

Limit Theorems for Multifractal Products of Geometric Stationary Processes

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Abstract: We investigate the properties of multifractal products of geometric Gaussian processes with possible long-range dependence and geometric Ornstein-Uhlenbeck processes driven by Lévy motion and their finite and infinite superpositions. We present the general conditions for the \mathcal{L}_q convergence of cumulative processes to the limiting processes and investigate their q -th order moments and Rényi functions, which are nonlinear, hence displaying the multifractality of the processes as constructed. We also establish the corresponding scenarios for the limiting processes, such as log-normal, log-gamma, log-tempered stable or log-normal tempered stable scenarios.

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1. Introduction

Multifractal models have been used in many applications in hydrodynamic turbulence, finance, genomics, computer network traffic, etc. (see, for example, Kolmogorov 1941, 1962, Kahane 1985, 1987, Novikov 1994, Frisch 1995, Mandelbrot 1997, Falconer 1997, Schertzer *et al.* 1997, Harte 2001, Riedi 2003). There are many ways to construct random multifractal models ranging from simple binomial cascades to measures generated by branching processes and the compound Poisson process (Kahane 1985, 1987, Falconer 1997, Schmitt and Marsan 2001, Harte 2001, Barral and Mandelbrot 2002, 2010, Bacry and Muzy 2003, Riedi 2003, Mörters and Shieh 2002, 2004, Shieh and Taylor 2002, Schmitt 2003, Schertzer *et al.* 1997, Barall *et al.* 2009, Ludena 2008, 2009). Jaffard (1999) showed that Lévy processes (except Brownian motion and Poisson processes) are multifractal; but since the increments of a Lévy process are independent, this class excludes the effects of dependence structures. Moreover, Lévy processes have a linear singularity spectrum while real data often exhibit a strictly concave spectrum.

Anh, Leonenko and Shieh (2008, 2009a,b, 2010) considered multifractal products of stochastic processes as defined in Kahane (1985, 1987) and Mannersalo, Norros and Riedi (2002). Especially Anh, Leonenko and Shieh (2008) constructed multifractal processes based on products of geometric Ornstein-Uhlenbeck (OU) processes driven by Lévy motion with inverse Gaussian or normal inverse Gaussian distribution. They also described the behavior of the q -th order moments and Rényi functions, which are nonlinear, hence displaying the multifractality of the processes as constructed. In these papers a number of scenarios were obtained for $q \in Q \cap [1, 2]$, where Q is a set of parameters of marginal distribution of an OU processes driven by Lévy motion. The simulations show that for q outside this range, the scenarios still hold (see Anh, Leonenko, Shieh and Taufer (2010)). In this paper we present a rigorous proof of these results and also construct new scenarios which generalize those corresponding to the inverse Gaussian and normal inverse Gaussian distributions obtained in Anh and Leonenko (2008), Anh, Leonenko and Shieh (2008). We use the theory of OU processes with

tempered stable law and normal tempered stable law for their marginal distributions (see Barndorff-Nielsen and Shephard 2002, Terdik and Woyczynski 2004, and the references therein). Note that the log-tempered stable distribution is also known (up to constants) as the Vershik-Yor subordinator (see Donati-Martin and Yor 2006, and the references therein).

The next section recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985, 1987) and Mannersalo, Norros and Riedi (2002). Section 3 contains the general \mathcal{L}_q bounds for cumulative process of multifractal products of stationary processes. Section 4 establishes the general results on the scaling moments of multifractal products of geometric OU processes in terms of the marginal distributions of OU processes and their Lévy measures. Similar results for the finite and infinite superpositions of OU processes are proved in the Section 7. The number of multifractal scenarios with exact forms of the Rényi function are given in the Sections 8-13.

Our exposition extends results of Mannersalo, Norros and Riedi (2002) on the basic properties of multifractal products of stochastic processes. We should also note some related results by Barndorff-Nielsen and Schmiegel (2004) who introduced some Lévy-based spatiotemporal models for parametric modelling of turbulence. Log-infinitely divisible scenarios related to independently scattered random measures were investigated in Schmitt and Marsan (2001), Schmitt (2003), Bacry and Muzy (2003), Rhodes and Vargas (2010), see also their references.

2. Multifractal products of stochastic processes

This section recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985, 1987) and Mannersalo, Norros and Riedi (2002). We provide an interpretation of their conditions based on the moment generating functions, which is useful for our exposition. Throughout the text the notation C, c is used for the generic constants which do not necessarily coincide.

We introduce the following conditions:

A'. Let $\Lambda(t)$, $t \in \mathbb{R}_+ = [0, \infty)$, be a measurable, separable, strictly stationary, positive stochastic process with $E\Lambda(t) = 1$.

We call this process the mother process and consider the following setting:

A''. Let $\Lambda(t) = \Lambda^{(i)}$, $i = 0, 1, \dots$ be independent copies of the mother process Λ , and $\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$:

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in \mathbb{R}_+, \quad i = 0, 1, 2, \dots,$$

where the scaling parameter $b > 1$, and $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions.

Moreover, in the examples, the stationary mother process satisfies the following conditions:

A'''. Let $\Lambda(t) = \exp\{X(t)\}$, $t \in \mathbb{R}_+$, where $X(t)$ is a strictly stationary process, such that there exist a marginal probability density function $\pi(x)$ and a bivariate probability density function $p(x_1, x_2; t_1 - t_2)$. Moreover, we assume that the moment generating function

$$M(\zeta) = E \exp\{\zeta X(t)\} \tag{1}$$

and the bivariate moment generating function

$$M(\zeta_1, \zeta_2; t_1 - t_2) = E \exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$$

exist.

The conditions **A'**-**A'''** yield

$$E\Lambda_b^{(i)}(t) = M(1) = 1;$$

$$\text{Var}\Lambda_b^{(i)}(t) = M(2) - 1 = \sigma_\Lambda^2 < \infty;$$

$$\text{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1, 1; (t_1 - t_2)b^i) - 1, \quad b > 1.$$

We define the finite product processes

$$\Lambda_n(t) = \prod_{i=0}^n \Lambda_b^{(i)}(t) = \exp\left\{\sum_{i=0}^n X^{(i)}(tb^i)\right\}, \quad t \in [0, 1], \tag{2}$$

and the cumulative processes

$$A_n(t) = \int_0^t \Lambda_n(s) ds, \quad n = 0, 1, 2, \dots, t \in [0, 1], \quad (3)$$

where $X^{(i)}(t), i = 0, \dots, n, \dots$, are independent copies of a stationary process $X(t), t \geq 0$.

We also consider the corresponding positive random measures defined on Borel sets B of \mathbb{R}_+ :

$$\mu_n(B) = \int_B \Lambda_n(s) ds, \quad n = 0, 1, 2, \dots \quad (4)$$

Kahane (1987) proved that the sequence of random measures μ_n converges weakly almost surely to a random measure μ . Moreover, given a finite or countable family of Borel sets B_j on \mathbb{R}_+ , it holds that $\lim_{n \rightarrow \infty} \mu_n(B_j) = \mu(B_j)$ for all j with probability one. The almost sure convergence of $A_n(t)$ in countably many points of \mathbb{R}_+ can be extended to all points in \mathbb{R}_+ if the limit process $A(t)$ is almost surely continuous. In this case, $\lim_{n \rightarrow \infty} A_n(t) = A(t)$ with probability one for all $t \in \mathbb{R}_+$. As noted in Kahane (1987), there are two extreme cases: (i) $A_n(t) \rightarrow A(t)$ in \mathcal{L}_1 for each given t , in which case $A(t)$ is not almost surely zero and is said to be fully active (non-degenerate) on \mathbb{R}_+ ; (ii) $A_n(1)$ converges to 0 almost surely, in which case $A(t)$ is said to be degenerate on \mathbb{R}_+ . Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987) (Eqs. (18) and (19) respectively.) The condition for complete degeneracy is detailed in Theorem 3 of Kahane (1987).

The Rényi function of a random measure μ , also known as the deterministic partition function, is defined for $t \in [0, 1]$ as

$$\begin{aligned} T(q) &= \liminf_{n \rightarrow \infty} \frac{\log \mathbb{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)} \right)}{\log \left| I_k^{(n)} \right|} \\ &= \liminf_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \log_2 \mathbb{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)} \right), \end{aligned} \quad (5)$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}]$, $k = 0, 1, \dots, 2^n - 1$, $\left| I_k^{(n)} \right|$ is its length, and \log_b is log to the base b .

Remark 1. The multifractal formalism for random cascades and other multifractal processes can be stated in terms of the Legendre transform of the Rényi function:

$$T^*(\alpha) = \min_{q \in \mathbb{R}} (q\alpha - T(q)).$$

In fact, let $f(\alpha)$ be the Hausdorff dimension of the set

$$C_\alpha = \left\{ t \in [0, 1] : \lim_{n \rightarrow \infty} \frac{\log \mu \left(I_k^{(n)}(t) \right)}{\log \left| I_k^{(n)} \right|} = \alpha \right\},$$

where $I_k^{(n)}(t)$ is a sequence of intervals $I_k^{(n)}$ that contain t . The function $f(\alpha)$ is known as the singularity spectrum of the measure μ , and we refer to μ as a multifractal measure if $f(\alpha) \neq 0$ for a continuum of α . In order to determine the function $f(\alpha)$, it was proposed to use the relationship

$$f(\alpha) = T^*(\alpha). \quad (6)$$

This relationship may not hold for a given measure. When the equality (6) is established for a measure μ , we say that the multifractal formalism holds for this measure.

We recapture for geometric stationary mother process the results of Mannersalo, Norros and Riedi (2002), in which the conditions for \mathcal{L}_2 -convergence and scaling of moments are established:

Theorem 2. (Mannersalo, Norros and Riedi 2002) Suppose that the conditions \mathbf{A}' - \mathbf{A}''' hold.

If, for some positive numbers δ and γ ,

$$\exp\{-\delta|\tau|\} \leq \text{Corr}(\Lambda(0), \Lambda(s)) = \frac{M(1, 1; \tau) - 1}{M(2) - 1} \leq |C\tau|^{-\gamma}, \quad (7)$$

then $A_n(t), t \in [0, 1]$ converges in L_2 if and only if

$$b > 1 + \sigma_\Lambda^2 = M(2).$$

If $A_n(t)$ converges in L_1 , then the limit process $A(t), t \in [0, 1]$, satisfies the recursion

$$A(t) = \frac{1}{b} \int_0^t \Lambda(s) d\tilde{A}(bs), \quad (8)$$

where the processes $\Lambda(t)$ and $\tilde{A}(t)$ are independent, and the processes $A(t)$ and $\tilde{A}(t)$ have identical finite-dimensional distributions.

If $A(t)$ is non-degenerate, the recursion (8) holds, $A(1) \in \mathcal{L}_q$ for some $q > 0$, and

$$\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty, \quad (9)$$

where

$$c(q, t) = \mathbb{E} \sup_{s \in [0, t]} |\Lambda^q(0) - \Lambda^q(s)|,$$

then there exist constants \overline{C} and \underline{C} such that

$$\underline{C}t^{q - \log_b \mathbb{E}\Lambda^q(t)} \leq \mathbb{E}A^q(t) \leq \overline{C}t^{q - \log_b \mathbb{E}\Lambda^q(t)}, \quad (10)$$

which will be written as

$$\mathbb{E}A^q(t) \sim t^{q - \log_b \mathbb{E}\Lambda^q(t)}, \quad t \in [0, 1].$$

If, on the other hand, $A(1) \in \mathcal{L}_q$, $q > 1$, then the Rényi function is given by

$$T(q) = q - 1 - \log_b \mathbb{E}\Lambda^q(t) = q - 1 - \log_b M(q). \quad (11)$$

If $A(t)$ is non-degenerate, $A(1) \in \mathcal{L}_2$, and $\Lambda(t)$ is positively correlated, then

$$\text{Var}A(t) \geq \text{Var} \int_0^t \Lambda(s) ds. \quad (12)$$

Remark 3. The result (10) means that the process $A(t)$, $t \in [0, 1]$ with stationary increments behaves as

$$\log \mathbb{E}[A(t + \delta) - A(t)]^q \sim K(q) \log \delta + C_q \quad (13)$$

for a wide range of resolutions δ with a nonlinear function

$$K(q) = q - \log_b \mathbb{E}\Lambda^q(t) = q - \log_b M(q),$$

where C_q is a constant. In this sense, the stochastic process $A(t)$ is said to be multifractal. The function $K(q)$, which contains the scaling parameter b and all the parameters of the marginal distribution of the stationary process $X(t)$, can be estimated by running the regression for a range of values of q . For the examples in Sections 8-14, the explicit form of $K(q)$ is obtained. Hence these parameters can be estimated by minimizing the mean square error between the $K(q)$ curve estimated from data and its analytical form for a range of values of q , see Ludena 2008, 2009.

3. \mathcal{L}_q convergence: general bound

This section contains a generalization of the basic results on multifractal products of stochastic processes developed in Kahane (1985, 1987) and Mannersalo, Norros and Riedi (2002).

Consider the cumulative process $A_n(t)$ defined in (3) For fixed t , the sequence $\{A_n(t), \mathcal{F}_n\}_{n=0}^\infty$ is a martingale. It is well known that for $q > 1$, \mathcal{L}_q convergence is equivalent to the finiteness of

$$\sup_n \mathbb{E} A_n^q(t) < \infty.$$

To illustrate the approach we consider separately a simpler case $q = 2$, which was studied in Mannersalo *et al.* (2002).

We have,

$$\begin{aligned} \mathbb{E} A_n^2(t) &= \mathbb{E} \int_0^t \int_0^t \Lambda_n(s_1) \Lambda_n(s_2) ds_1 ds_2 \\ &= \int_0^t \int_0^t \prod_{i=0}^n \mathbb{E} \Lambda^{(i)}(s_1) \Lambda^{(i)}(s_2) ds_1 ds_2. \end{aligned}$$

The process $\Lambda^{(i)}$ is stationary. Therefore,

$$\begin{aligned} \mathbb{E} A_n^2(t) &= 2 \int_0^t \int_{s_1}^t \prod_{i=0}^n \mathbb{E} \Lambda^{(i)}(0) \Lambda^{(i)}(s_2 - s_1) ds_1 ds_2 \\ &= 2 \int_0^t \int_0^{t-s_1} \prod_{i=0}^n \rho(b^i(s_2 - s_1)) ds_1 ds_2 \\ &\leq 2t \int_0^t \prod_{i=0}^n \rho(b^i u) du, \end{aligned}$$

where

$$\rho(u) = \mathbb{E} \Lambda(0) \Lambda(u). \quad (14)$$

Hence, to show \mathcal{L}_2 convergence it is sufficient to show that

$$\sup_n \int_0^t \prod_{i=0}^n \rho(b^i u) du < \infty.$$

Lemma 4. Assume that $\rho(u)$ as defined in (14) is monotone decreasing in u ,

$$b > \mathbb{E} \Lambda(0)^2 \quad (15)$$

and

$$\sum_{i=0}^{\infty} (\rho(b^i) - 1) < \infty. \quad (16)$$

Then $A_n(t)$ converges in \mathcal{L}_2 for every fixed $t \in [0, 1]$.

PROOF. Without loss of generality let $t = 1$. Let $n = n(u) = \lfloor -\log_b u \rfloor$ be the integer part of $-\log_b u$. Then, using monotonicity of ρ we obtain

$$\prod_{i=0}^n \sigma_\Lambda^2 \rho(b^i u) \leq \rho(0)^{n(u)} \prod_{i=n(u)}^n \rho(b^i u).$$

Using monotonicity of ρ again,

$$\prod_{i=n(u)}^n \rho(b^i u) \leq \prod_{i=0}^{\infty} \rho(b^{i+n(u)} u) \leq \prod_{i=0}^{\infty} \rho(b^i u) = \Pi.$$

Constant Π is finite due to the condition (16). For sufficiently small $\delta \in (0, 1)$, by the condition (15), $b^{1-\delta} > \rho(0) = \mathbb{E}\Lambda(0)^2$. Therefore,

$$\begin{aligned} \sup_n \int_0^1 \prod_{i=0}^n \rho(b^i u) du &\leq \Pi \int_0^1 \rho(0)^{n(u)} du \\ &\leq \Pi \int_0^1 b^{(1-\delta)n(u)} du \leq \Pi \int_0^1 \frac{1}{u^{1-\delta}} du < \infty. \end{aligned}$$

The proof of Lemma 4 is complete. \square

Now we are going to consider $q > 2$. Here, we assume additionally that $A_n(t)$ is a cadlag process. The starting point is the equality

$$\begin{aligned} \mathbb{E}A_n^q(t) &= \mathbb{E} \int_0^t \int_0^t \dots \int_0^t \Lambda_n(s_1)\Lambda_n(s_2) \dots \Lambda_n(s_q) ds_1 ds_2 \dots ds_q \\ &= q! \int_{0 < s_1 < \dots < s_q < t} \mathbb{E}\Lambda_n(s_1)\Lambda_n(s_2) \dots \Lambda_n(s_p) ds_1 ds_2 \dots ds_q. \end{aligned} \quad (17)$$

First we make change of variables

$$u_0 = s_1, u_1 = s_2 - s_1, \dots, u_{q-1} = s_q - s_{q-1},$$

which transform equality (17) into

$$\begin{aligned} \mathbb{E}A_n^q(t) &= q! \int_{0 < u_0, \dots, u_{q-1} \leq t} \mathbb{E}\Lambda_n(u_0)\Lambda_n(u_0 + u_1) \dots \Lambda_n(u_0 + \dots + u_{q-1}) du_0 \dots du_{q-1} \\ &\leq q! \int_{0 < u_0, \dots, u_{q-1} < t} \mathbb{E}\Lambda_n(u_0)\Lambda_n(u_0 + u_1) \dots \Lambda_n(u_0) du_0 du_1 \dots du_{q-1} \\ &= q! \int_{0 < u_1, \dots, u_{q-1} < t} \mathbb{E}\Lambda_n(0)\Lambda_n(u_1) \dots \Lambda_n(u_1 + \dots + u_{q-1}) du_1 \dots du_{q-1}, \end{aligned}$$

where we used stationary of the process $\Lambda(t)$ to obtain the latter inequality. Now let

$$\rho(u_1, \dots, u_{q-1}) = \mathbb{E}\Lambda(0)\Lambda(u_1) \dots \Lambda(u_1 + \dots + u_{q-1}) \quad (18)$$

and thus it is sufficient to prove that

$$\sup_n \int_{0 < u_1, \dots, u_{q-1} < t} \prod_{l=1}^n \rho(b^l u_1, \dots, b^l u_{q-1}) du_1 \dots du_{q-1} < \infty. \quad (19)$$

We require that the function $\rho(u_1, \dots, u_{q-1})$ satisfies certain mixing conditions. Namely, for any $m < q - 1$, $1 \leq i_1 < i_2 < \dots, i_m \leq q - 1$ and $u_j = 0, j \neq i_1, \dots, i_m$ and $u_j = A, j = i_1, \dots, i_m$, we assume that the following mixing condition holds:

$$\lim_{A \rightarrow \infty} \rho(u_1, \dots, u_{q-1}) = \mathbb{E}\Lambda(0)^{i_1} \mathbb{E}\Lambda(0)^{i_2 - i_1} \dots \mathbb{E}\Lambda(0)^{q - i_m}. \quad (20)$$

We are ready now to state the main result of this section.

Theorem 5. *Suppose that conditions \mathbf{A}' - \mathbf{A}'' hold. Assume that $\rho(u_1, \dots, u_{q-1})$ defined in (18) is monotone decreasing in all variables. Let*

$$b^{q-1} > \mathbb{E}\Lambda(0)^q \quad (21)$$

for some $q > 0$, and

$$\sum_{n=1}^{\infty} (\rho(b^n, \dots, b^n) - 1) < \infty. \quad (22)$$

Finally assume that the mixing condition (20) holds. Then,

$$\mathbb{E}A(t)^q < \infty, \quad (23)$$

and $A_n(t)$ converges to $A(t)$ in \mathcal{L}_q .

