N'\left(I_N\otimes\Gamma^{-1}+\Gamma^{-1}\otimes I_N\right)E_N z_t^{\infty}\right]_{\omega=\omega_0}\\
&= -\frac{1}{4}\left[L_N' P E_N z_t^{\infty}\right]_{\omega=\omega_0},
\end{aligned}$$

and trivially, since $\nabla\Gamma' = \partial\operatorname{vec}(\Gamma)/\partial\rho' = L_N$,

$$E\left[\frac{\partial^2 l_t^{\infty *}(\omega_0)}{\partial\rho\partial\rho'}|\mathcal{F}_{t-1}\right] = -\frac{1}{2}L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)L_N.$$

Substituting into the above expression for J_0^* yields,

$$\begin{aligned}
J_0^* &= \frac{1}{4}E\left\{\begin{bmatrix} z_t^{\infty\prime} & 0\\ 0 & I_{\frac{1}{2}N(N-1)}\end{bmatrix}\begin{bmatrix} E_N'E_N + E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N & E_N'PL_N\\ L_N'PE_N & 2L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)L_N\end{bmatrix}\begin{bmatrix} z_t^{\infty} & 0\\ 0 & I_{\frac{1}{2}N(N-1)}\end{bmatrix}\right\}_{\omega=\omega_0}\\
&\quad + E\begin{bmatrix} f_t^{\infty\prime}\Gamma^{-1}f_t^{\infty} & 0\\ 0 & 0\end{bmatrix}_{\omega=\omega_0}\\
&= \frac{1}{4}E\left\{\begin{bmatrix} z_t^{\infty\prime} & 0\\ 0 & I_{\frac{1}{2}N(N-1)}\end{bmatrix}\left[(E_N^*,L_N^*)'(E_N^*,L_N^*)\right]\begin{bmatrix} z_t^{\infty} & 0\\ 0 & I_{\frac{1}{2}N(N-1)}\end{bmatrix}\right\}_{\omega=\omega_0}\\
&\quad + E\begin{bmatrix} f_t^{\infty\prime}\Gamma^{-1}f_t^{\infty} & 0\\ 0 & 0\end{bmatrix}_{\omega=\omega_0},
\end{aligned}$$

where $E_N^* = \frac{1}{\sqrt{2}}(\Gamma^{1/2}\otimes\Gamma^{-1/2}+\Gamma^{-1/2}\otimes\Gamma^{1/2})E_N$, $L_N^* = \sqrt{2}(\Gamma^{-1/2}\otimes\Gamma^{-1/2})L_N$. It follows that (E_N^*,L_N^*) has full rank. To see this, consider the solution to $(E_N^*,L_N^*)\,x=0$, for any $\left(\frac{1}{2}N(N+1)\times 1\right)$ vector x. This can be expressed as $E_N^*x_1 + L_N^*x_2 = 0$, if and only if $\frac{1}{\sqrt{2}}(\Gamma^{1/2}\otimes\Gamma^{1/2})E_N^*x_1 + L_Nx_2 = 0$, which implies that $\frac{1}{\sqrt{2}}E_N'(\Gamma^{1/2}\otimes\Gamma^{1/2})E_N^* x_1=0$, .since $E_N' L_N = 0$. But

$$\frac{1}{\sqrt{2}}E_N'(\Gamma^{1/2}\otimes\Gamma^{1/2})E_N^* = \frac{1}{2}E_N'\left(\Gamma\otimes I_N + I_N\otimes\Gamma\right)E_N = \Gamma\odot I_N = I_N,$$

which implies $x_1=0$. But if $x_1=0$, then $x_2=0$, since L_N has full rank (recall $L_N' L_N = 2I_{\frac{1}{2}N(N-1)}$). Therefore (E_N^*, L_N^*) has full rank, $\left[(E_N^*, L_N^*)'(E_N^*, L_N^*)\right]$ is positive definite and, thus, J_0^* is positive definite (without requiring the condition imposed by Ling and McAleer (2003) that $\Gamma_A - I_N$ is positive semi-definite[1]).

