Hybrid modelling of a discontinuous dynamical system including switching control

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Abstract: Although discontinuous, non-smooth or switched dynamical systems have been broadly studied, there still exist dynamical behaviours which are not completely understood. In particular, those related to transitions between state-space regions and to crossings through discontinuity surfaces. The problem of specifying system transitions becomes crucial when multiple discontinuity surfaces or switching elements are present. In this case, a great deal of care has to be taken in order to simulate the system and to verify its properties. Obtaining a computational model appears to be an elegant way for the specification of the transitions and event-triggered phenomena involved in the dynamical behaviour of discontinuous systems. This is the goal of this paper. A class of discontinuous dynamical systems including control inputs and outputs is reinterpreted within the hybrid-automata framework, and what is referred as to *discontinuous-dynamical-system hybrid automaton* is proposed. An example is used, which corresponds to a 2-degrees-of-freedom electromechanical system including discontinuous friction and sliding-mode control. This system exhibits several discontinuity surfaces which give rise to different types of transitions and dynamical behaviours. New computational-kind phenomena may arise when the system is reinterpreted as a hybrid dynamical system. Nevertheless, this paper only deals with the modelling and specification of the system.

Keywords: Hybrid systems; Discontinuous systems; Hybrid automata; Computational models; Friction; Sliding motions; Switching control; Abstraction-based approach.

1. INTRODUCTION

Today's technological applications require the inevitable combination of multi-disciplinary methods. This is the spirit of hybrid dynamical systems, which have emerged as a framework with potential for improving the performance and capabilities of a wide variety of engineering systems. In addition, from the theoretical viewpoint, they provide alternative ways of interpreting and solving the abstraction, specification, verification and design of complex control systems.

The term of hybrid system has been used to label a wide variety of engineering problems, such as: heterogeneous systems, multi-modal systems, multi-controller systems, logic-based switching control systems, discrete-event systems or variable structure systems. Generally speaking, hybrid systems are dynamical systems consisting of continuous-type and discrete-event dynamics. They merge formal computational tools, dynamical systems theory and control engineering methodologies. Consequently, this framework gives rise to models, behaviour analysis tools, stability definitions, control schemes, numerical analysis methods and algorithms for simulation which are novel, and entail a better formulation of the system interaction with the environment.

There are several representations for hybrid dynamical systems. Each representation is oriented to specific types of problems and, indeed, reflects the background of the researchers behind it, being a computer scientist, a control engineer or an applied mathematician. Then, three main hybrid modelling frameworks can be distinguished. First, the automaton-based representation, which merges continuous dynamics and finite automata theories Alur et al. (1993); Henzinger (1996). It is a graph-related representation which is closer to computer science discrete representations and symbolic dynamics. Second, the equation-based and event-flow formulae representation Antsaklis et al. (1993); Branicky et al. (1998); Buss et al. (2002); Tavernini (1987), which is closer to the typical modelling frameworks for continuous systems. Indeed, in Branicky (1995), this framework is referred to as *systemization* due to the fact that it considers systems as interacting dynamical systems and focuses on the state-space changes. Here, the hybrid system is defined by a set of differential and algebraic equations, and the discrete transitions are specified by reset or transition functions added to the system dynamics. The basic ideas of this approach comes from the hybrid representation given in the seminal paper Witsenhausen (1966). Finally, we have the behavioural representation van der Schaft and Schumacher (2000); Ye et al. (1998). In this type of hybrid models, the system is defined by specifying its behaviour or time evolution in different manners, for example, by means of the set of trajectories on an arbitrary metric space. This approach can be too general for applications.

In this paper, the automaton-based framework is used. Hybrid control systems are studied, that is, systems obtained by interconnecting a hybrid plant and a hybrid controller. The basic hybrid automaton model is extracted from Johansson et al. (1999); Lygeros et al. (1999, 2003) which is a special case of the hybrid automata proposed in Alur et al. (1993). The hybrid automaton given is very similar to the *Hybrid State Model* (HSM) proposed in Buss et al. (2002). The main difference between the HSM and the hybrid model used here, is that the HSM uses an equation-based representation, and the discontinuity surfaces are defined by means of switching functions instead of guard sets.

The paper is focused on dynamical systems with discontinuous vector fields (Filippov systems) including inputs and outputs, and with several discontinuity or switching surfaces. The main contribution is the proposal of what is referred to as *discontinuous-dynamical-system* (DDS) *hybrid automaton*. That is, the reinterpretation of a DDS with inputs and outputs as a hybrid control system. Since the dynamics in a DDS change abruptly (discrete transitions) when the trajectories hit the discontinuity surfaces, it is reasonable to define such a system within the hybrid framework. The discrete transitions are induced depending on the continuous states and inputs.

