

# Recovering the isometry type of a Riemannian manifold from local boundary diffraction travel times<sup>☆</sup>

Maarten V. de Hoop<sup>a</sup>, Sean F. Holman<sup>b,\*</sup>, Einar Iversen<sup>c</sup>, Matti Lassas<sup>d</sup>,  
Bjørn Ursin<sup>e</sup>

<sup>a</sup>*Department of Mathematics, Purdue University, 150 N. University Street, West  
Lafayette IN 47907, USA.*

<sup>b</sup>*School of Mathematics, University of Manchester, Manchester, United Kingdom, M13  
9PL*

<sup>c</sup>*NORSAR, Gunnar Randers vei 15, P.O. Box 53, 2027 Kjeller, Norway*

<sup>d</sup>*Department of Mathematics and Statistics, Gustaf Hallstromin katu 2b, FI-00014  
University of Helsinki, Helsinki, Finland.*

<sup>e</sup>*Department of Petroleum Engineering and Applied Geophysics, Norwegian University of  
Science and Technology, S.P. Andersensvei 15A, NO-7491 Trondheim, Norway.*

---

## Abstract

We analyze the inverse problem, if a manifold and a Riemannian metric on it can be reconstructed from the sphere data. The sphere data consist of an open set  $U \subset \widetilde{M}$  and the pairs  $(t, \Sigma)$  where  $\Sigma \subset U$  is a smooth subset of a generalized metric sphere of radius  $t$ . This problem is an idealization of a seismic inverse problem, originally formulated by Dix [8], of reconstructing the wave speed inside a domain from boundary measurements associated with the single scattering of waves. In this problem, one considers a domain  $\widetilde{M}$  with a varying and possibly anisotropic wave speed which we model as a Riemannian metric  $g$ . For our data, we assume that  $\widetilde{M}$  contains a dense set

---

<sup>☆</sup>This research was supported by National Science Foundation grant CMG DMS-1025318, the members of the Geo-Mathematical Imaging Group at Purdue University, the Finnish Centre of Excellence in Inverse Problems Research, Academy of Finland project COE 250215, the Research Council of Norway, and the VISTA project. The research was initialized at the Program on Inverse Problems and Applications at MSRI, Berkeley, during the Fall of 2010.

\*Corresponding author.

*Email addresses:* mdehoop@math.purdue.edu (Maarten V. de Hoop),  
sean.holman@manchester.ac.uk (Sean F. Holman), Einar.Iversen@norsar.com  
(Einar Iversen), matti.lassas@helsinki.fi (Matti Lassas), bjorn.ursin@ntnu.no  
(Bjørn Ursin)

of point diffractors and that in a subset  $U \subset \widetilde{M}$ , we can measure the wave fronts of the waves generated by these. The inverse problem we study is to recover the metric  $g$  in local coordinates anywhere on a set  $M \subset \widetilde{M}$  up to an isometry (i.e. we recover the isometry type of  $M$ ). To do this we show that the shape operators related to wave fronts produced by the point diffractors within  $\widetilde{M}$  satisfy a certain system of differential equations which may be solved along geodesics of the metric. In this way, assuming that we know  $g$  as well as the shape operator of the wave fronts in the region  $U$ , we may recover  $g$  in certain coordinate systems (e.g. Riemannian normal coordinates centered at point diffractors). This generalizes the method of Dix to metrics which may depend on all spatial variables and be anisotropic. In particular, the novelty of this solution lies in the fact that it can be used to reconstruct the metric also in the presence of the caustics.

## Résumé

Nous analysons le problème inverse, si une variété riemannienne peut être reconstruit à partir des données sphère. Les données sphère sont constitués d'un ensemble ouvert  $U \subset \widetilde{M}$  et les paires  $(t, \Sigma)$ , où  $\Sigma \subset U$  set un sous ensemble lisse d'une sphère métrique généralisée. Ce problème est une idéalisation d'un problème sismique inverse, à l'origine formulé par Dix [8], consistant à reconstruire la vitesse d'onde dans un domaine à partir des mesures aux frontières associées à la dispersion simple des ondes sismiques. Nous considérons un domaine  $\widetilde{M}$  avec une vitesse d'onde variable et éventuellement anisotrope que nous modélisons par une métrique Riemannienne  $g$ . Nous supposons que  $\widetilde{M}$  contient une densité élevée de points diffractants et que dans un sous-ensemble  $U \subset \widetilde{M}$ , correspondant à un domaine contenant les instruments de mesure, nous pouvons mesurer les fronts d'onde de la diffusion simple des ondes diffractées depuis les points diffractant. Le problème inverse que nous étudions consiste à reconstruire la métrique  $g$  en coordonnées locales sur l'ensemble  $M \subset \widetilde{M}$  modulo une isométrie (i.e. nous reconstruisons le type d'isométrie). Pour ce faire nous montrons que l'opérateur de forme relatif aux fronts d'onde produits par les points diffractants dans  $M$  satisfait un certain système d'équations différentielles qui peut être résolu le long des géodésiques! de la métrique. De cette manière, en supposant que nous connaissons  $g$  ainsi que l'opérateur de forme des fronts d'onde dans la région  $U$ , nous pouvons retrouver  $g$  dans un certain système de coordonnées

(e.g. coordonnées normales Riemannienne centrées aux points diffractant). Ceci généralise la méthode géophysique de Dix à des métriques qui peuvent dépendre de toutes les variables spatiales et être anisotropes. En particulier, la nouveauté de cette solution est de pouvoir être utilisée pour reconstruire la métrique, même en présence des caustiques.

*Keywords:* geometric inverse problems, Riemannian manifold, shape operator

*2010 MSC:* 35R30, 86A15, 53C21

---

## 1. Introduction: Motivation of the problem

We consider a Riemannian manifold,  $(M, g)$ , of dimension  $n$  with boundary  $\partial M$ . We analyze the inverse problem, originally formulated by Dix [8] in reflection seismology, aimed at reconstructing  $g$  from boundary measurements associated with second-order expansions of diffraction travel times. When the waves produced by a source  $F$  are modeled by the solution of the wave equation  $(\partial_t^2 - \Delta_g)u(x, t) = F(x, t)$  on  $(M, g)$ , the geodesics  $\gamma_{x, \eta}$  on  $M$  correspond to the rays following the propagation of singularities by the parametrix corresponding with the wave operator on  $(M, g)$  and the metric distance  $d(x_1, x_2)$  of the points  $x_1, x_2 \in M$  corresponds to the travel time of the waves from the point  $x_1$  to the point  $x_2$ . The phase velocity in this case is given by  $v(x, \alpha) = [\sum_{j,k=1}^n g^{jk}(x)\alpha_j\alpha_k]^{1/2}$ , with  $\alpha$  denoting the phase or cotangent direction.

Below, we call the sets  $\Sigma_{t,y} = \{\gamma_{y,v}(t); v \in T_y M, \|v\|_g = 1\}$  generalized metric spheres (i.e. the images of the spheres  $\{\xi \in T_y M; \|\xi\|_g = t\}$  in the tangent space of radius  $t$  under the exponential map). We call these sets the generalized spheres in contrast to the metric spheres, that is, the boundaries of metric balls are the sets

$$\partial B(y, t) = \{x \in M; d(x, y) = t\}.$$

Since long geodesics on a Riemannian manifold may not be distance minimizing, we have  $\partial B(y, t) \subset \Sigma_{t,y}$  where the inclusion may not be equality (see Fig. 1).

The mathematical formulation of Dix' problem is then the following: Assume that

1. We are given an open set  $\Gamma \subset \partial M$ , the metric tensor  $g_{jk}|_\Gamma$  on the boundary, and the normal derivatives  $\partial_\nu^p g_{jk}|_\Gamma$  for all  $p \in \mathbb{Z}_+$ , where  $\nu$  is

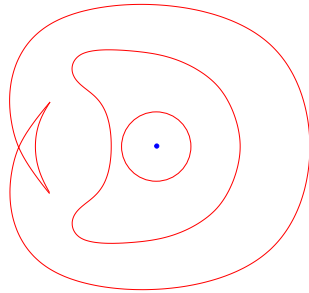


Figure 1: Generalized spheres centered at a point: For small radii the generalized spheres resemble the Euclidean spheres but for large radii cusps corresponding to caustics may appear. For example the outermost contour in the figure illustrates a generalized sphere containing two cusps. If we were to trace from this contour backwards to the center point along the rays we would see that some of the rays cross each other. The sphere data consists of smooth subsurfaces of these generalized spheres.

the normal vector of  $\partial M$  and  $g_{jk}$  is the metric tensor in the boundary normal coordinates.

2. For all  $x \in \Gamma$  and  $t > 0$ , we are given at the point  $x$  the second fundamental form of the generalized metric sphere of  $(M, g)$  having the center  $y_{x,t}$  and radius  $t$ . Here,  $y_{x,t} = \gamma_{x,\nu(x)}(t)$  is the end point of the geodesic that starts from  $x$  in the  $g$ -normal direction  $\nu(x)$  to  $\partial M$  and has the length  $t$ .

Dix' inverse problem is the question if one can use these data to determine uniquely the metric tensor  $g$  on the set  $W \subset M$  that can be connected to  $\Gamma$  with a geodesic that does not intersect with the boundary.

The above problem is the mathematical idealization of an inverse problem encountered in reflection seismology where the goal is to determine the speed of waves in the interior of a body from external measurements. In Dix' inverse problem, we assume that we observe the "curvatures" of wave fronts passing the boundary,  $\partial M$ , generated by interior diffraction points. In this paper we consider the case when the points form an almost dense set in  $M$ . Typically, the measurements are restricted to a subset  $\Gamma \subset \partial M$ . The observed wave fronts coincide with the generalized metric spheres of  $(M, g)$ . The "curvature" corresponds with the second fundamental form (or, equivalently, the shape operator) of the first wave front of the wave generated by the point diffractor at  $y_{x,t}$ , say, that is observed at  $x$  and propagates to the direction  $-\nu(x)$ .

There are different scenarios under which the mentioned data can be

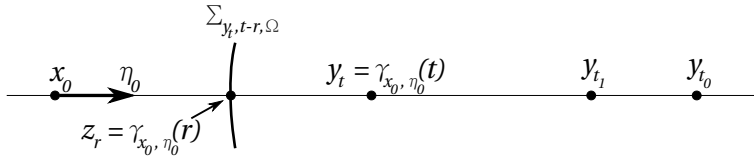


Figure 2: Notation used throughout the paper, following the geodesic  $\gamma$ .

obtained. Using the Boundary Control method, virtual diffraction point data can be formed from the hyperbolic Dirichlet-to-Neumann map (see [7, Theorem 7]). This is the subject of recent and ongoing research [23, 16]. In certain regions, microseismicity can lead to a large set of interior point sources which play the role of diffraction points. In reflection seismology, assuming a single scattering approximation, point diffractors are identified with point contrast sources. However, via data processing, one can obtain expansions of travel times associated with virtual point diffraction data using travel times associated with reflections from (many) smooth interfaces, that is, surface discontinuities, generated by point sources in the boundary. For a specific procedure, see [14] or for general background on seismic data processing see [4]. We note that the mathematically rigorous analysis of such a data processing and the precise conditions under which the data corresponding to reflections and diffractions are essentially equivalent is not yet done and is an interesting open question. The direction of rays can be controlled by synthesizing wave packets.

