MODEL THEORY OF SEPARABLY DIFFERENTIALLY CLOSED FIELDS

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Contents

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Abstract

We study a new class of differential fields called separably differentially closed fields as a differential analogue to separably closed fields. We define them in terms of existential closedness; that is, a differential field (K, δ) of arbitrary characteristic is said to be separably differentially closed if it is existentially closed in every differential field extension that is separable (in the field-theoretic sense).

We prove that this class of differential fields is first-order axiomatizable in the language of differential fields (we denote this theory by SDCF). We do this first by giving a full description of those prime differential ideals in $K\{x\}$ that are separable over K in terms of irreducible elements of $K\{x\}$ with nonzero separant (here (K, δ) is a differential field of arbitrary characteristic and $K\{x\}$ is the differential polynomial ring over K in one variable).

Assuming that the extension C_K/K^p is finite (here C_K denotes the constants of differential field (K, δ)), we then exhibit several characterizations of being separably differentially closed. In particular, one important characterization is in terms of being constrainedly closed (in the sense of Kolchin).

We then observe that even after specifying the characteristic $p > 0$, SDCF_p is not complete. It turns out that, in analogy to the algebraic counterpart SCF_p, one only needs to specify what we call the differential degree of imperfection to describe the completions. We do this in the case of the finite degree of imperfection; namely $\mathrm{SDCF}_{p,\epsilon}$ for finite ϵ. We also note that after adding the differential λ-functions to the language (these are the suitable analogue of algebraic λ-functions), one obtains a quantifier elimination result for $\mathrm{SDCF}^{\ell}_{p,\epsilon}$. Furthermore, we prove that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is stable with unique prime model extensions.

We note that our results generalize the work of Carol Wood [27], as her theory of differentially closed fields in characteristic $p > 0$, denoted by DCF_p, is a special case of ours when $\epsilon = 0$; in other words, $\mathrm{DCF}_p = \mathrm{SDCF}_{p,0}$.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

We study a new class of differential fields called separably differentially closed fields as a differential analogue to separably closed fields. We show that this class is elementary in the mathematical logic sense, denoted by SDCF. Generalizing the work of Wood [27] on the theory of differentially closed fields of positive characteristic DCF_p, we have established the foundational properties of the theory; namely, completeness, model-completeness, quantifier elimination, stability, and the existence of unique prime model extension.

Chapter 2 Preliminaries

In this chapter, we briefly explain the basic notions and facts of model theory, field theory in positive characteristic, model theory of separably closed fields, differential algebra, model theory of differentially closed fields of positive characteristic and intermediate fields in positive characteristic, which are used throughout this thesis.

In the model theory section, we emphasize three notions: existential closedness, quantifier elimination and prime model extensions. In the separably closed fields section, we review field theory in positive characteristic and model theory of separably closed fields; This is relevant for us as we will consider the differential analogue of separably closed fields as the main theme of this thesis. In the differentially closed fields of positive characteristic section, we review basic notions and facts of differential algebra and model theory of differentially closed fields of positive characteristic. In the last section, we make a few observations on the intermediate field theory in positive characteristic. This is relevant for us since the constant field C_K of a differential field (K, δ) of characteristic $p > 0$ is an intermediate field of K and K^p where K^p is a set of p-th powers of all element of K.

Chapter 3 Separable prime differential ideals

In this chapter, we study certain prime differential ideals called separable prime differential ideals in the differential polynomial ring $K\{x\}$ over the differential field (K, δ) of positive characteristic in one differential indeterminate and one derivation. Namely, we give a full description/characterization of such differential ideals in terms of irreducible differential polynomials with nonzero separant. (Recall that the separant of $f \in K\{x\}$, denoted by S_f, is the partial derivative of f with respect to its maximal order variable.) More precisely, we prove the following (see the details in Chapter 3).

Theorem A. (1) *Let $P \subset K\{x\}$ be a nonzero prime differential ideal. If P is separable over K, then $P = [f] : S_f^\infty$ for an irreducible and minimal f in P where $[f] : S_f^\infty = \{g \in K\{x\} \mid S_f^m g \in [f] \text{ for some } m \in \mathbb{N}\}$.*

(2) *Let $f \in K\{x\}$ be irreducible with $S_f \neq 0$. Then $P = [f] : S_f^\infty$ is a prime differential ideal separable over K and f is minimal rank in P.*

Chapter 4 Separably differentially closed fields

In this chapter, we study a new class of differential fields called separably differentially closed fields, a differential analogue to separably closed fields. Generalizing the work of Wood on the theory of differentially closed fields of positive characteristic DCF_p, we show that this class is elementary, denoted by SDCF. Namely, we characterize this class of differential fields in a first-order fashion in the sense of mathematical logic in the manner of Blum for characteristic zero. Moreover, we obtain several characterizations for it, particularly in terms of Kolchin's constrained extensions.

Definition 1.2. *A differential field (K, δ) is said to be separably differentially closed in the language of differential fields if it is existentially closed in every differential field extension that is separable.*

Our first-order axioms exhibiting that the class of separably differentially closed fields is elementary are as follows.

Theorem B. *Let (K, δ) be a differential field. Then the following are equivalent:*

(1) *(K, δ) is a separably differentially closed field.*

(2) *For any pair (f, g) in $K\{x\} \setminus \{0\}$ with $S_f \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f$, there exists an element a in K such that $f(a) = 0$ and $g(a) \neq 0$. We denote this theory by* SDCF.

Remark 1.3. (1) *In characteristic zero, S_f is always nonzero, and so we recover Blum's axioms of* DCF_0.

(2) *In characteristic $p > 0$, we note that if in condition (2) of the above theorem, one adds that (K, δ) is differentially perfect (i.e., $C_K = K^p$), we recover Woods's axioms of* DCF_p.

We then show several characterizations of separably differentially closed fields under a mild assumption on the extension C_K/K^p.

Theorem C. *Let (K, δ) be a differential field of arbitrary characteristic such that C_K/K^p is finite. Then the followings are equivalent:*

(1) *(K, δ) is separably differentially closed.*

(2) *(K, δ) is existentially closed in every differentially algebraic field extension that is separable.*

(3) *(K, δ) is constrainedly closed.*

(4) *Let $P \subset K\{\bar{x}\}$ be a prime differential ideal separable over K, and $g \in K\{\bar{x}\} \setminus P$. There exists $\bar{a}$ in K such that $f(\bar{a}) = 0$ (for all $f \in P$) and $g(\bar{a}) \neq 0$.*

(5) $(K, \delta) \models \mathrm{SDCF}$

Chapter 5 Model theoretic properties of separably differentially closed fields

In this chapter, we introduce the notions of differential degree of imperfection, differential p-basis and differential λ-functions (these are differential analogues of their algebraic counterparts). Then we establish the model theoretic properties of separably differentially closed fields in suitable expansions of the language of differential fields.

Definition 1.4. *Let (K, δ) be a differential field of characteristic $p > 0$.*

(1) *If $[C_K : K^p]$ is finite, there exists $\epsilon \in \mathbb{N}_0$ such that $[C_K : K^p] = p^\epsilon$ where ϵ is called the differential degree of imperfection of (K, δ), or we say that (K, δ) has a finite differential degree of imperfection.*

(2) *A tuple $\bar{a}$ from K is said to be differentially p-independent for (K, δ) if $\bar{a}$ is a tuple from C_K and the p-monomials of $\bar{a}$ are linearly independent over K^p.*

(3) *A tuple $\bar{a}$ from K is said to be a differential p-basis for (K, δ) if $\bar{a}$ is a tuple from C_K and the p-monomials of $\bar{a}$ form a linear basis of C_K over K^p.*

Note that the p-monomials of $\{a_1, \ldots, a_\epsilon\}$ is the set of form

$$\{a_1^{i_1} \cdots a_\epsilon^{i_\epsilon} \mid 0 \leq i_j \leq p-1\}.$$

After adding names to the languages for a differential p-basis, say $\bar{a}$, we are able to prove:

Theorem D. $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ *is the model completion of* $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ *(The latter denotes the theory of differential fields of characteristic $p > 0$ with the differential degree of imperfection ϵ and differential p-basis $\bar{a}$).*

To obtain the quantifier elimination result, we introduce the differential λ-functions, a differential analogue of λ-functions.

Definition 1.5. *Let (K, δ) be a differential field of characteristic $p > 0$ with a differential p-basis $\bar{a}$. The differential λ-functions ℓ_i's for $0 \leq i < p^\epsilon$ are defined by the conditions*

$$\begin{cases} \ell_i(b) = 0 & \text{if } b \notin C_K \\ b = \ell_0(b)^p m_0 + \cdots + \ell_s(b)^p m_s & \text{if } b \in C_K \end{cases}$$

for all b in K where m_i's are p-monomials of $\bar{a}$ and $s = p^\epsilon - 1$.

By adding new symbols for the differential λ-functions, we are able to prove:

Theorem E. (1) $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ *is the model completion of* $\mathrm{DF}^{\ell}_{p,\epsilon}$.

(2) $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ *has quantifier elimination and is complete. As a result,* $\mathrm{SDCF}_{p,\epsilon}$ *is complete.*

Chapter 6 Stability and prime model extensions in $\mathrm{SDCF}^{\ell}_{p,\epsilon}$

In the final chapter, we generalize Wood's work stating that DCF_p is stable but not superstable, and there exists a unique differential closure over each differentially perfect field. We show that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is stable but not superstable, and there exists a unique prime model extension over each model of $\mathrm{DF}^{\ell}_{p,\epsilon}$ Namely, we prove the following.

Theorem F. (1) $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ *is stable but not superstable.*

(2) *Over each* $F \models \mathrm{DF}^{\ell}_{p,\epsilon}$ *there exists a unique prime model extension* $K \models \mathrm{SDCF}^{l}_{p,\epsilon}$.

Chapter 2

Preliminaries

In this chapter, we explain the basic notions and facts of model theory, field theory in positive characteristic, model theory of separably closed fields, differential algebra, model theory of differentially closed fields of positive characteristic and intermediate fields of positive characteristic, which are used throughout this thesis.

In the model theory section, we emphasize three notions: existential closedness, quantifier elimination and prime model extension. In the separably closed fields section, we review field theory in positive characteristic and model theory of separably closed fields. In the differential closed fields of positive characteristic section, we review basic notions and facts of differential algebra and model theory of differential closed fields of positive characteristic. In the last section, we observe the intermediate field theory of positive characteristic. Since the fact that a constant field of a differential field is the intermediate field of the differential field and its p-th power, it is essential to observe that some field theory in positive characteristic is also applied in their intermediate fields.

2.1 Model theory

In this section, we explain the basic notions and facts of model theory that are used throughout this thesis. In particular, we emphasise three notions: existential closedness, quantifier elimination and prime model extension. We assume that the reader is familiar with basic model theoretic notions such as structures, models and formulas. There are many excellent references, in particular, Marker [9], Tent and Ziegler [23]. All facts stated in this section can be found in these references.

Existential Closedness

Let L be a first-order language, N an L-structure and M an L-substructure. We say that M is existentially closed in N if $N \models \exists \bar{x} \varphi(\bar{x})$ implies $M \models \exists \bar{x} \varphi(\bar{x})$ for every quantifier free L-formula $\varphi(\bar{x})$ with parameters from M. Now let T be an L-theory and suppose $M \models T$. We say that M is existentially closed (in T) if K is existentially closed in every L-extension N with $N \models T$. Note that every universal theory has existentially closed models.

For example, in the language of fields L_{fields},

$$L_{\text{fields}} = \{0, 1, +, -, \times, ^{-1}\},$$

which is the language of ring adding a unary function symbol $^{-1}$, the theory of fields T_{fields} is a universal theory, and hence T_{fields} has existentially closed models. Moreover, these are precisely the algebraically closed fields. More precisely, since every quantifier free L_{fields}-formula is equivalent to a Boolean combination of polynomial equations, a field K is existentially closed in T_{fields} if and only if every finite system of polynomial equations and inequations over K with a solution in a field extension has already a solution in K. The following characterization of algebraically closed fields is well known. Namely, the following are equivalent:

(1) K is existentially closed.

(2) K is algebraically closed; that is, K has no proper algebraic extension.

(3) For each f in $K[x] \setminus \{0\}$, there exists a in K such that $f(a) = 0$.

This scheme of conditions can be written in a first-order fashion, and we denote this theory by ACF.

The second example is that of differential fields of characteristic zero. In the language of differential fields L_δ,

$$L_\delta = \{0, 1, +, -, \times, ^{-1}, \delta\},$$

the theory of differential fields of characteristic zero DF_0 is a universal theory, and the existentially closed models of this theory are called differentially closed fields. Robinson [14] axiomatized the class of differentially closed fields, and Blum [1] simplified his axioms. More precisely, for a differential field (K, δ) of characteristic zero, the following are equivalent:

(1) (K, δ) is differentially closed

(2) For each pair (f, g) of differential polynomials in $K\{x\} \setminus \{0\}$ with $\mathrm{ord}g < \mathrm{ord}f$, there exists a in K such that $f(a) = 0$ and $g(a) \neq 0$.

These scheme of conditions can be expressed in a first-order fashion and thus denoted by DCF_0.

The last example, which is more relevant for us, considers the theory of differential fields of characteristic $p > 0$, denoted by DF_p. Again, the existentially closed models of this theory are called differentially closed fields. Wood [26] proved that a differential field (K, δ) of positive characteristic $p \neq 0$ is differentially closed if and only if: (K, δ) is differentially perfect (namely, the constant field C_K of K coincide with p-th power of K, that is, $C_K = K^p$) and "for every pair (f, g) in $K\{x\} \setminus \{0\}$ with $\mathrm{ord}g < \mathrm{ord}f$, $S_f \neq 0$ (here S_f denote the separant of f, see Section 2.3), there exists a in K such that $f(a) = 0$ and $g(a) \neq 0$". This theory is denoted by DCF_p.

Note that the difference between DCF_0 and DCF_p is that the latter requires the differential fields to be differentially perfect and that the separant of f to be nonzero. These conditions are trivially satisfied in characteristic zero.

Definition 2.1. *Let T be an L-theory. We say that T^* is a model companion of T if*

(1) *T^* is model complete, that is, every model of T^* is existentially closed*

(2) *Every model of T has an extension to a model of T^**

(3) *Every model of T^* has an extension to a model of T*

From the above observations, we see that ACF is a model companion of the theory of fields. Furthermore, DCF_0 and DCF_p are model companions of DF_0 and DF_p, respectively.

