Supplementary material: density of unidirectional vortex rings subject to reconnections

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I. INTRODUCTION

In these supplementary notes, we describe effects of collisions of unidirectional charged vortex rings (CVRs) due to the small variations, $\sim \delta v$, of their velocities around the mean value v. Any such collision of primary CVRs generally results in secondary CVRs of quite different size – hence depleting the number of the primary CVRs. We consider purely one-dimensional trajectories of CVRs, $x(t)$, between the injector at $x = 0$ and collector at $x = d$ [1]. We assume a uniform electric field U/d, a negligible initial radius of CVRs, $R_0 \to 0$, and a constant logarithmic parameter $\Lambda \equiv \ln \frac{8R}{a_0} \approx 13$ (where $a_0 \approx 1.3$ Å is the core radius). Within the time of flight from the injector to collector, τ_1, the lengthening of the beam, caused by the range of velocities δv, does not exceed the duration of injection $\Delta t = 0.2\,\mathrm{s}$ and is, hence, neglected.

The velocity and energy of a quantized vortex ring of radius R carrying one electron of charge e:

$$v = \frac{\kappa}{4\pi R}\left(\Lambda - \frac{1}{2}\right), \tag{1}$$

$$\mathcal{E} = \frac{\kappa^2 \rho R}{2}\left(\Lambda - \frac{3}{2}\right) = e\frac{U}{d}x, \tag{2}$$

where $\rho = 0.145\,\mathrm{g\,cm^{-3}}$ is the density of superfluid helium and $\kappa = 1.0 \times 10^{-3}\,\mathrm{cm^2\,s^{-1}}$ is the quantum of circulation.

The radius grows linearly with x,

$$R = R_1\frac{x}{d}, \tag{3}$$

where the radius of CVRs at collector, $x = d$, is

$$R_1 = \frac{2eU}{\rho\kappa^2(\Lambda - 3/2)}. \tag{4}$$

The time of flight between $x = 0$ and $x = x_1$ is

$$t(x) = \int_0^x v^{-1}dx' = \tau_1\frac{x^2}{d^2}, \tag{5}$$

i. e. the trajectory of primary CVRs is

$$x(t) = d\left(\frac{t}{\tau_1}\right)^{1/2}, \tag{6}$$

where the time of flight from injector to collector is

$$\tau_1 = \frac{4\pi e d}{\rho\kappa^3(\Lambda - 1)^2}U = \frac{2\pi d}{\kappa(\Lambda - 1/2)}R_1. \tag{7}$$

The velocity field, its gradient and pulse length in x-space can be conveniently expressed (including the explicit time dependence along the trajectory $x(t)$):

$$v \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{d^2}{2\tau_1}x^{-1} = \frac{d}{2}\tau_1^{-1/2}t^{-1/2}, \tag{8}$$

$$\nabla v = -\frac{d^2}{2\tau_1}x^{-2} = -\frac{1}{2}t^{-1}, \tag{9}$$

$$\Delta x = v\Delta t = \frac{d^2\Delta t}{2\tau_1}x^{-1} = \frac{d\Delta t}{2}\tau_1^{-1/2}t^{-1/2}. \tag{10}$$

II. EVOLUTION OF THE DENSITY OF CVRS

The number density of primary rings in the beam $n(x,t)$, along the trajectory $x(t)$, evolves according to:

$$\frac{\mathrm{d}n}{\mathrm{d}t} \equiv \frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} = \left(-\frac{\partial(nv)}{\partial x} - f\right) + v\frac{\partial n}{\partial x} = -n\frac{\mathrm{d}v}{\mathrm{d}x} - f, \tag{11}$$

where $-f$ is the rate of losses due to ring-ring collisions.

For small n, at which $f \to 0$, the number of CVRs is conserved, and the solution is $nv = const$, i. e. $n = Ax$. Experimentally, the value of constant A can be determined from the magnitude of the peak of the collector current due to primary CVRs,

$$I_m = \theta\pi r^2 e n_1 v_1 = \theta\pi r^2 e A d^2 (2\tau_1)^{-1}, \tag{12}$$

where r and θ are the radius and transparency of the collector grid, $n_1 \equiv n(d)$ and $v_1 \equiv v(d)$. We arrive at

$$n(x) = \frac{2\tau_1 I_m}{\theta\pi r^2 e d^2}x = \frac{8\pi I_m U}{\theta\pi r^2\rho\kappa^3(\Lambda - 1)^2 d}x. \tag{13}$$

The density at collector is

$$n_1 = \frac{8\pi I_m U}{\theta\pi r^2\rho\kappa^3(\Lambda - 1)^2}. \tag{14}$$

Thus, in this regime of conserved primary CVRs, their density can be varied by changing either the intensity of injection (as measured by I_m) or drive voltage U.