Earlier, Dix [8] developed a procedure, with a formula, for reconstructing wave speed profiles in a half space  $\mathbb{R} \times \mathbb{R}_+$  with an isotropic metric that is one-dimensional (i.e. depends only on the depth or boundary normal coordinate). We generalize this approach to the case of multi-dimensional manifolds with general non-Euclidean metrics. Before continuing we add that since Dix, various adaptations have been considered to admit more general wavespeed functions in a half space. Some of these adaptations include the work of Shah [27], Hubral & Krey [12], Ursin [33], Dubose, Jr. [9], Mann & Duveneck [22], Cameron et al [5], and Iversen & Tygel [13]. In the case that direct travel times are measured, rather than quantities related to point diffractors, the related mathematical formulations are either the boundary or lens rigidity problems. Much work has been done on these problems (see [28, 29] and the references contained therein).

## 2. Introduction: Definitions and main results

### 2.1. Description of the problem and results

We formulate the setting for a modification of the above Dix's inverse problem that we will study in this paper. Let  $(M, g)$  be a  $C^\infty$ -smooth Riemannian manifold with boundary. We introduce an extension,  $(\widetilde{M}, \widetilde{g})$ ,  $M \subset \widetilde{M}$ , of  $(M, g)$  which is a complete or closed manifold containing  $M$  so that  $\widetilde{g}|_M = g$ . For simplicity we simply write  $\widetilde{g} = g$  and assume that we are given  $U = \widetilde{M} \setminus M$  and the metric  $g$  on  $\widetilde{M} \setminus M$ . We note that if  $M$  is compact and the boundary of  $M$  is convex, the travel times between boundary points determine the normal derivatives, of all orders, of the metric in boundary normal coordinates [17]; hence, in the case of a convex boundary, the smooth extension can be constructed when the travel times between the boundary points are given.

In the following we consider a complete or closed Riemannian manifold  $\widetilde{M}$  with the measurement data given in an open subset  $U \subset \widetilde{M}$ . The tangent and cotangent bundles of  $\widetilde{M}$  at  $x \in \widetilde{M}$  are denoted by  $T_x \widetilde{M}$  and  $T_x^* \widetilde{M}$ , and the unit vectors at  $x$  are denoted by  $\Omega_x \widetilde{M} = \{v \in T_x \widetilde{M}; \|v\|_g = 1\}$ . We denote by  $\exp_x : T_x \widetilde{M} \rightarrow \widetilde{M}$  the exponential map of  $(\widetilde{M}, g)$  and when  $\eta \in \Omega_x \widetilde{M}$ , we denote the geodesics having the initial values  $(x, \eta)$  by  $\gamma_{x,\eta}(t) = \exp_x(t\eta)$ . Let  $B_{\widetilde{M}}(y, t) = \{x \in \widetilde{M}; d(x, y) < t\}$  denote the metric ball of radius  $t$  and center  $y$  and call the set  $\{\exp_y(\xi) \in \widetilde{M}; \|\xi\|_g = t\}$  the generalized metric sphere.

We now express the *data* described in the previous section in more precise terms. Eventually it will be related to the shape operators of so-called spherical surfaces.

**Definition 1.** *Let  $U \subset \widetilde{M}$  be an open set. The family  $\mathcal{S}_U^{or}$  of oriented spherical surfaces is the set of all triples  $(t, \Sigma, \nu)$  satisfying the following properties*

- (i)  $t > 0$ ,
- (ii)  $\Sigma \subset U$  is a non-empty connected  $C^\infty$ -smooth  $(n - 1)$ -dimensional submanifold,
- (iii) There exists a  $y \in \widetilde{M}$  and an open connected set  $\Omega \subset \Omega_y \widetilde{M}$  such that

$$\Sigma = \Sigma_{y,t,\Omega} = \{\gamma_{y,\eta}(t); \eta \in \Omega\}. \quad (1)$$

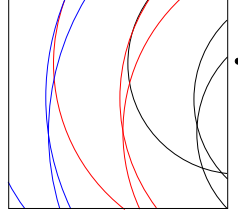


Figure 3: Part of sphere data in  $\mathbb{R}^2$  observed on an open square  $U = (-1, 1)^2$ : Spherical surfaces with three center points (black dots) where the radius of the sphere is coded with color. In  $\mathbb{R}^2$  the sphere data consist of the set  $U$  and all pairs  $(t, \Sigma)$ , where  $\Sigma \subset U$  is a circular arc of radius  $t$ .

(iv)  $\nu$  is the unit normal vector field on  $\Sigma$  given by

$$\nu(x) = \dot{\gamma}_{y,\eta}(t) \quad \text{at the point } x = \gamma_{y,\eta}(t). \quad (2)$$

The **sphere data** consist of the pair  $(U, \mathcal{S}_U)$  where  $U \subset M$  is an open set and  $\mathcal{S}_U$  is the collection of spherical surfaces, that is, the set

$$\mathcal{S}_U = \{(t, \Sigma); \text{ there exists } (t, \Sigma, \nu) \in \mathcal{S}_U^{or}\}.$$

Here, the collection  $\mathcal{S}_U$  contains the same information as  $\mathcal{S}_U^{or}$  but not the orientation of the spherical surfaces.

The set  $U \subset M$  is above considered as an open manifold, that is, as a set with the topological and differentiable structures. We note that if  $(t, \Sigma, \nu) \in \mathcal{S}_U^{or}$  then in (1) the set  $\Omega$  can be written in the form  $\Omega = \{\dot{\gamma}_{x,\nu(x)}(-t); x \in \Sigma\}$  and, as  $\Sigma$  is connected, thus also  $\Omega$  is connected.

Our main result is to prove that the sphere data determine uniquely the universal covering space.

**Theorem 2.** *Let  $\widetilde{M}$  and  $\widetilde{M}'$  be two smooth (compact or complete) Riemannian manifolds and  $U \subset \widetilde{M}$  and  $U' \subset \widetilde{M}'$  be non-empty open sets. Assume that the sphere data  $(U, \mathcal{S}_U)$  and  $(U', \mathcal{S}_{U'})$  coincide in the sense that there is a diffeomorphism  $\Phi : U \rightarrow U'$  and  $\mathcal{S}_{U'} = \{(t, \Phi(\Sigma)); (t, \Sigma) \in \mathcal{S}_U\}$ . Then there is a Riemannian manifold  $(N, g_N)$  such that there are Riemannian covering maps  $F : N \rightarrow \widetilde{M}$  and  $F' : N \rightarrow \widetilde{M}'$ , that is,  $\widetilde{M}$  and  $\widetilde{M}'$  have isometric universal covering spaces.*

We point out that, when a manifold is simply connected the reconstruction of the universal cover is the same thing as the reconstruction of the manifold. In Theorem 2 we formulate the result including the possibility of non-simply connected manifolds to consider the general case. Further, the determination of the metric up to a isometry means imaging the medium in the travel time coordinates. Similar imaging is done for example in medical ultrasound imaging where the reconstruction is typically shown in the travel time coordinates, not in the Euclidean coordinates. In the case when the wave speed is isotropic, it is possible to transform the reconstruction from the travel time coordinates to the Euclidean coordinates. This transformation is considered in [6]. We also note that as an intermediate step in the proof of Theorem 2, we show that the metric can be recovered from  $\mathcal{S}_U$  locally in Fermi coordinates along geodesics passing through  $U$ . This result is given in Theorem 4 below.

Later, in Example 1 in Section 4.2 we will show that the sphere data do not always determine the manifold  $(\widetilde{M}, g)$  but only its universal covering space. Hence Theorem 2 is an optimal result. In particular, Example 1 shows that  $\mathcal{S}_U^{or}$  contains less information than many other data sets used in inverse problems that determine the manifold  $(\widetilde{M}, g)$  uniquely such as the hyperbolic Dirichlet-to-Neumann map, the parabolic Dirichlet-to-Neumann map associated to the heat kernel considered in [3, 18, 19, 20, 23, 30, 31], the boundary distance representation  $R(M)$  (i.e., the boundary distance functions) considered in [2, 21, 19], or the broken scattering relation considered in [15]; see also related data sets in [32].

The idea of the proof is to consider the shape operators of generalized spheres along a geodesic  $\gamma_{x,\eta}$ , where  $x \in U$  and  $\eta$  is a unit vector at  $x$ . Let  $S(r, t) = S_{x,\eta,r,t}$  be the shape operator of the  $\gamma_{x,\eta}(t)$  centered sphere of radius  $(t-r)$  at the point  $\gamma_{x,\eta}(r)$ . The sphere data determines these shape operators when  $r \geq 0$  is so small that  $\gamma_{x,\eta}(r) \in U$  and  $t > r$ . Moreover, we have the Riccati equation

$$-\nabla_{\partial_r} S(r, t) + S(r, t)^2 = -R(r), \quad r \in [0, L] \quad (3)$$

where  $R(r) = R_{\partial_r}(r)$  is radial component of the curvature tensor along the geodesic  $\gamma_{x,\eta}$  and  $L$  is such that  $\gamma_{x,\eta}([0, L])$  has no conjugate points. Using asymptotic analysis in small spheres, we see the inverse operator  $K(r, t) = S(r, t)^{-1}$  has form

$$K(r, t) = (t - r) I + \frac{(t - r)^3}{3} R(t) + \mathcal{O}((t - r)^4),$$



so that

$$R(r) = \frac{1}{2} \partial_t^3 \left( S(r, t)^{-1} \right) \Big|_{t=r}. \quad (4)$$

The  $\partial_t^j$ -derivatives of equations (3) with  $j \leq 3$  and the equation (4) can be considered as a closed system of differential equations in the triangle

$$T = \{(r, t) \in \mathbb{R}^2; 0 \leq r \leq t \leq L\}$$

with initial data of  $S(r, t)$  given for  $r = 0$  and  $t \in [0, L]$ . Solving this system we find the curvature  $R(r)$  at  $\gamma_{x,\eta}(r)$  for  $r \in [0, L]$ . After this we explain a step-by-step construction along the geodesic  $\gamma_{x,\eta}$  that makes us able to find the curvature  $R(r)$  at  $\gamma_{x,\eta}(r)$  for all  $r \in \mathbb{R}_+$ . Further, using this data with  $(x, \eta)$  being in open set we can construct the metric tensor in Fermi coordinates in a neighborhood of a geodesic  $\gamma_{x_0, \eta_0}$ . Glueing these constructions together we can construct the universal covering space of  $(M, g)$ , see Fig. 4 below.

