Differentially Large Henselian Fields and Taylor Morphisms

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Contents

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Abstract

In this thesis, we present a variety of results on two topics relating to differentially large fields: on generalised functorial versions of the Taylor morphism, and on differentially henselian fields, which are an analogue of differentially large fields in the context of henselian valued fields.

The Taylor morphism is a tool used in differential algebra to construct differential morphisms from algebraic morphisms in a uniform way. We observe that this construction has certain functorial properties, and we generalise the notion of a Taylor morphism as a functor which satisfies the same properties. We demonstrate that generalised Taylor morphisms have corresponding applications to differentially large fields, and we also study the structure of these generalised Taylor morphisms in some detail.

We give a comprehensive overview of the model-theoretic properties of differentially henselian fields, including generalisations of classical results from valuation theory, e.g. Ax-Kochen/Ershov type results, quantifier elimination for equicharacteristic 0 fields with angular components, stable embeddedness properties, and more. We also prove various characterisations of differentially henselian fields analogous to those for differentially large fields. Finally, we adapt the machinery of the differential Weil descent to prove results about algebraic extensions of differentially henselian fields.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or institute of learning.

Copyright Statement

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1 Introduction

In 1996, Pop introduced the notion of *large fields* (also known as *ample fields*) in the paper [38], with applications to embedding problems in Galois theory. Large fields are fields with 'many points': a field K is said to be *large* if every absolutely irreducible K-variety (equivalently, K-curve) with a smooth K-rational point has a Zariski-dense set of K-rational points. Equivalently, K is large if and only if it is existentially closed in the field of formal Laurent series $K((t))$.

This class contains many of the classical model-theoretically "tame" classes of fields, such as algebraically closed, real closed and henselian fields. Large fields have found a number of model-theoretic applications, such as in the 2020 work [24] of Johnson, Tran, Walsberg and Ye, where the authors prove a version of the stable fields conjecture for large fields. In this thesis, we give a brief introduction to largeness in Section 2.3.1, but for a concise yet comprehensive overview of the state of research in the theory of large fields, we direct the reader to Bary-Soroker and Fehm's article [7]. For full details on the theory of large fields, we suggest Jarden's book [23].

León Sánchez and Tressl, in [33] and [32], developing on earlier work of Tressl in [46] on the uniform companion for large fields, introduce an analogue of largeness for differential fields, known as *di erential largeness*. A differential field (K, δ) is said to be *di erentially large* if, for every differential field extension (L, ∂) of (K, δ) where K is existentially closed in L as a pure field, (K, δ) is also existentially closed in (L, ∂) as a differential field. This roughly corresponds to the property that if a finite system of differential polynomials over K has an algebraic solution in K, then it also has a differential solution in K, modulo certain technical constraints.

This thesis will present a variety of results on two main topics relating to differentially large fields: the Taylor morphism and its generalisations, and differentially henselian fields. We will restrict ourselves to the case of fields in characteristic 0, and differential fields in one derivation, unless otherwise stated.

Taylor Morphisms

One of the main tools used to study differentially large fields is the ***twisted Taylor morphism*** which is constructed in [32]. This is a modified version of the ***classical Taylor morphism***, which takes a ring homomorphism $\varphi : A \to K$, where (A, δ) is a differential ring, and K is a field (or more generally, a $\mathbb{Q}$-algebra), to a differential ring homomorphism $T_\varphi : (A, \delta) \to (K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ by taking its Taylor series in the naïve way, i.e. for any $a \in A$,

$$T_\varphi(a) = \sum_{i<\omega} \frac{\varphi(\delta^i(a))}{i!} t^i.$$

The twisted Taylor morphism is an adaptation of the classical Taylor morphism to the case where K is not a constant field, i.e. K is equipped with a derivation $\partial \neq 0$. It takes a ring homomorphism $\varphi : A \to K$ as before, to a differential ring homomorphism $T^*_\varphi : (A, \delta) \to (K[[t]], \hat{\partial} + \frac{\mathrm{d}}{\mathrm{d}t})$. This allows us to construct differential $(K[[t]], \hat{\partial} + \frac{\mathrm{d}}{\mathrm{d}t})$ points of differential K-algebras with K-points. Using this construction, we can show that the differential field (K, ∂) is differentially large if and only if it is existentially closed in the field of formal Laurent series $K((t))$ equipped with the derivation $\hat{\partial} + \frac{\mathrm{d}}{\mathrm{d}t}$.

In Chapter 3, we observe that both the twisted and untwisted Taylor morphisms satisfy a functorial property, that is, if $\varphi : A \to K$, $\psi : B \to K$ and $\chi : A \to B$ are ring homomorphisms with χ differential such that $\varphi = \psi \circ \chi$, then we have that $T_\varphi = T_\psi \circ \chi$, where T is either the classical or twisted Taylor morphism. Further, T preserves differential maps, i.e. if φ is differential, then $T_\varphi = \varphi$. We introduce the generalised notion of a Taylor morphism as a functor satisfying these properties in both the twisted and untwisted cases (Definitions 3.1.1, 3.2.2). We introduce these generalisations and develop their basic theory in Sections 3.1 and 3.2.

In Section 3.3, we answer the following question: if K is a differentially large field, every differentially finitely generated K-algebra with an algebraic K-point also has a differential K-point. Can these be found in a 'functorial' way, i.e. in terms of some restricted form of the Taylor morphism? We answer this question in the negative. This is due to the fact that no differential field can admit a Taylor morphism over itself (Proposition 3.3.1), and any such restricted version of the Taylor morphism can be extended uniquely to a full Taylor morphism (Proposition 3.3.11).

We consider generalised versions of the twisting map in Section 3.4, and construct a concrete example of a twisting for a modified version of the classical Taylor morphism where we equip the ring $K[[t]]$ with a nonstandard derivation.

In Section 3.5, we demonstrate that differentially large fields can be characterised in terms of these abstract Taylor morphisms with existential closure properties in a similar way to the characterisations given in [32] (Proposition 3.5.3, Theorem 3.5.4). We also generalise a method of constructing differentially large fields by taking direct limits of a directed system differential fields which admit Taylor morphisms in a certain configuration (Proposition 3.5.5).

In Section 3.6, we construct a diagram in the category of differential K-algebras, whose cocones correspond to abstract K-Taylor morphisms (Proposition 3.6.3). From this, we prove that for any differential ring K, there is a 'universal' K-Taylor morphism through which every other K-Taylor morphism factors (Proposition 3.6.4).

The main results on the structure of Taylor morphisms are presented in Section 3.7, where we show that every abstract Taylor morphism over a differential $\mathbb{Q}$-algebra (K, δ) is precisely the result of embedding the standard twisted Taylor morphism in some differential $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$-algebra (Corollary 3.7.16). We do this by considering abstract versions of the evaluation map $K[[t]] \to K$, which are inverses to certain Taylor morphisms. We also consider the category of K-Taylor morphisms, which we show is isomorphic to the category of differential $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$-algebras (Corollary 3.7.17).

We conclude this chapter by briefly addressing further generalisations to the case of multiple derivations in Section 3.8.

Differentially Henselian Fields

In Chapter 4, we give a broad overview of the model theory of differentially henselian fields, which are an appropriate generalisation of the class of differentially large fields to henselian valued fields. In particular, they can be characterised in terms of an existential closure condition in a similar way to differentially large fields: a valued-differential field (K, v, δ) is differentially henselian if it is (nontrivially) henselian as a pure valued field, and for any valued-differential field extension (L, w, ∂) of (K, v, δ), if (K, v) is existentially closed in (L, w) as pure valued fields, then (K, v, δ) is existentially closed in (L, w, ∂) as valued-differential fields (Theorem 4.4.9).

This class of fields has been studied by a number of authors from various perspectives: by Guzy in [18] adapting the work of Tressl in [46] on the uniform companion for large fields to the henselian context; by Guzy and Point in

[19] in the context of topological differential fields; and by Cubides Kovacsics and Point in [12] in the context of topological differential fields with generic derivations. We provide a brief summary of their work on this topic in Section 4.1.

We approach the topic of differentially henselian fields mainly from the perspective of adapting results from differentially large fields to differentially henselian fields. We will also generalise a variety of classical model-theoretic results from the henselian to the differentially henselian case.

We discuss some of the basic properties of differentially henselian fields in Section 4.2, for example, properties of the constant field and various alternative axiomatisations. We also explore connections with differential largeness: we apply a theorem of Widawski to show that every non-algebraically closed differentially large field with a henselian valuation is differentially henselian (Theorem 4.2.12). From this, we obtain that the theory of a differentially henselian field is completely determined by the union of the theories of its underlying differential field and valued field (Corollary 4.2.17). We also consider the algebraically closed case (Propositions 4.2.13, 4.2.16), where we construct differentially closed fields with non-differentially henselian valuations, and also extract certain model-theoretic properties of the theory DCVF.

In Section 4.3, we adapt methods used to construct differentially large fields (those discussed in Section 2.3.4) to construct differentially henselian fields. We show that differentially henselian fields can be constructed by iterating power series (Proposition 4.3.1) and by constructing appropriate derivations on henselian valued fields of sufficiently large transcendence degree (Theorem 4.3.7).

Section 4.4 presents proofs of various 'existential lifting' properties of differentially henselian fields by showing that any quantifier-free valued-differential type with an 'algebraic realisation' has a realisation in a small neighbourhood around the algebraic realisation (Lemma 4.4.5). This recovers a theorem of Guzy in the one-derivative case (Corollary 4.4.7). We also obtain the main theorem of this chapter, the relative embedding theorem (Theorem 4.4.10), which we will exploit in the following sections.

In Sections 4.5, 4.6 and 4.7, we apply the relative embedding theorem to adapt a number of well-known results from the model theory of henselian valued fields to the differentially henselian case. These include various forms of the Ax-Kochen/Ershov (AKE) principle in the equicharacteristic 0 and unramified mixed characteristic cases, relative subcompleteness and similar results.

We recover certain results by Guzy and Point in [19] and special cases of results by Cubides Kovacsics and Point in [12], namely, the existential-closure form of the AKE principle for equicharacteristic 0 differentially henselian fields (Theorem 4.5.1), recovering [19, Theorem 8.3]; and relative completeness versions of the AKE principle for equicharacteristic 0 and unramified mixed characteristic differentially henselian fields (Theorems 4.5.8, 4.5.17), reproving [12, Corollary 2.4.7] for the above classes of fields.

We also generalise some results of Borrata's thesis [10] from the specific case of closed ordered differential valued fields to differentially henselian fields. In particular, we show that the residue field and value group of equicharacteristic 0 and unramified mixed characteristic differentially henselian fields are stably embedded (Theorem 4.7.2), generalising [10, Theorem 4.2.24]. We also show that the theory of differentially henselian fields of equicharacteristic 0 with an angular component eliminates field-sort quantifiers in the differential Pas language (Theorem 4.6.8), which is a generalisation of the classical result of Pas for equicharacteristic 0 henselian valued fields [37] to the differential context, and generalises [10, Theorem 4.3.8] from CODVF to the equicharacteristic 0 differentially henselian case.

In Section 4.8, we prove we prove a number of equivalent characterisations of differential henselianity in the style of [32, Theorem 4.3] (Theorems 4.8.5, 4.8.6 and 4.8.7). We achieve this by making a small modification to the notion of a Taylor morphism, and showing that the twisted Taylor morphism as constructed in [32] interacts well with valuations (Proposition 4.8.2), and also through applying various existential closure conditions.

Finally, in Section 4.9, we adapt the machinery of the differential Weil descent as introduced in [33] by León Sánchez and Tressl to the context of valued fields. In particular, we show in Theorem 4.9.4 that for a finite algebraic extension of valued fields $(K, v) \subseteq (L, w)$ and a finitely generated L-algebra A, if two K-points of the Weil descent $W(A)$ are 'close' with respect to the valuation topology on K, then the corresponding L-points of A are also close in L. We also show that the converse also holds in a restricted setting (Proposition 4.9.7). We apply the adapted differential Weil descent in this new setting to prove that algebraic extensions of differentially henselian fields are themselves differentially henselian (Theorem 4.9.10) which generalises [32, Theorem 5.11] from the differentially large to the differentially henselian context.

Prerequisites

We will assume that the reader has a working knowledge of both commutative algebra and model theory, which is covered in any standard graduate-level introductory course in these areas. We also assume familiarity with basic notions in category theory. For references covering this background material, we direct the reader to any standard reference text in these areas, for instance, Atiyah and MacDonald's [3] or Lang's [31] for commutative algebra, and any of Marker's [35], Tent and Ziegler's [43], or Chang and Keisler's [27] for the necessary model-theoretic background. For an introduction to category theory, we recommend Mac Lane's classic text [34].

We do not require familiarity with either valued fields or differential algebra. We briefly introduce the required notions and important results in Chapter 2. For full details and background, we advise that the reader consult Engler and Prestel's [14] for valuation theory, and Kolchin's comprehensive text [28] for differential algebra.

Notational Conventions

As notation can vary somewhat between model-theoretic texts, we fix here some of the standard notation that we will use throughout this thesis.

We denote by $\mathcal{L}$ a first-order language. If no confusion arises, we do not distinguish between an $\mathcal{L}$-structure and its underlying set, which we both denote by capital letters M, N, etc. If needed, we specify a structure by a tuple, for example, (K, v, δ) for a field K equipped with a valuation v and a derivation δ. For an L-structure M, write $\mathrm{Th}(M)$ for the $\mathcal{L}$-theory of M. For $\mathcal{L}$-structures M and N, write $M \equiv N$ when M is elementarily equivalent to N. In the case when $M \subseteq N$, we write $M \preccurlyeq N$ when M is an elementary substructure of N, and $M \preccurlyeq_{\exists} N$ when M is existentially closed in N. .

For a language $\mathcal{L}$ and a set of parameters A, we denote by $\mathcal{L}(A)$ the language $\mathcal{L}$ expanded by a constant symbol c_a for each $a \in A$. More generally, we denote expansions of $\mathcal{L}$ by other symbols in a similar way, for example, $\mathcal{L}(\delta)$ is the language $\mathcal{L}$ expanded by the symbol δ. For an $\mathcal{L}$-structure M, for any subset A, M_A is the $\mathcal{L}(A)$-structure on M, where the symbols of $\mathcal{L}$ as interpreted as in M, and for each $a \in A$, c_a is interpreted as a.

We denote elements of a set M by lower case letters a, b, c etc. Tuples (finite or infinite) are denoted by lower case letters with bars, e.g. $\bar{a}, \bar{b}, \bar{c}$. We may abuse notation and write $\bar{a} \in M$ for $\bar{a} \in M^{\alpha}$, where $\bar{a}$ is a tuple indexed by the ordinal α. We may specify the elements of a tuple by writing $\bar{a} = (a_i)_{i<\alpha}$ for an α-indexed tuple $\bar{a}$. The set of tuples of of elements of M of length at most α is denoted $M^{<\alpha}$. The letter ω denotes the least infinite ordinal, i.e. the set of natural numbers. Note in particular that $\bar{a}$ always denotes a tuple of elements, and not the residue of a (in the context of valued fields).

We denote $\mathcal{L}$-formulae by Greek letters φ, ψ, etc. If necessary, we write $\varphi(\bar{x})$ for a formula φ with free variables $\bar{x}$. For an $\mathcal{L}$-structure M, an L-formula $\varphi(\bar{x})$ (possibly with parameters from M) and $\bar{a} \in M$, we write $M \models \varphi(\bar{a})$ if $\varphi(\bar{a})$ holds in M. For a set A, and a tuple of variables $\bar{x}$ of length α, $A^{\bar{x}}$ denotes the set of tuples of length α with entries from A.

For a set A, we denote its cardinality by $|A|$. For a cardinal κ, κ^+ denotes its cardinal successor. For a function $f : A \to B$, for any subset $C \subseteq A$, the restriction of f to C is denoted by $f\restriction_C : C \to B$. For a tuple $\bar{a}$ of length at least n, we also denote the truncation of $\bar{a}$ to the first n entries by $\bar{a}\restriction_n$.

Assumptions

Every field, unless otherwise stated, has characteristic 0. All rings are commutative and unital.

2 | Background

In this chapter, we will give a concise overview of the background material which underpins the work later in this thesis. In particular, we present an introduction to basic topics in differential algebra, with a particular emphasis on differentially large fields.

We will then introduce the basic elements of valuation theory, and introduce some of the main results used to study the model theory of henselian valued fields, such as the Ax-Kochen/Ershov theorems.

2.1 Differential Rings, Fields and Algebras

We begin by introducing the main objects of study in differential algebra, which are differential rings, fields and algebras, and their corresponding notions of morphisms and maps. We will work towards introducing some of the tools with which we study differential fields, for instance the differential basis theorem and the primitive element theorem for differential fields.

We aim to give a general overview of the topic and state basic results, but will generally provide only a sketch of a proof or omit proofs of theorems entirely. For a more comprehensive introduction to the subject and for detailed proofs, we direct the reader to Marker's article [36] or Kolchin's classic text [28] on which we heavily base this section.

Definition 2.1.1. Let R be a ring. A *derivation* on R is a map $\partial : R \to R$, satisfying for any $a, b \in R$:

(i) $\partial(a+b) = \partial(a) + \partial(b)$;

(ii) $\partial(ab) = a\partial(b) + \partial(a)b$.

A *di erential ring* (respectively, *di erential field*) is a ring (respectively, field) R equipped with a derivation $\partial : R \to R$.

For (R, δ) a differential ring (respectively, field), a *di erential subring* (respectively *subfield*) is a subring (respectively, subfield) S of R such that S is closed under δ.

For (K, δ) a differential field, a *di erential field extension* of (K, δ) is a differential field (L, ∂) such that L/K is a field extension and the restriction of ∂ to K is δ, i.e. K is a differential subfield of L.

Definition 2.1.2. Let (A, ∂) and (B, δ) be differential rings. A map $\varphi : A \to B$ is a *differential ring homomorphism* if it is a ring homomorphism and, for any $a \in A$, $\varphi(\partial(a)) = \delta(\varphi(a))$. Equivalently, φ is a differential ring homomorphism if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \delta} \\ A & \xrightarrow{\varphi} & B \end{array}$$

If φ is a ring homomorphism which is also a differential ring homomorphism, we may say that φ *is differential*.

Definition 2.1.3. Let (R, δ) be a differential ring. A *differential R-algebra* is differential ring (A, ∂), where A is equipped with an R-algebra structure such that the structure map $\eta_A : R \to A$ is a differential ring homomorphism. For an arbitrary R-algebra A, we will usually denote its structure map by $\eta_A : R \to A$.

For differential R-algebras A, B with structure maps η_A, η_B, respectively, a *differential R-algebra homomorphism* $\varphi : A \to B$ is an R-algebra homomorphism which is differential.

Note. Let R be a differential ring, and A a differential R-algebra. If L is also a differential R-algebra which is a field, then, following the terminology in [32], a *(differential) L-rational point* of A is a (differential) R-algebra homomorphism $A \to L$. This terminology arises from algebraic geometry, where L-rational points of an affine variety V over a field K can be realised as K-algebra homomorphisms from the coordinate ring $K[V]$ to L.

Lemma 2.1.4. *Let (R, δ) be a differential ring which is a domain. Let K be the quotient field of R. Then, there is a unique derivation ∂ on K which extends the derivation δ on R.*

Proof. By direct computation, it is easy to show that for an arbitrary differential ring (R, δ), that the 'quotient rule' holds, i.e. for $a \in R$ and $b \in R^\times$,

$$\delta(ab^{-1}) = \frac{b\delta(a) - a\delta(b)}{b^2}.$$

It is then straightforward to check that this defines a derivation on the quotient field K, and any derivation on K which restricts to δ on R must satisfy the above relation. $\square$

Examples 2.1.5.
- Let R be any ring. The ***trivial derivation*** on R is the map $\partial : R \to R$ with $\partial(a) = 0$ for every $a \in R$. We denote the trivial derivation by $0 : R \to R$.
- Let R be a ring. The ring $R[t]$ of polynomials over R equipped with the formal derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is a differential ring.
- Let L/K be an extension of differential fields. Then L is a differential K-algebra, where the structure map is given by inclusion.

Note. All differential rings/fields/algebras, unless otherwise specified, will be in one derivation only. We may drop the reference to the derivation if no confusion arises.

Notation. For an element a in a differential ring (R, ∂) and an integer $i < \omega$, denote the i-times composition of ∂ by ∂^i. We may also sometimes denote the ith derivative of a as $a^{(i)}$, i.e. $a^{(i)} = \partial^i(a)$. We will also use a' to denote $\partial(a)$.

Definition 2.1.6. Let R be a differential ring, and let $a \in R$. For $n < \omega$, the n-***jet*** of a is the $(n+1)$-tuple consisting of a and its first n derivatives, i.e.

$$\mathrm{Jet}_n(a) = (a, a', ..., a^{(n)}).$$

The ***jet*** of a is the infinite tuple of all its derivatives, i.e.

$$\mathrm{Jet}(a) = (a^{(i)})_{i<\omega}.$$

For a tuple $\bar{a} = (a_i)_{i<\alpha}$, we write $\mathrm{Jet}_n(\bar{a})$ (respectively $\mathrm{Jet}(\bar{a})$) for the concatenation of the $\mathrm{Jet}_n(a_i)$ (respectively $\mathrm{Jet}(a_i)$).

Definition 2.1.7. Let (R, δ) be a differential ring. The ***constants of*** R, denoted C_R, is the kernel of δ, i.e.

$$C_R = \{a \in R : \delta(a) = 0\}.$$

The constants C_R form a subring of R. Further, if R is a field, then C_R is a subfield of R. Viewing R as a C_R module (or vector space), we see that $\delta : R \to R$ is a C_R-linear map. If $C_R = R$, that is, δ is trivial on R, we say that R is ***constant*** or R is a ***constant ring/field***.

Examples 2.1.8 (Formal Power Series)**.** Let K be a field of characteristic 0. The ***ring of formal power series over*** K, denoted $K[[t]]$ consists of formal sums of the form

$$a = \sum_{i<\omega} a_i t^i$$

where $a_i \in K$ for each i. For elements $a = \sum_i a_i t^i$, $b = \sum_i b_i t^i$, addition and multiplication are defined as follows:

$$a + b = \sum_{i<\omega} (a_i + b_i) t^i$$

$$ab = \sum_{i<\omega} \left(\sum_{j+k=i} a_j b_k \right) t^i.$$

It is also the t-adic completion of the polynomial ring $K[t]$, and is the inverse limit of the inverse system

$$\cdots \longrightarrow K[t]/(t^3) \longrightarrow K[t]/(t^2) \longrightarrow K[t]/(t) \longrightarrow 0$$

where the map $K[t]/(t^{n+1}) \to K[t]/(t^n)$ is given by the quotient modulo (t^n). The ring $K[[t]]$ can be equipped with a natural derivation $\frac{\mathrm{d}}{\mathrm{d}t}$, which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_i a_i t^i \right) = \sum_i (i+1) a_{i+1} t^i.$$

Its constant subring is precisely K. The ***field of formal Laruent series over*** K, denoted $K((t))$, is the quotient field of $K[[t]]$. It consists of formal series of the form

$$a = \sum_{i\in\mathbb{Z}} a_i t^i$$

where $a_i = 0$ for all $i < n$ for some $n \in \mathbb{Z}$. We can also equip $K((t))$ with the usual formal derivation $\frac{\mathrm{d}}{\mathrm{d}t}$, which extends $\frac{\mathrm{d}}{\mathrm{d}t}$ on $K[[t]]$.

Lemma 2.1.9. *Let* (R, δ) *be a di erential ring (or field), and let* $(S_i)_{i<\alpha}$ *be a family of di erential subrings (respectively, subfields) of* (R, δ)*. Then,* $S = \bigcap_{i<\alpha} S_i$ *is a di erential subring (respectively, subfield) of* (R, δ)*.*

Proof. Since the intersection of an arbitrary family of subrings (or fields) is a ring (or field), S is a subring (subfield) of R. Since each of the S_i is closed under δ, for any $a \in S$, $\delta(a) \in S_i$ for every i, so $\delta(a) \in S$ also. Thus S is a differential subring (subfield). □

Definition 2.1.10. Let $(R, \delta) \subseteq (S, \partial)$ be differential rings, and let $\bar{a} \in S$. The ***di erential ring generated by*** $\bar{a}$ ***over*** R is the intersection of all differential subrings of S containing R and $\bar{a}$. That is, $R\{\bar{a}\}$ is the smallest differential subring of R containing $\bar{a}$. Equivalently,

$$R\{\bar{a}\} = R[\mathrm{Jet}(\bar{a})].$$

We say that $\bar{a}$ is a *di erential generating set for S over R (as di erential rings)* if $S = R\{\bar{a}\}$. We say that S is *di erentially finitely generated (as a di erential ring) over R* if there is a finite generating set $\bar{a}$ of S over R.

Similarly, for $(K, \delta) \subseteq (L, \partial)$ differential fields and $\bar{a} \in L$, the *di erential field generated by $\bar{a}$ over K*, denoted $K\langle\bar{a}\rangle$, is the intersection of all differential subfields of L containing K and $\bar{a}$. Equivalently,

$$K\langle\bar{a}\rangle = K(\mathrm{Jet}(\bar{a})).$$

A *di erential generating set for L over K (as di erential fields)* is a tuple $\bar{a} \in L$ with $L = K\langle\bar{a}\rangle$. We say that L is *di erentially finitely generated (as a di erential field) over K* if there is a finite differential generating set for L over K.

Definition 2.1.11. Let (R, ∂) be a differential ring. The *ring of di erential polynomials* (in one variable x) over R is

$$R\{x\} = R[x, x', x^{(2)} ...].$$

That is, $R\{x\}$ consists of polynomials with coefficients in R, with variables being the derivatives of x. Setting $\partial(x^{(i)}) = x^{(i+1)}$ gives a natural extension of ∂ to $R\{x\}$.

For $f \in R\{x\} \setminus R$, the *order* of f is the largest n such that $x^{(n)}$ occurs in f. We denote this $\mathrm{ord}(f)$.

For f a differential polynomial of order $n \geqslant 0$, the *separant* of f is

$$s(f) = \frac{\partial f}{\partial x^{(n)}},$$

i.e. the formal partial derivative of f with respect to $x^{(n)}$.