Note that, since $E_N'\left(\Gamma^{-1}\otimes I_N\right)L_N = E_N'\left(I_N\otimes\Gamma^{-1}\right)L_N$ and $E_N' E_N + E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N = \Gamma_A$, we can also write

$$\begin{bmatrix} E_N' E_N + E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N & E_N' P L_N \\ L_N' P E_N & 2L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)L_N \end{bmatrix} = \begin{bmatrix} \Gamma_A & 2E_N'\left(\Gamma^{-1}\otimes I_N\right)L_N \\ 2L_N'\left(I_N\otimes\Gamma^{-1}\right)E_N & 2L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)L_N \end{bmatrix}.$$

From J_0^* we obtain $J_T^*(\omega)$ with h_{it} replacing h_{it}^∞, ω replacing ω_0 and $\frac{1}{T}\sum_{t=1}^T$ replacing "expectation", throughout, and noting that $L_N' P E_N \sum_{t=1}^T z_t = L_N' P E_N\left(I_N\otimes\iota_T'\right)Z = (L_N' P E_N\otimes\iota_T')Z$, $\sum_{t=1}^T z_t{}'\Gamma_A z_t = Z'\left(\Gamma_A\otimes I_T\right)Z$ and $\sum_{t=1}^T f_t'\Gamma^{-1}f_t = F'\left(\Gamma^{-1}\otimes I_T\right)F$. Alternatively, note that $J_T^*(\omega)$ can also be expressed as

$$\begin{aligned} J_T^*(\omega) &= \frac{1}{4}\frac{1}{T}\sum_{t=1}^T\left\{\begin{bmatrix} z_t^{\infty\prime} & 0 \\ 0 & I_{\frac{1}{2}N(N-1)}\end{bmatrix}\left[(E_N^*, L_N^*)'(E_N^*, L_N^*)\right]\begin{bmatrix} z_t^{\infty} & 0 \\ 0 & I_{\frac{1}{2}N(N-1)}\end{bmatrix}\right\} \\ &\quad + \frac{1}{T}\sum_{t=1}^T\begin{bmatrix} f_t^{\infty\prime}\Gamma^{-1}f_t^\infty & 0 \\ 0 & 0\end{bmatrix} \\ &= \frac{1}{4T}W'\left(Q_N^{*\prime}Q_N^*\otimes I_T\right)W + \frac{1}{T}\begin{bmatrix} F'\left(\Gamma^{-1}\otimes I_T\right)F & 0 \\ 0 & 0\end{bmatrix}, \end{aligned}$$

where $Q_N^* = (E_N^*, L_N^*)$ and $W = \begin{bmatrix} Z & 0 \\ 0 & I_{\frac{1}{2}N(N-1)}\otimes\iota_T\end{bmatrix}$. This follows from the general result that if $w_t = \mathrm{diag}(w_{it}')$, $(N\times Nq)$, with w_{it} being $(q\times 1)$, whilst $W=\mathrm{diag}(W_i)$ where W_i, $(T\times q)$, has rows w_{it}', then $\sum_{t=1}^T w_t{}'Aw_t = W'\left(A\otimes I_T\right)W$, for any $(N\times N)$ matrix A. Then $J_T^*(\hat\omega)$ is positive definite provided $\hat Z$ has full rank of $N(K+K^*)$.

The above results concur with the calculations given by Nakatani and Teräsvirta (2009, p.151), but allowing for regression parameters, φ_i. These imply that, with $\nabla D_t^\infty \equiv \frac{\partial\,\mathrm{vec}\,D_t^\infty}{\partial\theta'}$,

$$\begin{aligned} J_0^* &= \frac{1}{2}E\begin{bmatrix} \begin{matrix}\nabla D_t^\infty(2\left(D_t^{\infty-1}\otimes D_t^{\infty-1}\right) \\ +\left(H_t^{\infty-1}\otimes\Gamma\right)+\left(\Gamma\otimes H_t^{\infty-1}\right))\nabla D_t^{\infty\prime}\end{matrix} & \begin{matrix}\nabla D_t^\infty((\Gamma^{-1}D_t^{\infty-1}\otimes I_N) \\ +\left(I_N\otimes\Gamma^{-1}D_t^{\infty-1}\right))\nabla\Gamma'\end{matrix} \\ \begin{matrix}\nabla\Gamma(\left(\Gamma^{-1}D_t^{\infty-1}\otimes I_N\right) \\ +\left(I_N\otimes\Gamma^{-1}D_t^{\infty-1}\right))\nabla D_t^{\infty\prime}\end{matrix} & \nabla\Gamma\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)\nabla\Gamma' \end{bmatrix}_{\omega=\omega_0} \\ &\quad + E\begin{bmatrix} f_t^{\infty\prime}\Gamma^{-1}f_t^\infty & 0 \\ 0 & 0\end{bmatrix}_{\omega=\omega_0}. \end{aligned}$$