Note that in the definition of  $\mathcal{S}_U$  we consider arbitrary  $y \in \widetilde{M}$ , including points  $y$  in  $U$ . Due to this, we have the following result stating that  $\mathcal{S}_U$  determines both the metric  $g$  in  $U$  and the wave front data with orientation, that is,  $\mathcal{S}_U^{or}$ . Even though the determination of the metric in  $U$  is not very interesting from the point of view of applications, we state the proposition for mathematical completeness.

**Proposition 3.** *Assume that we are given the open set  $U$  as a differentiable manifold and the family of spherical surfaces  $\mathcal{S}_U$ . These data determine the metric  $g$  in  $U$  and the family of the oriented spherical surfaces  $\mathcal{S}_U^{or}$ .*

Proposition 3 is proven in the Appendix.

## 2.2. Background and notation

Before continuing, we briefly mention some general references to Riemannian geometry [10, 25, 26]. As this paper is intended also for researchers working on applied sciences, we recall some standard notations and constructions in local coordinates,  $(x^1, x^2, \dots, x^n)$ . The metric tensor is given by  $g_{jk}(x) dx^j dx^k$  and the inverse of the matrix  $[g_{jk}]$  is denoted by  $[g^{jk}]$ . Throughout the paper we use Einstein summation convention, summing over indexes

that appear both as sub- and super-indexes. The Riemannian curvature tensor,  $R_{ijkl}$ , is given in coordinates by

$$R_{jkl}^i = \frac{\partial}{\partial x^k} \Gamma_{jl}^i - \frac{\partial}{\partial x^l} \Gamma_{jk}^i + \Gamma_{jl}^p \Gamma_{pk}^i - \Gamma_{jk}^p \Gamma_{pl}^i, \quad \widetilde{R}_{jkl}^p = g^{pi} R_{ijkl},$$

where  $\Gamma_{jk}^i$  are the Christoffel symbols,

$$\Gamma_{jk}^i = \frac{1}{2} g^{pi} \left( \frac{\partial g_{jp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p} \right).$$

When  $X, Y \in T_x M$  are vectors, then the curvature operator  $R(X, Y) : T_x M \rightarrow T_x M$  is defined by the formula

$$g(R(X, Y)V, W) = R_{ijkl} X^i Y^j V^k W^l, \quad V, W \in T_x M.$$

Finally,  $\nabla_k = \nabla_{\partial_k}$  is the covariant derivative in the direction  $\partial_k = \frac{\partial}{\partial x^k}$ , which is defined for a (1,1)-tensor field  $A_l^j$  by

$$\nabla_k A_l^j = \frac{\partial}{\partial x^k} A_l^j - \Gamma_{kl}^p A_p^j + \Gamma_{kp}^j A_l^p,$$

and for a (1,0)-tensor field  $B^l$  and a (0,1)-tensor field  $B_l$  by

$$\nabla_k B^l = \frac{\partial}{\partial x^k} B^l + \Gamma_{kp}^l B^p, \quad \nabla_k B_l = \frac{\partial}{\partial x^k} B_l - \Gamma_{lk}^p B_p.$$

If  $f \in C^\infty$  then the gradient of  $f$  with respect to  $g$  is a (1,0)-tensor field (i.e. a vector field) given in coordinates by

$$(\nabla f)^l = g^{lj} \frac{\partial f}{\partial x^j}.$$

Let  $x \in U$  and  $\eta \in \Omega_x \widetilde{M}$ . We say that  $(t, \Sigma, \nu) \in \mathcal{S}_U^{or}$  is associated to the pair  $(x, \eta)$  if  $x \in \Sigma$  and  $\nu(x) = -\eta$ . It is easy to see that if  $(t, \Sigma, \nu) \in \mathcal{S}_U^{or}$  is associated to the pair  $(x, \eta)$  then we can represent  $\Sigma$  in the form (1) where  $y = \gamma_{x, \eta}(t)$  and  $\Omega \subset \Omega_y \widetilde{M}$  is such that  $\zeta = -\dot{\gamma}_{x, \eta}(t) \in \Omega$ .

Once again, suppose that  $x \in U$ ,  $\eta \in \Omega_x \widetilde{M}$ . Now we proceed with more geometrical constructions along the geodesics  $\gamma_{x, \eta}$ . Let  $F_k(r) = F_k(x, \eta, r)$ ,  $k = 1, 2, \dots, n$  be a linearly independent and parallel set of vector fields defined on  $\gamma_{x, \eta}(\mathbb{R})$ . This means that  $F_k(x, \eta, r) \in T_{\gamma_{x, \eta}(r)} \widetilde{M}$  and  $\nabla_{\dot{\gamma}_{x, \eta}(r)} F_k(r) = 0$ . We assume that  $F_n(x, \eta, r) = \dot{\gamma}_{x, \eta}(r)$ . Denote by  $\widehat{g}_{jk}$  the inner products

$$\widehat{g}_{jk} = g(F_j(r), F_k(r)). \quad (5)$$

Because the vector fields  $F_j$  are parallel,  $\widehat{g}_{jk}$  does not depend on  $r$ . Let  $f^j = f^j(x, \eta, r)$ ,  $j = 1, \dots, n$  be the co-frame dual to  $F_j$ . This means that

$$\langle f^j(r), F_k(r) \rangle = \delta_k^j$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual pairing of  $T_y \widetilde{M}$  and  $T_y^* \widetilde{M}$ . Let  $\Psi = \Psi_{x, \eta} : \mathbb{R}^n \rightarrow \widetilde{M}$  be the map

$$\Psi_{x, \eta}(s^1, s^2, \dots, s^{n-1}, r) = \exp_{q(r)} \left( \sum_{k=1}^{n-1} s^k F_k(x, \eta, r) \right), \quad \text{where } q(r) = \gamma_{x, \eta}(r).$$

For all  $r \in \mathbb{R}$  the point  $q(r)$  has a neighborhood  $B_{\widetilde{M}}(q(r), \varepsilon)$ ,  $\varepsilon > 0$  so that there exists an smooth inverse map  $\Psi_{x, \eta}^{-1} : B_{\widetilde{M}}(q(r), \varepsilon) \rightarrow \mathbb{R}^n$ . We call such inverse maps the Fermi coordinates.

Let  $R$  be the curvature operator of  $(\widetilde{M}, g)$ . Below, we denote

$$\mathbf{r}_j^k(x, \eta, r) = \langle f^k(x, \eta, r), R(F_j(x, \eta, r), \dot{\gamma}_{x, \eta}(r)) \dot{\gamma}_{x, \eta}(r) \rangle \quad (6)$$

and call  $\mathbf{r}_j^k(x, \eta, r)$  the *curvature coefficients* of the frame  $(F_j(x, \eta, r))_{j=1}^n$ .

For  $t \geq 0$ , let  $\mathcal{C}(x, \eta, t)$  be the set of those  $r \geq 0$  for which  $z_r = \gamma_{x, \eta}(r)$  and  $y_t = \gamma_{x, \eta}(t)$  are conjugate points on the geodesic  $\gamma_{x, \eta}$ . When  $t > r \geq 0$  and  $r \notin \mathcal{C}(x, \eta, t)$ , there is a spherical surface  $\Sigma_{r, t} := \Sigma_{t-r, y_t, \Omega}$  of the form (1) with some neighborhood  $\Omega \subset \Omega_{y_t} \widetilde{M}$  of  $\zeta = -\dot{\gamma}_{x, \eta}(t)$ . Then  $z_r = \gamma_{x, \eta}(r) \in \Sigma_{r, t}$ . In this case we write the shape operator of  $\Sigma_{r, t}$  at  $z_r$  as  $S_{x, \eta, r, t}$ . Thus  $S_{x, \eta, r, t} \in (T_1^1)_{z_r} \widetilde{M}$  is defined by

$$S_{x, \eta, r, t} X = \nabla_X \nu$$

for all  $X \in T_{z_r} \widetilde{M}$  where  $\nu$  is the normal vector field for  $\Sigma_{r, t}$  satisfying (2). Let us write the shape operator with respect to the parallel frame:

$$S_{x, \eta, r, t} = \mathbf{s}_j^k(x, \eta, r, t) f^j(x, \eta, r) \otimes F_k(x, \eta, r). \quad (7)$$

We say that  $\mathbf{s}_j^k(x, \eta, r, t)$  are the *coefficients of the second fundamental forms of the spherical surfaces* on the geodesic  $\gamma_{x, \eta}$  corresponding to the frame  $F_k(0)$ . The family of the oriented spherical surfaces  $\mathcal{S}_U^{or}$  and the metric tensor  $g$  in  $U$  (which are in fact determined by  $\mathcal{S}_U$  by Proposition 3)), determine all triples  $(t, \Sigma, \nu) \in \mathcal{S}_U^{or}$  that are associated to the pair  $(x, \eta)$ . Note that if  $t > 0$  is such that  $x$  and  $\gamma_{x, \eta}(t)$  are not conjugate along  $\gamma_{x, \eta}$ , there exists

at least one triple  $(t, \Sigma, \nu) \in \mathcal{S}_U^{or}$  that is associated to  $(x, \eta)$ . Note that all such surfaces  $\Sigma$  have the same the shape operator  $S_{x, \eta, r, t}$  with  $r = 0$  at  $x$ . Thus, by computing the shape operator of such surfaces  $\Sigma$  at  $x$  we can find the operator  $S_{x, \eta, 0, t}$  and furthermore the coefficients  $\mathbf{s}_j^k(x, \eta, 0, t)$  for all  $t > 0$  such that  $\gamma_{x, \eta}(t)$  is not conjugate point to  $x$ .

### 2.3. Determination of metric tensor in Fermi coordinates along a geodesic.

Our results are based on the following theorem stating that the radial curvature tensor can be constructed along a geodesic using the sphere data, and that this makes it possible to construct the metric tensor in Fermi coordinates near a given geodesic.