An example is considered, which is the torsional model of a conventional vertical oilwell drillstring of 2 degrees of freedom (DOF). This system has been widely studied by the author, for instance, in Navarro-López and D. Cortés (2007); Navarro-López (2008a,b); Navarro-López and Licéaga-Castro (2008).

The specification of DDS's given here entails a simulation method for these systems, for which integration methods used for ordinary differential equations normally fail due to the uncertainty introduced in the dynamics because of the discontinuity surfaces crossings. In the system description given, the events or discrete transitions are defined in order to clearly specify all the possible changes in the dynamics. As a consequence, the hybrid automaton proposed can be easily translate to a program or to any other description language (for instance, Brockett (1988); Forbus (1984); Kuipers (1986); Egerstedt and Brockett (2003); Woods (1991)).

Recently, there has been an effort on developing methods to simulate systems with multiple discontinuity surfaces. They eliminate the uncertainty when crossing these surfaces and the onset of different discontinuity-induced bifurcations is studied, for instance, Arango and Taborda (2008); Piiroinen and Kuznetsov (2008). In this paper, no study or continuation of discontinuous-induced bifurcations is made, the main goal is proposing a basic hybrid model to work with in the future.

2. HYBRID AUTOMATON MODEL TO USE

Having as a basic model the hybrid model presented in Johansson et al. (1999); Lygeros et al. (1999, 2003), the following hybrid automaton is proposed. Inputs and outputs are included due to the need of modifying the system dynamical behaviour in addition to exchanging information with the environment.

Definition 1. A hybrid automaton with inputs and outputs is a collection

$$H = (Q, E, \mathscr{X}, \Sigma, \mathscr{U}, O, \mathscr{Y}, Dom, \mathscr{F}, Init, G, R, h, r)$$

where:

- $Q = \{q_1, q_2, \ldots, q_N\}$ is a finite set of discrete states or locations.
- $\mathscr{X} \subseteq \mathbb{R}^n$ is the continuous state space.
- $E \subseteq Q \times Q$ is a finite set of edges called transitions or events.
- $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_M\}$ is a finite set of symbols labelling the edges and representing the discrete input events.
- $\mathscr{U} \subseteq \mathbb{R}^m$ is the continuous input space.
- $O = \{o_1, o_2, \ldots, o_K\}$ is a finite set of symbols representing the discrete output events.
- $\mathscr{Y} \subseteq \mathbb{R}^m$ is the continuous output space.
- $Dom : Q \to 2^{\mathscr{X} \times \mathscr{U}}$ is a location domain. It is a mapping from the locations Q to the set of all subsets of $\mathscr{X} \times \mathscr{U}$, that is, Dom assigns a set of continuous states and inputs to each discrete state $q_i \in Q$, thus, $Dom(q_i) \subset \mathscr{X} \times \mathscr{U}$.
- $\mathscr{F} = \{\mathbf{f}_{q_i}(\mathbf{x}, \mathbf{u}) : q_i \in Q\}$ is the collection of vector fields describing the continuous dynamics such that $f_{q_i} : \mathscr{X} \times \mathscr{U} \to \mathscr{X}$.
- $Init \subseteq Q \times \mathscr{X}$ is a set of initial states.
- $G : E \to 2^{\mathscr{X}}$ is a guard set. Funtion G assigns to each edge $e = (q_i, q_j) \in E$ a set of continuous states ($G(e) \subset \mathscr{X}$). The guard set is an enabling condition in order to change the location.
- $R : E \times \mathscr{X} \times \mathscr{U} \to 2^{\mathscr{X}}$ is a reset map for the continuous states for each edge.
- $h : Q \times \mathscr{X} \times \mathscr{U} \to \mathscr{Y}$ is the continuous output mapping, there is one for each location.
- $r : Q \times \mathscr{X} \times \Sigma \times \mathscr{U} \to \mathscr{O}$ is the discrete output map, there is one for each location.

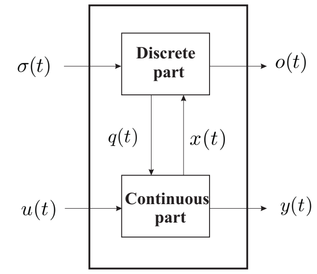

Fig. 1. Main interactions in a hybrid automaton with inputs and outputs: $q \in Q$, $\mathbf{x} \in \mathscr{X}$, $\sigma \in \Sigma$, $\mathbf{u} \in \mathscr{U}$, $o \in O$, $\mathbf{y} \in \mathscr{Y}$.

Figure 1 depicts the relationships of the main elements of the hybrid automaton proposed.