Proof of Theorem 5. It is sufficient to prove that equation (19) holds. To simplify notation we put $t = 1$. First represent the integral in (19) as the sum of the integrals over different regions

$$\begin{aligned} & \int_{0 \leq u_1, \dots, u_{q-1} \leq 1} \prod_{l=0}^n \rho(b^l u_1, \dots, b^l u_{q-1}) du_1 \dots du_{q-1} \\ &= \sum_{i_1, \dots, i_{q-1}} \int_{0 \leq u_{i_1} \leq u_{i_2} \leq \dots \leq u_{i_{q-1}} \leq 1} \prod_{l=0}^n \rho(b^l u_1, \dots, b^l u_{q-1}) du_1 \dots du_{q-1}, \end{aligned} \quad (24)$$

where the sum is taken over all possible permutations of numbers $(1, 2, \dots, q-1)$. Next we are going to bound the integrals on these separate regions. Put

$$u_{(1)} = u_{i_1}, u_{(2)} = u_{i_2}, \dots, u_{(q-1)} = u_{i_{q-1}}.$$

Fix a large number $A \geq 1$ which we define later and define an auxiliary function $n(u) = -[\log_b u/A]$. Note that this function is non-negative for $u \leq 1$. Now let

$$l_1 = n(u_{(1)}), l_2 = n(u_{(2)}), \dots, l_{q-1} = n(u_{(q-1)}).$$

These numbers are decreasing

$$l_1 \geq l_2 \geq \dots \geq l_{q-1}. \quad (25)$$

Then we can split the product as

$$\prod_{l=0}^n \rho(b^l u_1, \dots, b^l u_{q-1}) = \prod_{l=0}^{l_{q-1}-1} \prod_{l=l_{q-1}}^{l_{q-2}-1} \dots \prod_{l=l_2}^{l_1-1} \prod_{l=l_1}^n \rho(b^l u_1, \dots, b^l u_{q-1}). \quad (26)$$

Further, using monotonicity of the function ρ we can estimate for $l < l_{q-1}$,

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \leq \rho(0, \dots, 0) = \text{E}\Lambda(0)^q.$$

For $l \in [l_{q-1}, l_{q-2})$, we have

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \leq \rho(0, \dots, 0, A, 0, \dots, 0),$$

where i_{q-1} th argument of the function ρ is equal to A and all other arguments are equal to 0. Indeed this holds due to the fact that for $l > l_{q-1}$

$$b^l u_{(q-1)} \geq b^{l_{q-1}} u_{(q-1)} \geq \frac{A}{u_{(q-1)}} u_{(q-1)} = A$$

and the monotonicity of the function ρ . Here recall that $u_{(q-1)}$ corresponds to $u_{i_{q-1}}$. Fix a small number δ which we define later. Now we can note that mixing condition (20) implies that

$$\lim_{A \rightarrow \infty} \rho(0, \dots, 0, A, 0, \dots, 0) = \text{E}\Lambda(0)^{i_{q-1}} \text{E}\Lambda(0)^{q-i_{q-1}}$$

Hence we can pick $A = A(\delta)$ sufficiently large to ensure that

$$\rho(0, \dots, 0, A, 0, \dots, 0) \leq (1 + \delta) \text{E}\Lambda(0)^{i_{q-1}} \text{E}\Lambda(0)^{q-i_{q-1}}.$$

Function $g(x) = \ln \text{E}\Lambda(0)^x$ is convex. Hence we can apply Karamata majorization inequality [32] to obtain that

$$g(i_{q-1}) + g(q - i_{q-1}) \leq g(q - 1) + g(1).$$

Therefore,

$$\text{E}\Lambda(0)^{i_{q-1}} \text{E}\Lambda(0)^{q-i_{q-1}} \leq \text{E}\Lambda(0)^{q-1} \text{E}\Lambda(0) = \text{E}\Lambda(0)^{q-1}$$

and

$$\rho(0, \dots, 0, A, 0, \dots, 0) \leq (1 + \delta) \text{E}\Lambda(0)^{q-1}.$$

Similarly, for $l \in [l_{q-2}, l_{q-3}]$, we have

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \leq \rho(0, \dots, 0, A, \dots, 0, A, 0, \dots, 0),$$

where the arguments of the function ρ are equal to 0 except arguments i_{q-1} and i_{q-2} which are equal to A . Applying the mixing condition and increasing A if necessary we can ensure that for $l \in [l_{q-2}, l_{q-3}]$,

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \leq (1 + \delta) \text{E}\Lambda(0)^a \text{E}\Lambda(0)^{b-a} \text{E}\Lambda(0)^{q-b},$$

where $a = \min(i_{q-2}, i_{q-1})$, $b = \max(i_{q-2}, i_{q-1})$. We apply now Karamata's majorisation inequality twice. First application of the inequality gives

$$\text{E}\Lambda(0)^a \text{E}\Lambda(0)^{b-a} \leq \text{E}\Lambda(0)^{b-1}.$$

Second application of Karamata's inequality gives

$$\text{E}\Lambda(0)^{b-1} \text{E}\Lambda(0)^{q-b} \leq \text{E}\Lambda(0)^{q-2}.$$

Hence, for $l \in [l_{q-2}, l_{q-3})$ and sufficiently large A ,

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \leq (1 + \delta) \text{E}\Lambda(0)^a \text{E}\Lambda(0)^{b-a} \text{E}\Lambda(0)^{q-b} \leq (1 + \delta) \text{E}\Lambda(0)^{q-2}.$$

In exactly the same manner, using the mixing conditions and Karamata's majorisation inequality one can obtain for $l \in [l_j, l_{j-1})$ and $j = q-1, q-2, \dots, 2$

$$\rho(b^l u_1, \dots, b^l u_{q-1}) \leq (1 + \delta) \text{E}\Lambda(0)^j.$$

Hence,

$$\begin{aligned} \prod_{l=0}^{l_1-1} \rho(b^l u_1, \dots, b^l u_{q-1}) &= \prod_{l=0}^{l_{q-1}-1} \prod_{l=l_{q-1}}^{l_{q-2}-1} \dots \prod_{l=l_2}^{l_1-1} \rho(b^l u_1, \dots, b^l u_{q-1}) \\ &\leq (1 + \delta)^{l_1} \prod_{i=2}^q \prod_{l=l_i}^{l_{i-1}-1} \text{E}\Lambda(0)^i = (1 + \delta)^{l_1} \prod_{i=2}^q (\text{E}\Lambda(0)^i)^{l_{i-1}-l_i}, \end{aligned} \quad (27)$$

where $l_q = 0$. Rearranging the terms we can represent this product in a slightly different form

$$\prod_{i=2}^q (\text{E}\Lambda(0)^i)^{l_{i-1}-l_i} = \prod_{i=1}^{q-1} \left(\frac{\text{E}\Lambda(0)^i \text{E}\Lambda(0)^{i-2}}{(\text{E}\Lambda(0)^{i-1})^2} \right)^{l_{q-1} + \dots + l_i} \quad (28)$$

Now one can note that since l_i are decreasing, see (25),

$$l_{q-1} + \dots + l_i \leq \frac{q-i}{q-1} (l_1 + \dots + l_{q-1}),$$

for any $i = 1, \dots, q-1$. Indeed, the latter inequality is equivalent to

$$(i-1)(l_{q-1} + \dots + l_i) \leq (q-i)(l_{i-1} + \dots + l_1),$$

which follows from

$$\frac{l_{q-1} + \dots + l_i}{q-i} \leq l_i \leq l_{i-1} \leq \frac{l_{i-1} + \dots + l_1}{i-1}.$$

In addition, by the Karamata's majorization inequality,

$$\frac{\text{E}\Lambda(0)^i \text{E}\Lambda(0)^{i-2}}{(\text{E}\Lambda(0)^{i-1})^2} > 1.$$

Therefore,

$$\left(\frac{\text{E}\Lambda(0)^i \text{E}\Lambda(0)^{i-2}}{(\text{E}\Lambda(0)^{i-1})^2} \right)^{l_{q-1} + \dots + l_i} \leq \left(\frac{\text{E}\Lambda(0)^i \text{E}\Lambda(0)^{i-2}}{(\text{E}\Lambda(0)^{i-1})^2} \right)^{\frac{q-i}{q-1} (l_1 + \dots + l_{q-1})}$$

Hence we can continue (28) as follows

$$\begin{aligned} \prod_{i=2}^q (\text{EA}(0)^i)^{l_i - l_{i-1}} &\leq \prod_{i=2}^{q-1} \left(\frac{\text{EA}(0)^i \text{EA}(0)^{i-2}}{(\text{EA}(0)^{i-1})^2} \right)^{\frac{q-i}{q-1}(l_1 + \dots + l_{q-1})} \\ &= (\text{EA}(0))^{\frac{l_1 + \dots + l_{q-1}}{q-1}} \prod_{i=2}^{q-3} (\text{EA}(0)^{i-1})^{\frac{q-i+1-2(q-i)+q-i-1}{q-1}} = (\text{EA}(0))^{\frac{l_1 + \dots + l_{q-1}}{q-1}}. \end{aligned} \quad (29)$$

Plugging the latter estimate in (27) we arrive at

$$\prod_{l=0}^{l_1} \rho(b^l u_1, \dots, b^l u_{q-1}) \leq (1 + \delta)^{l_1} (\text{EA}(0))^{\frac{l_1 + \dots + l_{q-1}}{q-1}}.$$

We can now make use of the condition (21) and by taking δ sufficiently small we can ensure that

$$\prod_{l=0}^{l_1} \rho(b^l u_1, \dots, b^l u_{q-1}) \leq b^{(1-\varepsilon)(l_1 + \dots + l_{q-1})} = (u_1 u_2 \dots u_{q-1})^{-1+\varepsilon} A^{q(1-\varepsilon)} \quad (30)$$

for some small $\varepsilon > 0$. We are left to estimate the product $\prod_{l=l_1}^n$ uniformly in n . For that we are going to use finiteness of the series in (22). First note that for $l \geq l_1$,

$$b^l u_j \geq A b^{l-l_1}.$$

Then, by monotonicity of the function ρ , uniformly in n , for some $C > 0$

$$\begin{aligned} \prod_{l=l_1}^n \rho(b^l u_1, \dots, b^l u_{q-1}) &\leq \prod_{l=l_1}^n \rho(b^{l-l_1} A, \dots, b^{l-l_1} A) \\ &\leq \prod_{l=0}^{\infty} \rho(b^l, \dots, b^l) < C, \end{aligned} \quad (31)$$

according to the finiteness of the series. Together (30) and (31) give us

$$\begin{aligned} &\int_{0 \leq u_1, \dots, u_{q-1} \leq 1} \prod_{l=0}^n \rho(b^l u_1, \dots, b^l u_{q-1}) du_1 \dots du_{q-1} \\ &= \sum_{i_1, \dots, i_{q-1}} \int_{0 < u_{i_1} \leq u_{i_2} \leq \dots \leq u_{i_{q-1}} \leq 1} \prod_{l=0}^n \rho(b^l u_1, \dots, b^l u_{q-1}) du_1 \dots du_{q-1} \\ &\leq C \sum_{i_1, \dots, i_{q-1}} \int_{0 < u_{i_1} \leq u_{i_2} \leq \dots \leq u_{i_{q-1}} \leq 1} (u_1 u_2 \dots u_{q-1})^{-1+\varepsilon} du_1 \dots du_{q-1}, \\ &= C \int_{0 \leq u_1, \dots, u_{q-1} \leq 1} (u_1 u_2 \dots u_{q-1})^{-1+\varepsilon} du_1 \dots du_{q-1} \end{aligned} \quad (32)$$

which immediately gives a finite bound for $\text{EA}_n(1)^q$ uniform in n . □

Remark 6. It is not difficult to show that (21) is sharp. Indeed suppose that

$$b^{q-1} < \text{EA}(0)^q$$

and that $\rho(u_1, \dots, u_{q-1})$ is continuous at $(0, \dots, 0)$. Then, for $\varepsilon > 0$,

$$\begin{aligned}
EA_n^q(t) &= q! \int_{0 < u_0, \dots, u_{q-1} \leq t}^{u_0 + \dots + u_{q-1} \leq t} E\Lambda_n(0)\Lambda_n(u_1) \dots \Lambda_n(u_1 + \dots + u_{q-1}) du_0 \dots du_{q-1} \\
&= q! \int_{0 < u_0, \dots, u_{q-1} \leq t} \prod_{l=0}^{q-1} \rho(b^l u_1, \dots, b^l u_{q-1}) du_0 \dots du_{q-1} \\
&\geq q! \int_{0 < u_0 < 1/2, 0 < u_1, \dots, u_{q-1} \leq \varepsilon/b^n} \prod_{l=0}^{q-1} \rho(b^l u_1, \dots, b^l u_{q-1}) du_0 \dots du_{q-1} \\
&\geq \frac{q!}{2} \int_{0 < u_1, \dots, u_{q-1} \leq \varepsilon/b^n} \prod_{l=0}^{q-1} \rho(\varepsilon, \dots, \varepsilon) du_1 \dots du_{q-1} \\
&= \frac{q!}{2} \varepsilon^{q-1} \left(\frac{\rho(\varepsilon, \dots, \varepsilon)}{b^{q-1}} \right)^n
\end{aligned}$$

Since $\rho(\varepsilon, \dots, \varepsilon)$ can be made arbitrarily close to $\rho(0, \dots, 0) = E\Lambda(0)^q$, then, for sufficiently small $\varepsilon > 0$,

$$\rho(\varepsilon, \dots, \varepsilon) > b^{q-1}$$

and

$$EA_n(t) \geq \frac{q!}{2} \varepsilon^{q-1} \left(\frac{\rho(\varepsilon, \dots, \varepsilon)}{b^{q-1}} \right)^n \rightarrow \infty$$

as $n \rightarrow \infty$.

4. Scaling of moments

The aim of this Section is to establish the scaling property (10). For $q > 1$ let

$$\rho_q(s) = \inf_{u \in [0,1]} \left(\frac{E\Lambda(0)^{q-1} \Lambda(su)}{E\Lambda(0)^q} - 1 \right). \quad (33)$$

Note that $\rho_q(s) \leq 0$. For $q \in (0, 1)$ let

$$\rho_q(s) = \sup_{u \in [0,1]} \left(\frac{E\Lambda(0)^{q-1} \Lambda(su)}{E\Lambda(0)^q} - 1 \right). \quad (34)$$

For $q \leq 1$ it is easy to see that $\rho_q(s) \geq 0$.

Theorem 7. Assume that $A(t) \in \mathcal{L}_q$ and $\rho_q(s)$ defined in (33) and (34) is such that

$$\sum_{n=1}^{\infty} |\rho_q(b^{-n})| < \infty. \quad (35)$$

Then,

$$EA^q(t) \sim t^{q - \log_b E\Lambda^q(t)}, \quad t \in [0, 1]. \quad (36)$$

Proof of Theorem 7

Our strategy in proving of (36) is to use martingale properties of the sequence $A_n(t)$. We concentrate mainly on $q > 1$, as the case $q < 1$ is symmetric. For the upper bound we obtain uniform in n estimates from above for $EA_n(t)^q$. Then, since $A_n(t)$ converges to $A(t)$ in \mathcal{L}_q , the same estimates hold for $EA(t)^q$. For the lower bound, we use the fact that as $A_n(t) \in \mathcal{L}_q$ for $q > 1$ the martingale $A_n(t)$ is closable. Hence it can be represented as $A_n(t) = E(A(t)|A_1(t), \dots, A_n(t))$. Therefore, for $q > 1$, by the conditional Jensen inequality,

$$\begin{aligned}
EA_n(t)^q &= E(E(A(t)|A_1(t), \dots, A_n(t)))^q \\
&\leq E(E(A(t)^q|A_1(t), \dots, A_n(t))) = EA(t)^q.
\end{aligned}$$

Thus, we are going to obtain an estimate from below for $\mathbb{E}A_n(t)^q$ for a suitable choice of n . Clearly, by the latter inequality, this estimate will hold for $\mathbb{E}A(t)^q$ as well.

We start with a change of variable

$$\begin{aligned} A_n(t) &= \int_0^t \Lambda_n(s) ds \\ &= t \int_0^1 \Lambda_n(ut) du \equiv t\tilde{A}_n(t). \end{aligned}$$

Clearly $\tilde{A}_n(t)$ is a martingale for any fixed t .

We are going to treat the cases $q \geq 1$ and $q \leq 1$ separately. This is due to the fact that for $q \geq 1$, the sequences $\tilde{A}_n(t)^q$ and $A_n(t)^q$ are submartingales while for $q \in (0, 1)$ the sequences are supermartingales with respect to the filtration $\mathcal{F}_n = \sigma(\Lambda^{(1)}, \dots, \Lambda^{(n)})$.

We start with an upper bound for $q \geq 1$. Let $n_t = -[\log_b t]$ be the biggest integer such that $n_t \leq -\log_b t$. We use the Hölder inequality in the form,

$$\left(\int_0^1 |fg| \right)^q = \left(\int_0^1 |f||g|^{1/q}|g|^{1/p} \right)^q \leq \left(\int_0^1 |f|^q |g| \right) \left(\int_0^1 |g| \right)^{q/p},$$

where $1/q + 1/p = 1$. It follows from the latter inequality,

$$\begin{aligned} \left(\int_0^1 \prod_{k=0}^n \Lambda^{(k)}(ut) du \right)^q &\leq \left(\int_0^1 \left(\prod_{k=0}^{n_t-1} \Lambda^{(k)}(ut) \right)^q \prod_{k=n_t}^n \Lambda^{(k)}(ut) du \right) \left(\int_0^1 \prod_{k=n_t}^n \Lambda^{(k)}(ut) du \right)^{q/p}. \end{aligned}$$

Applying expectation to both sides we obtain, using independence of $\Lambda^{(k)}$ of each other,

$$\mathbb{E}\tilde{A}_n(t)^q \leq \left(\int_0^1 \prod_{k=0}^{n_t-1} \mathbb{E}(\Lambda^{(k)})^q(ut) \prod_{k=n_t}^n \mathbb{E} \left(\Lambda^{(k)}(ut) \left(\int_0^1 \prod_{k=n_t}^n \Lambda^{(k)}(vt) dv \right)^{q/p} \right) du \right).$$

By the stationarity of the process $\Lambda(t)$ we have

$$\prod_{k=0}^{n_t-1} \mathbb{E}(\Lambda^{(k)})^q(ut) = (\mathbb{E}\Lambda(0)^q)^{n_t} \leq (\mathbb{E}\Lambda(0)^q)^{-\log_b t} = t^{-\log_b \mathbb{E}\Lambda(0)^q}.$$

Therefore,

$$\begin{aligned} \mathbb{E}\tilde{A}_n(t)^q &\leq t^{-\log_b \mathbb{E}\Lambda(0)^q} \mathbb{E} \int_0^1 \prod_{k=n_t}^n \left(\Lambda^{(k)}(ut) \left(\int_0^1 \prod_{k=n_t}^n \Lambda^{(k)}(vt) dv \right)^{q/p} \right) du \\ &= t^{-\log_b \mathbb{E}\Lambda(0)^q} \mathbb{E} \left(\int_0^1 \prod_{k=n_t}^n \Lambda^{(k)}(ut) du \right)^{1+q/p} \\ &= t^{-\log_b \mathbb{E}\Lambda(0)^q} \mathbb{E} \left(\int_0^1 \prod_{k=n_t}^n \Lambda^{(k)}(b^k ut) du \right)^q \\ &= t^{-\log_b \mathbb{E}\Lambda(0)^q} \mathbb{E} \left(\int_0^1 \prod_{k=0}^{n-n_t} \Lambda^{(k)}(b^k u b^{-[\log_b t] + \log_b t}) du \right)^q \\ &= t^{-\log_b \mathbb{E}\Lambda(0)^q} \mathbb{E}\tilde{A}_{n-n_t-1}(b^{-[\log_b t] + \log_b t})^q. \end{aligned}$$

Now note that

$$\begin{aligned} \mathbb{E}\tilde{A}_{n-n_t-1}(b^{-[\log_b t]+\log_b t})^q &= b^{[\log_b t]-\log_b t}\mathbb{E}A(b^{-[\log_b t]+\log_b t})^q \\ &\leq b \sup_{s \in [0,1]} \mathbb{E}A(s)^q. \end{aligned}$$

This bound is uniform in n and therefore,

$$\mathbb{E}A(t)^q \leq bt^{q-\log_b \mathbb{E}\Lambda(0)^q} \sup_{s \in [0,1]} \mathbb{E}A(s)^q.$$

Now we turn to the lower bound for $q \geq 1$.

Since $\tilde{A}_n(t)$ is a submartingale,

$$\mathbb{E}\tilde{A}(t)^q \geq \mathbb{E}\tilde{A}_{n_t}(t)^q,$$

where $n_t = [-\log_b t]$.

We are going to obtain a recursive estimate for $\mathbb{E}\tilde{A}_n(t)$. First,

$$\begin{aligned} \mathbb{E}\tilde{A}_{n+1}(t)^q &= \mathbb{E} \left(\int_0^1 \Lambda_n(ut) \Lambda^{(n+1)}(b^{n+1}ut) du \right)^q \\ &= \mathbb{E} \left(\int_0^1 \Lambda_n(ut) (\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0)) du + \tilde{A}_n(t) \Lambda^{(n+1)}(0) \right)^q. \end{aligned}$$

Now we can use an elementary estimate of the form: if $a + b > 0$ and $b > 0$ then

$$(a + b)^q \geq qab^{q-1} + b^q \quad (37)$$

for $q \geq 1$. This estimate is easy to prove by analyzing the function $(1+t)^q - 1 - qt$ for $t \geq -1$. Applying (37) we obtain

$$\begin{aligned} \mathbb{E}\tilde{A}_{n+1}(t)^q &\geq q\mathbb{E} \left[\left(\tilde{A}_n(t) \Lambda^{(n+1)}(0) \right)^{q-1} \int_0^1 \Lambda_n(ut) (\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0)) du \right] \\ &\quad + \mathbb{E} \left(\tilde{A}_n(t) \Lambda^{(n+1)}(0) \right)^q \equiv E_1 + E_2. \end{aligned} \quad (38)$$

The second expectation is straightforward,

$$E_2 = \mathbb{E} \left(\int_0^1 \Lambda_n(ut) \Lambda^{(n+1)}(0) du \right)^q = \mathbb{E}\Lambda(0)^q \mathbb{E}\tilde{A}_n(t)^q, \quad (39)$$

where we use independence of Λ_n and $\Lambda^{(n+1)}$. For the first expectation, rearranging the terms, we have

$$\begin{aligned} E_1 &= q\mathbb{E} \left[\int_0^1 \tilde{A}_n(t)^{q-1} \Lambda_n(ut) (\Lambda^{(n+1)}(0))^{q-1} (\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0)) du \right] \\ &= q \int_0^1 \mathbb{E}\tilde{A}_n(t)^{q-1} \Lambda_n(ut) \mathbb{E}(\Lambda^{(n+1)}(0))^{q-1} (\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0)) du. \end{aligned}$$

By the definition of ρ_q , see (33), for all $u \in [0, 1]$,

$$\mathbb{E}(\Lambda^{(n+1)}(0))^{q-1} (\Lambda^{(n+1)}(b^{n+1}ut) - \Lambda^{(n+1)}(0)) \geq \mathbb{E}\Lambda(0)^q \rho_q(b^{n+1}t).$$

Therefore,

$$\begin{aligned} E_1 &\geq q \int_0^1 \mathbb{E}\tilde{A}_n(t)^{q-1} \Lambda_n(ut) du \mathbb{E}\Lambda(0)^q \rho_q(b^{n+1}t) \\ &\geq q\mathbb{E} \left[\tilde{A}_n(t)^{q-1} \int_0^1 \Lambda_n(ut) du \right] \mathbb{E}\Lambda(0)^q \rho_q(b^{n+1}t) \\ &= q\mathbb{E}\tilde{A}_n(t)^q \mathbb{E}\Lambda(0)^q \rho_q(b^{n+1}t) \end{aligned}$$

Therefore

$$E_1 \geq qE\tilde{A}_n(t)^q E\Lambda(0)^q \rho_q(b^{n-n_t}).$$

The latter inequality together with (38) and (39) gives us

$$E\tilde{A}_{n+1}(t)^q \geq E\tilde{A}_n(t)^q E\Lambda(0)^q (1 + q\rho_q(b^{n-n_t})) \quad (40)$$

Now we can iterate it. First fix N^* such that $|q\rho_q(b^{-n})| < 1$ for $n > N^*$. Then, iterating (40), we obtain

$$\begin{aligned} E\tilde{A}_{n_t-N^*}^q &\geq (E\Lambda(0)^q)^{n_t-N^*} \prod_{n=0}^{n_t-N^*} (1 + q\rho_q(b^{n-n_t})) \\ &\geq (E\Lambda(0)^q)^{n_t-N^*} \prod_{n=N^*}^{\infty} (1 + q\rho_q(b^{-n})) \end{aligned}$$

It is sufficient to note that the latter product is strictly positive due to (35). As $\tilde{A}_n(t)^q$ is a submartingale, we have $E\tilde{A}(t)^q \geq E\tilde{A}_{n_t-N^*}^q$ and the required lower bound for $q > 1$ follows.