**Theorem 4.** *Let  $(\widetilde{M}, g)$  be a complete or closed Riemannian manifold of dimension  $n$  and  $U \subset \widetilde{M}$  be open. Then*

(i) *Let  $x \in U$ ,  $\eta \in \Omega_x \widetilde{M}$  and  $F_k(r)$ ,  $k = 1, 2, \dots, n$  be linearly independent parallel vector fields along  $\gamma_{x, \eta}$ . Assume that we are given  $\widehat{g}_{jk} = g(F_j, F_k)$  for  $j, k = 1, 2, \dots, n$  and the coefficients of the second fundamental forms of the spherical surfaces corresponding to the frame  $F_k(0)$ , that is,  $\mathbf{s}_j^k(x, \eta, 0, t)$ , for all  $t \in \mathbb{R}_+ \setminus \mathcal{C}(x, \eta, 0)$ . Then we can determine uniquely  $\mathbf{s}_j^k(x, \eta, r, t)$  for all  $t > 0$  and  $r < t$ ,  $r \in \mathbb{R}_+ \setminus \mathcal{C}(x, \eta, t)$ , and the curvature coefficients  $\mathbf{r}_j^k(x, \eta, r)$  for all  $r \in \mathbb{R}_+$ .*

Consequently,

(ii) *Assume that we are given the open set  $U \subset \widetilde{M}$ , metric  $g$  on  $U$ ,  $x_0 \in U$  and a unit vector  $\eta_0 \in \Omega_{x_0} \widetilde{M}$  and let  $\mathcal{V}$  be a neighborhood of  $(x_0, \eta_0)$  in  $T\widetilde{M}$ . Moreover, assume we are given the set*

$$\mathcal{S}_{U, \mathcal{V}}^{or} := \{(t, \Sigma, \nu) \in \mathcal{S}_U^{or}; (x, -\nu(x)) \in \mathcal{V} \text{ for all } x \in \Sigma\}.$$

Then for all  $r > 0$  there is  $\rho = \rho(x_0, \eta_0, r)$  such that the Fermi coordinates  $\Psi_{x_0, \eta_0}^{-1}$  associated to the geodesic  $\gamma_{x_0, \eta_0}(\mathbb{R})$  are well defined in an open set

$$V_{x_0, \eta_0, r} = \Psi_{x_0, \eta_0}(B_{\mathbb{R}^{n-1}}(0, \rho) \times (r - \rho, r + \rho)),$$

and the above data determine uniquely  $(\Psi_{x_0, \eta_0})_* g$ , that is, the metric  $g$  in Fermi coordinates in  $V_{x_0, \eta_0, r}$ . This is the meaning of “reconstruction of the isometry type of the metric in the Fermi coordinates near  $\gamma_{x_0, \eta_0}(\mathbb{R}_+)$ .”

In the above theorem, (i) says that the shape operators  $S_{x,\eta,r,t}$  of the  $\gamma_{x,\eta}(t)$  centered generalized spheres with radius  $t - r$  can be uniquely determined from the shape operators  $S_{x,\eta,0,t}$  corresponding to  $r = 0$  when the metric in  $U$  is known. The claim (ii) says that the Riemannian metric near the geodesic  $\gamma_{x_0,\eta_0}(\mathbb{R}_+)$  can be determined from the knowledge of wave fronts propagating close to this geodesic. From the point of view of applications, it is particularly important that the reconstruction can be done past the first conjugate point, that is, beyond the caustics of the reflected waves.

We point out that the reconstruction method we develop in this paper is constructive and that it is based on solving a system of ordinary differential equations which are satisfied by  $\partial_t^p \mathbf{s}_j^k(x, \eta, r, t)$ ,  $p = 0, 1, 2, 3$  along each geodesic.

#### 2.4. Jacobi and Riccati equations

Before moving to the actual reconstruction procedure we collect a few more geometrical formulae that will be useful. First, if we fix the initial data  $(x, \eta)$  for the geodesic and a  $t > 0$ , then  $S(t, r) = S_{x,\eta,r,t}$  can be thought of as a  $(1, 1)$ -tensor field on the geodesic  $\gamma_{x,\eta}$ . Let  $y_t = \gamma_{x,\eta}(t)$ ,  $z_r = \gamma_{x,\eta}(r)$ , and  $\zeta = -(t - r)\dot{\gamma}_{x,\eta}(t)$  so that  $z_r = \exp_{y_t}(\zeta)$ . Assume that  $y_t$  and  $z_r$  are not conjugate points along  $\gamma_{x,\eta}(t)$ . Then the exponential function  $\exp_{y_t} : T_{y_t}\widetilde{M} \rightarrow \widetilde{M}$  has a local inverse  $F_t = \exp_{y_t}^{-1}$  in a neighborhood  $V$  of  $z_r$  and the function  $f_t(z) = \|F_t(z)\|_g$  is a generalized distance function (i.e.  $\|\nabla f_t(z)\|_g = 1$ ,  $z \in V$ ). As the spherical surface  $\Sigma_{y,t-r,\Omega}$  near  $z_r$  can be written as a level set of a generalized distance function  $f_t$ , it follows from the radial curvature (Riccati) equation [25, sect. 4.2, Thm 2] that

$$-\nabla_{\partial_r} S(t, r) + S(t, r)^2 = -R_{\partial_r}(r), \quad (8)$$

where  $R_{\partial_r}(r) : T_{z_r}\widetilde{M} \rightarrow T_{z_r}\widetilde{M}$ ,  $R_{\partial_r}(r) : V \mapsto R(V, \partial_r)\partial_r$  is the so-called directional curvature operator associated with the Riemannian curvature  $R$  of  $(\widetilde{M}, g)$  and  $\partial_r = -\nabla f_t$ . Note that at the point  $\gamma_{x,\eta}(r)$  we have  $\partial_r = \dot{\gamma}_{x,\eta}(r)$ . Let  $\mathbf{r}_j^k(r) = \mathbf{r}_j^k(x, \eta, r)$  be the coefficients defined in (6), that is,

$$\mathbf{r}_j^k(r) = \langle f^k, R(F_j, \partial_r)\partial_r \rangle.$$

If we also express  $S(t, r) = S_{x,\eta,r,t}$  in the parallel frame as in (7) with  $\mathbf{s}_j^k(r, t) = \mathbf{s}_j^k(x, \eta, r, t)$ , then equation (8) becomes

$$-\partial_r \mathbf{s}_j^k + \mathbf{s}_p^k \mathbf{s}_j^p = -\mathbf{r}_j^k. \quad (9)$$

Also there is an equation we will use relating Jacobi fields along  $\gamma_{x,\eta}$  to  $S$ . A Jacobi field  $J(r)$  along the geodesic  $\gamma_{x,\eta}$  is a vector field satisfying

$$\nabla_{\partial_r}^2 J + R(J, \partial_r)\partial_r = 0. \quad (10)$$

Writing

$$J = J^k F_k$$

this equation is

$$\partial_r^2 J^k + \mathbf{r}_j^k J^j = 0.$$

Finally, it follows from [11, p. 36] that if  $J|_{r=t} = 0$ , then  $\nabla_{-\partial_r} J = S J$ , that is,

$$-\nabla_{\partial_r} J(r) = S(t, r) J(r). \quad (11)$$

With respect to the parallel frame equation (11) reads as

$$-\partial_r J^k = \mathbf{s}_j^k J^j. \quad (12)$$

Our strategy for reconstruction is to show that from the data we can reconstruct shape operator  $S_{x,\eta,r,t}$  along each  $\gamma_{x,\eta}$  using the Riccati equation (8). Then the Jacobi fields may be calculated using (10), and since the Jacobi fields are the coordinate vectors for certain local coordinates, discussed in section 4, this then allows recovery of the metric with respect to those coordinates.

### 3. Reconstruction of the shape operator along one geodesic

The main purpose of this section is to prove claim (i) of Theorem 4. This part of the theorem only deals with the reconstruction of the shape operator and directional curvature along a single geodesic, and so we will here assume that  $x$  and  $\eta$  have been fixed. To simplify the notations we will write  $\gamma$  for  $\gamma_{x,\eta}$ , and use

$$\begin{aligned} \mathbf{s}_j^k(r, t) &= \mathbf{s}_j^k(x, \eta, r, t), \\ \mathbf{r}_j^k(r) &= \mathbf{r}_j^k(x, \eta, r). \end{aligned}$$

### 3.1. A lemma concerning curvature

Our first step toward the reconstruction will be to prove a lemma relating, roughly speaking, the inverse of the shape operator of spherical surfaces as their radius goes to zero with the directional curvature operator at the center of the spherical surfaces.

**Lemma 5.** *Let  $\mathbf{S}(r, t) = (\mathbf{s}_k^j(r, t))_{j,k=1}^{n-1}$  be given by the matrices defined in (7),  $t_1 > 0$  and  $i_1$  be the injectivity radius of  $(\widetilde{M}, g)$  at  $\gamma(t_1)$ . Let  $t, r \in [t_1 - i_1/2, t_1]$  with  $t > r$  and  $\mathbf{K}(r, t) = \mathbf{S}(r, t)^{-1}$ . Then*

$$\mathbf{K}(r, t) = (t - r) I + \frac{(t - r)^3}{3} \mathbf{R}(t) + \mathcal{O}((t - r)^4), \quad \mathbf{R}(t) = (\mathbf{r}_k^j(t))_{j,k=1}^{n-1}, \quad (13)$$

where  $\mathcal{O}((t - r)^4)$  is estimated in a norm on the space of matrices  $\mathbb{R}^{n \times n}$ .

PROOF. We use Jacobi fields  $\mathbf{j}_{(m)}^k(s; r, t) F_k(s)$  on  $\gamma([r, t])$  satisfying

$$\partial_s^2 \mathbf{j}_{(m)}^k(s; r, t) + \mathbf{r}_p^k(s) \mathbf{j}_{(m)}^p(s; r, t) = 0, \quad s \in [r, t], \quad (14)$$

supplemented with the boundary data

$$\mathbf{j}_{(m)}^k(s; r, t)|_{s=r} = \delta_m^k, \quad \mathbf{j}_{(m)}^k(s; r, t)|_{s=t} = 0. \quad (15)$$

Here the indices run from 1 to  $n - 1$ , and henceforth in this proof we will suppress these indices switching to the matrix notation. Since the directional curvature  $\mathbf{R}$  is uniformly bounded on finite intervals we find that  $\|\mathbf{j}(s; r, t)\|$ , where  $\|\cdot\|$  is any matrix norm, is also uniformly bounded for all  $s, r$ , and  $t$  such that  $t > r \geq t - i_1/2$ , and  $t \geq s \geq r$ . Using (14) and (15)

$$\begin{aligned} 0 &= \mathbf{j}(t; r, t) \\ &= \mathbf{j}(r; r, t) + \partial_s \mathbf{j}(r; r, t)(t - r) + \int_r^t \partial_s^2 \mathbf{j}(s; r, t)(t - s) ds \\ &= I + \partial_s \mathbf{j}(r; r, t)(t - r) - \int_r^t \mathbf{R}(s) \mathbf{j}(s; r, t)(t - s) ds. \end{aligned}$$

Therefore using the uniform bound on  $\mathbf{j}$

$$\partial_s \mathbf{j}(r; r, t) = -\frac{I}{(t - r)} + \mathcal{O}((t - r)). \quad (16)$$

On the other hand, taking the expansion above one step further we find

$$0 = I + \partial_s \mathbf{j}(r; r, t)(t-r) - \mathbf{R}(r) \frac{(t-r)^2}{2} - \int_r^t \left( \partial_s \mathbf{R}(s) \mathbf{j}(s; r, t) + \mathbf{R}(s) \partial_s \mathbf{j}(s; r, t) \right) \frac{(t-s)^2}{2} ds.$$

Using (16) and the uniform bound on  $\mathbf{j}$  again this becomes

$$\partial_s \mathbf{j}(r; r, t) = -\frac{I}{(t-r)} + \mathbf{R}(r) \frac{(t-r)}{2} - \int_r^t \frac{\mathbf{R}(s)}{(t-r)^2} \frac{(t-s)^2}{2} ds + \mathcal{O}((t-r)^2).$$

Expanding  $\mathbf{R}$  about  $t$  in two places and integrating finally shows

$$\begin{aligned} \partial_s \mathbf{j}(r; r, t) &= -\frac{I}{(t-r)} + \mathbf{R}(t) \frac{(t-r)}{2} - \mathbf{R}(t) \frac{(t-r)}{6} + \mathcal{O}((t-r)^2) \\ &= -\frac{I}{(t-r)} + \mathbf{R}(t) \frac{(t-r)}{3} + \mathcal{O}((t-r)^2). \end{aligned}$$

Next we have (cf. (12))

$$-\partial_s \mathbf{j}(r; r, t) = \mathbf{s}(r, t) \mathbf{j}(r; r, t) \Rightarrow -\mathbf{K}(r, t) \partial_s \mathbf{j}(r; r, t) = I.$$

Thus

$$\mathbf{K}(r, t) = (t-r) I + \frac{(t-r)^3}{3} \mathbf{R}(t) + \mathcal{O}((t-r)^4)$$

as claimed.  $\square$

### 3.2. Reconstruction

In this section we complete the proof of Theorem 4. The main part is the proof of claim (i) which is given in the following proposition.