By the properties of E_N

$$\nabla D_t^{\infty\prime} = \frac{1}{2}E_N D_t^\infty z_t^\infty = \frac{1}{2}\left(D_t^\infty\otimes I_N\right)E_N z_t^\infty = \frac{1}{2}\left(I_N\otimes D_t^\infty\right)E_N z_t^\infty,$$

[1] The redundancy of this condition on $\Gamma_A - I_N$ may arise from an error in the expression for the expected hessian gven by Ling and McAleer (2003, p.289). Specifically, the error arsises from writing $(\partial\,\mathrm{vec}(\Gamma)/\partial\rho')'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)\partial\,\mathrm{vec}(\Gamma)/\partial\rho'$as $P'P$ where, here, $P = (I_N\otimes\Gamma^{-1})\partial\,\mathrm{vec}(\Gamma)/\partial\rho'$.

so that

$$
\begin{aligned}
& \nabla D_t^{\infty}\left(2\left(D_t^{\infty-1}\otimes D_t^{\infty-1}\right)+\left(H_t^{\infty-1}\otimes\Gamma\right)+\left(\Gamma\otimes H_t^{\infty-1}\right)\right)\nabla D_t^{\infty\prime} \\
= & \frac{1}{2}z_t^{\infty\prime}E_N'\left(D_t^{\infty}\otimes I_N\right)\left(D_t^{\infty-1}\otimes D_t^{\infty-1}\right)\left(I_N\otimes D_t^{\infty}\right)E_N z_t^{\infty} \\
& +\frac{1}{4}z_t^{\infty\prime}E_N'\left(D_t^{\infty}\otimes I_N\right)\left(H_t^{\infty-1}\otimes\Gamma\right)\left(D_t^{\infty}\otimes I_N\right)E_N z_t^{\infty} \\
& +\frac{1}{4}z_t^{\infty\prime}E_N'\left(I_N\otimes D_t^{\infty}\right)\left(\Gamma\otimes H_t^{\infty-1}\right)\left(I_N\otimes D_t^{\infty}\right)E_N z_t^{\infty} \\
= & \frac{1}{2}z_t^{\infty\prime}E_N'E_N z_t^{\infty}+\frac{1}{4}E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N+\frac{1}{4}E_N'\left(\Gamma\otimes\Gamma^{-1}\right)E_N \\
= & \frac{1}{2}z_t^{\infty\prime}\{E_N'E_N+E_N'\left(\Gamma^{-1}\otimes\Gamma\right)E_N\}z_t^{\infty} \\
= & \frac{1}{2}z_t^{\infty\prime}\Gamma_A z_t^{\infty},
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla\Gamma\left(\left(\Gamma^{-1}D_t^{\infty-1}\otimes I_N\right)+\left(I_N\otimes\Gamma^{-1}D_t^{\infty-1}\right)\right)\nabla D_t^{\infty\prime} = & \frac{1}{2}L_N'\left(\Gamma^{-1}D_t^{\infty-1}\otimes I_N\right)\left(D_t^{\infty}\otimes I_N\right)E_N z_t^{\infty} \\
& +\frac{1}{2}L_N'\left(I_N\otimes\Gamma^{-1}D_t^{\infty-1}\right)\left(I_N\otimes D_t^{\infty}\right)E_N z_t^{\infty} \\
= & \frac{1}{2}L_N'PE_N z_t^{\infty},
\end{aligned}
$$

and, clearly, $\nabla\Gamma\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)\nabla\Gamma'=L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)L_N$.
To establish consistency, consider the following partitions of $J_T^*(\omega)$. Firstly, from $\frac{1}{4}\frac{1}{T}\sum_{t=1}^{T}z_t{}'\Gamma_A z_t+\frac{1}{T}\sum_{t=1}^{T}f_t'\Gamma^{-1}f_t$ we have the following sub-partitions

$$J_{\theta_i\theta_j T}^*(\omega)=\frac{1}{4}\left(\delta_{ij}+\rho^{ij}\rho_{ij}\right)\frac{1}{T}\sum_{t=1}^{T}z_{it}z_{jt}'+\rho^{ij}\frac{1}{T}\sum_{t=1}^{T}f_{it}f_{jt}'.$$