The proof for $q \in (0, 1)$ is symmetric. For these values of q and a fixed t , the process $\tilde{A}_n(t)^q$ is a supermartingale with respect to the natural filtration $\mathcal{F}_n = \sigma(\Lambda^{(1)}, \dots, \Lambda^{(n)})$. The bound from below is proved using the reverse Hölder inequality for $q \in (0, 1)$ and p such that $1/p + 1/q = 1$.

$$\left(\int_0^1 |fg| \right) \geq \left(\int_0^1 |f|^q \right)^{1/q} \left(\int_0^1 |g|^p \right)^{1/p}.$$

Note that p is negative. We are going to use this inequality in the form,

$$\left(\int_0^1 |fg| \right)^q = \left(\int_0^1 |f||g|^{1/q}|g|^{1/p} \right)^q \geq \left(\int_0^1 |f|^q |g| \right) \left(\int_0^1 |g| \right)^{q/p},$$

It follows from the latter inequality,

$$\begin{aligned} &\left(\int_0^1 \prod_{k=0}^n \Lambda_k(ut) du \right)^q \\ &\geq \left(\int_0^1 \left(\prod_{k=0}^{n_t} \Lambda_k(ut) \right)^q \prod_{k=n_t+1}^n \Lambda_k(ut) du \right) \left(\int_0^1 \prod_{k=n_t+1}^n \Lambda_k(ut) du \right)^{q/p}. \end{aligned}$$

The rest of the proof goes exactly as the proof of the upper bound for $q > 1$.

To prove the upper bound, we proceed similarly to the proof of the lower bound for $q > 1$. First we establish a recursive estimate. The elementary inequality (37) still holds (in the opposite direction), for $q \in (0, 1)$,

$$(a + b)^q \leq qab^{q-1} + b^q$$

for $a + b > 0, b > 0$. Repeating step by step the arguments for $q > 1$ we obtain an upper bound

$$E\tilde{A}_{n+1}(t)^q \leq E\tilde{A}_n(t)^q E\Lambda(0)^q (1 + q\rho_q(b^{n-n_t})).$$

Applying this bound recursively

$$\begin{aligned} E\tilde{A}_{n_t-N^*}^q &\leq (E\Lambda(0)^q)^{n_t-N^*} \prod_{n=0}^{n_t-N^*} (1 + q\rho_q(b^{n-n_t})) \\ &\leq (E\Lambda(0)^q)^{n_t-N^*} \prod_{n=N^*}^{\infty} (1 + q\rho_q(b^{-n})) \end{aligned}$$

It is sufficient to note that the latter product converge due to (35). As $\tilde{A}_n(t)^q$ is a supermartingale, we have $E\tilde{A}(t)^q \leq E\tilde{A}_{n_t-N^*}^q$ and the required upper bound for $q < 1$ follows. \square

5. Log-normal scenario with possible long-range dependence

The log-normal hypothesis of Kolmogorov [34] features prominently in turbulent cascades. In this section, we provide some related model, namely the log-normal scenario, for multifractal products of stochastic processes. In fact, this log-normal scenario has its origin in Kahane [30, 31], see also [1]. In this section we present a most general result on log-normal scenario for a models with possible long-range dependence.

We combine Theorems 5 and 7 for this special case in order to have a precise scaling law for the moments.

B'. Consider a mother process of the form

$$\Lambda(t) = \exp \left\{ X(t) - \frac{1}{2} \sigma_X^2 \right\}, \quad (41)$$

where $X(t)$, $t \in [0, 1]$ is a zero-mean Gaussian, measurable, separable stochastic process with covariance function

$$R_X(\tau) = \sigma_X^2 \text{Corr}(X(t), X(t + \tau))$$

Under condition **B'**, we obtain the following specifications of the moment generating functions (1) and (2):

$$\begin{aligned} M(\zeta) &= \mathbb{E} \exp \left\{ \zeta \left(X(t) - \frac{1}{2} \sigma_X^2 \right) \right\} = e^{\frac{1}{2} \sigma_X^2 (\zeta^2 - \zeta)}, \quad \zeta \in \mathbb{R}^1, \\ M(\zeta_1, \zeta_2; t_1 - t_2) &= \mathbb{E} \exp \left\{ \zeta_1 \left(X(t_1) - \frac{1}{2} \sigma_X^2 \right) + \zeta_2 \left(X(t_2) - \frac{1}{2} \sigma_X^2 \right) \right\} \\ &= \exp \left\{ \frac{1}{2} \sigma_X^2 [\zeta_1^2 - \zeta_1 + \zeta_2^2 - \zeta_2] + \zeta_1 \zeta_2 R_X(t_1 - t_2) \right\}, \quad \zeta_1, \zeta_2 \in \mathbb{R}^1, \end{aligned}$$

where $\sigma_X^2 \in (0, \infty)$. It turns out that, in this case,

$$\begin{aligned} M(1) &= 1; \quad M(2) = e^{\sigma_X^2}; \quad \sigma_\Lambda^2 = e^{\sigma_X^2} - 1; \\ \text{Cov}(\Lambda(t_1), \Lambda(t_2)) &= M(1, 1; t_1 - t_2) - 1 = e^{R_X(t_1 - t_2)} - 1 \end{aligned}$$

and

$$\log_b \mathbb{E} \Lambda(t)^q = \frac{(q^2 - q) \sigma_X^2}{2 \log b}, \quad q > 0.$$

Note that

$$e^{R_X(t_1 - t_2)} - 1 \geq R_X(t_1 - t_2).$$

Using Theorem 2, we obtain

Theorem 8. *Suppose that condition **B'** holds with the correlation function*

$$\text{Corr}(X(t), X(t + \tau)) \leq C \tau^{-\alpha}, \quad \alpha > 0, \quad (42)$$

for sufficiently large τ , and for some $a > 0$,

$$1 - \text{Corr}(X(t), X(t + \tau)) \leq C |\tau|^{-a}, \quad (43)$$

for sufficiently small τ . Assume that

$$b > \exp \{ q^* \sigma_X^2 / 2 \}, \quad (44)$$

where $q^* > 0$ is a fixed integer. Then the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n-1} \Lambda^{(j)}(sb^j) ds, \quad t \in [0, 1]$$

converge in $\mathcal{L}_q, 0 < q \leq q^*$ to the stochastic process $A(t), t \in [0, 1]$, as $n \rightarrow \infty$, such that

$$\mathbb{E}A(t)^q \sim t^{-aq^2+(a+1)q}, q \in [0, q^*], \quad (45)$$

and the Rényi function is given by

$$T(q) = -aq^2 + (a+1)q - 1, q \in (0, q^*),$$

where

$$a = \frac{\sigma_X^2}{2 \log b}.$$

Moreover, if

$$\text{Corr}(X(t), X(t+\tau)) = \frac{L(\tau)}{|\tau|^\alpha}, \alpha > 0,$$

where L is a slowly varying at infinity function, bounded on every bounded interval, then

$$\text{Var}A(t) \geq t^{2-\alpha} \sigma_X^2 \int_0^1 \int_0^1 \frac{L(t|u-v|)dudw}{L(t)|u-w|^\alpha}, 0 < \alpha < 1, \quad (46)$$

and

$$\text{Var}A(t) \geq 2t\sigma_X^2 \int_0^t \left(1 - \frac{\tau}{t}\right) \frac{L(\tau)}{|\tau|^\alpha} d\tau, \alpha \geq 1. \quad (47)$$

Remark 9. We interpret the inequality (46) as a form of long-range dependence of the limiting process in the following sense: one can replace the interval $[0, 1]$ into more general interval $[0, t]$, and for a large t we have the following:

$$\text{Var}A(t) \geq \lim_{t \rightarrow \infty} \int_0^t \int_0^t \frac{L(|u-v|)dudw}{|u-w|^\alpha} = \lim_{t \rightarrow \infty} t^{2-\alpha} \sigma_X^2 \int_0^1 \int_0^1 \frac{L(t|u-v|)dudw}{L(t)|u-w|^\alpha},$$

and

$$\lim_{t \rightarrow \infty} \int_0^1 \int_0^1 \frac{L(t|u-v|)dudw}{L(t)|u-w|^\alpha} = \int_0^1 \int_0^1 \frac{dudw}{|u-w|^\alpha} = \frac{2}{(1-\alpha)(2-\alpha)}, 0 < \alpha < 1,$$

while the inequality (47) has the form of short-range dependence of the limit process:

$$\lim_{t \rightarrow \infty} \int_0^t \left(1 - \frac{\tau}{t}\right) \frac{L(\tau)}{|\tau|^\alpha} d\tau = \int_0^\infty \frac{L(\tau)}{|\tau|^\alpha} d\tau < \infty, \alpha > 1.$$

For $\alpha = 1$, one can show that for some other slowly varying at infinity function $L_1(t)$, bounded on every bounded interval,

$$\lim_{t \rightarrow \infty} \left[\int_0^t \left(1 - \frac{\tau}{t}\right) \frac{L(\tau)}{|\tau|} d\tau \right] / L_1(t) = 1.$$

Remark 10. Note that the correlation function $\text{Corr}(X(t), X(t+\tau)) = (1+|\tau|^2)^{-\alpha/2}$, $\alpha > 0$, satisfies all assumptions of the Theorem 2, among the others.

Proof. To prove \mathcal{L}_q convergence we use Theorem 5. The moment generating function of the multidimensional normal distribution is given by the following expression

$$M(\zeta_1, \zeta_2, \dots, \zeta_q) = \mathbb{E}e^{\zeta_1 X(s_1) + \dots + \zeta_q X(s_q)} = \exp \left\{ \frac{1}{2} \zeta \mathbf{R} \zeta^t \right\},$$

where

$$\zeta = (\zeta_1, \dots, \zeta_q)$$

and \mathbf{R} is the covariance matrix

$$\mathbf{R} = \{R_X(|s_i - s_j|)\}_{i,j=1..q}$$

One can immediately see that

$$\begin{aligned}
\mathbb{E}(\Lambda(s_1)\Lambda(s_2)\dots\Lambda(s_q)) &= \mathbb{E}e^{X(s_1)-\frac{1}{2}\sigma_X^2}\dots e^{X(s_q)-\frac{1}{2}\sigma_X^2} \\
&= M(1,1,\dots,1)e^{-\frac{q}{2}\sigma_X^2} \\
&= e^{\frac{1}{2}\sum_{i=1}^q\sum_{j=1}^q R_X(|s_i-s_j|)}e^{-\frac{q}{2}\sigma_X^2} \\
&= e^{\sum_{1\leq i<j\leq q} R_X(s_j-s_i)}.
\end{aligned}$$

We can now substitute this into (18) and obtain

$$\rho(u_1, u_2, \dots, u_{q-1}) = \exp\left\{\sum_{1\leq i<j\leq q-1} R_X(u_i + \dots + u_j)\right\}.$$

Since the function $R_X(u)$ is monotone decreasing in u , function $\rho(u_1, \dots, u_{q-1})$ is monotone decreasing in all arguments. Next we need to check the mixing condition (20). Let $1 \leq i_1 < i_2 < \dots \leq i_m$ and $u_i = A$ if $i \in \{i_1, \dots, i_m\}$ and 0 otherwise. Then, as $A \rightarrow \infty$, and $i_0 = 0, i_{m+1} = q$

$$\begin{aligned}
\lim_{A \rightarrow \infty} \rho(u_1, \dots, u_{q-1}) &= \exp\left\{\sum_{1\leq k\leq m+1} \sum_{i_{k-1}<i<j<i_k} R_X(u_i + \dots + u_j)\right\} \\
&= \mathbb{E}\Lambda(0)^{i_1} \mathbb{E}\Lambda(0)^{i_2-i_1} \dots \mathbb{E}\Lambda(0)^{q-i_m}, \quad (48)
\end{aligned}$$

where we used that $\mathbb{E}\Lambda(0)^l = e^{\frac{l(l-1)}{2}\sigma_X^2}$. Finally, we should check the convergence of the series (22). We have,

$$\exp\{R_X(qb^n)\} \leq \rho(b^n, \dots, b^n) \leq \exp\left\{\frac{q(q-1)}{2}R_X(b^n)\right\}.$$

As $n \rightarrow \infty$, $R_X(b^n) \rightarrow 0$. Hence

$$\rho(b^n, \dots, b^n) - 1 \leq (1 + o(1))\frac{q(q-1)}{2}R_X(b^n)$$

and

$$\rho(b^n, \dots, b^n) - 1 \geq (1 + o(1))R_X(qb^n).$$

As both sums

$$\sum_{n=1}^{\infty} R_X(qb^n) < \infty, \quad \sum_{n=1}^{\infty} R_X(b^n) < \infty,$$

the convergence of the series (22) follows. Condition (21) becomes

$$b^{q-1} > \mathbb{E}\Lambda(0)^q = \exp\left\{\frac{q(q-1)}{2}\sigma_X^2\right\},$$

which is equivalent to (44).

Next we are going to prove scaling (45). For that we apply the results of Section 4. We need to show that (35) holds for ρ_q , where $q \in (0, q^*)$ and ρ_q is defined in (33) and (34). For $q > 1$ we have, for sufficiently small s ,

$$\begin{aligned}
|\rho_q(s)| &= -\inf_{u \leq 1} \left(\frac{\mathbb{E}\Lambda(0)^{q-1}\Lambda(su)}{\mathbb{E}\Lambda(0)^q} - 1 \right) = -\inf_{u \leq 1} \left(e^{\sigma_X^2((q-1)\rho_X(su)+1-q)} - 1 \right) \\
&\leq \sup_{u \leq 1} \left(1 - e^{(1-q)\sigma_X^2(su)^a} \right) \leq 1 - e^{(1-q)\sigma_X^2(s)^a} \leq (q-1)\sigma_X^2 s^a.
\end{aligned}$$

Thus using condition (43) one can immediately see that the series (35) converges. For $q < 1$, the same arguments give bound

$$\rho_q(s) \leq (1-q)\sigma_X^2 s^a.$$

Using condition (43) one can immediately see that the series (35) converges. Therefore, by the results of Section 4 scaling (45) holds. \square

6. Geometric Ornstein-Uhlenbeck processes

This section reviews a number of known results on Lévy processes (see Skorokhod 1991, Bertoin 1996, Kyprianou 2006) and OU type processes (see Barndorff-Nielsen 2001, Barndorff-Nielsen and Shephard 2001) The geometric OU type processes have been studied also by Matsui and Shieh 2009.

As standard notation we will write

$$\kappa(z) = C\{z; X\} = \log E \exp\{izX\}, \quad z \in \mathbb{R}$$

for the cumulant function of a random variable X , and

$$K\{\zeta; X\} = \log E \exp\{\zeta X\}, \quad \zeta \in D \subseteq \mathbb{C}$$

for the Lévy exponent or Laplace transform or cumulant generating function of the random variable X . Its domain D includes the imaginary axis and frequently larger areas.

A random variable X is infinitely divisible if its cumulant function has the Lévy-Khintchine form

$$C\{z; X\} = iaz - \frac{d}{2}z^2 + \int_{\mathbb{R}} (e^{izu} - 1 - izu\mathbf{1}_{[-1,1]}(u)) \nu(du), \quad (49)$$

where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, that is, a non-negative measure on \mathbb{R} such that

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}} \min(1, u^2) \nu(du) < \infty. \quad (50)$$

The triplet (a, d, ν) uniquely determines the random variable X . For a Gaussian random variable $X \sim N(a, d)$, the Lévy triplet takes the form $(a, d, 0)$.

A random variable X is self-decomposable if, for all $c \in (0, 1)$, the characteristic function $f(z)$ of X can be factorized as $f(z) = f(cz)f_c(z)$ for some characteristic function $f_c(z)$, $z \in \mathbb{R}$. A homogeneous Lévy process $Z = \{Z(t), t \geq 0\}$ is a continuous (in probability), càdlàg process with independent and stationary increments and $Z(0) = 0$ (recalling that a càdlàg process has right-continuous sample paths with existing left limits.) For such processes we have $C\{z; Z(t)\} = tC\{z; Z(1)\}$ and $Z(1)$ has the Lévy-Khintchine representation (49).

If X is self-decomposable, then there exists a stationary stochastic process $\{X(t), t \geq 0\}$, such that $X(t) \stackrel{d}{=} X$ and

$$X(t) = e^{-\lambda t} X(0) + \int_{(0,t]} e^{-\lambda(t-s)} dZ(\lambda s) \quad (51)$$

for all $\lambda > 0$ (see Barndorff-Nielsen 1998). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\{Z(t), t \geq 0\}$ is a Lévy process, independent of $X(0)$, such that $X(t)$ and $Z(t)$ satisfy the Itô stochastic differential equation

$$dX(t) = -\lambda X(t) dt + dZ(\lambda t) \quad (52)$$

for all $\lambda > 0$, then $X(t)$ is self-decomposable. A stationary process $X(t)$ of this kind is said to be an OU type process. The process $Z(t)$ is termed the background driving Lévy process (BDLP) corresponding to the process $X(t)$. In fact (51) is the unique (up to indistinguishability) strong solution to Eq. (52) (Sato 1999, Section 17). The meaning of the stochastic integral in (51) was detailed in Applebaum (2009, p. 214).

Let $X(t)$ be a square integrable OU process. Then $X(t)$ has the correlation function

$$\text{Corr}(X(0), X(t)) = r_X(t) = \exp\{-\lambda|t|\}. \quad (53)$$

The cumulant transforms of $X = X(t)$ and $Z(1)$ are related by

$$C\{z; X\} = \int_0^\infty C\{e^{-s}z; Z(1)\} ds = \int_0^z C\{\xi; Z(1)\} \frac{d\xi}{\xi}$$

and

$$C\{z; Z(1)\} = z \frac{\partial C\{z; X\}}{\partial z}.$$

Suppose that the Lévy measure ν of X has a density function $p(u)$, $u \in \mathbb{R}$, which is differentiable. Then the Lévy measure $\tilde{\nu}$ of $Z(1)$ has a density function $q(u)$, $u \in \mathbb{R}$, and p and q are related by

$$q(u) = -p(u) - up'(u) \quad (54)$$

(see Barndorff-Nielsen 1998).

The logarithm of the characteristic function of a random vector $(X(t_1), \dots, X(t_m))$ is of the form

$$\log \mathbb{E} \exp \{i(z_1 X(t_1) + \dots + z_m X(t_m))\} = \int_{\mathbb{R}} \kappa \left(\sum_{j=1}^m z_j e^{-\lambda(t_j-s)} \mathbf{1}_{[0,\infty)}(t_j-s) \right) ds, \quad (55)$$

where

$$\kappa(z) = \log \mathbb{E} \exp \{izZ(1)\} = C \{z; Z(1)\},$$

and the function (55) has the form (49) with Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$.