Quantifier elimination

We say that an L-theory T has quantifier elimination if for all L-formulas $\varphi(\bar{x})$, there exists a quantifier free L-formula $\psi(\bar{x})$, that is, a Boolean combination of atomic L-formulas such that for all $K \models T$, $K \models \forall \bar{x}[\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})]$.

Recall that T has the amalgamation property if $K, E_1, E_2 \models T$ with $f_1 : K \to E_1$ and $f_2 : K \to E_2$ L-embeddings, then there exists $L \models T$, $g_1 : E_1 \to L$ and $g_2 : E_2 \to L$ L-embeddings such that $g_1 \circ f_1 = g_2 \circ f_2$. And T^* is said to be a model completion of T if T^* is a model companion of T, and T^* plus the atomic diagram of K, $\mathrm{Diag}(K)$ is complete for all $K \models T$. Then the following criteria are used in Chapter 5.

Theorem 2.2. *(Chapter 3 of [9])*

(1) *Let T^* be a model companion of T. Then T has the amalgamation property if and only if T^* is a model completion of T.*

(2) *If T^* is the model completion of T and T is a universal theory, then T^* has quantifier elimination.*

Prime model extensions

Let T be a complete theory. Let M be a model T and A be a subset of M. Recall that the Stone topology of space of complete n-types $S_n(A)$ has basic open sets $[\varphi] = \{p \in S_n(A) \mid \varphi \in p\}$ for φ an L-formula with parameters from A. A type $p \in S_n(A)$ is isolated if $\{p\} = [\varphi]$ for some L-formula φ.

The subset $A \subseteq M$ is said to have a prime model extension $N \models T$ if for every $S \models T$ with $S \models \mathrm{Diag}_{el}^{M}(A)$, there exists an elementary A-embedding $f : N \to S$.

Morley [12] showed that the following criterion for the existence of prime model extension.

Theorem 2.3. *(Morley's criterion [12])*

Let T be a countable complete theory. Every subset of a model of T has a prime model extension if and only if the isolated types $S_1(A)$ are dense for every $A \subseteq M \models T$.

A theory T is λ-stable for some cardinal λ if $S(F) \leq \lambda$ for all $F \models T$ with $|F| \leq \lambda$. A theory T is stable if T is λ-stable for some infinite λ. It is well known that an ω-stable theory has unique prime model extensions by Morley [12] for existence and Shelah [21] for uniqueness.

For example, DCF_0 is ω-stable, and hence differential fields of characteristic zero have a unique (up to isomorphism) differential closure.

For a countable stable theory (not necessarily ω-stable), the uniqueness of prime models follows from their existence where a theory T is stable if T is λ-stable for some infinite λ.

Theorem 2.4. *(Chapter 9 of [24]) Let T be a countable stable theory. If a prime model extension exists, then it is unique up to isomorphism.*

2.2 Separably closed fields

In this section, we review field theory in positive characteristic and the model theory of separably closed fields. These are the basic algebraic and model theoretical facts that we will use throughout this thesis, and we will consider the differential analogue of separably closed fields in later Chapters.

Fix K a field of characteristic p. Let L be a field extension of K. We say that L is separable over K if K and L^p are linearly disjoint over K^p. K is separably closed if K has no proper separable algebraic extension. Note that by the notion of existential closedness, we have the following characterizations. Namely, the following are equivalent:

(1) K is separably closed.

(2) K is existentially closed in every field extension that is separable.

(3) K is existentially closed in every algebraic field extension that is separable.

(4) "for all f in $K[x]\setminus\{0\}$ with $\frac{df}{dx}\neq 0$, there exists a in K such that $f(a)=0$" We denote this theory by SCF.

We denote this theory by SCF_p when we specify the characteristic $p\neq 0$:

$$\underbrace{(1+\cdots+1)}_{\text{p-times}}=0.$$

If $[K:K^p]$ is finite, then there exists $e\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ such that

$$[K:K^p]=p^e$$

where e is called the degree of imperfection. When we specify the degree of imperfection, we denote this theory by $\mathrm{SCF}_{p,e}$. After specifying the characteristic

$p \neq 0$ and the degree of imperfection e, Ersov [4] showed that these determine the completion of SCF. More precisely,

Theorem 2.5. *(Ersov [4]) For $p \neq 0$ and finite e, $\mathrm{SCF}_{p,e}$ is complete.*

Remark 2.6. *When the degree of imperfection e is infinite, the theory $\mathrm{SCF}_{p,\infty}$ is also complete.*

For the remainder of this section, we assume that K is a field of characteristic $p \neq 0$. Given $a_1, \ldots, a_r \in K$, the p-monomials of $\{a_1, \ldots, a_r\}$ is the set of form of

$$\{a_1^{i_1} \cdots a_r^{i_r} \mid 0 \leq i_j \leq p-1\}.$$

Definition 2.7. *(See Jacobson [5])*

(1) *A tuple $\bar{a}$ from K is said to be p-independent for K if the p-monomials of $\bar{a}$ are linearly independent over K^p.*

(2) *A p-basis for K is a tuple of $\bar{a}$ from K such that the p-monomials of $\bar{a}$ form a linear basis of K over K^p.*

Lemma 2.8. *(See Jacobson [5]) K has the degree of imperfection e if and only if K has a p-basis of size e.*

Now we are ready to describe the following criterion for separability.

Proposition 2.9. *(See Jacobson [5])*

Let L be a field extension of K. Then the following are equivalent

(1) *L is separable over K.*

(2) *every p-independent set for K is p-independent set for L.*

(3) *Some p-basis for K is p-independent over L^p.*

Let $\bar{a} = (a_1, \ldots, a_e)$ be a tuple of constant symbols and consider the new language $L_{\bar{a}} = L_{fields} \cup \{a_1, \ldots, a_e\}$. While the theory $\mathrm{SCF}_{p,e}$ is not model complete, after specifying a p-basis, we overcome this.

Theorem 2.10. *(Ersov [4]) Let $\mathrm{SCF}_{p,e}^{\bar{a}}$ be the $L_{\bar{a}}$-theory obtained by adding to $\mathrm{SCF}_{p,e}$ $L_{\bar{a}}$-sentence specifying that $\bar{a} = \{a_1, \ldots, a_e\}$ is a p-basis. Then $\mathrm{SCF}_{p,e}^{\bar{a}}$ is model complete.*

Definition 2.11. *(See Delon [2]) Suppose K has the degree of imperfection e and $\bar{a} = \{a_1, \ldots, a_e\}$ is a p-basis. Let $s = p^e - 1$. Then the λ-functions $\lambda_i : K \to K$ for $0 \leq i < p^\epsilon$ are defined by:*

$$b = \lambda_0(b)^p m_0 + \cdots + \lambda_s(b)^p m_s$$

for all b in K where m_i's are the p-monomials of $\bar{a}$.

If we further expand the language $L_\lambda = L_{\bar{a}} \cup \{\lambda_1, \ldots \lambda_s\}$ by unary function symbols, and let $\mathrm{SCF}^{\bar{a}}_{p,e}$ denote the L_λ-theory obtained after adding to $\mathrm{SCF}^{\bar{a}}_{p,e}$ the L_λ-sentence specifying that the λ_i's are the λ-functions, we obtain further model-theoretic properties.

Recall that a theory T is stable if T is μ-stable for some infinite μ. T is superstable if there exists some cardinal κ such that T is μ-stable for all $\kappa \leq \mu$.

Theorem 2.12. *(see Delon [2] and Wood [29])*

(1) *$SCF^\lambda_{p,e}$ has quantifier elimination.*

(2) *If $e > 0$, $SCF^\lambda_{p,e}$ is stable and not superstable.*

Remark 2.13. *Let $K \models \mathrm{SCF}_{p,e}$ with $p \neq 0$. If $e = 0$, we have $K = K^p$, that is, K is perfect. Then we have $\mathrm{SCF}_{p,0} = \mathrm{ACF}_p$, and hence K is an algebraically closed field of characteristic p.*

2.3 Differentially closed fields in positive characteristic

2.3.1 Differential algebra

In this section, we review basic notions and facts of differential algebra, which are used throughout this thesis.

Let K be a field of arbitrary characteristic. A differential field (K, δ) is a field K with a derivation δ on K where a derivation on K is a additive homomorphism $\delta : K \to K$ satisfying Leibniz rule, that is for all $a, b \in K$,

$$\delta(ab) = \delta(a)b + a\delta(b).$$

The differential polynomial ring over (K, δ) in differential indeterminate x is a polynomial ring over K in a family of indeterminate $(x_i)_{i \in \mathbb{N}_0}$ by extending δ on K to the unique derivation on $K\{x\}$ which maps x_i to x_{i+1}, denoted by $K\{x\} = K[x_0, x_1, x_2, \ldots]$. We can also write $K\{x\} = K[x, \delta x, \delta^2 x, \ldots]$ where we identify $\delta^i x$ with x_i, and $\delta x_i = x_{i+1}$.

Let $f \in K\{x\} \setminus K$. The order of f, $\mathrm{ord} f$ is the highest i such that x_i appears in f. The degree of f, $\deg f$ is the degree of the indeterminate x_n when $n = \mathrm{ord} f$.

Let $f \in K\{x\} \setminus K$ with $\mathrm{ord} f = n$ and $\deg f = d$. Then $f(x)$ is of the form

$$f(x) = \sum_{k=0}^{d} g_k x_n^k$$

where $g_k \in K[x_0, x_1, \ldots, x_{n-1}]$. In particular g_d is called the initial of f denoted by I_f. The separant of f, S_f is the partial derivative of f with respect to x_n, that is

$$S_f = \frac{\partial f}{\partial x_n} = \sum_{k=1}^{d} k g_k x_n^{k-1}$$

The rank of f is denoted by $\mathrm{rank} f = (\mathrm{ord} f, \deg f)$, and we order differential polynomials lexicographically. In particular, $\mathrm{rank} S_f$ and $\mathrm{rank} I_f$ are smaller than $\mathrm{rank} f$.

Applying δ on f, we have the following facts,

$$\delta f = f^\delta + \sum_{k=0}^{n} \frac{\partial f}{\partial x_k} x_{k+1}$$

where f^δ is the polynomial obtained from f by taking derivatives of all coefficients of f. Also when $\mathrm{ord} f = n$, we have for each $l \geq 1$, there exists $f_i \in K\{x\}$ with $\mathrm{ord} f_l < n + l$ such that

$$\delta^l f = S_f x_{n+l} + f_l$$

A set $I \subseteq K\{x\}$ is a differential ideal if I is an ideal of $K\{x\}$ and $\delta f \in I$ for all $f \in I$. If I is a differential ideal and prime ideal, then I is called a prime differential ideal. Let B be a subset of $K\{x\}$. The ideal generated by B and the differential ideal generated by B are denoted by $(B)_{K\{x\}}$ and $[B]_{K\{x\}}$ respectively.

Theorem 2.14. *(Division Algorithm, Chapter 1 of [6])*

(1) *Let* $f \in K\{x\} \setminus \{0\}$ *and* $g \in K\{x\}$. *Then there exists* $m \in \mathbb{N}$ *and* $r \in K\{x\}$ *with* $\mathrm{ord} r \leq \mathrm{ord} f$ *such that*

$$S_f^m g \equiv r \mod [f]_{K\{x\}}$$

(2) *Let* $f \in K\{x\} \setminus \{0\}$ *and* $g \in K\{x\}$ *with* $\mathrm{ord} g \leq \mathrm{ord} f$. *Then there exists* $l \in \mathbb{N}$ *and* $r \in K\{x\}$ *with* $\mathrm{rank} r < \mathrm{rank} f$ *such that*

$$I_f^l g \equiv r \mod (f)_{K\{x\}}$$

Let (K, δ) be a differential field of characteristic $p \neq 0$. Then the set of constants of K denoted by $C_K = \{a \in K \mid \delta a = 0\}$ is a differential subfield of K. Moreover, $K^p = \{a^p \mid a \in K\}$ is a differential subfield of C_K.

Theorem 2.15. *(Differential Basis theorem, Chapter 3 of [6])*

(1) *(Characteristic zero) The differential polynomial ring* $K\{\bar{x}\}$ *has ACC on radical differential ideals.*

(2) *(Characteristic* $p > 0$*) The differential polynomial ring* $K\{\bar{x}\}$ *has ACC on radical differential ideals if and only if* $[C_K : K^p]$ *is finite.*

Theorem 2.16. *(Chapter 2 of [6]) Suppose* (E_1, δ) *and* (E_2, δ) *are differential field extensions of* (K, δ) *that are separable over* K. *Then there exists* (L, δ) *separable over* K *and* $f_1 : E_1 \to L$ *and* $f_2 : E_2 \to L$ *differential* K*-homomorphism such that* L *is the compositum of* $f_1(E_1)$ *and* $f_2(E_2)$.

2.3.2 Differentially closed fields

In this section, we review basic notions and facts of differential algebra and model theory of differential closed fields of positive characteristic. In particular, we will consider the generalization of the theory of differentially closed fields of positive characteristic in later Chapters.

A differential field (K, δ) of positive characteristic is said to be differentially closed if K is existentially closed in every differential field extension.

Definition 2.17. *(Wood [28])*

(1) (K, δ) *is differentially perfect if* $C_K = K^p$*; equivalently every differential field extension of* K *is a separable field extension.*

(2) *The theory of differentially perfect fields of characteristic* $p \neq 0$ *in* L_δ *is the theory of differential fields of characteristic* $p \neq 0$ *and* $C_k = K^p$*, that is*

$$\forall x \exists y (\delta x = 0 \Rightarrow y^p = x)$$

We denote this theory by DPF_p*.*

(3) *The theory of differentially closed fields in* L_δ *is* DPF_p *and*

"for every pair (f, g) *in* $K\{x\} \setminus \{0\}$ *with* $\mathrm{ord} g < \mathrm{ord} f$ *and* $S_f \neq 0$*, there exists* a *in* K *such that* $f(a) = 0$ *and* $g(a) \neq 0$*".*

We denote this theory by DCF_p*.*

The theory DCF_p is complete, and as we pointed out before, it is the model companion of DF_p. Furthermore it is the model completion of DPF_p (see Wood [27]).

Definition 2.18. *Suppose that* (K, δ) *is differentially perfect. The function* $r : K \to K$ *is defined by the conditions*

$$\begin{cases} r(a) = 0 & \text{if } a \notin C_K \\ r(a)^p = a & \text{if } a \in C_K \end{cases}$$

for all a *in* K*.*

Adding a function symbol r to the language L_δ to obtain $L_{\delta,r}$ and letting DCF_p^r denote the L_r-theory obtained by adding to DCF_p the L_r-sentence specifying that the r stands for the p-th root function on constants C_K, we have the follows.