Similarly

$$J_{\rho_{ij},\theta_k T}^*(\omega)=\frac{1}{2}\left(\delta_{jk}\rho^{ik}+\delta_{ik}\rho^{jk}\right)\frac{1}{T}\sum_{t=1}^{T}z_{kt}'.$$

Clearly, by consistency of $\hat\rho$, $L_N'\left(\Gamma^{-1}\otimes\Gamma^{-1}\right)L_N-L_N'\left(\hat\Gamma^{-1}\otimes\hat\Gamma^{-1}\right)L_N=o_p(1)$. For the remaining partitions of $J_T^*(\omega)$, define $q_{ij,t}^{\infty}=\left(1,f_{it}^{\infty\prime},f_{jt}^{\infty\prime},z_{it}^{\infty\prime},z_{jt}^{\infty\prime}\right)'$, for any pair $i>j$, and correspondingly $q_{ij,t}=\left(1,f_{it}',f_{jt}',z_{it}',z_{jt}'\right)'$. Then it is immediate from the results of, e.g., Halunga and Orme (2009, Proposition 4a and c) that (i) $E\sup_\omega\left\|q_{ij,t}^{\infty}\right\|^2<\infty$; and, (ii) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_\omega\left\|q_{ij,t}^{\infty}-q_{ij,t}\right\|=o_p(1)$. The first result ensures that

$$\frac{1}{T}\sum_{t=1}^{T}\left(q_{ij,t}^{\infty}q_{ij,t}^{\infty\prime}\right)_{\omega=\hat\omega}-E\left[q_{ij,t}^{\infty}q_{ij,t}^{\infty\prime}\right]_{\omega=\omega_0}=o_p(1).$$

Combined with (ii), and Proposition 1. we get

$$\sup_\omega\left\|\frac{1}{T}\sum_{t=1}^{T}q_{ij,t}^{\infty}q_{ij,t}^{\infty\prime}-\frac{1}{T}\sum_{t=1}^{T}q_{ij,t}q_{ij,t}'\right\|=o_p(1).$$

This, together with the triangle inequality is sufficient to ensure that $J_T^*(\hat\omega)-J_0^*=o_p(1)$.
(ii) $B_0^*=-E\left\{E\left[\frac{\partial m_t^{\infty}(\omega_0)}{\partial\omega'}\middle|\mathcal{F}_{t-1}\right]\right\}$, is obtained by stacking the matrices $E\left\{E\left[\frac{\partial m_{ij,t}^{\infty}(\omega_0)}{\partial\theta'}|\mathcal{F}_{t-1}\right]\right\}$.

First, f_{it}^{∞}, z_{it}^{∞}, $r_{ij,t}^{\infty}$ and $\frac{\partial r_{ij,t}^{\infty}}{\partial \theta_k'}$ and $\frac{\partial r_{ij,t}^{\infty}}{\partial \rho}$ are all $\mathcal{F}_{t-1}$ measurable. Second, $E\left[\zeta_{0jt}^{\infty 2}|\mathcal{F}_{t-1}\right] = 1$, $E\left[\zeta_{0jt}^{\infty} f_{0it}^{\infty}|\mathcal{F}_{t-1}\right] = E\left[\zeta_{0it}^{\infty} f_{0it}^{\infty}|\mathcal{F}_{t-1}\right] = 0$, and $E\left[\zeta_{0it}^{\infty}\zeta_{0jt}^{\infty}|\mathcal{F}_{t-1}\right] = \rho_{0ij}$. Then, from the derivatives (10)-(11), we obtain

$$E\left\{E\left[\frac{\partial m_{ij,t}^{\infty}(\omega_0)}{\partial \theta'}|\mathcal{F}_{t-1}\right]\right\} = -\frac{1}{2}E\left[\rho_{ij} r_{ij,t}^{\infty}\left(e_j \otimes z_{jt}^{\infty} + e_i \otimes z_{it}^{\infty}\right)'\right]_{\omega=\omega_0},$$

$$E\left\{E\left[\frac{\partial m_{ij,t}^{\infty}(\omega_0)}{\partial \rho'}|\mathcal{F}_{t-1}\right]\right\} = -E\left[r_{ij,t}^{\infty} e_{ij}'\right]_{\omega=\omega_0}.$$