Definition 2.1.12. Let (R, ∂) be a differential ring, and let $\bar{x} = (x_\alpha)_{\alpha<\kappa}$ be a (possibly infinite) tuple of indeterminates indexed by an ordinal κ. The *di erential polynomial ring over R in variables $\bar{x}$* is the differential ring

$$R\{\bar{x}\} = R[x_\alpha^{(i)} : \alpha < \kappa, i < \omega]$$

where the derivation on $R\{\bar{x}\}$ is given by setting $x_\alpha^{(i)} \mapsto x_\alpha^{(i+1)}$.

Definition 2.1.13. Let R be a differential ring, and let $f(x) \in R\{x\}$ be a differential polynomial of order n. We define $f_{\text{alg}}(\bar{x})$ to be the polynomial in variables $x_0, ..., x_n$ obtained by replacing every instance of $x^{(i)}$ with x_i. That is, f_{alg} is the image of f under the isomorphism (of pure R-algebras) $R\{x\} \to R[x_0, x_1, ...]$ defined by setting $x^{(i)} \mapsto x_i$ for every $i < \omega$.

For a differential polynomial $R(x) \in R\{x\}$ of order n, we say that $\bar{a} \in R^{n+1}$ is an *algebraic solution* (or *root*) of f if $f_{\text{alg}}(\bar{a}) = 0$.

Definition 2.1.14. Let (R, δ) be a differential ring. A *di erential ideal* of R is an ideal $I \subseteq R$ which is closed under δ. That is, for any $a \in I$, $\delta(a) \in I$. A *prime* (respectively *maximal*, *radical*) *di erential ideal* is a differential ideal which is prime (respectively maximal, radical).

It should be clear that quotients of differential rings by differential ideals should yield differential rings:

Lemma 2.1.15. *Let* (R, δ) *be a di erential ring, and* $I \subseteq R$ *be a di erential ideal. Then,* $\delta : R/I \to R/I$ *given by* $\delta(a + I) = \delta(a) + I$ *is well defined and is a derivation on* R/I.

Conversely, the kernel of a differential ring homomorphism is a differential ideal:

Lemma 2.1.16. *Let* $\varphi : (R, \delta) \to (S, \partial)$ *be a di erential ring homomorphism. Then,* $\ker \varphi$ *is a di erential ideal of* R.

Definition 2.1.17. Let (R, δ) be a differential ring, and let (A, ∂) be a differential R-algebra. We say that A is a *di erentially finitely generated* (as a differential R-algebra) if there is some $n < \omega$ and a surjective differential R-algebra homomorphism $R\{x_0, ..., x_{n-1}\} \to A$. Equivalently, there is some $n < \omega$ and a differential ideal $I \subseteq R\{x_0, ..., x_{n-1}\}$ such that A is isomorphic to $R\{x_0, ..., x_{n-1}\}/I$.

Remark. Every differential R-algebra can be realised as a quotient of a differential polynomial ring over R (not necessarily in finitely many variables): take a generating set $(a_\alpha)_{\alpha<\kappa} \subseteq A$ (over the image of R in A), and consider the map $\pi : R\{x_\alpha : \alpha < \kappa\} \to A$ which evaluates x_α at a_α for each α. This map is a differential R-algebra homomorphism by construction, and realises A as the quotient $R\{x_\alpha : \alpha < \kappa\}/\ker(\pi)$.

Remark. Some standard results from commutative algebra do not apply in the differential case, and care should be taken. For example, not every differential ring which is not a field has proper nonzero ideals, for example, where K is a field of characteristic 0, $(K[t], \frac{\mathrm{d}}{\mathrm{d}t})$ has no proper nontrivial differential ideals: if f is a nonzero polynomial, then there is some n such that $(\frac{\mathrm{d}}{\mathrm{d}t})^n f \in K$, so any differential ideal containing a nonzero element is $K[t]$.

A common case of a differential ideal that we will consider is the vanishing ideal of an element of a field extension.

Example 2.1.18. Let L/K be an extension of differential fields, and let $\alpha \in L$. The set $I(\alpha/K)$ of differential polynomials with coefficents in K which vanish at α, i.e.

$$I(\alpha/K) = \{f \in K\{x\} : f(a) = 0\}$$

is a prime differential ideal.

Definition 2.1.19. Let R be a differential ring, and $f(x) \in R\{x\}$. The *di erential ideal generated by* f is

$$\langle f \rangle = (f, f', f^{(2)}, ...).$$

Equivalently, $\langle f \rangle$ is the smallest differential ideal of $R\{x\}$ containing f.

We note that even if f is irreducible, the differential ideal $\langle f \rangle$ is not necessarily prime. Consider the differential polynomial $f(x) = (x')^2 - 2x$. The differential ideal generated by f contains $f' = 2x'(x'' - 1)$, but contains neither x' nor $x'' - 1$. We address this with a modified definition:

Definition 2.1.20. Let $f \in R\{x\}$ be a differential polynomial. Define the differential ideal $I(f)$ by

$$I(f) = \{g \in R\{x\} : s(f)^k g \in \langle f \rangle \text{ for some } k\}.$$

Proposition 2.1.21 ([36, Lemma 1.8])**.** *Every nonzero prime di erential ideal of* $R\{x\}$ *is of the form* $I(f)$ *for some irreducible di erential polynomial* f*.*

Definition 2.1.22. Let $I \subseteq R\{x\}$ be a nonzero prime differential ideal. A *di erential minimal polynomial of* I is a differential polynomial $f \in R\{x\}$ such that $I = I(f)$.

Let L/K be a differential field extension, and $\alpha \in L$. Define $I(\alpha/K)$ as in Example 2.1.18. If $I(\alpha/K)$ is not zero, i.e. α satisfies a nonzero differential polynomial over K, then we say that α is ***di erentially algebraic over*** K, and a ***minimal polynomial of*** α ***over*** K is a minimal polynomial of $I(\alpha/K)$. If $I(\alpha/K) = \{0\}$, then we say that α is ***di erentially transcendental over*** K.

If every $\alpha \in L$ is differentially algebraic over K, then we say that the differential field extension L/K is ***di erentially algebraic***.

Lemma 2.1.23. *Let* L/K *be a di erential field extension, and let* $a \in L$*. Suppose that* a *is di erentially algebraic over* K*, and has di erential minimal polynomial of order* n*. Then,*

$$K\langle a\rangle = K(a, a', ..., a^{(n)}).$$

Definition 2.1.24. Let L/K be a differential field extension. We say that a (possibly infinite) tuple of elements $\bar{a}$ is ***di erentially algebraically independent over*** K if, for any nonzero differential polynomial $f(\bar{x}) \in K\{\bar{x}\}$ (involving only finitely many indeterminates), $f(\bar{a}) \neq 0$. Equivalently, S is differentially algebraically independent over K if $\mathrm{Jet}(\bar{a})$ is algebraically independent over K. The ***di erential transcendence degree*** of the extension L/K is the cardinality of a maximal differentially algebraically independent tuple in L over K. Such a maximal differentially algebraically independent tuple is called a ***di erential transcendence basis of*** L ***over*** K. [1]

We may drop the enumeration and regard such tuples as pure sets. The usual properties of transcendence bases for pure field extensions extend to differential transcendence bases:

Theorem 2.1.25 ([28, Theorem II.4, p.105] in characteristic 0)**.** *Let* $K \subseteq L$ *be an extension of di erential fields. Then,*

(a) *Let* $\Sigma \subseteq T \subseteq L$*, and suppose that* Σ *is di erentially algebraically independent over* K *and* L *is di erentially algebraic over* $K\langle T\rangle$*. Then, there is a di erential transcendence basis* B *of* L *over* K *such that* $\Sigma \subseteq B \subseteq T$*.*

(b) *There exists a di erential transcendence basis of* L *over* K*.*

(c) *Every di erential transcendence basis of* L *over* K *has the same cardinality.*

In particular, part (a) of the above theorem implies the following:

Corollary 2.1.26. *Let* K *be a di erential field, and let* $L = K\langle\bar{a}\rangle$ *be a differentially finitely generated di erential field extension of* K*. Then, there are*

[1] This definition suffices in the characteristic 0 case, however, in positive characteristic care must be taken with regards to separability.

sub-tuples $\bar{a}_0$, $\bar{a}_1$ partitioning $\bar{a}$ such that $\bar{a}_0$ is a di erential transcendence basis of L over K, and $L = K\langle\bar{a}_0\rangle\langle\bar{a}_1\rangle$ is di erentially algebraic over $K\langle\bar{a}_0\rangle$.

We now state the Ritt-Raudenbush basis theorem, which is a finiteness condition for radical differential ideals of differential polynomial rings. It is a partial generalisation of the Hilbert basis theorem for polynomial rings.

Theorem 2.1.27 (Ritt-Raudenbush Basis Theorem, [36, Theorem 1.16])**.** *Let R be a di erential $\mathbb{Q}$-algebra such that every radical di erential ideal is finitely generated (as a radical di erential ideal). Then, every radical di erential ideal of $R\{x\}$ (equivalently, for $n < \omega$, $R\{x_0, ..., x_n\}$) is finitely generated.*

We conclude this section by stating a few results on the structure of differential field extensions.

Proposition 2.1.28. *Let (K, δ) be a di erential field. Let L be an algebraic extension of K as pure fields. Then δ admits a unique extension to L such that (L, δ) is a di erential field extending (K, δ).*

Proof. Take $a \in L$, and let $f(x) \in K[x]$ be the minimal polynomial of a over K. Writing $f(x) = \sum_i \alpha_i x^i$ and applying δ to the equation $f(x) = 0$, we obtain

$$f^\delta(x) + \delta(x)f'(x) = 0$$

where $f^\delta = \sum_i \delta(\alpha_i)x^i$ and f' is the formal derivative. Rearranging and substituting, we see that $\delta(a)$ is uniquely determined by

$$\delta(a) = \frac{f^\delta(a)}{f'(a)}.$$ □

In particular, from the above proof, we can harvest the following corollary:

Corollary 2.1.29. *Let (K, δ) be a di erential field. Then C_K is relatively algebraically closed in K.*

Proof. Let $a \in K$ be algebraic over C_K, with minimal polynomial f. By the previous proof,

$$\delta(a) = \frac{f^\delta(a)}{f'(a)} = 0$$

as $f^\delta = 0$. □

Finally, we state a differential version of the primitive element theorem for finitely differentially generated differentially algebraic field extensions:

Theorem 2.1.30 (Differential Primitive Element Theorem, [28, Proposition II.9])**.** *Let (K, δ) be a differential field, where δ is nontrivial. Let L/K be a differentially finitely generated differential field extension. Then, $L = K\langle\alpha\rangle$ for some $\alpha \in L$.*

2.2 Model Theory of Some Differential Fields

We devote the following section to discussing two model-theoretically important classes of differential fields and their model theory: *differentially closed fields* and *closed ordered differential fields*. These will occur as examples of differentially large fields in later sections. We will give a brief exposition of the model-theoretic properties of these theories of differential fields.

Definition 2.2.1. The language of differential rings, $\mathcal{L}_{\delta-\mathrm{ring}}$, consists of the language of rings $\mathcal{L}_{\mathrm{ring}}$ and a unary function symbol δ which is interpreted as the derivation. The *theory of differential rings (respectively, fields)* is the theory of rings (fields) along with the axioms which state that δ is a derivation. Denote the $\mathcal{L}_{\delta-\mathrm{ring}}$-theory of differential fields (of characteristic 0) by DF.

We begin by introducing the notion of a differentially closed field, which are an analogue of algebraically closed fields in the differential context. Differentially closed fields were introduced by Robinson in [41]; and in his thesis [9], Blum gives a simple axiomatisation of this theory and, among other results, shows that it is the model completion of the theory of differential fields (in characteristic 0).

Definition 2.2.2. A differential field (K, δ) is *differentially closed* if it is existentially closed in the class of differential fields, i.e. for every differential field extension (L, ∂) of (K, δ), (K, δ) is existentially closed in (L, ∂).

Theorem 2.2.3 (Blum, [9])**.** *A differential field (K, δ) is differentially closed if and only if it satisfies the following axiom scheme:*

For every pair of differential polynomials $f, g \in K\{x\}$ with $\mathrm{ord}(f) > \mathrm{ord}(g)$, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

We denote the theory of a differentially closed field (of characteristic 0) by DCF_0.

Proposition 2.2.4 ([36, Lemma 2.2])**.** *Every differential field (K, δ) admits a differential field extension (L, ∂) which is differentially closed.*

Proof Sketch. Follows from a standard inductive construction, adding solutions to differential polynomial systems of the form $f(x) = 0$ and $g(x) \neq 0$ as required in each step. $\square$

In fact, there is a unique 'minimal' such extension, known as the *di erential closure*, which follows from ω-stability below.

Theorem 2.2.5 (Blum, [8, Theorem $*$, p. 44]). *The theory of di erentially closed fields is the model completion of the theory of di erential fields.*

Corollary 2.2.6 ([9, Corollaries 3.2.20, 3.2.21, 3.2.24]). *The theory* DCF_0 *is complete, model complete, decidable and eliminates quantifiers in the language* $\mathcal{L}_{\delta-\mathrm{ring}}$.

Corollary 2.2.7 ([36, Lemmas 2.7, 2.8]). *The theory* DCF_0 *is* ω*-stable.*

Proof Sketch. From quantifier elimination, for a countable differential subfield K of a differentially closed field, one may construct a bijection between $S_1(K)$ and the prime differential ideals by setting $p \mapsto I_p$, where

$$I_p = \{f \in K\{x\} : \text{`}f(x) = 0\text{'} \in p\}.$$

By Proposition 2.1.21, prime differential ideals over K are determined by irreducible differential polynomials, which are countable. $\square$

The following model theoretic result gives the existence of differential closures:

Theorem 2.2.8 (Morley, Shelah [36, Theorem 2.9]). *Let* T *be an* ω*-stable theory. Then, for any substructure* A *of a model of* T*, there is a model* $M \models T$ *such that* $A \subseteq M$ *and* M *is prime and atomic over* A*. Further, if* M *and* N *are prime over* A*, then there is an isomorphism* $M \to N$ *over* A*.*

Corollary 2.2.9 ([36, Corollary 2.10]). *Let* (K, δ) *be a di erential field. Then, there is a di erential field extension* (L, ∂) *such that* (L, ∂) *is di erentially closed, and for any di erentially closed di erential field extension* (F, d) *of* (K, δ)*, there is a embedding* $(L, \partial) \to (F, \mathrm{d})$ *over* (K, δ)*. Further, for any two such extensions* $(L_0, \partial_0), (L_1, \partial_1)$*, there is an isomorphism* $(L_0, \partial_0) \to (L_1, \partial_1)$ *over* K*. The di erential field* (L, ∂) *is known as the* differential closure of (K, δ)*.*

The notion of a *closed ordered differential field* is similarly an analogue for real closed fields for ordered differential fields. These were introduced by Singer in [42], where he shows that the theory CODF is the model completion of the theory of ordered differential fields.

Definition 2.2.10. An *ordered differential field* is a differential field (K, δ) equipped with a field ordering $<$. The *language of ordered differential rings* is $\mathcal{L}_{\delta-\text{ring}}(<)$, where $<$ is a binary relation symbol which is interpreted as the field ordering. Note in particular that there is no interaction specified between δ and $<$. Denote the $\mathcal{L}_{\delta-\text{ring}}(<)$-theory of ordered differential fields by ODF.

Definition 2.2.11 (Singer [42]). A *closed ordered differential field* is an ordered differential field $(K, \delta, <)$ satisfying the following axiom scheme:

Let $f, g_1, ..., g_m \in K\{x\}$ be differential polynomials, such that $n = \text{ord}(f) \geqslant \text{ord}(g_i)$ for every i. If there is $\bar{a} \in K^{n+1}$ such that $f_{\text{alg}}(\bar{a}) = 0$, $s(f)_{\text{alg}}(\bar{a}) \neq 0$ and $(g_i)_{\text{alg}}(\bar{a}) > 0$ for each i, then there is $b \in K$ with $f(b) = 0$ and $g_i(b) > 0$ for each i.

Denote the $\mathcal{L}_{\delta-\text{ring}}(<)$-theory of closed ordered differential fields by CODF.

Theorem 2.2.12 (Singer [42, p. 85]). *The theory* CODF *is the model completion of* ODF*.*

Corollary 2.2.13. *The closed ordered differential fields are precisely the existentially closed ordered differential fields.*

Corollary 2.2.14. *The theory* CODF *is complete, model complete and has quantifier elimination in the language* $\mathcal{L}_{\delta-\text{ring}}(<)$*.*

2.3 Differentially Large Fields

The notion of a large field was introduced by Pop in [38] with applications to Galois theory, and has since found applications in model theory and beyond. Many ‘tame’ theories of fields fall into this class, such as algebraically, separably, real and p-adically closed fields.

Differential largeness is an notion for differential fields introduced by León Sánchez and Tressl in [32] as an analogue for largeness in pure fields. We begin this section by presenting a brief introduction to the theory of large fields.

2.3.1 Large Fields

For a more thorough treatment with full proofs of large fields, we direct the reader to [23, Chapter 5]. We also recommend [7] for a concise but comprehensive outline of the current state of research into large fields.

Definition 2.3.1 ([23, Lemma 5.3.1, Definition 5.3.2])**.** A field K is said to be ***large*** (alternatively, ***ample***), if it satisfies any of the following equivalent conditions:

(a) For any absolutely irreducible polynomial $f \in K[x, y]$, if there is $(a, b) \in K^2$ such that $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$, then there are infinitely many such points;

(b) Every absolutely irreducible K-curve C with a smooth K-rational point has infinitely many K-rational points;

(c) Every absolutely irreducible K-variety with a smooth K-rational point has a Zariski-dense set of K-rational points;

(d) K is existentially closed in the field of formal Laurent series $K((t))$;

(e) K is existentially closed in the henselisation of $K(t)$ with respect to the t-adic valuation.[2]

We clearly see from criterion (a) above that the class of large fields is indeed an elementary class.

Examples 2.3.2.

1. Algebraically closed fields: algebraically closed fields are the existentially closed models of the theory of fields, i.e. if K is an algebraically closed field, it is existentially closed in any field extension. In particular, K is existentially closed in $K((t))$.
2. Real closed fields: if K is a real closed field, then $K((t))$ can be equipped with the Hahn ordering extending the ordering on K. As K is existentially closed in the class of ordered fields, it is in particular existentially closed in $K((t))$.
3. Henselian fields[3]: criterion (a) is an immediate consequence of the implicit function theorem for henselian fields (Theorem 2.5.7). See Proposition 2.5.8.
4. Pseudo algebraically closed (PAC) fields: for example, pseudofinite fields [38, Proposition 3.1].

Non-Examples 2.3.3.

1. Finite fields [23, Proposition 5.3.3].

[2]See Theorem 2.5.5 for the definition of the henselisation.

[3]A pure field is said to be *henselian* if it admits a nontrivial henselian valuation.

2. Any number field: $\mathbb{Q}$ and any finite extension of $\mathbb{Q}$ [23, Proposition 6.2.5]. This is a consequence of Falting's Theorem, formerly 'Mordell's conjecture'.
3. Any function field, i.e. a finitely generated transcendental extension of any field K [23, Proposition 6.1.8].

An interesting class of non-examples known as ***curve-excluding fields*** was recently constructed by Johnson and Ye in [25], where the authors construct a model complete field over which a given curve has only finitely many points.

Lemma 2.3.4 ([23, Proposition 5.5.2])**.** *Let $K \subseteq L$ be a field extension. Suppose that K is existentially closed in L and that L is large. Then, K is large.*

Proof. By the transitivity of existential closure, and the largeness of L, we have that K is existentially closed in $L((t))$. As $K \subseteq K((t)) \subseteq L((t))$, we have that K is also existentially closed in $K((t))$. □

Corollary 2.3.5. *Let K be a field. Then, K is large if and only if it is existentially closed in some henselian field.*

Proof. The forward direction is a direct consequence of Definition 2.3.1(d), as $K((t))$ is henselian. The converse follows from Lemma 2.3.4, and the fact that every henselian field is large. □

2.3.2 The Twisted Taylor Morphism

One of the main tools that is constructed in [32] to study differentially large fields is known as the ***twisted Taylor morphism***. The twisted Taylor morphism provides a uniform method of constructing differential points in the ring of power series over K from algebraic points of differential K-algebras. We first recall the classical, non-twisted construction.

Definition 2.3.6 (The Classical Taylor Morphism)**.** Let K be a $\mathbb{Q}$-algebra, and let (A, δ) be a differential ring. Let $\varphi : A \to K$ be a K-point, i.e. a ring homomorphism into K (without regard for the derivation δ on A). Then, the (classical) Taylor morphism of φ is the differential ring homomorphism $T_\varphi : (A, \delta) \to (K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$

$$T_\varphi(a) = \sum_{i<\omega} \frac{\varphi(\delta^i(a))}{i!} t^i.$$

Example 2.3.7. Consider the differential $\mathbb{R}$-algebra $C^\infty(\mathbb{R})$ which consists of smooth functions $f : \mathbb{R} \to \mathbb{R}$ with pointwise addition and multiplication, equipped with the usual derivation. Let $\varphi : C^\infty(\mathbb{R}) \to \mathbb{R}$ be the 'evaluation at 0' map, i.e. $\varphi(f) = f(0)$. Then, $T_\varphi : C^\infty(\mathbb{R}) \to \mathbb{R}[[t]]$ is precisely the differential $\mathbb{R}$-algebra homomorphism sending a smooth function to its Taylor expansion at 0.

The main constraint which arises from working with the classical Taylor morphism is the assumption that K is a constant field (or $\mathbb{Q}$-algebra). However, León Sánchez and Tressl show in [32] that it is possible to 'twist' the Taylor morphism to correct for a differential structure on K. In the remainder of this section, we will present a one-derivation version of the construction of the twisted Taylor morphism from [32].

We introduce notation to differentiate between the Taylor morphisms of a ring homomorphism $\varphi : A \to B$, for varying derivations on A.

Notation. Let (A, δ) be a differential ring, and let B be a $\mathbb{Q}$-algebra. Let $\varphi : A \to B$ be a ring homomorphism. Denote by $T_\varphi^\delta : (A, \delta) \to (B[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ the Taylor morphism of φ, where A is equipped with the derivation δ. For a $\mathbb{Q}$-algebra A, the evaluation map $\mathrm{ev} : A[[t]] \to A$ is the map which sends a power series to its evaluation at 0.

We denote by $\hat{\delta}$ the unique derivation on $A[[t]]$ which extends δ on A, sends t to 0 and is strongly additive. Explicitly,

$$\hat{\delta}\left(\sum_i a_i t^i\right) = \sum_i \delta(a_i) t^i.$$

That is, $\hat{\delta}$ acts by δ on the coefficients of a series.

One can show with a direct computation the following result:

Theorem 2.3.8 ([32, Theorem 3.2]). *Let A be a $\mathbb{Q}$-algebra, and let δ and ω be commuting derivations on A. Then,*

$$T_{\mathrm{ev}}^{\hat{\delta}+\hat{\omega}+\frac{\mathrm{d}}{\mathrm{d}t}} = T_{\mathrm{ev}}^{\hat{\delta}+\frac{\mathrm{d}}{\mathrm{d}t}} \circ T_{\mathrm{ev}}^{\hat{\omega}+\frac{\mathrm{d}}{\mathrm{d}t}}.$$

The reader may refer to Lemma 3.4.3, where we perform a similar computation for a modified version of the classical Taylor morphism. From this result, we obtain the following 'twisting map':

Corollary 2.3.9 ([32, Corollary 3.3]). *Let A be a $\mathbb{Q}$-algebra, and let δ be a derivation on A. Then,*

$$T_{\mathrm{ev}}^{\hat{\delta}+\frac{\mathrm{d}}{\mathrm{d}t}} : \left(A[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}\right) \to \left(A[[t]], \frac{\mathrm{d}}{\mathrm{d}t}\right)$$

is an isomorphism of di erential rings, with compositional inverse $T_{\mathrm{ev}}^{-\hat{\delta}+\frac{\mathrm{d}}{\mathrm{d}t}}$.

This gives us the necessary components with which we construct the twisted Taylor morphism.

Definition 2.3.10 (The Twisted Taylor Morphism [32, Section 3.4]). Let (A, δ) be a differential ring, and (B, ∂) be a differential $\mathbb{Q}$-algebra. Let $\varphi : A \to B$ be a ring homomorphism. The ***twisted Taylor morphism of*** φ is the differential ring homomorphism

$$T_{\varphi}^{*} = T_{\mathrm{ev}}^{-\hat{\partial}+\frac{\mathrm{d}}{\mathrm{d}t}} \circ T_{\varphi}^{\delta} : (A, \delta) \to \left(B[[t]], \hat{\partial} + \frac{\mathrm{d}}{\mathrm{d}t}\right).$$

We can view this composition as follows:

$$(A, \delta) \xrightarrow{T_{\varphi}} \left(B[[t]], \tfrac{\mathrm{d}}{\mathrm{d}t}\right) \xrightarrow{T_{\mathrm{ev}}^{-\hat{\partial}+\frac{\mathrm{d}}{\mathrm{d}t}}} \left(B[[t]], \hat{\partial} + \tfrac{\mathrm{d}}{\mathrm{d}t}\right), \qquad (A,\delta) \xrightarrow{T_{\varphi}^{*}} \left(B[[t]], \hat{\partial} + \tfrac{\mathrm{d}}{\mathrm{d}t}\right)$$

Note that by Corollary 2.3.9, $T_{\mathrm{ev}}^{-\hat{\partial}+\frac{\mathrm{d}}{\mathrm{d}t}}$ is an isomorphism of differential rings. Writing $T_{\varphi}^{*}(a) = \sum_i b_i t^i$, we may explicitly compute the b_i as

$$b_i = \frac{1}{i!} \sum_{j \leqslant i} (-1)^{i-j} \binom{i}{j} \partial^{i-j}(\varphi(\delta^j(a))).$$

In later chapters, we may refer to the above as the 'classical' or 'standard' twisted Taylor morphism. We generalise this construction in Chapter 3 with the introduction of generalised Taylor morphisms which satisfy a certain categorical condition.