**Proposition 6.** *For fixed  $t_0 > 0$ , the functions  $\mathbf{s}_k^j(0, t)$ ,  $t_0 > t > 0$ , determine uniquely functions  $\mathbf{r}_k^j(r)$  for  $r \in [0, t_0]$ , and functions  $\mathbf{s}_k^j(r, t)$  for  $r, t \in [0, t_0]$  where they are defined.*



PROOF. We are given the matrices  $\mathbf{S}(0, t) = (\mathbf{s}_k^j(0, t))_{j,k=1}^{n-1}$ ,  $t > 0$ . Using Lemma 5, it follows that the curvature matrix,  $\mathbf{R}(r) = (\mathbf{r}_k^j(r))_{j,k=1}^{n-1}$ , satisfies

$$\mathbf{R}(r) = \frac{1}{2} \partial_t^3 \mathbf{K}(r, t)|_{t=r}; \quad (17)$$

similarly,  $\mathbf{R}(r) = -\frac{1}{2} \partial_r^3 \mathbf{K}(r, t)|_{r=t}$ .

Using

$$\partial_r \mathbf{S}(r, t) = \mathbf{S}(r, t)^2 + \mathbf{R}(r)$$

(cf. (9)) we find that

$$\begin{aligned} \partial_r \mathbf{K}(r, t) &= -(\mathbf{S}(r, t))^{-1} \partial_r \mathbf{S}(r, t) (\mathbf{S}(r, t))^{-1} \\ &= -(\mathbf{S}(r, t))^{-1} (\mathbf{S}(r, t)^2 + \mathbf{R}(r)) (\mathbf{S}(r, t))^{-1} \\ &= -I - (\mathbf{S}(r, t))^{-1} \mathbf{R}(r) (\mathbf{S}(r, t))^{-1} \\ &= -I - \mathbf{K}(r, t) \mathbf{R}(r) \mathbf{K}(r, t). \end{aligned}$$

We let  $\partial_t$  act on the final equation above, and obtain

$$\begin{aligned} \partial_r ((\partial_t \mathbf{K})(r, t)) &= \partial_t (-I - \mathbf{K}(r, t) \mathbf{R}(r) \mathbf{K}(r, t)) \\ &= -((\partial_t \mathbf{K})(r, t) \mathbf{R}(r) \mathbf{K}(r, t) + \mathbf{K}(r, t) \mathbf{R}(r) (\partial_t \mathbf{K})(r, t)). \end{aligned}$$

Computing the second and third  $t$ -derivatives in a similar manner, and denoting  $V = V(r, t) = (V^j(r, t))_{j=0}^3$ ,  $V^j(r, t) = \partial_t^j \mathbf{K}(r, t)$  and  $\mathbf{R} = \mathbf{R}(r)$ , we obtain the equations

$$\partial_r V^0 = -I - V^0 \mathbf{R} V^0, \quad (18)$$

$$\partial_r V^1 = -(V^1 \mathbf{R} V^0 + V^0 \mathbf{R} V^1), \quad (19)$$

$$\partial_r V^2 = -(V^2 \mathbf{R} V^0 + V^0 \mathbf{R} V^2 + 2V^1 \mathbf{R} V^1), \quad (20)$$

$$\partial_r V^3 = -(V^3 \mathbf{R} V^0 + V^0 \mathbf{R} V^3 + 3V^2 \mathbf{R} V^1 + 3V^1 \mathbf{R} V^2). \quad (21)$$

Since  $\mathbf{R}$  depends on  $V^3$ , this system is “closed”. We define the operator  $\mathcal{T}$  by

$$(\mathcal{T}V)(r) = V^3(r, r) \quad (22)$$

so that, with (17),  $\mathbf{R} = \mathbf{R}(r) = \frac{1}{2}(\mathcal{T}V)(r)$ . Hence,

$$\begin{aligned}\partial_r V^0 &= -I - \frac{1}{2}V^0(\mathcal{T}V)V^0, \\ \partial_r V^1 &= -\frac{1}{2}(V^1(\mathcal{T}V)V^0 + V^0(\mathcal{T}V)V^1), \\ \partial_r V^2 &= -\frac{1}{2}(V^2(\mathcal{T}V)V^0 + V^0(\mathcal{T}V)V^2 + 2V^1(\mathcal{T}V)V^1), \\ \partial_r V^3 &= -\frac{1}{2}(V^3(\mathcal{T}V)V^0 + V^0(\mathcal{T}V)V^3 + 3V^2(\mathcal{T}V)V^1 + 3V^1(\mathcal{T}V)V^2).\end{aligned}\tag{23}$$

We write this as

$$\partial_r V(r, t) = F(V(r, t), (\mathcal{T}V)(r)),$$

where the map  $F$  is a polynomial of its variables. We then introduce

$$\mathcal{F} : W(r, t) \mapsto F(W(r, t), (\mathcal{T}W)(r))$$

so that the system (23) of nonlinear differential equations attains the form

$$\partial_r V(r, t) = (\mathcal{F}V)(r, t).\tag{24}$$

Assuming that we are given  $\mathbf{S}(0, t)$  with  $t > 0$ , we know the initial data

$$V_0(t) = V(0, t) = (\partial_t^j(\mathbf{S}(0, t))^{-1})_{j=0}^3.\tag{25}$$

We now address whether the initial value problem (24)-(25) has a unique solution.

Let us now take  $T$  and  $t_1$  such that  $0 < t_1 < T < i_1/2$  where, as in Lemma 5,  $i_1$  is the injectivity radius of  $(\widetilde{M}, g)$ . Define the function space

$$Y_{t_1} = C\left(\{(r, t) : 0 \leq r \leq t \leq t_1\} ; (\mathbb{R}^{(n-1) \times (n-1)})^4\right)$$

equipped with the norms

$$\|V\|_{Y_{t_1}} = \sup_{0 \leq r \leq t \leq t_1} \max_{j \in \{0, \dots, 3\}} \|V^j(r, t)\|_{\mathbb{R}^{(n-1) \times (n-1)}}.$$

It is immediate that

$$|V^3(r, r)| \leq \|V\|_{Y_{t_1}}$$

for  $0 \leq r \leq t_1$ . If  $B_{t_1}(\mathcal{R}) \subset Y_{t_1}$  is the zero centered ball of radius  $\mathcal{R} \geq 1$  in  $Y_{t_1}$ , because  $\mathcal{F}$  contains no differentiation, we find that

$$\mathcal{F} : \overline{B}_{t_1}(\mathcal{R}) \rightarrow Y_{t_1}$$

is (locally) Lipschitz, with Lipschitz constant  $L(\mathcal{R})$ , that does not depend on  $t_2$ .

We reformulate the initial value problem (24)-(25) in integral form,  $HV = V$ , with

$$H : Y_{t_1} \rightarrow Y_{t_1}, \quad (HW)(r, t) = V_0(t) + \int_0^r \mathcal{F}(W(r', t)) dr', \quad r, t \in [0, t_1].$$

Clearly,  $H : \overline{B}_{t_1}(\mathcal{R}) \rightarrow Y_{t_1}$  is (locally) Lipschitz, with Lipschitz constant  $t_1 L(\mathcal{R})$ . For  $H$  to be a contraction, we need that

$$t_1 L(\mathcal{R}) < 1.$$

On the other hand, to guarantee that  $H(\overline{B}_{t_1}(\mathcal{R})) \subset \overline{B}_{t_1}(\mathcal{R})$ , we need that

$$\|V_0\|_{C([0, T]; \mathbb{R}^{(n-1) \times (n-1)})^4} + t_1 (1 + 4\mathcal{R}^3) < \mathcal{R}.$$

Thus if we set

$$\mathcal{R} = \|V_0\|_{C([0, T]; \mathbb{R}^{(n-1) \times (n-1)})^4} + 2$$

and specify that  $t_1$  must satisfy

$$0 < t_1 < \min \left( T, \frac{1}{L(\mathcal{R})}, \frac{1}{1 + 4\mathcal{R}^3} \right) \quad (26)$$

then  $H : \overline{B}_{t_2}(\mathcal{R}) \rightarrow \overline{B}_{t_2}(\mathcal{R})$  is a contraction. Thus (24)-(25) has a unique solution  $V \in Y_{t_2}$  provided that  $(\partial_t^j(\mathbf{S}(0, t)^{-1}))_{j=0}^3 \in C([0, T]; \mathbb{R}^{(n-1) \times (n-1)})^4$ . This last is true by Lemma 5 since we asked that  $T < i/2$  and therefore we have proven that  $\mathbf{K}(r, t)$ ,  $\mathbf{S}(r, t) = \mathbf{K}(r, t)^{-1}$ , and  $\mathbf{R}(t) = -(1/2)\partial_r^3 \mathbf{K}(r, t)|_{r=t}$  are uniquely determined for  $0 \leq r < t \leq t_2$  by the data  $\mathbf{S}(0, t)$ . At the moment we have only proven the result for  $t_1$  sufficiently small. The rest of the proof is now dedicated to showing that this may be extended beyond  $t_1$  using a stepping procedure.

