TWO-DIMENSIONAL SELF-AFFINE SETS WITH INTERIOR POINTS, AND THE SET OF UNIQUENESS

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ABSTRACT. Let M be a 2×2 real matrix with both eigenvalues less than 1 in modulus. Consider two self-affine contraction maps from $\mathbb{R}^2 \to \mathbb{R}^2$,

$$T_m(v) = Mv - u \quad \text{and} \quad T_p(v) = Mv + u,$$

where $u \neq 0$. We are interested in the properties of the attractor of the iterated function system (IFS) generated by T_m and T_p, i.e., the unique non-empty compact set A such that $A = T_m(A) \cup T_p(A)$. Our two main results are as follows:

- If both eigenvalues of M are between $2^{-1/4} \approx 0.8409$ and 1 in absolute value, and the IFS is non-degenerate, then A has non-empty interior.
- For almost all non-degenerate IFS, the set of points which have a unique address is of positive Hausdorff dimension – with the exceptional cases fully described as well.

This paper continues our work begun in [11].

1. Introduction

Consider two self-affine linear contraction maps $T_m, T_p : \mathbb{R}^2 \to \mathbb{R}^2$:

$$T_m(v) = Mv - u \quad \text{and} \quad T_p(v) = Mv + u, \tag{1.1}$$

where M is a 2×2 real matrix with both eigenvalues less than 1 in modulus and $u \neq 0$. Here "m" is for "minus" and "p" is for "plus". We are interested in the iterated function system (IFS) generated by T_m and T_p. Then, as is well known, there exists a unique non-empty compact set A such that $A = T_m(A) \cup T_p(A)$.

The properties that we are interested in (non-empty interior of A and the set of uniqueness) do not change if we consider a conjugate of

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T_m and T_p. That is, if we consider $g \circ T_i \circ g^{-1}$ instead of the T_i where g is any invertible linear map from $\mathbb{R}^2 \to \mathbb{R}^2$. As such, we can assume that M is a 2×2 matrix in one of three forms:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We will call the first of these *the real case*, the second *the Jordan block case*, and the last *the complex case*.

We will say that the IFS is *degenerate* if it is restricted to a one-dimensional subspace of $\mathbb{R}^2$. This will occur if any of the eigenvalues are 0. It will also occur if $\lambda = \mu$ in the real case, or (equivalently) if $b = 0$ in the complex case. If our IFS is non-degenerate, then u can be chosen to be a cyclic vector for M, i.e., such that the span of $\{M^n u \mid n \geq 0\}$ is all of $\mathbb{R}^2$ (which we will assume henceforth).

In [11] the authors studied the real case with $\lambda, \mu > 0$. Properties of the complex case have been studied extensively since the seminal paper [2] - see, e.g., [3] and references therein. Note that most authors concentrate on the connectedness locus, i.e., pairs (a, b) such that the attractor A is connected.

In the present paper we study all three of the above cases, allowing us to make general claims. Our main result is

Theorem 1.1. *If all eigenvalues of M are between $2^{-1/4} \approx 0.8409$ and 1 in absolute value, and the IFS is non-degenerate, then the attractor of the IFS has non-empty interior. More precisely,*

- *If $0.832 < \lambda < \mu < 1$ then $A_{\lambda,\mu}$, the attractor for the (positive) real case, has non-empty interior* [11, Corollary 1.3].
- *If $2^{-1/2} \approx 0.707 < \lambda < \mu < 1$ then $A_{-\lambda,\mu}$, the attractor for the (mixed) real case, has non-empty interior.*
- *If $0.832 < \nu < 1$ then A_ν, the attractor for the Jordan block case, has non-empty interior.*
- *If $2^{-1/4} \approx 0.841 < |\kappa| < 1$ with $\kappa = a + bi \notin \mathbb{R}$ then A_κ, the attractor for the complex case, has non-empty interior.*

The remaining three cases are shown in Section 3. The last case relies upon an argument of V. Kleptsyn [14].

Some non-explicit results are known in the complex case. Let $\kappa = a + bi$ and consider the attractor A'_κ satisfying $A'_\kappa = \kappa A'_\kappa \cup (\kappa A'_\kappa + 1)$ which is clearly similar to A_κ.

Theorem 1.2 (Z. Daróczy and I. Kátai [6])**.** *Let $\kappa \notin \mathbb{R}$ be sufficiently close to 1 in absolute value. Then $A'_\kappa \supset \{z : |z| \leq 1\}$.*

Remark 1.3. Since A'_κ tends to a segment in $\mathbb{R}$ as the imaginary part of κ tends to 0 in the Hausdorff metric (with any fixed real part), it is clear that there cannot be an absolute bound in a result like this. In fact, a detailed analysis of the proof indicates that the actual condition the authors use is $|\kappa| > 1 - C|\arg(\kappa)|$ with some absolute constant $C > 0$. That is, "sufficiently close to 1" means "for any $\theta \in (0, \pi) \cup (\pi, 2\pi)$ there exists δ such that A_κ contains the closed unit disc for all κ with $\arg(\kappa) = \theta$ and $|\kappa| > 1 - \delta$" [1]. Theorem 1.1 overcomes this obstacle.

Given the two maps T_m and T_p, there is a natural projection map from the set of all $\{m, p\}$ sequences to points on A. We define $\pi : \{m, p\}^{\mathbb{N}} \to A$ by $\pi(a_0 a_1 a_2 \dots) = \lim_{n\to\infty} T_{a_0} \circ T_{a_1} \circ \dots \circ T_{a_n}(0, 0)$. Note that because both T_m and T_p are contraction maps, this yields a well defined point in A. We call $a_0 a_1 \dots$ an *address* for $(x, y) \in A$ if $\pi(a_0 a_1 \dots) = (x, y)$. We say that a point $(x, y) \in A$ is a *point of uniqueness* if it has a unique address.

The question on when this IFS has a large number of points of uniqueness depends somewhat on the nature of the eigenvalues. If M has two complex eigenvalues, κ and $\overline{\kappa}$ where $\arg(\kappa)/\pi \in \mathbb{Q}$ then it is possible for the IFS to have a small number of points of uniqueness (see Theorem 4.16). With the exception of this case, all other IFS will have a continuum of points of uniqueness.

Our second result is

Theorem 1.4. *For all non-degenerate IFS not explicitly mentioned in Theorem 4.16, the set of points of uniqueness is uncountable, and with positive Hausdorff dimension.*

Again, the real case where $\lambda > 0, \mu > 0$ has been shown in [11]. We prove the remaining cases in Section 4.

Example 1.5. As an example, consider the famous Rauzy fractal introduced in [18]. Let κ be one of the complex roots of $z^3 - z^2 - z - 1$, i.e., $\kappa \approx -0.419 + 0.606i$. Consider the attractor A_κ satisfying $A_\kappa = (\kappa A_\kappa - 1) \cup (\kappa A_\kappa + 1)$. It follows from the results of [17] that the unique addresses in this case are precisely those which do not contain three consecutive identical symbols.

It is easy to show by induction that the number of m-p words of length n with such a property which start with m is the nth Fibonacci number. Consequently, the set of unique addresses has topological entropy equal to $\log \tau$, where $\tau = \frac{1+\sqrt{5}}{2}$. Hence the Hausdorff dimension of the set of uniqueness is $-\frac{\log \tau}{\log |\kappa|} \approx 1.579354467$.

[1] In [16] V. Komornik and P. Loreti obtained a similar result for the condition $A'_\kappa \supset \{z : |z| \le R\}$ for an arbitrary $R > 1$.

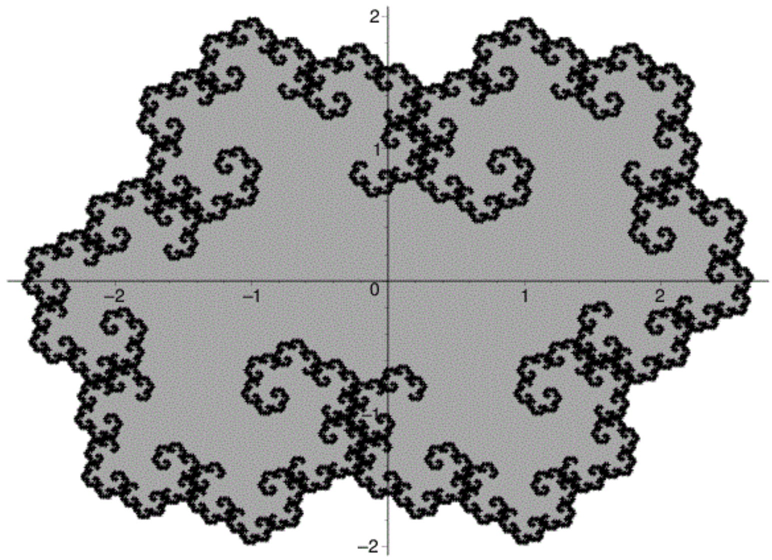

FIGURE 1. Points of uniqueness for the Rauzy fractal

See Figure 1 for the attractor (grey) and points of uniqueness (black). It is interesting to note that since the Hausdorff dimension of the boundary here is approximately 1.093 (see [13]), “most” points of uniqueness of the Rauzy fractal are interior points, whereas our general construction only uses boundary points - see Section 4.

Example 1.6. Another famous complex fractal is the twin dragon curve which in our notation is A_κ with $\kappa = \frac{1+i}{2}$ – see Figure 2. The grey half corresponds to all points in A_κ whose address begins with m and the black half – with p. Their intersection is a part of the boundary of either half, which has the same Hausdorff dimension as the boundary of A_κ, approximately 1.524 (see, e.g., [5]).

Clearly, if a point in A_κ has a non-unique address $a_0a_1\dots$, then $\pi(a_na_{n+1}\dots)$ must lie in the aforementioned intersection for some n. This means that the complement of the set of uniqueness in this case has dimension ≈ 1.524; on the other hand, it is well known that A_κ has non-empty interior (see, e.g. [8] and references therein). Consequently, a.e. point of A_κ has a unique address.