Based on these expressions, the partitions of $B_T^*(\omega)$ which correspond to the above partitions of B_0^* are

$$\begin{aligned} B_{ijT}^*(\omega) &= \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{2}\rho_{ij} r_{ij,t}\left(e_j' \otimes z_{jt}' + e_i' \otimes z_{it}'\right),\ r_{ij,t} e_{ij}'\right] \\ &= \left[\frac{1}{2}\rho_{ij} R_{ij}'(e_j' \otimes Z_j + e_i' \otimes Z_i),\ R_{ij}'(e_{ij}' \otimes \iota_T)\right]. \end{aligned}$$

Finally, to establish consistency of $B_T^*(\hat{\omega})$, define $q_{ij,t}^{\infty} = \left(1, z_{it}^{\infty\prime}, r_{ij,t}^{\infty\prime}\right)'$, for any pair $i > j$, and correspondingly $q_{ij,t} = \left(1, z_{it}', r_{ij,t}'\right)'$. Then it is immediate from B4 and Remark 1(ii) that:(i) $E\sup_{\omega}\left\|q_{ij,t}^{\infty}\right\|^2 < \infty$; and, (ii) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left\|q_{ij,t}^{\infty} - q_{ij,t}\right\| = o_p(1)$. The result then follows by Lemma 1.■

Remark 3 *For the Robust FQMLE Tse's test we can also obtain* $D_T^*(\omega)$, *the consistent estimator of* $-E\left\{E\left[\frac{\partial m_t^{\infty LM}(\omega_0)}{\partial \omega'}\middle|\mathcal{F}_{t-1}\right]\right\}$, *directly from the derivations for the estimated hessian,* $J_T^*(\omega)$, *above. Specifically, we can write* $\bar{m}_T^{LM} = \frac{1}{T}\sum_{t=1}^{T} m_t^{LM}(\omega) = \frac{1}{T}\sum_{t=1}^{T}\phi_t' \operatorname{vecl}\left(\varepsilon_t^*\varepsilon_t^{*\prime} - \Gamma^{-1}\right)$, $(q^C \times 1)$, *where* $\phi_t = \operatorname{diag}(\phi_{kl,t}')$, $(\frac{1}{2}N(N-1) \times q^C)$, *with indices* (k,l) *ordered according to* s_N^C *and* $q^C = \sum_{k>l} q_{kl}$ *with* $\phi_{kl,t}$ *being a* $(q_{kl} \times 1)$ *vector of test variables. Given previous calculations for* $\partial l_t^{\infty *}(\omega)/\partial \rho'$, *this can be expressed as* $\bar{m}_T^{LM} = \frac{1}{2}\frac{1}{T}\sum_{t=1}^{T}\phi_t' L_N'\left(\Gamma^{-1} \otimes \Gamma^{-1}\right)\operatorname{vec}\left(\zeta_t^{\infty}\zeta_t^{\infty\prime} - \Gamma\right)$ *and it follows immediately that, for the joint Tse test of all* $\frac{1}{2}N(N-1)$ *constant conditional correlations,*

$$\begin{aligned} D_T^*(\omega) &= \frac{1}{4}\frac{1}{T}\sum_{t=1}^{T}\left[\phi_t' L_N' P E_N z_t^{\infty},\ 2\phi_t' L_N'\left(\Gamma^{-1} \otimes \Gamma^{-1}\right) L_N\right] \\ &= \frac{1}{4}\frac{1}{T}\left[\Phi'(L_N' P E_N \otimes I_T) Z,\ 2\Phi'(L_N'\left(\Gamma^{-1} \otimes \Gamma^{-1}\right) L_N \otimes \iota_T)\right] \\ &= \frac{1}{4}\frac{1}{T}\left[\Phi'(L_N^{*\prime} E_N^* \otimes I_T) Z,\ \Phi'(L_N^{*\prime} L_N \otimes I_T)(I_{\frac{1}{2}N(N-1)} \otimes \iota_T)\right] \\ &= \frac{1}{4}\frac{1}{T}\Phi'(L_N^{*\prime} Q_N^* \otimes I_T) W, \end{aligned}$$