Remark. Observe that if the derivation ∂ on B is trivial, the twisted Taylor morphism is precisely the classical Taylor morphism. This property can be seen as a special case of the phenomenon observed in Proposition 3.2.7.

2.3.3 Characterisations of Differential Largeness

We now introduce the notion of a differentially large field. These are differential fields which are large as fields, and satisfy a certain genericity condition. We will discuss their definition, and present a number of equivalent characterisations of this class of fields.

Definition 2.3.11 ([32, Definition 4.1])**.** Let (K, δ) be a differential field. We say that (K, δ) is ***di erentially large*** if K is large as a pure field, and for any differential field extension (L, ∂) of (K, δ), if K is existentially closed in L as a pure field, then (K, δ) is existentially closed in (L, ∂) as differential fields.

Intuitively, this condition roughly says that a differential field is differentially large if every finite system of differential polynomial equations which has an algebraic solution also has a differential solution.

We can interpret existential closure of differential fields in terms of points of differential algebras in the following sense:

Lemma 2.3.12 ([32, 2.1(ii)])**.** *Let $K \subseteq L$ be an extension of di erential fields. Then, $K \preccurlyeq_{\exists} L$ if and only if every di erentially finitely generated K-algebra with a di erential point $A \to L$ also has a di erential point $A \to K$.*

Examples of differentially large fields include the two classes of differential fields we introduced earlier:

Examples 2.3.13.

1. Let (K, δ) be differentially closed. Since K is algebraically closed, it is large as a pure field. Further, since a differentially closed field is existentially closed in any differential field extension, (K, δ) is differentially large.
2. Let (K, δ) be a closed ordered differential field. It is real closed as a pure field, thus large. Let (L, ∂) be a differential field extension in which (K, δ) is existentially closed as a pure field. Since -1 is not a sum of squares in K, it is also not a sum of squares in L by existential closure. Thus, L is orderable, i.e. there is an field ordering $<$ on L (necessarily extending the unique field ordering on K). Since $(K, \delta, <)$ is existentially closed in the class of ordered differential fields, it is existentially closed in $(L, \partial, <)$. Taking reducts shows the required result.

An important notion which we will need is that of a ***composite*** differential K-algebra. These are differentially finitely generated differential K-algebras which satisfy a certain tameness condition:

Definition 2.3.14 ([32, Definition 2.3]). Let (K, δ) be a differential field, and let S be a differentially finitely generated differential K-algebra which is a domain. A ***decomposition of*** S consists of (not necessarily differential) K-subalgebras A and P of S such that:

1. the subalgebra A is finitely generated as a K-algebra, and P is a polynomial K-algebra (i.e. $P \cong K[T]$ for some possibly infinite set of indeterminates T);
2. the natural map $A \otimes_K P \to S$ given by multiplication is an isomorphism.

If S admits a decomposition, then say that S is ***composite***, and write $S = A \otimes P$.

These arise in the structure theorem for finitely generated differential algebras:

Theorem 2.3.15 ([45, Theorem 1]). *Let* (K, δ) *be a di erential field, and let* S *be a di erentially finitely generated di erential* K*-algebra and a domain. There are* K*-subalgebras* A *and* P *of* S *and an element* $h \in A \setminus \{0\}$ *such that* A *is a finitely generated* K*-algebra,* P *is a polynomial* K*-algebra and the natural homomorphism* $A_h \otimes_K P \to S_h$ *given by multiplication is an isomorphism.*

Remark. That is, every differentially finitely generated differential K-algebra which is a domain admits (is contained in) a localisation which is composite.

León Sanchez and Tressl prove a number of equivalent algebraic and geometric characterisations of differentially large fields in [32, Theorem 4.3], which we reproduce in part below:

Theorem 2.3.16. *Let* (K, δ) *be a di erential field. The following are equivalent:*

(i) K *is di erentially large.*

(ii) (K, δ) *is existentially closed in* $(K((t)), \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$.

(iii) (K, δ) *is existentially closed in* $(K((t_1))...((t_k)), \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t_1} + ... + \frac{\mathrm{d}}{\mathrm{d}t_n})$ *for every* $k \geqslant 1$.

(iv) K *is large as a field and every di erentially finitely generated* K*-algebra that has a* K*-rational point has a di erential* K*-rational point.*

(v) K *is large as a field and every composite* K*-algebra* A *with* K *existentially closed in* A *has a di erential* K*-rational point.*

(vi) K *is large as a field, and for every composite* K*-algebra* $S = A \otimes_K P$*, if* A *has a* K*-rational point, then* S *has a di erential* K*-rational point.*

A useful alternative characterisation in the one derivation case is the following:

Proposition 2.3.17 (León Sanchez-Tressl, [44]). *Let K be a di erential field, large as a field. Then, K is di erentially large if and only if K satisfies the following condition: for any di erential polynomials $f, g \in K\{x\}$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$, if there is $\bar{a} \in K$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$, $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$ and $g_{\mathrm{alg}}(\bar{a}) \neq 0$, then there is $b \in K$ with $f(b) = 0$ and $g(b) \neq 0$.*

The class of differentially large fields can also be realised as the models of the uniform companion for large fields as introduced in [46]. We note that in the paper [46], Tressl proves this for differential fields with multiple commuting derivations, however, we shall restrict ourselves to the one-derivation version.

Theorem 2.3.18 ([46, Main Theorem 6.2, Proposition 6.3]). *There is an inductive theory UC in the language of di erential rings satisfying the following:*

(I) If $L, M \models \mathrm{UC}$, and A is a common di erential subring of L, M such that L and M have the same universal theory over A as pure fields, then L and M also have the same universal theory over A as di erential fields.

(II) Every di erential field K which is large as a pure field admits a di erential field extension L with $L \models \mathrm{UC}$, and $K \preccurlyeq L$ as pure fields.

Proposition 2.3.19 ([32, Proposition 4.7]). *A di erential field K is di erentially large if and only if it is large as a field and a model of UC.*

Proof. Let K be a differentially large field. Then, by Theorem 2.3.18(II), there is a differential field L extending K such that $L \models \mathrm{UC}$ and $K \preccurlyeq L$ as pure fields. In particular, K is existentially closed in L as pure fields, and as K is differentially large K is also existentially closed in L as differential fields. Since UC is inductive, we conclude that K is also a model of UC.

Conversely, let $K \models \mathrm{UC}$. Let L be a differential field extension such that K is existentially closed in L as a pure field. Then, there is field extension F of K such that $K \prec F$ as fields, and L embeds in M over K as pure fields. Equip F with an arbitrary derivation extending the derivation on L.

Theorem 2.3.18(II), there is a differential field extension M of F such that $M \models \mathrm{UC}$ and $F \preccurlyeq M$ as pure fields. Then, $K \preccurlyeq F \preccurlyeq F$ as pure fields. In particular, K and M have the same universal theory over K as pure fields, and by Theorem 2.3.18(I), they have the same universal theory over K as differential fields. Thus, K is existentially closed in M as differential fields. □

2.3.4 Constructing Differentially Large Fields

We now present two methods of constructing differentially large fields, both due to León Sánchez and Tressl. The first is a consequence of the existence of the twisted Taylor morphism, and involves taking a direct limit of a directed system of differential fields.

Proposition 2.3.20 ([32, Proposition 5.1]). *Let $((K_i,\delta_i),f_{ij})_{i<j\in I}$ be a directed system of di erential fields and embeddings of di erential fields with the following properties:*

(a) Each K_i is large as a pure field.

(b) For each map $f_{ij}:K_i\to K_j$, $f_{ij}(K_i)$ is existentially closed in K_j as a field.

(c) For each $i\in I$, there is $j\geqslant i$ such that there is an embedding

$$(K_i[[t]],\hat{\delta}_i+\tfrac{\mathrm{d}}{\mathrm{d}t})\to(K_j,\delta_j)$$

extending f_{ij}.

Then, the direct limit (L,∂) of the directed system is a di erentially large field.

This gives a concrete construction of examples of differentially closed and closed ordered differential fields:

Example 2.3.21 ([32, Example 5.2(ii)]). Let (K,δ) be a differential field, large as a field, and inductively define the differential field (K_n,δ_n) for $n<\omega$ as follows: let $(K_0,\delta_0)=(K,\delta)$, and suppose we have constructed (K_n,δ_n) for some $n<\omega$. Define

$$K_{n+1}=K_n((t_n^{\mathbb{Q}}))\text{ and }\delta_{n+1}=\hat{\delta_n}+\tfrac{\mathrm{d}}{\mathrm{d}t_n}.$$

For details on the construction of the field $K((t^{\mathbb{Q}}))$, see the construction of Hahn series fields in Section 2.5.1.

For each $i<j<\omega$, let $f_{ij}:K_i\to K_j$ be the inclusion. Clearly, the directed system $((K_i,\delta_i),f_{ij})_{i<j<\omega}$ satisfies the conditions (a) and (c) of Proposition 2.3.20.

In the case where K is an algebraically closed, real closed or p-adically closed field, so is $K((t^{\mathbb{Q}}))$. Thus, by model completeness of these theories, we have that condition (b) holds also in these cases. Defining (K_∞,δ_∞) to be the union of the chain (K_i,δ_i), we obtain that (K_∞,δ_∞) is a differentially large field which is algebraically closed, real closed, or p-adically closed, respectively.

Since every differentially large field which is algebraically closed, real closed or p-adically closed is a differentially closed field, closed ordered differential field or an existentially closed model of the theory of differential p-valued fields, respectively, we obtain that (K_∞,δ_∞) is a model of these respective theories. We will later adapt this construction by iterated power series to the differentially henselian case in Proposition 4.3.1.

The second method involves constructing a derivation on an arbitrary large field of sufficiently large transcendence degree such that the resulting differential field satisfies the condition in Proposition 2.3.17 and thus is differentially large.

Definition 2.3.22. Let K be a differential field. A *di erentially large problem* over K is a pair (f, g) of differential polynomials from $K\{x\}$ such that f is irreducible, and $\mathrm{ord}(g) < \mathrm{ord}(f) = n$, for which there is $\bar{a} \in K^{n+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$, $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$ and $g_{\mathrm{alg}}(\bar{a}) \neq 0$. The *order* of a differentially large problem (f, g) is $\mathrm{ord}(f)$.

A *solution* of the differentially large problem (f, g) over K in a differential field extension L of K is an element $a \in L$ such that $f(a) = 0$ and $g(a) \neq 0$.

Proposition 2.3.23 (León Sanchez-Tressl [44])**.** *Let (K, δ) be a di erential field, and let L be a field extension of K which is large, with $\mathrm{trdeg}(L/K) \geqslant n$. Let (f, g) be a di erentially large problem of order n over K. Then, there is a subfield $F \subseteq L$, finitely generated over K as a field, and a derivation ∂ on F extending δ, such that there is a solution $a \in (F, \partial)$ of (f, g) such that $\mathrm{Jet}_{n-1}(a)$ is algebraically independent over K.*

Proof Idea. Find a suitable tuple $\bar{a} = (a_0, ..., a_n) \in L$ such that $a_0, ..., a_{n-1}$ are algebraically independent over K, $f_{\mathrm{alg}}(\bar{a}) = 0$, $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$ and $g(\bar{a}) \neq 0$. Extend δ to a derivation on $F = K(\bar{a})$ by setting $\partial(a_i) = a_{i+1}$ for $i < n$. Then (F, ∂) has the required properties. □

Theorem 2.3.24 (León Sanchez-Tressl [44])**.** *Let (K, δ) be a di erential field, and let L be a large field extending K with $\mathrm{trdeg}(L/K) \geqslant |K|$. Then, there is a derivation ∂ on L extending δ such that (L, δ) is di erentially large.*

Proof Idea. Construct a chain of differential fields $(K_i, \delta_i)_{i<\omega}$ with $K_i \subseteq L$ for each i, such that (K_{i+1}, δ_{i+1}) contains solutions to all differentially large problems over K_i, and $\bigcup_{i<\omega} K_i = L$. Then, setting $\partial = \bigcup \delta_i$, (L, ∂) solves all differentially large problems over itself, i.e. (L, ∂) is differentially large. □

Corollary 2.3.25 (León Sanchez-Tressl [44])**.** *Every large field K of infinite transcendence degree admits a derivation δ such that (K, δ) is di erentially large.*

Proof. Apply the above theorem with $(K, \delta) = (\mathbb{Q}, 0)$. □

In Section 4.3, we will adapt both of these methods to the valued field context to construct differentially henselian fields.

2.4 An Introduction to Valued Fields

In this section, we give a brief introduction to the notion of a ***valued field*** and their associated structures. We can think of a valued field in a similar way to fields with absolute values: a valuation on a field is an abstract notion of the 'size' of field elements, and as such induces a field topology.

Definition 2.4.1. Let K be a field. A ***valuation*** on K is a surjective map $v : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group and $\infty \notin \Gamma$, satisfying the following for any $a, b \in K$:

(i) $v(a) = \infty$ if and only if $a = 0$;

(ii) $v(ab) = v(a)+v(b)$, where we set $\gamma+\infty = \infty+\gamma = \infty$ for any $\gamma \in \Gamma\cup\{\infty\}$;

(iii) $v(a + b) \geqslant \min(v(a), v(b))$, where we let $\infty > \gamma$ for all $\gamma \in \Gamma$.

A ***valued field*** is a field K equipped with a valuation v, denoted as the pair (K, v).

We sometimes think of elements $a \in K$ with zero valuation as 'finite', and positive and negative valuation as 'infinitesimal' and 'infinite', respectively. The reader may refer to [2, Section 3.1] for more on this viewpoint. Note in particular that elements with large valuation are regarded as 'small', and vice versa.

A valuation v on a field K naturally defines a field topology on K.

Definition 2.4.2. Let (K, v) be a valued field. Let $a \in K$ and $\gamma \in vK$. The ***(open) ball of radius γ with centre a*** is

$$B_\gamma(a) = \{b \in K : v(a - b) > \gamma\}.$$

The balls $B_\gamma(a)$ for $a \in K$ and $\gamma \in vK$ form a basis for a field topology on K, called the ***valuation topology***. For any element $\bar{a} = (a_0, ..., a_{n-1}) \in K^n$, we write

$$B_\gamma(\bar{a}) = \prod_{i<n} B_\gamma(a_i)$$

for the ball of radius γ around $\bar{a}$ in K^n.

Definition 2.4.3. Let (K, v) be a valued field. We define the following associated objects:

- The ***value group*** vK is the ordered abelian group $v(K^\times)$.
- The ***valuation ring*** $\mathcal{O}_v \subseteq K$ is the subring consisting of elements of nonnegative valuation, i.e.

$$\mathcal{O}_v = \{a \in K : v(a) \geqslant 0\}.$$

The valuation ring $\mathcal{O}_v$ is a local ring with unique maximal ideal $\mathfrak{m}_v$ consisting of the elements of strictly positive valuation, i.e.

$$\mathfrak{m}_v = \{a \in K : v(a) > 0\}.$$

- The ***residue field*** Kv is the quotient field $\mathcal{O}_v/\mathfrak{m}_v$. The ***residue map***, denoted res (or res_v if the valuation needs to be specified) is the quotient map $\mathcal{O}_v \to Kv$.

Intuitively, we can think of the valuation ring as consisting of the 'finite or infinitesimal' elements of the field, and the residue field as the field consisting of 'finite elements modulo infinitesimals'. Example 2.4.11 gives a concrete demonstration of this point of view.

Notation. The residue map $\mathrm{res} : \mathcal{O}_v \to Kv$ induces a map $\mathcal{O}_v[x] \to Kv[x]$ by acting on the coefficients of polynomials. If no confusion arises, we denote this map also by $\mathrm{res} : \mathcal{O}_v[x] \to Kv$.

Definition 2.4.4. Let Γ be an ordered abelian group. The ***rank*** of Γ is the order type of the set of proper convex subgroups of Γ ordered by inclusion. For (K, v) a valued field, the ***rank*** of v is the rank of vK. If vK is the trivial group, we say that v is ***trivial***.

Definition 2.4.5. Let (K, v) be a valued field. The ***characteristic*** of (K, v) is the pair $(\mathrm{char}(K), \mathrm{char}(Kv))$. We say that (K, v) is ***equicharacteristic*** p if $\mathrm{char}(K) = \mathrm{char}(Kv) = p$, where p is a prime or 0, and of ***mixed characteristic*** if $\mathrm{char}(K) = 0$ and $\mathrm{char}(Kv) = p > 0$.

Example 2.4.6 (The p-adics)**.** Let p be a prime. We first define the ring of p-adic integers $\mathbb{Z}_p$ as the inverse limit

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z},$$

with the inverse system given by

$$\cdots \xrightarrow{\bmod p^3} \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{\bmod p^2} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\bmod p} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0 \ .$$

We can express an element $a \in \mathbb{Z}_p$ as a tuple $(a_i)_{i<\omega}$, where $a_i \in \mathbb{Z}/p^i\mathbb{Z}$, and a_i is the image of a_{i+1} modulo p^i.

The valuation v_p is defined on $\mathbb{Z}_p$ by

$$v_p((a_i)_{i<\omega}) = \min\{i \in \omega : a_i \neq 0\}.$$

Equivalently, if we let $\pi_i : \mathbb{Z}_p \to \mathbb{Z}/p^i\mathbb{Z}$ be the natural projection,

$$v_p(a) = \min\{i \in \omega : \pi_i(a) \neq 0\}.$$

We now define the field of p-adic numbers $\mathbb{Q}_p$ to be the field of fractions of $\mathbb{Z}_p$. The valuation v_p extends naturally to $\mathbb{Q}_p$ by setting

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b).$$

It is easy to see that $\mathbb{Z}_p$ is precisely the valuation ring of $\mathbb{Q}_p$, the value group is $\mathbb{Z}$, and the residue field is $\mathbb{F}_p$.

We have seen that every valuation v on a field K gives rise to a valuation ring $\mathcal{O}_v$. It turns out that the converse is true, i.e. every valuation ring $\mathcal{O}$ on K determines (in fact, uniquely up to equivalence) a valuation v on K with valuation ring $\mathcal{O}$.

Definition 2.4.7. Let K be a field. A ***valuation ring*** on K is a subring $\mathcal{O} \subseteq K$ such that for any $a \in K^\times$, $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$.

For any valuation v on K, it is evident that $\mathcal{O}_v$ is a valuation ring in the above sense.

Proposition 2.4.8 ([14, Proposition 2.1.2])**.** *Let K be a field, and $\mathcal{O} \subseteq K$ be a valuation ring. Then, there is a valuation v on K such that $\mathcal{O} = \mathcal{O}_v$.*

Proof. We let $\Gamma = K^\times/\mathcal{O}^\times$, and define an ordering $\leqslant$ on Γ by setting

$$a\mathcal{O}^\times \leqslant b\mathcal{O}^\times \iff \frac{a}{b} \in \mathcal{O}.$$

It is easy to verify by a routine computation that this is well defined and endows Γ with an ordered group structure. We then define a valuation v on K by setting, for $a \in K^\times$, $v(a) = a\mathcal{O}^\times \in \Gamma$ and $v(0) = \infty$. It is now easy to verify that v is indeed a valuation on K, with $\mathcal{O}_v = \mathcal{O}$. □

Definition 2.4.9. Let K be a field, and v, w valuations on K. We say that v and w are ***equivalent*** if $\mathcal{O}_v = \mathcal{O}_w$.

Proposition 2.4.10 ([14, Proposition 2.1.3]). *Let K be a field, and v, w valuations on K. Then, v and w are equivalent if and only if there is an isomorphism $\varphi : vK \to wK$ of ordered abelian groups such that $\varphi \circ v = w$. (We slightly abuse notation and consider valuations to be maps from $K^\times$ instead of K.)*

We see that this gives a bijective correspondence between the valuation rings of a field K and the valuations of K, up to equivalence. We give an example of a valuation which can be more easily described in this way:

Example 2.4.11. Let $\mathbb{R}^*$ be a proper elementary extension of the real ordered field $\mathbb{R}$. Let $\mathcal{O} = \operatorname{conv}(\mathbb{R}) \subseteq \mathbb{R}^*$ be the convex hull of $\mathbb{R}$, i.e. $\mathcal{O}$ consists of every element $a \in \mathbb{R}^*$ such that there exists $b < c \in \mathbb{R}$ with $b < a < c$. Then, $\mathcal{O}$ is a (convex) valuation ring of $\mathbb{R}^*$. Indeed, if $a \in \mathbb{R}^* \setminus \mathcal{O}$, i.e. $|a| > r$ for every $r \in \mathbb{R}_{>0}$, then as $\mathbb{R}^*$ is an ordered field, $|a^{-1}| < \varepsilon$ for every $\varepsilon \in \mathbb{R}_{>0}$. This is an example of what is known as a *real closed valued field*.

The maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ consists of precisely the 'infinitesimal' elements, i.e. the elements $a \in \mathcal{O}$ with $|a| < \varepsilon$ for every $\varepsilon \in \mathbb{R}_{>0}$. The residue field $\mathcal{O}/\mathfrak{m}$ is isomorphic to $\mathbb{R}$, and the residue map $\operatorname{res} : \mathcal{O} \to \mathbb{R}$ (after composition with an appropriate section) is commonly known as the 'standard part map'.

This gives a convenient way to define the appropriate notion of an extension of valued fields:

Definition 2.4.12. Let (K, v), (L, w) be valued fields with $K \subseteq L$. We say that $(K, v) \subseteq (L, w)$ is an *extension of valued fields* if $\mathcal{O}_v = K \cap \mathcal{O}_w$.

An extension of valued fields $(K, v) \subseteq (L, w)$ induces natural inclusions of the value group $vK \subseteq wL$ and residue field $Kv \subseteq Lw$. We say that the extension $(K, v) \subseteq (L, w)$ is *immediate* if $vK = wL$ and $Kv = Lw$.

If a field K has valuation rings $\mathcal{O}_1 \subseteq \mathcal{O}_2$, we say that $\mathcal{O}_2$ is *coarser* than $\mathcal{O}_1$. Conversely, we say that $\mathcal{O}_1$ is *finer* than $\mathcal{O}_2$. There is a natural correspondence between the coarsenings of a valuation and convex subgroups of its value group:

Lemma 2.4.13 ([14, Proposition 2.3.1]). *Let (K, v) be a valued field. There is a bijective correspondence between:*

(a) The overrings of $\mathcal{O}_v$;

(b) The prime ideals of $\mathcal{O}_v$; and

(c) The convex subgroups of vK.

The correspondence between prime ideals $\mathfrak{p}$ and overrings $\mathcal{O}$ of $\mathcal{O}_v$ is given by

$$\mathfrak{p} \mapsto (\mathcal{O}_v)_{\mathfrak{p}}$$
$$\mathcal{O} \mapsto \mathfrak{m} = \mathcal{O} \setminus \mathcal{O}^{\times} \subseteq \mathcal{O}_v$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. The correspondence between prime ideals $\mathfrak{p}$ of $\mathcal{O}_v$ and convex subgroups Δ of vK is given by:

$$\Delta \mapsto \mathfrak{p}_{\Delta} = \{x \in K : v(x) > \delta \text{ for all } \delta \in \Delta\}$$
$$\mathfrak{p} \mapsto \Delta_{\mathfrak{p}} = \{\gamma \in \Gamma : \gamma, -\gamma < v(x) \text{ for all } x \in \mathfrak{p}\}.$$

We now consider two standard constructions of valuations: quotients by convex subgroups and the composition of valuations.

Definition 2.4.14. Let (K, v) be a valued field, and let Δ be a convex subgroup of vK. Define the *quotient of v by Δ*, to be the valuation $v_{\Delta} : K^{\times} \to vK/\Delta$ given by the composition

$$K^{\times} \xrightarrow{v} vK \xrightarrow{\pi} vK/\Delta$$

where π is the quotient map. Denote the valuation ring and residue map corresponding to v_{Δ} by $\mathcal{O}_{\Delta}$ and res_{Δ}, respectively.

We observe that the valuation v induces a valuation $\bar{v}_{\Delta}$ on the residue field Kv_{Δ}, with value group Δ. That is, for any $a \in \mathcal{O}_{\Delta}$, we set $\bar{v}_{\Delta}(\mathrm{res}_{\Delta}(a)) = v(a)$. This is well defined, as $\mathrm{res}_{\Delta}(a) = \mathrm{res}_{\Delta}(b)$ if and only if $v(a - b) > \delta$ for all $\delta \in \Delta$.

We are able to reverse this process by 'composing' valuations.

Definition 2.4.15. Let (K, v) be a valued field, and let w be a valuation on the residue field Kv. Define the *composition* of v with w as the valuation $w \circ v$ on K given by the valuation ring

$$\mathcal{O}_{w \circ v} = \mathrm{res}_v^{-1}(\mathcal{O}_w).$$

These are indeed inverse to each other:

Lemma 2.4.16. *Let (K, v) be a valued field, and let $\Delta \subseteq vK$ be a convex subgroup. Write $w = \bar{v}_{\Delta} \circ v_{\Delta}$. Then, $\mathcal{O}_v = \mathcal{O}_w$.*

Proof. By definition, $\mathcal{O}_w = \mathrm{res}_{\Delta}^{-1}(\mathcal{O}_{\bar{v}_{\Delta}})$. Thus, $a \in \mathcal{O}_w$ if and only if $v_{\Delta}(a) \geqslant 0$, and $\bar{v}_{\Delta}(\mathrm{res}_{\Delta}(a)) \geqslant 0$. $\square$

It is always possible to extend a subring R of a field K to a valuation ring of K. This is Chevalley's extension theorem:

Theorem 2.4.17 (Chevalley's Extension Theorem, [14, Theorem 3.1.1]). *Let K be a field, and $R \subseteq K$ be a subring. Let $\mathfrak{p} \subset R$ be a prime ideal of R. Then, there is a valuation ring $\mathcal{O}$ of K such that $R \subseteq \mathcal{O}$, and $\mathfrak{m} \cap R = \mathfrak{p}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$.*

As a corollary, we obtain the existence of extensions of valuation rings to field extensions:

Corollary 2.4.18 ([14, Theorem 3.1.2]). *Let K be a field, and $\mathcal{O}$ a valuation ring of K. Let L be a field extension of K. Then, there exists a valuation ring $\mathcal{O}'$ of L such that $K \cap \mathcal{O}' = \mathcal{O}$, i.e. $(K, \mathcal{O}) \subseteq (L, \mathcal{O}')$ is an extension of valued fields.*

2.5 Henselian Valued Fields

In this section, we introduce the class of henselian valued fields, which are the valued fields which satisfy a version of Hensel's Lemma. These are a 'tame' class of valued fields where the model theory is relatively well understood, at least in certain subclasses of henselian fields, through what are known as Ax-Kochen/Ershov principles. We will begin by discussing various algebraic properties of henselian fields, then we will consider these fields from a model-theoretic perspective, especially in the equicharacteristic 0 and unramified mixed characteristic cases.