To make the extension beyond  $t_1$  we will first show that  $\mathbf{S}(t_1, t)$  can be reconstructed for all  $t > t_1$  such that  $t$  is not in the set  $\mathcal{C}_{t_1}$  of those  $t \in [0, \infty)$  for which  $\gamma(t_1)$  and  $\gamma(t)$  are conjugate along  $\gamma$ . To do this let us start by

taking any  $t > t_1$  such that  $t \notin \mathcal{C}_0$ . Then based on what we have already done we know  $\mathbf{R}(r)$  for  $r \in [0, t_1]$  and the matrices  $\mathbf{S}(0, t)$ . Therefore we can find the solutions  $\mathbf{j}^k(r, t)$  of the Cauchy problems for the Jacobi equations

$$\begin{aligned} \partial_r^2 \mathbf{j}_l^k(r, t) + \mathbf{r}_p^k(r) \mathbf{j}_l^p(r, t) &= 0, \quad r \in [0, t_1], \\ \mathbf{j}_l^k(r, t)|_{r=0} &= \delta_l^k, \quad \partial_r \mathbf{j}_l^k(r, t)|_{r=0} = -\mathbf{s}_l^k(0, t). \end{aligned} \quad (27)$$

Now, for all  $r \in [0, t_1] \setminus \mathcal{C}_t$  the vectors  $\{\mathbf{j}_l^k(r, t)\}_{l=1}^n$  are linearly independent; thus, the equations  $-\partial_r \mathbf{j}_l^j(r, t) = \mathbf{s}_k^j(r, t) \mathbf{j}_l^k(r, t)$  (cf. (12)) determine  $\mathbf{S}(r, t)$ . Now, as  $t \mapsto \mathbf{S}(r, t)$  is continuous for  $t \in \mathbb{R}_+ \setminus \mathcal{C}_r$ , we see that we can find  $\mathbf{S}(r, t)$  for all  $r \in [0, t_1]$  and  $t > r$  such that  $t \in \mathbb{R}_+ \setminus \mathcal{C}_r$ . In particular we can determine  $\mathbf{S}(t_1, t)$  for all  $t > t_1$  such that  $t \in \mathbb{R}_+ \setminus \mathcal{C}_{t_1}$ . This yields a new dataset in the interior of  $M$  at the point  $\gamma(t_1)$ . We now repeat the above argument with 0 replaced by  $t_1$  to recover  $\mathbf{R}$  and  $\mathbf{S}(r, t)$  on another interval  $[t_1, t_2]$ .

The size of the interval,  $t_2 - t_1$ , is determined by the injectivity radius  $i_1$ , and bounds on

$$\|V_0\|_{C([t_1, t_1+T]; \mathbb{R}^{(n-1) \times (n-1)})^4} = \|(\partial_t^j \mathbf{K}(t_1, \cdot))_{j=0}^3\|_{C([t_1, t_1+T]; \mathbb{R}^{(n-1) \times (n-1)})^4} \quad (28)$$

(cf. (26), the  $t_1$  there should be replaced by  $t_2 - t_1$  in the second step). By the proof of Lemma 5 (28) can be bounded uniformly in terms of the directional curvature and a finite number of its derivatives, and therefore (28) can be bounded independent of  $t_1$ . Thus the size of the step  $t_2 - t_1$  can be uniformly bounded away from 0, and so, continuing the same procedure we only require a finite number of steps to cover the entire interval  $[0, t_0]$ . This completes the proof of the claim (i) of Theorem 4. □

#### 4. Reconstruction of the metric in Fermi coordinates and the reconstruction the universal covering space

We are now ready to prove claim (ii) of Theorem 4.

PROOF. For any  $x_0 \in U = \widetilde{M} \setminus M$ ,  $\eta_0 \in T_{x_0} \widetilde{M}$  and  $t_0 > 0$  let  $\Sigma_0 \subset U$  be the spherical surface with center  $y_{t_0} = \gamma_{x_0, \eta_0}(t_0)$ . On  $\Sigma_0$  suppose we have a coordinate map  $\widehat{X} : \widehat{V} \rightarrow \mathbb{R}^{n-1}$  in a neighborhood  $\widehat{V} \subset \Sigma_0$  of  $x_0$ , and for  $\widehat{x} \in \widehat{V}$  let  $\nu = \nu(\widehat{x})$  be the normal vector field of  $\Sigma_0$  oriented so

that  $\nu(x_0) = -\eta_0$ . Also, denote the geodesic starting normally to  $\Sigma_0$  from  $\widehat{X}^{-1}(\widehat{x}) \in \Sigma_0$  by  $\gamma_{\widehat{x}}(t) = \gamma_{\widehat{x}, -\nu(\widehat{x})}(t)$ . Let  $\iota : \Sigma_0 \rightarrow \widetilde{M}$  be the identical embedding, and for  $\widehat{x} \in \widehat{X}(\widehat{V})$  define at the point  $x = \widehat{X}^{-1}(\widehat{x})$  the vectors

$$F_j(\widehat{x}) = \iota_* \left( \frac{\partial}{\partial \widehat{X}^j} \right), \quad j \leq n-1, \quad \text{and} \quad F_n(\widehat{x}) = -\nu(\widehat{X}^{-1}(\widehat{x}))$$

Then  $(F_j(\widehat{x}))_{j=1}^n$  is a basis for  $T_x \widetilde{M}$  and we obtain, for each  $r > 0$ , a basis  $(F_j(\widehat{x}, r))_{j=1}^n$  for  $T_{\gamma_{\widehat{x}}(r)} \widetilde{M}$  by parallel translation along the geodesic  $\gamma_{\widehat{x}}$ .

Let  $\mathcal{C}(\widehat{x})$  be the set of those  $r \in \mathbb{R}_+$  for which  $x = \gamma_{\widehat{x}}(0) \in \Sigma$  and  $\gamma_{\widehat{x}}(r)$  are conjugate points on the geodesic  $\gamma_{\widehat{x}}$ , and define the sets

$$\begin{aligned} \mathcal{W} &= \{(\widehat{x}, r) \in X(\widehat{V}) \times \mathbb{R}_+; r \in \mathbb{R}_+ \setminus \mathcal{C}(\widehat{x})\}, \\ \mathcal{W} &= \{\gamma_{\widehat{x}}(r) \in \widetilde{M}; (\widehat{x}, r) \in \mathcal{W}\}. \end{aligned}$$

Then the map

$$X_{\widehat{V}} : \mathcal{W} \rightarrow W \subset \widetilde{M}, \quad X_{\widehat{V}}(\widehat{x}, r) = \gamma_{\widehat{x}}(r) \quad (29)$$

is a local diffeomorphism. Below, we use the local inverse maps of  $X_{\widehat{V}}$  as local coordinates on  $W$ . The  $(\widehat{x}, r)$  coordinates, basically, are Riemannian normal coordinates centered at  $y_{t_0}$ , but parametrized in a particular way:  $\widehat{x}$  can be thought of as a parametrization of part of the sphere of radius  $t_0$  in  $T_{y_{t_0}} \widetilde{M}$ , and then  $r$  corresponds to the radial variable in  $T_{y_{t_0}} \widetilde{M}$ . Note also that the coordinate vectors in these coordinates are Jacobi fields along the geodesics  $\gamma_{\widehat{x}}$ .

Note that as  $\Sigma_0 \subset U$  and we know the metric tensor  $g$  on  $U$ , we can determine the inner products

$$\widehat{g}_{jk}(\widehat{x}) = g(F_j(\widehat{x}), F_k(\widehat{x})), \quad (30)$$

and  $g(F_j(\widehat{x}, r), F_k(\widehat{x}, r)) = \widehat{g}_{jk}(\widehat{x})$  for all  $r \geq 0$ .

By the proof of claim (i) of Theorem 4, for any  $\widehat{x} \in \widehat{X}(\widehat{V})$  we can determine the coefficients  $\mathbf{j}_l^k(r, t_0) = \mathbf{j}_l^k(\widehat{x}, r, t_0)$  given in (27). Then

$$J_j(\widehat{x}, r; t_0) = \mathbf{j}_j^m(\widehat{x}, r, t_0) F_m(\widehat{x}, r)$$

are the Jacobi fields along the geodesic  $\gamma_{\widehat{x}}(r)$  that satisfy

$$J_j(\widehat{x}, t_0; t_0) = 0, \quad J_j(\widehat{x}, 0; t_0) = F_m(\widehat{x}, 0).$$

Let us now consider the set  $W \subset \widetilde{M}$  that is a neighborhood of  $\gamma_{x_0, \eta_0}((0, t_0) \setminus \mathcal{C}(x_0, \eta_0, t_0))$  and the map  $X_{\widehat{\nu}} : \mathcal{W} \rightarrow W$  given in (29), a point  $(\widehat{x}_0, r_0) \in \mathcal{W}$  and its small neighborhood  $\mathcal{V} \subset \mathcal{W}$  so that  $X_{\widehat{\nu}}|_{\mathcal{V}} : \mathcal{V} \rightarrow V = X_{\widehat{\nu}}(\mathcal{V})$  is a diffeomorphism. The inverse of this map defines local coordinates  $x \mapsto (\widehat{x}, r) = (X_{\widehat{\nu}}|_{\mathcal{V}})^{-1}(x)$  in the set  $V$ . We see that the Jacobi fields  $J_j(\widehat{x}, r)$  are in fact the coordinate vectors for the  $(\widehat{x}, r)$  coordinates. Therefore the metric  $g$  with respect to these coordinates can be recovered by

$$g(\partial_{\widehat{x}^j}, \partial_{\widehat{x}^k})|_{(\widehat{x}, r)} = \widehat{g}_{ml}(\widehat{x}) \mathbf{j}_j^m(\widehat{x}, r; t_0) \mathbf{j}_k^l(\widehat{x}, r; t_0).$$

Note that here we are in fact varying  $\widehat{x}$ , and performing the entire recovery of  $\mathbf{r}$  and  $\mathbf{s}$  along each geodesic in order to calculate the Jacobi fields along that geodesic and can then compute the metric tensor  $g$  in the set  $W$  in the local  $(\widehat{x}, r)$  coordinates. Moreover, as we know the coefficients of the Jacobi fields  $J_j(\widehat{x}, r)$  represented in the parallel frame  $F_m(\widehat{x}, r)$  along  $\gamma_{\widehat{x}}(r)$ , we can also find the coefficients of the vectors  $F_m(\widehat{x}, r)$  in the basis given by the Jacobi fields  $J_j(\widehat{x}, r)$ . Thus we change the local  $(\widehat{x}, r)$  coordinates to the Fermi coordinates and determine the metric tensor  $g$  in the Fermi coordinates in some neighborhood  $W_{x_0, \eta_0, t_0}^{fermi} \subset W$  of the set  $\gamma_{x_0, \eta_0}((0, t_0) \setminus \mathcal{C}(x_0, \eta_0, t_0))$ .