where $\Phi = \operatorname{diag}(\Phi_{kl})$, $\left(\frac{T}{2}N(N-1) \times q^C\right)$ *with* Φ_{kl}, $(T \times q_{kl})$ *having rows* $\phi_{kl,t}$, $t = 1, ..., T$.
For the PQMLE case, it is clear that $D_T(\omega) = \frac{1}{4}\frac{1}{T}\bar{\Phi}'(L_N^{*\prime} E_N^* \otimes I_T) Z$.

Proposition 7 *Define* $\bar{m}_T(\omega) = T^{-1}\sum_{t=1}^{T} m_t(\omega)$ *constructed from the* $(q_{ij} \times 1)$ *sub-vectors* $\bar{m}_{ij,T}(\omega) = \frac{1}{T}\sum_{t=1}^{T}(\zeta_{it}\zeta_{jt} - \rho_{ij})\left(r_{ij,t} - \bar{r}_{ijT}(\omega)\right)$ *and* $\bar{n}_T^{\infty}(\omega) = T^{-1}\sum_{t=1}^{T} n_t^{\infty}(\omega)$ *constructed from the* $(q_{ij} \times 1)$ *sub-vectors* $\bar{n}_{ij,T}^{\infty}(\omega) = \frac{1}{T}\sum_{t=1}^{T}(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{0,ij})\left(r_{ij,t}^{\infty} - \mu_{ij}(\omega_0)\right)$, *where* $\mu_{ij}(\omega_0) = E\left[r_{ij,t}^{\infty}\right]_{\omega=\omega_0}$ *and* $\left\|\mu_{ij}(\omega)\right\| < \infty$, *by B4. Under Assumptions A and B1, B2:*

(i) $E\sup_{\theta}\left\|g_t^{\infty}(\theta)\right\|^2 < \infty$;

(ii) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\theta}\left\|g_t^{\infty}(\theta) - g_t(\theta)\right\| = o_p(1)$.

In addition, and adding B3 and B4:

(iii) $E\sup_{\omega}\left\|n_t^{\infty}(\omega)\right\|^2 < \infty$;

(iv) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\|n_t^{\infty}(\tilde{\omega}) - m_t(\tilde{\omega})\| = o_p(1)$, *where* $\tilde{\omega}$ *is the PQML estimator.*

Proof. *It is readily shown that (i) and (ii) hold, and (iii) follows from Proposition 6(iii). For (iv), let* $\Delta m_{ij,t}^{\infty}(\omega) = (\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{0,ij})r_{ij,t}^{\infty} - (\zeta_{it}\zeta_{jt} - \rho_{0,ij})r_{ij,t}$ *and write* $\sqrt{T}\left(\bar{n}_{ij,T}^{\infty}(\tilde{\omega}) - \bar{m}_{ij,T}(\tilde{\omega})\right) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} a_{ij,t}(\tilde{\omega})$, *where*

$$\begin{aligned}
a_{ij,t}(\omega) &= \left(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \rho_{0,ij}\right)\left(r_{ij,t}^{\infty} - \mu_{ij}(\omega_0)\right) - \left(\zeta_{it}\zeta_{jt} - \rho_{0,ij}\right)\left(r_{ij,t} - \bar{r}_{ijT}(\omega)\right) \\
&= \Delta m_{ij,t}^{\infty}(\omega) + \left(\zeta_{it}\zeta_{jt} - \rho_{0,ij}\right)\left(\bar{r}_{ijT}(\omega) - \mu_{ij}(\omega_0)\right) - \left(\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \zeta_{it}\zeta_{jt}\right)\mu_{ij}(\omega_0).
\end{aligned}$$