Definition 2.5.1. Let (K, v) be a valued field. We say that (K, v) is *henselian* if, for every $f(x) \in \mathcal{O}_v[x]$ and $a \in \mathcal{O}_v$ such that $\mathrm{res}(f(a)) = 0$ and $\mathrm{res}(f'(a)) \neq 0$, where f' denotes the formal derivative of f with respect to x, there is $b \in \mathcal{O}_v$ such that $f(b) = 0$ and $\mathrm{res}(a) = \mathrm{res}(b)$.

Remark. For any field K, the trivial valuation v on K satisfies the condition above trivially, as the residue map $\mathrm{res} : K \to Kv = K$ is the identity. Therefore we will assume, unless otherwise specified, that a henselian valuation is nontrivial.

Example 2.5.2. Let K be an algebraically closed field. Then, for any nontrivial valuation v on K, (K, v) is henselian.

We state a few equivalent formulations of henselianity:

Proposition 2.5.3 ([14, Lemma 4.1.1, Theorem 4.1.3]). *Let (K, v) be a valued field of characteristic 0. The following are equivalent:*

(i) (K, v) *is henselian.*

(ii) For any algebraic extension $K \subseteq L$, there is a unique valuation w on L extending v.

(iii) For any $f, g, h \in \mathcal{O}_v[x]$ with $\operatorname{res} f = \operatorname{res} g \operatorname{res} h$, with $\operatorname{res} g$ and $\operatorname{res} h$ relatively prime in $Kv[x]$, there are $g_1, g_1 \in \mathcal{O}_v[x]$ such that $f = g_1 h_1$, $\operatorname{res} g_1 = \operatorname{res} g$, $\operatorname{res} h_1 = \operatorname{res} h$ *and* $\deg(g_1) = \deg(\operatorname{res}(g))$.

(iv) For any $f \in \mathcal{O}_v[x]$ and $a \in \mathcal{O}_v$ with $v(f(a)) > 2v(f'(a))$, where f' denotes the formal derivative of f, there is $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$ and $v(a - \alpha) > v(f'(a))$. *(Newton's Lemma)*

We will need the following fact:

Lemma 2.5.4 ([14, Corollary 4.1.5]). *Let $(K, v) \subseteq (L, w)$ be an extension of valued fields. Suppose that (L, w) is henselian and K is relatively algebraically closed in L. Then, (K, v) is also (possibly trivially) henselian.*

Proof. Let $f \in \mathcal{O}_v[x]$, $a \in \mathcal{O}_v$ such that $\operatorname{res}(f(a)) = 0$ and $\operatorname{res}(f'(a)) \neq 0$. As (L, w) is henselian, there is $b \in L$ such that $f(b) = 0$ and $\operatorname{res}(a) = \operatorname{res}(b)$. As K is algebraically closed in L, and b is algebraic over K, we also have that $a \in K$. Thus K is henselian. □

It turns out that, for any valued field (K, v), there exists a 'minimal' henselian extension called its *henselisation*.

Theorem 2.5.5 ([14, Theorem 5.2.2]). *Let (K, v) be a valued field. Then, there exists a henselian valued field extension (K^h, v^h) called the* henselisation *of (K, v), satisfying the following universal property:*

For any henselian valued field extension (L, w) of (K, v), there is a unique embedding $(K^h, v^h) \to (L, w)$ over (K, v).

Clearly, the henselisation is unique up to unique K-preserving isomorphism. An important property of the henselisation is that it is an immediate extension:

Theorem 2.5.6 ([14, Theorem 5.2.5]). *Let (K, v) be a valued field and (K^h, v^h) its henselisation. Then, $(K, v) \subseteq (K^h, v^h)$ is an algebraic and immediate extension.*

A crucial property of henselian fields that we will use repeatedly is that a version of the implicit function theorem holds for systems of polynomial equations.

Theorem 2.5.7 (Implicit Function Theorem for Henselian Fields, [29, p. 390]). *Let* $f_1, ..., f_n \in K[x_1, ..., x_m, y_1, ..., y_n]$. *Set*

$$\bar{z} = (x_1, ..., x_m, y_1, ..., y_n)$$

and

$$J(\bar{z}) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\bar{z}) & \cdots & \frac{\partial f_1}{\partial y_n}(\bar{z}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(\bar{z}) & \cdots & \frac{\partial f_n}{\partial y_n}(\bar{z}) \end{pmatrix}.$$

Suppose that $\bar{f} = (f_1, ...f_n)$ *has a zero* $(\bar{a}, \bar{b}) = (a_1, ..., a_m, b_1, ..., b_n) \in K^{n+m}$*, and the determinant of* $J(\bar{a}, \bar{b})$ *is nonzero. Then, there is some* $\alpha \in vK$ *such that for all* $\bar{c} = (c_1, ..., c_m) \in K^m$ *with* $v(a_i - c_i) > 2\alpha$ *for each* i*, there is a unique* $\bar{d} = (d_1, ..., d_n) \in K^n$ *such that* $(\bar{c}, \bar{d})$ *is a zero of* $\bar{f}$*, and* $v(b_i - d_i) > \alpha$ *for each* i*.*

The reader may also refer to [40, Theorem 7.4], where Prestel and Ziegler prove this result more generally for t-henselian fields. In particular, we note that the function $\bar{c} \mapsto \bar{d}$ as defined above is continuous with respect to the valuation topology. The implicit function theorem gives a direct way to see that henselian fields are large:

Proposition 2.5.8. *Let* (K, v) *be a henselian valued field, where* v *is nontrivial. Then* K *is large.*

Proof. We apply the criterion in Definition 2.3.1(a). Let $f \in K[x, y]$ be an absolutely irreducible polynomial, and let $(a, b) \in K^2$ such that $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) \neq 0$. Form the matrix J as in Theorem 2.5.7, and observe that it consists of a single entry $\frac{\partial f}{\partial y}(x, y)$, which does not vanish at (a, b) by assumption. Then, there is some $\alpha \in vK$ such that for any $c \in B_{2\alpha}(a)$, there is a unique $d \in K$ such that $f(c, d) = 0$. In particular, there are infinitely many such points as $B_{2\alpha}(a)$ is infinite, and we conclude that K is large. □

We also note that the a composition of valuations is henselian if and only if the components are henselian, i.e.

Proposition 2.5.9 ([14, Corollary 4.1.4]). *Let* (K, v) *be a valued field, let* w *be a valuation on* Kv*, and let* $w \circ v$ *denote the composition valuation on* K*. Then,* $(K, w \circ v)$ *is henselian if and only if both* (K, v) *and* (Kv, w) *are henselian. In particular, if* (K, v) *is a henselian valued field, then so is* $(K((t)), v \circ v_t)$*.*

An interesting property of non-algebraically closed henselian fields is that their henselian valuation topology is unique. This is because every henselian valuation ring on a non-algebraically closed field is dependent:

Definition 2.5.10 ([14, p. 42])**.** Let $\mathcal{O}_1, \mathcal{O}_2$ be valuation rings on a field K. We say that $\mathcal{O}_1$ and $\mathcal{O}_2$ are *dependent* if their product $\mathcal{O}_1\mathcal{O}_2$, i.e. the smallest subring of K containing $\mathcal{O}_1$ and $\mathcal{O}_2$ is not K. We say that $\mathcal{O}_1$ and $\mathcal{O}_2$ are *independent* if they are not dependent.

Dependence of valuation rings forms an equivalence relation on the set of valuation rings on a field K. The crucial fact is that dependent valuation rings induce the same valuation topology:

Theorem 2.5.11 ([14, Theorem 2.3.4])**.** *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be nontrivial valuation rings on a field K. Then, $\mathcal{O}_1$ and $\mathcal{O}_2$ are dependent if and only if they induce the same topology on K.*

A classical theorem of Schmidt then gives the following result:

Theorem 2.5.12 ([14, Theorem 4.4.1])**.** *Let K be a field (of arbitrary characteristic), and suppose that K admits two independent henselian valuation rings. Then, K is separably closed.*

That is, every non-separably closed (i.e. in the case of characteristic 0 fields, algebraically closed) henselian field admits a unique henselian valuation topology.

2.5.1 Hahn Fields

In the following section, we introduce an important family of henselian valued fields, known as *Hahn series fields* or *generalised power series fields*. These are generalisations of the usual field of formal Laurent series $K((t))$ with the t-adic valuation, where the powers of t are taken from an arbitrary ordered abelian group Γ. This content is based on the construction given in Section 3 of [2].

Definition 2.5.13. Let $(\Gamma, +, <)$ be an ordered abelian group. We say that a subset $A \subseteq \Gamma$ is *well-based* if A is well-ordered with respect to the induced ordering. Equivalently, a subset is well-based if it contains no infinite descending chain.

We think of the elements of Γ as 'exponents' or 'monomials'. We require the following preliminary lemma:

Lemma 2.5.14 ([2, Lemma 3.1.2])**.** *Let* Γ *be an ordered abelian group, and let* $A_1, A_2 \subseteq \Gamma$ *be well-based. Then, for any* $\gamma \in \Gamma$*, there are finitely many pairs* $(\gamma_1, \gamma_2 \in A_1 \times A_2$ *such that* $\gamma_1 + \gamma_2 = \gamma$*, and the set*

$$A_1 \cdot A_2 \coloneqq \{\gamma_1 + \gamma_2 : \gamma_1 \in A_1, \gamma_2 \in A_2\}$$

is well-based.

Definition 2.5.15. Let K be a field, and Γ be an ordered abelian group. A ***series*** (with coefficients in K and monomials in Γ) is a function $f : \Gamma \to K \in K^\Gamma$. We think of such a series as a formal sum

$$f = \sum_{\gamma \in \Gamma} f_\gamma \gamma \text{ with } f_\gamma = f(\gamma)$$

and denote it accordingly. The ***support*** of a series $f \in K^\Gamma$ is the set

$$\operatorname{supp}(f) = \{\gamma \in \Gamma : f_\gamma \neq 0\}.$$

We are now able to construct fields of Hahn series.

Definition 2.5.16 (Hahn series)**.** Let K be a field, and Γ be an ordered abelian group. The ***Hahn field with coe cients in*** K ***and monomials from*** Γ is

$$K[[\Gamma]] = \{f \in K^\Gamma : \operatorname{supp}(f) \text{ is well-based}\}.$$

Addition is defined pointwise, and multiplication is given by the 'Cauchy product':

$$\left(\sum_{\gamma \in \Gamma} f_\gamma \gamma\right)\left(\sum_{\gamma \in \Gamma} g_\gamma \gamma\right) = \sum_{\gamma \in \Gamma}\left(\sum_{\gamma_1 + \gamma_2 = \gamma} f_{\gamma_1} g_{\gamma_2}\right)\gamma$$

which is well defined by Lemma 2.5.14.

Lemma 2.5.17 ([2, Lemma 3.1.3])**.** *For any field* K *and ordered abelian group* Γ*,* $K[[\Gamma]]$ *is a field.*

We will generally use the notation $K((t^\Gamma))$ to denote the Hahn series field $K[[\Gamma]]$, where we think of the elements of Γ as exponents of an indeterminate t. That is, we write a series instead as

$$f = \sum_{\gamma \in \Gamma} f_\gamma t^\gamma.$$

We easily see from this that the field of formal Laurent series $K((t))$ is a special case of a Hahn series field, in particular it is isomorphic to $K((t^{\mathbb{Z}})) \cong K[[\mathbb{Z}]]$.

A Hahn series field $K((t^{\Gamma}))$ admits a natural valuation directly analogous to the degree (t-adic) valuation on the field of formal Laurent series $K((t))$.

Definition 2.5.18. Let K be a field and Γ be an ordered abelian group. The *t-adic valuation* on the Hahn series field $K((t^{\Gamma}))$ is the valuation

$$\begin{aligned} v_t : K((t^{\Gamma})) &\to \Gamma \cup \{\infty\} \\ f &\mapsto \min(\mathrm{supp}(f)) \end{aligned}$$

which is well defined, as $\mathrm{supp}(f)$ is well ordered for any $f \in K((t^{\Gamma}))^{\times}$, and $v_t(0)$ is defined to be ∞.

We note a few properties of the valued field $(K((t^{\Gamma})), v_t)$:

- The valuation ring $\mathcal{O}_{v_t}$ is

$$\mathcal{O}_{v_t} = \{f \in K((t^{\Gamma})) : \min(\mathrm{supp}(f)) \geqslant 0\}$$

  i.e. $\mathcal{O}_{v_t}$ consists precisely of series with non-negative support. The maximal ideal $\mathfrak{m}_{v_t}$ consists precisely of series with strictly positive support, i.e. elements $f \in \mathcal{O}_{v_t}$ with $f_0 = 0$.
- The value group $v_t K((t^{\Gamma}))$ is precisely Γ.
- The residue field $K((t^{\Gamma}))v_t$ is precisely K.

This gives a useful way to construct an equicharacteristic $\mathrm{char}(K)$ valued field with a given residue field K and value group Γ. We will see that all (nontrivial) Hahn series fields are henselian with respect to the t-adic topology. For this, we need the notion of *spherical completeness*:

Definition 2.5.19. Let (K, v) be a valued field. We say that (K, v) is *spherically complete* if every chain of balls in K has nonempty intersection.

It is easy to verify that any Hahn series field has this property:

Lemma 2.5.20. *Every Hahn series field $(K((t^{\Gamma})), v_t)$ is spherically complete.*

Now, we apply a standard result from valuation theory:

Lemma 2.5.21. *Every spherically complete valued field (K, v) is henselian.*

This gives the desired conclusion:

Corollary 2.5.22. *Every Hahn series field $(K((t^{\Gamma})), v_t)$ is henselian.*

2.5.2 Some Model Theory of Henselian Fields

In this section, we discuss briefly the model-theoretic properties of certain 'tame' classes of henselian valued fields, with a particular emphasis on the equicharacteristic 0 case and unramified mixed characteristic cases.

In particular, we will discuss the ***Ax-Kochen/Ershov theorems*** (or ***principles***), a class of results which allow certain model-theoretic properties to be lifted from the value group and residue field to the valued field. Versions of this theorem in the equicharacteristic 0 case were proved independently by Ax and Kochen in their series of papers [4, 5, 6] and by Ershov in [15, 16, 17]. Further developments were made by a number of other authors, for example by Anscombe and Jahnke prove several Ax-Kochen/Ershov principles for unramified mixed characteristic henselian valued fields via Cohen rings.

There are many reasonable choices for the language of a valued field, for example, the language of rings with a unary predicate V for the valuation ring; a three-sorted structure with sorts for the valued field, residue field and value group, along with function symbols for the residue and valuation maps, etc.

For the purposes of this thesis, we fix the following choice:

Definition 2.5.23. The ***language of valued fields***, denoted $\mathcal{L}_{\mathrm{vf}}$, consists of the language of rings $\mathcal{L}_{\mathrm{ring}}$, along with a binary relation symbol $|$ called 'valuation divisibility', which is interpreted in all valued fields (K, v) by

$$K \models a|b \text{ if and only if } v(a) \leqslant v(b).$$

The valuation ring and its maximal ideal are 0-definable in this language, by the formulae $1|x$ and $(1|x) \wedge \neg(x|1)$, respectively. The residue field and value group are clearly interpretable, along with the valuation and residue maps. We now take a brief detour to talk about the special case of algebraically closed valued fields, which is particularly tame.

Definition 2.5.24. The theory of algebraically closed valued fields, denoted ACVF, is the $\mathcal{L}_{\mathrm{vf}}$-theory of an algebraically closed field equipped with a non-trivial valuation. The theory $\mathrm{ACVF}_{(p,q)}$, where $(p, q) = (0, 0), (0, p)$ or (p, p), where p is a prime, denotes the $\mathcal{L}_{\mathrm{vf}}$-theory of an algebraically closed valued field of characteristic p with residue characteristic q.

Theorem 2.5.25 ([39, Theorem 4.4.2, Corollary 4.4.3])**.** *The theory* ACVF *admits quantifier elimination, is model complete, and is the model companion of the theory of valued fields.*

From this, we can obtain completeness of ACVF after fixing the characteristic of the field and residue field:

Corollary 2.5.26 ([39, Corollary 4.4.4]). *The theory* $\mathrm{ACVF}_{(p,q)}$ *for any fixed* (p,q) *is complete.*

Proof. It suffices to find a prime substructure for each such theory: these are $\mathbb{Q}$ with the trivial derivation, $\mathbb{Q}$ with the p-adic valuation, and $\mathbb{F}_p$ with the trivial valuation, for $(0,0)$, $(0,p)$ and (p,p), respectively. □

We now return to introduce the other class of 'tame' henselian valued field which we are mainly concerned with:

Definition 2.5.27. Let (K,v) be a henselian valued field of mixed characteristic $(0,p)$. We say that (K,v) is *unramified* if vK has a minimal positive element 1, and $v(p)=1$.

We finally state various forms of the classical Ax-Kochen/Ershov theorem for equicharacteristic 0 and unramified mixed characteristic henselian valued fields.

In equicharacteristic 0, the classical result by Ax, Kochen and Ershov states the following:

Theorem 2.5.28 (Ax-Kochen/Ershov, equicharacteristic 0 version [4][5][15]). *Let* (K,v) *and* (L,w) *be henselian valued fields of equicharacteristic 0. Then:*

(i) $(K,v)\equiv(L,w)$ *if and only if* $vK\equiv wL$ *as ordered abelian groups and* $Kv\equiv Lw$ *as pure fields.*

(ii) Suppose that $(K,v)\subseteq(L,w)$*. Then,* $(K,v)\preccurlyeq(L,w)$ *if and only if* $vK\preccurlyeq wL$ *as ordered abelian groups and* $Kv\preccurlyeq Lw$ *as pure fields.*

(iii) Suppose that $(K,v)\subseteq(L,w)$*. Then,* $(K,v)\preccurlyeq_\exists(L,w)$ *if and only if* $vK\preccurlyeq_\exists wL$ *as ordered abelian groups and* $Kv\preccurlyeq_\exists Lw$ *as pure fields.*

Sometimes the above properties are referred to as $\mathrm{AKE}_\equiv$, $\mathrm{AKE}_\preccurlyeq$ and $\mathrm{AKE}_\exists$ principles, respectively. In the unramified mixed characteristic case, Anscombe and Jahnke show the following versions:

Theorem 2.5.29 (Ax-Kochen/Ershov, unramified version [1]). *Let* (K,v) *and* (L,w) *unramified henselian valued fields. Then:*

(i) $(K,v)\equiv(L,w)$ *if and only if* $vK\equiv wL$ *as ordered abelian groups and* $Kv\equiv Lw$ *as pure fields.*

(ii) *Suppose that* $(K, v) \subseteq (L, w)$*. Then,* $(K, v) \preccurlyeq (L, w)$ *if and only if* $vK \preccurlyeq wL$ *and* $Kv \preccurlyeq Lw$ *as ordered abelian groups and pure fields, respectively.*

(iii) *Suppose that* $(K, v) \subseteq (L, w)$ *and that* Kv *and* Lw *have the same finite degree of imperfection. Then,* $(K, v) \preccurlyeq_\exists (L, w)$ *if and only if* $vK \preccurlyeq_\exists wL$ *and* $Kv \preccurlyeq_\exists Lw$*.*

These results hold in some more general classes of 'tame' valued fields, for which the reader may refer to [30] for further details. Various versions of relative quantifier elimination results are available, for example, for equicharacteristic 0 henselian valued fields with angular components, which we discuss in Section 4.6.

3 | Taylor Morphisms

In this chapter, we explore in greater detail the classical and twisted Taylor morphisms which we first discussed in Section 2.3.2, and introduce a generalised notion of a Taylor morphism as a functor between certain categories. We develop the basic theory of these generalised Taylor morphisms, and adapt a number of theorems relating to differentially large fields to this generalised context.

Recall from Definition 2.3.6 that given a differential ring (A, δ) and a $\mathbb{Q}$-algebra B, the classical Taylor morphism T_φ of φ is the differential ring homomorphism $(A, \delta) \to (B[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ defined by

$$a \mapsto \sum_{i<\omega} \frac{\varphi(\delta^i(a))}{i!} t^i.$$

The main constraint which arises from working with this construction is that the classical Taylor morphism T does not necessarily preserve differential B-algebra structures. To illustrate this, suppose that B is equipped with a nontrivial derivation δ, (A, ∂) is a differential (B, δ)-algebra, and $\varphi : A \to B$ is a B-algebra homomorphism. Then, the Taylor morphism of φ, $T_\varphi : (A, \delta) \to (B[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ is a differential ring homomorphism, but no longer a B-algebra homomorphism. This is clear to see, as the standard structure map of $B[[t]]$ as a B-algebra is not differential if B is nonconstant. Indeed, if we let $\mathrm{id} : B \to B$ be the identity, the Taylor morphism of id, $T_{\mathrm{id}} : (B, \delta) \to (B[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ endows $(B[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ with a new B-algebra structure which is now differential, but here T_{id} is not the natural inclusion.

In [32], the authors develop the notion of a 'twisted' Taylor morphism (Definition 2.3.10) for applications to differential largeness. Essentially, where (B, δ) is a differential $\mathbb{Q}$-algebra, we obtain differential ring homomorphism $T^*_\varphi : (A, \partial) \to (B[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$, which preserves the derivation on B. That is, when (A, ∂) is equipped with a differential (B, δ)-algebra structure, and $\varphi : A \to B$ is a B-algebra homomorphism, then T^*_φ remains a differential (B, δ)-algebra homomorphism $(A, \partial) \to (B[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$.

3.1 Abstract Taylor Morphisms

As a convention for the remainder of this chapter, if K is a pure ring, a ***di erential** K**-algebra*** means a differential $(K, 0)$-algebra, where 0 is the trivial derivation on K. More generally, a pure ring K is considered to be a differential ring with the trivial derivation.

Let K be a $\mathbb{Q}$-algebra. We observe that the classical Taylor morphism T satisfies the following property: for differential rings A and B, and ring homomorphisms $\varphi : A \to K$, $\psi : B \to K$ and differential ring homomorphism $\chi : A \to B$ such that $\varphi = \psi \circ \chi$, we have that $T_\varphi = T_\psi \circ \chi$. This is easy to verify by direct computation: let $a \in A$, and consider $T_\psi \circ \chi(a)$:

$$T_\psi(\chi(a)) = \sum_{i<\omega} \frac{\psi(\delta^i(\chi(a)))}{i!} t^i.$$

As χ is differential, $\delta^i(\chi(a)) = \chi(\partial(a))$. Finally, as $\varphi = \psi \circ \chi$, we have that

$$T_\psi(\chi(a)) = \sum_{i<\omega} \frac{\varphi(\delta^i(a))}{i!} t^i = T_\varphi(a),$$

as required.

Consider the category $\mathcal{C}$ whose objects consist of pairs of the form (A, φ), where A is a differential ring and $\varphi : A \to K$ is a ring homomorphism (not necessarily differential); and a morphism $\chi : (A, \varphi) \to (B, \psi)$ is a commuting triangle of the form

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & K \\ {\scriptstyle\chi}\downarrow & \nearrow_{\psi} & \\ B & & \end{array}$$

where $\chi : A \to B$ is a differential ring homomorphism. This is an instance of a more general categorical construction known as a ***comma category***, which we introduce later in this chapter.

Now, similarly construct the category $\mathcal{D}$ whose objects are pairs of the form (A, φ), where A is a differential ring and $\varphi : A \to (K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ is a differential ring homomorphism; and morphisms $\chi : (A, \varphi) \to (B, \psi)$ are the commuting triangles of the form

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & (K[[t]], \frac{\mathrm{d}}{\mathrm{d}t}) \\ {\scriptstyle\chi}\downarrow & \nearrow_{\psi} & \\ B & & \end{array}$$

where $\chi : A \to B$ is a differential ring homomorphism. This is the slice category over $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ in the category of differential rings. We can now observe that the earlier property of the classical Taylor morphism T says that T is a functor $T : \mathcal{C} \to \mathcal{D}$.

We also note that the classical Taylor morphism also preserves constants, i.e. if C is a constant ring, and $\varphi : C \to K$ is a ring homomorphism, then the Taylor morphism of φ is precisely the composition $\iota \circ \varphi : C \to K[[t]]$, where $\iota : K \to K[[t]]$ is the inclusion. We may also translate this property into categorical terms as above: let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ whose objects are precisely pairs (C, φ), where C is a constant ring. Then, this implies that the action of T on the subcategory $\mathcal{C}_0$ is equal to that of the functor $\iota \circ - : \mathcal{C}_0 \to \mathcal{D}$ which sends and object (C, φ) to $(C, \iota \circ \varphi)$, and is identity on morphisms.

In our definition of an abstract Taylor morphism, we capture these properties of the classical Taylor morphism as follows:

Definition 3.1.1. Let K be a ring, and let L be a differential K-algebra. A *K-Taylor morphism for L* is a map T which sends pairs of the form (A, φ), where A is a differential ring and $\varphi : A \to K$ is a ring homomorphism, to a differential ring homomorphism $T_\varphi : A \to L$ satisfying the following:

(TM1) $T_\varphi{\restriction}_{C_A} = \eta_L \circ \varphi{\restriction}_{C_A}$ where C_A denotes the constants of A, and $\eta_L : K \to L$ is the structure map.

(TM2) Suppose A, B are differential rings, $\varphi : A \to K$ and $\psi : B \to K$ are ring homomorphisms, and $\chi : A \to B$ is a differential ring homomorphism such that the following triangle commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & K \\ {\scriptstyle\chi}\downarrow & \nearrow_{\psi} & \\ B & & \end{array}$$

Then, the following triangle commutes:

$$\begin{array}{ccc} A & \xrightarrow{T_\varphi} & L \\ {\scriptstyle\chi}\downarrow & \nearrow_{T_\psi} & \\ B & & \end{array}$$

If a K-Taylor morphism for L exists, then we say that L *admits* a K-Taylor morphism.