The logarithms of the moment generation functions (if they exist) take the forms

$$\log \mathbb{E} \exp \{\zeta X(t)\} = \zeta a + \frac{d}{2} \zeta^2 + \int_{\mathbb{R}} (e^{\zeta u} - 1 - \zeta u \mathbf{1}_{[-1,1]}(u)) \nu(du),$$

where the triplet (a, d, ν) is the Lévy triplet of $X(0)$, or in terms of the Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$

$$\begin{aligned} \log \mathbb{E} \exp \{\zeta X(t)\} &= \tilde{a} \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)) ds + \frac{\tilde{d}}{2} \zeta^2 \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))^2 ds \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp \{u \zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\} - 1 - u (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))] \mathbf{1}_{[-1,1]}(u) \tilde{\nu}(du) ds, \end{aligned} \quad (56)$$

and

$$\begin{aligned} &\log \mathbb{E} \exp \{\zeta_1 X(t_1) + \zeta_2 X(t_2)\} \\ &= \tilde{a} \int_{\mathbb{R}} \left(\sum_{j=1}^2 \zeta_j e^{-\lambda(t_j-s)} \mathbf{1}_{[0,\infty)}(t_j-s) \right) ds + \frac{\tilde{d}}{2} \zeta^2 \int_{\mathbb{R}} \left(\sum_{j=1}^2 \zeta_j e^{-\lambda(t_j-s)} \mathbf{1}_{[0,\infty)}(t_j-s) \right)^2 ds \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\exp \left\{ u \sum_{j=1}^2 \zeta_j e^{-\lambda(t_j-s)} \mathbf{1}_{[0,\infty)}(t_j-s) \right\} - 1 \right. \\ &\left. - u \left(\sum_{j=1}^2 \zeta_j e^{-\lambda(t_j-s)} \mathbf{1}_{[0,\infty)}(t_j-s) \right) \right] \mathbf{1}_{[-1,1]}(u) \tilde{\nu}(du) ds. \end{aligned} \quad (57)$$

Let us consider a geometric OU-type process as the mother process:

$$\Lambda(t) = e^{X(t)-c_X}, \quad c_X = \log \mathbb{E} e^{X(0)}, \quad M(\zeta) = \mathbb{E} e^{\zeta(X(t)-c_X)}, \quad M_0(\zeta) = \mathbb{E} e^{\zeta X(t)}$$

where $X(t)$, $t \in \mathbb{R}_+$, is the OU-type stationary process (51). Note that

$$\frac{M_0(q)}{M_0(1)^q} = \frac{M(q)}{M(1)^q}.$$

Then the correlation function of the mother process is of the form.

$$\text{Corr}(\Lambda(t), \Lambda(t+\tau)) = \frac{M(1, 1; \tau) - 1}{M(2) - 1}, \quad (58)$$

where now

$$\begin{aligned} M(\zeta_1, \zeta_2; \tau) &= \mathbb{E} \exp \{\zeta_1 (X(t_1) - c_X) + \zeta_2 (X(t_2) - c_X)\} \\ &= \exp \{-(\zeta_1 + \zeta_2) c_X\} \mathbb{E} \exp \{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}, \end{aligned} \quad (59)$$

and $\mathbb{E} \exp \{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$ is defined by (57).

To prove that a geometric OU process satisfies the covariance decay condition (35) in Theorem 7, the expression given by (57) is not ready to yield the decay as $t_2 - t_1 \rightarrow \infty$.

The following result plays a key role in multifractal analysis of geometric OU processes.

Theorem 11. Let $X(t), t \in \mathbb{R}_+$ be an OU-type stationary process (51) such that the Lévy measure ν in (49) of the random variable $X(0)$ satisfies the condition: for a positive integer $q^* \in \mathbf{N}$,

$$\int_{|x| \geq 1} x e^{q^* x} \nu(dx) < \infty. \quad (60)$$

Then, for any fixed b such that

$$b > \left\{ \frac{M_0(q^*)}{M_0(1)^{q^*}} \right\}^{\frac{1}{q^*-1}}, \quad (61)$$

stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^n \Lambda^{(j)}(sb^j) ds, t \in [0, 1]$$

converge in \mathcal{L}_q to the stochastic process $A(t) \in \mathcal{L}_q$, as $n \rightarrow \infty$, for every fixed $t \in [0, 1]$. The limiting process $A(t), t \in [0, 1]$ satisfies

$$\mathbb{E}A^q(t) \sim t^{q - \log_b \mathbb{E}\Lambda^q(t)}, \quad q \in [0, q^*].$$

The Rényi function is given by

$$T(q) = q - 1 - \log_b \mathbb{E}\Lambda^q(t) = q \left(1 + \frac{c_X}{\log b} \right) - \log_b M_0(q) - 1, \quad q \in [0, q^*]. \quad (62)$$

In addition,

$$\text{Var}A(t) \geq 2t \int_0^t \left(1 - \frac{\tau}{t} \right) (M(1, 1; \tau) - 1) d\tau, \quad (63)$$

where the bivariate moment generating function $M(\zeta_1, \zeta_2; t_1 - t_2)$ is given by (59)

Proof of Theorem 11 We are starting with \mathcal{L}_q convergence. To show the convergence we apply Theorem 5. It is sufficient to show the convergence for $q = q^*$ since the convergence for $q < q^*$ immediately follows from the convergence for $q = q^*$. First we will derive a suitable explicit expression for $\rho(u_1, \dots, u_{q-1})$. Put $s_1 = 0 \leq s_2 = u_1 \leq s_3 = u_1 + u_2, \dots, s_q = u_1 + \dots + u_{q-1}$. Then,

$$\rho(u_1, \dots, u_{q-1}) = \mathbb{E}\Lambda(s_1) \dots \Lambda(s_q) = \mathbb{E} \exp \{ X(s_1) + \dots + X(s_q) - qc_X \}.$$

Using representation (51) one can obtain

$$X(s_q) = e^{-\lambda(s_q - s_{q-1})} X(s_{q-1}) + \int_{(s_{q-1}, s_q]} e^{-\lambda(s_q - s)} dZ(\lambda s).$$

Then, using independence of $X(s_{q-1})$ and the integral $\int_{(s_{q-1}, s_q]} e^{-\lambda(s_q - s)} dZ(\lambda s)$ we obtain

$$\begin{aligned} & \mathbb{E} \exp \{ X(s_1) + \dots + X(s_q) \} \\ &= \mathbb{E} \exp \left\{ X(s_1) + \dots + (1 + e^{-\lambda(s_q - s_{q-1})}) X(s_{q-1}) \right\} \mathbb{E} e^{\int_{(s_{q-1}, s_q]} e^{-\lambda(s_q - s)} dZ(\lambda s)} \\ &= \mathbb{E} \exp \left\{ X(s_1) + \dots + (1 + e^{-\lambda(s_q - s_{q-1})}) X(s_{q-1}) \right\} \frac{\mathbb{E} e^{X(s_q)}}{\mathbb{E} e^{-\lambda(s_q - s_{q-1})} X(0)} \\ &= \mathbb{E} \exp \left\{ X(s_1) + \dots + (1 + e^{-\lambda(s_q - s_{q-1})}) X(s_{q-1}) \right\} \frac{M_0(1)}{M_0(e^{-\lambda(s_q - s_{q-1})})}. \end{aligned}$$

Proceeding further by induction we obtain

$$\begin{aligned} & \mathbb{E} \exp \{ X(s_1) + \dots + X(s_q) \} \\ &= M_0(1) \frac{M_0(1 + e^{-\lambda u_{q-1}}) M_0(1 + e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1} + u_{q-2})}) \dots M_0(1 + e^{-\lambda u_1} + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})})}{M_0(e^{-\lambda u_{q-1}}) M_0(e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1} + u_{q-2})}) \dots M_0(e^{-\lambda u_1} + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})})}. \end{aligned}$$

Hence

$$\rho(u_1, \dots, u_{q-1}) = \frac{M_0(1 + e^{-\lambda u_{q-1}})}{M_0(1)M_0(e^{-\lambda u_{q-1}})} \frac{M_0(1 + e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1}+u_{q-2})})}{M_0(1)M_0(e^{-\lambda u_{q-2}} + e^{-\lambda(u_{q-1}+u_{q-2})})} \cdots \times \frac{M_0(1 + e^{-\lambda u_1 + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})}})}{M_0(1)M_0(e^{-\lambda u_1 + \dots + e^{-\lambda(u_1 + \dots + u_{q-1})}})}. \quad (64)$$

This representation allows us to show monotonicity of $\rho(u_1, \dots, u_{q-1})$. For that we use the following inequality

$$\frac{M_0(1+s)}{M_0(s)} \leq \frac{M_0(1+t)}{M_0(t)} \quad (65)$$

for $s \leq t$. This inequality follows from the fact that $\ln M_0(t)$ is a convex function and the Karamata majorisation inequality. Hence

$$\frac{M_0(1+s)}{M_0(1)M_0(s)}$$

is monotone increasing in s . Since $e^{-\lambda u}$ is monotone decreasing in u the representation (64) implies that $\rho(u_1, \dots, u_{q-1})$ is monotone decreasing in all variables.

Condition (21) becomes

$$b^{q-1} > \text{E}\Lambda(0) = \frac{M_0(q)}{M_0(1)^q},$$

which is equivalent to (61).

To show the finiteness of the series (22) we are going to use the following statement.

Lemma 12. For $s \in [0, 1]$, the following estimate holds

$$\frac{M_0(1+s)}{M_0(1)M(s)} \leq \left(\frac{M_0(2)}{M_0(1)e^{\text{E}X(1)}} \right)^s. \quad (66)$$

Proof of Lemma 12 Function $\ln M_0(t)$ is convex. Therefore,

$$\ln M_0(1+s) = \ln M_0((1-s) + 2s) \leq (1-s) \ln M_0(1) + s \ln M_0(2).$$

In addition, by the Jensen inequality,

$$M_0(s) = \text{E}e^{sX(1)} \geq e^{s\text{E}X(1)}.$$

Together these inequalities imply,

$$\frac{M_0(1+s)}{M_0(1)M_0(s)} \leq \frac{M_0(1)^{1-s} M_0(2)^s}{M_0(1)e^{s\text{E}X(1)}} = \left(\frac{M_0(2)}{M_0(1)e^{\text{E}X(1)}} \right)^s.$$

□

Now, using (64) and monotone decrease of $M_0(1+s)/M_0(s)$

$$1 \leq \rho(b^n, \dots, b^n) \leq \left(\frac{M_0(1 + e^{-\lambda b^n})}{M_0(1)M_0(e^{-\lambda b^n})} \right)^q \leq C^q e^{-\lambda b^n} \leq 1 + o(1) \ln C q e^{-\lambda b^n}, \quad (67)$$

where the former inequality follows from Lemma 12 with $C = M_0(2)/(M_0(1)e^{\text{E}X(1)})$. Then, convergence of the series (64) follows from the finiteness of the series $\sum_{n=1}^{\infty} e^{-\lambda b^n}$.

Finally we need to check the mixing condition (20). Let $1 \leq i_1 < i_2 < \dots < i_m$ and $u_i = A$ if $i \in \{i_1, \dots, i_m\}$ and 0 otherwise. In this context it is convenient to use (64) in the form

$$\rho(u_1, \dots, u_{q-1}) = \prod_{j=1}^{q-1} \frac{M_0(1 + \sum_{k=j}^{q-1} e^{-\lambda \sum_{t=j}^k u_t})}{M_0(1)M_0(\sum_{k=j}^{q-1} e^{-\lambda \sum_{t=j}^k u_t})}$$

Then, as $A \rightarrow \infty$, and $i_0 = 0, i_{m+1} = q$

$$\begin{aligned} \lim_{A \rightarrow \infty} \rho(u_1, \dots, u_{q-1}) &= \prod_{\alpha=1}^m \prod_{j=i_{\alpha+1}}^{i_{\alpha+1}-1} \frac{M_0(1 + \sum_{k=j}^{i_{\alpha+1}-1} 1)}{M_0(1)M_0(\sum_{k=j}^{i_{\alpha+1}-1} 1)} \\ &= \prod_{\alpha=1}^m \frac{M_0(i_{\alpha+1} - i_{\alpha})}{M_0(1)^{i_{\alpha+1} - i_{\alpha}}} = \prod_{\alpha=1}^m \text{E}\Lambda(0)^{i_{\alpha+1} - i_{\alpha}}. \end{aligned} \quad (68)$$

This proves (20). Therefore Theorem 5 gives \mathcal{L}_q convergence of $A_n(t)$.

To prove the scaling property we are going to use the results of Theorem 7. First using representation (51), we have for any q ,

$$\begin{aligned} \text{E}\Lambda(t)\Lambda(0)^{q-1} &= \text{E} \exp\{(q-1)X(0) + X(t) - qc_X\} \\ &= \text{E} \exp\{(q-1 + e^{-\lambda t})X(0) + \int_{(0,t)} e^{-\lambda(t-s)} dZ(\lambda s) - qc_X\} \\ &= \text{E} \exp\{(q-1 + e^{-\lambda t})X(0) - qc_X\} \frac{\text{E} \exp\{e^{-\lambda t}X(0) + \int_{(0,t)} e^{-\lambda(t-s)} dZ(\lambda s)\}}{\text{E} e^{-\lambda t}X(0)} \\ &= \text{E} \exp\{(q-1 + e^{-\lambda t})X(0) - qc_X\} \frac{\text{E} \exp\{X(t)\}}{\text{E} e^{-\lambda t}X(0)} \\ &= \frac{M_0(q-1 + e^{-\lambda t})}{M_0(1)^{q-1} M(e^{-\lambda t})}. \end{aligned}$$

Then,

$$\frac{\text{E}\Lambda(t)^{q-1}\Lambda(0)^{q-1}}{\text{E}\Lambda(0)^q} = \frac{M_0(q-1 + e^{-\lambda t})M_0(1)}{M_0(q)M_0(e^{-\lambda t})}.$$

For $q > 1$ the latter function is monotone decreasing in t , as follows from the Karamata motorization inequality. Hence,

$$|\rho_q(s)| = \sup_{u \in [0,1]} \left(1 - \frac{\text{E}\Lambda(su)\Lambda(0)^{q-1}}{\text{E}\Lambda(0)^q} \right) = 1 - \frac{M_0(q-1 + e^{-\lambda s})M_0(1)}{M_0(q)M_0(e^{-\lambda s})}. \quad (69)$$

Function $f(x) = \ln M(x)$ is convex. Condition (60) ensures that the derivative $f'(q)$ exists for $q \leq q^*$. Then, for any $x \leq q$,

$$f(x) - f(q) \geq (x - q)f'(q).$$

In particular for $x = q - 1 + e^{-\lambda s}$,

$$f(q - 1 + e^{-\lambda s}) - f(q) \geq (-1 + e^{-\lambda s})f'(q).$$

In addition, by the Jensen inequality,

$$M_0(e^{-\lambda s}) = \text{E} e^{-\lambda s X(0)} \leq (\text{E} e^{X(0)}) e^{-\lambda s} = M_0(1) e^{-\lambda s}.$$

The latter two inequalities give

$$\begin{aligned} |\rho_q(s)| &\leq 1 - e^{(-1 + e^{-\lambda s})(f'(q) - f(1))} \\ &\leq (1 - e^{-\lambda s})(f'(q) - f(1)) \leq \lambda s(f'(q) - f(1)). \end{aligned} \quad (70)$$

Then

$$0 \leq \sum_{n=1}^{\infty} |\rho_q(b^{-n})| \leq \lambda(f'(q) - f(1)) \sum_{n=1}^{\infty} b^{-n} < \infty.$$

Since we have already shown that $A(t) \in \mathcal{L}_q$ we can apply Theorem 7. □

7. Superpositions of geometric Ornstein-Uhlenbeck processes

The correlation structures found in applications may be more complex than the exponential decreasing autocorrelation of the form (53). Barndorff-Nielsen (1998) (see also Barndorff-Nielsen and Sheppard (2001)) proposed to consider the following class of autocovariance functions:

$$R_m(t) = \sum_{j=1}^m \sigma_j^2 \exp\{-\lambda_j |t|\}, \quad (71)$$

which is flexible and can be fitted to many autocovariance functions arising in applications.

In order to obtain models with dependence structure (71) and given marginal density with finite variance, we consider stochastic processes defined by

$$dX_j(t) = -\lambda_j X_j(t) dt + dZ_j(\lambda_j t), \quad j = 1, 2, \dots, m, \dots$$

and their finite superposition

$$X_{m \text{ sup}}(t) = X_1(t) + \dots + X_m(t), \quad t \geq 0, \quad (72)$$

where Z_j , $j = 1, 2, \dots, m, \dots$ are mutually independent Lévy processes. Then the solution $X_j = \{X_j(t), t \geq 0\}$, $j = 1, 2, \dots, m$, is a stationary process. Its correlation function is of the exponential form (assuming finite variance of the components).

The superposition (72) has its marginal density given by that of the random variable

$$X_{m \text{ sup}}(0) = X_1(0) + \dots + X_m(0), \quad (73)$$

and autocovariance function (71). One can generalize the Theorem 10 to the case of finite superposition process (72).

We are interested in the case when the distribution of (73) is tractable, for instance when $X_m(0)$ belongs to the same class as $X_j(0)$, $j = 1, \dots, m$ (see the examples in Sections 8–13 below).

Define the mother process as geometric process

$$\Lambda(t) = e^{X_{m \text{ sup}}(t) - c_X t}, \quad c_X = \log \mathbb{E} e^{X_{m \text{ sup}}(0)}, \quad M(\zeta) = \mathbb{E} e^{\zeta(X_{m \text{ sup}}(t) - c_X t)}, \quad M_0(\zeta) = \mathbb{E} e^{\zeta X_{m \text{ sup}}(t)},$$

where $X_{m \text{ sup}}(t)$, $t \in \mathbb{R}_+$, is the finite superposition process (72). Note that

$$\log \mathbb{E} \exp\{\zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2)\} = \sum_{j=1}^m \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

where $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, \dots, m$ are given by (57).

Denote

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\} \mathbb{E} \exp\{\zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2)\} \quad (74)$$

We can formulate the following theorem which can be proved similar to Theorem 11.

Theorem 13. *Let $X_{m \text{ sup}}(t)$, $t \in \mathbb{R}_+$ be a finite superposition of OU-type stationary processes (72) such that the Lévy measure ν in (49) of the random variable $X_m(t)$ satisfies the condition that for a positive integer $q^* \in \mathbf{N}$,*

$$\int_{|x| \geq 1} x e^{q^* x} \nu(dx) < \infty. \quad (75)$$

Then, for any fixed b such that

$$b > \left\{ \frac{M_0(q^*)}{M_0(1)q^*} \right\}^{\frac{1}{q^*-1}}, \quad (76)$$

stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^n \Lambda^{(j)}(sb^j) ds,$$

converge in \mathcal{L}_q to the stochastic process $A(t) \in \mathcal{L}_q$, as $n \rightarrow \infty$. The limiting process $A(t)$ satisfies

$$EA^q(t) \sim t^{q-\log_b EA^q(t)}, \quad q \in [0, q^*], t \in [0, 1].$$

The Rényi function is given by

$$T(q) = q - 1 - \log_b EA^q(t) = q\left(1 + \frac{c_X}{\log b}\right) - \log_b M_0(q) - 1, \quad q \in [0, q^*]. \quad (77)$$

In addition,

$$\text{Var}A(t) \geq 2t \int_0^t \left(1 - \frac{u}{t}\right) M(\zeta_1, \zeta_2; u) du, \quad (78)$$

where the bivariate moment generating function $M(\zeta_1, \zeta_2; t_1 - t_2)$ is given by (74)

We are interesting to generalization of the above result to the case of infinite superposition of OU-type processes which has a long-range dependence property.

Note that an infinite superposition ($m \rightarrow \infty$) gives a complete monotone class of covariance functions

$$R_{\text{sup}}(t) = \int_0^\infty e^{-tu} dU(u), t \geq 0,$$

for some finite measure U , which display long-range dependence (see Barndorff-Nielsen 1998, 2001, Barndorff-Nielsen and Leonenko 2005 for possible covariance structures and spectral densities and Barndorff-Nielsen, Eiler and Stelzer, R., 2011 for multivariate generalizations).

We are going to consider an infinite superposition of the OU processes, which corresponds to $m = \infty$, that is now

$$X_{\text{sup}}(t) = \sum_{j=1}^\infty X_j(t), \quad (79)$$

assuming that

$$\sum_{j=1}^\infty EX_j(t) < \infty, \sum_{j=1}^\infty \text{Var}X_j(t) < \infty, \quad (80)$$

In this case

$$R_{\text{sup}}(t) = \sum_{j=1}^\infty \sigma_j^2 \exp\{-\lambda_j |t|\}, \quad (81)$$

and if we assume that for some $\delta_j > 0$

$$EX_j(t) = \delta_j C_1, \text{Var}X_j(t) = \sigma_j^2 = \delta_j C_2, \delta_j = j^{-(1+2(1-H))}, \frac{1}{2} < H < 1, \quad (82)$$

where the constants $C_1 \in \mathbf{R}$ and $C_2 > 0$ represent some other possible parameters (see examples in the Sections 8–13 below), then

$$EX_{\text{sup}}(t) = C_1 \sum_{j=1}^\infty \delta_j = C_1 \zeta(1 + 2(1 - H)) < \infty, \quad (83)$$

where $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, $\text{Res} > 1$, is the Riemann zeta-function, and with $\lambda_j = \lambda/j$, we have

$$R_{\text{sup}}(t) = \sum_{j=1}^\infty \sigma_j^2 \exp\{-\lambda_j |t|\} = C_2 \sum_{j=1}^\infty \delta_j \exp\{-\lambda |t|/j\} = \frac{L_2(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1, \quad (84)$$

where L_2 is a slowly varying at infinity function, bounded on every bounded interval. Thus we obtain a long range dependence property:

$$\int_{\mathbf{R}} R_{\text{sup}}(t) dt = \infty.$$

We are going to make an additional assumption that there exists parameters δ_j such that

$$\mathbb{E}e^{\zeta X_j(0)} = \mathbb{E}e^{\zeta \delta_j Y} \quad (85)$$

for some random variable Y . The sum

$$\sum_{j=1}^{\infty} \delta_j < \infty \quad (86)$$

must be finite. When we specialize (81) to this situation we obtain

$$R_{\text{sup}}(t) = C_2 \sum_{j=1}^{\infty} \delta_j \exp\{-\lambda_j |t|\}, \quad (87)$$

for some $C_2 > 0$. This approach allows also to treat the case of several parameters.