As  $W_{x_0, \eta_0, t_0}^{fermi}$  is a neighborhood of  $\gamma_{x_0, \eta_0}((0, t_0) \setminus \mathcal{C}(x_0, \eta_0, t_0))$  we have not yet reconstructed  $g$  in the whole neighborhood of  $\gamma_{x_0, \eta_0}$ . To do this, let  $s_1 > 0$  be so small that  $\widetilde{x}_0 = \gamma_{x_0, \eta_0}(-s_1)$  and  $\widetilde{\eta}_0 = \dot{\gamma}_{x_0, \eta_0}(-s_1)$  satisfy  $\widetilde{x}_0 \in U$  and repeat the above construction by replacing  $x_0$  by  $\widetilde{x}_0$ ,  $\eta_0$  by  $\widetilde{\eta}_0$  and  $t_0$  by arbitrary  $\widetilde{t}_0 > s_1$  and the spherical surface  $\Sigma_0$  by the corresponding surface  $\widetilde{\Sigma}_0$ . Then, we can determine the metric tensor in local coordinates  $W_{\widetilde{x}_0, \widetilde{\eta}_0, \widetilde{t}_0}^{fermi}$ . By varying  $s_1$  and  $\widetilde{t}_0$  and using the fact that on a given geodesic the conjugate points of a given point form a discrete set, we see that the whole geodesic  $\gamma_{x_0, \eta_0}(\mathbb{R}_+)$  can be covered by neighborhoods of the form  $W_{\widetilde{x}_0, \widetilde{\eta}_0, \widetilde{t}_0}^{fermi}$ . This completes the proof of claim (ii) of Theorem 4.  $\square$

We finish this section by proving Theorem 2.

PROOF. Let  $\exp_x : T_x \widetilde{M} \rightarrow \widetilde{M}$  and  $\exp'_{x'} : T_{x'} \widetilde{M}' \rightarrow \widetilde{M}'$  be the exponential maps of  $(\widetilde{M}, g)$  and  $(\widetilde{M}', g')$ , correspondingly.

Let  $p \in U$  and  $p' \in U'$  be such that  $p' = \Phi(p)$  and let

$$\ell = d\Phi|_p : T_p \widetilde{M} \rightarrow T_{p'} \widetilde{M}'$$

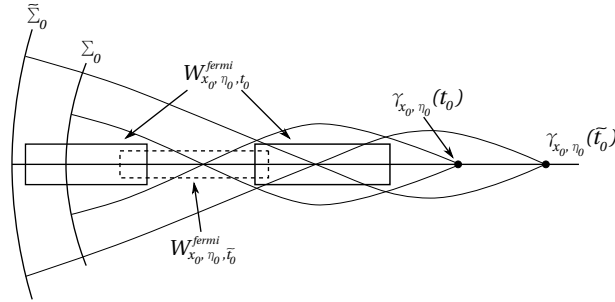


Figure 4: Reconstruction procedure in the case of conjugate points.

be the differential of  $\Phi$  at  $p$ . For  $v = tv^0 \in T_p\tilde{M}$ ,  $\|v^0\|_g = 1$ ,  $t \geq 0$ , let  $\tau_v : T_p\tilde{M} \rightarrow T_q\tilde{M}$  denote the parallel transport along the geodesic  $\gamma_{p,v^0}([0, t])$ , where  $q = \gamma_{p,v^0}(t)$  in  $(\tilde{M}, g)$  and let  $\tau'_{v'} : T_{p'}\tilde{M}' \rightarrow T_{q'}\tilde{M}'$  denote the corresponding operation on  $(\tilde{M}', g')$ . For  $v, w \in T_p\tilde{M}$  let the curve  $\mu_{v,w} : [0, 2] \rightarrow \tilde{M}$  be the broken geodesic, that is defined by

$$\begin{aligned} \mu_{v,w}(s) &= \exp_p(sv), & \text{for } 0 \leq s \leq 1, \\ \mu_{v,w}(s) &= \exp_q((s-1)\tau_v w), & \text{for } 1 \leq s \leq 2, \end{aligned}$$

where  $q = \exp_p(v)$ . When  $r = \mu_{v,w}(2)$  is the end point of the broken geodesic, we denote by  $\tau_{v,w} : T_p\tilde{M} \rightarrow T_r\tilde{M}$  the parallel transport of vectors along the curve  $\mu_{v,w}([0, 2])$ . For all  $v \in T_p\tilde{M}$  let  $\rho(p, v, r)$  be the function in Theorem 4 for the geodesic  $\gamma_{p,v}(r)$ , that is, we can determine the Riemannian metric in the Fermi coordinates in the tubular neighborhood  $V_{x_0, \eta_0, r}$  that contains the ball  $B_{\tilde{M}}(\exp_p(v), \rho(p, v, r))$ . Let  $\rho_0(v) > 0$  be such that  $\rho_0(v) < \rho(p, v, r)$  and the ball  $B_{\tilde{M}}(\exp_p(v), \rho_0(v))$  is geodesically convex. Let  $\rho'_0(v') > 0$  be the corresponding function for  $(\tilde{M}', g')$  and  $p'$ . Finally, we define  $f(v) = \min(\rho_0(v), \rho'_0(\ell(v)))$ .

Let  $v \in T_p\tilde{M}$ . When  $q = \exp_p(v)$  and  $q' = \exp'_{p'}(\ell(v))$ , let  $E_v : T_p\tilde{M} \rightarrow \tilde{M}$  be the map  $E_v(\xi) = \exp_q(\tau_v \xi)$  and  $E'_v : T_{p'}\tilde{M}' \rightarrow \tilde{M}$  be  $E'_v(\xi') = \exp'_{q'}(\tau'_{v'} \xi')$ . We see that

$$E'_{\ell(v)} \circ \ell \circ E_v^{-1} : B_{\tilde{M}}(q, f(v)) \rightarrow B_{\tilde{M}'}(q', f(v))$$

is an isometry. In particular, if  $v, w \in T_p\tilde{M}$  are such that  $\|w\|_g < f(v)$ , and  $r = \mu_{v,w}(2) \in \tilde{M}$  and  $r' = \mu'_{\ell(v), \ell(w)}(2) \in \tilde{M}'$  are the end points of the broken

geodesics, the above implies that the linear isometry

$$\ell_{v,w} = \tau'_{\ell(v),\ell(w)} \circ \ell \circ \tau_{v,w}^{-1} : T_r \widetilde{M} \rightarrow T_{r'} \widetilde{M}'$$

preserves the sectional curvature, i.e.,  $\text{Sec}_r(\xi, \eta) = \text{Sec}_{r'}(\ell_{v,w}(\xi), \ell_{v,w}(\eta))$ . Thus from the proof of the Ambrose theorem given in [24] (for the original reference, see [1]) it follows that  $(\widetilde{M}, g)$  and  $(\widetilde{M}', g')$  have isometric covering spaces.  $\square$

Next we show that  $\mathcal{S}_U$  does not always determine the manifold  $(\widetilde{M}, g)$  but only its universal covering space and that hence Theorem 2 is an optimal result.

**Example 1.** Let  $(\widetilde{M}, g)$  be the flat torus  $\mathbb{R}^2/\mathbb{Z}^2$ , and  $(\widetilde{M}', g')$  be the flat torus  $\mathbb{R}^2/(2\mathbb{Z} \times 2\mathbb{Z})$ . Below, we consider  $(\widetilde{M}, g)$  and  $(\widetilde{M}', g')$  as the squares  $[0, 1]^2$  and  $[0, 2]^2$  with the parallel sides being glued together. Both  $(\widetilde{M}, g)$  and  $(\widetilde{M}', g')$  have the universal covering space  $\mathbb{R}^2$  with the Euclidean metric. Let  $U$  and  $U'$  be the disc  $B(p, \frac{1}{4})$  of the radius  $\frac{1}{4}$  and the center  $p = (\frac{1}{2}, \frac{1}{2})$ . We see that both the collection  $\mathcal{S}_U^{or}$  for  $(\widetilde{M}, g)$  and the collection  $\mathcal{S}_{U'}^{or}$  for  $(\widetilde{M}', g')$  consist of triples  $(\Sigma, t, \nu)$  where  $t > 0$ ,  $\Sigma$  is a connected circular arc having radius  $t$  that is a subset of the disc  $B(p, \frac{1}{4})$ , that is,  $\Sigma = \{(t \sin \alpha + x, t \cos \alpha + y) \in U; |\alpha - \alpha_0| < c_0\}$ , and  $\nu$  is the exterior normal vector of  $\Sigma$ . Note that the circular arc  $\Sigma$  can also be a whole circle if  $t$  is small enough. This shows that the knowledge of  $U$  and  $\mathcal{S}_U^{or}$  is not always enough to determine uniquely the manifold  $(\widetilde{M}, g)$  but only its universal covering space.

## Appendix A: Proof of Proposition 3

We begin with the reconstruction of the Riemannian metric  $g|_U$  if  $\mathcal{S}_U$  is given. If  $x \in U$  then in a sufficiently small neighborhood  $U' \subset U$  of  $x$  all points  $z$  can be connected to  $x$  with a geodesic of a given length (travel time) contained in  $U$ . As a consequence, the distance between  $x$  and  $z$  can be found as

$$d(x, z) = \inf \left\{ \sum_{j=1}^N 2t_j; (t_j, \Sigma_j) \in \mathcal{S}_U, x_j, x_{j+1} \in \Sigma_j \right. \\ \left. \text{for } j = 1, \dots, N \text{ such that } x_1 = x, x_{N+1} = z \right\}. \quad (31)$$



Indeed, we observe that the infimum is obtained when  $\Sigma_j$  are the boundaries of sufficiently small balls (which are always smooth),  $B_{\widetilde{M}}(y_j, t_j)$ , where  $y_j$  are points on the shortest geodesic connecting  $x$  to  $z$ . Thus we can determine the distance function  $(y, y') \mapsto d(y, y')$  between two arbitrary points in  $y, y' \in U'$ .

Now, if  $r > 0$  is small enough and  $z_1, \dots, z_n \in U'$  are disjoint points so that  $d(x, z_j) = r$ , then the function  $y \mapsto (d(y, z_j))_{j=1}^n \in \mathbb{R}^n$  defines local coordinates near the point  $x \in U'$ . So, in  $U'$ , we can find the differentiable structure inherited from the manifold  $\widetilde{M}$ . Using the distances between points  $y, y' \in U'$ , we can determine the Riemannian metric in these coordinates in  $U'$ . But then we can find the Riemannian metric  $g|_U$  if  $\mathcal{S}_U$  is given.

For  $(t, \Sigma) \in \mathcal{S}_U$  and  $x_0 \in \Sigma$ , let  $N(x_0, \Sigma, t)$  be the set consisting of the two unit normal vectors of  $\Sigma$  at  $x_0$ . Let  $N_1(x_0, \Sigma, t)$  be the set of those  $\nu_0 \in N(x_0, \Sigma, t)$  for which the point  $x_0$  has a neighborhood  $U' \subset U$  such that  $\Sigma \cap U'$  has the representation

$$\Sigma \cap U' = \{\gamma_{y, \eta}(t); \eta \in \Omega\}, \quad (32)$$

where  $y = \gamma_{x_0, \nu_0}(-t)$  and  $\Omega \subset \Omega_y \widetilde{M}$  is a neighborhood of  $\eta_0 = -\dot{\gamma}_{x_0, \nu_0}(-t)$ . Note that it is possible for  $N_1(x_0, \Sigma, t)$  to contain both normal vectors in  $N(x_0, \Sigma, t)$ . An example of a case in which this occurs is when  $\widetilde{M}$  is  $S^2$  and  $\Sigma$  is a subset of the Equator.