Example 3.1.2. For a $\mathbb{Q}$-algebra K, the classical Taylor morphism T is a K-Taylor morphism for $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ in this new sense.

An interesting example of such a mapping arises from the Hurwitz series construction. In particular, it also applies in the case when the characteristic of K is not necessarily 0.

Definition 3.1.3 ([26, p. 1846])**.** Let K be a ring (of arbitrary characteristic). We define the ***ring of Hurwitz series over*** K, denoted $H(K)$ consists of elements of the form

$$a = (a_i)_{i<\omega} \in K^\omega$$

with addition defined termwise, i.e. $(a_i)+(b_i) = (a_i+b_i)$ and for any $(a_i), (b_i) \in H(K)$, their product (c_i) is defined by

$$c_i = \sum_{j=0}^{i} \binom{i}{j} a_j b_{i-j}.$$

The additive and multiplicative identities are given by the series $(0, 0, 0, ...)$ and $(1, 0, 0, ...)$, respectively.

If we identify a series (a_i) with the formal power series $\sum_i a_i t^i \in K[[t]]$, We see that $H(K)$ is isomorphic to $K[[t]]$ as an additive group, but the multiplication on $H(K)$ is given by a binomial convolutions rather than the usual Cauchy product.

The ring of Hurwitz series may be equipped with a natural derivation denoted $\partial_K : H(K) \to H(K)$, which is given by the shift operator

$$\partial_K(a_i) = (a_{i+1}),$$

i.e. $\partial_K(a_0, a_1, a_2, ...) = (a_1, a_2, a_3, ...)$. The inclusion $K \to H(K)$ defined by $a \mapsto (a, 0, 0, ...)$ defines a natural K-algebra structure on $H(K)$.

Example 3.1.4. Let K be a ring (of arbitrary characteristic). Let (A, δ) be a differential ring, and $\varphi : A \to K$ be a ring homomorphism. The map $H_\varphi : (A, \delta) \to (H(K), \partial_K)$ defined by

$$H_\varphi(a) = (\varphi(\delta^i(a))$$

for any $a \in A$ is a differential ring homomorphism by [26, Proposition 2.1]. We will refer to this map as the ***Hurwitz morphism of*** φ. We can verify simply that H is a K-Taylor morphism for $(H(K), \partial_K)$:

For (TM1), let $a \in A$ be a constant. Then, $H_\varphi(a) = (\varphi(a), 0, 0, ...)$, which is the image of a under the natural K-algebra structure map of $H(K)$. For

(TM2), simply observe that if $(A, \delta), (B, \partial)$ are differential rings, $\varphi : A \to K$, $\psi : B \to K$ are ring homomorphisms and $\chi : A \to B$ is a differential ring homomorphism with $\varphi = \psi \circ \chi$, then for any $a \in A$,

$$(H_\psi \circ \chi)(a) = (\psi(\partial^i(\chi(a))))_i = (\psi(\chi(\delta_i(a))))_i = (\varphi(\delta^i(a))) = H_\varphi(a)$$

as χ is differential.

Note ([26, Proposition 2.4]). In the case where K is a $\mathbb{Q}$-algebra, the differential rings of Hurwitz series $(H(K), \partial_K)$ and formal power series $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ are in fact isomorphic. We will recover this result later in this chapter from work on evaluation maps as Corollary 3.7.13. This isomorphism is given by the map $H_{\mathrm{ev}_0} : (K[[t]], \frac{\mathrm{d}}{\mathrm{d}t}) \to (H(K), \partial_K)$, where $\mathrm{ev}_0 : K[[t]] \to K$ is the 'evaluation at 0' map which sends a series $\sum a_i t^i$ to $a_0 \in K$.

Further, for any differential ring A and ring homomorphism $\varphi : A \to K$, the Hurwitz morphism of φ, H_φ is precisely equal to the composition $H_{\mathrm{ev}_0} \circ T_\varphi$, where T is the classical Taylor morphism.

Remark. The condition on K being a $\mathbb{Q}$-algebra is important - in the case when K has positive characteristic, for instance, $H(K)$ is not even a domain.

An important property of K-Taylor morphisms is that they respect the K-algebra structure of differential K-algebras (where K is constant):

Lemma 3.1.5. *Let A be a di erential K-algebra with structure map $\eta_A : K \to A$. Then, if $\varphi : A \to K$ is a K-algebra homomorphism, $T_\varphi : A \to L$ is a di erential K-algebra homomorphism.*

Proof. Applying (TM1) on the identity map $\mathrm{id}_K : K \to K$, we see clearly that $T_{\mathrm{id}_K} = \eta_L$ as K is constant. Consider the following diagram:

$$\begin{array}{ccc} K & \xrightarrow{\mathrm{id}_K} & K \\ {\scriptstyle \eta_A}\big\downarrow & \nearrow_{\varphi} & \\ A & & \end{array}$$

which commutes as φ is a K-algebra homomorphism. Now, we observe that the triangle

$$\begin{array}{ccc} K & \xrightarrow{\eta_L} & L \\ {\scriptstyle \eta_A}\big\downarrow & \nearrow_{T_\varphi} & \\ A & & \end{array}$$

commutes by (TM2). □

Remark. In other words, for a K-point $\varphi : A \to K$, the result of applying a Taylor morphism T, $T_\varphi : A \to L$ is a differential K-point in L.

We now present a characterisation of differential largeness in terms of existence of Taylor morphisms.

Lemma 3.1.6. *Let (K, δ) be a di erential field that is large as a field. Let (L, ∂) be a di erential field extension of $(K, 0)$, and suppose that (L, ∂) admits a K-Taylor morphism T. Let $\mathrm{id} : K \to K$ be the identity map, and suppose that $T_{\mathrm{id}}(K)$ is existentially closed in (L, ∂) as a di erential field. Let $S = A \otimes P$ be a composite di erential K-algebra. Suppose that A has a K-rational point $\varphi : A \to K$. Then, there exists a di erential K-rational point $S \to K$.*

Proof. Since P has a K-rational point $P \to K$, there is a point $\psi : S \to K$ extending φ. Then, $T_\psi : S \to L$ is a differential ring homomorphism. Since T is a Taylor morphism, the following diagram commutes:

$$\begin{array}{ccc} K & \hookrightarrow & S \\ {\scriptstyle T_{\mathrm{id}}}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle T_\psi} \\ T_{\mathrm{id}}(K) & \hookrightarrow & L \end{array}$$

Observe that $T_\psi(S)$ is a differentially finitely generated $T_{\mathrm{id}}(K)$ subalgebra of L, and it now suffices to find a differential point $\gamma : T_\psi(S) \to T_{\mathrm{id}}(K)$.

Since the differential polynomial ring over $T_{\mathrm{id}}(K)$ in finitely many variables is differentially Noetherian, $T_\psi(S)$ is differentially finitely presented as a $T_{\mathrm{id}}(K)$-algebra, i.e. $T_\psi(S)$ is isomorphic to a differential $T_{\mathrm{id}}(K)$ algebra of the form $T_{\mathrm{id}}(K)\{X_1, ..., X_k\}/I$, where I is a differential ideal of $T_{\mathrm{id}}(K)\{X_1, ..., X_k\}$ finitely generated as a radical differential ideal.

Since $T_\psi(S)$ has a differential L-rational point, and I is finitely generated (as a radical differential ideal), by existential closure, there is a differential point $\chi : T_\psi(S) \to T_{\mathrm{id}}(K)$. Then, $T_{\mathrm{id}}^{-1} \circ \chi \circ T_\psi : S \to K$ is a differential K-point, as required. □

Corollary 3.1.7. *Let (K, δ) be a di erential field that is large as a field. Let (L, ∂) be a di erential field extension of $(K, 0)$ which admits a K-Taylor morphism T. Let $\mathrm{id} : K \to K$ be the identity, and suppose that $T_{\mathrm{id}}(K)$ is existentially closed in (L, ∂) as a di erential field. Then, (K, δ) is di erentially large.*

Proof. Follows from the previous lemma and Theorem 2.3.16(vi). □

Remark. Generally, it is difficult to prove the condition $T_{\mathrm{id}}(K) \preccurlyeq_{\exists} L$, as the structure of $T_{\mathrm{id}}(K)$ can be quite complicated. We resolve this issue by applying *twisted* Taylor morphisms, which we introduce in the following section.

3.2 Twisted Taylor Morphisms

As previously discussed, it is possible to construct a 'twisting' of the classical Taylor morphism in order to preserve the differential structure of K in the differential K-algebra $K[[t]]$.

The twisted Taylor morphism T^* constructed in Definition 2.3.10 has a similar functorial property to the one we described for the classical Taylor morphism: construct the category $\mathcal{C}$ as in the previous section, and construct the category $\mathcal{D}$ as before except replacing $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ with $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$. We can then view T^* as a functor which sends objects $(A, \varphi) \in \mathcal{C}$ to $(A, T^*_{\varphi}) \in \mathcal{D}$, and is the identity on morphisms.

Instead of preserving constant rings, the twisted Taylor morphism preserves differential ring homomorphisms: let $\mathcal{C}_\delta$ be the full subcategory of $\mathcal{C}$ which consists of objects of the form (A, φ), where $\varphi : A \to K$ is a differential ring homomorphism. Then, the restriction of T^* to the subcategory $\mathcal{C}_\delta$ is equal to the functor which sends objects by $(A, \varphi) \mapsto (A, \eta \circ \varphi)$, where $\eta : K \to K[[t]]$ is the structure map, and is the identity on morphisms.

We will define a generalised twisted K-Taylor morphism to be similarly a functor which has such properties. To do this, we begin with a preparatory lemma.

Lemma 3.2.1. *Let (A, δ) and (B, ∂) be di erential rings, and $\varphi : A \to B$ a ring homomorphism. Then, there is a unique maximal di erential subring A_φ of A such that $\varphi\restriction_{A_\varphi}$ is a homomorphism of di erential rings.*

Proof. Let $A_\varphi := \{a \in A : \varphi\restriction_{\mathbb{Z}\{a\}} \text{ is differential}\}$. We prove that A_φ is a differential subring of A. Clearly, A_φ contains both 1 and 0. Let $a_1, a_2 \in A_\varphi$. By definition, $\varphi\restriction_{\mathbb{Z}\{a_i\}}$ are differential morphisms for each i. Equivalently, for every $n < \omega$, $\varphi(\delta^n a_i) = \partial^n(\varphi(a_i))$.

Consider $\varphi(\delta^n(a_1 + a_2))$. Since both φ and δ^n are additive, we have that

$$\varphi(\delta^n(a_1 + a_2)) = \varphi(\delta^n a_1) + \varphi(\delta^n a_2) = \partial^n(\varphi(a_1)) + \partial^n(\varphi(a_2)).$$

Thus A_φ is closed under addition. Since -1 is a constant and is preserved by φ, A_φ is also closed under additive inverse. Now consider $\varphi(\delta^n(a_1 a_2))$.

Expanding, we obtain that

$$\begin{aligned}\varphi(\delta^n(a_1a_2)) &= \varphi\left(\sum_{i=0}^{n}\binom{n}{i}\delta^i a_1\delta^{n-i}a_2\right) = \sum_{i=0}^{n}\binom{n}{i}\varphi(\delta^i a_1)\varphi(\delta^{n-i}a_2)\\ &= \sum_{i=0}^{n}\binom{n}{i}\partial^i\varphi(a_1)\partial^{n-i}\varphi(a_2) = \partial^n(\varphi(a_1)\varphi(a_2)) = \partial^n(\varphi(a_1a_2)).\end{aligned}$$

by assumption. Clearly, A_φ is closed under δ, and if C is any other differential subring of A where $\varphi\restriction_C$ is differential, then C is contained in A_φ. □

We call the differential subring A_φ the ***di erential part of*** φ.

Remark. Let K be a differential ring. If A, B are differential K-algebras, and $\varphi : A \to B$ is a differential K-algebra homomorphism, then A_φ is a differential K-subalgebra as any differential K-algebra homomorphism respects the structure map.

Definition 3.2.2. Let (K, δ) be a differential field, and L be a differential K-algebra. A ***twisted K-Taylor morphism*** T^* for L is a map that assigns to each differential ring (A, ∂) and ring homomorphism $\varphi : A \to K$ a differential ring homomorphism $T^*_\varphi : A \to L$ satisfying the following properties:

(TTM1) $T^*_\varphi\restriction_{A_\varphi} = \eta_L \circ (\varphi\restriction_{A_\varphi})$, where $\eta_L : K \to L$ is the structure map;

(TTM2) For every commutative triangle of the form

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & K \\ {\scriptstyle\chi}\downarrow & \nearrow_{\psi} & \\ B & & \end{array}$$

where A, B are differential rings, φ, ψ are ring homomorphisms, and χ is a differential ring homomorphism, the triangle

$$\begin{array}{ccc} A & \xrightarrow{T^*_\varphi} & L \\ {\scriptstyle\chi}\downarrow & \nearrow_{T^*_\psi} & \\ B & & \end{array}$$

commutes.

Remark. We will make precise the functorial properties of Taylor morphisms later in Proposition 3.2.13.

Lemma 3.2.3. *For any di erential $\mathbb{Q}$-algebra (K, δ), the twisted Taylor morphism constructed in Definition 2.3.10 is a twisted K-Taylor morphism for*

$(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$.

Proof. We verify this by direct computation: Let (A, ∂) be a differential ring, and $\varphi : A \to K$ be a ring homomorphism. Then, recall that for any $a \in A$, writing $T^*_\varphi(a) = \sum_i b_i t^i$, we have

$$b_i = \frac{1}{i!}\sum_{j\leqslant i}(-1)^{\alpha-\beta}\binom{i}{j}\delta^{i-j}(\varphi(\partial^j(a))).$$

For (TTM1), let $a \in A_\varphi$. Then, we have that $\delta^{i-j}(\varphi(\partial^j(a))) = \varphi(\partial^i(a))$ for any i, j, as φ restricted to A_φ is differential. We then have that

$$b_i = \varphi(\partial^i(a))\frac{1}{i!}\sum_{j\leqslant i}(-1)^{i-j}\binom{i}{j}.$$

The alternating sum of binomial coefficients is 0 for all $i \neq 0$ and 1 otherwise. Thus,

$$b_i = \begin{cases} \varphi(a) & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and so $T^*_\varphi(a) = a$ (after identifying K with its image in $K[[t]]$ under the usual structure map) as required.

For (TTM2), differential rings $(A, \partial), (B, \mathrm{d})$, and ring homomorphisms $\varphi : A \to K$, $\psi : B \to K$ and $\chi : A \to B$ with χ differential, such that $\varphi = \psi \circ \chi$. Let $a \in A$, and again write $T^*_\varphi(a) = \sum_i b_i t^i$. Then,

$$\begin{aligned} b_i &= \frac{1}{i!}\sum_{j\leqslant i}(-1)^{\alpha-\beta}\binom{i}{j}\delta^{i-j}((\psi\circ\chi)(\partial^j(a))) \\ &= \frac{1}{i!}\sum_{j\leqslant i}(-1)^{\alpha-\beta}\binom{i}{j}\delta^{i-j}(\psi(\mathrm{d}^j(\chi(a)))) \end{aligned}$$

as χ is differential. Observe that this is the coefficient of t^i in $T^*_\psi(\chi(a))$, which gives that $T^*_\varphi = T^*_\psi \circ \chi$, as required. □

It is often more convenient to avoid reference to restrictions of maps, and we can reformulate our axiomatisation to do this. Fix a differential ring K and differential K-algebra L, and let T be a map as in Definition 3.2.2, i.e. for any differential ring A and ring homomorphism $\varphi : A \to K$, $T_\varphi : A \to L$ is a differential ring homomorphism. Axiom (TTM1') states:

If $\varphi : A \to K$ is a differential ring homomorphism, then $T^*_\varphi = \eta_L \circ \varphi$, where $\eta_L : K \to L$ is the structure map.

In other words, (TTM1') states that T^* "preserves differential maps".

Lemma 3.2.4. *A map T^* as in Definition 3.2.2 satisfies (TTM1) and (TTM2) if and only if it satisfies (TTM1') and (TTM2).*

Proof. Suppose T^* satisfies (TTM1) and (TTM2). Let $\varphi : A \to K$ be a differential ring homomorphism. Then, $A_\varphi = A$, and by (TTM1),

$$T^*_\varphi = T^*_\varphi\restriction_{A_\varphi} = \eta_L \circ (\varphi\restriction_{A_\varphi}) = \eta_L \circ \varphi.$$

Thus T^* satisfies (TTM1'). Conversely, suppose that T^* satisfies (TTM1'). Let $\varphi : A \to K$ be a ring homomorphism. Let $\iota : A_\varphi \to A$ be the inclusion map. Then the following commutes:

$$\begin{array}{ccc} A_\varphi & \xrightarrow{\varphi\restriction_{A_\varphi}} & K \\ {\scriptstyle\iota}\downarrow & \nearrow_{\varphi} & \\ A & & \end{array}$$

In particular, we note that ι is a differential ring homomorphism. Thus, by (TTM2),

$$\begin{array}{ccc} A_\varphi & \xrightarrow{T^*_{\varphi\restriction_{A_\varphi}}} & K \\ {\scriptstyle\iota}\downarrow & \nearrow_{T^*_\varphi} & \\ A & & \end{array}$$

commutes also. We observe that $T^*_\varphi \circ \iota = T^*_\varphi\restriction_{A_\varphi}$, thus $T^*_{\varphi\restriction_{A_\varphi}} = T^*_\varphi\restriction_{A_\varphi}$. As $\varphi\restriction_{A_\varphi}$ is differential by the definition of A_φ, we may apply (TTM1') to obtain that

$$T^*_\varphi\restriction_{A_\varphi} = T^*_{\varphi\restriction_{A_\varphi}} = \eta_L \circ (\varphi\restriction_{A_\varphi})$$

as required. □

Remark. Viewing a twisted K-Taylor morphism T as a functor again, the axiom (TTM1') corresponds to saying that, when restricted to the subcategory consisting of objects of the form (A, φ), where φ is differential, T acts by postcomposition with η_L, where $\eta_L : K \to L$ is the structure map of L as a differential K-algebra.

As in the untwisted case, a twisted K-Taylor morphism preserves differential K-algebra structures:

Lemma 3.2.5. *Let K be a di erential ring, L a di erential K-algebra and T a twisted K-Taylor morphism for L. Let A be a di erential K-algebra, and let*

$\varphi : A \to K$ be a K-algebra homomorphism. Then, $T^*_\varphi : A \to L$ is a di erential K-algebra homomorphism.

Proof. This is the same as the proof for the untwisted case, Lemma 3.1.5. □

We should now also make the following obvious observation:

Lemma 3.2.6. *Let T^* be a twisted K-Taylor morphism for L. Let $\theta : L \to F$ be a di erential K-algebra homomorphism. Then, defining $S_\varphi = \theta \circ T^*_\varphi$ for every di erential ring homomorphism $\varphi : A \to K$, S is a twisted K-Taylor morphism for F.*

We now claim that every K-Taylor morphism is a twisted $(K, 0)$-Taylor morphism.

Proposition 3.2.7. *Let K be a field, considered to be equipped with the trivial derivation, and let L be a di erential K-algebra. Then, T is a K-Taylor morphism for L if and only if T is a twisted $(K, 0)$-Taylor morphism for L.*

Proof. Let T^* be a twisted K-Taylor morphism for L. As K is constant, we have that for any differential ring A and ring homomorphism $\varphi : A \to K$, $C_A \subseteq A_\varphi$. Thus, T^* satisfies (TM1), and (TM2) follows immediately from (TTM2).

Conversely, let T be a K-Taylor morphism for L. As above, (TTM2) follows immediately from (TM2), thus it remains to show that (TTM1) holds for T. Let A be a differential ring, and $\varphi : A \to K$ be a ring homomorphism. Let I be the maximal differential ideal of A contained in $\ker \varphi$. In particular, I is the ideal generated by all $a \in \ker \varphi$ such that $a^{(n)} \in \ker \varphi$ for every $n < \omega$. Let $\pi : A \to A/I$ be the quotient map, and φ/I be the map $A/I \to K$ induced by φ. Then, by construction, the following triangle commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & K \\ {\scriptstyle\pi}\downarrow & \nearrow_{\varphi/I} & \\ A/I & & \end{array}$$

As I is a differential ideal, π is a differential K-algebra homomorphism. Applying (TM2), we obtain that the following also commutes:

$$\begin{array}{ccc} A & \xrightarrow{T_\varphi} & L \\ {\scriptstyle\pi}\downarrow & \nearrow_{T_{\varphi/I}} & \\ A/I & & \end{array}$$

Now let $a \in \pi^{-1}(C_{A/I})$. By commutativity of the second diagram, we observe that $T_\varphi(a) = T_{\varphi/I} \circ \pi(a)$. As $\pi(a) \in C_{A/I}$, we apply (TM1) to obtain that $T_{\varphi/I}(\pi(a)) = \eta_L \circ \varphi/I(\pi(a))$. Thus, $T_\varphi(a) = \eta_L \circ \varphi(a)$.

Finally, we claim that $\pi^{-1}(C_{A/I}) = A_\varphi$. For the forward inclusion, suppose that $a \in \pi^{-1}(C_{A/I})$. To show that $a \in A_\varphi$, it suffices to show that $\varphi(\delta^n(a)) = \delta^n\varphi(a)$ for every n. The $n = 0$ case holds trivially. As $\pi(a) \in C_{A/I}$, we have that $\delta^n(a) \in \ker\varphi$ for every $n > 0$, and thus $\varphi(\delta^n(a)) = 0$ for every $n > 0$. Finally, as δ is trivial on K, $\delta^n(\varphi(a)) = 0$ for every $n > 0$ also.

For the reverse inclusion, Suppose that $a \in A_\varphi$. Then, as remarked above, we have that $\varphi(\delta^n(a)) = \delta^n(\varphi(a))$ for every n. As $\delta^n(\varphi(a)) = 0$ for every $n > 0$, we have that $\delta^n(a) \in \ker\varphi$ for every $n > 0$. Thus $\delta(a) \in I$, and $\pi(a) \in C_{A/I}$, as required. ☐

From the proof above, we can harvest the following fact:

Corollary 3.2.8. *Let A be a di erential ring, K a constant ring and $\varphi : A \to K$ a ring homomorphism. Let I be the maximal di erential ideal of A contained in $\ker\varphi$, and $\pi : A \to A/I$ be the quotient map. Then, $\pi^{-1}(C_{A/I})$ is the di erential part of φ.*

In this light, we will drop the adjective 'twisted' for Taylor morphisms, as non-twisted Taylor morphisms are simply special cases of twisted Taylor morphisms. That is, from this point, we simply write 'K-Taylor morphism' for a twisted K-Taylor morphism.

As previously stated, Taylor morphisms can be viewed as functors. In the remainder of this section, we make this translation more precise. We begin by defining a few categories, functors, and recalling a standard category theoretic construction.

Definition 3.2.9. We define the following:

- The category RING is the category of (commutative) rings, with objects being rings and morphisms ring homomorphisms.
- The category DRING denotes the category of differential rings, with objects differential rings and morphisms differential ring homomorphisms.
- The category $\mathbf{1}$ is the category containing a unique object $\bullet$ with the only morphism being the identity $\mathrm{id} : \bullet \to \bullet$.
- The functor U : DRING $\to$ RING is the forgetful functor, which sends a differential ring to its underlying ring.
- For an object K in an arbitrary category $\mathcal{C}$, the functor $1_K : \mathbf{1} \to \mathcal{C}$ sends the unique object in $\mathbf{1}$ to K.

- For an arbitrary category $\mathcal{C}$, $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor on $\mathcal{C}$.

Definition 3.2.10 (Comma Category)**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. The ***comma category*** $(F \downarrow G)$ is the category with objects given by triples of the form (c, d, φ), where $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $\varphi : F(c) \to G(d)$ is a morphism in $\mathcal{E}$. A morphism from (c_1, d_1, φ_1) to (c_1, d_1, φ_1) in $(F \downarrow G)$, is a pair (α, β), where $\alpha : c_1 \to c_2$ is a morphism in $\mathcal{C}$, $\beta : d_1 \to d_2$ is a morphism in $\mathcal{D}$, such that the following square commutes:

$$\begin{array}{ccc} F(c_1) & \xrightarrow{F(\alpha)} & F(c_2) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ G(d_1) & \xrightarrow{G(\beta)} & G(d_2) \end{array}$$

The composition of morphisms $(\alpha_1, \beta_1) : (c_1, d_1, \varphi_1) \to (c_2, d_2, \varphi_2)$ and $(\alpha_2, \beta_2) : (c_2, d_2, \varphi_2) \to (c_3, d_3, \varphi_3)$ is defined by

$$(\alpha_2, \beta_2) \circ (\alpha_1, \beta_1) = (\alpha_2 \circ \alpha_1, \beta_2 \circ \beta_1).$$

The ***domain functor*** $D : (F \downarrow G) \to \mathcal{C}$ is the functor which maps objects by $(c, d, \varphi) \mapsto c$ and morphisms by $(\alpha, \beta) \mapsto \alpha$.

Examples 3.2.11.
1. For an object K in a category $\mathcal{C}$, the slice category $\mathcal{C}/K$ is the category with objects being morphisms in $\mathcal{C}$ with codomain K, and morphisms being the appropriate commuting triangles. Then, $\mathcal{C}/K$ is (isomorphic to) the comma category $(\mathrm{id}_{\mathcal{C}} \downarrow 1_K)$.
2. Let K be a differential ring. The category $\mathcal{C}$ constructed on page 51 is (isomorphic to) the comma category $(U \downarrow 1_K)$: its objects are triples $(A, \bullet, \varphi)$, where A is a differential ring, and $\varphi : A \to K$ is a ring homomorphism, and the morphisms $(A, \bullet, \varphi) \to (B, \bullet, \psi)$ in $(U \downarrow 1_K)$ are pairs (χ, id), where $\chi : A \to B$ is a differential ring homomorphism such that the following square commutes:

$$\begin{array}{ccc} U(A) & \xrightarrow{U(\chi)} & U(B) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ 1_K(\bullet) & \xrightarrow{1_K(\mathrm{id})} & 1_K(\bullet) \end{array}$$

That is, the following triangle commutes in the category of rings:

$$\begin{array}{ccc} U(A) & \xrightarrow{\varphi} & K \\ {\scriptstyle U(x)}\downarrow & \nearrow_{\psi} & \\ U(B) & & \end{array}$$

which is precisely the construction of the category $\mathcal{C}$.