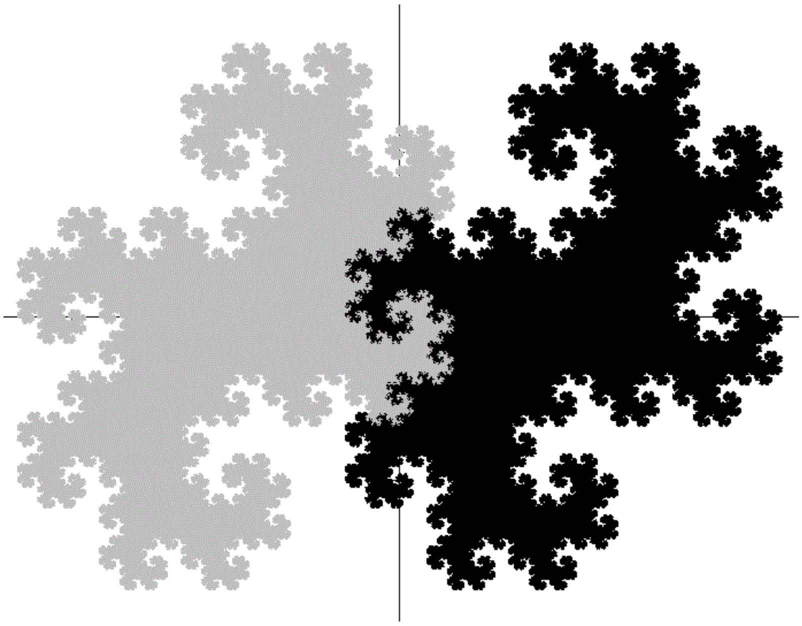

FIGURE 2. The twin dragon curve $A_{\frac{1+i}{2}}$

2. NOTATION

For the real case we will consider two subcases. Let $0 < \lambda \leq \mu < 1$ and consider

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

This we will call the *positive real case*. This was the case considered in [11]. The second subcase is

$$M = \begin{pmatrix} -\lambda & 0 \\ 0 & \mu \end{pmatrix},$$

which we will call the *mixed real case*. In both cases we take $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is clearly cyclic.

In the real positive case, $\pi : \{m, p\}^{\mathbb{N}} \to A_{\lambda,\mu}$, we have $\pi(a_0a_1a_2\dots) = (\sum_{i=0}^{\infty} a_i\lambda^i, \sum_{i=0}^{\infty} a_i\mu^i) \in \mathbb{R}^2$, whereas in the real mixed case, $\pi : \{0, 1\}^{\mathbb{N}} \to A_{-\lambda,\mu}$, we have

$$\pi(a_0a_1a_2\dots) = \left(\sum_{i=0}^{\infty} a_i(-\lambda)^i, \sum_{i=0}^{\infty} a_i\mu^i\right).$$

It is easy to see that all other real cases can be reduced to one of these two. For example, there is a symmetry from $(-\lambda, \mu)$ to $(\lambda, -\mu)$.

To see this, write $(x,y) = (\sum a_i(-\lambda)^i, \sum a_i\mu^i) \in A_{-\lambda,\mu}$. Taking $a_i' = (-1)^i a_i \in \{\pm 1\}$ we see that $(x,y) = (\sum a_i'\lambda^i, \sum a_i'(-\mu)) \in A_{\lambda,-\mu}$.

For the Jordan block case we will assume that $0 < \nu < 1$. In this case we take $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which is, again, clearly a cyclic vector. We have

$$\pi(a_0a_1a_2\dots) = \left(\sum_{i=0}^{\infty} i a_i \nu^{i-1}, \sum_{i=0}^{\infty} a_i\nu^i\right)$$

(see Lemma 3.1 below). There is a symmetry to the $\nu < 0$ case such that A_ν and $A_{-\nu}$ share all of the desired properties. To see this, write $(x,y) = (\sum i a_i \nu^{i-1} \sum a_i\nu^i) \in A_{-\nu}$. Taking $a_i' = (-1)^i a_i \in \{m,p\}$, we see that $(-x,y) = (\sum a_i' i(-\nu)^i, \sum a_i'(-\nu)) \in A_{-\nu}$. Hence A_ν and $A_{-\nu}$ are reflections of each other across the y-axis.

For the complex case, we let $\kappa = a + bi$ and consider $v = \begin{pmatrix} x \\ y \end{pmatrix}$ as $z = x + yi$. We see that the maps in (1.1) with $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, are equivalent to the maps in $\mathbb{C}$, namely,

$$T_m(z) = \kappa z - 1 \quad \text{or} \quad T_p(z) = \kappa z + 1.$$

In the complex case we have $\pi(a_0a_1a_2\dots) = \sum_{j=0}^{\infty} a_j\kappa^j \in \mathbb{C}$, i.e., the attractor A_κ is the set of expansions in complex base κ with "digits" 0 and 1. Note that if $\kappa \in \mathbb{R}$ then the resulting IFS is real (and degenerate).

Throughout we will refer to $[i_1 \dots i_k]$ as the *cylinder* of all $(a_i)_0^\infty \in \{m,p\}^{\mathbb{N}}$ such that $a_j = i_j$ for $j = 1,\dots,k$. We note that this is a compact subset of $\{m,p\}^{\mathbb{N}}$ under the usual product topology.

3. Attractors with interior

The first question that we are interested in is, when does A have interior. For the real and Jordan block case we look at a related, albeit somewhat easier, question: when is $(0,0)$ contained in the interior of A? We will say that $(-\lambda,\mu)$ for the mixed real case is in $\mathcal{Z}_{\mathbb{R}}$ if $(0,0) \in \operatorname{int}(A_{-\lambda,\mu})$. An equivalent definition is given for $\mathcal{Z}_J$ for the Jordan block case.

In fact, the real case (both mixed and positive) and the Jordan block case are both special cases of a more general result – see Theorem 3.3 below.

Consider a contraction matrix M with all real eigenvalues such that any duplicate eigenvalue is within the same Jordan block. That is, let

$J_{\lambda,k}$ be the $k \times k$ Jordan block

$$J_{\lambda,k} = \begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

and write M as

$$M = \begin{pmatrix} J_{\lambda_1,k_1} & & & 0 \\ & J_{\lambda_2,k_2} & & \\ & & \ddots & \\ 0 & & & J_{\lambda_r,k_r} \end{pmatrix}, \tag{3.1}$$

where all λ_i are distinct and $0 < |\lambda_i| < 1$ for all i. Then M will have dimensions $N \times N$ where $N = k_1 + k_2 + \cdots + k_r$.

We consider the two affine maps

$$T_m(v) = Mv - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T_p(v) = Mv + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Here there are $k_1 - 1$ copies of 0s follows by one 1, then $k_2 - 1$ copies of 0s follows by one 1, and so on.

Consider the case with M as a single $k \times k$ Jordan block $J_{\lambda,k}$.

Lemma 3.1. *We have*

$$\pi(a_0 a_1 a_2 \ldots) = \begin{pmatrix} \frac{1}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} \sum_{i=0}^{\infty} a_i \lambda^i \\ \frac{1}{(k-2)!} \frac{d^{k-2}}{d\lambda^{k-2}} \sum_{i=0}^{\infty} a_i \lambda^i \\ \vdots \\ \frac{d}{d\lambda} \sum_{i=0}^{\infty} a_i \lambda^i \\ \sum_{i=0}^{\infty} a_i \lambda^i \end{pmatrix}.$$

Proof. It suffices to show that

$$T_{a_0}\dots T_{a_n}\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix}\sum_{i=0}^n \binom{i}{k-1}a_i\lambda^{i-k+1}\\ \sum_{i=0}^n \binom{i}{k-2}a_i\lambda^{i-k+2}\\ \vdots\\ \sum_{i=0}^n ia_i\lambda^{i-1}\\ \sum_{i=0}^n a_i\lambda^i\end{pmatrix}, \tag{3.2}$$

with the usual convention that $\binom{i}{j}=0$ if $i<j$. We prove this by induction: for $n=0$ we have

$$T_{a_0}\begin{pmatrix}0\\ \vdots\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ a_0\end{pmatrix},$$

which is what we need. Assume (3.2) holds for $n-1$; then, given that $T_{a_0}(v)=Mv+a_0(0,0,\dots,0,1)^T$,

$$T_{a_0}\begin{pmatrix}\sum_{i=0}^{n-1} \binom{i}{k-1}a_{i+1}\lambda^{i-k+1}\\ \sum_{i=0}^{n-1} \binom{i}{k-2}a_{i+1}\lambda^{i-k+2}\\ \vdots\\ \sum_{i=0}^{n-1} ia_{i+1}\lambda^{i-1}\\ \sum_{i=0}^{n-1} a_{i+1}\lambda^i\end{pmatrix}=\begin{pmatrix}\sum_{i=0}^{n-1} \left(\binom{i}{k-1}+\binom{i}{k-2}\right)a_{i+1}\lambda^{i-k+2}\\ \sum_{i=0}^{n-1} \left(\binom{i}{k-2}+\binom{i}{k-3}\right)a_{i+1}\lambda^{i-k+3}\\ \vdots\\ \sum_{i=0}^{n-1}(i+1)a_{i+1}\lambda^{i}\\ \sum_{i=0}^{n-1} a_{i+1}\lambda^{i+1}+a_0\end{pmatrix}$$

$$=\begin{pmatrix}\sum_{i=0}^n \binom{i}{k-1}a_i\lambda^{i-k+1}\\ \sum_{i=0}^n \binom{i}{k-2}a_i\lambda^{i-k+2}\\ \vdots\\ \sum_{i=0}^n ia_i\lambda^{i-1}\\ \sum_{i=0}^n a_i\lambda^i\end{pmatrix},$$

as required. □

Remark 3.2. It is easy to see how this would generalize to multiple Jordan blocks.

Return to the general case of M given by (3.1). The following theorem is along the lines of [11, Theorem 3.1] and is based on the ideas from [10] (originally) and [4].