Theorem 2.19. *(Wood, Shelah, and Macintyre see [28])*

(1) DCF_p^r *is the model completion of* DPF_p^r*, and* DCF_p^r *has quantifier elimination.*

(2) DCF_p^r *is stable, not superstable, and there exists a unique prime model extension over each model of* DPF_p^r*.*

2.4 Intermediate fields in positive characteristic

In this section, we make a few observations on the intermediate field theory of positive characteristic. Let (K, δ) be a differential field with characteristic $p \neq 0$. Since the fact that a constant field C_K of K is an intermediate field of K and K^p, it is essential to observe that some basic field theory in positive characteristic also applies for intermediate fields. Thus, for the following lemmas, the reader might wish to think of E as C_K. We also point out that the following lemmas are well known when $E = K$ (see Jacobson [5]) and our arguments are adaptations of this case.

Lemma 2.20. *Let E be an intermediate field of K and K^p. If $[E : K^p]$ is finite, then we have $[E : K^p] = p^e$ for some $e \in \mathbb{N}_0$.*

Proof. Let $a_1 \in E \setminus K^p$. Since $f_1 = t^p - a_1^p \in K^p[t]$ is irreducible, we have

$$[K^p(a_1) : K^p] = p.$$

Let $a_2 \in E \setminus K^p(a_1)$. Since $f_2 = t^p - a_2^p \in K^p(a_1)[t]$ is irreducible, we have

$$[K^p(a_1, a_2) : K^p] = [K^p(a_1, a_2) : K^p(a_1)][K^p(a_1) : K^p] = p \cdot p = p^2.$$

Since $[E : K^p]$ is finite, by iterating this process, there exists $e \in \mathbb{N}$ such that $[E : K^p] = p^e$. □

Lemma 2.21. *Let L be a field extension of K and E be an intermediate field of K and K^p. Then the following are equivalent:*

(1) *E and L^p are linearly disjoint over K^p*

(2) *Every subset of E which is p-independent over K^p is also p-independent over L^p*

(3) *There exists a p-basis of E over K^p which is p-independent over L^p*

Proof. For (1) → (2). Let $A \subseteq E$ be p-independent over K^p. Then p-monomials of A is linearly independent over K^p. Since E and L^p are linearly disjoint over K^p, it is also linearly independent over L^p.

For (2) $\to$ (3). Let $A \subseteq E$ be a p-basis of E over K^p. Then p-monomials of A is a basis of E over K^p, and hence it is linearly independent over K^p. By the assumption, it is also linearly independent over L^p.

For (3) $\to$ (1). Let $A = \{a_1^p, \ldots, a_n^p\} \subseteq L^p \setminus \{0\}$ be linearly dependent over E. We claim that it is linearly dependent over K^p.

Then there exists $b_1, \ldots, b_n \in E$ not all zero such that

$$a_1^p b_1 + \cdots + a_n^p b_n = 0$$

Let $B \subseteq E$ be a p-basis of E over K^p, and the p-monomials $\{m_i\}_{i \in I}$ of B be a basis of E over K^p. Then for each $b_i \in E$, there exists $c_{i1}, \ldots, c_{is}$ in K such that

$$b_i = c_{i1}^p m_1 + \cdots + c_{is}^p m_s$$

By substituting a_i's, we have

$$(c_{11}^p m_1 + \cdots + c_{1s}^p m_s) a_1^p + \cdots + (c_{n1}^p m_1 + \cdots + c_{ns}^p m_s) a_n^p = 0$$

Associating it with m_i's, we have

$$(c_{11}^p a_1^p + \cdots + c_{n1}^p a_n^p) m_1 + \cdots + (c_{1s}^p a_1^p + \cdots + c_{ns}^p a_n^p) m_s = 0$$

Since by the assumption, $m_1, \ldots, m_s$ are linearly independent over L^p, each coefficient of $m_i's$ are zero. Hence we have

$$(c_{11}^p a_1^p + \cdots + c_{n1}^p a_n^p) = \cdots = (c_{1s}^p a_1^p + \cdots + c_{ns}^p a_n^p) = 0$$

Since $b_1, \ldots, b_n$ in E are not all zero, there exists $1 \leq i \leq n$ and $1 \leq j \leq s$ such that $0 \neq c_{ij} \in K$, and hence we have

$$c_{1j}^p a_1^p + \cdots + c_{ij}^p a_i^p + \cdots + c_{nj}^p a_n^p = 0$$

Therefore, $a_1^p, \ldots, a_n^p$ is linearly dependent over K^p. □

Lemma 2.22. *Let $K/E/K^p$ be field extensions. Then every element of K that is separable algebraic over E is in E.*

Proof. Let $a \in K$ that is separable algebraic over E. Considering $x^p - a^p \in E[x]$ and $g \in E[x]$ a minimal polynomial of a over E. Then we have $x^p - a^p = h(x)g(x)$. Since g has no repeated roots, only possible form of g is $x - a$, and hence we have $a \in E$. □

Chapter 3

Separable prime differential ideals

In this chapter, we study certain prime differential ideals called separable prime differential ideals in the differential polynomial ring $K\{x\}$ over the differential field (K, δ) of positive characteristic in one differential indeterminate and one derivation. Namely, we give a full description/characterization of such differential ideals in terms of irreducible differential polynomials with nonzero separant.

First, we review some notions of separability in a field extension. Fix K a field of characteristic $p > 0$. Let L be a field extension of K and $(a)_{i \in I}$ be a family of elements of L. The family $(a)_{i \in I}$ is said to be separably dependent over K if there exists $f \in K[x_i]_{i \in I}$ such that f vanishes at $(a)_{i \in I}$ and for some $i \in I$ the partial derivative of f with respect to x_i does not vanish at the family. A family $(a)_{i \in I}$ is said to be separably independent in the contrary case. Kolchin (see page 4 of [6]) obtained that L is separable over K if and only if every family of elements of L that is algebraically dependent over K is separably dependent over K.

Let B is a subset of L. $B \subseteq L$ is said to be a separating transcendence basis over K if B is algebraically independent over K, and L is separable over $K(B)$. Then we have the criterion for separability.

Theorem 3.1. *(Chapter 0 of [6]) A finitely generated field extension is separable if and only if it has a separating transcendence basis.*

3.1 Separable prime differential ideals

A prime ideal P of $K\{x\}$ is said to be separable over K if the fraction field $\mathrm{Frac}(K\{x\}/P)$ is separable over K.

Let $I \subseteq K\{x\}$ be an ideal and $h \in K\{x\}$. The saturated ideal I over h is the set

$$I : h^{\infty} = \{g \in K\{x\} \mid h^m g \in I \text{ for some } m \in \mathbb{N}\}.$$

The following technical lemma will be useful in the proof of Lemma 3.3.

Lemma 3.2. *Let $f \in K\{x\}$ with* $\mathrm{ord} f = n$. *Then for each* $0 \leq i \leq n$,

$$\frac{\partial \delta f}{\partial x_i} = \delta \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_{i-1}}$$

with the convention that $\frac{\partial f}{\partial x_{-1}} = 0$.

Proof. Let $f \in K\{x\}$ with $\mathrm{ord} f = n$ and $0 \leq i \leq n$. Then we have

$$\delta f = f^{\delta} + \sum_{k=0}^{n} \frac{\partial f}{\partial x_k} x_{k+1}$$

By differentiating both sides with respect to x_i, we have

$$\begin{aligned} \frac{\partial \delta f}{\partial x_i} &= \frac{\partial f^{\delta}}{\partial x_i} + \sum_{k=0}^{n} \frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_k} x_{k+1}\Big) \\ &= \Big(\frac{\partial f}{\partial x_i}\Big)^{\delta} + \sum_{k=0}^{n} \Big(\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_k}\Big) x_{k+1} + \frac{\partial f}{\partial x_k}\frac{\partial}{\partial x_i}\Big(x_{k+1}\Big)\Big) \end{aligned}$$

Since if $k = i-1$, $\frac{\partial}{\partial x_i}(x_{k+1}) = 1$ (or 0 if otherwise), we have

$$\sum_{k=0}^{n} \frac{\partial f}{\partial x_k}\frac{\partial}{\partial x_i}\Big(x_{k+1}\Big) = \frac{\partial f}{\partial x_{i-1}}$$

Since $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_k}$ commute, we have

$$\begin{aligned}\frac{\partial \delta f}{\partial x_i} &= \Big(\frac{\partial f}{\partial x_i}\Big)^{\delta} + \sum_{k=0}^{n}\Big(\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_k}\Big)x_{k+1}\Big) + \frac{\partial f}{\partial x_{i-1}} \\ &= \delta\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_{i-1}}\end{aligned}$$

□

Lemma 3.3. *Let $P \subset K\{x\}$ be a nonzero prime differential ideal and let $f \in K\{x\}$ be an irreducible and minimal in P.*

(1) *If $S_f \in P$, then all partial derivatives of f are in P.*

(2) *If all partial derivatives of f are in P, then P is not separable over K.*

(3) *If P is separable over K, then $S_f \neq 0$.*

(4) *If $S_f \neq 0$, then $P = [f]_{K\{x\}} : S_f^{\infty}$.*

Proof. (1) Prove this by downwards induction on i for $0 \leq i \leq \text{ord}f = n$.

If $i = n$, then $\frac{\partial f}{\partial x_n} = S_f \in P$. Note that since $\text{rank}S_f < \text{rank}f$ and f is minimal rank in P, we have $S_f = 0$.

Let $\frac{\partial f}{\partial x_i} \in P$. Since $S_f = 0$, we have $\text{ord}(\delta f) \leq \text{rank}f$. By Theorem 2.14 (2), there exists $l \in \mathbb{N}$ and $r \in K\{x\}$ with $\text{rank}r < \text{rank}f$ such that

$$I_f^l \delta f \equiv r \mod (f)$$

Since $\delta f \in P$, we have $r \in P$. And since $\text{rank}r < \text{rank}f$ and f is minimal rank in P, we have $r = 0$. Hence we have $I_f^l \delta f \in (f)$. Since f is irreducible and $\text{ord}I_f^l < \text{ord}f$, we have $\delta f \in (f)$.

Then there exists $g \in K\{x\}$ such that $\delta f = gf$. By differentiating both sides with respect to x_i, by Lemma 3.2, we have

$$\frac{\partial \delta f}{\partial x_i} = \delta\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_{i-1}}$$

$$\frac{\partial g f}{\partial x_i} = \frac{\partial g}{\partial x_i} f + g\frac{\partial f}{\partial x_i}$$

Hence we have

$$\delta\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_{i-1}} = \frac{\partial g}{\partial x_i} f + g\frac{\partial f}{\partial x_i}$$

By considering each term of the equation, we have $\frac{\partial f}{\partial x_{i-1}}$ is in P.

(2) Let $\mathrm{ord} f = n$. Let $K\langle a\rangle = \mathrm{Frac}(K\{x\}/P)$ for $a = x + P$. Then we have $f(a) = 0$. We claim that the tuple $(a, \delta a, \ldots, \delta^n a)$ witnesses that P is not separable, that is, for each $g \in K\{x\}$ with $\mathrm{ord} g \leq n$ and $g(a) = 0$, we will show that $\frac{\partial g}{\partial x_i}(a) = 0$ for all i.

By Lemma 2.14 (1), there exists $l \in \mathbb{N}$ and $r \in K\{x\}$ with $\mathrm{rank} r < \mathrm{rank} f$ such that

$$I_f^l g \equiv r \mod (f)$$

Since $g \in P$, we have $r \in P$. Since f is minimal rank in P and $\mathrm{rank} r < \mathrm{rank} f$, we have $r = 0$, and hence we have $I_f^l g \in (f)$.

Since $\mathrm{ord} I_f^l < \mathrm{ord} f$, we have $I_f^l \notin P$. Since (f) is prime, we have $g \in (f)$.

Hence there exists $h \in K\{x\}$ such that $g = hf$. By taking partial derivatives in both sides with respect to x_i, we have

$$\begin{aligned}\frac{\partial g}{\partial x_i} &= \frac{\partial hf}{\partial x_i} \\ &= \frac{\partial h}{\partial x_i} f + h\frac{\partial f}{\partial x_i}\end{aligned}$$

By the assumption, we have $\frac{\partial f}{\partial x_i} \in P$, and hence we have $\frac{\partial g}{\partial x_i} \in P$. Therefore we have $\frac{\partial g}{\partial x_i}(a) = 0$.

(3) If $S_f = 0$, we have $S_f \in P$ and then by Lemma (1), all partial derivatives of f are in P. By Lemma (2), P is not separable over K. This contradicts to the assumption.

(4) Let $g \in [f] : S_f^\infty$. There exists $k \in \mathbb{N}$ such that $S_f^k g \in [f]$. Since $S_f \neq 0$ and f is minimal in P, we have $S_f \notin P$, and hence we have $S_f^k \notin P$. Since $[f] \subseteq P$ and P is prime, we have $g \in P$.

Let $g \in P \setminus \{0\}$. By Theorem 2.14 (1), there exists $m \in \mathbb{N}$ and $r_1 \in K\{x\}$ with $\mathrm{ord} r_1 \leq \mathrm{ord} f$ such that

$$S_f^m g \equiv r_1 \mod [f]$$

Then since $S_f^m g \in P$, we have $r_1 \in P$.

If $\mathrm{ord} r_1 < \mathrm{ord} f$, since f is minimal rank in P, we have $r_1 = 0$, and hence we have $S_f^m g \in [f]$. Therefore we have $g \in [f] : S_f^\infty$.

If $\mathrm{ord} r_1 = \mathrm{ord} f$ with $r_1 \neq 0$, by Theorem 2.14 (2), there exists $l \in \mathbb{N}$ and $r_2 \in K\{x\}$ with $\deg r_2 < \deg f$ such that

$$I_f^l r_1 \equiv r_2 \mod (f)$$

Then since $r_1 \in P$, we have $r_2 \in P$. Since $\deg r_2 < \deg f$ and f is minimal rank in P, we have $r_2 = 0$, and hence $I_f^l r_1 \in (f)$. Since (f) is prime and $\mathrm{ord} I_f^l < \mathrm{ord} f$, we have $r_1 \in (f) \subseteq [f]$. Therefore we have $S_f^m g \in [f]$.