Some additional statements have to be done on Definition 1:

- The state of H is $(q, \mathbf{x}) \in Q \times \mathscr{X}$.
- $\mathscr{X}_i \subseteq \mathscr{X}$, $\mathscr{U}_i \subseteq \mathscr{U}$ and $\mathscr{Y}_i \subseteq \mathscr{Y}$ are the continuous state, input and output spaces corresponding to location q_i.
- Each $\mathbf{f}_{q_i}(\mathbf{x}, \cdot)$ is assumed to be Lipschitz continuous on the location domain for q_i in order to ensure that the solution within $\mathscr{X}_i$ exists and is unique.
- As long as the system is within location q_i, the continuous state $\mathbf{x}$ must satisfy $\mathbf{x} \in Dom(q_i)$. It can be interpreted as an enforcing condition in the hybrid automaton.
- The transition from a discrete state q_i to q_j is enabled when the continuous state $\mathbf{x}$ reaches the guard $G(q_i, q_j) \subset \mathscr{X}$ of some edge $(q_i, q_j) \in E$. Then, the discrete state changes to q_j and at the same time, the continuous state $\mathbf{x}$ is reset to the value specified by $R(q_i, q_j, \mathbf{x}, \mathbf{u}) \subset \mathscr{X}$. In the seminal hybrid model proposed in Witsenhausen (1966), the guard conditions are written as transitions sets together with discrete state transition functions.
- It is assumed that $\forall e \in E$, $G(e) \neq \emptyset$ and $\forall \mathbf{x} \in G(e)$, $R(e, \mathbf{x}, \mathbf{u}) \neq \emptyset$.

In order to define the evolution of the state of the hybrid automaton, it is necessary to establish the hybrid time evolution and trajectories. The main feauture is that time consists of a set of times containing continuous intervals and discrete points at which discrete transitions occur. Such a set of time is called a hybrid time trajectory or hybrid time set, and according to it the hybrid trajectory, typically referred to as execution, is defined.

For details, the reader is referred to Johansson et al. (1999); Lygeros et al. (1999, 2003).

In the following, the hybrid automaton proposed will be represented as a directed graph (Q,E) with vertices Q and edges E. For each vertex $q_i \in Q$, a set of initial conditions, a vector field and a domain are given. On the other hand, a guard, a label and a reset function are associated with each edge, $e \in E$.

In this paper, closed-loop systems will be considered as the feedback series connection of a hybrid plant (H_p) and a hybrid controller (H_c). The result will be referred as to hybrid control system. This was previously proposed in Malmborg (1998); Pettersson (1999) and recently revisited in Sedghi (2003). Either the plant or the controller is considered as a hybrid automaton. The result is presented in Fig. 2. A new signal is introduced, ν, which represents the reference input. This reference may consist of discrete and continuous inputs. The p and c subscripts stand for the plant and controller variables, respectively.

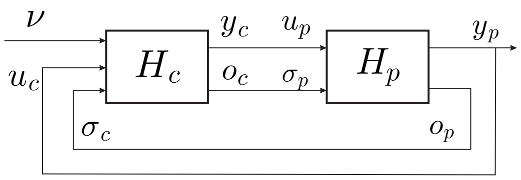

Fig. 2. Hybrid control system scheme.

3. DISCONTINUOUS DYNAMICAL SYSTEMS REINTERPRETED AS HYBRID SYSTEMS

A DDS will be specified as a hybrid automaton. A particular case is considered, which is a piecewise-smooth (PWS) dynamical system including one surface of discontinuity or switching surface. This system will be the basis for building the hybrid model for the controlled PWS system with multiple switching surfaces considered in the next section.

A state-dependent input control $\mathbf{u}(\mathbf{x})$ will be considered in this section, consequently, the vector fields can be written as $\mathbf{f}_{q_i}(\mathbf{x})$. A system of the following form is considered:

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^+(\mathbf{x}) \text{ if } \mathbf{x} \in S^+, \\ \mathbf{f}^-(\mathbf{x}) \text{ if } \mathbf{x} \in S^-, \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathscr{X} \subseteq \mathbb{R}^n$ is the state vector, $\mathbf{f}^+$ and $\mathbf{f}^-$ are continuous and smooth, and $S^+ = \{\mathbf{x} \in \mathscr{X} : s(\mathbf{x}) > 0\}$, $S^- = \{\mathbf{x} \in \mathscr{X} : s(\mathbf{x}) < 0\}$, with s a smooth scalar function with nonvanishing gradient. The switching or discontinuity surface which divides the state space is S: $S = \{\mathbf{x} \in \mathscr{X} : s(\mathbf{x}) = 0\}$. Consequently, $\mathscr{X} = S^+ \bigcup S^- \bigcup S$. On S, $\mathbf{f}^+(\mathbf{x})$ and $\mathbf{f}^-(\mathbf{x})$ do not agree.