Given B1 and B2, and the similar derivations to these employed in the Proof of Proposition 2, it is readily shown that $T^{-1}/2\sum_{t=1}^{T}\tilde{\zeta}_{it}\tilde{\zeta}_{jt} - \rho_{0,ij} = O_p(1)$, *and by B3 and B4, and the triangle inequality,*

$$\sup_{\omega}\left\|\bar{r}_{ijT}(\omega) - \mu_{ij}(\omega_0)\right\| \leq \sup_{\omega}\left\|\bar{r}_{ijT}^{\infty}(\omega) - \mu_{ij}(\omega_0)\right\| + \sup_{\omega}\left\|\bar{r}_{ijT}^{\infty}(\omega) - \bar{r}_{ijT}(\omega)\right\| = o_p(1),$$

so that $\bar{r}_{ijT}(\tilde{\omega}) - \mu_{ij}(\omega_0) = o_p(1)$. *Therefore,*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\|a_{ij,t}(\tilde{\omega})\| = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\|\Delta m_{ij,t}^{\infty}(\tilde{\omega}))\right\| + \left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left|\tilde{\zeta}_{it}^{\infty}\tilde{\zeta}_{jt}^{\infty} - \tilde{\zeta}_{it}\tilde{\zeta}_{jt}\right|\right\}\left\|\mu_{ij}(\omega_0)\right\| + o_p(1).$$

By Proposition 6(iv), B3 and the triangle inequality, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\|\Delta m_{ij,t}^{\infty}(\tilde{\omega})\right\| = o_p(1)$, *and (by similar reasoning)* $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\sup_{\omega}\left|\zeta_{it}^{\infty}\zeta_{jt}^{\infty} - \zeta_{it}\zeta_{jt}\right| = o_p(1)$, *so that* $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left|\tilde{\zeta}_{it}^{\infty}\tilde{\zeta}_{jt}^{\infty} - \tilde{\zeta}_{it}\tilde{\zeta}_{jt}\right| = o_p(1)$. *Thus since* $\left\|\mu_{ij}(\omega_0)\right\| < \infty$, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\|a_{ij,t}(\tilde{\omega})\| = o_p(1)$, *so that* $\sqrt{T}\,\|\bar{n}_{T}^{\infty}(\tilde{\omega}) - \bar{m}_T(\tilde{\omega})\| = o_p(1)$. ■

Proof of Proposition 4: Σ is finite by Proposition Proposition 7(i) and (iii). As in Ling and McAleer (2003, Lemma 5.2), (i) follows from a Martingale Central Limit Theorem. Part (ii), follows from Proposition 7 and Remark 2.■

Proof of Proposition 5: Firstly, by Proposition 7, $\sqrt{T}\bar{m}_T(\tilde{\omega}) = \sqrt{T}\bar{n}_T^{\infty}(\tilde{\omega}) + o_p(1)$, and we work with $\sqrt{T}\bar{n}_T^{\infty}(\tilde{\omega})$. Second, from the consistency and asymptotic normality of $\tilde{\theta}$, $\sqrt{T}(\tilde{\theta} - \theta_0) = J_0^{-1}\sqrt{T}\bar{g}_T^{\infty}(\theta_0) + o_p(1)$, where $\sqrt{T}\bar{g}_T^{\infty}(\theta) = \sqrt{T}\bar{g}_T(\theta) + o_p(1)$, by Proposition 7. Similar to proof of Proposition 2, it is readily shown that $E\sup_{\omega}\left\|\frac{\partial n_t^{\infty}(\omega)}{\partial\theta}\right\| < \infty$ so that $\frac{\partial\bar{n}_T^{\infty}(\omega_T)}{\partial\theta'} \xrightarrow{p} -B_0 = E\left[\frac{\partial n_t^{\infty}(\omega_0)}{\partial\theta'}\right]$, for any sequence $\omega_T = \omega_0 + o_p(1)$. Thus taking a mean value expansion of $\sqrt{T}\bar{n}_T^{\infty}(\tilde{\omega})$ about ω_0, and ignoring asymptotically negligible terms, yields $\sqrt{T}\bar{n}_T^{\infty}(\tilde{\omega}) = A_0\sqrt{T}\bar{u}_T^{\infty}(\omega_0) + o_p(1)$, and the result follows from Proposition 4. ■

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