□

As a result of the Lemma, we have the following.

Theorem 3.4. *Let $P \subset K\{x\}$ be a nonzero prime differential ideal separable over K, then $P = [f] : S_f^\infty$ for any irreducible and minimal f in P.*

Assuming $S_f \neq 0$, we will show that for an irreducible f, $[f] : S_f^\infty$ is a prime differential ideal.

Lemma 3.5. *Let $f \in K\{x\}$ be irreducible with $S_f \neq 0$ and $\mathrm{ord} f = n$.*

(1) $(f) : S_f^\infty \cap K[x_0, \dots, x_n] = [f] : S_f^\infty \cap K[x_0, \dots, x_n]$

(2) $[f] : S_f^\infty$ *is a prime differential ideal*

Proof. (1) The inclusion $\subseteq$ is clear.

Let $g \in [f] : S_f^\infty \cap K[x_0, \dots, x_n]$. Then we have $S_f^m g \in [f] \cap K[x_0, \dots, x_n]$ for some $m \in \mathbb{N}$, and hence there exists $l \in \mathbb{N}$ and $h_0, \dots, h_l \in K\{x\}$ such that $S_f^m g = h_0 f + \cdots + h_l \delta^l f$. We prove the desired result by induction on l.

If $l = 0$, then $S_f^m g = h_0 f$. Therefore we have $S_f^m g \in (f) \cap K[x_0, \ldots, x_n]$, and hence $g \in (f) : S_f^\infty \cap K[x_0, \ldots, x_n]$.

If $l = k$ holds, considering $S_f^m g = h_0 f + \cdots + h_k \delta^k f + h_{k+1} \delta^{k+1} f$ with $h_{k+1} \neq 0$, (see Section 2.3.1) we have

$$\delta^{k+1} f = S_f \delta^{n+k+1} x + f_{k+1}$$

where $f_{k+1} \in K\{x\}$ with $\mathrm{ord} f_{k+1} < n + k + 1$. Hence we have

$$S_f^m g = h_0 f + \cdots + h_{k+1}(S_f \delta^{n+k+1} x + f_{k+1})$$

Let $K\{x\}_{S_f}$ be the localization of $K\{x\}$ at S_f. By $S_f \neq 0$ we let a map $\phi : K\{x\} \to K\{x\}_{S_f}$ be ring homomorphism defined by $\delta^{n+k+1} x \mapsto \frac{-f_{k+1}}{S_f}$. Then by applying the map ϕ for $S_f^m g$, we have

$$S_f^m g = \phi(h_0) f + \cdots + \phi(h_k) \delta^k f$$

Let $d_i = \deg h_i$. Then there exists $g_0, \ldots, g_{d_i} \in K\{x\}$ where each g does not contain variable $\delta^{n+k+1} x$ such that

$$h_i = g_0 + g_1(\delta^{n+k+1} x) + \cdots + g_{d_i}(\delta^{n+k+1} x)^{d_i}$$

Since we have $\phi(g_i(\delta^{n+k+1} x)^i) = g_i(\frac{-f_{k+1}}{S_f})^i$, we have

$$S_f^{d_i} \phi(h_i) = g_0 S_f^{d_i} + g_1 S_f^{d_i - 1}(-f_{k+1}) \cdots + g_{d_i}(-f_{k+1}^{d_i})$$

Let $d = \max\{d_0, \ldots, d_k\}$ and $\bar{h}_i = \phi(h_i) S_f^d$. Then we have $\bar{h}_i \in K\{x\}$. Hence we have

$$S_f^m g = \frac{1}{S_f^d}(\bar{h}_0 f + \cdots + \bar{h}_k \delta^k f)$$

and hence we have

$$S_f^{m+d} g = \bar{h}_0 f + \cdots + \bar{h}_k \delta^k f$$

Since $\bar{h}_0, \ldots, \bar{h}_k \in K\{x\}$, by induction hypothesis there exists $r \in \mathbb{N}$ such that $S_f^r g \in (f) \cap K[x, \ldots, \delta^n x]$ and hence $g \in (f) : S_f^\infty \cap K[x, \ldots, \delta^n x]$.

(2) Let $f \in K\{x\}$ be irreducible and $g_1 g_2 \in [f] : S_f^\infty$ where $g_1, g_2 \in K\{x\}$. By Theorem 2.14 (1), there exists $m_1, m_2 \in \mathbb{N}$ and $h_1, h_2 \in K\{x\}$ with their order less than or equal to n such that

$$S_f^{m_1} g_1 \equiv h_1 \mod [f]$$

$$S_f^{m_2} g_2 \equiv h_2 \mod [f]$$

Hence we have

$$S_f^{m_1+m_2} g_1 g_2 \equiv h_1 h_2 \mod [f]$$

Since $g_1 g_2 \in [f] : S_f^\infty$, we have $h_1 h_2 \in [f] : S_f^\infty$, and ord$h_1$ and ordh_2 are less than ord$f$, we have ord$h_1 h_2 \leq$ ord$f$, then by (1), we have $h_1 h_2 \in (f) : S_f^\infty$. Since f is irreducible, by $S_f \neq 0$ we have $h_1 \in (f)$ or $h_2 \in (f)$. If $h_1 \in (f)$, then we have $S_f^{m_1} g_1 \in (f)$. Therefore we have $g_1 \in [f] : S_f^\infty$.

□

Theorem 3.6. *Let $f \in K\{x\}$ be irreducible with $S_f \neq 0$. Then $P = [f] : S_f^\infty$ is a prime differential ideal separable over K and f is minimal rank in P.*

Proof. Let $P = [f] : S_f^\infty$ for irreducible f with $S_f \neq 0$ and ord$f = n$. By Lemma 3.5 (2), P is a prime differential ideal.

We now argue that f is minimal rank in P. Indeed, if $g \in P$ with ord$g \leq$ ord$f$, then $g \in [f] : S_f^\infty \cap K[x, \ldots, \delta^n x]$. Then by Lemma 3.5 (1), we have $g \in (f) \cap K[x, \ldots, \delta^n x]$, and since rank$S_f <$ rank$f$ and (f) is prime, we have $g \in (f)$. Hence we have ord$g =$ ord$f$ and $\deg f \leq \deg g$. In other words, rank$f \leq$ rank$g$.

For $a = x + P$, we have $K\langle a \rangle = \mathrm{Frac}(K\{x\}/P)$. We claim that $K\langle a \rangle = K(a, \delta a, \ldots, \delta^n a)$. Since we have $\delta f \in P$, we have $\delta f(a) = 0$. Then we have

$$0 = \delta f(a) = f^\delta(a) + \frac{\partial f}{\partial x}(a)\delta a + \cdots + \frac{\partial f}{\partial \delta^{n-1} x}(a)\delta^n a + \frac{\partial f}{\partial \delta^n x}(a)\delta^{n+1} a$$

By $S_f \notin P$, we have $S_f(a) \neq 0$, and

$$\delta^{n+1}a = \frac{-1}{S_f(a)}\big(f^{\delta}(a) + \frac{\partial f}{\partial x}(a)\delta a + \cdots + \frac{\partial f}{\partial \delta^{n-1}x}(a)\delta^n a\big)$$

Considering each term of the right hand side of the equation, we have $\delta^{n+1}a \in K(a, \delta a, \ldots, \delta^n a)$. By induction step, we have $\delta^k a \in K(a, \delta a, \ldots, \delta^n a)$ for all $k > n$. Hence we have $K\langle a\rangle = K(a, \delta a, \ldots, \delta^n a)$.

We claim that $B = \{a, \delta a, \ldots, \delta^{n-1}a\}$ is separating transcendence basis over K, that is, $K(a, \delta a, \ldots, \delta^n a)$ is separable over K.

B is algebraic independent over K since if not, there exist $g \in K\{x\} \setminus \{0\}$ with ord$g < n$ such that $g(a, \delta a, \ldots, \delta^{n-1}a) = 0$. Then we have $g \in P$ and it contradicts to the minimality of f in P.

Next we claim that $K(a, \delta a, \ldots, \delta^n a)$ is separable over $K(B)$. It is enough to show that $\delta^n a$ is separably algebraic over K. Let $h(x_n) = f(a, \delta a, \ldots, \delta^{n-1}a, x_n) \in K(a, \ldots, \delta^{n-1}a)[x_n]$. Then we have $h(\delta^n a) = 0$, and

$$\begin{aligned}\frac{\partial h}{\partial x_n}(\delta^n a) &= \frac{\partial f(a, \ldots, \delta^{n-1}a, x_n)}{\partial x_n}(\delta^n a)\\ &= S_f(a)\\ &\neq 0\end{aligned}$$

□

By the proof above, we have the following.

Corollary 3.7. *Let* $P = [f] : S_f^{\infty}$ *with irreducible* f *and* $S_f \neq 0$. *Then* Frac$(K\{x\}/P)$ *is separable over* K.

Chapter 4

Separably differentially closed fields

In this chapter, we study a new class of differential fields called separably differentially closed fields, a differential analogue to separably closed fields. Generalizing the work of Wood on the theory of differentially closed fields of positive characteristic DCF_p, we show that this class is elementary, denoted by SDCF. Namely, we characterize this class of differential fields in a first-order fashion in the sense of mathematical logic in the manner of Blum for characteristic zero. Moreover, we show several characterizations for it, particularly, in terms of Kolchin's constrained extensions.

4.1 Separability in a differential field extension

In this section, we will recall the notions on the separability of a differential field extension and some classical results that are used in this chapter.

Fix (K, δ) a differential field of arbitrary characteristic. Let $(a_i)_{i \in I}$ be a family of elements of an extension of (K, δ). $(a_i)_{i \in I}$ is differentially algebraically dependent over (K, δ) if $(\delta^j a_i)_{i \in I, j \in \mathbb{N}_0}$ is algebraically dependent over K, and it is said to be differentially algebraically independent in the contrary case. $(a_i)_{i \in I}$ is said to be differentially separably dependent over (K, δ) if $(\delta^j a_i)_{i \in I, j \in \mathbb{N}_0}$ is separably dependent over K, and it is differentially separably independent over (K, δ) in the contrary case, (in particular, we say that it is differentially separable or differentially inseparable over (K, δ) respectively if the index set I consists of a single element).

Note that if $(a_i)_{i\in I}$ is algebraically dependent over (K,δ), it is differentially algebraically dependent over (K,δ). A differential field extension (L,δ) of (K,δ) is said to be differentially separable if every element of L is differentially separable over K.

Given a differential field extension $(L,\delta)/(K,\delta)$, and a subset B of L, we denote the smallest differential δ-subfield of L containing K and B by $K\langle B\rangle$.

A differential field (K,δ) is said to be nondegenerate if for all nonzero $f\in K\{x\}$, there exists $a\in K$ such that $f(a)\neq 0$.

Lemma 4.1. *(Seidenberg [18]) (K,δ) is nondegenerate if and only if $[K:C_K]$ is infinite.*

We have a primitive element theorem for a differential field of arbitrary characteristic assuming the nondegenerate property, that is, every finitely generated differentially separable extension of (K,δ) is generated by a single element of K.

Theorem 4.2. *(Primitive element theorem, [18]) Let (K,δ) be nondegenerate. If $L=K\langle a_1,\ldots,a_n\rangle$ with a_i differentially separable over K, then there exists $a\in L$ such that $L=K\langle a\rangle$.*

Let $\bar{a}$ be a n-tuple of elements of an extension of (K,δ). The defining differential ideal of $\bar{a}$ over K denoted by $I_K^{\delta}(\bar{a})$ is the differential polynomials in $K\{\bar{x}\}$ that vanish at $\bar{a}$. Every defining differential ideal is prime, and conversely any prime differential ideal is the defining differential ideal of some tuple of elements.

Let $\bar{b}$ be another n-tuple of an extension of (K,δ). $\bar{a}$ is said to be a differential specialization of $\bar{b}$ over K if the defining differential ideal of $\bar{a}$ over K contains that of $\bar{b}$ over K. If $\bar{a}$ is a differential specialisation of $\bar{b}$ over K and $\bar{b}$ is also a differential specialization of $\bar{a}$, then the specialization is said to be a generic differential specialization.

Definition 4.3. *(Chapter 3 of [6]) Let $\bar{\alpha}$ be a finite tuple of elements of an extension of (K,δ). Then $\bar{\alpha}$ is constrained over (K,δ) if $K\langle\bar{\alpha}\rangle$ is separable over K, and there exists $B\in K\{\bar{x}\}$ with $B(\bar{\alpha})\neq 0$ such that for all differential specialisations $\bar{\beta}$ of $\bar{\alpha}$ over K with $K\langle\bar{\beta}\rangle/K$ separable, if $B(\bar{\beta})\neq 0$ then $\bar{\beta}$ is the generic differential specialisation. Such B is said to be a constraint of $\bar{\alpha}$ over K.*

The following theorem is a key fact that we will use throughout the thesis.

Theorem 4.4. *(Chapter 3 of [6]) Let $\bar{a}$ be a finite tuple of elements of an extension of (K, δ). If $K\langle\bar{a}\rangle$ is separable over K, then for all $B \in K\{\bar{x}\}$ with $B(\bar{a}) \neq 0$, there exists a differential specialization $\bar{\alpha}$ of $\bar{a}$ such that $\bar{\alpha}$ is constrained over (K, δ) by constraint $B(\bar{\alpha}) \neq 0$.*

Proposition 4.5. *(Chapter 3 of [6]) Let $\bar{a}$ and $\bar{b}$ be finite tuples of elements of an extension of (K, δ). Let $K\langle\bar{a}, \bar{b}\rangle$ be separable over $K\langle\bar{a}\rangle$. If $(\bar{a}, \bar{b})$ is constrained over (K, δ), then $\bar{a}$ is constrained over K.*

Let (L, δ) be a differential field extension of (K, δ) and B is a subset of L. Then $B \subseteq L$ is said to be a differential inseparability basis of (L, δ) over (K, δ) if B is differentially separably independent over (K, δ), and (L, δ) is differentially separable over $(K\langle B\rangle, \delta)$. $B \subseteq L$ is said to be a differential transcendence basis of (L, δ) over (K, δ) if B is differential inseparability basis of (L, δ) over (K, δ), and it is differentially algebraically independent over (K, δ). $B \subseteq L$ is said to be a separating differential transcendence basis of (L, δ) over (K, δ) if B is differentially transcendence basis of (L, δ) over (K, δ) and (L, δ) is separable over $K\langle B\rangle$.