The discontinuity surface is divided into two regions, the sliding set S_s, which is closed, and the crossing set S_c, which is open. Then $S = S_s \bigcup S_c$. S_s is the set where a sliding motion can take place and S_c is the set of S within which the system trajectory crosses S without sliding. The system dynamics on S have the form $\dot{\mathbf{x}} = \mathbf{f}_s(\mathbf{x})$, where $\mathbf{f}_s$ is the equivalent dynamics on S Filippov (1988); Utkin (1992).

In this paper, the Utkin's equivalent control method is used Utkin (1992), and

$$\mathbf{f}_s(\mathbf{x}) = \frac{\mathbf{f}^+(\mathbf{x}) + \mathbf{f}^-(\mathbf{x})}{2} + u_{eq}(\mathbf{x}) \frac{\mathbf{f}^-(\mathbf{x}) - \mathbf{f}^+(\mathbf{x})}{2} \tag{2}$$

with $u_{eq}(\mathbf{x}) \in [-1,1]$ a scalar function playing the role of the equivalent control. From (2), and taking into account that $\mathbf{f}_s$ must be tangential to S, one yields to,

$$u_{eq}(\mathbf{x}) = -\frac{\langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^+(\mathbf{x}) \rangle + \langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^-(\mathbf{x}) \rangle}{\langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^-(\mathbf{x}) \rangle - \langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^+(\mathbf{x}) \rangle}, \tag{3}$$

where $\bigtriangledown s$ is the gradient of s and $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors. The sliding set has the following form,

$$S_s := \{\mathbf{x} \in S : \ -1 \le u_{eq}(\mathbf{x}) \le 1\}. \tag{4}$$

The crossing set S_c is the complement set of S_s in S. It is assumed that there are no points on S_s at which both f^+ and f^- are tangent to S. Furthermore, the sliding set is attractive for $\mathbf{x}$ such that $\langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^-(\mathbf{x}) \rangle - \langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^+(\mathbf{x}) \rangle > 0$ and repulsive for $\mathbf{x}$ such that $\langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^-(\mathbf{x}) \rangle - \langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^+(\mathbf{x}) \rangle < 0$.

The hybrid automaton describing the dynamical behaviour of system (1) is the following one, which will be referred to as DDS hybrid automaton:

- $Q = \{q_1, q_2, q_3\} = \{slip^+, slip^-, stick\}$, $\mathscr{X} \subseteq \mathbb{R}^n$.
- $E = \{(q_1,q_2), (q_1,q_3), (q_2,q_1), (q_2,q_3), (q_3,q_1), (q_3,q_2)\}$.
- $\Sigma = \{a,b,c\}$, one edge label is assigned to each type of guard. For the systems considered here, these discrete inputs can be considered redundant and unnecessary for the specification problem.
- $Dom(q_1) = S^+$, $Dom(q_2) = S^-$, $Dom(q_3) = S$.
- $f_{q_1}(\mathbf{x}) = f^+(\mathbf{x})$, $f_{q_2}(\mathbf{x}) = f^-(\mathbf{x})$, $f_{q_3}(\mathbf{x}) = f_s(\mathbf{x})$.
- $Init = Q \times \mathscr{X} / U_s$, with $U_s := \{\mathbf{x} \in S_s : \ \langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^-(\mathbf{x}) \rangle - \langle \bigtriangledown s(\mathbf{x}), \mathbf{f}^+(\mathbf{x}) \rangle < 0\}$. Then, the problem of nonuniqueness of solutions starting at unstable sliding sets is avoided. Indeed, in most cases, solutions starting away from unstable sliding surfaces do not usually reach them.
- $G(q_1,q_3) = G(q_2,q_3) = S_s$, $G(q_1,q_2) = G(q_3,q_2) = \{\mathbf{x} \in S : u_{eq}(\mathbf{x}) > 1\}$, $G(q_2,q_1) = G(q_3,q_1) = \{\mathbf{x} \in S : u_{eq}(\mathbf{x}) < -1\}$.
- $R(q_i,q_j,\mathbf{x}) = \{\mathbf{x}\}, \forall i,j \in \{1, 2, 3\}$.
- $y = h(q_1,x) = h(q_2,x) = h(q_3,x)$ is the continuous output, which is the same for all the locations. No discrete outputs are considered.

The transition system behaviour is the following one. Assume that $Init = (q_1, \mathbf{x}_0)$, with $\mathbf{x}_0 \in S^+$, then the state vector is calculated by using $\dot{\mathbf{x}} = \mathbf{f}^+(\mathbf{x})$. If the trajectory evolves towards the switching surface, at some time, it will hit S (switching to location q_3) and according to (4), if $-1 \le u_{eq}(\mathbf{x}) \le 1$, the trajectory enters the sliding set and will remain on S until it crosses the boundaries of the sliding set and enters the crossing set, that is, until $u_{eq}(\mathbf{x}) > 1$ (switching to location q_2) or $u_{eq}(\mathbf{x}) < -1$ (switching to location q_1). The trajectory switches directly from q_1 to q_2 (or from q_2 to q_1), if the trajectory hits S in the crossing set. All phenomena associated with the equilibria that may exist on the switching surface S Kuznetsov et al. (2003) are considered in the DDS hybrid automaton.