We are starting our considerations with \mathcal{L}_q convergence. Let

$$\rho_j(u_1, \dots, u_{q-1}) = \mathbb{E}e^{X_j(0) + X_j(u_1) + X_j(u_1 + u_2) + \dots + X_j(u_1 + \dots + u_{q-1})}$$

corresponds to the process $X_j(t)$. Then, since $X_j(\cdot)$ are independent of each other,

$$\rho(u_1, \dots, u_{q-1}) = \prod_{j=1}^{\infty} \rho_j(u_1, \dots, u_{q-1}). \quad (88)$$

We have shown above that $\rho_j(u_1, \dots, u_{q-1})$ is monotone decreasing in u_1, \dots, u_{q-1} . Therefore $\rho(u_1, \dots, u_{q-1})$, being a product of monotone decreasing functions is monotone decreasing itself. Next we prove finiteness of the series. For that recall estimate (67)

$$1 \leq \rho_j(b^n, \dots, b^n) \leq \left(\frac{M_0(1 + e^{-\lambda_j b^n})}{M_0(1)M_0(e^{-\lambda_j b^n})} \right)^q \leq \left(\frac{\mathbb{E}e^{2X_j(0)}}{\mathbb{E}e^{X_j(0)}\mathbb{E}e^{X_j(0)}} \right)^{qe^{-\lambda_j b^n}} = C^{q\delta_j} e^{-\lambda_j b^n}$$

where $C = \frac{\mathbb{E}e^{2Y}}{\mathbb{E}e^Y\mathbb{E}e^Y}$, and we denote

$$M_0(\zeta) = \mathbb{E}e^{\zeta X_j(t)}.$$

Then, using (87), we obtain

$$1 \leq \rho(b^n, \dots, b^n) \leq C^q \sum_{j=1}^{\infty} \delta_j e^{-\lambda_j b^n} = C^{\frac{q}{\sigma^2}} R_{\text{sup}}(b^n) \leq 1 + o(1) \frac{q}{\sigma^2} \ln C R_{\text{sup}}(b^n).$$

Then $\sum_{n=1}^{\infty} \rho(b^n, \dots, b^n)$ is finite if the sum

$$\sum_{n=1}^{\infty} R_{\text{sup}}(b^n) < \infty$$

is finite.

We are left to check the mixing condition (20). But this condition follows from the fact that it holds for ρ_j , representation (88) and monotonicity of ρ_j . Indeed let $1 \leq i_1 < i_2 < \dots \leq i_m$ and put $\delta_{i_1, \dots, i_m}(A) = (u_1, \dots, u_{q-1})$, where $u_i = A$ if $i \in \{i_1, \dots, i_m\}$ and 0 otherwise. Let N be a number which we let to ∞ later. Then, for fixed N ,

$$\prod_{j=1}^N \rho_j(\delta_{i_1, \dots, i_m}(A)) \rightarrow \prod_{j=1}^N \mathbb{E}e^{i_1 X_j(0)} \mathbb{E}e^{(i_2 - i_1) X_j(0)} \dots \mathbb{E}e^{(q - i_m) X_j(0)}$$

by the corresponding property of the geometric Ornstein-Uhlenbeck process, see (68). The product,

$$1 \leq \prod_{j=N}^{\infty} \rho_j(\delta_{i_1, \dots, i_m}(A)) \leq \prod_{j=N}^{\infty} \rho_j(\delta_{i_1, \dots, i_m}(1)) \rightarrow 1,$$

as $N \rightarrow \infty$, uniformly in $A > 1$. Therefore, the mixing property holds.

Now we turn to proving the scaling property. For that we use Theorem 7. Let $M_0(\zeta) = \mathbb{E}e^{\zeta Y}$ be the moment generating function of Y and $f(\zeta) = \ln M_0(\zeta)$. Then, similarly to (69),

$$\begin{aligned} |\rho_q(s)| &= 1 - \prod_{j=1}^{\infty} \left(\frac{M_0(q-1 + e^{-\lambda_j s}) M_0(1)}{M_0(q) M_0(e^{-\lambda_j s})} \right)^{\delta_j} \leq 1 - \prod_{j=1}^{\infty} e^{(-1 + e^{-\lambda_j s})(f'(q) - f(1))\delta_j} \\ &\leq 1 - \prod_{j=1}^{\infty} e^{-\lambda_j s (f'(q) - f(1))\delta_j} \leq s(f'(q) - f(1)) \sum_{j=1}^{\infty} \lambda_j \delta_j. \end{aligned}$$

The convergence of the series immediately follows from this estimate and we can apply Theorem 7.

Note that

$$\log \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\} = \sum_{j=1}^{\infty} \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

where $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, 2, \dots$ are given by (57).

Define the mother process as geometric process

$$\Lambda(t) = e^{X_{\text{sup}}(t) - c_X}, \quad c_X = \log \mathbb{E} e^{X_{\text{sup}}(0)}, \quad M(\zeta) = \mathbb{E} e^{\zeta (X_{\text{sup}}(0) - c_X)},$$

where $X_{\text{sup}}(t)$, $t \in \mathbb{R}$, is the infinite superposition process (79).

Denote

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\} \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\}. \quad (89)$$

We arrive to the following result.

Theorem 14. *Let $X_{\text{sup}}(t)$, $t \in \mathbb{R}_+$ be an infinite superposition of OU-type stationary processes (79) such that (80), (82), (85) are satisfied as well as (86). Assume that the Lévy measure ν in (49) of the random variable $X_{\text{sup}}(t)$ satisfies the condition that for a positive integer $q^* \in \mathbf{N}$,*

$$\int_{|x| \geq 1} x e^{q^* x} \nu(dx) < \infty. \quad (90)$$

Then, for any fixed b such that

$$b > \left\{ \frac{M(q^*)}{M(1)^{q^*}} \right\}^{\frac{1}{q^*-1}}, \quad (91)$$

stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^n \Lambda^{(j)}(sb^j) ds, \quad t \in [0, 1]$$

converge in \mathcal{L}_q to the stochastic process $A(t) \in \mathcal{L}_q$, $t \in [0, 1]$, as $n \rightarrow \infty$. The limiting process $A(t)$ satisfies

$$\mathbb{E} A^q(t) \sim t^{q - \log_b \mathbb{E} \Lambda^q(t)}, \quad q \in [0, q^*].$$

The Rényi function is given by

$$T(q) = q - 1 - \log_b \mathbb{E} \Lambda^q(t), \quad q \in [0, q^*], \quad t \in [0, 1]. \quad (92)$$

In addition,

$$\text{Var} A(t) \geq \int_0^t \int_0^t M(\zeta_1, \zeta_2; t_1 - t_2) dt ds, \quad (93)$$

where the bivariate moment generating functions $M(\zeta_1, \zeta_2; t_1 - t_2)$ is given by (89).

8. Log-tempered stable scenario

This section introduces a scenario which generalize the log-inverse Gaussian scenario obtained in Anh, Leonenko and Shieh (2008). Note that the tempered stable distribution (up to constants) arises in the theory of Vershik-Yor subordinator (see Donati-Martin and Yor 2006, and the references therein). This section constructs a multifractal process based on the geometric tempered stable OU process. In this case, the mother process takes the form $\Lambda(t) = \exp\{X(t) - c_X t\}$, where $X(t), t \geq 0$ is a stationary OU type process (51) with tempered stable marginal distribution and c_X is a constant depending on the parameters of its marginal distribution. This form is needed for the condition $E\Lambda(t) = 1$ to hold. The log-tempered stable scenario appeared in Novikov (1994) in a physical setting and under different terminology. We provide a rigorous construction for this scenario here.

We consider the stationary OU process whose marginal distribution is the tempered stable distribution $TS(\kappa, \delta, \gamma)$ (see, for example, Barndorff-Nielsen and Shephard 2002). This distribution is the exponentially tilted version of the positive κ -stable law $S(\kappa, \delta)$ whose probability density function (pdf) is

$$s_{\kappa, \delta}(x) = \frac{\delta^{-1/\kappa}}{2\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \sin(k\pi\kappa) \frac{\Gamma(k\kappa+1)}{k!} 2^{k\kappa+1} \left(\frac{x}{\delta^{1/\kappa}}\right)^{-k\kappa-1}, \quad x > 0. \quad (94)$$

The pdf of the tempered stable distribution $TS(\kappa, \delta, \gamma)$ is

$$\pi(x) = \pi(x; \kappa, \delta, \gamma) = e^{\delta\gamma} s_{\kappa, \delta}(x) e^{-\frac{x}{2}\gamma^{1/\kappa}}, \quad x > 0, \quad (95)$$

where the parameters satisfy

$$\kappa \in (0, 1), \delta > 0, \gamma > 0.$$

It is clear that $TS(\frac{1}{2}, \delta, \gamma) = IG(\delta, \gamma)$, the inverse Gaussian distribution with pdf

$$\pi(x) = \frac{1}{\sqrt{2\pi}} \frac{\delta e^{\delta\gamma}}{x^{3/2}} \exp\left\{-\left(\frac{\delta^2}{x} + \gamma^2 x\right) \frac{1}{2}\right\} \mathbf{1}_{[0, \infty)}(x), \delta > 0, \gamma \geq 0,$$

and $TS(\frac{1}{3}, \delta, \gamma)$ has the pdf

$$\pi(x) = \frac{\sqrt{2}}{\pi} \delta^{3/2} e^{\delta^{3/2}\gamma} x^{-3/2} K_{\frac{1}{3}}\left(\left(\frac{2\delta}{3}\right)^{3/2} \frac{1}{\sqrt{x}}\right) e^{-\frac{x\gamma^3}{2}}, \quad x > 0,$$

where here and below

$$K_{\nu}(z) = \int_0^{\infty} e^{-z \cosh(u)} \cosh(\nu z) du, \quad z > 0, \quad (96)$$

is the modified Bessel function of the third kind of index ν , $\mathbf{Re}\nu > 0$.

The cumulant transform of a random variable X with pdf (95) is equal to

$$\log Ee^{\zeta X} = \delta\gamma - \delta \left(\gamma^{\frac{1}{\kappa}} - 2\zeta\right)^{\kappa}, \quad 0 < \zeta < \frac{\gamma^{1/\kappa}}{2}. \quad (97)$$

Note that

$$EX(t) = 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}}, \quad \text{Var}X(t) = 4\kappa(1-\kappa)\delta\gamma^{\frac{\kappa-2}{\kappa}}.$$

We will consider a stationary OU type process (52) with marginal distribution $TS(\kappa, \delta, \gamma)$. This distribution is self-decomposable (and hence infinitely divisible) with the Lévy triplet $(a, 0, \nu)$, where

$$\begin{aligned} \nu(du) &= b(u)du, \\ b(u) &= 2^{\kappa}\delta \frac{\kappa}{\Gamma(1-\kappa)} u^{-1-\kappa} e^{-\frac{u\gamma^{1/\kappa}}{2}}, \quad u > 0. \end{aligned}$$

The BDLP $Z(t)$ in (52) has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, with

$$\tilde{\nu}(du) = \lambda\omega(u)du, \quad (98)$$

$$\omega(u) = 2^{\kappa}\delta \frac{\kappa}{\Gamma(1-\kappa)} \left(\frac{\kappa}{u} + \frac{\gamma^{1/\kappa}}{2}\right) u^{-\kappa} e^{-\frac{u\gamma^{1/\kappa}}{2}}, \quad u > 0, \quad (99)$$

that is, the BDLP of the distribution $TS(\kappa, \delta, \gamma)$ is the sum of a Lévy process with density

$$2^\kappa \delta \frac{\kappa^2}{\Gamma(1-\kappa)} u^{-1-\kappa} e^{-\frac{u\gamma^{1/\kappa}}{2}}, u > 0,$$

and a compound Poisson process $Z(t) = \sum_{k=1}^{P(t)} Z_k$, which has Lévy density

$$\delta\gamma\kappa \frac{(\gamma^{1/\kappa}/2)^{1-\kappa}}{\Gamma(1-\kappa)} u^{-\kappa} e^{-\frac{u\gamma^{1/\kappa}}{2}}, u > 0,$$

meaning that the summands Z_k have gamma distribution $\Gamma(1-\kappa, \frac{\gamma^{1/\kappa}}{2})$, while the independent Poisson process $P(t)$ has rate $\delta\gamma\kappa$.

B'. Consider a mother process of the form

$$\Lambda(t) = \exp \{X(t) - c_X\}$$

with

$$c_X = \left[\delta\gamma - \delta \left(\gamma^{1/\kappa} - 2 \right)^\kappa \right], \gamma > 2^\kappa, \quad (100)$$

where $X(t)$, $t \in \mathbb{R}$ of OU processes with $TS(\kappa, \delta_j, \gamma)$ marginal distributions and covariance function

$$R_X(t) = 4\kappa(1-\kappa)\delta\gamma^{\frac{\kappa-2}{\kappa}} \exp\{-\lambda|t|\}.$$

The correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1, 1; \tau) - 1}{M(2) - 1}, \gamma > 4^\kappa, \quad (101)$$

where

$$M(\zeta) = e^{-\zeta c_X} \mathbb{E} e^{\zeta X(t)},$$

and the bivariate moment generating function $M(1, 1, \tau)$ is given by (59), in which the Lévy measure $\tilde{\nu}$ is defined by (98), (99), and c_X is given by (100).

Condition (60) becomes

$$\int_1^\infty u e^{q^* u} u^{-1-\kappa} e^{-\frac{u\gamma^{1/\kappa}}{2}} du = \int_1^\infty u^{-\kappa} e^{-u(\frac{\gamma^{1/\kappa}}{2} - q^*)} du < \infty,$$

if $0 < q^* < \frac{\gamma^{1/\kappa}}{2}$, $\kappa \in (0, 1)$. Note that $M(q)$ exists if $(2q)^\kappa < \gamma$.

We can formulate the following

Theorem 15. *Suppose that condition **B'** holds, $\lambda > 0$ and*

$$Q = \left\{ q : 0 < q < \frac{\gamma^{1/\kappa}}{2}, \gamma \geq \max\{(2q^*)^\kappa, 4^\kappa\}, \kappa \in (0, 1), \delta > 0 \right\} \cap (0, q^*),$$

where q^* is a fixed integer. Then, for any

$$b > \exp \left\{ -\gamma\delta + \frac{\delta}{1-q^*} \left(\gamma^{1/\kappa} - 2q^* \right)^\kappa - \frac{q^*}{1-q^*} \delta \left(\gamma^{1/\kappa} - 2 \right)^\kappa \right\}, \gamma \geq \max\{(2q^*)^\kappa, 4^\kappa\},$$

the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^n \Lambda^{(j)}(sb^j) ds, t \in [0, 1]$$

converge in \mathcal{L}_q to the stochastic process $A(t)$ for each fixed $t \in [0, 1]$ as $n \rightarrow \infty$ such that, $A_q(1) \in \mathcal{L}_q$, for $q \in Q$, and

$$\mathbb{E} A^q(t) \sim t^{T(q)+1},$$

where the Rényi function $T(q)$ is given by

$$T(q) = q \left(1 + \frac{\delta\gamma}{\log b} - \frac{\delta}{\log b} \left(\gamma^{\frac{1}{\kappa}} - 2 \right)^\kappa \right) + \frac{\delta}{\log b} \left(\gamma^{\frac{1}{\kappa}} - 2q \right)^\kappa - \frac{\delta\gamma}{\log b} - 1, q \in Q.$$

Moreover,

$$\text{Var}A(t) \geq \int_0^t \int_0^t [M(1, 1; u - w) - 1] dudw,$$

where $M(1, 1, \tau)$ is given by (57), in which the Lévy measure $\tilde{\nu}$ is defined by (98), (99).

Theorem 15 follows from the Theorem 11. Note that for $\kappa = 1/2$ Theorem 15 is an extension of the Theorem 4 of Anh, Leonenko and Shieh (2008).

In this partial case we arrive to log-inverse Gaussian scenario where the Rényi function is of the form:

$$T(q) = q \left(1 + \frac{\delta \left[\gamma - \sqrt{\gamma^2 - 2} \right]}{\log b} \right) + \frac{\delta}{\log b} \sqrt{\gamma^2 - 2q} - \frac{\gamma\delta}{\log b} - 1, q \in Q,$$

and

$$Q = \left\{ q : 0 < q < \frac{\gamma^2}{2}, \gamma \geq 2, \delta > 0 \right\} \cap (0, q^*) \quad (102)$$

if

$$b > \exp \left\{ -\gamma\delta - \frac{\delta}{1 - q^*} \sqrt{\gamma^2 - 2q} - \frac{q^*}{1 - q^*} \delta \sqrt{\gamma^2 - 2q} \right\}$$

and q^* is a fixed integer.

Note that the set (102) is an extension of the log-inverse Gaussian scenario in the Theorem 4 of Anh, Leonenko and Shieh (2008) which is obtained for the set

$$Q = \left\{ q : 0 < q < \frac{\gamma^2}{2}, \gamma \geq 2, \delta > 0 \right\} \cap [1, 2].$$

We can construct log-tempered stable scenarios for a more general class of finite superpositions of stationary tempered stable OU-type processes:

$$X_{m \text{ sup}}(t) = \sum_{j=1}^m X_j(t), t \in [0, 1],$$

where $X_j(t), j = 1, \dots, m$, are independent stationary processes with marginals $X_j(t) \sim TS(\kappa, \delta_j, \gamma), j = 1, \dots, m$, and parameters $\delta_j, j = 1, \dots, m$. Then $X_{m \text{ sup}}(t)$ has the marginal distribution $TS(\kappa, \sum_{j=1}^m \delta_j, \gamma)$, and covariance function

$$R_{m \text{ sup}}(t) = 4\kappa(1 - \kappa) \gamma^{\frac{\kappa-2}{\kappa}} \sum_{j=1}^m \delta_j \exp \{ -\lambda_j |t| \}.$$

It follows from the Theorem 13 that the statement of Theorem 15 can be reformulated for $X_{m \text{ sup}}(t)$ with $\delta = \sum_{j=1}^m \delta_j$, λ being replaced by λ_j , and

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp \{ -c_X(\zeta_1 + \zeta_2) \} \mathbb{E} \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2) \},$$

where

$$\log \mathbb{E} \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2) \} = \sum_{j=1}^m \log \mathbb{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \},$$

and $\log \mathbb{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \}, j = 1, \dots, m$ are given by (57).

Moreover, one can construct log-tempered stable scenarios for a more general class of infinite superpositions of stationary tempered stable OU-type processes:

$$X_{\text{sup}}(t) = \sum_{j=1}^{\infty} X_j(t), t \in [0, 1], \quad (103)$$

where $X_j(t)$, $j = 1, \dots, m, \dots$ are independent stationary processes with correlation functions $\exp\{-\lambda_j |t|\}$, and the marginals $X_j(t) \sim TS(\kappa, \delta_j, \gamma)$, $j = 1, \dots, m, \dots$, where parameters $\delta_j = j^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, and $\lambda_j = \lambda/j$.

Then $X_{\text{sup}}(t)$ has the marginal distribution $TS(\kappa, \sum_{j=1}^{\infty} \delta_j, \gamma)$, an expectation

$$\mathbb{E}X_{\text{sup}}(t) = 2\kappa\gamma^{\frac{\kappa-1}{\kappa}} \sum_{j=1}^{\infty} \delta_j$$

and the long-range dependent covariance function

$$R_{\text{sup}}(t) = 4\kappa(1-\kappa)\gamma^{\frac{\kappa-2}{\kappa}} \sum_{j=1}^{\infty} \delta_j \exp\{-\lambda_j |t|\} = \frac{L_3(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1$$

where L_3 is a slowly varying at infinity function, bounded on every bounded interval. It follows from the Theorem 14 that the statement of the Theorem 15 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j$, and

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \exp\{-c_X(\zeta_1 + \zeta_2)\} \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\},$$

where

$$\log \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\} = \sum_{j=1}^{\infty} \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, \dots, m$ are given by (57) with Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$ given by (98) and (99).