Theorem 3.3. *Let* $P(x)=x^n+b_{n-1}x_{n-1}+\dots+b_0$ *with* $n\ge N$. *Assume that*

(i) $P(1/\lambda_i)=P'(1/\lambda_i)=\dots=P^{(k_i-1)}(1/\lambda_i)=0$ *for* $i=1,\dots,r$.
(ii) $\sum_{j=0}^{n-1}|b_j|\le 2$.

(iii) *There exists a non-singular $N \times N$ submatrix of the matrix B (defined by (3.3) below).*

Then there exists a neighbourhood of $(\underbrace{0, 0, \ldots, 0}_{N})$ contained in A.

Proof. Let

$$B_t(y) = \sum_{k=0}^{t} b_k y^{t-k}$$

for $t = 0, 1, \ldots, n-1$.

Define the matrix B as follows:

$$B := \begin{pmatrix} B_0^{(k_1-1)}(\lambda_1) & B_1^{(k_1-1)}(\lambda_1) & \ldots & B_{n-1}^{(k_1-1)}(\lambda_1) \\ \vdots & \vdots & & \vdots \\ B_0^{(1)}(\lambda_1) & B_1^{(1)}(\lambda_1) & \ldots & B_{n-1}^{(1)}(\lambda_1) \\ B_0(\lambda_1) & B_1(\lambda_1) & \ldots & B_{n-1}(\lambda_1) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ B_0^{(k_r-1)}(\lambda_r) & B_1^{(k_r-1)}(\lambda_r) & \ldots & B_{n-1}^{(k_r-1)}(\lambda_r) \\ \vdots & \vdots & & \vdots \\ B_0^{(1)}(\lambda_r) & B_1^{(1)}(\lambda_r) & \ldots & B_{n-1}^{(1)}(\lambda_r) \\ B_0(\lambda_r) & B_1(\lambda_r) & \ldots & B_{n-1}(\lambda_r) \end{pmatrix}. \tag{3.3}$$

Here $B_t^{(s)}(y) = \frac{1}{s!} \frac{d^s}{dy^s} B_t(y)$. Notice that B is an $N \times n$ matrix.

Let P have the required properties and let $u_{-n}, \ldots, u_{-1}$ satisfy

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = B \begin{pmatrix} u_{-n} \\ u_{-n+1} \\ \vdots \\ u_{-1} \end{pmatrix}. \tag{3.4}$$

So long as some $N \times N$ sub-matrix of B has non-zero determinant, we have that for all x_i sufficiently close to 0, there is a solution of (3.4) with small u_j. Specifically, we can choose δ such that if $|x_i| < \delta$, then there is a solution with $|u_j| \leq 1$.

Fix a vector $(x_1, x_2, \ldots, x_N)$ in the neighbourhood of $(0, \ldots, 0)$, where each $|x_i| < \delta$. We will construct a sequence (a_j) with $a_j \in \{-1, 1\}$ such that

$$\pi(a_1 a_2 a_3 \ldots) = (x_1, x_2, \ldots, x_n),$$

which will prove the result. To do this, we first solve equation (3.4) for $u_{-n}, \ldots, u_{-1}$ with $|u_i| \leq 1$. We will then choose the u_j and a_j for

$j = 0, 1, 2, 3, \dots$ by induction, such that

$$u_j := a_j - \sum_{k=0}^{n-1} b_k u_{j+k-n}$$

and such that $u_j \in [-1, 1]$ and $a_j \in \{-1, +1\}$. We see that this is possible, as, by induction, all u_j with $j < 0$ are such that $|u_j| \le 1$. Furthermore,

$$\begin{aligned} \left| \sum_{k=0}^{n-1} b_j u_{j+k-n} \right| &\le \sum_{k=0}^{n-1} |b_k u_{j+k-n}| \\ &\le \sum_{k=0}^{n-1} |b_k| \\ &\le 2, \end{aligned}$$

by our assumption on the b_k. Hence there is a choice of a_j, either $+1$ or -1, such that $u_j = a_j - \sum_{k=0}^{n-1} b_k u_{j+k-n} \in [-1, 1]$.

We claim that this sequence of a_j has the desired properties. To see this, we first consider the base case (we put $b_n = 1$ for ease of notation). Observe that:

$$\begin{aligned} \sum_{j=0}^{\infty} a_j y^j &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{n} b_k u_{j+k-n} \right) y^j \\ &= \sum_{k=0}^{n} b_k y^{-k} \sum_{j=0}^{\infty} u_{j+k-n} y^{j+k} \\ &= \sum_{k=0}^{n} b_k y^{-k} \left(\sum_{t=k}^{n-1} u_{t-n} y^t + \sum_{t=n}^{\infty} u_{t-n} y^t \right) \\ &= \sum_{k=0}^{n} \sum_{t=k}^{n-1} b_k y^{t-k} u_{t-n} + P(y^{-1}) \sum_{t=n}^{\infty} u_{t-n} y^t. \end{aligned}$$

Evaluating at $y = \lambda_i$ and observing that $P(\lambda_i^{-1}) = 0$, this simplifies to

$$\sum_{t=0}^{n-1} u_{t-n} B_t(\lambda_i). \tag{3.5}$$

We further see that

$$\frac{1}{s!} \frac{d^s}{dy^s} \left(\sum_{j=0}^{\infty} a_j y^j \right) = \frac{1}{s!} \frac{d^s}{dy^s} \left(\sum_{k=0}^{n} \sum_{t=k}^{n-1} b_k y^{t-k} u_{t-n} + P(y^{-1}) \sum_{t=n}^{\infty} u_{t-n} y^t \right).$$

Taking derivatives and evaluating at λ_i, this simplifies to

$$\sum_{t=0}^{n-1} u_{t-n} B_t^{(s)}(\lambda_i), \tag{3.6}$$

Combining equations (3.5), (3.6) with Lemma 3.1 gives

$$\pi(a_0a_1a_2\dots) = \begin{pmatrix} \frac{1}{(k_1-1)!}\frac{d^{k_1-1}}{d\lambda_1^{k_1-1}}\sum_{i=0}^{\infty} a_i\lambda_1^i \\ \frac{1}{(k_1-2)!}\frac{d^{k_1-2}}{d\lambda_1^{k_1-2}}\sum_{i=0}^{\infty} a_i\lambda_1^i \\ \vdots \\ \frac{d}{d\lambda_1}\sum_{i=0}^{\infty} a_i\lambda_1^i \\ \sum_{i=0}^{\infty} a_i\lambda_1^i \\ \vdots \\ \frac{1}{(k_r-1)!}\frac{d^{k_r-1}}{d\lambda_r^{k_r-1}}\sum_{i=0}^{\infty} a_i\lambda_r^i \\ \frac{1}{(k_r-2)!}\frac{d^{k_r-2}}{d\lambda_r^{k_r-2}}\sum_{i=0}^{\infty} a_i\lambda_r^i \\ \vdots \\ \frac{d}{d\lambda_r}\sum_{i=0}^{\infty} a_i\lambda_r^i \\ \sum_{i=0}^{\infty} a_i\lambda_r^i \end{pmatrix} = \begin{pmatrix} \sum_{t=0}^{n-1} u_{t-n}B_t^{(k_1-1)}(\lambda_1) \\ \sum_{t=0}^{n-1} u_{t-n}B_t^{(k_1-2)}(\lambda_1) \\ \vdots \\ \sum_{t=0}^{n-1} u_{t-n}B_t^{(1)}(\lambda_1) \\ \sum_{t=0}^{n-1} u_{t-n}B_t(\lambda_1) \\ \vdots \\ \sum_{t=0}^{n-1} u_{t-n}B_t^{(k_1-1)}(\lambda_r) \\ \sum_{t=0}^{n-1} u_{t-n}B_t^{(k_1-2)}(\lambda_r) \\ \vdots \\ \sum_{t=0}^{n-1} u_{t-n}B_t^{(1)}(\lambda_r) \\ \sum_{t=0}^{n-1} u_{t-n}B_t(\lambda_r) \end{pmatrix}$$

$$= B\begin{pmatrix} u_{-n} \\ u_{-n+1} \\ \vdots \\ u_{-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix},$$

which proves the desired result. □

Remark 3.4. It is worth observing that if M is an $N \times N$ matrix with distinct eigenvalues sufficiently close to (but less than) 1 in absolute value, then the N-dimensional attractor A will have non-empty interior. Here "sufficiently close" depends only on N. This follows from essentially the same proof as in [11, Theorem 3.4] using the polynomial

$$P(x) = x^{mn+2} - x^{nm} + b_{m-1}x^{(m-1)n} + b_{m-2}x^{(m-2)n} + \cdots + b_0$$

and n even. We can choose the b_i of this polynomial such that $\sum |b_i| < 2$ and $(x^2-1)^m | P(x)$. Letting $P(x) = Q(x)(x^2-1)^m$, we have for λ_i sufficiently close to 1 that

$$\begin{aligned} P^*(x) &= Q(x)(x^2 - 1/\lambda_1^2)(x^2 - 1/\lambda_2^2)\cdots(x^2 - 1/\lambda_m^2) \\ &= x^{mn+2} + b_{nm+1}^* x^{nm+1} + \cdots + b_0^* \end{aligned}$$

will also have $\sum |b_i^*| < 2$.

It seems highly likely that the same would be true for the case where M contains non-trivial Jordan blocks, although the analysis becomes much messier.