The directed graph associated with the DDS hybrid automaton is shown in Fig. 3. A similar hybrid model is obtained in Sedghi (2003), the difference with this work is that the hybrid framework used in Sedghi (2003) is an equation-based representation.

The type of discontinuous controllers presented in this paper can be also considered as a DDS hybrid automaton. The combined model of the DDS hybrid plant and the DDS hybrid controller will be considered for an example in Section 5.

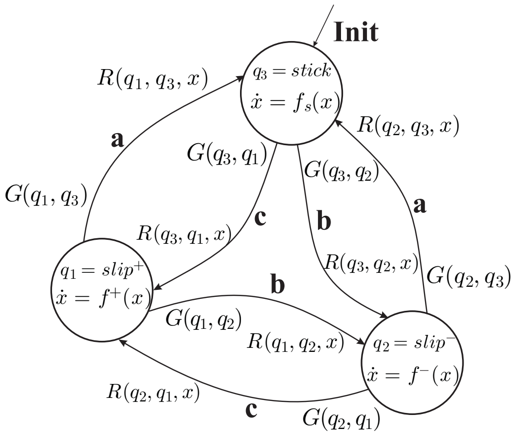

Fig. 3. Graphical representation of the DDS hybrid automaton.

4. A DISCONTINUOUS CONTROLLED DYNAMICAL SYSTEM

A PWS system is presented, corresponding to the simplified torsional model of a conventional vertical oilwell drillstring. A model of 2 DOF is considered, which is a particular case of the n-DOF model proposed in Navarro-López and D. Cortés (2007). All the results are valid for the n-DOF model, nevertheless, the 2-DOF case is given for the sake of clarity in the presentation of results.

The drillstring torsional behaviour is described by a simple torsional pendulum driven by an electrical motor, and the bit-rock contact is described by a dry friction model. The drill pipes are represented by a linear spring with torsional stiffness k_t and a torsional damping c_t, which connect the inertias J_r and J_b.

The system state vector is $\mathbf{x} = (\dot{\varphi}_r, \varphi_r - \varphi_b, \dot{\varphi}_b)^T$, with φ_i, $\dot{\varphi}_i$ ($i \in \{r, b\}$) the angular displacements and angular velocities of the drillstring elements, respectively. At the top-drive system, a viscous damping torque is considered ($c_r x_1$). T_m is the torque applied by the electrical motor at the surface. The actuator dynamics is not considered and $T_m = u$, with u one of the control inputs. $T_b(x_3) = c_b x_3 + T_{f_b}(x_3)$ is the torque on the bit with $c_b x_3$ approximating the influence of the mud drilling on the bit behaviour. $T_{f_b}(x_3)$ is the friction modelling the bit-rock contact,

$$T_{f_b}(x_3) = W_{ob} R_b \left[\mu_{c_b} + (\mu_{s_b} - \mu_{c_b}) \exp^{-\frac{\gamma_b}{v_f}|x_3|} \right] \text{sign}(x_3), \quad (5)$$

with $W_{ob} > 0$ the weight on the bit (considered as a constant input), $R_b > 0$ the bit radius; μ_{s_b}, $\mu_{c_b} \in (0,1)$ the static and Coulomb friction coefficients associated with J_b, $0 < \gamma_b < 1$ and $v_f > 0$. In addition, the Coulomb and static friction torque is T_{c_b} and T_{s_b}, respectively, with $T_{c_b} = W_{ob} R_b \mu_{c_b}$, $T_{s_b} = W_{ob} R_b \mu_{s_b}$. In equation (5), the sign function is considered as:

$$\text{sign}(x_3) = x_3/|x_3| \ \text{ if } x_3 \neq 0, \ \text{sign}(x_3) \in [-1, 1] \ \text{ if } x_3 = 0. \quad (6)$$

Finally, the drillstring behaviour is described by:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{J_r} \left[-(c_t + c_r) x_1 - k_t x_2 + c_t x_3 + u \right], \ \dot{x}_2 = x_1 - x_3, \\ \dot{x}_3 &= \frac{1}{J_b} \left[c_t x_1 + k_t x_2 - (c_t + c_b) x_3 - T_{f_b}(x_3) \right], \end{aligned} \quad (7)$$

or in a compact form, $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{T}_f(\mathbf{x}(t))$, where $\mathbf{A}$, $\mathbf{B}$ are constant matrices depending on system parameters and $\mathbf{T_f}$ represents the torque on the bit.