III. COLLISIONS OF PRIMARY CVRS

For each CVR, the probability of a collision with another CVR per unit time is $n\sigma_1\delta v$, where $\delta v = v\delta R/R$ (as $v \propto R^{-1}$), and the cross-section is $\sigma = \sigma_1' R^2$, where $\sigma_1' \sim 1$. Naive geometric considerations suggest $\sigma_1' = 4\pi$, although further research into this problem is necessary [2]. If each collision effectively removes two primary CVRs from the coherent beam, the total number of removed CVRs per unit time and unit volume:

$$f_a = n^2\sigma_1\delta v = \frac{\Lambda - 1/2}{4\pi}\kappa\sigma_1'\delta R n^2. \tag{15}$$

After using Eqs. 9&7, and $f = f_a$ from the above expression, Eq. 11 becomes:

$$\frac{\mathrm{d}n}{\mathrm{d}t} - \frac{n}{2t} + \frac{d\sigma_1'\delta R R_1}{2\tau_1}n^2 = 0. \tag{16}$$

Its asymptotic solution is

$$n = \frac{3}{\sigma_1'\delta R d}R_1^{-1}\frac{\tau_1}{t}. \tag{17}$$

At $t = \tau_1$,

$$n_1 R_1 = \frac{3}{\sigma_1'\delta R d}. \tag{18}$$

The RHS is independent of both the injection intensity and U. For $\sigma' = 4\pi$ and $\delta R = 0.5\,\mu$m, it is equal to $4 \times 10^3\,\mathrm{cm}^{-2}$, which is a factor of 8 greater than the experimental value of $5 \times 10^2\,\mathrm{cm}^{-2}$. There might be several reasons for such a discrepancy. Firstly, the cross-section for the effective removal of CVRs from the coherent beam of similar orientation and velocities might be several times greater than the geometric guess $\sigma = 4\pi R^2$. Secondly, in this model we restricted all CVRs to motion along the x-axis with only a spread of radii; in reality they also have a small random spread of directions of motion caused by the conserved transverse component of the impulse of the initial CVRs – our estimates show that its contribution to the frequency of collisions is comparable with that calculated here.

IV. MULTIPLE PILE-UPS OF PRIMARY CVRS

Another process that should further limit the density of primary CVRs is the removal of all CVRs that bump into the secondary vortex rings formed upon any binary collision discussed above. Large secondary CVRs appear with each primary collisions at a rate, per unit volume and time,

$$\frac{f_a}{2} = \frac{(\Lambda - 1/2)\kappa\sigma_1'\delta R}{8\pi}n^2. \tag{19}$$

As these larger rings are considerably slower than the primary ones, they block all the primary CVRs from behind within the effective cross-section $\sigma_2 = \sigma_2' R^2$, where $\sigma_2' \sim 1$, and the upper limit on its geometrical value is $\sigma_2' = 4\sqrt{2}\pi \approx 18$ [3]. The typical number of primary CVRs lost in such a capture (the "multiplication factor") is $N_2 \sim n\sigma_2\frac{\Delta x}{2}$. The rate of losses per unit volume and time is hence

$$f_b = N_2\frac{f_a}{2} = \frac{(\Lambda - 1/2)\kappa\sigma_1'\sigma_2'\delta R R_1^2\Delta t d}{32\pi\tau_1^{3/2}}t^{1/2}n^3. \tag{20}$$

With $f = f_b$, Eq. 11 becomes

$$\frac{\mathrm{d}n}{\mathrm{d}t} - \frac{n}{2t} + \frac{(\Lambda - 1/2)\kappa\sigma_1'\sigma_2'\delta R R_1^2\Delta t d}{32\pi\tau_1^{3/2}}t^{1/2}n^3 = 0. \tag{21}$$

Its asymptotic solution is:

$$n = \left(\frac{40\pi}{(\Lambda - 1/2)\kappa\sigma_1'\sigma_2'\delta R\Delta t d}\right)^{1/2} R_1^{-1}\left(\frac{\tau_1}{t}\right)^{3/4}. \tag{22}$$

For $t = \tau_1$ this gives

$$n_1 R_1 = \left(\frac{40\pi}{(\Lambda - 1/2)\kappa\sigma_1'\sigma_2'\delta R\Delta t d}\right)^{1/2}, \tag{23}$$

independent of the injection intensity and U. For $\sigma_1' = 4\pi$, $\sigma_2' = 4\sqrt{2}\pi$, $\delta R = 0.5\,\mu$m, $\Delta t = 0.2\,$s and $d = 4.5\,$cm, this produces $n_1 R_1 = 1.0 \times 10^3\,\mathrm{cm}^{-2}$, which is only factor of 2 greater than the experimental value. Furthermore, as we only considered the effect of longitudinal variations of the velocities δv of primary CVRs but disregarded the comparable effect from their transverse velocities, the agreement is quite good.

V. CONCLUSION

To conclude, the solutions, $n_1 R_1 = const$, found for either of the simple models of removal of primary vortex rings upon collisions, Eq. 18 and Eq. 23, qualitatively agree with our experimental observations. Moreover, the latter is actually in a reasonable quantitative agreement. Thus multiple pile-ups explain the observed universal value of the product $n_1 R_1$ when CVRs collide frequently due to their large numbers and large radii. Further experiments with varyable flight path d, duration of injection Δt and distribution of CVRs' radii δR, as well as numerical simulations of the interaction between CVRs, will help improve our understanding of the microscopic processes that occur within beams of unidirectional vortex rings.

[1] In our experiments, CVRs' motion very near the injector is not quite one-dimensional, but the number of reconnections is small there, hence the end result should not be strongly affected by this approximation.

[2] Our preliminary calculations of the interaction between two parallel vortex rings of slightly different radii show that reconnections occur with the cross-section $\sigma_1 \approx 0.6R^2$, while a noticible perturbation of the directions of motion and velocities (even without reconnections) occurs with the cross-section even larger than $\sigma_1 = 4\pi R^2$.

[3] We assume that, upon a collision of two primary CVRs of radii R each carrying impulse $\rho\kappa\pi R^2$, they merge into a circular loop with impulse $2\rho\kappa\pi R^2$ and hence radius $\sqrt{2}R$.