3.1. **The mixed real case.** Here we apply Theorem 3.3 with roots $-\lambda$ and μ and $k_1 = k_2 = 1$. The polynomial we use is ($N = n = 2$):

$$P(x) = x^2 + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right) x - \frac{1}{\mu\lambda}.$$

Observe that $P(-1/\lambda) = P(1/\mu) = 0$. The matrix B in this case is

$$B = \begin{pmatrix} B_0(-\lambda) & B_1(-\lambda) \\ B_0(\mu) & B_1(\mu) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\lambda\mu} & \frac{1}{\lambda} \\ -\frac{1}{\lambda\mu} & -\frac{1}{\mu} \end{pmatrix}.$$

We see that this has determinant $\frac{\lambda+\mu}{\lambda^2\mu^2} \neq 0$, as we are assuming both $\lambda, \mu > 0$. Since

$$\left|\frac{1}{\lambda} - \frac{1}{\mu}\right| + \frac{1}{|\mu\lambda|} \leq 2, \quad \frac{1}{\sqrt{2}} \leq \lambda, \mu \leq 1,$$

we infer

Corollary 3.5. *For all $\frac{1}{\sqrt{2}} \leq \lambda, \mu \leq 1$ we have that $(0,0)$ lies in the interior of $A_{-\lambda,\mu}$.*

The above gives us a sufficient condition for checking whether a point $(-\lambda, \mu) \in \mathcal{Z}_{\mathbb{R}}$. To show a point $(-\lambda, \mu) \notin \mathcal{Z}_{\mathbb{R}}$ it suffices to show that $(0,0) \notin A$. This can be done utilizing information about the convex hull of A and using the techniques described in [11]. In particular, let $K = K_0$ be the convex hull of A and let $K_n = T_p(K_{n-1}) \cup T_m(K_{n-1})$. It is easy to see that $A \subset K_n$ for all n. Hence if there exists an n such that $(0,0) \notin K_n$ then $(0,0) \notin A$. A precise description of K is given in Section 4. See Figure 3 for illustration.

3.2. **The Jordan block case.** Consider the polynomial ($n = 8, N = 2$)

$$P(x) = x^8 - \frac{8}{7\nu}x^7 + \frac{1}{7\nu^8}.$$

A quick check shows that $P(1/\nu) = P'(1/\nu) = 0$. Furthermore, for all $\nu \geq 0.831458513$ then we have

$$\left|\frac{8}{7\nu}\right| + \left|\frac{1}{7\nu^8}\right| \leq 2.$$

In this case, the matrix from the Proof is

$$B = \begin{pmatrix} 0 & \frac{1}{7\nu^8} & \frac{2}{7\nu^7} & \frac{3}{7\nu^6} & \frac{4}{7\nu^5} & \frac{5}{7\nu^4} & \frac{6}{7\nu^3} & \frac{7}{7\nu^2} \\ \frac{1}{7\nu^8} & \frac{1}{7\nu^7} & \frac{1}{7\nu^6} & \frac{1}{7\nu^5} & \frac{1}{7\nu^4} & \frac{1}{7\nu^3} & \frac{1}{7\nu^2} & -\frac{1}{\nu} \end{pmatrix}.$$

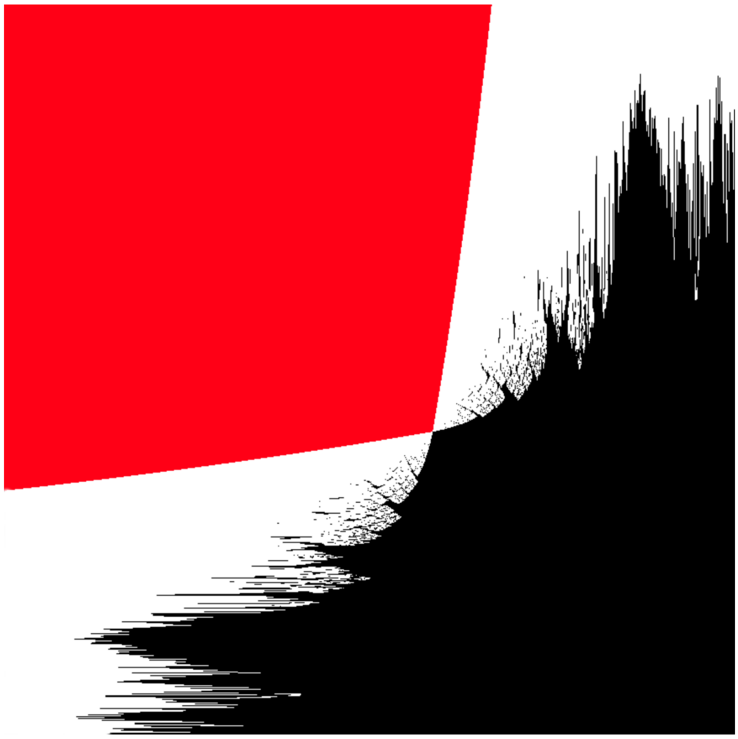

FIGURE 3. Points in $\mathcal{Z}_{\mathbb{R}}$ (red) and points not in $\mathcal{Z}_{\mathbb{R}}$ (black)

Clearly, the first 2×2 minor of B in this case is non-zero.

It is shown in [20, Theorem 2.6] that if $\nu < 0.6684$ that A_ν is disconnected, and hence totally disconnected, whence $\nu \notin \mathcal{Z}_J$. Here we have that if $\nu > 0.8315$ then $\nu \in \mathcal{Z}_J$. Where exactly this dividing line is between these two conditions is still unclear. For that matter, it is not even clear if $\mathcal{Z}_J$ is a connected set, so the term "dividing line" might not be an accurate description of the boundary.

3.3. **The complex case.** Theorem 3.3 does not seem to be applicable here, so we use a different method. Notice that this method works for the other three cases as well (and even higher-dimensional ones – see [12]) but gives worse bounds.

Theorem 3.6. *If A_{κ^2} is connected, then A_κ has non-empty interior. In particular, this is the case if $|\kappa| \geq 2^{-1/4}$.*

Proof. Note first that if $|\kappa^2| \geq 1/\sqrt{2}$ then A_{κ^2} is connected – see [2, Proposition 1]. Moreover, by the Hahn-Mazurkiewicz theorem, A_{κ^2} is path connected.

Let $a, b \in A_{\kappa^2}$ and γ the path connecting them. Consider $\kappa a, \kappa b \in \kappa A_{\kappa^2}$ and let γ' be the path between them. As $\kappa \notin \mathbb{R}$, we see that γ

and γ' cannot be parallel lines. By observing that $\sum a_i\kappa^i = \sum a_{2i}\kappa^{2i} + \kappa\sum a_{2i+1}\kappa^{2i}$, we have $A_\kappa = A_{\kappa^2} + \kappa A_{\kappa^2}$ (the Minkowski sum). In particular, A_κ will contain $\gamma + \gamma'$. By Theorem 3.8 below, $\gamma + \gamma'$ contains points in its interior, whence so does A_κ.

Hence if $|\kappa| \geq 2^{-1/4}$, then A_κ has non-empty interior. □

Remark 3.7. A great deal of information is known about the set $\mathcal{M}$ of all κ for which A_κ is connected – see [3] and references therein.

The following proof is by V. Kleptsyn (via Mathoverflow [14]).

Theorem 3.8 (V. Kleptsyn)**.** *If γ and γ' are two paths in $\mathbb{R}^2$, not both parallel lines, then $\gamma + \gamma'$ has non-empty interior.*

Proof. See Appendix. □

4. Unique addresses and convex hulls

Recall that a point $(x, y) \in A$ has a unique address (notation: $(x, y) \in \mathcal{U}$) if there is a unique sequence $(a_i)_0^\infty \in \{p, m\}^{\mathbb{N}}$ such that $(x, y) = \pi(a_1a_2a_3\dots)$. These have been studied in [11] for the positive real case and in [9] for the one-dimensional real case. We say the set of all such points in A is the *set of uniqueness* and denote it by $\mathcal{U}_{-\lambda,\mu}$, $\mathcal{U}_\nu$ and $\mathcal{U}_\kappa$ for the mixed real case the Jordan block case, and the complex case respectively.

The purpose of this section is to provide a proof of Theorem 1.4 by considering all three cases.

The main outline of all three of these proofs are the same:

- find the vertices for the convex hull of A;
- show that these vertices have unique addresses;
- using these vertices, in combination with Lemma 4.1 below, construct a set of points with unique addresses that have positive Hausdorff dimension.

Lemma 4.1. *Denote $\overline{m} = p, \overline{p} = m$ and assume that $u = a_1a_2\dots a_\ell$, $v = b_1b_2\dots b_k$ and $w = c_1c_2\dots c_n$ satisfy*

- $\pi[a_ia_{i+1}\dots a_\ell b_1b_2\dots b_k a_1a_2\dots a_\ell] \cap \pi[\overline{a_i}] = \varnothing$;
- $\pi[b_jb_{j+1}\dots b_k a_1a_2\dots a_\ell] \cap \pi[\overline{b_j}] = \varnothing$;
- $\pi[a_ia_{i+1}\dots a_\ell c_1c_2\dots c_n a_1a_2\dots a_\ell] \cap \pi[\overline{a_i}] = \varnothing$;
- $\pi[c_jc_{j+1}\dots c_n a_1a_2\dots a_\ell] \cap \pi[\overline{c_j}] = \varnothing$.

Then the images of $\{uv, uw\}^*$ under π all have unique addresses. That is, the images of all infinite words of the form $t_1t_2t_3\dots$ with $t_i \in \{uv, uw\}$ under π all have unique addresses.

Proof. We see that any shift of a word from $\{uv, uw\}^*$ is such that it's prefix will be of one of the four forms listed above. Further, by assumption, the first term is uniquely determined. By applying T_m^{-1} or T_p^{-1} as appropriate, we get that all terms are uniquely determined, which proves the result. □

Corollary 4.2. *If the conditions of Lemma 4.1 are satisfied and* $\{uv, uw\}^*$ *is unambiguous, then* $\dim_H \mathcal{U} > 0$.

We recall that $\{uv, uw\}$ is ambiguous if there exists two sequences $(t_1, t_2, t_3, \dots) \neq (s_1, s_2, s_3, \dots)$ with $t_i, s_i \in \{uv, uw\}$ where $t_1 t_2 t_3 \cdots = s_1 s_2 s_3 \dots$. If no such sequence exists, then this language is unambiguous. For example, $\{mpmp, mp\}^*$ would be ambiguous, whereas $\{m, pp\}^*$ would be unambiguous.

Proof. This is completely analogous to [11, Corollary 4.3]. We say that a language $\mathcal{L}$ has positive topological entropy if the size of the set of prefixes of length n of $\mathcal{L}$ grows exponentially in n. In brief, if we consider closure of all the shifts of sequences from $\{uv, uw\}^*$, then this set will clearly have positive topological entropy, and the injective projection π of this set will have positive Hausdorff dimension. □

4.1. **The mixed real case.** We first assume that $\lambda \neq \mu$. The case when they are equal is considered in subsection 4.4 below.