We naturally define two more functors that will be required in the definition of the Taylor morphism as a functor:

Definition 3.2.12. Let K be a differential ring, and L a differential K-algebra. By an abuse of notation, we may regard the forgetful functor U as a functor

$$U : (\mathrm{id}_{\mathrm{DRing}} \downarrow 1_K) \to (U \downarrow 1_K)$$

which simply forgets the differential structure on objects and acts as the identity on morphisms. Precisely, $U(A, \bullet, \varphi) = (A, \bullet, \varphi)$, where A is now regarded as a pure ring.

Let $\eta_L : K \to L$ denote the structure map. Define

$$\eta_L \circ - : (\mathrm{id}_{\mathrm{DRing}} \downarrow 1_K) \to (\mathrm{id}_{\mathrm{DRing}} \downarrow 1_L)$$

by setting $(A, \bullet, \varphi) \mapsto (A, \bullet, \eta_L \circ \varphi)$ for objects and the identity for morphisms.

Now, we can restate the axioms for Taylor morphisms in terms of categories and functors:

Proposition 3.2.13. *Let K be a di erential ring, and L a di erential K-algebra. A K-Taylor morphism for L is precisely a functor*

$$T : (U \downarrow 1_K) \to (\mathrm{id}_{DRing} \downarrow 1_L)$$

such that the following two triangles commute:

$$\begin{array}{ccc} (U \downarrow 1_K) & \xrightarrow{T} & (\mathrm{id}_{DRing} \downarrow 1_L) \\ & \searrow_{D} & \downarrow{\scriptstyle D} \\ & & DRing \end{array} \qquad \begin{array}{ccc} (\mathrm{id}_{DRing} \downarrow 1_K) & \xrightarrow{U} & (U \downarrow 1_K) \\ & \searrow_{\eta_L \circ -} & \downarrow{\scriptstyle T} \\ & & (\mathrm{id}_{DRing} \downarrow 1_L) \end{array}$$

The correspondence between K-Taylor morphisms for L and such functors is given as follows: let T be a K-Taylor morphism for L. Then, the functor

$T : (U \downarrow 1_K) \to (\mathrm{id}_{DRING} \downarrow 1_L)$ acts on objects by $(A, \bullet, \varphi) \mapsto (A, \bullet, T_\varphi)$, and on morphisms by the identity. Conversely, if S is such a functor, for any di erential ring A and ring homomorphism $\varphi : A \to K$, we set $S_\varphi = \psi$, where $S(A, \bullet, \varphi) = (A, \bullet, \psi)$.

Remark. The commutativity of the first triangle essentially just says 'T does not move the domain of maps', which is necessary in this setting. The second is a restatement of axiom (TTM1') - it says that the action of T on the subcategory which consists of differential rings with differential maps into K (i.e. the slice category over K in the category of differential rings) is equal to that of postcomposition by the structure map of L.

Proof. Let T be a K-Taylor morphism for L. As above, we define the functor $T : (U \downarrow 1_K) \to (\mathrm{id}_{\mathrm{DRING}} \downarrow 1_L)$. We show that T is a functor satisfying the above commutativity properties. By definition, $T_\varphi : A \to L$ is a morphism in DRING, thus $T(A, \bullet, \varphi) = (A, \bullet, T_\varphi)$ is an object in $(\mathrm{id}_{\mathrm{DRING}} \downarrow 1_L)$. Now, suppose that $(\alpha, \mathrm{id}) : (A, \bullet, \varphi) \to (B, \bullet, \psi)$ is a morphism in $(U \downarrow 1_K)$. Then, by definition, the following diagram commutes:

$$\begin{array}{ccc} U(A) & \xrightarrow{U(\alpha)} & U(B) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ K & \xrightarrow{\mathrm{id}} & K \end{array}$$

Thus, applying (TTM2), we have that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle T_\varphi} & & \downarrow{\scriptstyle T_\psi} \\ L & \xrightarrow{\mathrm{id}} & L \end{array}$$

also commutes. Thus, (α, id) is a morphism $(A, \bullet, T_\varphi) \to (B, \bullet, T_\psi)$ in the category $(\mathrm{id}_{\mathrm{DRING}} \downarrow 1_L)$.

We now check the two commutativity conditions. As every functor in the diagram acts as the identity on morphisms, it suffices to check that the diagrams commute for objects.

For the first diagram, let $(A, \bullet, \varphi) \in (U \downarrow 1_K)$. Then,

$$D(T(A, \bullet, \varphi)) = D(A, \bullet, T_\varphi) = A = D(A, \bullet, \varphi).$$

For the second, let $(A, \bullet, \varphi) \in (\mathrm{id}_{\mathrm{DRING}} \downarrow 1_K)$. Then, $\varphi : A \to K$ is a differential ring homomorphism, and thus by (TTM1'), we have that $T_\varphi = \eta_L \circ \varphi$. Hence,

$$T(U(A, \bullet, \varphi)) = T(A, \bullet, \varphi) = (A, \bullet, T_\varphi) = (A, \bullet, \eta_L \circ \varphi) = (\eta_L \circ -)(A, \bullet, \varphi)$$

as required.

Conversely, let $S : (U \downarrow 1_K) \to (\mathrm{id}_{\mathrm{DRing}} \downarrow 1_L)$ be a functor satisfying the above commutativity conditions. For a differential ring A and ring homomorphism $\varphi : A \to K$ define S_φ as above.

By commutativity of the first diagram, $S(A, \bullet, \varphi) = (A, \bullet, S_\varphi)$, i.e. S_φ is a differential ring homomorphism $A \to L$. We check (TTM1') holds for S. Suppose that $\varphi : A \to K$ is differential. Then, $(A, \bullet, \varphi)$ is an object in $(\mathrm{id}_{\mathrm{DRing}} \downarrow 1_K)$. By commutativity of the second diagram, we have that $S(A, \bullet, \varphi) = (A, \bullet, \eta_L \circ \varphi)$, thus $S_\varphi = \eta_L \circ \varphi$, as required.

Finally, for (TTM2), let A, B be differential rings, $\varphi : A \to K$ and $\psi : B \to K$ be ring homomorphisms, and $\chi : A \to B$ be a differential ring homomorphism such that $\varphi = \psi \circ \chi$. Then, $(\psi, \mathrm{id}) : (A, \bullet, \varphi) \to (B, \bullet, \psi)$ is a morphism in $(U \downarrow 1_K)$. By functoriality, $S(\chi, \mathrm{id}) : S(A, \bullet, \varphi) \to S(B, \bullet, \psi)$ is a morphism in $(\mathrm{id}_{\mathrm{DRing}} \downarrow 1_L)$. Now, by definition, $S(A, \bullet, \varphi) = (A, \bullet, S_\varphi)$, and similarly, $S(B, \bullet, \psi) = (B, \bullet, S_\psi)$. Thus, we have that $S_\varphi = S_\psi \circ \chi$, as required. □

3.3 Restrictions of Taylor Morphisms

In this section, we will show that Taylor morphisms are completely determined by their restrictions to finitely differentially generated K-algebras, and in doing so answer a question of León Sánchez in the negative.

We begin by showing that a differential field cannot admit a K-Taylor morphism into itself.

Proposition 3.3.1. *Let (K, δ) be a di erential field. Then K does not admit a K-Taylor morphism.*

Proof. Suppose that (K, δ) admits a K-Taylor morphism T. Consider the evaluation map (at 0) $\mathrm{ev} : K[[t]] \to K$, where $K[[t]]$ is equipped with the standard derivation $\partial := \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}$. Then, $T_{\mathrm{ev}} : K[[t]] \to K$ is a differential K-algebra homomorphism. Since T_{ev} is surjective onto K, $\ker T_{\mathrm{ev}}$ is a maximal ideal of $K[[t]]$. As $K[[t]]$ is a local ring with maximal ideal $tK[[t]]$, we find that $\ker T_{\mathrm{ev}} = tK[[t]]$, and in particular, $T_{\mathrm{ev}}(t) = 0$. This is a contradiction, as $T_{\mathrm{ev}}(\partial(t)) = T^*_{\mathrm{ev}}(1) = 1$, and $\delta(T_{\mathrm{ev}}(t)) = \delta(0) = 0$, so T_{ev} is not a differential morphism. □

We will now answer a question of León Sánchez negatively: it was asked whether differential largeness could be characterised by a form of Taylor morphism from K to itself, perhaps after restricting to differentially finitely generated K-algebras.

We know by Theorem 2.3.16(iv) that if K is a differentially large field, then every differentially finitely generated K-algebra A with a K-point $\varphi : A \to K$ has a differential K-point as well. We ask whether such points can be found in a 'uniform' or 'functorial' way similarly to in the case of Taylor morphisms.

Definition 3.3.2. Let K be a differential ring, and L a differential K-algebra. A *finite K-Taylor morphism for L* is a map $\hat{T}$ which sends pairs (A, φ), where A is a differentially finitely generated K-algebra, and $\varphi : A \to K$ is a K-algebra homomorphism, to a differential K-algebra homomorphism $\hat{T}_\varphi : A \to L$, satisfying the axioms (TTM1) and (TTM2), with the appropriate restriction to differentially finitely generated K-algebras and K-algebra homomorphisms.

Question 3.3.3. *Let K be a di erential field. Is it true that K is di erentially large if and only if there is a finite K-Taylor morphism for K?*

Example 3.3.4. Let T be any K-Taylor morphism for a differential K-algebra L, and let $\hat{T}$ be its restriction to differentially finitely generated K-algebras. Then, $\hat{T}$ is a finite Taylor morphism.

We begin by showing that if we have a finite K-Taylor morphism $\hat{T}$ for L, then we can extend its domain uniquely to all differential K-algebras.

Definition 3.3.5. A *restricted K-Taylor morphism for L* is a map $\tilde{T}$ which sends pairs (A, φ), where A is a differential K-algebra and $\varphi : A \to K$ is a K-algebra homomorphism, to a differential K-algebra homomorphism $\tilde{T}_\varphi : A \to K$, which satisfies the axioms (TTM1) and (TTM2) restricted to differential K-algebras and K-algebra homomorphisms.

Proposition 3.3.6. *Let K be a di erential field, and L a di erential K-algebra. Suppose that L admits a finite K-Taylor morphism $\hat{T}$. Then, L admits a restricted K-Taylor morphism $\tilde{T}$ whose restriction to di erentially finitely generated K-algebras is $\hat{T}$. Further, if L admits restricted K-Taylor morphisms $\tilde{T}$ and $\tilde{S}$ such that their restrictions $\hat{T}$ and $\hat{S}$ to di erentially finitely generated K-algebras is equal, then $\tilde{T} = \tilde{S}$.*

Proof. Let A be a differential K-algebra, and let $\varphi : A \to K$ be a K-algebra homomorphism. Consider the directed system $B = (B_\alpha, f_{\alpha\beta})$ of finitely generated K-subalgebras $B_\alpha \subseteq A$ and inclusion maps $f_{\alpha\beta} : B_\alpha \to B_\beta$. For each α, the restriction $\varphi_\alpha := \varphi\restriction_{B_\alpha} : B_\alpha \to K$ is a K-algebra homomorphism. Further, for any α, β with $B_\alpha \subseteq B_\beta$, the triangle

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\varphi_\alpha} & K \\ {\scriptstyle f_{\alpha\beta}}\downarrow & \nearrow_{\varphi_\beta} & \\ B_\beta & & \end{array}$$

commutes. We note also that $f_{\alpha\beta}$ is a differential K-algebra homomorphism. Thus, as $\hat{T}$ satisfies (TTM2), we conclude that the diagram

$$\begin{array}{ccc} B_\alpha & \xrightarrow{\hat{T}_{\varphi_\alpha}} & L \\ {\scriptstyle f_{\alpha\beta}}\downarrow & \nearrow_{\hat{T}_{\varphi_\beta}} & \\ B_\beta & & \end{array}$$

of differential morphisms also commutes. Set $\tilde{T}_\varphi$ to be the union $\bigcup_\alpha \hat{T}_{\varphi_\alpha}$ which is well defined by the commutativity of the above diagram. We now claim that $\tilde{T}$ is a restricted K-Taylor morphism for L.

We establish that (TTM1) holds for $\tilde{T}$. Let $a \in A_\varphi$. In particular, we have that $\varphi\restriction_{K\{a\}}$ is a differential K-algebra homomorphism. Since $B := K\{a\}$ is a differentially finitely generated K-algebra, we may apply $\hat{T}$ to $\varphi_B := \varphi\restriction_B$. Further, since φ_B is differential on B, by (TTM1) for $\hat{T}$, we have that $\hat{T}_{\varphi_B} = \eta_L \circ \varphi_B$. Finally, since B is a differentially finitely generated K-subalgebra of A, we have, by the construction of $\tilde{T}$, that $\tilde{T}_\varphi\restriction_B = \hat{T}_{\varphi_B} = \eta_L \circ \varphi_B$. Thus, $T_\varphi(a) = (\eta_L \circ \varphi)(a)$, as required.

For (TTM2), let B be another differential K-algebra, $\psi : B \to K$ be a K-algebra homomorphism and $\chi : A \to B$ be a differential K-algebra homomorphism such that

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & K \\ {\scriptstyle \chi}\downarrow & \nearrow_{\psi} & \\ B & & \end{array}$$

commutes. Let $a \in A$, and let $C := K\{a\}$ be the differential K-subalgebra of A generated by a. Let $D := K\{\chi(a)\}$ be the differential K-subalgebra of B generated by $\chi(a)$. Denote the restrictions $\varphi\restriction_C$, $\chi\restriction_C$ and $\psi\restriction_D$ by φ_C, χ_C and

ψ_D, respectively. Then, the following commutes:

$$\begin{array}{ccc} C & \xrightarrow{\varphi_C} & K \\ {\scriptstyle\chi_C}\big\downarrow & \nearrow_{\psi_D} & \\ D & & \end{array} .$$

Since C and D are finitely generated differential K-algebras, and χ_C is a differential K-algebra homomorphism, we also have that

$$\begin{array}{ccc} C & \xrightarrow{\hat{T}_{\varphi_C}} & L \\ {\scriptstyle\chi_C}\big\downarrow & \nearrow_{\hat{T}_{\psi_D}} & \\ D & & \end{array} .$$

commutes by (TTM2) for $\hat{T}$. Thus, for any $a \in A$, we have by construction that $\tilde{T}_\varphi(a) = T_\psi \circ \chi(a)$, and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tilde{T}_{\varphi}} & L \\ {\scriptstyle\chi}\big\downarrow & \nearrow_{\tilde{T}_{\psi}} & \\ B & & \end{array}$$

commutes, as required. It is clear that the restriction of T to differentially finitely generated K-algebras is indeed $\hat{T}$.

Now suppose that $\tilde{T}$ and $\tilde{S}$ are restricted K-Taylor morphisms for L such that their restrictions $\hat{T}$ and $\hat{S}$ to differentially finitely generated K-algebras are equal. Let A be a differential K-algebra and $\varphi : A \to K$ a K-algebra homomorphism. Let $a \in A$ and let $B = K\langle a\rangle$ be the differential K-subalgebra of A generated by a, and $\psi := \varphi\restriction_B$. Since B is differentially finitely generated, $T_\psi = S_\psi$ by assumption. Since the triangle

$$\begin{array}{ccc} B & \xrightarrow{\psi} & K \\ \big\downarrow & \nearrow_{\varphi} & \\ A & & \end{array}$$

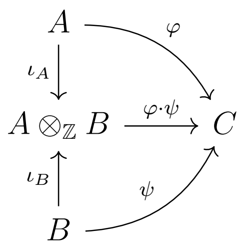
commutes, and the inclusion map $B \subseteq A$ is differential, we have that

$$\tilde{T}_\varphi(a) = \tilde{T}_\psi(a) = S_\psi(a) = S_\varphi(a)$$

as required. □

Notation. For differential rings A, B, $A \otimes_{\mathbb{Z}} B$ denotes their tensor product equipped with the standard derivation on the tensor product given by $\delta(a\otimes b) = \delta(a)\otimes b+a\otimes\delta(b)$. This is the coproduct in the category of differential rings. The canonical maps from A, B into their coproduct are denoted by $\iota_A : A \to A\otimes_{\mathbb{Z}}B$ and $\iota_B : B \to A \otimes_{\mathbb{Z}} B$.

For $\varphi : A \to C$ and $\psi : B \to C$ (differential) ring homomorphisms, by the universal property of the coproduct, there is a unique (differential) ring homomorphism denoted by $\varphi \cdot \psi : A \otimes_{\mathbb{Z}} B \to C$ such that



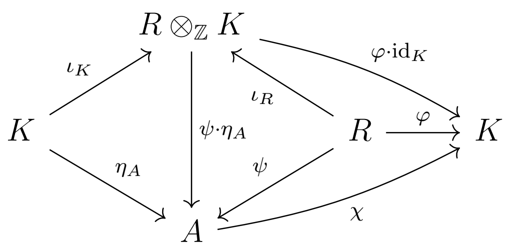
commutes. Concretely, $(\varphi \cdot \psi)(a \otimes b) = \varphi(a)\psi(b)$.

We will now extend our restricted K-Taylor morphism to all differential rings and ring homomorphisms. For the following, we let K be a differential ring, L a differential K-algebra, and $\tilde{T}$ a restricted K-Taylor morphism for L. For an arbitrary differential K-algebra A, we denote its structure map by $\eta_A : K \to A$.

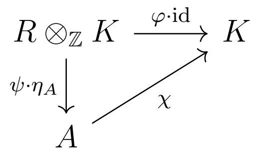
We begin by constructing suitable differential homomorphisms for each pair (R, φ), where R is an arbitrary differential ring, and $\varphi : R \to K$ is a ring homomorphism.

Lemma 3.3.7. *Let R be a di erential ring, and $\varphi : R \to K$ a ring homomorphism. There is a unique di erential ring homomorphism $T_\varphi : R \to L$ such that for any di erential K-algebra A with a di erential ring homomorphism $\psi : R \to A$ and a K-algebra homomorphism $\chi : A \to K$ with $\chi \circ \psi = \varphi$, we have that $T_\varphi = \tilde{T}_\chi \circ \psi$.*

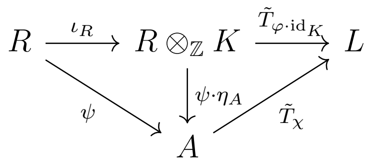
Proof. Let A, φ, ψ satisfy the above hypotheses. Observe that the following diagram commutes:



The left square commutes by the universal property of the coproduct. Now, the lower right triangle commutes by assumption, and the upper right triangle commutes again by the universal property of coproducts. We now observe that we have the following commutative triangle of K-algebra homomorphisms:



where $\psi \cdot \eta_A$ is differential. Applying $\tilde{T}$ and (TTM2), and adding maps from R, we obtain that



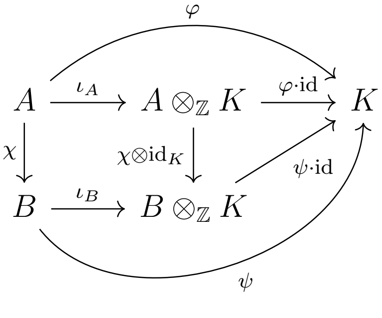
also commutes. Define $T_\varphi : R \to L$ to be $\tilde{T}_{\varphi\cdot\mathrm{id}_K} \circ \iota_R$. Thus, we have shown that for any A, ψ, χ satisfying the hypothesis in the lemma, we have that $\tilde{T}_\chi \circ \psi = \tilde{T}_{\varphi\cdot\mathrm{id}} \circ \iota_R = T_\varphi$, as required.

For uniqueness, let T_φ satisfy the above condition. Then, $R \otimes_{\mathbb{Z}} K$ is a differential K-algebra, $\iota_R : R \to R \otimes_{\mathbb{Z}} K$ is a differential ring homomorphism and $\varphi{\cdot}\mathrm{id}_K : R \otimes_{\mathbb{Z}} K \to K$ is a K-algebra homomorphism such that $(\varphi{\cdot}\mathrm{id}) \circ \iota_R = \varphi$. Thus by the above condition, $T_\varphi = \tilde{T}_{\varphi\cdot\mathrm{id}_K} \circ \iota$, as required. □

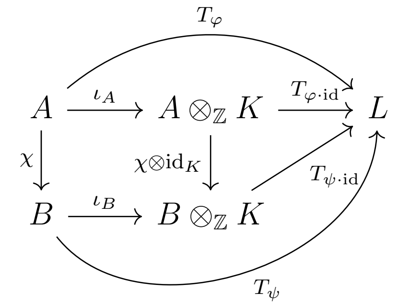
Now, for any differential ring A and ring homomorphism $\varphi : A \to K$, we define T_φ as in Lemma 3.3.7. We now show that (TTM2) holds for T.

Lemma 3.3.8. *Let* A, B *be di erential rings,* $\varphi : A \to K$*,* $\psi : B \to K$ *ring homomorphisms, and* $\chi : A \to B$ *a di erential ring homomorphism. If* $\varphi = \psi \circ \chi$*, then* $T_\varphi = T_\varphi \circ \chi$*.*

Proof. Let $\iota_A : A \to A \otimes_{\mathbb{Z}} K$ and $\iota_B : B \to B \otimes_{\mathbb{Z}} K$ denote the canonical maps into the tensor product. We form the following commutative diagram:



Applying T to the right triangle and applying (TTM2), and by the definition of T_φ, T_ψ, we obtain the following commutative diagram:



In particular, we have that $T_\varphi = T_\psi \circ \chi$. $\square$

Finally, we show that (TTM1') holds for T.

Lemma 3.3.9. *Let A be a di erential ring, $\varphi : A \to K$ a di erential ring homomorphism. Let $\eta_L : K \to L$ denote the structure map. Then, $T_\varphi = \eta_L \circ \varphi$.*

Proof. By definition, $T_\varphi = T_{\varphi\cdot\mathrm{id}_K} \circ \iota_R$. Since both φ and id_K are differential, $\varphi \cdot \mathrm{id}_K : A \otimes_{\mathbb{Z}} K \to K$ is a differential ring homomorphism. In particular, it is a differential K-algebra homomorphism. Applying (TTM1'), we obtain that $T_{\varphi\cdot\mathrm{id}_K} = \eta_L \circ (\varphi \cdot \mathrm{id}_K)$. Composing, we get that

$$T_\varphi = \eta_L \circ (\varphi \cdot \mathrm{id}_K) \circ \iota_R = \eta_L \circ \varphi$$

as required. $\square$

Proposition 3.3.10. *Suppose L admits a restricted K-Taylor morphism $\tilde{T}$. Then it admits a unique K-Taylor morphism T whose restriction to di erential K-algebras is $\tilde{T}$.*

Proof. Existence is by Lemmas 3.3.7, 3.3.8 and 3.3.9. Uniqueness follows from the uniqueness condition in Lemma 3.3.7 as any such Taylor morphism must satisfy the hypothesis in this lemma. $\square$

Proposition 3.3.11. *Suppose L admits a finite K-Taylor morphism $\hat{T}$. Then, it admits a unique K-Taylor morphism T whose restriction to finitely di erentially generated K-algebras is $\hat{T}$.*

Proof. By Proposition 3.3.6 and 3.3.10. $\square$

This gives a negative answer to Question 3.3.3:

Corollary 3.3.12. *Let* (K, δ) *be a di erential field. Then* K *does not admit a finite* K*-Taylor morphism.*

Proof. This follows from Proposition 3.3.10 and Proposition 3.3.1. □

Remark. We recall that if K is a differentially large field, then every differentially finitely generated K-algebra A with an algebraic K-point also has a differential K-point. Corollary 3.3.12 simply states that there is no 'functorial' way to find these points.

3.4 Generalised Twistings

In this section, we consider a generalisation of the 'twisting' used to construct the twisted Taylor morphism in [32], and show that the same construction can be applied to $K[[t]]$ with non-standard derivations.

Definition 3.4.1 (Generalised Twistings)**.** Let (K, δ) be a differential field. Let L be a K-algebra, and let ∂ and d be derivations on L satisfying the following:

- K is contained in the ∂-constants of L; and
- (L, d) is a differential K-algebra.

Let T be a $(K, 0)$-Taylor morphism for (L, ∂). A *twisting* $\tau : (L, \partial) \to (L, \mathrm{d})$ (of T) is a differential ring homomorphism such that $\tau \circ T_{\mathrm{id}_K} = \iota_K$, where $\iota_K : K \to L$ is the inclusion.

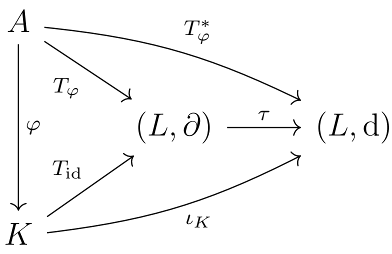
Proposition 3.4.2. *Let* (K, δ) *be a di erential field, and let* L *be a* K*-algebra with derivations* ∂ *and* d *such that:*

- K *is contained in the* ∂*-constants of* L*; and*
- (L, d) *is a di erential* K*-algebra.*

Let T *be a* $(K, 0)$*-Taylor morphism for* (L, ∂)*, and let* $\tau : (L, \partial) \to (L, \mathrm{d})$ *be a twisting of* T*. For a di erential* (K, δ)*-algebra* A *and* K*-algebra homomorphism* $\varphi : A \to K$*, define* $T^*_\varphi = \tau \circ T_\varphi$*. Then,* T^* *is a* (K, δ)*-Taylor morphism for* (L, d)*.*

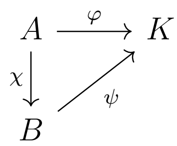
Proof. By construction, it is clear that T^*_φ is a differential K-algebra homomorphism $A \to (K, \mathrm{d})$. For (TTM1), let A be a differential ring, and $\varphi : A \to K$

a ring homomorphism. Consider the following commutative diagram:

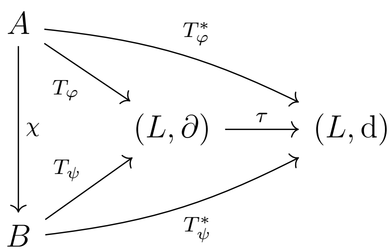


Thus, identifying K with its inclusion in L, we find that $\varphi\restriction_{A_\varphi} = T_\varphi^*\restriction_{A_\varphi}$, as required.