For (7) with (5), $s(\mathbf{x}) = s^b(\mathbf{x}) = x_3$, the switching surface is $S^b = \{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\}$, and the sliding set is $S^b_s = \{\mathbf{x} \in S^b : |c_t x_1 + k_t x_2| \leq T_{s_b}\}$.

The control strategy consists in inserting an attractive surface of discontinuity, $s^r = 0$, along which the system exhibits the desired dynamics. For this purpose, it is proposed a discontinuous control so that the system trajectory reaches this surface and enters a sliding motion. Thus, the following functions are proposed Navarro-López and Licéaga-Castro (2008); Navarro-López (2008a):

$$\begin{aligned} s^r(\mathbf{x}, t) &= (x_1 - \Omega) + \lambda \int_0^t [x_1(\tau) - \Omega] \, d\tau + \\ &+ \lambda \int_0^t [x_1(\tau) - x_3(\tau)] \, d\tau, \ \lambda > 0, \\ u &= c_t(x_1 - x_3) + k_t x_2 + c_r x_1 - J_r [\lambda(x_1 - \Omega) + \\ &+ \lambda(x_1 - x_3) + \eta \, \text{sign}(s^r)], \ \eta > 0, \end{aligned} \quad (8)$$

where $\Omega > 0$ is the desired top-rotary velocity. Furthermore, $s^r(\mathbf{x}, t)$ becomes zero in a finite time interval $t_{sr} = \frac{|s^r(\mathbf{x}, t_0)|}{\eta}$. Two new states x_4, x_5 are defined, such that, $\dot{x}_4 = x_1 - \Omega$ and $\dot{x}_5 = x_1 - x_3$. The following switching surface is defined: $S^r := \{\mathbf{x} \in \mathbb{R}^5 : s^r(\mathbf{x}, t) = 0\}$. This surface has been designed in such a way to be attractive for all $\mathbf{x}$ and to be a sliding set for all $\mathbf{x} \in S^r$. Control u is of swiched type, with the form:

$$u = \begin{cases} u^+ & \text{if } s^r > 0 \\ u^- & \text{if } s^r < 0 \end{cases}, \quad (9)$$

obtaining u^+ and u^- by changing the sign of s^r in (8). From (8), the equivalent control associated with S^r is $u^- < u^r_{eq} < u^+$, with:

$$u^r_{eq}(\mathbf{x}) = c_t(x_1 - x_3) + k_t x_2 + c_r x_1 - J_r [\lambda(x_1 - \Omega) + \lambda(x_1 - x_3)]. \quad (10)$$

Consequently, the dynamics on S^r has the following form:

$$\dot{\mathbf{x}} = f^r_s(\mathbf{x}, u)|_{u = u^r_{eq}} = \begin{pmatrix} -\lambda(x_1 - \Omega) - \lambda(x_1 - x_3) \\ x_1 - x_3 \\ \frac{1}{J_b} \left[c_t x_1 + k_t x_2 - (c_t + c_b) x_3 - T_{f_b}(x_3) \right] \\ x_1 - \Omega \\ x_1 - x_3 \end{pmatrix}.$$

In addition, control u has modified the dynamics on S^b, and:

$$\dot{\mathbf{x}} = f^b_s(\mathbf{x}) = \begin{pmatrix} -2\lambda x_1 + \lambda \Omega - \eta \, \text{sign}(s^r) \\ x_1 \\ 0 \\ x_1 - \Omega \\ x_1 \end{pmatrix}.$$

5. DRILLSTRING DDS HYBRID AUTOMATON

The inputs of the system are u and W_{ob}. The input W_{ob} is considered constant. The outputs are the angular velocities x_1 and x_3, and the reference signal is Ω.

The open-loop plant represented in (7) is an example of the DDS hybrid automaton shown in Fig. 3, with:

$$\mathbf{f}_{q1}(\mathbf{x},W_{ob},u) = \mathbf{A}\mathbf{x}+\mathbf{B}u+\mathbf{T}_f(\mathbf{x})|_{T_{f_b}=T^+_{f_b}},$$
$$\mathbf{f}_{q2}(\mathbf{x},W_{ob},u) = \mathbf{A}\mathbf{x}+\mathbf{B}u+\mathbf{T}_f(\mathbf{x})|_{T_{f_b}=T^-_{f_b}},$$
$$\mathbf{f}_{q3}(\mathbf{x},u) = \begin{pmatrix} \frac{1}{J_r}[-(c_t+c_r)x_1-k_tx_2+u] \\ x_1 \\ 0 \end{pmatrix}, \quad (11)$$
$$G(q_1,q_3)=G(q_2,q_3)=\{\mathbf{x}\in S^b: |c_tx_1+k_tx_2|\le T_{s_b}\},$$
$$G(q_1,q_2)=G(q_3,q_2)=\{\mathbf{x}\in S^b: c_tx_1+k_tx_2< -T_{s_b}\},$$
$$G(q_2,q_1)=G(q_3,q_1)=\{\mathbf{x}\in S^b: c_tx_1+k_tx_2> T_{s_b}\},$$
$$h(q_1,\mathbf{x})=h(q_2,\mathbf{x})=h(q_3,\mathbf{x})=\begin{pmatrix} x_1 \\ x_3\end{pmatrix},$$

where $T^+_{f_b}$ and $T^-_{f_b}$ are $T_{f_b}(x_3)$ for $x_3>0$ and $x_3<0$, respectively.