Proposition 4.3. *Let* $0 < \lambda < \mu < 1$. *The vertices of the convex hull of* $A_{-\lambda,\mu}$ *are given by* $\pi((pm)^k p^\infty), \pi((mp)^k p^\infty), \pi((pm)^k m^\infty)$, *and* $\pi((mp)^k m^\infty)$, *where* $k \geq 0$.

Proof. It suffices to show that the lines from $\pi((pm)^k p^\infty)$ to $\pi((pm)^{k+1} p^\infty)$, and similarly from $\pi((mp)^k p^\infty)$ to $\pi((mp)^{k+1} p^\infty)$, from $\pi((pm)^k m^\infty)$ to $\pi((pm)^{k+1} m^\infty)$, and from $\pi((mp)^k m^\infty)$ to $\pi((mp)^{k+1} m^\infty)$ are support lines for $A_{-\lambda,\mu}$ and that their union is homeomorphic to a circle. We will do the first case only. The other cases are similar.

We will proceed by induction. Consider first the line from $\pi(p^\infty)$ to $\pi(pmp^\infty)$. This will be in the direction $\pi(p^\infty) - \pi(pmp^\infty) = (2\lambda, -2\mu)$, with slope $-\mu/\lambda$. Consider now the line from $\pi(p^\infty)$ to any other point

$(x,y)=\pi(a_0a_1\dots)\in A_{-\lambda,\mu}$. This will have a direction of the form

$$\begin{aligned}\pi(p^\infty)-\pi(a_0a_1\dots)&=\left(\sum_{i=0}^{\infty}(1-a_i)(-\lambda)^i,\sum_{i=0}^{\infty}(1-a_i)\mu^i\right)\\&=\left(\sum_{i\text{ even}}(1-a_i)\lambda^i,\sum_{i\text{ even}}(1-a_i)\mu^i\right)\\&+\left(-\sum_{i\text{ odd}}(1-a_i)\lambda^i,\sum_{i\text{ odd}}(1-a_i)\mu^i\right).\end{aligned}$$

Clearly, no point in $A_{-\lambda,\mu}$ can have larger y-coordinate that p^∞, whence the second coordinate is always non-negative. If the first coordinate is positive as well, then we are done; so, let us assume that it is negative. We notice that the slope has the form:

$$\frac{\sum_{i\text{ even}}(1-a_i)\mu^i+\sum_{i\text{ odd}}(1-a_i)\mu^i}{\sum_{i\text{ even}}(1-a_i)\lambda^i-\sum_{i\text{ odd}}(1-a_i)\lambda^i}\le-\frac{\sum_{i\text{ odd}}(1-a_i)\mu^i}{\sum_{i\text{ odd}}(1-a_i)\lambda^i}$$

(since $\lambda<\mu$). We want

$$\frac{\sum_{i\text{ odd}}(1-a_i)\mu^i}{\sum_{i\text{ odd}}(1-a_i)\lambda^i}\le-\frac{\mu}{\lambda}.$$

Cross multiplying, this will occur if

$$\lambda\sum_{i\text{ odd}}(1-a_i)\mu^i\ge\mu\sum_{i\text{ odd}}(1-a_i)\lambda^i$$

or, equivalently,

$$\sum_{i\text{ odd}}(1-a_i)\mu^{i-1}\ge\sum_{i\text{ odd}}(1-a_i)\lambda^{i-1}.$$

This is clearly true, as $\lambda<\mu$. This proves the base case $k=0$.

Assume the line from $\pi((pm)^jp^\infty)$ to $\pi((pm)^{j+1}p^\infty)$ is a support hyperplane for $A_{-\lambda,\mu}$ for all $j<k$. Consider the line from $\pi((pm)^kp^\infty)$ to $\pi((pm)^{k+1}p^\infty)$. This will have slope $-\frac{\mu^k}{\lambda^k}$ Consider any $(x,y)=\pi(a_0a_1\dots)\in A_{-\lambda,\mu}$. Note that without loss of generality we can assume that $a_0a_1\dots a_{2k}=(pm)^k$, in view of the fact that the sequence of slopes, $-\frac{\mu^k}{\lambda^k}$, is is a decreasing negative sequence, so if $a_0\dots a_{2k}\ne(pm)^k$, then we can apply the inductive hypothesis for some $j<k$.

As before, we see that the slope of this point is

$$\frac{\sum_{i\text{ even}}(1-a_i)\mu^i+\sum_{i\text{ odd}}(\varepsilon_i-a_i)\mu^i}{\sum_{i\text{ even}}(1-a_i)\lambda^i-\sum_{i\text{ odd}}(\varepsilon_i-a_i)\lambda^i}<-\frac{\sum_{i\text{ odd}}(\varepsilon_i-a_i)\mu^i}{\sum_{i\text{ odd}}(\varepsilon_i-a_i)\lambda^i},$$

where $\varepsilon_i=-1$ if $i<2k$ and 1 otherwise.

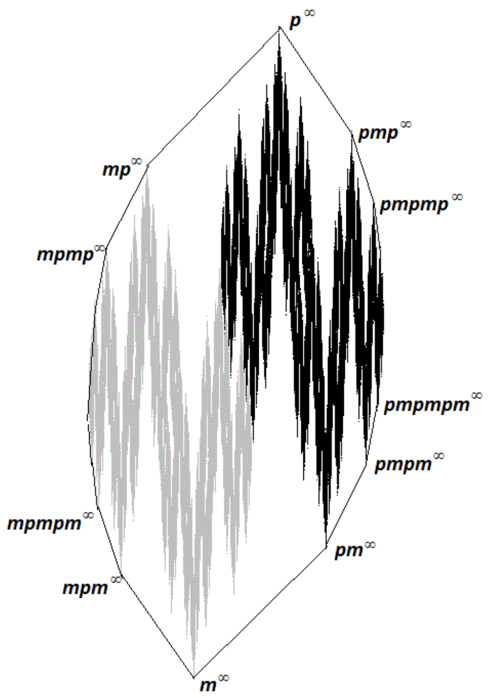

FIGURE 4. Convex hull for $A_{-0.55,0.8}$

We want

$$\frac{\sum_{i\text{ odd}}(\varepsilon_i - a_i)\mu^i}{\sum_{i\text{ odd}}(\varepsilon_i - a_i)\lambda^i} < -\frac{\mu^{2k}}{\lambda^{2k}}.$$

Cross multiplying, this will occur if

$$\lambda^{2k}\sum_{i\text{ odd}}(\varepsilon_i - a_i)\mu^i \geq \mu^{2k}\sum_{i\text{ odd}}(\varepsilon_i - a_i)\lambda^i$$

or, equivalently,

$$\sum_{i\text{ odd}}(\varepsilon_i - a_i)\mu^{i-2k} \geq \sum_{i\text{ odd}}(\varepsilon_i - a_i)\lambda^{i-2k}.$$

We see that $\lambda < \mu$ and hence $1/\mu < 1/\lambda$, from which it follows that

$$\sum_{i\text{ odd},i<2k}(\varepsilon_i - a_i)\mu^{i-2k} \geq \sum_{i\text{ odd},i<2k}(\varepsilon_i - a_i)\lambda^{i-2k}$$

and

$$\sum_{i\text{ odd},i>2k}(\varepsilon_i - a_i)\mu^{i-2k} \geq \sum_{i\text{ odd},i>2k}(\varepsilon_i - a_i)\lambda^{i-2k}.$$

Thus, we have shown that the line from $\pi((pm)^k p^\infty)$ to $\pi((pm)^{k+1} p^\infty)$ is a support line for $A_{-\lambda,\mu}$ and that $A_{-\lambda,\mu}$ lies below it. The remaining three cases (see the beginning of the proof) are similar, and once it is established whether $A_{-\lambda,\mu}$ lies below or above these, the claim about their union being a topological circle becomes trivial. We leave the details to the reader. □

See Figure 4 for illustration.

Proposition 4.4. *There exists an L such that for all $k_1, k_2 > 0$ we have $u = mp^L$, $v = p^{k_1}$ and $w = p^{k_2}$ satisfy the conditions of Lemma 4.1.*

Proof. We claim that there exists an L such that for all $k \geq 0$ and $1 \leq i \leq L + k$ we have

$$\pi[mp^{L+k} mp^L] \cap \pi[p] = \varnothing. \tag{4.1}$$

and

$$\pi[p^i mp^L] \cap \pi[m] = \varnothing. \tag{4.2}$$

Consequently, using $u = mp^L$, $v = p^{k_1}$ and $w = p^{k_2}$ with $k_1, k_2 \geq 0$ in Lemma 4.1 proves the result.

To prove (4.1), we observe that $\pi(mp^\infty)$ is a point of uniqueness. Therefore, there exists an L_1 such $\pi[mp^{L_1}]$ will be disjoint from $\pi[p]$.

To establish (4.2), we observe that the point in $\pi[m]$ with maximal second coordinate is $\pi(mp^\infty)$. Denote this maximal second coordinate by e. We also observe that $\pi(pmp^\infty)$ has second coordinate strictly greater than e. Hence there exists an L_2 such that the minimal second coordinate of $\pi[pmp^{L_1}]$ is greater than e. By observing that the minimal second coordinate of $\pi[p^{i+1} mp^{L_2}]$ is always greater than that of $\pi[p^i mp^{L_1}]$, we see that $\pi[p^i mp^{L_2}]$ is disjoint from $\pi[m]$ for all i.