Theorem 4.6. *(Chapter 2 of [6]) Let (L, δ) be a differential field extension of (K, δ).*

(1) *$B \subseteq L$ is a differential inseparability basis of (L, δ) over (K, δ) if and only if B is a maximal subset of L such that B is differentially separably independent over (K, δ).*

(2) *Let $\Sigma \subset T \subset L$ such that Σ is differentially separably independent over (K, δ), and (L, δ) is differentially separable over $(K\langle T\rangle, \delta)$. Then there exists a differential inseparability basis B of (L, δ) over (K, δ) with $\Sigma \subset B \subset T$.*

(3) *If L is separable over K, then every differential inseparability basis of (L, δ) over (K, δ) is a separating differential transcendence basis of (L, δ) over (K, δ).*

4.2 Separably differentially closed fields

In this section, we define separably differentially closed fields by the way of existential closedness, and characterise it in several manners.

Lemma 4.7. *Let (K, δ) be a differential field and $\bar{a}$ be a tuple of elements of an extension of (K, δ).*

(1) *If $\bar{a}$ is differentially algebraically independent over K, then $\bar{a}$ is not constrained over K.*

(2) *If $\bar{a}$ is constrained over K, then each entry of $\bar{a}$ is differentially separable over K.*

Proof. (1) We assume that $\bar{a}$ is constrained over K with constraint $g(a_1, \dots, a_n) \neq 0$ for some $n \in \mathbb{N}$ for contradiction. Since $\bar{a}$ is differentially algebraically independent over (K, δ), $I_K^\delta(\bar{a})$ is a zero ideal. Let i such that g has order $k \geq 0$ in x_i. Let $n > k$. We will consider the following system of a differential equation and an inequation

$$\delta^n x_i = 0 \land g_i(x_i) \neq 0$$

where $g_i \in K\langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\rangle\{x_i\}$, and g_i is given by $g(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$.

Since $\delta^n x_i$ is irreducible and $S_{\delta^n x_i} = 1 \neq 0$, by Theorem 3.6, we have that $P := [\delta^n x]_{K\{x\}}$ is a separable prime differential ideal over (K, δ), and hence $\text{Frac}(K\{x_i\}/P)$ is separable over K. Let $b = x_i + P$. Then we have $\text{Frac}(K\{x_i\}/P) \cong K\langle b\rangle$. Since $\text{ord} g_i < n$, we have

$$\delta^n b = 0 \land g_i(b) \neq 0.$$

Since $g_i(b) \neq 0$, we have $g(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \neq 0$.

Thus, the tuple $(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ is a generic specialization of $\bar{a}$ over K. However we have $\delta^n x_i \in I_K^\delta(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ which contradicts that $I_K^\delta(\bar{a})$ is the zero ideal.

(2) Let $\bar{a} = (a_1, \dots, a_n)$ and $A = \{a_1, \dots, a_n\}$. We assume that at least one of the entries of $\bar{a}$ is differentially inseparable over (K, δ) for contradiction.

By reordering the tuple $\bar{a}$, we may assume that $B = \{a_1, \dots, a_l\}$ is a maximal subset of A such that all element of B is differentially inseparable over (K, δ). Hence by Theorem 4.6 (1), B is a differential inseparability basis of $(K\langle\bar{a}\rangle, \delta)$ over (K, δ).

Since $K\langle\bar{a}\rangle$ is separable over K, by Theorem 4.6 (3), B is a separating differential transcendence basis of $(K\langle\bar{a}\rangle, \delta)$ over (K, δ). Hence B is differentially algebraically independent over (K, δ).

Since $K\langle\bar{a}\rangle$ is separable over $K\langle B\rangle$ and $\bar{a}$ is constrained over K, by Theorem 4.4 we have $(a_1, \ldots, a_l)$ is constrained over (K, δ). Hence by the above Lemma, $(a_1, \ldots, a_l)$ is differentially algebraically dependent over (K, δ) which is a contradiction.

□

We define a new class of differential fields called separably differentially closed fields, a differential analogue to separably closed fields.

Definition 4.8. *A differential field (K, δ) is said to be separably differentially closed if it is existentially closed in every differential field extension that is separable.*

Generalizing the work of Wood [27] on the theory of differentially closed fields of positive characteristic DCF_p, we show that this class is elementary, denoted by SDCF in the manner of Blum [1] for characteristic zero.

Theorem 4.9. *Let (K, δ) be a differential field. Then the followings are equivalent*

(1) *(K, δ) is a separably differentially closed field.*

(2) *For any pair (f, g) in $K\{x\} \setminus \{0\}$ with $S_f \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f$, there exists an element a in K such that $f(a) = 0$ and $g(a) \neq 0$. We denote this theory by* SDCF.

Proof. Let (f, g) be a pair in $K\{x\} \setminus \{0\}$ with $S_f \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f$. We factorize f into irreducible factors $f_1 \cdots f_s$. By Leibniz rule, we have

$$\begin{aligned} S_f &= \frac{\partial f_1 \cdots f_s}{\partial \delta^n x} \\ &= \frac{\partial f_1}{\partial \delta^n x} f_2 \cdots f_s + \cdots + \frac{\partial f_s}{\partial \delta^n x} f_1 \cdots f_{s-1} \end{aligned}$$

Since $S_f \neq 0$, one of the terms has to be nonzero. Hence we have an irreducible f_i with $S_{f_i} \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f_i$ for some i. By Theorem 3.6, $P = [f_i] : S_{f_i}$ is a separable prime differential ideal over (K, δ). Let $a = x + P$. Since ord$g <$ ordf_i, we have

$$K\langle a\rangle \models \exists x(f_i(x) = 0 \wedge g(x) \neq 0)$$

Since by Corollary 3.7, $K\langle a\rangle$ is separable over K and (K, δ) is separably differentially closed, we have

$$K \models \exists x(f_i(x) = 0 \wedge g(x) \neq 0)$$

Since $f_i(a) = 0$, we have $f(a) = 0$. Therefore we have

$$K \models \exists x(f(x) = 0 \wedge g(x) \neq 0)$$

So we have shown (1) $\rightarrow$ (2).

Now we show that (2) $\rightarrow$ (1). Let $(L, \delta)/(K, \delta)$ be a differential field extension that is separable. Let $(L, \delta) \models \exists \bar{x}\varphi(\bar{x})$ where φ is L_δ-quantifier free formula with parameter from K. Then we have $(L, \delta) \models \varphi(\bar{a})$ for some $\bar{a}$ in L. We may assume that $L = K\langle \bar{a}\rangle$.

Let $B \in K\{\bar{x}\}$ with $B(\bar{a}) \neq 0$. By Theorem 4.4, there exists a differential specialization $\bar{\alpha}$ of $\bar{a}$ such that $\bar{\alpha}$ is constrained over (K, δ) by constraint $B(\bar{\alpha}) \neq 0$. Since $\bar{\alpha}$ is constrained over (K, δ), by Lemma 4.7 (2) each entry of $\bar{b}$ is differentially separable over K. The assumption readily imples that (K, δ) is nondegenerate, and hence by Differential primitive element theorem 4.6, we have $K\langle\bar{\alpha}\rangle = K\langle\beta\rangle$ for some element β in some extension of K.

Since $\varphi(\bar{x})$ is quantifier free formula, it is of the form

$$f_1(\bar{x}) = 0 \wedge \cdots \wedge f_r(\bar{x}) = 0 \wedge B(\bar{x}) \neq 0$$

Since $(K\langle\beta\rangle, \delta) \models \exists\bar{x}\varphi(\bar{x})$, we have

$$(K\langle\beta\rangle, \delta) \models \exists\bar{x}(f_1(\bar{x}) = 0 \wedge \cdots \wedge f_r(\bar{x}) = 0 \wedge B(\bar{x}) \neq 0)$$

By $K\langle\bar{\alpha}\rangle = K\langle\beta\rangle$, we will transform $\bar{x}$ into x in single variable.

Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i = p_i(\beta)/q_i(\beta)$ with $p_i, q_i \in K\{x\}$. Then we have

$$(K\langle\beta\rangle,\delta) \models \exists x(f_1(\frac{p_1(x)}{q_1(x)},\ldots,\frac{p_n(x)}{q_n(x)}) = 0 \wedge \cdots$$

$$\cdots \wedge f_r(\frac{p_1(x)}{q_1(x)},\ldots,\frac{p_n(x)}{q_n(x)}) = 0 \wedge B(\frac{p_1(x)}{q_1(x)},\ldots,\frac{p_n(x)}{q_n(x)}) \neq 0$$

Then we let

$$\frac{H_i(x)}{Q(x)} = f_i(\frac{p_1(x)}{q_1(x)},\ldots,\frac{p_n(x)}{q_n(x)})$$

and

$$\frac{E(x)}{Q(x)} = g(\frac{p_1(x)}{q_1(x)},\ldots,\frac{p_n(x)}{q_n(x)})$$

where $H_1,\ldots,H_r$, Q, E are in $K\{x\}$ and $Q = q_1^{m_1}\cdots q_n^{m_n} \neq 0$ with m_i's are in $\mathbb{N}$. Then we have

$$(K\langle\beta\rangle,\delta) \models \exists x\big(\frac{H_1(x)}{Q(x)} = 0 \wedge \cdots \wedge \frac{H_r(x)}{Q(x)} = 0 \wedge \frac{E(x)}{Q(x)} \neq 0\big)$$

and the element β satisfies the formula with $H_1(\beta) = 0,\ldots,H_r(\beta) = 0$, $Q(\beta) \neq 0, E(\beta) \neq 0$.

Since $K\langle\beta\rangle/K$ is separable, by Theorem 3.6 we have a separable prime differential ideal $I^\delta(\beta) = [f] : S_f$ such that

$$f(\beta) = 0 \wedge S_f(\beta) \neq 0 \wedge g(\beta) \neq 0$$

where f is irreducible and minimal in $I_K^\delta(\beta)$, $S_f \neq 0$ and $\text{ord} g < \text{ord} f$.

Let $\Phi(x) = \{f(x) = 0 \wedge g_i(x) \neq 0 \mid g_i \in K\{x\} \setminus \{0\}$ such that $\text{ord} g_i < \text{ord} f\}$ which is finitely satisfiable by the assumption with $S_f \neq 0$. Then there exists K^+-saturated elementary extension $(\mathbb{U},\delta)$ of (K,δ) such that $\Phi(x)$ is realized by some $u \in \mathbb{U}$.

We claim that for all $h \in K\{x\}$ with $\text{rank} h < \text{rank} f$, we have $h(u) \neq 0$. If $\text{ord} h < \text{ord} f$, we heve $h(u) \neq 0$ since u is a realization of Φ. If $\text{ord} h = \text{ord} f = m$ with $\deg h < \deg f$, then we assume that $h(\alpha) = 0$ for contradiction. Since $\alpha, \delta\alpha, \ldots, \delta^m\alpha$ are algebraically independent over K and $\deg h < \deg f$, h divide f. That contradicts to the irreduciblility of f.

By differential division algorithm, there exists $k \in \mathbb{N}$ and $\hat{Q}$ with $\text{rank}\hat{Q} < \text{rank} f$ such that

$$S_f^k Q \equiv \hat{Q} \mod [f]_{K\{x\}}$$

and there exists $l \in N$ and $\hat{E}$ with rank$\hat{E} <$ rank$f$ such that

$$S_f^l E \equiv \hat{E} \mod [f]_{K\{x\}}$$

By the claim above, we have $\hat{Q}(u) \neq 0$ and $\hat{E}(u) \neq 0$, and hence we have $Q(u) \neq 0$ and $E(u) \neq 0$.

Since $H_1, \ldots, H_r \in I_K^\delta(\beta) = [f]_{K\{x\}} : S_f^\infty$ and rank$S_f(u) \neq 0$, we have $H_1(u) = 0, \ldots, H_r(u) = 0$. Therefore we have

$$(\mathbb{U}, \delta) \models \exists x \big(H_1(x) = 0 \wedge \cdots \wedge H_r(x) = 0 \wedge Q(x) \neq 0 \wedge E(x) \neq 0\big)$$

Then we have

$$(\mathbb{U}, \delta) \models \exists x \big(\frac{H_1(x)}{Q(x)} = 0 \wedge \cdots \wedge \frac{H_r(x)}{Q(x)} = 0 \wedge \frac{E(x)}{Q(x)} \neq 0\big)$$

and hence we have

$$(\mathbb{U}, \delta) \models \exists \bar{x}(f_1(\bar{x}) = 0 \wedge \cdots \wedge f_r(\bar{x}) = 0 \wedge B(\bar{x}) \neq 0)$$

Since $(\mathbb{U}, \delta)$ is elementary extension of (K, δ) , we have

$$(K, \delta) \models \exists \bar{x}(f_1(\bar{x}) = 0 \wedge \cdots \wedge f_r(\bar{x}) = 0 \wedge B(\bar{x}) \neq 0)$$

Hence we have $(K, \delta) \models \exists \bar{x} \varphi(\bar{x})$ as required. □

Definition 4.10. (1) *The theory of differential fields of characteristic p is the theory of field together with*

$$\delta(a + b) = \delta(a) + \delta(b)$$

$$\delta(ab) = \delta(a)b + a\delta(b)$$

We denote this theory by DF.

(2) *The theory of differential fields of characteristic p is* DF *together with*

$$\underbrace{(1 + \cdots + 1)}_{p \text{ times}} = 0.$$

We denote this theory by DF_p.

(3) *The theory of separably differentially closed fields in the language of* L_δ *is* DF *together with*

"for any pair (f, g) *in* $K\{x\}$ *with* $S_f \neq 0$ *and* $g \neq 0$ *and* $\mathrm{ord} g < \mathrm{ord} f$*, there exists an element* a *in* K *such that* $f(a) = 0$ *and* $g(a) \neq 0$*. " We denote this theory by* SDCF.

(4) *The theory of separably differentially closed fields of characteristic* $p > 0$ *is* $\mathrm{SDCF} \cup \mathrm{DF}_p$*. We denote this theory by* SDCF_p.

Recall that a differential field (K, δ) is nondegenerate if for all nonzero $f \in K\{x\}$, there exists $a \in K$ such that $f(a) \neq 0$. Seidenburg [18] show that (K, δ) is nondegenerate if and only if $[K : C_K] = \infty$.

Lemma 4.11. *Let* $(K, \delta) \models \mathrm{SDCF}_p$

(1) (K, δ) *is nondegenerate, or equivalently* $[K : C_K] = \infty$.

(2) $K \models \mathrm{SCF}_{p,\infty}$ *and* $[K : K^p] = \infty$.