The hybrid automaton corresponding to controller (8) will not be presented. The hybrid automaton for the closed-loop system will be given as follows. This hybrid automaton has nine modes or locations.

The switching surfaces S^b and S^r divide the state space in four regions where the system is smooth. For each of these regions, a location is defined:

$$q_1=\{slip^+_b, slip^+_r\},\ q_2=\{slip^+_b, slip^-_r\},$$
$$q_3=\{slip^-_b, slip^+_r\},\ q_4=\{slip^-_b, slip^-_r\}.$$

The domains and vector fields for each of these locations are:

$$Dom(q_1)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})>0, s^r(\mathbf{x},t)>0\},$$
$$Dom(q_2)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})>0, s^r(\mathbf{x},t)<0\},$$
$$Dom(q_3)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})<0, s^r(\mathbf{x},t)>0\},$$
$$Dom(q_4)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})<0, s^r(\mathbf{x},t)<0\},$$

$$\mathbf{f}_{q_1}=\begin{pmatrix}\varphi_1(\mathbf{x})-\eta \\ x_1-x_3 \\ \varphi_2(\mathbf{x})-\frac{T^+_{f_b}(x_3)}{J_b} \\ x_1-\Omega \\ x_1-x_3\end{pmatrix}, \mathbf{f}_{q_2}=\begin{pmatrix}\varphi_1(\mathbf{x})+\eta \\ x_1-x_3 \\ \varphi_2(\mathbf{x})-\frac{T^+_{f_b}(x_3)}{J_b} \\ x_1-\Omega \\ x_1-x_3\end{pmatrix},$$
$$\mathbf{f}_{q_3}=\begin{pmatrix}\varphi_1(\mathbf{x})-\eta \\ x_1-x_3 \\ \varphi_2(\mathbf{x})-\frac{T^-_{f_b}(x_3)}{J_b} \\ x_1-\Omega \\ x_1-x_3\end{pmatrix}, \mathbf{f}_{q_4}=\begin{pmatrix}\varphi_1(\mathbf{x})+\eta \\ x_1-x_3 \\ \varphi_2(\mathbf{x})-\frac{T^-_{f_b}(x_3)}{J_b} \\ x_1-\Omega \\ x_1-x_3\end{pmatrix},$$

with $\varphi_1(\mathbf{x})=-\lambda(x_1-\Omega)-\lambda(x_1-x_3)$ and $\varphi_2(\mathbf{x})=\frac{1}{J_b}[c_tx_1+ +k_tx_2-(c_t+c_b)x_3]$. The second group of locations corresponds to dynamics on some of the switching surfaces:

$q_5=\{slip^+_b, stick_r\}$, $q_6=\{stick_b, stick_r\}$, $q_7=\{slip^-_b, stick_r\}$, $q_8=\{stick_b, slip^+_r\}$, $q_9=\{stick_b, slip^-_r\}$.

The domains and vector fields for each of these locations are:

$$Dom(q_5)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})>0, s^r(\mathbf{x},t)=0\},$$
$$Dom(q_6)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})=0, s^r(\mathbf{x},t)=0\},$$
$$Dom(q_7)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})<0, s^r(\mathbf{x},t)=0\},$$
$$Dom(q_8)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})=0, s^r(\mathbf{x},t)>0\},$$
$$Dom(q_9)=\{\mathbf{x}\in\mathbb{R}^5: s^b(\mathbf{x})=0, s^r(\mathbf{x},t)<0\},$$

$$\mathbf{f}_{q_5}=\begin{pmatrix}\varphi_1(\mathbf{x}) \\ x_1-x_3 \\ \varphi_2(\mathbf{x})-\frac{T^+_{f_b}(x_3)}{J_b} \\ x_1-\Omega \\ x_1-x_3\end{pmatrix}, \mathbf{f}_{q_6}=\begin{pmatrix}-2\lambda x_1+\lambda\Omega \\ x_1 \\ 0 \\ x_1-\Omega \\ x_1\end{pmatrix},$$
$$\mathbf{f}_{q_7}=\begin{pmatrix}\varphi_1(\mathbf{x}) \\ x_1-x_3 \\ \varphi_2(\mathbf{x})-\frac{T^-_{f_b}(x_3)}{J_b} \\ x_1-\Omega \\ x_1-x_3\end{pmatrix}, \mathbf{f}_{q_8}=\begin{pmatrix}-2\lambda x_1+\lambda\Omega-\eta \\ x_1 \\ 0 \\ x_1-\Omega \\ x_1\end{pmatrix},$$
$$\mathbf{f}_{q_9}=\begin{pmatrix}-2\lambda x_1+\lambda\Omega+\eta \\ x_1 \\ 0 \\ x_1-\Omega \\ x_1\end{pmatrix}.$$

The dynamics within locations q_6, q_8 and q_9 do not depend on the input W_{ob}, and within the other six locations, the dynamics depend on $\mathbf{x}$ and the input W_{ob}.