9. Log-normal tempered stable scenario

This subsection constructs a multifractal process based on the geometric normal tempered stable (NTS) OU process. We consider a random variable $X = \mu + \beta Y + \sqrt{Y}\epsilon$, where the random variable Y follows the $TS(\kappa, \delta, \gamma)$ distribution, ϵ has a standard normal distribution, and Y and ϵ are independent. We then say that X follows the normal tempered stable law $NTS(\kappa, \gamma, \beta, \mu, \delta)$ (see, for example, Barndorff-Nielsen and Shephard 2002). In particular, for $\kappa = 1/2$ we have that $NTS(\frac{1}{2}, \gamma, \beta, \mu, \delta)$ is the same as the normal inverse Gaussian law $NIG(\alpha, \beta, \mu, \delta)$ with $\alpha = \sqrt{\beta^2 + \gamma^2}$ (see Barndorff-Nielsen 1998). Assuming $\mu = 0$ for simplicity, the pdf of $NTS(\kappa, \gamma, \beta, \mu, \delta)$ may be written as a mixture representation

$$p(x; \kappa, \gamma, \beta, 0, \delta) = \frac{1}{\sqrt{2\pi}} e^{\delta\gamma + \beta x} \int_0^{\infty} z^{-1/2} e^{-\frac{1}{2}(\frac{x^2}{z} + \beta^2 z)} \pi(z; \kappa, \delta, \gamma) dz, \quad x \in \mathbb{R}, \quad (104)$$

where $\pi(z; \kappa, \delta, \gamma)$ is the pdf of the $TS(\kappa, \delta, \gamma)$ distribution. We assume that

$$\mu \in \mathbb{R}, \delta > 0, \gamma > 0, \beta > 0, \kappa \in (0, 1).$$

Let $\alpha = \sqrt{\beta^2 + \gamma^2}$ and using the substitution $s = 1/z$ we obtain

$$p(x; \kappa, \gamma, \beta, 0, \delta) = \frac{1}{\sqrt{2\pi}} e^{\delta\gamma + \beta x} \int_0^{\infty} s^{-3/2} e^{-\frac{1}{2}(x^2 s + \alpha^2/s)} s_{\kappa, \delta}(1/s) ds, \quad x \in \mathbb{R},$$

where $s_{\kappa, \delta}(x)$ is given by (94). As an example, the pdf of $NTS(\frac{1}{3}, \alpha, \beta, 0, 1)$ has the form

$$p(x; \frac{1}{3}, \alpha, \beta, 0, 1) = \left(\frac{\delta}{\pi}\right)^{\frac{3}{2}} e^{\delta\gamma + \beta x} \int_0^{\infty} e^{-\frac{1}{2}(x^2 s + \alpha^2/s)} K_{\frac{1}{3}} \left(\left(\frac{2\delta}{3}\right)^{3/2} \sqrt{s} \right) ds.$$

It was pointed out by Barndorff-Nielsen and Shephard (2002) that $NTS(\kappa, \gamma, \beta, \mu, \delta)$ is self-decomposable. Thus, there exists a stationary OU-type process $X(t), t \geq 0$, with stationary $NTS(\kappa, \gamma, \beta, \mu, \delta)$ marginal distribution, correlation function

$$r_X(t) = \exp\{-\lambda|t|\},$$

and

$$\begin{aligned} EX(t) &= \mu + 2\kappa\beta\delta\gamma^{\frac{\kappa-1}{\kappa}}, \\ \text{Var}X(t) &= 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa\beta^2\delta(\kappa-1)\gamma^{\frac{\kappa-2}{\kappa}}. \end{aligned}$$

We see that the variance can be factorized in a similar manner as in Section 7, and thus superposition can be used to create multifractal scenarios with more elaborate dependence structures.

Also one can obtain the following semi-heavy tail behavior:

$$p(x; \kappa, \gamma, \beta, 0, \delta) \sim 2^{\kappa+1}\delta e^{\delta\gamma} \frac{\Gamma(1+\kappa)}{\Gamma(\kappa)\Gamma(1-\kappa)} \alpha^{\kappa+\frac{1}{2}} |x|^{-\kappa-1} e^{-\alpha|x|+\beta x}$$

as $x \rightarrow \pm\infty$.

The cumulant transform of the random variable $X(t)$ with pdf (104) is equal to

$$\log Ee^{\zeta X(t)} = \mu\zeta + \delta\gamma - \delta(\alpha^2 - (\beta + \zeta)^2)^\kappa, \quad |\beta + \zeta| < \alpha = \sqrt{\beta^2 + \gamma^{1/\kappa}}. \quad (105)$$

The Lévy triplet of $NTS(\kappa, \gamma, \beta, \mu, \delta)$ is $(a, 0, \nu)$, where

$$\nu(du) = b(u)du, \quad (106)$$

$$b(u) = \frac{\delta}{\sqrt{2\pi}} \alpha^{\kappa+\frac{1}{2}} \frac{\kappa 2^{\kappa+1}}{\Gamma(1-\kappa)} |u|^{-(\kappa+\frac{1}{2})} K_{\kappa+\frac{1}{2}}(\alpha|u|) e^{\beta u}, \quad u \in \mathbb{R}, \quad (107)$$

the modified Bessel function $K_\nu(z)$ being given by (96).

From (54), (105) and the formulae

$$\begin{aligned} K_\nu(x) &= K_\nu(-x), \quad K_{-\nu}(x) = K_\nu(x), \\ \frac{d}{dx} K_\nu(x) &= -\frac{\lambda}{x} K_\nu(x) - K_{\nu-1}(x), \end{aligned}$$

we obtain that the BDLP $Z(t)$ in (46) has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, with

$$\tilde{\nu}(du) = \lambda\omega(u)du,$$

where

$$\begin{aligned} \omega(u) &= -b(u) - ub'(u) = \frac{\delta}{\sqrt{2\pi}} \alpha^{\kappa+\frac{1}{2}} \frac{\kappa 2^{\kappa+1}}{\Gamma(1-\kappa)} \\ &\times \left\{ \left(\kappa - \frac{1}{2}\right) |u|^{-(\kappa+\frac{1}{2})} K_{\kappa+\frac{1}{2}}(\alpha|u|) e^{\beta u} + |u|^{-(\kappa-\frac{1}{2})} \left[-\frac{\kappa+\frac{1}{2}}{|u|} K_{\kappa+\frac{1}{2}}(\alpha|u|) e^{\beta u} \right. \right. \\ &\left. \left. - K_{\kappa-\frac{1}{2}}(\alpha|u|) e^{\beta u} \alpha + K_{\kappa+\frac{1}{2}}(\alpha|u|) e^{\beta u} \beta \right] \right\}. \quad (108) \end{aligned}$$

B''. Consider a mother process of the form

$$\Lambda(t) = \exp\{X(t) - c_X\},$$

with

$$c_X = \mu + \delta\gamma - \delta \left(\beta^2 + \gamma^{1/\kappa} - (\beta + 1)^2 \right)^\kappa, \quad \beta < \frac{\gamma^{1/\kappa} - 1}{2},$$

where $X(t), t \in [0, 1]$ is a stationary $NTS(\kappa, \gamma, \beta, \mu, \delta)$ OU-type process.

Under condition B'', we obtain the following moment generating function

$$M(\zeta) = E \exp\{\zeta(X(t) - c_X)\} = e^{-c_X\zeta} e^{\mu\zeta + \delta\gamma - \delta(\beta^2 + \gamma^{1/\kappa} - (\beta + \zeta)^2)^\kappa}, \quad |\beta + \zeta| < \alpha, \quad (109)$$

and bivariate moment generating function

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbf{E} \exp \{ \zeta_1 (X(t_1) - c_X) + \zeta_2 (X(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbf{E} \exp \{ \zeta_1 X(t_1) + \zeta_2 (X(t_2)) \}, \end{aligned} \quad (110)$$

and $\mathbf{E} \exp \{ \zeta_1 X(t_1) + \zeta_2 (X(t_2)) \}$ is given by (57) with Lévy measure $\tilde{\nu}$ having density (108). Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1, 1; \tau) - 1}{M(2) - 1}, \quad (111)$$

where we assumed that $\beta < (\gamma^{1/\kappa} - 4)/4$.

Note that as $z \rightarrow \infty$ the modified Bessel function of the third kind of index ν .

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \dots \right), z > 0,$$

Condition (60) now becomes

$$\int_{|u|>1} u e^{q^* u} |u|^{-(\kappa + \frac{1}{2})} K_{\kappa + \frac{1}{2}}(\alpha |u|) e^{\beta u} du < \infty,$$

if $|\beta + q^*| < \alpha = \sqrt{\beta^2 + \gamma^{1/\kappa}}$.

Theorem 16. *Suppose that condition \mathbf{B}'' holds, $\lambda > 0$ and*

$$q \in Q = \left\{ q : 0 < q < q^* \leq \sqrt{\beta^2 + \gamma^{1/\kappa}} - \beta, \beta < (\gamma^{1/\kappa} - 1)/2, \mu \in \mathbb{R}, \delta > 0, \kappa \in (0, 1) \right\},$$

where q^* is a fixed integer.

Then, for any

$$b > \exp \left\{ -\delta \gamma + \frac{\delta (\beta^2 + \gamma^{1/\kappa} - (\beta + q^*)^2)^\kappa - q^* \delta (\beta^2 + \gamma^{1/\kappa} - (\beta + 1)^2)^\kappa}{1 - q^*} \right\},$$

the stochastic processes, the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^n \Lambda^{(j)}(sb^j) ds, t \in [0, 1]$$

converge in \mathcal{L}_q to the stochastic process $A(t)$ for each fixed $t \in [0, 1]$ as $n \rightarrow \infty$ such that $A(1) \in \mathcal{L}_q$ for $q \in Q$, and

$$\mathbf{E} A^q(t) \sim t^{T(q)+1}, q \in Q,$$

where the Rényi function $T(q)$ is given by

$$\begin{aligned} T(q) &= \left(1 - \frac{\delta \left[(\beta^2 + \gamma^{1/\kappa} - (\beta + 1)^2)^\kappa \right] - \gamma}{\log b} \right) q + \\ &\quad + \frac{\delta}{\log b} \left(\beta^2 + \gamma^{1/\kappa} - (\beta + q)^2 \right)^\kappa - \frac{\delta \gamma}{\log b} - 1, q \in Q. \end{aligned}$$

Moreover,

$$\text{Var} A(t) \geq \int_0^t \int_0^t [M(1, 1; u - w) - 1] dudw,$$

where M is given by (110).

Theorem 16 follows from the Theorem 11.

Note that, for $\kappa = 1/2$, Theorem 16 is an extension to Theorem 5 of Anh, Leonenko and Shieh (2008), which is now extended to present a log-normal inverse Gaussian scenarios with Rényi function:

$$T(q) = \left(1 - \frac{\delta \left[\sqrt{\beta^2 + \gamma^2 - (\beta + 1)^2} \right] - \gamma}{\log b} \right) q + \frac{\delta}{\log b} \sqrt{\beta^2 + \gamma^2 - (\beta + q)^2} - \frac{\delta \gamma}{\log b} - 1, q \in Q,$$

where we use the notation $NTS(\frac{1}{2}, \gamma, \beta, \mu, \delta) = NIG(\alpha, \beta, \delta, \mu)$, where $\gamma = \sqrt{\alpha^2 - \beta^2}$.

$$Q = \left\{ q : 0 < q < q^* \leq \sqrt{\beta^2 + \gamma^2} - \beta, \beta < (\gamma^2 - 1)/2, \mu \in \mathbb{R}, \delta > 0 \right\},$$

and

$$b > \exp \left\{ -\delta \gamma + \frac{\delta \sqrt{\beta^2 + \gamma^2 - (\beta + q^*)^2} - q^* \delta \sqrt{\beta^2 + \gamma^2 - (\beta + 1)^2}}{1 - q^*} \right\}.$$

If $X_j(t)$, $j = 1, \dots, m$, are independent so that $X_j(t) \sim NTS(\kappa, \gamma, \beta, \mu_j, \delta_j)$, $j = 1, \dots, m$, then we have that

$$X_1(t) + \dots + X_m(t) \sim NTS(\kappa, \gamma, \beta, \sum_{j=1}^m \mu_j, \sum_{j=1}^m \delta_j).$$

We can construct log-normal tempered stable scenarios for a more general class of finite superpositions of stationary normal tempered stable OU-type processes:

$$X_{m \text{ sup}}(t) = \sum_{j=1}^m X_j(t), t \in [0, 1],$$

where $X_j(t)$, $j = 1, \dots, m$, are independent stationary processes with marginals $X_j(t) \sim NTS(\kappa, \gamma, \beta, \mu_j, \delta_j)$, $j = 1, \dots, m$, and parameters δ_j, μ_j , $j = 1, \dots, m$. Then $X_{m \text{ sup}}(t)$, $t \in [0, 1]$ has the marginal distribution $NTS(\kappa, \gamma, \beta, \sum_{j=1}^m \mu_j, \sum_{j=1}^m \delta_j)$, and covariance function

$$R_{m \text{ sup}}(t) = \left[2\kappa \gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa \beta^2 (\kappa - 1) \gamma^{\frac{\kappa-2}{\kappa}} \right] \sum_{j=1}^m \delta_j \exp \{-\lambda_j |t|\}.$$

$$E X_{m \text{ sup}}(t) = \mu + 2\kappa \beta \delta \gamma^{\frac{\kappa-1}{\kappa}},$$

$$\text{Var} X_{m \text{ sup}}(t) = 2\kappa \delta \gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa \beta^2 \delta (\kappa - 1) \gamma^{\frac{\kappa-2}{\kappa}}.$$

Then the statement of the Theorem 16 can be reformulated for $X_{m \text{ sup}}$ with $\delta = \sum_{j=1}^m \delta_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= E \exp \{ \zeta_1 (X_{m \text{ sup}}(t_1) - c_X) + \zeta_2 (X_{m \text{ sup}}(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} E \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 (X_{m \text{ sup}}(t_2)) \}, \end{aligned}$$

where

$$\log E \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2) \} = \sum_{j=1}^m \log E \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \},$$

and $\log E \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \}$, $j = 1, \dots, m$ are given by (57) with Lévy measure $\tilde{\nu}$ having density (108).

Moreover, one can construct log-tempered stable scenarios for a more general class of infinite superpositions of stationary normal tempered stable OU-type processes:

$$X_{\text{sup}}(t) = \sum_{j=1}^{\infty} X_j(t), t \in [0, 1], \quad (112)$$

where $X_j(t)$, $j = 1, \dots, m$, are independent stationary processes with correlation functions $\exp\{-\lambda_j |t|\}$, and the marginals $X_j(t) \sim X_j(t) \sim NTS(\kappa, \gamma, \beta, \mu_j, \delta_j)$, $j = 1, \dots, m$, where parameters $\delta_j = j^{-(1+2(1-H))}$, $\mu_j = j^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, and $\lambda_j = \lambda/j$.

Then $X_{\text{sup}}(t)$, $t \in [0, 1]$ has the marginal distribution $NTS(\kappa, \gamma, \beta, \sum_{j=1}^{\infty} \mu_j, \sum_{j=1}^{\infty} \delta_j)$, an expectation

$$\mathbb{E}X_{\text{sup}}(t) = 2\kappa\beta\gamma \sum_{j=1}^{\infty} \delta_j + \sum_{j=1}^{\infty} \mu_j,$$

and the long-range dependent covariance function

$$R_{\text{sup}}(t) = \left[2\kappa\gamma^{\frac{\kappa-1}{\kappa}} - 4\kappa\beta^2(\kappa-1)\gamma^{\frac{\kappa-2}{\kappa}} \right] \sum_{j=1}^{\infty} \delta_j \exp\{-\lambda_j |t|\} = \frac{L_4(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1,$$

where L_4 is a slowly varying at infinity function, bounded on every bounded interval. Then the statement of Theorem 16 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j$, $\mu = \sum_{j=1}^{\infty} \mu_j$ and

$$\begin{aligned} M(\zeta_1, \zeta_2; t_1 - t_2) &= \mathbb{E} \exp\{\zeta_1 (X_{\text{sup}}(t_1) - c_X) + \zeta_2 (X_{\text{sup}}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 (X_{\text{sup}}(t_2))\}, \end{aligned}$$

where

$$\log \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\} = \sum_{j=1}^{\infty} \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, \dots, m, \dots$ are given by (56) with Lévy measure $\tilde{\nu}$ having density (108).

10. Log-gamma scenario

The log-gamma distribution is well-known in the theory of turbulence and multiplicative cascades (Saito 1992). In this section, we propose a stationary version of the log-gamma scenario. We will use a stationary OU type process (46) with marginal gamma distribution $\Gamma(\beta, \alpha)$, which is self-decomposable, and hence infinitely divisible. The probability density function (pdf) of $X(t)$, $t \in \mathbb{R}_+$, is given by

$$\pi(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \mathbf{1}_{[0, \infty)}(x), \alpha > 0, \beta > 0, \quad (113)$$

with the Lévy triplet of the form $(0, 0, \nu)$, where

$$\nu(du) = \frac{\beta e^{-\alpha u}}{u} \mathbf{1}_{[0, \infty)}(u) du,$$

while the BDLP $Z(t)$ in (46) is a compound Poisson subordinator

$$Z(t) = \sum_{n=1}^{P(t)} Z_n,$$

Z_n , $n = 1, 2, \dots$, being independent copies of the random variable $\Gamma(1, \alpha)$, and $P(t)$, $t \geq 0$, being a homogeneous Poisson process with intensity β . The logarithm of the characteristic function of $Z(1)$ is

$$\kappa(z) = \log \mathbb{E} e^{izZ(1)} = \frac{i\beta z}{\alpha - iz}, z \in \mathbb{R},$$

and the (finite) Lévy measure $\tilde{\nu}$ of $Z(1)$ is

$$\tilde{\nu}(du) = \alpha\beta e^{-\alpha u} \mathbf{1}_{(0, \infty)}(u) du. \quad (114)$$

The correlation function is then

$$r_X(t) = \exp(-\lambda|t|).$$

\mathbf{B}''' . Consider a mother process of the form

$$\Lambda(t) = \exp(X(t) - c_X), \quad c_X = \log \frac{1}{\left(1 - \frac{1}{\alpha}\right)^\beta}, \alpha > 1,$$

where $X(t), t \in \mathbb{R}_+$ is a stationary gamma OU type stochastic process with marginal density (113) and covariance function

$$R_X(t) = \frac{\beta}{\alpha^2} \exp(-\lambda|t|).$$

Under condition \mathbf{B}' , we obtain the following moment generating function:

$$M(\zeta) = \mathbb{E} \exp(\zeta(X(t) - c_X)) = \frac{e^{-c_X \zeta}}{\left(1 - \frac{\zeta}{\alpha}\right)^\beta}, \quad \zeta < \alpha, \alpha > 1, \quad (115)$$

and the bivariate moment generating function is given by the formula (57), in which the measure $\tilde{\nu}$ is given by (114), since

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp(\zeta_1(X(t_1) - c_X) + \zeta_2(X(t_2) - c_X)) \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp(\zeta_1 X(t_1) + \zeta_2 X(t_2)) \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \exp\left(\int_{\mathbb{R}} \frac{\beta \sum_{j=1}^2 \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0, \infty)}(t_j - s)}{\alpha - \sum_{j=1}^2 \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0, \infty)}(t_j - s)} ds\right), \end{aligned} \quad (116)$$

or

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \exp(-c_X(\zeta_1 + \zeta_2)) \\ &\times \exp\left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left(\exp\left(u \sum_{j=1}^2 \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0, \infty)}(t_j - s)\right) - 1\right. \right. \\ &\left. \left. - u \left(\sum_{j=1}^2 \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0, \infty)}(t_j - s)\right) \mathbf{1}_{[-1, 1]}(u)\right) \alpha \beta e^{-\alpha u} \mathbf{1}_{(0, \infty)}(u) dud s\right). \end{aligned} \quad (117)$$

Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1, 1; \tau) - 1}{M(2) - 1},$$

where $M(2)$ is given by (115) and $M(1, 1; \tau)$ is given by (117). It turns out that, in this case,

$$\log_b \mathbb{E} \Lambda(t)^q = \frac{1}{\log b} \left(-q \log \frac{1}{\left(1 - \frac{1}{\alpha}\right)^\beta} - \beta \log \left(1 - \frac{q}{\alpha}\right) \right),$$

and the condition (60) of Theorem 11 holds, since

$$\int_{|u| \geq 1} u e^{q^* u} \nu(du) = \frac{\alpha^\beta \beta}{\Gamma(\beta)} \int_1^\infty e^{q^* u} e^{-\alpha u} du < \infty, \quad q^* < \alpha.$$

We formulate the following

Theorem 17. *Suppose that condition \mathbf{B}''' holds and let $Q = \{q : 0 < q < q^* \leq \alpha, \alpha > 2, \beta > 0\}$, where q^* is a fixed integer. Then, for any*

$$b > \left[\frac{\left(1 - \frac{1}{\alpha}\right)^{\beta q^*}}{\left(1 - \frac{q^*}{\alpha}\right)^\beta} \right]^{\frac{1}{q^* - 1}},$$

the stochastic processes $A_n(t)$ defined by (3) for the mother process as in condition \mathbf{B}''' converge in \mathcal{L}_q to the stochastic process $A(t)$ as $n \rightarrow \infty$, such that $A(t) \in \mathcal{L}_q$ and

$$\mathbb{E}A(t)^q \sim t^{T(q)+1},$$

where the Rényi function $T(q)$ is given by

$$T(q) = q \left(1 + \frac{1}{\log b} \log \frac{1}{\left(1 - \frac{1}{\alpha}\right)^\beta} \right) + \frac{\beta}{\log b} \log \left(1 - \frac{q}{\alpha} \right) - 1, q \in Q.$$

Moreover,

$$\text{Var}A(t) \geq \int_0^t \int_0^t (M(1, 1; u - w) - 1) dudw,$$

where M is given by (116) or (117).

Proof. Theorem 17 follows from the Theorem 11 □

Note that the Theorem 17 is an extension of the log-gamma scenario in the Theorem 3 of Anh, Leonenko and Shieh (2008) which is obtained for the set

$$Q = \{q : 0 < q < \alpha, \alpha \geq 2, \beta > 0\} \cap [1, 2].$$

We can construct log-gamma scenarios for a more general class of finite superpositions of stationary gamma OU type processes defined in (72), where $X_j(t), j = 1, \dots, m$, are independent OU type gamma stationary processes with marginals $\Gamma(\beta_j, \alpha), j = 1, \dots, m$, and parameters $\lambda_j, j = 1, \dots, m$. Then $X_{m \text{ sup}}(t), t \in \mathbb{R}_+$ has the marginal distribution $\Gamma(\sum_{j=1}^m \beta_j, \alpha)$ and covariance function

$$R_{m \text{ sup}}(t) = \frac{1}{\alpha^2} \sum_{j=1}^m \beta_j \exp(-\lambda_j |t|), .$$

Theorem 17 can be reformulated for the process of superposition $X_{m \text{ sup}}$ with $\beta = \sum_{j=1}^m \beta_j$ and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp \{ \zeta_1 (X_{m \text{ sup}}(t_1) - c_X) + \zeta_2 (X_{m \text{ sup}}(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 (X_{m \text{ sup}}(t_2)) \}, \end{aligned}$$

where

$$\log \mathbb{E} \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2) \} = \sum_{j=1}^m \log \mathbb{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \},$$

and $\log \mathbb{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \}, j = 1, \dots, m$ are given by (116) or (117).