(3) $C_K \models \mathrm{SCF}_{p,\infty}$ *and* $[C_K : C_K^p] = \infty$.

Proof. (1). Let $g \in K\{x\}$ with $\mathrm{ord} g = n$. Let f be a form $\delta^{n+1}x$. Then we have a pair (f, g) in $K\{x\} \setminus \{0\}$ such that $S_f = 1 \neq 0$. By the assumption, there exists $a \in K$ such that $g(a) \neq 0$. □

Proof. (2). Since $(K, \delta) \models \mathrm{SDCF}_p$, we have $[K : C_k] = \infty$. Since K^p is a subfield of C_K, we have

$$\infty = [K : C_k] < [K : K^p]$$

We claim that K is separably closed. Let $f \in K[x] \setminus K$ be separable, hence $S_f \neq 0$. Let $g = 1$. By the assumption for a pair (f, g), there exists $a \in K$ such that $f(a) = 0$. □

Proof. (3). Let $f \in C_K[x] \setminus C_K$ be separable, then $S_f \neq 0$. Let $g = 1$. By the assumption for a pair (f, g), we have $f(a) = 0$ for some $a \in K$. Since C_K is separably algebraically closed in K, we have $f(a) = 0$ for some $a \in C_K$. Hence C_K is separably closed.

We claim that $[C_K : C_K^p] = \infty$. Since (K, δ) is nondegenerate, there exists infinite linear basis $a_1, a_2 \dots$ of K over C_K. It is enough to show that $a_1^p, a_2^p \dots$ of K^p is linearly independent over C_K^p. If not, we have

$$a_1^p b_1^p + \dots + a_n^p b_n^p = 0$$

where $b_1, \dots, b_n$ in C_K are not all zero. By the Frobenius property, we have

$$a_1 b_1 + \dots + a_n b_n = 0$$

which contradicts to the assumption. □

4.3 Characterizations of separably differentially closed fields

We defined separably differentially closed fields and characterized it in manner of Blum. Moreover, we obtain several characterizations for it, particularly in terms of Kolchin's constrained extensions.

Definition 4.12. *A differential field (K, δ) is said to be constrainedly closed if every finite tuple of elements of an extension of (K, δ) that is constrained over (K, δ) is in (K, δ).*

Theorem 4.13. *Let (K, δ) be a differential field of arbitrary characteristic such that $[C_K : K^p]$ is finite. Then the following are equivalent*

(1) *(K, δ) is separably differentially closed.*

(2) *(K, δ) is existentially closed in every differentially algebraic field extension of (K, δ) that is separable.*

(3) *(K, δ) is constrainedly closed.*

(4) *Let* $P \subset K\{\bar{x}\}$ *be a nonzero prime differential ideal separable over* K *and* $g \in K\{\bar{x}\} \setminus P$. *There exists* $\bar{a}$ *in* K *such that* $f(\bar{a}) = 0$ *for all* $f \in P$, *and* $g(\bar{a}) \neq 0$.

(5) $(K, \delta) \models$ SDCF

Proof. (1) → (2). It is clear since every differentially algebraic field extension is differential field extension. □

Proof. (2) → (3). Let $\bar{a}$ be constrained over K. Then $K\langle\bar{a}\rangle/K$ is separable, and by Lemma 4.7 (1), $K\langle\bar{a}\rangle$ is differentially algebraic field extension.

Let g be a constraint for $\bar{a}$, and by Differential Basis theorem (Theorem 2.15 (2)), we have $I^{\delta}_{K}(\bar{a}) = \{f_1, \ldots, f_s\}$. Considering the system $f_1 = 0 \wedge \cdots \wedge f_s = 0 \wedge g \neq 0$, by the assumption there exists $\bar{b}$ in K satisfying the system. However then $I^{\delta}_{K}(\bar{a}) = I^{\delta}_{K}(\bar{b})$. Hence $\bar{a}$ is in K. □

(3) → (4). Let $P \subseteq K\{\bar{x}\}$ be a nonzero separable prime differential ideal over (K, δ) and $g \in K\{\bar{x}\} \setminus P$. Then $\mathrm{Frac}(K\{\bar{x}\}/P) \cong K\langle\bar{a}\rangle$ for $\bar{a} = \bar{x} + P$ is a separable differential field extension over (K, δ) and $f(\bar{a}) = 0$ for all $f \in P$ and $g(\bar{a}) \neq 0$.

By Theorem 4.4, there exists a differential specialisation $\bar{\alpha}$ of $\bar{a}$ such that $\bar{\alpha}$ is constrained over (K, δ) with constraint $g(\bar{\alpha}) \neq 0$. Hence we have $f(\bar{\alpha})$ and $g(\bar{\alpha}) \neq 0$. Since (K, δ) is constrainedly closed, we are done. □

(4) → (5). Let (f, g) be a pair in $K\{x\}$ with f irreducible and $S_f \neq 0$, $g(x) \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f$. Then by Theorem 3.6, we have a separable prime differential ideal $[f] : S_f^{\infty}$ over (K, δ). Since $\mathrm{ord} g < \mathrm{ord} f$, we have $g \notin P$. By the assumption, there exists $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$. □

(5) → (1). We show this in Theorem 4.9. □

Remark 4.14. *The condition* $[C_K : K^p]$ *being finite is usually referred as* K *is differentially quasi-perfect (see Chapter 2 of [6]).*

Chapter 5

Model theoretic properties of separably differentially closed fields

In this chapter, we introduce the notions of differential degree of imperfection, differential p-basis and differential λ-function. Then we establish several model theoretic properties of separably differentially closed fields in natural expansions of the language of differential fields. In particular, we prove that the theory of separably differentially closed fields of characteristic $p > 0$ with specified differential degree of imperfection is complete. Furthermore after adding names for differential p-basis and differential λ-functions, we obtain quantifier elimination.

As we will be working with different languages, we will write $F \subseteq_L K$ to mean that F is an L-substructure of K. All fields in this chapter are of characteristic $p > 0$.

5.1 Differential p-basis

In this section, we introduce two notions: the differential degree of imperfection and the differential p-basis. These are the differential analogues of the degree of imperfection and p-basis. (See algebraic degree of imperfection and p-basis in Section 2.2).

We recall that the theory of separably differentially closed fields, denoted by SDCF: for each pair (f, g) in $K\{x\} \setminus \{0\}$ with $S_f \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f$, there exists an element a in K such that $f(a) = 0$ and $g(a) \neq 0$. And SDCF_p is the theory of separably differentially closed fields of positive characteristic $p > 0$.

To understand the completion of SDCF_p, we need to specify the differential analogue of the degree of imperfection. If $[C_K : K^p]$ is finite, by Lemma 2.20 there exists $\epsilon \in \mathbb{N}_0$ such that

$$[C_K : K^p] = p^\epsilon$$

then we call ϵ the differential degree of imperfection of (K, δ), or we say that (K, δ) has a finite differential degree of imperfection denoted by

$$\epsilon(K) = \epsilon.$$

On the other hand, if $[C_K : K^p]$ is infinite, we say that (K, δ) has an infinite differential degree of imperfection.

Assumption 1. *From now on, throughout this chapter, we assume that differential fields have a finite differential degree of imperfection.*

Definition 5.1. (1) *The theory of differential fields of characteristic $p > 0$ with a differential degree of imperfection $\epsilon \in \mathbb{N}_0$ in the language of L_δ is DF_p together with*

"there exist $a_1, \cdots, a_\epsilon \in C_K$ such that for all elements $c \in C_K$, there exist unique elements $b_0, \cdots, b_s \in K$ such that

$$c = b_0^p m_0 + \cdots + b_s^p m_s$$

where $m_i's$ are the p-monomials of $\{a_1, \cdots, a_\epsilon\}$." We denote this theory by $\mathrm{DF}_{p,\epsilon}$.

(2) *The theory of separably differentially closed fields of characteristic $p > 0$ with a differential degree of imperfection ϵ in the language of L_δ is $\mathrm{SDCF} \cup \mathrm{DF}_{p,\epsilon}$. We denote this theory by $\mathrm{SDCF}_{p,\epsilon}$.*

Remark 5.2. *When $\epsilon = 0$, we have $C_K = K^p$; that is, K is differentially perfect. Thus we have $\mathrm{DF}_{p,0} = \mathrm{DPF}_p$ where DPF_p is the theory of differentially perfect fields of characteristic p (see Section 2.3.2). Furthermore, we have $\mathrm{SDCF}_{p,0} = \mathrm{DCF}_p$ where DCF_p is the theory of differentially closed fields of characteristic p (see Section 2.3.2).*

We will prove that $\mathrm{SDCF}_{p,\epsilon}$ is complete in L_δ (Corollary 5.24). To this end, we introduce the differential analogue of the p-basis.

Let $\bar{a} = (a_1, \ldots, a_n)$ be a tuple of elements from K. Recall that the p-monomials of $\bar{a}$ is the set of the form

$$\{a_1^{i_1} \cdots a_n^{i_n} \mid 0 \leq i_j \leq p-1\}.$$

Definition 5.3. (1) *A tuple $\bar{a}$ from K is said to be differentially p-independent for (K, δ) if $\bar{a}$ is a tuple from C_K and the p-monomials of $\bar{a}$ are linearly independent over K^p.*

(2) *A tuple $\bar{a}$ from K is said to be a differential p-basis for (K, δ) if $\bar{a}$ is a tuple from C_K and the p-monomials of $\bar{a}$ form a linear basis of C_K over K^p.*

Lemma 5.4. *Let $\epsilon > 0$. A differential field (K, δ) has differential degree of imperfection ϵ if and only if it has a differential p-basis of size ϵ.*

Proof. Let $\bar{a} = \{a_1, \ldots, a_\epsilon\}$ be a differential p-basis. Then the p-monomials $m_0, \ldots, m_{p^\epsilon - 1}$ of $\bar{a}$ form a linear basis for C_K over K^p. Thus, $[C_K : K^p] = p^\epsilon$.

On the other hand, let $[C_K : K^p] = p^\epsilon$. Let $a_1 \in C_K \setminus K^p$. Since $f_1 = t^p - a_1^p \in K^p[t]$ is irreducible,

$$\{a_1, a_1^2, \ldots, a_1^{p-1}\}$$

form a linear basis of $K^p(a_1)$ over K^p. Let $a_2 \in C_K \setminus K^p(a_1)$. Since $f_2 = t^p - a_2^p \in K^p(a_1)[t]$ is irreducible,

$$\{a_2, a_2^2, \ldots, a_2^{p-1}\}$$

form a linear basis of $K^p(a_1, a_2)$ over $K^p(a_1)$. By the Tower law,

$$\{a_1^{i_1} a_2^{i_2} \mid 0 \leq i_1, i_2 \leq p-1\}$$

form a linear basis for $K^p(a_1, a_2)$ over K^p. Continuing this process, we build $a_1, a_2, \ldots, a_\epsilon$ such that the p-monomials are a basis of $K^p(a_1, \ldots, a_\epsilon) = C_K$ over K^p. □

Remark 5.5. *Let K be equipped with the trivial derivation $\delta \equiv 0$. Then a tuple $\bar{a}$ is a differential p-basis for (K, δ) if and only if it is an algebraic p-basis for K.*

In particular, the differential degree of imperfection agrees with the field theoretic degree of imperfection.

Definition 5.6. (1) *The language of differential fields with a differential p-basis is denoted by*

$$L_{\delta,\bar{a}} = L_\delta \cup \{a_1, \cdots, a_\epsilon\}$$

where the constant symbols $\{a_1, \cdots, a_\epsilon\}$ *stand for a differential* p*-basis.*

(2) *The theory of differential fields of characteristic* p *with a differential* p*-basis* $\{a_1, \ldots, a_\epsilon\}$ *in the language of* $L_{\delta,\bar{a}}$ *is* $\mathrm{DF}_{p,\epsilon}$ *together with*

"for each element $c \in C_K$*, there exists unique elements* $b_0, \ldots, b_s \in K$ *with* $s = p^\epsilon - 1$ *such that*

$$c = b_0^p m_0 + \ldots + b_s^p m_s$$

where $m_i's$ *are the* p*-monomials of* $\{a_1, \ldots, a_\epsilon\}$*." We denote this theory by* $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$*.*

(3) *The theory of separably differentially closed fields of characteristic* p *with differential* p*-basis* $\{a_1, \ldots, a_\epsilon\}$ *in the language of* $L_{\delta,\bar{a}}$ *is* $\mathrm{SDCF} \cup \mathrm{DF}^{\bar{a}}_{p,\epsilon}$*. We denote this theory by* $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$*.*

We give a criterion for separability in a differential field extension (Proposition 5.8). This is the differential version of separability in a field extension (See Proposition 2.9). But first we need a result of Kolchin:

Lemma 5.7. *(Chapter 2 of [6]) Let* (L, δ) *be a differential extension of* (K, δ)*. Then* L/K *is separable if and only if* C_K *and* L^p *are linearly disjoint over* K^p*.*

Proposition 5.8. *Let* $(L, \delta)/(K, \delta)$ *be a differential field extension. Then the following are equivalent:*

(1) L/K *is separable*

(2) C_K *and* L^p *are linearly disjoint over* K^p

(3) *Every differential* p*-independent set for* (K, δ) *is differential* p*-independent set for* (L, δ)

(4) *Some differential* p*-basis for* (K, δ) *is differential* p*-independent for* (L, δ)

Proof. By the above lemma, we have (1) $\Leftrightarrow$ (2). Taking $E = C_K$ as intermediate field of K and K^p in Lemma 2.21, we have (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). □

Corollary 5.9. *If K and L are models of $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ and $K \subseteq_{L_{\delta,\bar{a}}} L$, then L is separable over K.*

Proof. Since K and L share the same differential p-basis, by the above Proposition, the extension is separable. □

Theorem 5.10. *The theory of $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ is model complete.*

Proof. Let $(K, \delta) \subseteq_{L_{\delta,\bar{a}}} (L, \delta)$ be models of $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$. We claim that (K, δ) is existentially closed in (L, δ). By the definition of SDCF (see Theorem 4.9), it is enough to show that the extension is separable. By Corollary 5.9, we are done. □

We are interested in what kind of extension preserves the differential degree of imperfection. We will see in Proposition 5.13 that a differential field extension generated by a constrained element preserves the differential degree of imperfection. However first, we need a result of Kolchin as follows.