Some features of the system will be considered as constraints in order to reduce the number of feasible transitions between locations. The most important characteristic is that the surface S^r is attractive for all $\mathbf{x}$ and that all trajectories reach S^r in a finite time, and once the trajectory reaches S^r, it remains there. Because of these facts, once the system reaches locations q_5, q_6 and q_7, its future transitions will be restricted to these three locations. This can be considered as a desired *recurrent loop*. In addition, unfeasible transitions are also the transitions involving a cross through S^r. Finally, we assume that when the trajectory intersects S^r and the velocity x_3 is not zero, the sign of x_3 is maintained, which has physical meaning. For instance, from q_1, if $s^r(\mathbf{x})=0$, the two possible transitions are to q_5 or q_6. Consequently, there are 28 feasible edges or transitions:

$$\begin{aligned}E_{fea}=\{&(q_1,q_8),(q_1,q_3),(q_3,q_1),(q_3,q_8),(q_8,q_1),(q_8,q_3),\\&(q_5,q_6),(q_5,q_7),(q_7,q_5),(q_7,q_6),(q_6,q_5),(q_6,q_7),\\&(q_2,q_9),(q_2,q_4),(q_4,q_2),(q_4,q_9),(q_9,q_2),(q_9,q_4),\\&(q_1,q_5),(q_8,q_6),(q_3,q_7),(q_2,q_5),(q_9,q_6),(q_4,q_7),\\&(q_1,q_6),(q_2,q_6),(q_3,q_6),(q_4,q_6)\}.\end{aligned}$$

In Fig. 4, the guards for the feasible transitions (for space reasons) are shown in the graphical representation of the resulting hybrid control system. The guards are located next to the departure locations, and $G^+ := \{x_3=0 \text{ and } \varphi(\mathbf{x})>T_{s_b}\}$, $G^- := \{x_3=0 \text{ and } \varphi(\mathbf{x})<-T_{s_b}\}$, $G^0 := \{x_3=0 \text{ and } |\varphi(\mathbf{x})|\le T_{s_b}\}$, with $\varphi(\mathbf{x})=c_tx_1+k_tx_2$.

As it is appreciated from the figure, the hybrid control system consists of three DDS hybrid automata represented by these three groups of locations: $\{q_1,q_8,q_3\}$, $\{q_5,q_6,q_7\}$, $\{q_2,q_9,q_4\}$.

6. CONCLUSION

The DDS hybrid automaton is defined in this paper in order to rewrite a DDS with several switching surfaces, inputs and ouputs into the hybrid-automaton framework. An example is used to illustrate the model proposed. It is a simplified torsional model of a drillstring including discontinuous friction and sliding-mode control. This paper is the first step of hybrid modelling of DDS's. Under the framework proposed, all the complex behaviours associated with DDS's can be reinterpreted

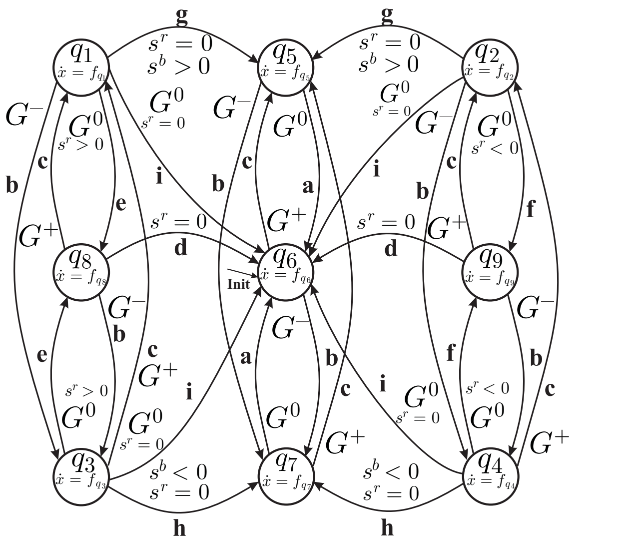

Fig. 4. Graphical representation of the hybrid automaton for the closed-loop drillstring.

and redefined from a computational viewpoint. Simulations are not included due to limited space reasons.

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