Taking $L = \max(L_1, L_2)$ proves the claim. □

Corollary 4.5. *The set $\mathcal{U}_{-\mu,\lambda}$ has positive Hausdorff dimension.*

4.2. **The Jordan block case.**

Proposition 4.6. *The vertices of the convex hull of A_ν are given by $\pi(m^k p^\infty)$, and $\pi(p^k m^\infty)$, where $k \geq 0$.*

Proof. Recall that $\pi(a_0 a_1 a_2 \dots) = (\sum i a_i \nu^{i-1}, \sum a_i \nu^i)$. Consider the map taking an address $a_0 a_1 \dots$ to $(x + y, y)$, because it will simplify our argument. Thus, we have

$$\widetilde{\pi}(a_0 a_1 \dots) = \left(\sum_{i=1}^{\infty} i(a_i + 1)\nu^{i-1}, \sum_{i=0}^{\infty} a_i \nu^i \right).$$

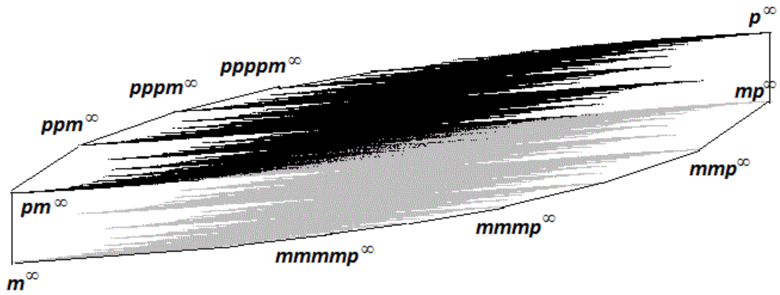

FIGURE 5. Convex hull for $A_{0.7}$

Note first that

$$\widetilde{\pi}(pm^\infty) - \widetilde{\pi}(m^\infty) = (0, 2)$$

and for $w = a_0a_1\ldots$,

$$\widetilde{\pi}(w) - \widetilde{\pi}(m^\infty) = \left(\sum_{i=1}^{\infty} i(a_i+1)\nu^{i-1}, \sum_{i=0}^{\infty}(a_i+1)\nu^i\right).$$

We notice that the first coordinate of $\widetilde{\pi}(w) - \widetilde{\pi}(m^\infty)$ is clearly nonnegative, which is enough to prove that w is to the right of the vertical line from $\widetilde{\pi}(m^\infty)$ to $\widetilde{\pi}(pm^\infty)$.

Proceed by induction and assume that for all $j < k$, the straight line passing through $\widetilde{\pi}(p^jm^\infty)$ and $\widetilde{\pi}(p^{j+1}m^\infty)$ is a support hyperplane for A_ν which lies to the left of the attractor – see Figure 5.

Consider now the case $j = k$; we have

$$\widetilde{\pi}(p^{k+1}m^\infty) - \widetilde{\pi}(p^km^\infty) = (2k\nu^{k-1}, 2\nu^k).$$

This sequence has the slopes ν/k, which is clearly decreasing. Thus, we can assume that $a_i \equiv p$, $0 \le i \le k-1$, otherwise we appeal to a case $j < k$. We see that the desired result is true if the slope of $\widetilde{\pi}(w) - \widetilde{\pi}(p^km^\infty)$ is less than or equal to ν/k. After simplifying, this is equivalent to

$$\frac{\sum_{i=k+1}^{\infty}(a_i+1)\nu^{i-k}}{\sum_{i=k+1}^{\infty} i(a_i+1)\nu^{i-k-1}} \le \frac{\nu}{k},$$

which is clearly true, since $i > k$. □

The following claim is trivial.

Lemma 4.7. *There exists an L such that for all $k_1, k_2 \ge L$ we have $u = m^{k_1}$, $v = p^{k_1}$ and $w = p^{k_2}$ satisfy the conditions of Theorem 4.1.*

Corollary 4.8. *The set of uniqueness A_ν has positive Hausdorff dimension.*

4.3. **The complex case.** For each $\phi \in [0, 2\pi)$ define $p_\phi : \mathbb{C} \to \mathbb{R}$ by $p_\phi(z) = \Re(z\mathrm{e}^{-i\phi})$. This measures the distance of z in the $\mathrm{e}^{i\phi}$ direction. We define the set Z_ϕ as those $z \in A$ such that $p_\phi(z)$ is maximized. We note that this set is well defined as A is a compact set. If points $z \in Z_\phi$ then $z = \pi(a_1 a_2 a_3 \dots) = \sum_{j=0}^{\infty} a_j^{(\phi)} \kappa^j$, where

$$a_j^{(\phi)} = \begin{cases} -1 & \text{if } \Im(\kappa^j \mathrm{e}^{i\phi}) < 0 \\ +1 & \text{if } \Im(\kappa^j \mathrm{e}^{i\phi}) > 0 \\ -1 \text{ or } +1 & \text{if } \Im(\kappa^j \mathrm{e}^{i\phi}) = 0 \end{cases} \quad \text{for all } j.$$

These have been studied in [17, Sections 5–7] in the cases of the Rauzy fractal and the twin dragon curve (see Examples 1.6 and 1.5 above).

We will distinguish two cases, depending on whether $\arg(\kappa)/\pi$ is irrational or rational.

4.3.1. *Case 1 – irrational.* Let $\mathcal{E}_\phi = \{(a_1 a_2 a_3 \dots) \mid \pi(a_1 a_2 a_3 \dots) \in Z_\phi\}$. We see that $|\mathcal{E}_\phi| = 1$ or 2, as there is at most one j where $\Im(\kappa^j \mathrm{e}^{i\phi}) = 0$. All points $z \in \mathcal{Z}_\phi$ are points of uniqueness.

Let $\bar{\mathcal{E}}_\phi$ denote the closure of the orbit of $\mathcal{E}_\phi$ under the shift transformation. Notice that any $z \in \pi(\bar{\mathcal{E}}_\phi)$ has a unique address, since for any $w \in A_\kappa$ we have $\Im(we^{i\phi}) \le \Im(ze^{i\phi})$, with the equality if and only if $w \in \bar{\mathcal{E}}_\phi$ (whose elements are all distinct).

Proposition 4.9. *Put $\mathcal{E} = \bigcup_\phi \bar{\mathcal{E}}_\phi$. We have*

- *$\mathcal{E}$ is closed under the standard product topology.*
- *$\mathcal{E}$ is uncountable.*
- *$\mathcal{E}$ is a shift-invariant.*
- *For each (a_i) in $\mathcal{E}$ we have that (a_i) is recurrent.*
- *The image $\pi(\mathcal{E})$ is a closed compact subset of A_κ.*

Proof. Notice that $(a_j^{(\phi)})_0^\infty$ is closely related to the irrational rotation of the circle $\mathbb{R}/\mathbb{Z}$ by $\arg(\kappa)/2\pi$, namely,

$$a_j^{(\phi)} = \begin{cases} +1, & j\arg(\kappa)/2\pi \in \left(-\frac{\phi}{2\pi} - \frac{1}{4}, -\frac{\phi}{2\pi} + \frac{1}{4}\right) \bmod 1, \\ -1, & j\arg(\kappa)/2\pi \in \left(-\frac{\phi}{2\pi} + \frac{1}{4}, -\frac{\phi}{2\pi} - \frac{1}{4}\right) \bmod 1, \\ +1 \text{ or } -1, & \text{otherwise} \end{cases}$$

(the third case can only occur for one j). In other words, each $(a_j^{(\phi)})$ is a hitting sequence for some semi-circle. Since our rotation is irrational, it is uniquely ergodic, whence $\bar{\mathcal{E}}_\phi$ is recurrent. The remaining properties are obvious. ☐

Remark 4.10. The sequences $(a_j^{(\phi)})$ are known to have subword complexity $2n$ (for n large enough). Such sequences are studied in detail in [19]. In particular, $\pi(\mathcal{E}_\phi)$ has zero Hausdorff dimension for all ϕ.

By the last property in Proposition 4.9, there exists $d > 0$ such that

$$\operatorname{dist}(\pi(\mathcal{E} \cap [m]), \pi[p]) > d$$

and

$$\operatorname{dist}(\pi(\mathcal{E} \cap [p]), \pi[m]) > d.$$

By taking K such that $\frac{|\kappa|^{K+1}}{1-|\kappa|} < d$, we observe that for all $(a_i) \in \mathcal{E}$,

$$\pi[a_0 a_1 \dots a_K] \cap \pi[\overline{a_0}] = \varnothing.$$

As the sequence is recurrent, for any $(a_i) \in \mathcal{E}$ there will exist two subwords of length $L > K$ of the form $b_1 b_2 \dots b_L b_{L+1}$ and $b_1 b_2 \dots b_L \overline{b_{L+1}}$. (If two such words did not exist, then the sequence would necessarily be periodic.) Since this sequence is recurrent, there exist $c_1 \dots c_m$ and $d_1 \dots d_n$ such that

$$b_1 b_2 \dots b_L b_{L+1} c_1 \dots c_m b_1 b_2 \dots b_L$$

and

$$b_1 b_2 \dots b_L \overline{b_{L+1}} d_1 \dots d_n b_1 \dots b_L$$

are both subwords of (a_i).

It is easy to see that $u = b_1 \dots b_L$, $v = b_{L+1} c_1 \dots c_m$ and $w = \overline{b_{L+1}} d_1 \dots d_n$ satisfy the conditions of Lemma 4.1, from which it follows that the images of $\{uv, uw\}^*$ will all have unique address. As this set has positive topological entropy, we have that the set of uniqueness has positive Hausdorff dimension.

Remark 4.11. Thus, in this case the points z_ϕ are all points of uniqueness. Furthermore, they are the vertices of the convex hull of A_κ. The proof is essentially the same as that of [17, Théorème 7], so we omit it.

4.3.2. *Case 2 – rational.* Let now $\kappa = \rho e^{2\pi i p/q}$ with $(p, q) = 1$. Put

$$q' = \begin{cases} q, & q \text{ odd}, \\ q/2, & q \text{ even} \end{cases}$$

and

$$\beta = \rho^{-q'} > 1. \tag{4.3}$$

Lemma 4.12. *If $\beta \le 2$, then A_κ is a convex polygon.*

Proof. Put

$$J = \left\{ \sum_{k=0}^{\infty} b_k \beta^{-k} \mid b_k \in \{\pm 1\} \right\}.$$

Since $\beta \le 2$, we have $J = \left[-\frac{\beta}{\beta-1}, \frac{\beta}{\beta-1}\right]$. Now the claim follows from the fact that A_κ can be expressed as the following Minkowski sum:

$$A_\kappa = J + \kappa J + \cdots + \kappa^{q'-1} J.$$

□

Let U_β denote the set of all unique addresses for $x = \sum_{k=0}^\infty b_k \beta^{-k}$ with $b_k \in \{\pm 1\}$.