Lemma 5.11. *(Chapter 3 of [6]) If $\bar{\alpha}$ is constrained over K, then $C_{K\langle\bar{\alpha}\rangle}$ is separably algebraic over $K\langle\bar{\alpha}\rangle^p C_K$.*

We now note that the above lemma can be improved as follows.

Lemma 5.12. *If $\bar{\alpha}$ is constrained over K, then $C_{K\langle\bar{\alpha}\rangle} = K\langle\bar{\alpha}\rangle^p C_K$.*

Proof. Let $\bar{\alpha}$ be constrained over K. By the previous lemma, $C_{K\langle\bar{\alpha}\rangle}$ is separable algebraic over $K\langle\bar{\alpha}\rangle^p C_K$. Since $K\langle\bar{\alpha}\rangle^p C_K$ is an intermediate field of $C_{K\langle\bar{\alpha}\rangle}$ and $K\langle\bar{\alpha}\rangle^p$, by Lemma 2.22 every element of $C_{K\langle\bar{\alpha}\rangle}$ that is separable algebraic over $K\langle\bar{\alpha}\rangle^p C_K$ is in $K\langle\bar{\alpha}\rangle^p C_K$. Hence we have $C_{K\langle\bar{\alpha}\rangle} = K\langle\bar{\alpha}\rangle^p C_K$. □

Proposition 5.13. *Let $\bar{\alpha}$ be constrained over K. Then a differential p-basis for (K, δ) remains a differential p-basis for $(K\langle\bar{\alpha}\rangle, \delta)$.*

Proof. Let $\{a_1, \ldots, a_\epsilon\}$ be a differential p-basis for (K, δ). Since $K\langle\bar{\alpha}\rangle$ is separable over K, it is also differentially p-independent for $(K\langle\bar{\alpha}\rangle, \delta)$.

We claim that $\{a_1, \ldots, a_\epsilon\}$ p-spans $C_{K\langle\bar{\alpha}\rangle}$ over $K\langle\bar{\alpha}\rangle^p$. By the above lemma, we have $C_{K\langle\bar{\alpha}\rangle} = K\langle\bar{\alpha}\rangle^p C_K$, and since $\{a_1, \ldots, a_\epsilon\}$ p-spans C_K over K, we have

$$\begin{aligned} C_{K\langle\bar{\alpha}\rangle} &= K\langle\bar{\alpha}\rangle^p C_K \\ &= K\langle\bar{\alpha}\rangle^p Span_{K^p}\{m_1, \cdots, m_s\} \\ &= Span_{K\langle\bar{\alpha}\rangle^p}\{m_1, \cdots, m_s\} \end{aligned}$$

where $m_i's$ are p-monomials of $\{a_1, \ldots, a_\epsilon\}$. □

Theorem 5.14. (1) *Every model of* $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ *has an extension to a model of* $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$.

(2) $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ *is the model companion of* $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$.

Proof (1). $K \models \mathrm{DF}^{\bar{a}}_{p,\epsilon}$. Let (f, g) in $K\{x\} \setminus \{0\}$ with $S_f \neq 0$ and $\mathrm{ord} g < \mathrm{ord} f$.

Let an element a be in a field extension that satisfies $f(a) = 0$ and $g(a) \neq 0$ and $K\langle a\rangle/K$ is separable. Such a exists because $P = [f] : S_f^\infty$ is a prime differential ideal of $K\{x\}$, which is separable over K with $g \notin P$, one can take $a = x + P$ in $\mathrm{Frac}(K\{x\}/P)$.

Then by Theorem 4.4, there exists a differential specialization α of a such that α is constrained over K by constraint g, and hence we have $f(\alpha) = 0$ and $g(\alpha) \neq 0$. By the above Proposition, K and $K\langle\alpha\rangle$ share the same differential p-basis. By iterating this process, we can build L as an extension of K with $L \models \mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$. □

Proof (2). Since $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ is model complete, and every model of $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ has an extension to a model of $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$, we are done. □

Fact 1.1 (Wood [27]). Let (K, δ) be a differential field. Let $b \in C_K$ and c the p-th root of b in an algebraic closure of K. Then, there is a unique derivation on $K(c)$ extending that on K such that $\delta(c) = 0$.

Lemma 5.15. *Let (K, δ) be a differential field and A and B disjoint subsets of K. Assume $A \cup B$ is a differential p-basis for K. Then, there is a differential field extension $K \subseteq_{L_\delta} K'$ such that A is a differential p-basis for for K' (with the convention that if A is empty, then A being a differential p-basis of K' means $C_{K'} = (K')^p$).*

Proof. We assume $B = \{b\}$ (the general case follows by a standard transfinite induction).

Let b_1 be a p-th root of b. By the above fact, there is a unique extension of the derivation to $K_1 = K(b_1)$ with $\delta(b_1) = 0$. One can readily check that $A \cup \{b_1\}$ is a differential p-basis for K_1. Repeating this process, we set b_{i+1} to be a p-th root of b_i and extend the derivation to $K_i(b_{i+1})$ such that $\delta(b_{i+1}) = 0$. Again, it follows that $A \cup \{b_{i+1}\}$ is a differential p-basis for K_{i+1}.

Now let $K' = \bigcup_i K_i$. Since A is differentially p-independent in each K_i, we get that A is differentially p-independent in K'. Furthermore, from the construction we see that A differentially p-spans K'. In other words, A is a differential p-basis for K', as desired.

□

Proposition 5.16. (1) $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ *has the amalgamation property.*

(2) $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ *is the model completion of* $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$.

Proof. (1) Let $K, E_1, E_2 \models \mathrm{DF}^{\bar{a}}_{p,\epsilon}$ with $L_{\delta,\bar{a}}$-embeddings $f_1 : K \to E_1$, and $f_2 : K \to E_2$. Then by Corollary 5.9, E_1/K and E_2/K are separable. By Theorem 2.16, there exists L ($L \models \mathrm{DF}^{\bar{a}}_{p,\epsilon}$ by the lemma above) and $g_1 : E_1 \to L$ and $g_2 : E_2 \to L$ $L_{\delta,\bar{a}}$-embeddings such that $g_1 \circ f_1 = g_2 \circ f_2$.

(2) Since $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ is the model companion of $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ and $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$ has the amalgamation property, by Theorem 2.2 (1) we have that $\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ is the model completion of $\mathrm{DF}^{\bar{a}}_{p,\epsilon}$.

□

5.2 Differential λ-function

$\mathrm{SDCF}^{\bar{a}}_{p,\epsilon}$ does not admit quantifier elimination. Indeed for any model K, quantifier free definable subsets of C_K are finite or cofinite. Now since K is the model, the subfield K^{p^2} is an infinite subfield of C_K and also coinfinite as it a proper subfield. Thus, the definable subfield K^{p^2} is not quantifier free definable.

In this section, we introduce a notion called the differential λ-functions that is a differential analogue of the algebraic λ-function. (See algebraic λ-function in Section 2.2). By expanding the language $L_{\delta,\bar{a}}$ to include symbols for these differential λ-functions, we obtain quantifier elimination.

We fix a differential field (K, δ) with a differential p-basis $\bar{a} = \{a_1, \cdots, a_\epsilon\}$.

Definition 5.17. *The differential λ-functions $\ell_i : K \to K$ for $0 \leq i \leq p^\epsilon - 1$ are defined by the conditions:*

$$\begin{cases} \ell_i(a) = 0 & \text{if } a \notin C_K \\ a = \ell_0(a)^p m_0 + \cdots + \ell_s(a)^p m_s & \text{if } a \in C_K \end{cases}$$

for all a in K where m_i's are p-monomials for $\bar{a}$.

Remark 5.18. *Note that δ and ℓ_i do not generally commute. Indeed, Let $\epsilon = 0$. Then we may take ℓ_0 to be pth root. Let (K, δ) have a nontrivial derivation δ. Let $a \notin C_K$. Then we have $\ell_0 \delta(a^p) = \ell_0(0) = 0$, however $\delta \ell_0(a^p) = \delta(a) \neq 0$.*

Definition 5.19. (1) *The language of differential fields with a differential p-basis and differential λ-functions is denoted by*

$$L_{\delta,\ell} = L_\delta \cup \{a_1, \cdots, a_\epsilon\} \cup \{\ell_i \mid 0 \leq i \leq p^\epsilon - 1\}$$

where the unary function symbols ℓ_i's stand for the differential λ-functions.

(2) *The theory of differential fields of characteristic p with a differential p-basis $\{a_1, \cdots, a_\epsilon\}$ in the language of $L_{\delta,\ell}$ is DF_p together with*

$$\begin{cases} \ell_i(c) = 0 & \text{if } \delta(c) \neq 0 \\ c = \ell_0(c)^p m_0 + \cdots + \ell_s(c)^p m_s & \text{if } \delta(c) = 0 \end{cases}$$

for all $c \in K$ where m_i's are the p-monomials of $\{a_1, \cdots, a_\epsilon\}$ and $s = p^\epsilon - 1$. We denote this theory by $\mathrm{DF}^\ell_{p,\epsilon}$.

(3) *The theory of separably differentially closed fields of characteristic p with a differential p-basis $\{a_1, \cdots, a_\epsilon\}$ and differential λ-functions $\{\ell_0, \ldots, \ell_{p^\epsilon - 1}\}$ in the language of $L_{\delta,\ell}$ is $\mathrm{SDCF} \cup \mathrm{DF}^\ell_{p,\epsilon}$. We denote this theory by $\mathrm{SDCF}^\ell_{p,\epsilon}$.*

Remark 5.20. (1) *Note that $\mathrm{DF}^\ell_{p,\epsilon}$ is a universal theory.*

(2) *Let $\epsilon = 0$ and $a \in C_K$. Then we have $a = \ell_0(a)^p$ and hence ℓ_0 is the pth root function on C_K. Wood ([28]) denotes ℓ_0 as r, and we have $\mathrm{SDCF}^\ell_{p,0} = \mathrm{DCF}^r_p$ (see Section 2.3.2).*

Proposition 5.21. *Let $(F, \delta) \models \mathrm{DF}^\ell_{p,\epsilon}$ If $(E, \delta) \subseteq_{L_{\delta,\ell}} (F, \delta)$, then they share the same differential p-basis $\{a_1, \cdots, a_\epsilon\}$, and hence F is separable over E (by Proposition 5.8).*

Proof. Let $\{a_1, \ldots, a_\epsilon\}$ be a differential p-basis for (F, δ). We claim that it is also a differential p-basis for (E, δ). Let $a \in C_E$. Since $a \in C_F$, we have

$$a = b_0^p m_0 + \cdots + b_s^p m_s$$

where b_i's are in F and m_i's are p-monomials for $\{a_1, \ldots, a_\epsilon\}$. Since $b_i = \ell_i(a)$ for $0 \leq i \leq s$ and E is closed under the ℓ_i's, we deduce $b_i \in E$. Hence $\{a_1, \ldots, a_\epsilon\}$ differentially p-spans C_E over E^p.

Since $\{a_1, \ldots, a_\epsilon\}$ is differentially p-independent for (F, δ), it is also differentially p-independent for (E, δ). Hence we are done. □

The proof of Proposition 5.14 (1) and Theorem 5.16 (1) for $\mathrm{DF}_{p,\epsilon}^{\bar{a}}$ can be easily adapted to prove the following.

Proposition 5.22. (1) *Every model of* $\mathrm{DF}_{p,\epsilon}^{\ell}$ *has an extension to a model of* $\mathrm{SDCF}_{p,\epsilon}^{\ell}$.

(2) $\mathrm{DF}_{p,\epsilon}^{\ell}$ *has the amalgamation property.*

We now aim to prove that $\mathrm{SDCF}_{p,\epsilon}^{\ell}$ has quantifier elimination. To this end, we first find an equivalence between quantifier free $L_{\delta,\ell}$-formulas and $L_{\delta,\bar{a}}$-formulas.

Let $T_{\delta,\ell}$ be the set of $L_{\delta,\ell}$-terms in variables $x_1, x_2, \ldots$. For each $t \in T_{\delta,\ell}$ and $0 \leq i \leq s$ with $s = p^\epsilon - 1$, we introduce new variables $y_{0t}, \ldots, y_{st}$.

Given a quantifier free $L_{\delta,\ell}$-formula φ. Going from left to right in φ, for each occurrence of $\ell_i(t)$, replace φ with the following:

$$\exists y_{0t} \ldots y_{st} \big[\ \varphi(\ell_i(t)/y_{it}) \wedge$$
$$\Big(\big(\delta(t) = 0 \wedge (t = y_{0t}^p m_0 + \cdots + y_{st}^p m_s)\big) \vee \big(\delta(t) \neq 0 \wedge (y_{it} = 0)\big)\Big)\big]$$

where $\varphi(\ell_i(t)/y_{it})$ is obtained by replacing $\ell_i(t)$ by y_{it} in φ. Continuing this process by induction on the complexity of t, we replace φ by quantifier free $L_{\delta,\bar{a}}$-formula.

Theorem 5.23. (1) $\mathrm{SDCF}_{p,\epsilon}^{\ell}$ *is model complete.*

(2) $\mathrm{SDCF}_{p,\epsilon}^{\ell}$ *is the model companion of* $\mathrm{DF}_{p,\epsilon}^{\ell}$.

(3) $\mathrm{SDCF}_{p,\epsilon}^{\ell}$ *is the model completion of* $\mathrm{DF}_{p,\epsilon}^{\ell}$.

(4) $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ *has quantifier elimination.*

Proof. (1) Let $(K,\delta) \subseteq_{\delta,\ell} (L,\delta)$ be models of $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ and $\varphi(\bar{x})$ a quantifier free $L_{\delta,\ell}$-formula with parameters from K such that there is $\bar{b}$ from L with $L \models \varphi(\bar{b})$. We can find a finite set $\{t_1,\dots,t_r\} \subseteq T_{\delta,\ell}$ and a quantifier free $L_{\delta,\bar{a}}(K)$-formula $\tilde{\varphi}$ such that

$$L \models \exists y_{0t_1}\cdots\exists y_{st_1},\dots,\exists y_{0t_r}\cdots\exists y_{st_r}\tilde{\varphi}(\bar{b},\bar{y}).$$

In other words there is a tuple $\bar{c}$ from L such that $L \models \tilde{\varphi}(\bar{b},\bar{c})$. Now, by Proposition 5.21, L/K is separable, and thus since (K,δ) is separably differentially closed, we can find $(\bar{d},\bar{e})$ from K such that $K \models \tilde{\varphi}(\bar{d},\bar{e})$. Therefore we have $K \models \varphi(\bar{d})$ as required.