**Lemma 7.** *If  $U$  and  $\mathcal{S}_U$  are given, we can determine  $N_1(x_0, \Sigma, t)$  for any  $(t, \Sigma) \in \mathcal{S}_U$  and  $x_0 \in \Sigma$ .*

PROOF. For given  $(t, \Sigma) \in \mathcal{S}_U$  and  $x_0 \in U$  let  $\zeta_0 \in N(x_0, \Sigma, t)$  be one of the two unit normal vectors to  $\Sigma$  at  $x_0$ , and let  $\zeta(x)$  be a smooth normal vector field on  $\Sigma \cap U'$  such that  $\zeta(x_0) = \zeta_0$ . We introduce the notation

$$\Sigma_{U'}^\pm(s) = \{\gamma_{x, \pm \zeta(x)}(s); x \in \Sigma \cap U'\};$$

$\Sigma_{U'}^\pm(s)$  will be smooth for  $s \in (-\varepsilon, \varepsilon)$ ,  $\varepsilon > 0$  sufficiently small since  $\zeta(x)$  is always normal to  $\Sigma$ .

Assume next that  $\zeta_0 \in N_1(x_0, \Sigma, t)$ . Then representation (32) is valid with  $y = \gamma_{x_0, \zeta(x_0)}(-t)$ , and for  $p(x, s) = \gamma_{x, \zeta(x)}(s)$  and  $\eta(x, s) = \dot{\gamma}_{x, \zeta(x)}(s)$ , we have  $\gamma_{p(x, s), \eta(x, s)}(-t - s) = \gamma_{x, \zeta(x)}(s - t - s) = y$ . Hence, there is a neighborhood  $U'' \subset U$  of  $x_0$  such that for all  $\varepsilon > 0$  small enough

$$(t + s, \Sigma_{U''}^+(s)) \in \mathcal{S}_U \text{ for all } s \in (-\varepsilon, +\varepsilon). \quad (33)$$

Let us consider the following condition

(C) There exists  $y'$  such that  $\gamma_{x,\zeta(x)}(+t) = y'$  for all  $x \in \Sigma$  close to  $x_0$ .

If condition (C) is valid, then both  $\zeta_0$  and  $-\zeta_0$  are in  $N_1(x_0, \Sigma, t)$ . If condition (C) is not valid, then  $N_1(x_0, \Sigma, t)$  contains the vector  $\zeta_0$  but not  $-\zeta_0$ .

In the case when condition (C) is valid, we see that (33) holds as well as the analogous identity with the minus sign, that is, we have

$$(t + s, \Sigma_{U''}^-(s)) \in \mathcal{S}_{U''} \text{ for all } s \in (-\varepsilon, +\varepsilon). \quad (34)$$

Next, consider the case when the condition (C) is not valid. Our aim is show that then (34) can not hold. For this end, let us assume that the condition (C) is not valid but we have (34). Then we see that for all  $s \in (-\varepsilon, \varepsilon)$  one of the sets

$$\begin{aligned} A_+ &= \{\gamma_{x,\zeta(x)}(-s + (t + s)); x \in \Sigma \cap U''\}, \\ A_- &= \{\gamma_{x,\zeta(x)}(-s - (t + s)); x \in \Sigma \cap U''\}, \end{aligned}$$

would consist of a single point. Now, if  $A_+$  consisted of a single point then this point would satisfy the condition required for the point  $y'$  in condition (C). As we assumed that condition (C) is not valid, we conclude that  $A_+$  cannot be a single point. If  $A_-$  consisted of a single point for all  $s \in (-\varepsilon, \varepsilon)$ , then for all  $x_1, x_2 \in \Sigma \cap U''$  we would have  $\gamma_{x_1,\zeta(x_1)}(-t-2s) = \gamma_{x_2,\zeta(x_2)}(-t-2s)$  for all  $s \in (-\varepsilon, \varepsilon)$  and hence for all  $s \in \mathbb{R}$ . With  $s = -t/2$  we would see that  $x_1 = x_2$  for all  $x_1, x_2 \in \Sigma \cap U''$  but that is not possible. Hence equation (34) can not be true when the condition (C) is not valid

Summarizing the above, we can find the set  $N_1(x_0, \Sigma, t)$  using the fact that it contains the vector  $\pm\zeta_0$  if and only if there are  $U''$  and  $\varepsilon > 0$  such that  $(t + s, \Sigma_{U''}^\pm(s)) \in \mathcal{S}_U$  holds for all  $s \in (-\varepsilon, \varepsilon)$ .

Lemma 7 and the considerations above it prove Proposition 3.

## Bibliography

- [1] W. AMBROSE, *Parallel translation of Riemannian curvature*. Ann. of Math. 64 (1956), 337–363.

- [2] M. ANDERSON, A. KATSUDA, Y. KURYLEV, M. LASSAS, M. TAYLOR, *Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand inverse boundary problem*, *Invent. Math.* 105 (2004), 261–321.
- [3] M. BELISHEV, Y. KURYLEV, *To the reconstruction of a Riemannian manifold via its spectral data*. *Comm. PDE* 17 (1992), 767–804.
- [4] N. BLEISTEIN, J. K. COHEN, J. W. STOCKWELL JR., *Mathematics of multidimensional seismic imaging, migration, and inversion*, Springer, New York, 2001.
- [5] M. K. CAMERON, S. B. FOMEL, J. A. SETHIAN, *Seismic velocity estimation from time migration*, *Inverse Problems* 23 (2007), pp. 1329–1369.
- [6] M. DE HOOP, E. IVERSEN, M. LASSAS, B. URSIN, *Reconstruction of a conformally Euclidean metric from local boundary diffraction travel times*, arXiv:1211.6132.
- [7] M. DAHL, A. KIRPICHNIKOVA, M. LASSAS, *Focusing waves in unknown media by modified time reversal iteration*, *SIAM Journal on Control and Optimization* 48 (2009), Issue 2, pp. 839–858.
- [8] C.H. DIX, *Seismic velocities from surface measurements*, *Geophysics*, 20 (1955), pp. 68–86.
- [9] J.B. DUBOSE JR., *A technique for stabilizing interval velocities from the Dix equation*, *Geophysics*, 53 (1988), pp. 1241–1243.
- [10] L. EISENHART, *Riemannian Geometry*, Princeton University Press, Princeton, 1977.
- [11] D. GROMOLL AND G. WALSHAP, *Metric foliations and curvature*, *Progress in Mathematics*, 268, Birkhäuser-Verlag, Basel, 2009.
- [12] P. HUBRAL AND T. KREY, *Interval velocities from seismic reflection time measurements*, in *Expanded Abstracts*, Society of Exploration Geophysicists, 1980.

- [13] E. IVERSEN AND M. TYGEL, *Image-ray tracing for joint 3D seismic velocity estimation and time-to-depth conversion*, *Geophysics*, 73 (2008), pp. S99–S114.
- [14] E. IVERSEN, M. TYGEL, B. URSIN, AND M. V. DE HOOP, *Kinematic time migration and demigration of reflections in pre-stack seismic data*, *Geophys. J. Int.* 189 (2012), pp. 1635–1666.
- [15] Y. KURYLEV, M. LASSAS, G. UHLMANN, *Rigidity of broken geodesic flow and inverse problems*. *Amer. J. Math.* 132 (2010), 529–562.
- [16] Y. KURYLEV, M. LASSAS, G. UHLMANN, *Determination of structures in the space-time from local measurements: a detailed exposition*, arXiv:1305.1739.
- [17] M. LASSAS, V. SHARAFUTDINOV, AND G. UHLMANN, *Semiglobal boundary rigidity for Riemannian metrics*, *Math. Ann.*, 325 (2003), pp. 767–793.
- [18] A. KATCHALOV, Y. KURYLEV, *Multidimensional inverse problem with incomplete boundary spectral data*. *Comm. PDE* 23 (1998), 55–95.
- [19] A. KATCHALOV, Y. KURYLEV, M. LASSAS, *Inverse Boundary Spectral Problems*, *Monographs and Surveys in Pure and Applied Mathematics* 123, Chapman Hall/CRC-press, 2001, xi+290 pp.
- [20] A. KATCHALOV, Y. KURYLEV, M. LASSAS, N. MANDACHE, *Equivalence of time-domain inverse problems and boundary spectral problem*, *Inverse problems* 20 (2004), 419–436.
- [21] Y. KURYLEV, *Multidimensional Gel'fand inverse problem and boundary distance map*, *Inverse Problems related with Geometry*, Ibaraki Univ. (1997), 1–16.
- [22] J. MANN AND E. DUVERNECK, *Event-consistent smoothing in generalized high-density velocity analysis*, in *Expanded Abstracts, Society of Exploration Geophysicists*, 2004, pp. 2176–2179.
- [23] L. OKSANEN, *Solving an inverse problem for the wave equation by using a minimization algorithm and time-reversed measurements*. *Inverse Probl. Imaging* 5 (2011), 731–744.

- [24] B. O'NEILL, *Construction of Riemannian coverings*. Proc. Amer. Math. Soc., 19 (1968) 1278–1282.
- [25] P. PETERSEN, *Riemannian geometry*, Graduate Texts in Mathematics, 171, Springer-Verlag, New York, 1998.
- [26] ———, *Aspects of global Riemannian geometry*, Bull. Amer. Math. Soc. (N.S.), 36 (1999), pp. 297–344.
- [27] P.M. SHAH, *Use of wavefront curvature to relate seismic data with subsurface parameters*, Geophysics, 38 (1973), pp. 812–825.
- [28] P. STEFANOV, G. UHLMANN, *Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds*. J. Diff. Geom. 82(2) (2009), 383–409.
- [29] P. STEFANOV, G. UHLMANN, *Boundary and Lens Rigidity, Tensor Tomography and Analytic Microlocal Analysis*. in Algebraic Analysis of Differential Equations, Festschrift in Honor of Takahiro Kawai, edited by T. Aoki, H. Majima, Y. Katei and N. Tose, 275–293 (2008).
- [30] P. STEFANOV, G. UHLMANN, *Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map*. Int. Math. Res. Not. 2005, no. 17, 1047–1061.
- [31] P. STEFANOV, G. UHLMANN, *Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media*. J. Funct. Anal. 154 (1998) 330–358.
- [32] G. UHLMANN, *Inverse boundary value problems for partial differential equations*. Proceedings of the International Congress of Mathematicians. Vol. III (Berlin, 1998). Doc. Math. Vol. III, 77–86.
- [33] B. URSIN, *Quadratic wavefront and travel time approximations in inhomogeneous layered media with curved interfaces*, Geophysics, 47 (1982), pp. 1012–1021.