For (TTM2), suppose that we have A, B differential rings, $\varphi : A \to K$ and $\psi : B \to K$ ring homomorphisms and $\chi : A \to B$ a differential ring homomorphism such that



commutes. Then, by (TM2) and definition of T^*, we have that the following commutes:



In particular, the outer triangle commutes, as required. □

We consider an example in a special case. Let (K, δ) be a differential field, and let L denote the differential ring $(K[[t]], \hat{\delta} + \partial_a)$, where $a \in C_K$, and $\partial_a = a\frac{\mathrm{d}}{\mathrm{d}t}$. We construct an (untwisted) K-Taylor morphism for $(K[[t]], \partial_a)$ similarly to the classical Taylor morphism, by setting, for any differential ring A, ring homomorphism $\varphi : A \to K$, and any $a \in A$

$$T_\varphi(a) = \sum_{i<\omega} \frac{\varphi(a^{(i)})}{a^i i!} t^i.$$

This can be easily verified to be a K-Taylor morphism for $(K[[t]], \partial_a)$. With this, we now perform construction analogous to Theorem 2.3.8.

Lemma 3.4.3. *Let K be a field, and let δ and ω be commuting derivations on K. Fix $a \in K^*$ with $\delta a = 0$ and $\omega a = 0$. Denote by $T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}$, $T_{\mathrm{ev}}^{\hat{\omega}+\partial_a}$ and $T_{\mathrm{ev}}^{\hat{\delta}+\hat{\omega}+\partial_a}$*

the maps given by the Taylor morphism T applied to the evaluation map $\mathrm{ev}:K[[t]]\to K$ *at 0, where $K[[t]]$ is equipped with the derivations $\hat{\delta}+\partial_a$, $\hat{\omega}+\partial_a$ and $\hat{\delta}+\hat{\omega}+\partial_a$, respectively. Then, considered as pure ring endomorphisms of $K[[t]]$,*

$$T_{\mathrm{ev}}^{\hat{\delta}+\hat{\omega}+\partial_a}=T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}\circ T_{\mathrm{ev}}^{\hat{\omega}+\partial_a}.$$

Proof. The result follows by simply verifying the equality by computation. Let $\alpha=\sum_i\alpha_i t^i\in K[[t]]$. Consider the coefficient of t^j on the left hand side. By definition, this is given by

$$\begin{aligned}
\alpha_j &= \frac{1}{j!a^j}\mathrm{ev}\left((\hat{\delta}+\hat{\omega}+\partial_a)^j\left(\sum_i\alpha_i t^i\right)\right)\\
&= \frac{1}{j!a^j}\mathrm{ev}\left(\sum_{n+m\leqslant j}\binom{j}{n+m}\binom{n+m}{n}\delta^n\omega^m\partial_a^{j-n-m}\left(\sum_i\alpha_i t^i\right)\right)\\
&= \mathrm{ev}\left(\sum_{n+m\leqslant j}\frac{1}{a^{n+m}(j-n-m)!n!m!}\delta^n\omega^m\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j-n-m}\left(\sum_i\alpha_i t^i\right)\right)\\
&= \mathrm{ev}\left(\sum_{n+m\leqslant j}\frac{1}{a^{n+m}(j-n-m)!n!m!}\delta^n\omega^m\right.\\
&\qquad\qquad\left.\left(\sum_i\frac{i!}{(i-j+n+m)!}\alpha_i t^{i-j+n+m}\right)\right)\\
&= \sum_{n+m\leqslant j}\frac{1}{a^{n+m}n!m!}\delta^n\omega^m(\alpha_{j-n-m}).
\end{aligned}$$

Similarly, we compute $T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}(\sum_i\alpha_i t^i)$. By definition, the coefficient of t^j is given by:

$$\begin{aligned}
\frac{1}{j!a^j}\mathrm{ev}\left((\hat{\delta}+\partial_a)^j\left(\sum_i\alpha_i t^i\right)\right) &= \frac{1}{j!a^j}\mathrm{ev}\left(\sum_{n\leqslant j}\binom{j}{n}\hat{\delta}^n\partial_a^{j-n}\left(\sum_i\alpha_i t^i\right)\right)\\
&= \frac{1}{j!a^j}\sum_{n\leqslant j}\binom{j}{n}(j-k)!a^{j-n}\delta^n(\alpha_{j-k})\\
&= \sum_{n\leqslant j}\frac{1}{a^n n!}\delta^n(\alpha_{j-n}).
\end{aligned}$$

Note that by considering the case where δ is trivial, we also obtain that $T_{\mathrm{ev}}:(K,\partial_a)\to(K,\partial_a)$ is precisely the identity map.

Let us now compute $T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}\circ T_{\mathrm{ev}}^{\hat{\omega}+\partial_a}(\sum_i\alpha_i t^i)$. By considering the previous expression for the coefficient of t^j in $T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}(\sum_i\alpha_i t^i)$, and replacing $\hat{\delta}$ with $\hat{\omega}$, we obtain that the coefficient of t^j in $T_{\text{ev}}^{\hat{\delta}+\partial_a} \circ T_{\text{ev}}^{\hat{\omega}+\partial_a}(\sum_i \alpha_i t^i)$ is:

$$\sum_{n\leqslant j} \frac{1}{a^n n!}\delta^n \left(\sum_{m\leqslant j-k} \frac{1}{a^m m!}\omega^m(\alpha_{j-n-m}) \right)$$
$$= \sum_{n+m\leqslant j} \frac{1}{a^{n+m} n! m!}\delta^n \omega^m(\alpha_{j-n-m})$$

as required. □

As an immediate consequence, we obtain that $T_{\text{ev}}^{\hat{\delta}+\partial_a} : (K[[t]], \hat{\delta} + \partial_a) \to (K[[t]], \partial_a)$ is an isomorphism of differential rings (although not of K-algebras), as $T_{\text{ev}}^{\hat{\delta}+\partial_a}$ has a compositional inverse $T_{\text{ev}}^{-\hat{\delta}+\partial_a}$.

We now show by direct computation that the 'twisting map' $T_{\text{ev}}^{-\delta+\partial_a}$ obtained in this way allows us to retrieve a (K, δ)-Taylor morphism for $(K[[t]], \hat{\delta}+ \partial_a)$, i.e. is an example of a generalised twisting as defined previously. This is analogous to the construction in Definition 2.3.10.

Proposition 3.4.4. *For a di erential ring (A, ω), and ring homomorphism $\varphi : A \to K$, define $T_\varphi^* : (A, \omega) \to (K[[t]], \hat{\delta} + \partial_a)$ by the composition*

$$T_\varphi^* = T_{\text{ev}}^{-\hat{\delta}+\partial_a} \circ T_\varphi.$$

Then, T^* is a (K, δ)-Taylor morphism for $(K[[t]], \hat{\delta} + \partial_a)$.

Proof. Since $T_\varphi : (A, \omega) \to (K[[t]], \partial_a)$ and $T_{\text{ev}}^{-\hat{\delta}+\partial_a} : (K[[t]], \partial_a) \to (K[[t]], \hat{\delta} + \partial_a)$ are differential homomorphisms, the composition T_φ^* is also a differential homomorphism. Further, (TTM2) is directly inherited from (TM2) of T. It suffices now to show that T^* satisfies (TTM1).

For (TTM1), it suffices to show that, for an element $a \in A$ such that $\varphi\restriction_{\mathbb{Z}\{a\}}$ is differential, then $T_\varphi^*\restriction_{\mathbb{Z}\{a\}} = \varphi\restriction_{\mathbb{Z}\{a\}}$. In particular, we will show that, if $a \in A$ such that $\varphi(\omega^n(a)) = \delta^n(\varphi(a))$ for every n, then $T_\varphi^*(\omega^n(a)) = \varphi(\omega^n(a))$ for every n.

We first explicitly compute an expression for $T_\varphi^*(a)$ for arbitrary $a \in A$. Recall that $T_\varphi(a) = \sum_i \frac{\varphi(\omega^i(a))}{i! a^i} t^i$, and by previous computations, we have that

$$T_{\text{ev}}^{-\hat{\delta}+\partial_a} \left(\sum_i \alpha_i t^i \right) = \sum_j \left(t^j \sum_{n\leqslant j} \frac{1}{a^n n!}(-\delta)^n(\alpha_{j-n}) \right).$$

Composing, we obtain that:

$$T_\varphi^*(a) = \sum_j \left(t^j \sum_{n \leqslant j} \frac{(-1)^n}{a^n n!} \delta^n \left(\frac{\varphi(\omega^{j-n}(a))}{(j-n)! a^{j-n}} \right) \right)$$
$$= \sum_j \left(t^j \sum_{n \leqslant j} \frac{(-1)^n}{a^j j!} \binom{j}{n} \delta^n (\varphi(\omega^{j-n}(a)) \right).$$

Now suppose that $a \in A$ is such that $\varphi(\omega^n(a)) = \delta^n(\varphi(a))$ for every n. Then,

$$T_\varphi^*(\omega^k(a)) = \sum_j \left(t^j \sum_{n \leqslant j} \frac{(-1)^n}{a^j j!} \binom{j}{n} \delta^j (\varphi(\omega^k(a))) \right).$$

Notice that for every $j \geqslant 1$, the coefficient of t^j is 0, thus we obtain that $T_\varphi^*(\omega^k(a)) = \varphi(\omega^k(a))$ for every k as required. Thus, T^* is a (K, δ)-Taylor morphism for $(K[[t]], \hat{\delta} + \partial_a)$. □

Remark. By considering the sum in an alternative way, we can express our 'twisting map' $T_{\text{ev}}^{\delta+\partial_a}$ in terms of simpler maps. Let $\alpha = \sum_i \alpha_i t^i \in K[[t]]$, and observe the following:

$$T_{\text{ev}}^{\delta+\partial_a}(\alpha) = \sum_j \left(\sum_{n \leqslant j} \frac{1}{a^n n!} \delta^n(\alpha_{j-n}) \right) t^j$$

$$= \begin{array}{cccccccc} \alpha_0 & + & & & & & & \\ \alpha_1 t & + & \frac{1}{a}\delta(\alpha_0)t & + & & & & \\ \alpha_2 t^2 & + & \frac{1}{a}\delta(\alpha_1)t^2 & + & \frac{1}{2a^2}\delta^2(\alpha_0)t^2 & + & & \\ \vdots & + & \vdots & + & \vdots & + & \cdots \end{array}$$

By summing the columns first, we get that

$$T_{\text{ev}}^{\hat{\delta}+\partial_a}(\alpha) = \sum_i \frac{t^i}{a^i i!} \hat{\delta}^i(\alpha).$$

Now, consider the ring $K[[t, x]]$ as a $K[[t]]$-algebra, equipped with the derivation $\frac{\partial}{\partial x}$. This differential ring admits the classical $K[[t]]$-Taylor morphism S where we consider $K[[t, x]]$ as the ring of formal power series in the variable x over the $\mathbb{Q}$-algebra $K[[t]]$. Considering the identity map id of the differential ring $K[[t]]$ equipped with the derivation $\hat{\delta}$, we get that

$S_{\mathrm{id}} : K[[t]] \to K[[t,x]]$ is the map defined by

$$\alpha \mapsto \sum_{i<\omega} \frac{x^i}{i!} \hat{\delta}^i(\alpha)$$

for each $\alpha \in K[[t]]$. Evaluating x at t/a, we see that the composition of S_{id} with this evaluation map is precisely the map $T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}$.

In other words, the following diagram commutes:

$$\begin{array}{ccc} K[[t]] & \xrightarrow{S_{\mathrm{id}}} & K[[t,x]] \\ & {\scriptstyle T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}} \searrow & \downarrow {\scriptstyle \mathrm{ev}_{x,\frac{t}{a}}} \\ & & K[[t]] \end{array}$$

where $\mathrm{ev}_{x,\frac{t}{a}}$ is the map $K[[t,x]] \to K[[t]]$ evaluating x at $\frac{t}{a}$.

Note. By direct computation, one can verify that if $a \in K^\times$ is nonconstant with respect to δ, then the composition $\mathrm{ev}_{x,\frac{t}{a}} \circ S_{\mathrm{id}}$ is not (generally) a differential map. Thus, when $a \in K^\times \setminus C_K$, we cannot in general express $T_{\mathrm{ev}}^{\hat{\delta}+\partial_a}$ in this form (for any suitably chosen Taylor morphism T).

3.5 Taylor Morphisms and Differential Largeness

We now take a short interlude from discussing Taylor morphisms themselves, and consider a few applications of generalised Taylor morphisms to differentially large fields.

Proposition 3.5.1. *Let K be a di erential field that is large as a field. Let L be a di erential field extension of K, which admits a K-Taylor morphism T^*. Suppose that K is existentially closed in L as a di erential field. Then, K is di erentially large.*

Proof. Follows from the proof of Lemma 3.1.6, by replacing every instance of $T_{\mathrm{id}}(K)$ with K, and applying Theorem 2.3.16(vi). □

An immediate corollary of the above result, and Proposition 3.4.4 is the following:

Corollary 3.5.2. *Let (K,δ) be a di erential field, and $a \in C_K^\times$. Then, (K,δ) is di erentially large if and only if (K,δ) is existentially closed in $(K[[t]], \hat{\delta}+\partial_a)$.*

Proposition 3.5.3. *Let K be a di erential field, and L be a di erential field extension of K, admitting a twisted K-Taylor morphism T^*. Suppose that K is existentially closed in L as a pure field. Then, K is di erentially large if and only if K is existentially closed in L as a di erential field.*

Proof. Follows immediately from Proposition 3.5.1. □

Remark. We should point out that condition on K being existentially closed in L as a pure field is necessary for the above proposition to hold. As an easy counterexample, let k be a model of CODF (which is differentially large). Let (K, δ) be a DCF extending k as a differential field, and $L = K((t))$ equipped with the derivation $\hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}$. Note that k is not existentially closed in K, even as a pure field (K is algebraically closed, and k is not). Then, L admits a twisted K-Taylor morphism (and hence a twisted k-Taylor morphism by taking inclusions). However, k is not existentially closed in L as it is not existentially closed in $K \subseteq L$.

Theorem 3.5.4. *Let K be a di erential field that is large as a field. Then, K is di erentially large if and only if there is an elementary extension K^* of K such that K^* admits a K-Taylor morphism.*

Proof. First, suppose there is an elementary extension K^* of K which admits a twisted K-Taylor morphism. Then, since K is existentially closed in K^*, by Proposition 3.5.1, K is differentially large.

Conversely, suppose that K is differentially large. Then, by Theorem 2.3.16(ii), K is existentially closed in $K((t))$, equipped with the natural derivation extending the derivation on K. Since K is existentially closed in $K((t))$, there is an elementary extension K^* of K containing $K((t))$. Since $K((t))$ admits a twisted K-Taylor morphism, K^* also admits a twisted K-Taylor morphism. □

We will now adapt Proposition 2.3.20 for generalised Taylor morphisms. We will follow the proof given in [32] closely.

Proposition 3.5.5. *Let $(K_i, f_{ij})_{i,j \in I}$ be a directed system of di erential fields and di erential embeddings with the following properties:*

(a) Each K_i is large as a field.

(b) For each pair $i < j$, $f_{ij}(K_i)$ is existentially closed in K_j as a field.

(c) For each $i \in I$, there is $j \geqslant i$ such that K_j admits a K_i-Taylor morphism.

Then, the direct limit L of the directed system is di erentially large.

Proof. For each $i < j \in I$, we identify K_i with its image in K_j under the embedding f_{ij}. Similarly, we identify K_i with its image under the natural embedding $K_i \to L$. Following the same argument as in [32], L is large as a field. Let S be a differentially finitely generated L-algebra which is a domain, and suppose that S has an (algebraic) point $S \to L$. Since S is differentially finitely generated, $S = L\{\bar{x}\}/\mathfrak{p}$ for some $\bar{x} = (x_0, ..., x_r)$ and differential prime ideal $\mathfrak{p} \subseteq L\{\bar{x}\}$.

By the Ritt-Raudenbush basis theorem, there is a finite set $\Sigma \subseteq \mathfrak{p}$ such that $\mathfrak{p}$ is generated as a differential radical ideal by Σ. We will show that Σ has a differential zero in L. Take $i \in I$ with $\Sigma \subseteq K_i\{\bar{x}\}$, let $\mathfrak{p} = K_i\{\bar{x}\} \cap \mathfrak{p}$ and let $S_0 = K_i\{\bar{x}\}/\mathfrak{p}_0$. Then, S_0 is a finitely differentially generated K_i-algebra, and the composition of the natural embedding $S_0 \to S$ with the point $S \to L$ gives a point $S_0 \to L$.

It is easy to see that K_i is existentially closed in L as a field. Thus, we have that there is an algebraic point $S_0 \to L_i$. Since there is $j > i$ admitting a twisted K_i-Taylor morphism, there is a differential point $S_0 \to L_j$. Thus, Σ has a differential solution in L, as required. □

3.6 Constructing Taylor Morphisms

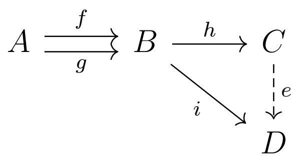
We work towards building a diagram $\mathcal{D}_K$ in the category of differential K-algebras such that the cocones of $\mathcal{D}_K$ correspond precisely to (finite) K-Taylor morphisms. In doing so, we will show that for any differential ring K, there exists a differential K-algebra L and a K-Taylor morphism T for L such that every other K-Taylor morphism factors through T.

We begin by showing that the category of differential K-algebras is cocomplete:

Proposition 3.6.1. *Let K be a di erential ring. Then the category of di erential K-algebras is cocomplete.*

Proof. By [34, Corollary V.2.2], it suffices to show that the category of differential K-algebras has coequalisers of all pairs of arrows and all small coproducts.

We first show that for all differential K-algebras A, B and pairs of differential K-algebra homomorphisms $f, g : A \to B$, there is a differential K-algebra C and morphism $h : B \to C$ such that $h \circ f = h \circ g$, and, for any differential K-algebra D and morphism $i : B \to D$ such that $i \circ f = i \circ g$, there is a unique morphism $e : C \to D$ such that the following commutes:



This is a generalisation of the analogous result for commutative rings. Let I be the ideal of B generated by all elements of the form $f(a) - g(a)$, where $a \in A$. Then, I is a differential ideal of B, as $\delta(f(a) - g(a)) = f(\delta(a)) - g(\delta(a))$. Now, let C be the differential K-algebra B/I and h be the quotient map. Clearly, by construction, $h \circ f = h \circ g$.

Now, for any element $b + I \in C$, define $e(b + I) = i(b)$. This is well defined as $i(f(a)) = i(g(a))$ for all $a \in A$, and thus $i(b) = 0$ for any $b \in I$. By commutativity of the diagram, e is the unique morphism with this property.

For the second part, we observe that for pairs of differential K-algebras A, B, their coproduct is given by the tensor product $A \otimes_K B$, with the derivation given by

$$\delta(a \otimes b) = \delta a \otimes b + a \otimes \delta b.$$

To see this, suppose that C is a differential K-algebra with morphisms $f : A \to C$ and $g : B \to C$, and let ι_A, ι_B denote the canonical morphisms of A, B into their tensor product. Then, defining $h : A \otimes_K B \to C$ by setting $h(a \otimes b) = f(a)g(b)$, we obtain the unique morphism $A \otimes_K B \to C$ such that $f = h \circ \iota_A$ and $g = h \circ \iota B$. To verify that this is differential, observe the following:

$$h(\delta(a \otimes b)) = h(\delta a \otimes b + a \otimes \delta b) = \delta f(a)g(b) + f(a)\delta g(b) = \delta(h(a \otimes b)).$$

Now, consider an arbitrary family $(A_i : i \in I)$ of differential K-algebras. For any finite subset $S \subseteq I$, let A_S be the finite coproduct $\bigotimes_{i \in S} A_S$ with the derivation defined above.

Let A be the directed limit of the family

$$\left(\bigotimes_{i \in S} A_i : S \subseteq I \text{ finite} \right)$$

where the ordering is given by inclusion of finite subsets of I. By [34, Theorem IX.1.1], A is the coproduct of the family $(A_i : i \in I)$. □

We now build the diagram $\mathcal{D}_K$ as follows:

Definition 3.6.2. Let K be a differential ring. The diagram $\mathcal{D}_K$ is the diagram in the category of differential K-algebras which contains, for each differentially finitely generated K-algebra A and each K-algebra homomorphism $\varphi : A \to K$, a copy of A with a label φ, denoted A^φ. The morphisms in the diagram $\mathcal{D}_K$ consist of all differential K-algebra homomorphisms $\chi : A^\varphi \to B^\psi$ such that $\varphi = \psi \circ \chi$.

Remark. We restrict our diagram to only consider differentially finitely generated K-algebras, as this diagram is small (set-sized), and by Proposition 3.3.11, constructing a finite Taylor morphism is equivalent to constructing a full Taylor morphism.

Notation. We will denote a cocone of $\mathcal{D}_K$ by a pair (L, τ), where L is the sink of the cocone, and for each object A^φ in $\mathcal{D}_K$, the component at A^φ is denoted $\tau_{A^\varphi} : A^\varphi \to L$.

Proposition 3.6.3. *Let K be a di erential ring, and let (L, τ) be a cocone of $\mathcal{D}_K$. For each pair (A, φ), where A is a di erentially finitely generated K-algebra and $\varphi : A \to K$ is a K-algebra homomorphism, set $T_\varphi = \tau_{A^\varphi}$. Then, T is a finite K-Taylor morphism for L.*

Conversely, if T is a finite K-Taylor morphism for L, letting $\tau_{A^\varphi} = T_\varphi : A^\varphi \to L$ for each A^φ, then (L, τ) is a cocone for the diagram $\mathcal{D}_K$.

This gives a bijective correspondence between cocones of $\mathcal{D}_K$ and finite K-Taylor morphisms (and hence also K-Taylor morphisms).

Proof. We begin with the forward direction. Let (L, τ) be a cocone for $\mathcal{D}_K$, and define T as above. Let A be a differentially finitely generated K-algebra, and $\varphi : A \to K$ a K-algebra homomorphism.

We first verify that (TTM1) holds. Let $a \in A_\varphi$, and let $B \subseteq A_\varphi$ be a differentially finitely generated K-subalgebra containing a, and let $\psi : B \to K$ denote the restriction $\varphi\restriction_B$. In particular, since B is a differentially finitely generated K-algebra, and $\psi : B \to K$ is a differential K-algebra homomorphism, B^ψ is an object in $\mathcal{D}_K$. In addition, $\psi : B^\psi \to K^{\mathrm{id}}$ is a morphism in $\mathcal{D}_K$.

Further, as the inclusion map $\iota : B \to A$ is a differential K-algebra homomorphism such that $\psi = \varphi \circ \iota$, we have that $\iota : B^\psi \to A^\varphi$ is a morphism in $\mathcal{D}_K$. Thus, observe that the following diagram commutes:

$$\begin{array}{ccc} B^\psi & \xrightarrow{\ \iota\ } & A^\varphi \\ {\scriptstyle\psi}\big\downarrow & \searrow^{T_\psi} & \big\downarrow{\scriptstyle T_\varphi} \\ K^{\mathrm{id}} & \xrightarrow[T_{\mathrm{id}}]{} & L \end{array}$$

As T_{id} is a K-algebra homomorphism, it preserves structure maps, thus $T_{\text{id}} = \eta_L$ is the structure map of L as a differential K-algebra. Now, we have that $T_\varphi(a) = T_\psi(a) = (\eta_L \circ \psi)(a)$, by the commutativity of the above diagram.

For (TTM2), let A, B be differentially finitely generated K-algebras, and let $\varphi : A \to K$, $\psi : B \to K$ be K-algebra homomorphisms, and $\chi : A \to B$ a differential K-algebra homomorphism such that $\varphi = \psi \circ \chi$. Then, by construction, $\chi : A^\varphi \to B^\psi$ is a morphism in $\mathcal{D}_K$, and the triangle

$$\begin{array}{ccc} A^\varphi & \xrightarrow{T_\varphi} & L \\ \downarrow{\scriptstyle \chi} & \nearrow_{T_\psi} & \\ B^\psi & & \end{array}$$

commutes by construction as τ is a cocone.

For the backwards direction, let T be a (finite) Taylor morphism for L. Let τ be as defined above. We need to show that for any morphism $\chi : A^\varphi \to B^\psi$ in the diagram $\mathcal{D}_K$, the triangle

$$\begin{array}{ccc} A^\varphi & \xrightarrow{\chi} & B^\psi \\ & {}_{\tau_{A^\varphi}}\searrow & \downarrow{\scriptstyle \tau_{B^\psi}} \\ & & L \end{array}$$

commutes. This is immediate from the axiom (TTM2), as by assumption, χ is differential with $\varphi = \psi \circ \chi$, and so $T_\varphi = T_\psi \circ \chi$, i.e. $\tau_{A^\varphi} = \tau_{B^\psi} \circ \chi$, as required.

The identifications between finite K-Taylor morphisms and cocones of $\mathcal{D}_K$ are clearly mutually inverse to one another, thus we obtain the desired bijective correspondence. Further, by Proposition 3.3.11, there is a bijection between K-Taylor morphisms and finite K-Taylor morphisms. Thus, we obtain a bijective correspondence between K-Taylor morphisms and cocones of $\mathcal{D}_K$, as required.

□

By taking the colimit of the diagram $\mathcal{D}_K$, we are able to find a 'universal' K-Taylor morphism T^* such that every other K-Taylor morphism factors through T^*.