(2) By Proposition 5.22 (1) every model of $\mathrm{DF}^{\ell}_{p,\epsilon}$ extends to a model of $\mathrm{SDCF}^{\ell}_{p,\epsilon}$, and by (1) $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is model complete, which imply that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is the model companion of $\mathrm{DF}^{\ell}_{p,\epsilon}$.

(3) Since $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is the model companion of $\mathrm{DF}^{\ell}_{p,\epsilon}$ with amalgamation, by Theorem 2.2 we have that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is the model completion of $\mathrm{DF}^{\ell}_{p,\epsilon}$.

(4) Since $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is the model completion of $\mathrm{DF}^{\ell}_{p,\epsilon}$ with $\mathrm{DF}^{\ell}_{p,\epsilon}$ is a universal theory, by Theorem 2.2 it follows that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ has quantifier elimination.

□

Corollary 5.24. $\mathrm{SDCF}_{p,\epsilon}$ *is complete for finite* ϵ.

Proof. It suffices to show that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is complete. By quantifier elimination it is enough to show that there is a prime substructure.

Considering the structure $(\mathbb{F}_p(x_1,\dots,x_\epsilon),\delta)$ with the trivial derivation δ, since δ is trivial, an algebraic p-basis $\{x_1,\dots,x_\epsilon\}$ for $\mathbb{F}_p(x_1,\dots,x_\epsilon)$ is a differential p-basis for $\big(\mathbb{F}_p(x_1,\dots,x_\epsilon),\delta\big)$ (see Remark 5.5).

Furthermore, it naturally embeds into any model $K \models \mathrm{SDCF}^{\ell}_{p,\epsilon}$. Indeed, the differential p-basis $\{a_1,\dots,a_\epsilon\}$ for (K,δ) is algebraically independent over $\mathbb{F}_p$ as $\mathbb{F}_p(a_1,\dots,a_\epsilon)/\mathbb{F}_p$ is separable (recall that $\mathbb{F}_p$ is perfect).

□

Chapter 6

Stability and prime model extensions in $\mathrm{SDCF}^{\ell}_{p,\epsilon}$

In the final chapter, we generalize Wood's work [28], [29] stating that DCF_p is stable but not superstable, and there exists a unique differential closure over each differentially perfect field. We show that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is stable but not superstable, and there exists a unique prime model extension over each model of $\mathrm{DF}^{\ell}_{p,\epsilon}$

Throughout this chapter we assume that every differential field is of characteristic $p > 0$ and has a finite differential degree of imperfection ϵ.

6.1 Stability

Remark 6.1. *We point out situations in which we have $C_L = L^p \cdot C_K$.*

(1) *When L is generated by a constrained (finite) tuple, we have seen that $C_L = L^p \cdot C_K$.*

(2) *Assume L/K is separable and $\bar{a}$ is a differential p-basis for K. Then $\bar{a}$ is a differential p-basis for L if and only if $C_L = L^p \cdot C_K$.*

(3) *A reformulation of (2) is as follows: Assume L/K is separable and $K \models \mathrm{DF}^{\bar{a}}_{p,\epsilon}$. Then, $L \models \mathrm{DF}^{\bar{a}}_{p,\epsilon}$ with $K \subseteq_{L_{\delta,\bar{a}}} L$ if and only if $C_L = L^p \cdot C_K$.*

In general, for a differential field extension L/K, the constants C_L are not equal to $L^p{\cdot}C_K$. However, in the case when L is finitely generated (as a differential extension), the following result of Kolchin says that C_L is not much bigger than

$L^p \cdot C_K$ (see [6]).

Fact 1.4 [6]. Let L/K be a differential field extension. If L is finitely generated as a differential field extension over K, then C_L is finitely generated as a field extension over $L^p \cdot C_K$.

Theorem 6.2. $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ *is stable but not superstable.*

Proof. We claim that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is not superstable. Since every infinite superstable field is algebraically closed (see [23]), if we assume that the theory is superstable, that contradicts to $[K : K^p] = \infty$ for all model K by Lemma 4.11.

Let $F \subseteq_{\delta,\ell} K$ be models of $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ with K sufficiently saturated, and let $a \in K$. Let L be the $L_{\delta,\ell}$-structure generated by a over F. We claim that L is countably generated as a differential field extension over F.

Let $F_0 = F\langle a\rangle$. By the above fact, we can find a finite tuple b_0 such that

$$C_{F_0} = F_0^p \cdot C_F(b_0).$$

Let $F_1 = F_0\langle \ell(b_0)\rangle$ where $\ell(b_0)$ denotes the finite tuple obtained by applying the ℓ_i's to each entry of b_0. Again, the above fact yields a finite tuple b_1 such that

$$C_{F_1} = F_1^p \cdot C_{F_0}(b_1).$$

Continuing in this fashion we build $F_{i+1} = F_i\langle \ell(b_i)\rangle$ and also obtain a finite tuple b_{i+1} such that

$$C_{F_{i+1}} = F_{i+1}^p \cdot C_{F_i}(b_{i+1}).$$

One then readily checks that $\bigcup_i F_i$ is a $L_{\delta,\ell}$-structure and clearly the smallest one containing a and F (by construction). Thus, $L = \bigcup_i F_i$, and so L is countably generated as a differential extension of F.

By Lemma 6.3 (below), the $L_{\delta,\ell}$-isomorphism type of L is determined by the L_δ-isomorphism type. The latter is determined by the defining differential ideal (over F) of the countably-many generators of L. This yields a (prime) differential ideal in a differential polynomial ring in countable many variables.

By the differential basis theorem, there are at most $|F|^{\aleph_0}$ differential ideals in such differential polynomial rings. Hence, there are at most $|F|^{\aleph_0}$ isomorphism types for L. In other words, there are at most $|F|^{\aleph_0}$-many 1-types over F. It follows that $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ is κ-stable for any κ with $\kappa^{\aleph_0} = \kappa$. □

6.2 Prime model extensions

In this section, we prove that the theory $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ has a unique prime model extension. First we will show the existence of such prime model extensions. By Morley's criterion [12], it is enough to show that for $F \models \mathrm{DF}^{\ell}_{p,\epsilon}$, the isolated types in $S_1(F)$ are dense where a type $p \in S_1(F)$ is isolated if $\{p\} = [\varphi]$ for some $L_{\delta,\ell}$-formula φ where $[\varphi] = \{p \in S_1(F) \mid \varphi \in p\}$. We work within the complete theory $\mathrm{SDCF}^{\ell}_{p,\epsilon}$, so we write $\mathrm{tp}^{\mathrm{SDCF}^{\ell}_{p,\epsilon}}(a/F)$ as $\mathrm{tp}(a/F)$.

Lemma 6.3. *Let $E_1, E_2 \models \mathrm{DF}^{\ell}_{p,\epsilon}$ with $K \subseteq_{L_{\delta,\ell}} E_1$ and $K \subseteq_{L_{\delta,\ell}} E_2$. Let $f : E_1 \to E_2$ be a L_δ-isomorphism fixing K. Then f is an $L_{\delta,\ell}$-isomorphism.*

Proof. Let $K \subseteq_{L_{\delta,\ell}} E_1$ and $K \subseteq_{L_{\delta,\ell}} E_2$ such that $f : E_1 \to E_2$ is a L_δ-isomorphism fixing K. Then we have $f(C_{E_1}) = C_{E_2}$. Let $a \in E_1$. We claim that $f \circ \ell_i^{E_1}(a) = \ell_i^{E_2} \circ f(a)$.

If $a \notin C_{E_1}$, we have $\ell_i(a) = 0$, and hence $f \circ \ell_i(a) = 0$. Since $f(a) \notin C_{E_2}$, we have $\ell_i \circ f(a) = 0$. Therefore, we have $f \circ \ell_i^{E_1}(a) = \ell_i^{E_2} \circ f(a)$.

On the other hand, if $a \in C_{E_1}$, then we have

$$a = \ell_0^{E_1}(a)^p m_0 + \cdots + \ell_s^{E_1}(a)^p m_s$$

Applying f at a, since $m_i's \in K$ we have

$$f(a) = (f \circ \ell_0^{E_1}(a))^p m_0 + \cdots + (f \circ \ell_s^{E_1}(a))^p m_s$$

On the other hand, since $f(a) \in C_{E_2}$, we have

$$f(a) = \left(\ell_0^{E_2}(f(a))\right)^p m_0 + \cdots + \left(\ell_s^{E_2}(f(a))\right)^p m_s$$

By the uniqueness of the coefficients, we have $f \circ \ell_i^{E_1}(a) = \ell_i^{E_2} \circ f(a)$. □

Lemma 6.4. *Let $(K, \delta) \models \mathrm{SDCF}^{\ell}_{p,\epsilon}$ be sufficiently saturated and $F \subseteq_{L_{\delta,\ell}} K$. Let $\bar{\alpha}$ be a tuple from K. If $\bar{\alpha}$ is constrained over F, then $\mathrm{tp}(\bar{\alpha}/F)$ is isolated.*

Proof. Let $\bar{\alpha}$ be a tuple of elements from $(K, \delta) \models \mathrm{SDCF}^{\ell}_{p,\epsilon}$ such that $\bar{\alpha}$ is constrained over $F \subseteq_{L_{\delta,\ell}} K$ with constraint g. By the differential basis theorem (Theorem 2.15 (2)) using the fact that $\epsilon(F)$ is finite, there exists $f_1, \ldots, f_s \in F\{\bar{x}\}$ such that $I^{\delta}(\bar{\alpha}) = \{f_1, \ldots, f_s\}_{F\{\bar{x}\}}$.

Let $\varphi(\bar{x}) \in L_{\delta,\ell}(F)$ be the following system of differential equations and inequation

$$f_1(\bar{x}) = 0 \wedge \cdots \wedge f_s(\bar{x}) = 0 \wedge g(\bar{x}) \neq 0,$$

then we claim that $\{\mathrm{tp}(\bar{\alpha}/F)\} = [\varphi]$.

Let $p \in [\varphi]$. Then by the saturation of K, there exists $\bar{\beta}$ from K such that $p = \mathrm{tp}(\bar{\beta}/F)$. Since $F \subseteq_{L_{\delta,\ell}} K$, $F\langle\bar{\beta}\rangle$ is separable over F. Moreover since $(K, \delta) \models \varphi(\bar{\beta})$, we have $I^{\delta}(\bar{\alpha}/F) \subseteq I^{\delta}(\bar{\beta}/F)$, and hence by $\bar{\alpha}$ constrained over F, we have

$$I^{\delta}(\bar{\alpha}/F) = I^{\delta}(\bar{\beta}/F)$$

which implies

$$F\langle\bar{\alpha}\rangle \cong_{L_{\delta}} F\langle\bar{\beta}\rangle$$

By L_{δ}-isomorphism, $\bar{\beta}$ is also constrained over F. Hence we have that $F\langle\bar{\alpha}\rangle$ and $F\langle\bar{\beta}\rangle$ are $L_{\delta,\ell}$-extension of F. By the above lemma, we have

$$F\langle\bar{\alpha}\rangle \cong_{L_{\delta,\ell}} F\langle\bar{\beta}\rangle$$

By the quantifier elimination of $\mathrm{SDCF}^{\ell}_{p,\epsilon}$, we have

$$\mathrm{tp}(\bar{\alpha}/F) = \mathrm{tp}(\bar{\beta}/F)$$

Hence we have $[\varphi] \subseteq \{\mathrm{tp}(\bar{\alpha}/F)\}$.

On the other hands, since $(K, \delta) \models \varphi(\bar{\alpha})$, we have $\varphi \in \mathrm{tp}(\bar{\alpha}/F)$, and hence $\{\mathrm{tp}(\bar{\alpha}/F)\} \subseteq [\varphi]$. □

Theorem 6.5. *Over each $F \models \mathrm{DF}^{\ell}_{p,\epsilon}$ there exists a prime model extension $K \models \mathrm{SDCF}^{\ell}_{p,\epsilon}$.*

Proof. It is enough to show that for all $F \subseteq_{L_{\delta,\ell}} K$ with $K \models \mathrm{SDCF}^{\ell}_{p,\epsilon}$ sufficiently saturated, the isolated types in $S_1(F)$ are dense. We claim that $[\varphi]$ contains an isolated type for any $\varphi \in L_{\delta,\ell}(F)$ with $[\varphi]$ a nonempty open subset of $S_1(F)$.

Let $\varphi(x) \in L_{\delta,\ell}(F)$ with $[\varphi]$ a nonempty open subset of $S_1(F)$ and $a \in K$ such that $K \models \varphi(a)$. By quantifier elimination, there exists a finite set $\{t_1, \ldots, t_r\} \subseteq T_{\delta,\ell}$ and a quantifier free $L_{\delta,\bar{a}}(F)$-formula $\tilde{\varphi}$ with parameter from F such that

$$K \models \varphi(a) \leftrightarrow \big(\exists y_{0t_1} \cdots \exists y_{st_1}, \ldots, \exists y_{0t_r} \cdots \exists y_{st_r} \tilde{\varphi}(a, \bar{y})\big).$$

We may assume that $\tilde{\varphi}(a, \bar{y})$ is of the form

$$f_1(a, \bar{y}) = 0 \wedge \cdots \wedge f_n(a, \bar{y}) = 0 \wedge g(a, \bar{y}) \neq 0.$$

Then, there exists $\bar{b}$ from K such that

$$f_1(a, \bar{b}) = 0 \wedge \cdots \wedge f_n(a, \bar{b}) = 0 \wedge g(a, \bar{b}) \neq 0.$$

Since $F\langle a, \bar{b}\rangle/F$ is separable over F, by Theorem 4.4, there exists differential specialization $(\alpha, \bar{\beta})$ of $(a, \bar{b})$ such that $(\alpha, \bar{\beta})$ is constrained over F with constraint g. By Lemma 6.3, $\mathrm{tp}((\alpha, \bar{\beta})/F)$ is isolated, and then $\mathrm{tp}(\alpha/F)$ is also isolated. Since α satisfies $\varphi(x)$, the isolated type $\mathrm{tp}(\alpha/F)$ is in $[\varphi]$. $\square$

In a countable stable theory with prime model extensions, prime models are unique up to isomorphism (see Theorem 2.4 in Preliminaries). This implies the following.

Corollary 6.6. *Over each $F \models \mathrm{DF}^{\ell}_{p,\epsilon}$ there exists a unique (up to isomorphism) prime model extension $K \models \mathrm{SDCF}^{\ell}_{p,\epsilon}$.*

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