Moreover, one can construct log-tempered stable scenarios for a more general class of infinite superpositions of stationary gamma OU type processes:

$$X_{\text{sup}}(t) = \sum_{j=1}^{\infty} X_j(t), t \in [0, 1],$$

where $X_j(t), j = 1, \dots, m, \dots$ are independent stationary processes with correlation functions $\exp \{ -\lambda_j |t| \}$, and the marginals $X_j(t) \sim \Gamma(\beta_j, \alpha), j = 1, \dots, m, \dots$ where parameters $\beta_j = j^{-(1+2(1-H))}, \frac{1}{2} < H < 1$, and $\lambda_j = \lambda/j$.

Then $X_{\text{sup}}(t), t \in [0, 1]$ has the marginal distribution $\Gamma(\sum_{j=1}^{\infty} \beta_j, \alpha)$, an expectation

$$\mathbb{E}X_{\text{sup}}(t) = \frac{1}{\alpha} \sum_{j=1}^{\infty} \beta_j$$

and the long-range dependent covariance function

$$R_{\text{sup}}(t) = \frac{1}{\alpha^2} \sum_{j=1}^{\infty} \beta_j \exp \{ -\lambda_j |t| \} = \frac{L_5(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1,$$

where L_5 is a slowly varying at infinity function, bounded on every bounded interval. Then the statement of the Theorem 17 remains true with $\beta = \sum_{j=1}^{\infty} \beta_j$ and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp \{ \zeta_1 (X_{\text{sup}}(t_1) - c_X) + \zeta_2 (X_{\text{sup}}(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp \{ \zeta_1 X_{\text{sup}}(t_1) + \zeta_2 (X_{\text{sup}}(t_2)) \}, \end{aligned}$$

where

$$\log \mathbb{E} \exp \{ \zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2) \} = \sum_{j=1}^{\infty} \log \mathbb{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \},$$

and $\log \mathbb{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \}, j = 1, \dots, m$ are given by (116) or (117).

11. Log-variance gamma scenario

The next example of a hyperbolic OU process is based on the variance-gamma distribution (see, for example, Madan *et al.* 1998, Finlay and Seneta 2006, Carr *et al.* 2007). We will use a stationary OU type process (46) with marginal variance gamma distribution $VG(\lambda, \alpha, \beta, \mu)$, which is self-decomposable, and hence infinitely divisible. The pdf of $X(t), t \in \mathbb{R}_+$ is

$$\pi(x) = \frac{\gamma^{2\kappa}}{\sqrt{\pi} \Gamma(\kappa) (2\alpha)^{\kappa-1/2}} |x - \mu|^{\kappa-1/2} K_{\kappa-1/2}(\alpha |x - \mu|) e^{\beta(x-\mu)}, \quad x \in \mathbb{R}, \quad (118)$$

where $K_\lambda(z)$ is defined by (96) and the set of parameters is

$$\gamma^2 = \alpha^2 - \beta^2, \kappa > 0, \alpha > |\beta| > 0, \mu \in \mathbb{R}.$$

Note that

$$\mathbb{E}X(t) = \mu + \frac{2\beta\kappa}{\gamma^2}, \quad \text{Var}X(t) = \frac{2\kappa}{\gamma^2} \left(1 + 2 \left(\frac{\beta}{\gamma} \right)^2 \right),$$

and $VG(\kappa, \alpha, \beta, \mu)$ has semi-heavy tails. The moment generating function of $VG(\kappa, \alpha, \beta, \mu)$ is given by

$$\log \mathbb{E} e^{\zeta X(t)} = \mu\zeta + 2\kappa \log \left(\gamma / \sqrt{\alpha^2 - (\beta + \zeta)^2} \right), \quad |\beta + \zeta| < \alpha.$$

Thus, if $X_j(t), j = 1, \dots, m$, are independent so that $X_j \sim VG(\kappa_j, \alpha, \beta, \mu_j), j = 1, \dots, m$, then we have that

$$X_1(t) + \dots + X_m(t) \sim VG(\kappa_1 + \dots + \kappa_m, \alpha, \beta, \mu_1 + \dots + \mu_m).$$

The Lévy measure ν of $X(t)$ has density

$$p(u) = \frac{\kappa}{|u|} e^{\beta u - \alpha|u|}, \quad u \in \mathbb{R}. \quad (119)$$

By (54) the Lévy measure $\tilde{\nu}$ of the BDLP $Z(t)$ in (46) has density

$$q(u) = -p(u) - up'(u), \quad (120)$$

$$p'(u) = \begin{cases} -\frac{\kappa}{u} e^{u(\beta+\alpha)}(\beta+\alpha) + \frac{\kappa}{u^2} e^{u(\beta+\alpha)}, & u < 0, \\ \frac{\kappa}{u} e^{u(\beta-\alpha)}(\beta-\alpha) - \frac{\kappa}{u^2} e^{u(\beta-\alpha)}, & u > 0. \end{cases}$$

The correlation function of the stationary process with marginal density (118) is then

$$r_X(t) = \exp(-\lambda|t|).$$

B''''. Consider a mother process of the form

$$\Lambda(t) = \exp(X(t) - c_X), \quad c_X = \mu + 2\kappa \log \left(\gamma / \sqrt{\alpha^2 - (\beta + 1)^2} \right), \quad |\beta + 1| < \alpha,$$

where $X(t)$, $t \in \mathbb{R}_+$ is a stationary $VG(\kappa, \alpha, \beta, \mu)$ OU type process with marginal density (118) and covariance function

$$R_X(t) = \frac{2\kappa}{\gamma^2} \left(1 + 2 \left(\frac{\beta}{\gamma} \right)^2 \right) \exp(-\lambda|t|).$$

Under condition \mathbf{B}'''' , we obtain the moment generating function

$$M(\zeta) = \mathbb{E} \exp(\zeta(X(t) - c_X)) = e^{-c_X \zeta} e^{\mu \zeta + 2\kappa \log(\gamma / \sqrt{\alpha^2 - (\beta + \zeta)^2})}, \quad |\beta + \zeta| < \alpha, \quad (121)$$

and the bivariate moment generating function

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp(\zeta_1(X(t_1) - c_X) + \zeta_2(X(t_2) - c_X)) \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp(\zeta_1 X(t_1) + \zeta_2(X(t_2))), \end{aligned} \quad (122)$$

where $\mathbb{E} \exp(\zeta_1 X(t_1) + \zeta_2(X(t_2)))$ is given by (57) with Lévy measure $\tilde{\nu}$ having density (120). Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1, 1; \tau) - 1}{M(2) - 1},$$

where $M(2)$ is given by (121) and $M(1, 1; \tau)$ is given by (122).

The condition (60) of Theorem 11 holds for $q < \alpha - |\beta|$.

Theorem 18. *Suppose that condition \mathbf{B}'''' holds and let*

$$Q = \{q : 0 < q < q^* \leq |\alpha| - |\beta|, |\beta + 1| < \alpha, \kappa > 0\},$$

where q^* is a fixed integer.

Then, for any

$$b > \exp \left\{ 2\kappa \left[\frac{1}{1 - q^*} \log \frac{\gamma}{\sqrt{\alpha^2 - (\beta + q^*)^2}} \right] - \frac{q^*}{1 - q^*} \log \frac{\gamma}{\sqrt{\alpha^2 - (\beta + 1)^2}} \right\},$$

the stochastic processes $A_n(t)$ defined by (3) for the mother process as in condition \mathbf{B}'''' converge in \mathcal{L}_q to the stochastic process $A(t)$ as $n \rightarrow \infty$ such that, if $A(1) \in \mathcal{L}_q$ for $q \in Q$,

$$\mathbb{E} A(t)^q \sim t^{T(q)+1},$$

where the Rényi function is given by

$$T(q) = q \left(1 + \frac{2\kappa}{\log b} \log \frac{\gamma}{\sqrt{\alpha^2 - (\beta + 1)^2}} \right) + \frac{2\kappa}{\log b} \log \sqrt{\alpha^2 - (\beta + q)^2} - \frac{2\kappa}{\log b} \log \gamma - 1.$$

Moreover,

$$\text{Var} A(t) \geq \int_0^t \int_0^t (M(1, 1; u - w) - 1) dudw,$$

where M is given by (122).

Proof. Theorem 18 follows from the Theorem 11. \square

We can construct log-variance gamma scenarios for a more general class of finite superpositions of stationary variance gamma OU type processes of the form (72), where $X_j(t)$, $j = 1, \dots, m$, are independent variance gamma stationary processes with marginals

$X_j \sim VG(\kappa_j, \alpha, \beta, \mu_j)$, $j = 1, \dots, m$, and parameters λ_j , $j = 1, \dots, m$. Then $X_{m \text{ sup}}(t)$, $t \in \mathbb{R}_+$ has the marginal distribution $VG(\kappa_1 + \dots + \kappa_m, \alpha, \beta, \delta, \mu_1 + \dots + \mu_m)$ and covariance function

$$R_{m \text{ sup}}(t) = \frac{2}{\gamma^2} \left(1 + 2 \left(\frac{\beta}{\gamma} \right)^2 \right) \sum_{j=1}^m \kappa_j \exp(-\lambda_j |t|), \quad t \in \mathbb{R}.$$

The generalization of Theorem 18 remains true for this situation with $\lambda = \sum_{j=1}^m \lambda_j$, $\mu = \sum_{j=1}^m \mu_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbf{E} \exp \{ \zeta_1 (X_{m \sup}(t_1) - c_X) + \zeta_2 (X_{m \sup}(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbf{E} \exp \{ \zeta_1 X_{m \sup}(t_1) + \zeta_2 (X_{m \sup}(t_2)) \}, \end{aligned}$$

where

$$\log \mathbf{E} \exp \{ \zeta_1 X_{m \sup}(t_1) + \zeta_2 X_{m \sup}(t_2) \} = \sum_{j=1}^m \log \mathbf{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \},$$

and $\log \mathbf{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \}$, $j = 1, \dots, m$ are given by (122).

Moreover, one can construct log-variance gamma scenarios for a more general class of infinite superpositions of stationary gamma OU type processes:

$$X_{\sup}(t) = \sum_{j=1}^{\infty} X_j(t), t \in [0, 1],$$

where $X_j(t)$, $j = 1, \dots, m, \dots$ are independent stationary processes with correlation functions $\exp \{ -\lambda_j |t| \}$, and the marginals $X_j(t) \sim VG(\kappa_j, \alpha, \beta, \mu_j)$, $j = 1, \dots, m, \dots$ where parameters $\kappa_j = j^{-(1+2(1-H))}$, $\mu_j = j^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, and $\lambda_j = \lambda/j$.

Then $X_{\sup}(t)$, $t \in [0, 1]$ has the marginal distribution $VG(\sum_{j=1}^{\infty} \kappa_j, \alpha, \beta, \sum_{j=1}^{\infty} \mu_j)$, an expectation

$$\mathbf{E} X_{\sup}(t) = \sum_{j=1}^{\infty} \mu_j + \frac{2\beta}{\gamma^2} \sum_{j=1}^{\infty} \kappa_j$$

and the long-range dependent covariance function

$$R_{\sup}(t) = \frac{2}{\gamma^2} \left(1 + 2 \left(\frac{\beta}{\gamma} \right)^2 \right) \sum_{j=1}^{\infty} \kappa_j \exp \{ -\lambda_j |t| \} = \frac{L_6(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1,$$

where L_6 is a slowly varying at infinity function, bounded on every bounded interval. Then the statement of the Theorem 15 remains true with $\mu = \sum_{j=1}^{\infty} \mu_j$, $\kappa = \sum_{j=1}^{\infty} \kappa_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbf{E} \exp \{ \zeta_1 (X_{\sup}(t_1) - c_X) + \zeta_2 (X_{\sup}(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbf{E} \exp \{ \zeta_1 X_{\sup}(t_1) + \zeta_2 (X_{\sup}(t_2)) \}, \end{aligned}$$

where

$$\log \mathbf{E} \exp \{ \zeta_1 X_{\sup}(t_1) + \zeta_2 X_{\sup}(t_2) \} = \sum_{j=1}^{\infty} \log \mathbf{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \},$$

and $\log \mathbf{E} \exp \{ \zeta_1 X_j(t_1) + \zeta_2 X_j(t_2) \}$, $j = 1, \dots, \dots$ are given by (122).

12. Log-Euler's gamma multifractal scenario

This section presents a new scenario which is based on Euler's gamma distribution (see, for example, Grigelionis 2003). We consider the random variable Y with the gamma distribution $\Gamma(\beta, \alpha)$ having pdf (113) and the random variable

$$X_\gamma = \gamma \log Y, \gamma \neq 0,$$

which has the pdf

$$\pi(x) = \frac{\alpha^\beta}{|\gamma| \Gamma(\beta)} \exp \left\{ \frac{\beta x}{\gamma} - \alpha e^{\frac{x}{\gamma}} \right\}, \quad x \in \mathbb{R}, \quad (123)$$

where the parameters satisfy

$$\alpha > 0, \beta > 0, \gamma \neq 0.$$

The characteristic function of random variable X with pdf (123) is

$$\mathbb{E}e^{izX_\gamma} = \frac{\Gamma(\beta + i\gamma z)}{\Gamma(\beta) \alpha^{i\gamma z}}, z \in \mathbb{R}.$$

Grigelionis (2003) proved that for

$$\delta > 0, \alpha > 0, \beta > 0, \gamma \neq 0, \tag{124}$$

the function

$$\mathbb{E}e^{izX} = \left(\frac{\Gamma(\beta + i\gamma z)}{\Gamma(\beta) \alpha^{i\gamma z}} \right)^\delta, z \in \mathbb{R} \tag{125}$$

is a self-decomposable characteristic function. We denote the distribution of the random variable X by $\Gamma(\gamma, \alpha, \beta, \delta)$.

We note that $\Gamma(\gamma, e^{-\frac{\theta}{\gamma}}, 1, 1), \theta \in \mathbb{R}$, is the Gumbel distribution with location parameter θ and scale parameter $|\gamma|$, since with

$$\Lambda(x) = \exp\{-e^{-x}\}, \bar{\Lambda}(x) = 1 - \Lambda(-x), x \in \mathbb{R},$$

$$P\{X \leq x\} = \begin{cases} \Lambda\left(\frac{x-\theta}{|\gamma|}\right), & \gamma < 0, \\ \bar{\Lambda}\left(\frac{x-\theta}{\gamma}\right), & \gamma > 0 \end{cases}, x \in \mathbb{R}.$$

We will use a stationary OU-type process (46) with marginal distribution $\Gamma(\gamma, \alpha, \beta, \delta)$, which is self-decomposable, and hence infinitely divisible. It means that the characteristic function of $X(t), t \in \mathbf{R}^+$ is of the form (125) under the set of parameters (124). Note that, for $\beta > 0$, we have

$$\begin{aligned} \Gamma(\beta + iz) = \Gamma(\beta) \exp \left\{ iz \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-\beta x}}{1 - e^{-x}} 1_{\{0 \leq x \leq 1\}} \right) dx \right. \\ \left. + \int_{-\infty}^0 (e^{izx} - 1 - izx 1_{\{-1 \leq x < 0\}}) \frac{e^{\beta x}}{|x|(1 - e^x)} dx \right\}, \end{aligned}$$

and thus the distribution corresponding to the characteristic function (125) has the Lévy triplet $(\delta a, 0, \nu)$, where

$$a = \gamma \int_0^{\frac{1}{|\gamma|}} \left(\frac{e^{-x}}{x} - \frac{e^{-\beta x}}{1 - e^{-x}} \right) dx + \gamma \int_{\frac{1}{|\gamma|}}^\infty \frac{e^{-x}}{x} dx - \gamma \log \alpha,$$

and

$$\nu(du) = \delta b(u) du, \tag{126}$$

$$b(u) = \begin{cases} \frac{e^{\frac{\beta}{\gamma} u}}{|u| \left(1 - e^{\frac{1}{\gamma} u} \right)}, & u < 0, \gamma > 0, \\ \frac{e^{\frac{\beta}{\gamma} u}}{u \left(1 - e^{\frac{1}{\gamma} u} \right)}, & u > 0, \gamma < 0. \end{cases}$$

Thus, if $X_j(t), j = 1, \dots, m$, are independent so that $X_j(t) \sim \Gamma(\gamma, \alpha, \beta, \delta_j), j = 1, \dots, m$, then we have that

$$X_1(t) + \dots + X_m(t) \sim \Gamma(\gamma, \alpha, \beta, \delta_1 + \dots + \delta_m)$$

and if $X_j(t), j = 1, \dots, m$, are independent so that $X_j(t) \sim \Gamma(\gamma, \alpha_j, \beta, \delta), j = 1, \dots, m$, then

$$X_1(t) + \dots + X_m(t) \sim \Gamma(\gamma, \prod_{j=1}^m \alpha_j, \beta, \delta).$$

The BDLP $Z(t)$ in (52) has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, where

$$\tilde{a} = \gamma\lambda\delta\frac{d}{d\beta}\frac{\Gamma(\beta)}{\Gamma(\beta)} + \gamma\lambda\delta\log\alpha - \lambda\delta\int_{|x|>1}x\omega(x)dx,$$

with the density of $\tilde{\nu}$ given by

$$\begin{aligned} \tilde{\nu}(du) &= \lambda\delta\omega(u)du, \\ \omega(u) &= \begin{cases} \frac{\beta}{\gamma}e^{\frac{\beta}{\gamma}u}\left(1 - e^{\frac{1}{\gamma}u} + \frac{1}{\beta}e^{\frac{1}{\gamma}u}\right)\frac{1}{\left(1 - e^{\frac{1}{\gamma}u}\right)^2}, & \gamma > 0, u < 0, \\ \frac{\beta}{|\gamma|}e^{\frac{\beta}{\gamma}u}\left(1 - e^{\frac{1}{\gamma}u} + \frac{1}{\beta}e^{\frac{1}{\gamma}u}\right)\frac{1}{\left(1 - e^{\frac{1}{\gamma}u}\right)^2}, & \gamma < 0, u > 0. \end{cases} \end{aligned} \quad (127)$$

The correlation function of the stationary process $X(t)$ then takes the form

$$r_X(t) = \exp\{-\lambda|t|\}.$$

Note that

$$\mathbb{E}X(t) = \gamma\delta\frac{d}{d\beta}\frac{\Gamma(\beta)}{\Gamma(\beta)} - \gamma\delta\log\alpha, \quad \text{Var}X(t) = \delta\gamma^2\int_0^\infty\frac{xe^{-\beta x}}{1 - e^{-x}}dx.$$

B^V. Consider a mother process of the form

$$\Lambda(t) = \exp\{X(t) - c_X\}, \quad t \in \mathbf{R}^+$$

with

$$c_X = \delta\log\frac{\Gamma(\beta + \gamma)}{\Gamma(\beta)e^\gamma}, \quad \beta > 0, \gamma < 0, \beta > -\gamma.$$

where $X(t), t \in [0, 1]$ is a stationary $\Gamma(\gamma, \alpha, \beta, \delta)$ OU-type stochastic process with covariance function

$$R_X(t) = \text{Var}X(t)\exp\{-\lambda|t|\}.$$

The logarithm of the moment generating function of $\Gamma(\gamma, \alpha, \beta, \delta)$ is

$$\log\mathbb{E}e^{\zeta X(t)} = \delta\log\frac{\Gamma(\beta + \gamma\zeta)}{\Gamma(\beta)e^{\gamma\zeta}}, \quad 0 < \zeta < -\frac{\beta}{\gamma}, \beta > 0, \gamma < 0.$$

Under condition **B^V**, we obtain the following moment generating function

$$M(\zeta) = \mathbb{E}\exp\{\zeta(X(t) - c_X)\} = e^{-c_X\zeta}e^{M(\zeta)}, \quad 0 < \zeta < -\frac{\beta}{\gamma}, \quad (128)$$

and bivariate moment generating function

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E}\exp\{\zeta_1(X(t_1) - c_X) + \zeta_2(X(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1 + \zeta_2)}\mathbb{E}\exp\{\zeta_1X(t_1) + \zeta_2(X(t_2))\}, \end{aligned} \quad (129)$$

where $\mathbb{E}\exp\{\zeta_1X(t_1) + \zeta_2(X(t_2))\}$ is given by (57) with Lévy measure $\tilde{\nu}$ having density (127). Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M_\theta(1, 1; \tau) - 1}{M_\theta(2) - 1}, \quad (130)$$

where $M_\theta(2)$ is given by (128) and $M_\theta(1, 1; \tau)$ is given by (129).

Theorem 19. *Suppose that condition \mathbf{B}^V holds and*

$$q \in Q = \left\{ q : 0 < q < q^* \leq -\frac{\beta}{\gamma}, \beta > 0, \gamma < 0, \beta > -\gamma, \delta > 0, \alpha > 0 \right\}.$$

where q^* is a fixed integer. Then, for any

$$b > \left[\frac{\Gamma^\delta(\beta + \gamma q^*) \Gamma^{q^*}(\beta)}{\Gamma(\beta) \Gamma^{q^*}(\beta + \gamma)} \right]^{\frac{1}{q^* - 1}}$$

the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^n \Lambda^{(j)}(sb^j) ds, t \in [0, 1]$$

converge in \mathcal{L}_q to the stochastic process $A(t)$, $t \in [0, 1]$ as $n \rightarrow \infty$ such that $A(1) \in \mathcal{L}_q$, and $q \in Q$,

$$\mathbb{E}A^q(t) \sim t^{T(q)+1},$$

where the Rényi function is given by

$$T(q) = q \left(1 + \frac{\delta}{\log b} \log \Gamma(\beta + \gamma) - \frac{\delta}{\log b} \log \Gamma(\beta) \right) - \frac{\delta}{\log b} \log \Gamma(\beta + q\gamma) - 1, q \in Q.$$

Moreover,

$$\text{Var}A(t) \geq \int_0^t \int_0^t [M_\theta(1, 1; u - w) - 1] dudw,$$

where M_θ is given by (129).