Lemma 4.13. *We have:*

(i) *if* $(a_k)_{k=0}^\infty$ *is a unique address in* A_κ, *then* $(a_{q'j+\ell})_{j=0}^\infty \in U_\beta$ *for all* $\ell \in \{0, 1, \dots, q'-1\}$;

(ii) *if* $(a_{q'j})_{j=0}^\infty$ *belongs to* U_β, *then there exists* $(b_k)_{k=0}^\infty$ *such that* $b_{q'j} = a_{q'j}$ *for all* $j \ge 0$, *and* $(b_k)_{k=0}^\infty$ *is a unique address in* A_κ.

Proof. (i) If $(a_{q'j+\ell})$ were not unique, there would exist $(b_{q'j+\ell})$ such that $\sum_{j=0}^\infty a_{q'j+\ell}\beta^{-j} = \sum_{j=0}^\infty b_{q'j+\ell}\beta^{-j}$, i.e., $\sum_{j=0}^\infty a_{q'j+\ell}\kappa^{q'j} = \sum_{j=0}^\infty b_{q'j+\ell}\kappa^{q'j}$, whence (a_k) could not be a unique address.

(ii) Let q be odd; the even case is similar. Put for $k \not\equiv 0 \bmod q$,

$$b_k = \begin{cases} +1, & \Im(\kappa^k) > 0, \\ -1, & \Im(\kappa^k) < 0. \end{cases}$$

Clearly, this sequence is well defined, since $\Im(\kappa^k) \ne 0$ if $k \not\equiv 0 \bmod q$. Now put $b_{qj} = a_{qj}$ for all $j \ge 0$. We claim that the resulting sequence $(b_k)_{k=0}^\infty$ is a unique address.

Indeed, by our construction, $\Im(\sum_{k=0}^\infty b'_k\kappa^k) \le \Im(\sum_{k=0}^\infty b_k\kappa^k)$ for any (b'_k), with the equality if and only if $b'_k \equiv b_k$ for all $k \not\equiv 0 \bmod q$. If such an equality takes place, then $\sum_{k=0}^\infty (b'_k - b_k)\kappa^k$ is real. Moreover, $\sum_{k=0}^\infty (b'_k - b_k)\kappa^k = \sum_{j=0}^\infty (b'_{qj} - a_{qj})\beta^{-j} \ne 0$, since $(a_{qj})_{j=0}^\infty \in U_\beta$. □

This yields the following result.

Lemma 4.14. *The set of uniqueness* $\mathcal{U}_\kappa$ *is finite if and only if* U_β *is. If these sets are infinite, then their cardinalities are equal. Furthermore,* $\dim_H \mathcal{U}_\kappa > 0$ *if and only if the topological entropy of* U_β *is positive.*

Proof. By Lemma 4.13 part (i), we see that the cardinality of $\mathcal{U}_\kappa$ is bounded above by the cardinality of $\underbrace{U_\beta \times \cdots \times U_\beta}_{q'}$, and hence by the cardinality of U_β. By part (ii) we see that the cardinality of $\mathcal{U}_\kappa$ is bounded below by the cardinality of U_β. This proves the first two statements.

If $\dim \mathcal{U}_\kappa > 0$ then $\mathcal{U}_\kappa$ has positive topological entropy, and hence so does $\underbrace{U_\beta \times \cdots \times U_\beta}_{q'}$. The other direction is similar. □

Let $\beta_* = 1.787231650\ldots$ denote the *Komornik-Loreti constant* introduced by V. Komornik and P. Loreti in [15], which is defined as the unique solution of the equation $\sum_{n=1}^{\infty} \mathfrak{m}_n x^{-n+1} = 1$, where $\mathfrak{m} = (\mathfrak{m}_n)_1^\infty$ is the Thue-Morse sequence

$$\mathfrak{m} = 0110\ 1001\ 1001\ 0110\ 1001\ 0110\ldots,$$

i.e., the fixed point of the substitution $0 \to 01,\ 1 \to 10$. Put $G = \frac{1+\sqrt{5}}{2}$. The following result gives a complete description of the set U_β.

Theorem 4.15 ([7, 9])**.** *The set U_β is:*

(i) $\left\{-\frac{\beta}{\beta-1}, \frac{\beta}{\beta-1}\right\}$ *if $\beta \in (1, G]$;*
(ii) *infinite countable for $\beta \in (G, \beta_*)$;*
(iii) *an uncountable set of zero Hausdorff dimension if $\beta = \beta_*$; and*
(iv) *a set of positive Hausdorff dimension for $\beta \in (\beta_*, \infty)$.*

Lemma 4.14 and Theorem 4.15 yield

Theorem 4.16. *Let β be given by (4.3). Then the set of uniqueness $\mathcal{U}_\kappa$ for the rational case is:*

(i) *finite non-empty if $\beta \in (1, G]$;*
(ii) *infinite countable for $\beta \in (G, \beta_*)$;*
(iii) *an uncountable set of zero Hausdorff dimension if $\beta = \beta_*$; and*
(iv) *a set of positive Hausdorff dimension for $\beta \in (\beta_*, \infty)$.*

Remark 4.17. Note that if $\arg(\kappa)/\pi \in \mathbb{Q}$ and $\beta > 2$, then the convex hull of A_κ is still a $2q'$-gon. This follows directly from [22, Theorem 4.1]. See also [23] for further discussion on the convex hull of A.

Remark 4.18. If $\beta \le G$, then we have a bound $\#\mathcal{U}_\kappa \le 2^{q'}$. In fact, one can show that $\#\mathcal{U}_\kappa = 2q'$ - more precisely, only the extreme points of A_κ are points of uniqueness. This is completely analogous to [21, Theorem 2.7] which deals with the self-similar IFS without rotations. We leave a proof to the interested reader.

See Figure 6 for illustration.

4.4. **The remaining mixed real case.** Finally let

$$M = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

for $0 < \lambda < 1$.

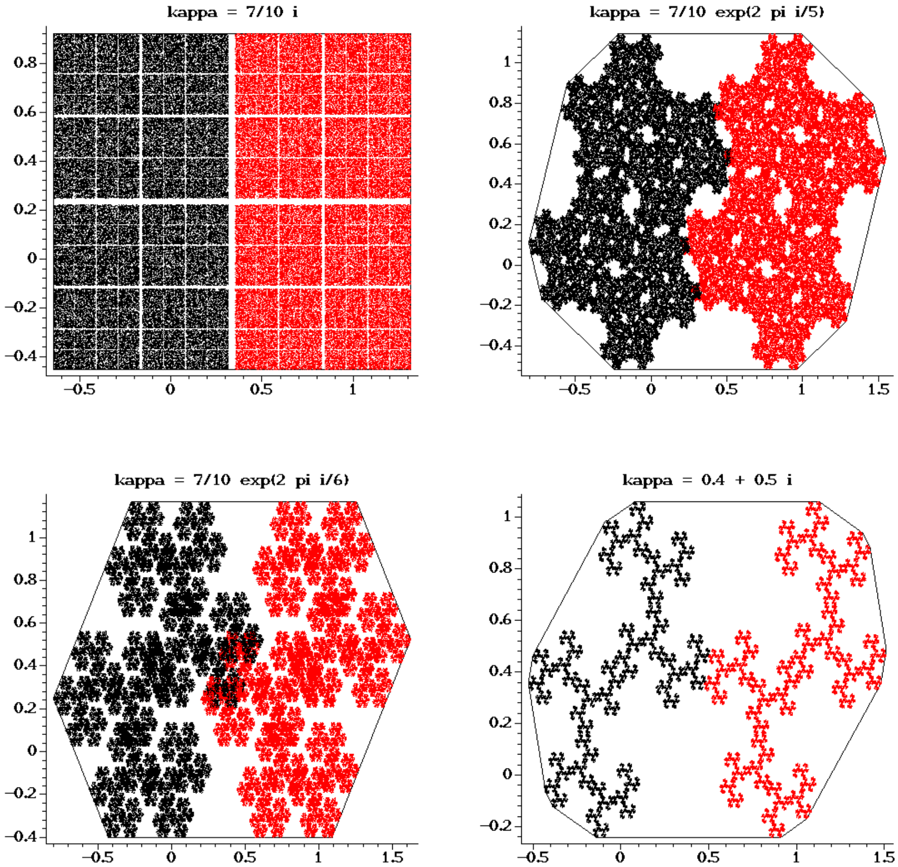

FIGURE 6. Convex hulls for $A_{0.7i}$ (a square), $A_{0.7e^{2\pi i/5}}$ (a decagon), $A_{0.7e^{\pi i/3}}$ (a hexagon) and $A_{0.4+0.5i}$ (an "infinite polygon")

Lemma 4.19. (i) *If* $\lambda < 1/\sqrt{2}$ *then* $A_{-\lambda,\lambda}$ *is totally disconnected.*
(ii) *If* $\lambda \ge 1/\sqrt{2}$ *then* $A_{-\lambda,\lambda}$ *is a parallelogram.*

Proof. We have that $x = \sum_{k=0}^{\infty} a_k(-\lambda)^k$ and $y = \sum_{k=0}^{\infty} a_k\lambda^k$. Make a change of coordinates $(x, y) \to \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$. Then

$$x = \sum_{k=0}^{\infty} a_{2k}\lambda^{2k}$$
$$y = \sum_{k=0}^{\infty} a_{2k+1}\lambda^{2k+1},$$

where $a_j \in \{0,1\},\ j \geq 0$. If $\lambda < \frac{1}{\sqrt{2}}$ then the set of x's and y's are both Cantor sets, and hence $A_{-\lambda,\lambda}$ is disconnected. If $\lambda > \frac{1}{\sqrt{2}}$ then $x \in \left[\frac{-1}{1-\lambda^2}, \frac{1}{1-\lambda^2}\right]$ and $y \in \left[\frac{-\lambda}{1-\lambda^2}, \frac{\lambda}{1-\lambda^2}\right]$, with x and y independent, and taking all values in these intervals. Thus, under this change of variables, the attractor is a rectangle. Inverting the change of variables proves the result. □

Remark 4.20. Lemma 4.19 implies that the bound $1/\sqrt{2}$ in Theorem 1.1 is sharp for the mixed real case.