Proof. Theorem 19 follows from the Theorem 11 □

We can construct $\log\text{-}\Gamma(\gamma, \alpha, \beta, \delta)$ scenario for a more general class of finite superpositions of stationary OU-type processes:

$$X_{m \text{ sup}}(t) = \sum_{j=1}^m X_j(t), t \in [0, 1],$$

where $X_j(t)$, $j = 1, \dots, m$, are independent stationary processes with marginals $X_j(t) \sim \Gamma(\gamma, \alpha, \beta, \delta_j)$, $j = 1, \dots, m$, and parameters δ_j , $j = 1, \dots, m$. Then $X_{m \text{ sup}}(t)$ has the marginal distribution $\Gamma(\gamma, \alpha, \beta, \sum_{j=1}^m \delta_j)$, covariance function

$$R_{\text{sup}}(t) = \left[\gamma^2 \int_0^\infty x \frac{e^{-\beta x}}{1 - e^{-x}} dx \right] \sum_{j=1}^m \delta_j \exp\{-\lambda_j |t|\}.$$

and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp\{\zeta_1(X_{m \text{ sup}}(t_1) - c_X) + \zeta_2(X_{m \text{ sup}}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp\{\zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2(X_{m \text{ sup}}(t_2))\}, \end{aligned}$$

where

$$\log \mathbb{E} \exp\{\zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2)\} = \sum_{j=1}^m \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, \dots, m$ are given by (129).

Moreover, one can construct $\log\text{-}\Gamma(\gamma, \alpha, \beta, \delta)$ scenarios for a more general class of infinite superpositions of stationary gamma OU type processes:

$$X_{\text{sup}}(t) = \sum_{j=1}^{\infty} X_j(t), \tag{131}$$

where $X_j(t), j = 1, \dots, m, \dots$ are independent stationary processes with correlation functions $\exp\{-\lambda_j |t|\}$, and the marginals $X_j(t) \sim \Gamma(\gamma, \alpha, \beta, \delta_j), j = 1, \dots, m, \dots$ where parameters $\delta_j = j^{-(1+2(1-H))}, \frac{1}{2} < H < 1$, and $\lambda_j = \lambda/j$.

Then $X_{\text{sup}}(t), t \in [0, 1]$ has the marginal distribution $\Gamma(\gamma, \alpha, \beta, \sum_{j=1}^{\infty} \delta_j)$, variance $\text{Var}X(t) = \delta\gamma^2 \int_0^{\infty} \frac{xe^{-\beta x}}{1-e^{-x}} dx$, expectation

$$\mathbb{E}X(t) = \left(\gamma \frac{\frac{d}{d\beta}\Gamma(\beta)}{\Gamma(\beta)} - \gamma \log \alpha\right) \sum_{j=1}^{\infty} \delta_j,$$

and the long-range dependent covariance function

$$R_{\text{sup}}(t) = \gamma^2 \int_0^{\infty} \frac{xe^{-\beta x}}{1-e^{-x}} dx \sum_{j=1}^{\infty} \delta_j \exp\{-\lambda_j |t|\} = \frac{L_7(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1,$$

where L_7 is a slowly varying at infinity function, bounded on every bounded interval. Then the statement of the Theorem 19 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j$ and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp\{\zeta_1 (X_{\text{sup}}(t_1) - c_X) + \zeta_2 (X_{\text{sup}}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 (X_{\text{sup}}(t_2))\}, \end{aligned}$$

where

$$\log \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\} = \sum_{j=1}^{\infty} \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}, j = 1, \dots, \dots$ are given by (129).

13. Log-z scenario

The next scenario is based on the z-distribution (see, for example, Grigelionis 2001). We consider a pdf of the form

$$\pi(x) = \frac{2\pi \exp\left(\frac{2\pi\beta_1}{\alpha}(x - \mu)\right)}{\alpha B(\beta_1, \beta_2) \left(1 + \exp\left(\frac{2\pi}{\alpha}(x - \mu)\right)\right)^{\beta_1 + \beta_2}}, \quad x \in \mathbb{R}, \quad (132)$$

where the set of parameters is

$$\alpha > 0, \beta_1 > 0, \beta_2 > 0, \mu \in \mathbb{R}$$

(see Prentice 1975, Barndorf-Nielsen *et al.* 1982). The characteristic function of a random variable X with pdf (132) is given by

$$\mathbb{E}e^{izX} = \frac{B(\beta_1 + \frac{i\alpha z}{2\pi}, \beta_2 - \frac{i\alpha z}{2\pi})}{B(\beta_1, \beta_2)} e^{iz\mu}, \quad z \in \mathbb{R}.$$

This distribution has semi-heavy tails and is known to be self-decomposable (Barndorf-Nielsen *et al.* 1982), hence is infinitely divisible. Due to this infinite divisibility of the z-distribution, the following generalization can be suggested.

We will use a stationary OU type process (46) with marginal generalized z-distribution $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$. The characteristic function of $X(t), t \in \mathbb{R}_+$ is then of the form

$$\mathbb{E}e^{izX} = \left(\frac{B(\beta_1 + \frac{i\alpha z}{2\pi}, \beta_2 - \frac{i\alpha z}{2\pi})}{B(\beta_1, \beta_2)}\right)^{2\delta} e^{iz\mu}, \quad z \in \mathbb{R}, \quad (133)$$

where the set of parameters is

$$\alpha > 0, \beta_1 > 0, \beta_2 > 0, \delta > 0, \mu \in \mathbb{R}.$$

This distribution is self-decomposable, hence infinitely divisible, with the Lévy triplet $(a, 0, \nu)$, where

$$a = \frac{\alpha\delta}{\pi} \int_0^{\frac{2\pi}{\alpha}} \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1 - e^{-x}} dx + \mu,$$

and

$$\nu(du) = b(u)du, \quad (134)$$

$$b(u) = \begin{cases} \frac{2\delta e^{-\frac{2\pi\beta_2}{\alpha}u}}{u(1 - e^{-\frac{2\pi}{\alpha}u})}, & u > 0, \\ \frac{2\delta e^{\frac{2\pi\beta_1}{\alpha}u}}{|u|(1 - e^{\frac{2\pi}{\alpha}u})}, & u < 0. \end{cases}$$

Thus, if $X_j(t)$, $j = 1, \dots, m$, are independent so that $X_j(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_j, \mu_j)$, $j = 1, \dots, m$, then we have that

$$X_1(t) + \dots + X_m(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_1 + \dots + \delta_m, \mu_1 + \dots + \mu_m).$$

The BDLP $Z(t)$ has a Lévy triplet $(\tilde{a}, 0, \tilde{\nu})$, where

$$\tilde{a} = \lambda\mu + \frac{\alpha\lambda\delta}{\pi} \int_0^\infty \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1 - e^{-x}} dx - \lambda \int_{|x|>1} x\omega(x)dx,$$

with the density of $\tilde{\nu}$ being given from

$$\tilde{\nu}(du) = \lambda\omega(u)du,$$

$$\omega(u) = \begin{cases} \frac{4\pi\delta}{\alpha} \left(\beta_2 e^{-\frac{2\pi\beta_2}{\alpha}u} \left(1 - e^{-\frac{2\pi}{\alpha}u} \right) + e^{-\frac{2\pi(\beta_2+1)}{\alpha}u} \right) \frac{1}{\left(1 - e^{-\frac{2\pi}{\alpha}u} \right)^2}, & u > 0, \\ \frac{4\pi\delta}{\alpha} \left(\beta_1 e^{\frac{2\pi\beta_1}{\alpha}u} \left(1 - e^{\frac{2\pi}{\alpha}u} \right) + e^{\frac{2\pi(\beta_1+1)}{\alpha}u} \right) \frac{1}{\left(1 - e^{\frac{2\pi}{\alpha}u} \right)^2} & u < 0. \end{cases} \quad (135)$$

The correlation function of the stationary process with marginal density (132) is then

$$r_X(t) = \exp(-\lambda|t|).$$

The pdf of the generalized z-distribution $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$ has semi-heavy tails:

$$\pi(x) \sim C_\pm |x|^{\rho_\pm} e^{-\sigma_\pm|x|}, |x| \rightarrow \pm\infty,$$

where

$$\rho_\pm = 2\delta - 1, \sigma_+ = \frac{2\pi\beta_2}{\alpha}, \sigma_- = \frac{2\pi\beta_1}{\alpha}, C_\pm = \left(\frac{2\pi}{\alpha B(\beta_1, \beta_2)} \right)^{2\delta} \frac{e^{\pm\mu\sigma_\pm}}{\Gamma(2\delta)}.$$

Note that

$$\mathbb{E}X(t) = \frac{\alpha\delta}{\pi} \int_0^\infty \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1 - e^{-x}} dx + \mu, \text{Var}X(t) = \frac{2\alpha^2\delta}{(2\pi)^2} \int_0^\infty x \frac{e^{-\beta_2 x} + e^{-\beta_1 x}}{1 - e^{-x}} dx.$$

In particular, the generalized z-distribution $Z(\alpha, \frac{1}{2} + \frac{\beta}{2\pi}, \frac{1}{2} - \frac{\beta}{2\pi}, \delta, \mu) = M(\alpha, \beta, \delta, \mu)$ is known as the Meixner distribution (Schoutens and Teugels 1998, Grigelionis 1999, Morales and Schoutens 2003). The density function of a Meixner distribution is given by

$$\pi(x) = \frac{\left(2 \cos \frac{\beta}{2} \right)^{2\delta}}{2\pi\alpha\Gamma(2\delta)} \exp\left(\frac{\beta}{\alpha}(x - \mu) \right) \left| \Gamma\left(\delta + i \frac{x - \mu}{\alpha} \right) \right|^2, \quad x \in \mathbb{R},$$

where

$$\alpha > 0, -\pi < \beta < \pi, \delta > 0, \mu \in \mathbb{R}.$$

Note that

$$|\Gamma(x + iy)|^2 \sim \sqrt{2\pi} |y|^{x-1/2} e^{-\pi|y|/2} \text{ as } |y| \rightarrow \infty.$$

This distribution is infinitely divisible and self-decomposable with triplet $(a, 0, \nu)$, where

$$a = \alpha\delta \tan \frac{\beta}{2} - 2\delta \int_1^\infty \frac{\sinh(\beta x/2)}{\sinh(\pi x/2)} dx + \mu$$

and

$$\nu(du) = \frac{\delta \exp\left(\frac{\beta u}{\alpha}\right)}{u \sinh\left(\frac{\pi u}{\alpha}\right)} du.$$

The cumulant function is

$$C\{z; X(t)\} = i\mu z + 2\delta \log \frac{\cos \beta/2}{\cosh((\alpha z - i\beta)/2)}, \quad z \in \mathbb{R}.$$

In particular, the hyperbolic cosine distribution $Z(\alpha, \frac{1}{2}, \frac{1}{2}, 0, \mu) = M(\alpha, 0, \frac{1}{2}, \mu)$ has the pdf

$$\pi(x) = \frac{1}{\alpha \cosh\left(\frac{\pi}{\alpha}(x - \mu)\right)}, \quad x \in \mathbb{R}$$

and characteristic function

$$\mathbb{E}e^{izX(t)} = e^{iz\mu} \frac{1}{\cosh\left(\frac{\alpha z}{2}\right)}, \quad z \in \mathbb{R},$$

while the logistic distribution $Z(\alpha, 1, 1, 0, \mu)$ has the pdf

$$\pi(x) = \frac{2\pi \exp\left(\frac{\pi}{\alpha}(x - \mu)\right)}{\alpha \left(1 + \cosh\left(\frac{\pi}{\alpha}(x - \mu)\right)\right)}, \quad x \in \mathbb{R}$$

and characteristic function

$$\mathbb{E}e^{izX(t)} = e^{iz\mu} \frac{\alpha z}{2 \sinh\left(\frac{\alpha z}{2}\right)}, \quad z \in \mathbb{R}.$$

Another example is the z-distribution $Z(2\pi, \frac{k_1}{2}, \frac{k_2}{2}, 0, \log \frac{k_1}{k_2})$, which is the log F_{k_1, k_2} distribution, where F_{k_1, k_2} is the Fisher distribution (Barndorff-Nielsen *et al.* 1982). Note that the generalized z-distributions and generalized hyperbolic distributions form non-intersecting sets. However, one can show that some Meixner distributions and corresponding Lévy processes can be obtained by subordination, that is, by random time change in the Brownian motion (see, for instance, Morales and Schoutens 2003).

B^{VI}. Consider a mother process of the form

$$\Lambda(t) = \exp(X(t) - c_X),$$

with

$$c_X = 2\delta \left(\log \Gamma\left(\beta_1 + \frac{\alpha}{2\pi}\right) + \log \Gamma\left(\beta_2 - \frac{\alpha}{2\pi}\right) - \log \frac{\Gamma(\beta_1)}{\Gamma(\beta_2)} \right) + \mu,$$

where $X(t)$, $t \in \mathbb{R}_+$ is a stationary $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$ OU-type process with covariance function

$$R_X(t) = (\text{Var}X(t)) \exp(-\lambda|t|).$$

The logarithm of the moment generating function of $Z(\alpha, \beta_1, \beta_2, \delta, \mu)$ is

$$\log \mathbb{E}e^{\zeta X(t)} = 2\delta \left(\log \Gamma\left(\beta_1 + \frac{\alpha\zeta}{2\pi}\right) + \log \Gamma\left(\beta_2 - \frac{\alpha\zeta}{2\pi}\right) - \log \frac{\Gamma(\beta_1)}{\Gamma(\beta_2)} \right) + \mu\zeta, \quad \zeta \in \left(-\frac{2\pi\beta_2}{\alpha}, \frac{2\pi\beta_1}{\alpha}\right).$$

Under condition **B^{VI}**, we obtain the moment generating function

$$M(\zeta) = \mathbb{E} \exp(\zeta(X(t) - c_X)) = e^{-c_X\zeta} e^{K\{\zeta; X(t)\}}, \quad (136)$$

and the bivariate moment generating function

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbf{E} \exp(\zeta_1 (X(t_1) - c_X) + \zeta_2 (X(t_2) - c_X)) \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbf{E} \exp(\zeta_1 X(t_1) + \zeta_2 (X(t_2))), \end{aligned} \quad (137)$$

where $\mathbf{E} \exp(\zeta_1 X(t_1) + \zeta_2 (X(t_2)))$ is given by (56) with Lévy measure $\tilde{\nu}$ having density (135). Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M(1, 1; \tau) - 1}{M(2) - 1}, \quad (138)$$

where $M(2)$ is given by (136) and $M(1, 1; \tau)$ is given by (137).

Theorem 20. *Suppose that condition \mathbf{B}^{VI} holds and let $Q = \left\{ q \in (0, q^*) : 0 < \frac{2\pi\beta_1}{\alpha} < q^* < \frac{2\pi\beta_2}{\alpha}, \beta_1 < \beta_2, \beta_1 + \frac{\alpha}{2\pi} > 0, \beta_2 \right.$*
where q^ is a fixed integer.*

Then, for any

$$b > \left[\frac{\Gamma(\beta_1)/\Gamma(\beta_2)}{\Gamma(\beta_1 + \frac{q^*\alpha}{2\pi})\Gamma(\beta_2 - \frac{q^*\alpha}{2\pi})} \right]^{2\delta},$$

the stochastic processes $A_n(t)$ defined by (3) for the mother process (95) converge in \mathcal{L}_q to the stochastic process $A(t)$ as $n \rightarrow \infty$ such that $A(1) \in \mathcal{L}_q$ for $q \in Q$,

$$\mathbf{E}A(t)^q \sim t^{T(q)+1},$$

where the Rényi function is given by

$$\begin{aligned} T(q) &= q \left(1 + \frac{2\delta \left(\log \Gamma\left(\beta_1 + \frac{\alpha}{2\pi}\right) + \log \Gamma\left(\beta_2 - \frac{\alpha}{2\pi}\right) - \log \frac{\Gamma(\beta_1)}{\Gamma(\beta_2)} \right)}{\log b} \right) \\ &\quad - \frac{2\delta}{\log b} \left(\log \Gamma\left(\beta_1 + \frac{q\alpha}{2\pi}\right) + \log \Gamma\left(\beta_2 - \frac{q\alpha}{2\pi}\right) \right) + \frac{1}{\log b} 2\delta \log \frac{\Gamma(\beta_1)}{\Gamma(\beta_2)} - 1. \end{aligned}$$

Moreover,

$$\text{Var}A(t) \geq \int_0^t \int_0^t (M(1, 1; u - w) - 1) dudw,$$

where M is given by (137).

Proof. Theorem 20 follows from the Theorem 11. □

We can construct log-z scenarios for a more general class of finite superpositions of stationary OU-type processes of the form (72), where $X_j(t), j = 1, \dots, m$, are independent stationary processes with marginals

$X_j(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_j, \mu_j), j = 1, \dots, m$ and parameters $\delta_j, \mu_j, j = 1, \dots, m$. Then $X_{m \text{ sup}}(t), t \in \mathbb{R}_+$ has the marginal distribution $Z(\alpha, \beta_1, \beta_2, \sum_{j=1}^m \delta_j, \sum_{j=1}^m \mu_j)$ and covariance function

$$R_{m \text{ sup}}(t) = \left(\frac{2\alpha^2}{(2\pi)^2} \int_0^\infty x \frac{e^{-\beta_2 x} + e^{-\beta_1 x}}{1 - e^{-x}} dx \right) \sum_{j=1}^m \delta_j \exp(-\lambda_j |t|).$$

The statement of Theorem 20 can be reformulated for the process of superposition $X_{m \text{ sup}}(t)$ with $\delta = \sum_{j=1}^m \delta_j, \mu = \sum_{j=1}^m \mu_j$, and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbf{E} \exp \{ \zeta_1 (X_{m \text{ sup}}(t_1) - c_X) + \zeta_2 (X_{m \text{ sup}}(t_2) - c_X) \} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbf{E} \exp \{ \zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 (X_{m \text{ sup}}(t_2)) \}, \end{aligned}$$

where

$$\log \mathbb{E} \exp\{\zeta_1 X_{m \text{ sup}}(t_1) + \zeta_2 X_{m \text{ sup}}(t_2)\} = \sum_{j=1}^m \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, \dots, m$ are given by (137).

Moreover, one can construct log- Z scenarios for a more general class of infinite superpositions of stationary OU type processes with marginal z -distribution:

$$X_{\text{sup}}(t) = \sum_{j=1}^{\infty} X_j(t), t \in [0, 1],$$

where $X_j(t)$, $j = 1, \dots, m, \dots$ are independent stationary processes with correlation functions $\exp\{-\lambda_j |t|\}$, and the marginals $X_j(t) \sim Z(\alpha, \beta_1, \beta_2, \delta_j, \mu_j)$, $j = 1, \dots, m, \dots$ where parameters $\delta_j = j^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, and $\lambda_j = \lambda/j$.

Then $X_{\text{sup}}(t)$, $t \in [0, 1]$ has the marginal distribution $Z(\alpha, \beta_1, \beta_2, \sum_{j=1}^{\infty} \delta_j, \sum_{j=1}^{\infty} \mu_j)$, an expectation

$$\mathbb{E} X_{\text{sup}}(t) = \left(\frac{\alpha}{\pi} \int_0^{\infty} \frac{e^{-\beta_2 x} - e^{-\beta_1 x}}{1 - e^{-x}} dx \right) \sum_{j=1}^{\infty} \delta_j + \sum_{j=1}^{\infty} \mu_j,$$

and the long-range dependent covariance function

$$R_{\text{sup}}(t) = \left(\frac{2\alpha^2}{(2\pi)^2} \int_0^{\infty} x \frac{e^{-\beta_2 x} + e^{-\beta_1 x}}{1 - e^{-x}} dx \right) \sum_{j=1}^{\infty} \delta_j \exp\{-\lambda_j |t|\} = \frac{L_8(|t|)}{|t|^{2(1-H)}}, \frac{1}{2} < H < 1,$$

where L_8 is a slowly varying at infinity function, bounded on every bounded interval. Then the statement of Theorem 20 remains true with $\delta = \sum_{j=1}^{\infty} \delta_j \cdot \mu = \sum_{j=1}^{\infty} \mu_j$ and

$$\begin{aligned} M(\zeta_1, \zeta_2; (t_1 - t_2)) &= \mathbb{E} \exp\{\zeta_1 (X_{\text{sup}}(t_1) - c_X) + \zeta_2 (X_{\text{sup}}(t_2) - c_X)\} \\ &= e^{-c_X(\zeta_1 + \zeta_2)} \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\}, \end{aligned}$$

where

$$\log \mathbb{E} \exp\{\zeta_1 X_{\text{sup}}(t_1) + \zeta_2 X_{\text{sup}}(t_2)\} = \sum_{j=1}^{\infty} \log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\},$$

and $\log \mathbb{E} \exp\{\zeta_1 X_j(t_1) + \zeta_2 X_j(t_2)\}$, $j = 1, \dots, \dots$ are given by (137).

In principle, it is possible to obtain log-hyperbolic scenarios for which there exist exact forms of Lévy measures of the OU process and the BDLP Lévy process; however some analytical work is still to be carried out. This will be done elsewhere.

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