Proposition 4.21. *Let $\beta = \lambda^{-2}$. The set $U_{-\lambda,\lambda}$ is:*

(i) *finite non-empty if $\beta \in (1, G]$;*
(ii) *infinite countable for $\beta \in (G, \beta_*)$;*
(iii) *an uncountable set of zero Hausdorff dimension if $\beta = \beta_*$; and*
(iv) *a set of positive Hausdorff dimension for $\beta \in (\beta_*, \infty)$.*

Proof. Notice that if $(a_{2k})_0^\infty \in U_\beta$, then $(a_k)_0^\infty \in U_{-\lambda,\lambda}$ with $a_{2k+1} \equiv -1, k \geq 0$. The rest of the proof goes exactly like in the previous subsection, so we omit it. □

5. Appendix: proof of Theorem 3.8

Lemma 5.1 (V. Kleptsyn)**.** *Let γ and γ' be two paths in $\mathbb{C}$. Let δ be the diameter of $\gamma([s_1, s_2])$, and assume that there is no point with nonzero index with respect to the loop $\sigma = \{\gamma(s)+\gamma'(t) : s,t \in \partial([s_1, s_2] \times [0,1])$. Then the sets $\gamma(s_1) + \gamma'([0,1])$ and $\gamma(s_2) + \gamma'(([0,1])$ coincide outside δ-neighbourhoods of $\gamma([s_1, s_2]) + \gamma'(0)$ and $\gamma([s_1, s_2]) + \gamma'(1)$.*

Proof. Assume the contrary and let z be a point of the curve $\widetilde{\gamma} := \gamma([s_1, s_2]) + \gamma(t_1)$ that lies outside the above neighbourhoods and that does not belong to the $\gamma([s_1, s_2]) + \gamma(t_2)$. By continuity, there is ε-neighbourhood of z that the latter curve does not intersect.

Now, by the Jordan curve Theorem, in this neighbourhood one can find two points "on different sides" with respect to $\widetilde{\gamma}$.

These two points have thus different indices with respect to the loop σ. Hence, for at least one of them this index is non-zero. □

Proof of Theorem 3.8. Let γ and γ' be two paths in $\mathbb{C}$ with $\gamma(0) = a$, $\gamma(1) = b$, $\gamma'(0) = c$ and $\gamma'(1) = d$. Consider the loop

$$\omega := \{\gamma(s) + \gamma'(t) : (s,t) \in \partial([0,1] \times [0,1])\}.$$

Any point not on ω that has non-zero index with respect to this loop is contained in $\gamma([0,1]) + \gamma'([0,1])$. This yields a point in the interior of $\gamma([0,1]) + \gamma'([0,1])$. Hence it suffices to show that there exists a point of non-zero index.

Let $\delta = \delta(s_1, s_2)$ be the diameter of $\gamma([s_1, s_2])$ for $s_1, s_2 \in [0, 1]$. Clearly, $\delta \to 0$ as $s_1 \to s_2$. Pick s_1 and s_2 sufficiently close so the diameter of $\gamma'([0, 1])$ is greater than 2δ. Hence there exists a point on the curve $\gamma(s_1) + \gamma'([0, 1])$ that is neither in the δ-neighbourhood of $\gamma([s_1, s_2]) + \gamma'(0)$ nor in the δ-neighbourhood of $\gamma([s_1, s_2]) + \gamma'(1)$. By Lemma 5.1, either there exists a point not on this curve of non-zero index, or $\gamma(s_1) + \gamma'([0, 1])$ and $\gamma(s_2) + \gamma'([0, 1])$ coincide outside the δ-neighbourhoods of $\gamma([s_1, s_2]) + \gamma'(0)$ and $\gamma([s_1, s_2]) + \gamma'(1)$.

Taking $s_1 \to s_2$ and assuming that there is never a point of non-zero index gives that $\gamma'([0, 1])$ admits an arbitrarily small translation symmetry outside its endpoints, and hence is a straight line. Reversing the roles of γ and γ' gives that either there is a point of non-zero index, or $\gamma([0, 1])$ is also a straight line.

If γ and γ' are both straight lines, then $\gamma + \gamma'$ is a parallelogram, and will only have empty interior if γ and γ' are parallel. By assumption, γ and γ' are not parallel lines, and hence $\gamma + \gamma'$ contains a point in its interior. □

6. Open questions

1. Let $d \geq 3$ and let M be a $d \times d$ real matrix whose eigenvalues are all less than 1 in modulus. Denote by A_M the attractor for the contracting self-affine iterated function system (IFS) $\{Mv - u, Mv + u\}$, where u is a cyclic vector. The following result is proved in our most recent paper on the subject to date [12].

Theorem 6.1. *If*

$$|\det M| \geq 2^{-1/d},$$

then the attractor A_M has non-empty interior. In particular, this is the case when each eigenvalue of M is greater than $2^{-1/d^2}$ in modulus.

Clearly, this is generalisation of Theorem 1.1 to higher dimensions (albeit with different constants).

Is it true that A_M contains no holes if all the eigenvalues are close enough to 1?

2. Is there a closed description of $\mathcal{B} := \partial A$? In particular, does $\mathcal{B}$ always have Hausdorff dimension greater than 1? The known examples for the complex case involve κ which are Galois conjugates of certain Pisot numbers (algebraic integers greater than 1 whose other conjugates are less than 1 in modulus) – e.g. the Rauzy fractal for the tribonacci number or the fractal associated with the smallest Pisot number [1] – in which case one can generate the boundary via a self-similar IFS.

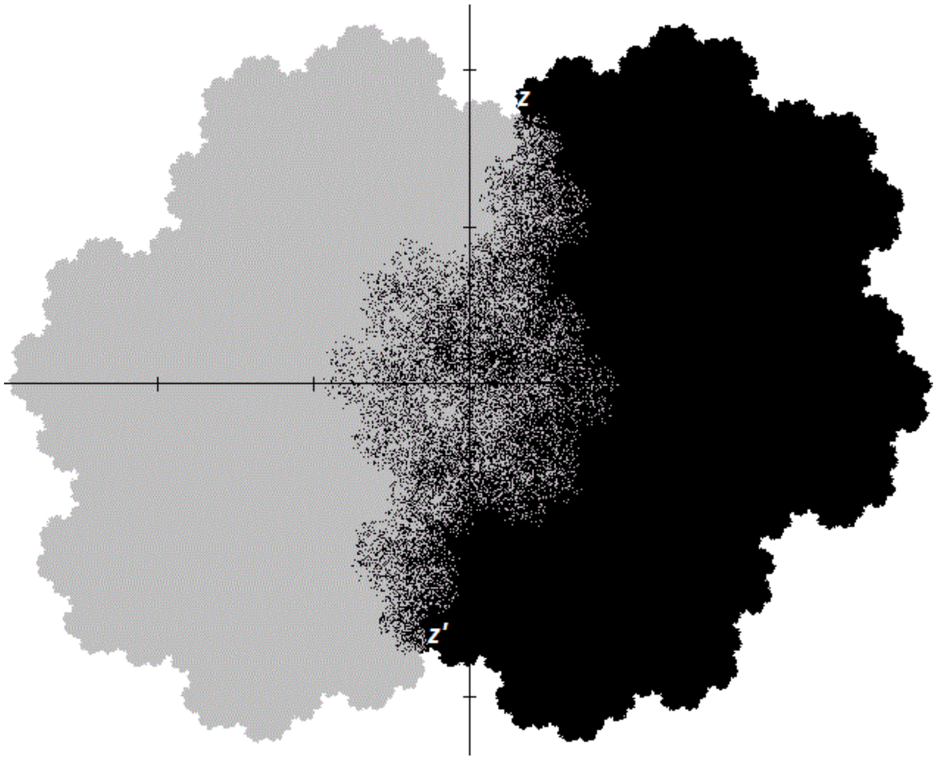

FIGURE 7. The attractor $A_{0.5+0.58i}$. It appears that z and z' are the only points in $\mathcal{B}_0$. If this is indeed the case, then all except a countable set of points of the boundary have a unique address.

3. Denote $\mathcal{B}_m = \partial T_m(A), \mathcal{B}_p = \partial T_p(A)$ and $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{B}_m \cap \mathcal{B}_p$. If $z \in \mathcal{B}_0$, then clearly, $z \notin \mathcal{U}$.

Proposition 6.2. *The set*

$$\mathcal{B}'_0 := \bigcup_{\substack{n \geq 0 \\ (i_1,\ldots,i_n) \in \{m,p\}^n}} T_{i_1} \ldots T_{i_n}(\mathcal{B}_0)$$

lies in $\mathcal{B}$. *Moreover,* $\mathcal{B} \setminus \mathcal{B}'_0 \subset \mathcal{U}$.

Proof. Note first that $T_i(\mathcal{B}) \subset \mathcal{B}$ for $i \in \{m, p\}$, whence follows the first claim. Now, suppose $z \in \mathcal{B} \setminus \mathcal{B}_p$, say. Then the first symbol of any address of z has to be m. Let us shift this address, which corresponds to applying T_m^{-1} to z in the plane. If the resulting point is in $\mathcal{B} \setminus \mathcal{B}_p$ or $\mathcal{B} \setminus \mathcal{B}_m$, then the first symbol of its address is also unique, etc. Hence follows the second claim. □

Thus, if we could somehow determine that the set $\mathcal{B}_0$ is "small" – countable, say – then "almost every" point of the boundary would be a point of uniqueness. See Figure 7 for an example.

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