Proposition 3.6.4. *For any di erential ring K, there is a di erential K-algebra K^* and K-Taylor morphism T^* for K^* such that for any di erential K-algebra L admitting a K-Taylor morphism T, there is a unique K-algebra homomorphism $\tau : K^* \to L$ such that for any di erential K-algebra A and K-algebra homomorphism $\varphi : A \to K$, we have that $T_\varphi = \tau \circ T^*_\varphi$.*

Proof. As the category of differential K-algebras is cocomplete, the diagram $\mathcal{D}_K$ has a initial cone (K^*, τ^*), and let $\hat{T}^*$ be the finite K-Taylor morphism associated to this cocone given by Proposition 3.6.3. By Proposition 3.3.11, there is a unique K-Taylor morphism T^* whose restriction to differentially finitely generated K-algebras is $\hat{T}^*$.

Let L be any differential K-algebra admitting a K-Taylor morphism T. Let (L, τ) be the cocone of $\mathcal{D}_K$ associated to T. As (K^*, τ^*) is the initial cocone, (L, τ) factors through (K^*, τ^*). In particular, there is a unique K-algebra morphism $\theta : K^* \to L$ such that for any differentially finitely generated K-algebra A and K-algebra homomorphism $\varphi : A \to K$, we have that $T_\varphi = \theta \circ T^*_\varphi$.

Now, observe that for any differential K-algebra A (not necessarily finitely generated) and K-algebra homomorphism φ, T_φ is completely determined by all of its restrictions to differentially finitely generated K-subalgebras of A. Thus we still have $T_\varphi = \theta \circ T^*_\varphi$ in this case.

Finally, recall from Lemma 3.3.7 that for any differential ring R and ring homomorphism $\varphi : R \to K$, we have that $T_\varphi = T_\chi \circ \psi$, where $\psi : R \to A$ is a differential ring homomorphism, A is a differential K-algebra, and $\chi : A \to K$ satisfies $\chi \circ \psi = \varphi$. Since A is a differential K-algebra, we can apply the above to obtain that $T_\chi = \theta \circ T^*_\chi$, thus $T_\varphi = \theta \circ T^*_\chi \circ \psi$. By (TTM2) for T^*, $T^*_\chi \circ \psi = T^*_\varphi$, therefore we recover that $T_\varphi = \theta \circ T^*_\varphi$, as required. □

Definition 3.6.5. For a differential ring K, the ***universal*** K***-Taylor morphism*** is the K-Taylor morphism T^* for K^* as constructed in Corollary 3.6.4.

3.7 Evaluation Maps

In this section, we discuss the existence of inverses for certain Taylor morphisms. These take the form of a generalised 'evaluation map', which capture certain properties of the 'evaluation at 0' map $\mathrm{ev}_0 : K[[t]] \to K$.

Definition 3.7.1. Let K be a differential ring, L a differential K-algebra, and T a K-Taylor morphism for L. An ***evaluation map for*** T is a K-algebra homomorphism $\mathrm{ev} : L \to K$ satisfying the following:

(EV1) For any differential ring A and ring homomorphism $\varphi : A \to K$,

$$\mathrm{ev} \circ T_\varphi = \varphi.$$

(EV2) $T_{\mathrm{ev}} : L \to L$ is the identity on L.

If an evaluation map for T exists, we say that T ***admits an evaluation map***.

Examples 3.7.2. 1. Let K be a constant $\mathbb{Q}$-algebra, and let T be the classical untwisted Taylor morphism for $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$. Then, the 'evaluation at 0' map, $\mathrm{ev}_0 : \sum_i a_i t^i \mapsto a_0$ is an evaluation map for T. This is easy to verify: for (EV1), let (A, δ) be a differential ring, $\varphi : A \to K$ be a ring homomorphism and $a \in A$. Then,

$$\mathrm{ev}_0(T_\varphi(a)) = \mathrm{ev}_0\left(\sum_i \frac{\varphi(\delta^i(a)}{i!} t^i\right) = \varphi(a).$$

For (EV2), let $a = \sum_i a_i t^i \in K[[t]]$, and compute $T_{\mathrm{ev}_0}(a)$:

$$\begin{aligned} T_{\mathrm{ev}_0}(a) &= \sum_i \frac{1}{i!}\mathrm{ev}_0\left(\frac{\mathrm{d}^i}{\mathrm{d}t^i}a\right)t^i \\ &= \sum_i \frac{1}{i!}\mathrm{ev}_0\left(\sum_n \frac{(n+i)!}{n!}a_{n+i}t^n\right)t^i \\ &= \sum_i \frac{i!a_i}{i!}t^i = \sum_i a_i t^i = a \end{aligned}$$

so $T_{\mathrm{ev}_0} = \mathrm{id}$, as required.

2. Let (K, δ) be a $\mathbb{Q}$-algebra, and let T^* be the twisted Taylor morphism for $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$. Then $\mathrm{ev}_0 : \sum_i a_i t^i \mapsto a_0$ is also an evaluation map for T^*. We verify this as follows: to see that (EV1) holds, let A be a differential ring, $\varphi : A \to K$ be a ring homomorphism and $a \in A$. Observe by applying the explicit formula in Definition 2.3.10, that $\mathrm{ev}_0 \circ T^*_\varphi(a)$, i.e. the coefficient of t^0, is precisely $\varphi(a)$. For (EV2), let $\sum_i a_i t^i \in K[[t]]$, and write $T^*_{\mathrm{ev}_0}(\sum_i a_i t^i) = \sum_i b_i t^i$. We compute b_i:

$$b_i = \frac{1}{i!}\sum_{j \leqslant i}(-1)^{i-j}\binom{i}{j}\left(\mathrm{ev}_0\left(\left(\hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}\right)^j\left(\sum_n a_n t^n\right)\right)\right).$$

Observe that

$$\begin{aligned} \left(\hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}\right)^j\left(\sum_n a_n t^n\right) &= \sum_{k \leqslant j}\binom{j}{k}\hat{\delta}^{j-k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\sum_n a_n t^n\right) \\ &= \sum_{k \leqslant j}\sum_n \frac{(n+k)!}{n!}\binom{j}{k}\delta^{j-k}(a_{n+k})t^n. \end{aligned}$$

As $\mathrm{ev}_0((\hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})^j(\sum_n a_n t^n))$ is the coefficient of t^0 in the above sum, we see that

$$\mathrm{ev}_0\left(\left(\hat{\delta}+\frac{\mathrm{d}}{\mathrm{d}t}\right)^j\left(\sum_n a_n t^n\right)\right)=\sum_{k\leqslant j} k!\binom{j}{k}\delta^{j-k}(a_k).$$

Substituting this into the equation for b_i, we obtain that

$$\begin{aligned}b_i &= \frac{1}{i!}\sum_{j\leqslant i}(-1)^{i-j}\binom{i}{j}\left(\sum_{k\leqslant j} k!\binom{j}{k}\delta^{j-k}(a_k)\right)\\ &= \sum_{j\leqslant i}\sum_{k\leqslant j}\frac{(-1)^{i-j}k!}{i!}\binom{i}{j}\binom{j}{k}\delta^{i-k}(a_k).\end{aligned}$$

Fix some $k \leqslant i$. Then, the coefficient $c_{i,k}$ of $\delta^{i-k}(a_k)$ in the above sum is

$$\begin{aligned}\sum_{j=k}^{i}\frac{(-1)^{i-j}k!}{i!}\binom{i}{j}\binom{j}{k} &= \sum_{j=k}^{i}\frac{(-1)^{i-j}}{(i-j)!(j-k)!}\\ &= \frac{(-1)^{i+k}}{(i-k)!}\sum_{j=k}^{i}\frac{(-1)^{j-k}(i-k)!}{(j-k)!((i-k)-(j-k))!}.\end{aligned}$$

Let $l = j-k$ and reindex the sum, obtaining:

$$c_{i,k}=\frac{(-1)^{i+k}}{(i-k)!}\sum_{l=0}^{i-k}(-1)^l\binom{i-k}{l}.$$

The alternating sum of binomial coefficients is 0 unless $i = k$, in which case it is 1. Observe then, that $c_{i,k} = 0$ if $i \neq k$ and $c_{i,i} = 1$. Finally, we see the following:

$$b_i=\sum_{k\leqslant i} c_{i,k}\delta^{i-k}(a_k)=c_{i,i}a_i=a_i$$

and $T^*_{\mathrm{ev}_0}(\sum_i a_i t^i)=\sum_i a_i t^i$, and $T^*_{\mathrm{ev}_0}=\mathrm{id}$, as required.

3. Let K be an arbitrary constant ring, and let H be the Hurwitz morphism for $(H(K), \partial_K)$ as constructed in Example 3.1.4. Then, the map $\varepsilon_K : H(K) \to K$ given by $(a_i)_{i<\omega} \mapsto a_0$ is a evaluation map for H. We verify this by direct computation: let (A, δ) be a differential ring, and let $\varphi : A \to K$ be a ring homomorphism and let $a \in A$. Then,

$$\varepsilon_K(H_\varphi(a))=\varepsilon_K((\varphi(\delta^i a)))=\varphi(a).$$

Thus (EV1) holds. (This is also the consequence of Proposition 2.1 of [26].) Now, for (EV2), let $(a_i)_{i<\omega} \in H(K)$. Then, writing $H_{\varepsilon_K}((a_i)_{i<\omega})$ as $(b_i)_{i<\omega}$, we have that for any fixed $j < \omega$,

$$b_j = \varepsilon_K(\partial_K^j((a_i)_{i<\omega})) = \varepsilon_K((a_{i+j})_{i<\omega}) = a_j$$

So $(a_i)_{i<\omega} = (b_i)_{i<\omega}$ and $H_{\varepsilon_K} = \mathrm{id}_{H(K)}$, as required.

For the remainder of this section, we let K be a differential ring, L be a differential K-algebra and T a K-Taylor morphism for L.

Lemma 3.7.3. *Let* $\mathrm{ev} : L \to K$ *be a* K*-algebra homomorphism. Then,* $T_{\mathrm{ev}} = \mathrm{id}_L$ *if and only if for any di erential ring* B *and di erential ring homomorphism* $\psi : B \to L$*,* $T_{\mathrm{ev}\circ\psi} = \psi$*.*

Proof. For the forward direction, let $T_{\mathrm{ev}} = \mathrm{id}_L$. Let B be a differential ring, and $\psi : B \to L$ be a differential ring homomorphism. Then, the following commutes:

$$\begin{array}{ccc} B & \xrightarrow{\mathrm{ev}\circ\psi} & K \\ {\scriptstyle\psi}\downarrow & \nearrow_{\mathrm{ev}} & \\ L & & \end{array}$$

As ψ was assumed to be differential, we apply (TTM2) to obtain that the following also commutes:

$$\begin{array}{ccc} B & \xrightarrow{T_{\mathrm{ev}\circ\psi}} & K \\ {\scriptstyle\psi}\downarrow & \nearrow_{T_{\mathrm{ev}}} & \\ L & & \end{array}$$

By assumption, $T_{\mathrm{ev}} = \mathrm{id}_L$. Thus, $T_{\mathrm{ev}\circ\psi} = \mathrm{id}_L \circ \psi = \psi$.

Conversely, assume that for any differential ring B and differential ring homomorphism $\psi : B \to L$, $T_{\mathrm{ev}\circ\psi} = \psi$. Then, consider the following commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{evoid}_L} & K \\ {\scriptstyle\mathrm{id}_L}\downarrow & \nearrow_{\mathrm{ev}} & \\ L & & \end{array}$$

As id_L is differential, we apply (TTM2) to obtain that

$$\begin{array}{ccc} L & \xrightarrow{T_{\mathrm{evoid}_L}} & L \\ {\scriptstyle\mathrm{id}_L}\downarrow & \nearrow_{T_{\mathrm{ev}}} & \\ L & & \end{array}$$

also commutes. By assumption, $T_{\mathrm{ev} \circ \mathrm{id}_L} = \mathrm{id}_L$. Thus, $T_{\mathrm{ev}} \circ \mathrm{id}_L = T_{\mathrm{ev}} = T_{\mathrm{ev} \circ \mathrm{id}_L} = \mathrm{id}_L$, as required. $\square$

Lemma 3.7.4. *If T admits an evaluation map, then it is unique.*

Proof. Suppose both ev and ev′ are evaluation maps for T. Then, by (EV1) on ev, we have that $\mathrm{ev} \circ T_{\mathrm{ev}'} = \mathrm{ev}'$. By (EV2) on ev′, we have that $T_{\mathrm{ev}'} = \mathrm{id}_L$. Thus,

$$\mathrm{ev}' = \mathrm{ev} \circ T_{\mathrm{ev}'} = \mathrm{ev} \circ \mathrm{id}_L = \mathrm{ev}$$

as required. $\square$

Proposition 3.7.5. *Suppose T admits an evaluation map* ev. *Define the functor*

$$\mathrm{ev} \circ - : (\mathrm{id}_{DR\text{ING}} \downarrow 1_L) \to (U \downarrow 1_K)$$

by setting

$$(\mathrm{ev} \circ -)(A, \bullet, \varphi) = (A, \bullet, \mathrm{ev} \circ \varphi)$$

on objects, and $(\mathrm{ev} \circ -)$ *to be the identity on morphisms. Then,* $(\mathrm{ev} \circ -)$ *and* T *are inverse functors, and* $(U \downarrow 1_K)$ *and* $(\mathrm{id}_{DR\text{ING}} \downarrow 1_L)$ *are isomorphic as categories.*

Proof. We begin by verifying that $(\mathrm{ev} \circ -)$ is a functor. We check that if $(\alpha, \mathrm{id}) : (A, \bullet, \varphi) \to (B, \bullet, \psi)$ is a morphism in $(\mathrm{id}_{\mathrm{DRING}} \downarrow 1_L)$, then it is also a morphism $(A, \bullet, \mathrm{ev} \circ \varphi) \to (B, \bullet, \mathrm{ev} \circ \psi)$ in $(U \downarrow 1_K)$. Consider the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & L & \xrightarrow{\mathrm{ev}} & K \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_L} & & \downarrow{\scriptstyle \mathrm{id}_K} \\ B & \xrightarrow{\psi} & L & \xrightarrow{\mathrm{ev}} & K \end{array}$$

The left square commutes by definition of morphisms in $(\mathrm{id}_{\mathrm{DRING}} \downarrow 1_L)$. The right square is clearly commutative also. Thus, the large square commutes (considered as a diagram in RING), and (α, id) is a morphism $(A, \bullet, \mathrm{ev} \circ \varphi) \to (B, \bullet, \mathrm{ev} \circ \psi)$ in $(U \downarrow 1_K)$. Clearly $(\mathrm{ev} \circ -)$ preserves composition.

Now, let $(A, \bullet, \varphi)$ be an object in $(\mathrm{id}_{\mathrm{DRING}} \downarrow 1_L)$. Then,

$$T(\mathrm{ev} \circ -)(A, \bullet, \varphi) = (A, \bullet, T_{\mathrm{ev} \circ \varphi}) = (A, \bullet, \varphi)$$

by Lemma 3.7.3. Conversely, let $(B, \bullet, \psi)$ be an object in $(U \downarrow 1_K)$. Then,

$$(\mathrm{ev} \circ -)T(B, \bullet, \psi) = (B, \bullet, \mathrm{ev} \circ T_{\psi}) = (B, \bullet, \psi)$$

by (EV1). As both T and $(\mathrm{ev} \circ -)$ are the identity on morphisms, we obtain the desired result. $\square$

Proposition 3.7.6. *Let K be a di erential $\mathbb{Q}$-algebra, and let T^* be the universal K-Taylor morphism for K^*. Then, T^* admits an evaluation map ev^*.*

Proof. Let T denote the standard twisted Taylor morphism for $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$, and let $\tau : K^* \to K[[t]]$ be the unique differential K-algebra homomorphism such that $T = \tau \circ T^*$. Let $\mathrm{ev}^* = \mathrm{ev}_0 \circ \tau$.

For (EV1), let A be a differential ring and let $\varphi : A \to K$ be a ring homomorphism. Then,

$$\mathrm{ev}^* \circ T^*_\varphi = \mathrm{ev}_0 \circ \tau \circ T^*_\varphi = \mathrm{ev}_0 T_\varphi.$$

As ev_0 is an evaluation map for T, by (EV1) we have that $\mathrm{ev}_0 \circ T_\varphi = \varphi$.

We now claim that, for any A and φ as above, we have that $T^*_{\mathrm{ev}^*} \circ T^*_\varphi = T_\varphi$. We consider the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & K \\ {\scriptstyle T^*_\varphi}\downarrow & \overset{\mathrm{ev}^*}{\nearrow} & \uparrow{\scriptstyle \mathrm{ev}_0} \\ K^* & \xrightarrow{\tau} & K[[t]] \end{array}$$

The outer square commutes as $\tau \circ T^*_\varphi = T_\varphi$ and by (EV1) for ev_0. The triangles commute by definition of ev^*.

Applying T^* to the upper triangle, we obtain that

$$\begin{array}{ccc} A & \xrightarrow{T^*_\varphi} & K^* \\ {\scriptstyle T^*_\varphi}\downarrow & \underset{T^*_{\mathrm{ev}^*}}{\nearrow} & \\ K^* & & \end{array}$$

also commutes, as T^*_φ is differential. This establishes the claim. Thus, $T^*_{\mathrm{ev}^*}$ is an automorphism of the cone of $\mathcal{D}_K$ corresponding to the Taylor morphism T^*. As T^* is the initial cone, by the universal property of colimits, we have that $T^*_{\mathrm{ev}^*} = \mathrm{id}_{K^*}$ as it has no nontrivial automorphisms. $\square$

We recall that we can represent K-Taylor morphisms as cocones of the diagram $\mathcal{D}_K$. In this light, we define a morphism of K-Taylor morphisms to correspond with the notion of a morphism of cocones:

Definition 3.7.7. Let T, S be K-Taylor morphisms for L, F, respectively. A *morphism of K-Taylor morphisms* $\theta : T \to S$ is a differential K-algebra homomorphism $\theta : L \to F$ such that for any differential ring A and ring homomorphism $\varphi : A \to K$, we have

$$S_\varphi = \theta \circ T_\varphi.$$

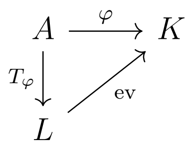
If θ is a differential K-algebra isomorphism, we say that θ is an ***isomorphism of K-Taylor morphisms***.

We form the ***category of K-Taylor morphisms*** with objects being K-Taylor morphisms and arrows consisting of morphisms of K-Taylor morphisms.

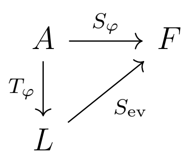
Remark. The above definition coincides precisely with the notion of a morphism in the category of cocones of $\mathcal{D}_K$.

Lemma 3.7.8. *Let T, S be K-Taylor morphisms for L, F, respectively. Suppose T admits an evaluation map $\mathrm{ev} : L \to K$. Then, $S_{\mathrm{ev}} : L \to F$ is a morphism of K-Taylor morphisms $T \to S$. Further, if $\theta : T \to S$ is any other morphism, then $\theta = S_{\mathrm{ev}}$.*

Proof. Let A be a differential ring, and $\varphi : A \to K$ be a differential ring homomorphism. By (EV1), $\mathrm{ev} \circ T_\varphi = \varphi$. That is, the following triangle commutes:



Applying S and (TTM2), we obtain that



also commutes, and thus $S_{\mathrm{ev}} \circ T_\varphi = S_\varphi$, as required.

For uniqueness, let $\theta : T \to S$ be any other morphism of K-Taylor morphisms. Then, by definition, for any differential ring A and ring homomorphism $\varphi : A \to K$, we have that $S_\varphi = \theta \circ T_\varphi$. In particular, applying this to $\mathrm{ev} : L \to K$, we have that

$$S_{\mathrm{ev}} = \theta \circ T_{\mathrm{ev}} = \theta \circ \mathrm{id}_L = \theta$$

by (EV2), as required. $\square$

We have in fact proven that:

Theorem 3.7.9. *Suppose there is a K-Taylor morphism T for L which admits an evaluation map $\mathrm{ev} : L \to K$. Then, T is the initial object in the category of K-Taylor morphisms.*

Theorem 3.7.10. *Let T, T' be K-Taylor morphisms for L, L', respectively. Suppose $\mathrm{ev} : L \to K$, $\mathrm{ev}' : L' \to K$ are evaluation maps for T, T', respectively. Then, $T'_{\mathrm{ev}} : L \to L'$ is an isomorphism $T \to T'$ with inverse $T_{\mathrm{ev}'}$.*

Proof. By the previous lemma, $T'_{\mathrm{ev}} : L \to L'$ and $T_{\mathrm{ev}'} : L' \to L$ are morphisms $T \to T'$ and $T' \to T$, respectively.

By (EV1) for ev', we have that $\mathrm{ev}' \circ T'_{\mathrm{ev}} = \mathrm{ev}$. That is, the following commutes:

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{ev}} & K \\ {\scriptstyle T'_{\mathrm{ev}}}\downarrow & \nearrow_{\mathrm{ev}'} & \\ L' & & \end{array}$$

Applying (TTM2) for T, we then have that

$$\begin{array}{ccc} L & \xrightarrow{T_{\mathrm{ev}}} & L \\ {\scriptstyle T'_{\mathrm{ev}}}\downarrow & \nearrow_{T_{\mathrm{ev}'}} & \\ L' & & \end{array}$$

also commutes. By (EV2) for ev, $T_{\mathrm{ev}} = \mathrm{id}_L$. Thus, $T_{\mathrm{ev}'} \circ T'_{\mathrm{ev}} = \mathrm{id}_L$. By symmetry we also have that $T'_{\mathrm{ev}} \circ T_{\mathrm{ev}'} = \mathrm{id}_{L'}$ also. Hence, T'_{ev} is an isomorphism of K-Taylor morphisms with inverse $T_{\mathrm{ev}'}$, as required. □

This gives us that for a differential $\mathbb{Q}$-algebra K, the universal Taylor morphism is the standard twisted Taylor morphism:

Corollary 3.7.11. *Let (K, δ) be a di erential $\mathbb{Q}$-algebra. Then, the universal Taylor morphism T^* for K^* is isomorphic to the standard twisted Taylor morphism T for $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$.*

Proof. By Proposition 3.7.6, the universal Taylor morphism T^* admits an evaluation map ev^*. As noted previously, the usual evaluation at 0 map, $\mathrm{ev} : K[[t]] \to K$ is an evaluation map for the standard twisted Taylor morphism for $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$. Thus, by Theorem 3.7.10, T^* and T are isomorphic as K-Taylor morphisms. □

By the same argument, we also have:

Corollary 3.7.12. *Let R be an arbitrary constant ring. Then, the universal Taylor morphism T^* for R^* is isomorphic to the Hurwitz morphism H for $(H(R), \partial_R)$.*

Applying the above two results, we recover [26, Proposition 2.4], which gives an isomorphism between the Hurwitz series ring and power series ring over a $\mathbb{Q}$-algebra K, and in particular, between the Hurwitz morphism and classical Taylor morphism:

Corollary 3.7.13. *Let K be a constant $\mathbb{Q}$-algebra. Then, the Hurwitz series ring $(H(K), \partial_K)$ and the ring of formal power series $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$ are isomorphic as di erential K-algebras. Further, the Hurwitz morphism H and classical Taylor morphism T are isomorphic.*

Proof. By Examples 3.7.2(1) and (3), the maps $\mathrm{ev}_0 : K[[t]] \to K$ and $\varepsilon_K : H(K) \to K$ are evaluation maps for T and H, respectively. By Theorem 3.7.10, the maps $T_{\varepsilon_K} : H(K) \to K[[t]]$ and $H_{\mathrm{ev}_0} : K[[t]] \to H(K)$ are mutually inverse isomorphisms of Taylor morphisms, and in particular of differential K-algebras. □

Corollary 3.7.14. *Let (K, δ) be a di erential $\mathbb{Q}$-algebra. Let S be a K-Taylor morphism for a di erential K-algebra L. Then, there is a unique di erential K-algebra homomorphism $\theta : (K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}) \to L$ such that $\theta : T \to S$ is a morphism of K-Taylor morphisms.*

Rephrased in terms of the category of K-Taylor morphisms, we can state:

Corollary 3.7.15. *Let (K, δ) be a di erential $\mathbb{Q}$-algebra. Then, the standard twisted Taylor morphism for $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ is the initial object in the category of K-Taylor morphisms.*

Corollary 3.7.16. *Let (K, δ) be a di erential $\mathbb{Q}$-algebra, and let L be a differential K-algebra admitting a K-Taylor morphism S. Then, L contains $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ as a di erential K-subalgebra, and S is given by the standard K-Taylor morphism T into $K[[t]]$ composed with the inclusion of $K[[t]]$ into L.*

Proof. By the previous corollary, there is a differential K-algebra homomorphism $\theta : (K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t}) \to L$ such that $S = \theta \circ T$. As every ideal of $K[[t]]$ is of the form (t^n) for some $n < \omega$, no proper ideal of $K[[t]]$ is differential. Thus, θ is injective, as required. □

Again, we can rephrase this to say:

Corollary 3.7.17. *Let K be a di erential $\mathbb{Q}$-algebra. Then, there is a bijective correspondence between di erential $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$-algebras and K-Taylor morphisms. In fact, the category of K-Taylor morphisms is isomorphic to the category of $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$-algebras.*

Proof. The obvious identification suffices. Let S be a K-Taylor morphism for some differential K-algebra L, and let T^* denote the standard twisted K-Taylor morphism for $K^* := (K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$, and let $\mathrm{ev} : K^* \to K$ be evaluation at 0. Let T, S be arbitrary K-Taylor morphisms for differential K-algebras L, F, respectively.

Then, by Lemma 3.7.8, $T_{\mathrm{ev}} : K^* \to L$ is a morphism of K-Taylor morphisms $T^* \to T$. This map endows L with a differential K^*-algebra structure. Let S be any other K-Taylor morphism for another differential K-algebra F. Similarly, S_{ev} endows F with a differential K^*-algebra structure.

Let $\theta : T \to S$ be a morphism of K-Taylor morphisms. Then, we claim that $S_{\mathrm{ev}} = \theta \circ T_{\mathrm{ev}}$, i.e. θ is a differential K^*-algebra homomorphism. This is clear, as $\theta \circ T_{\mathrm{ev}}$ is a morphism of K-Taylor morphisms, and as T^* is initial in the category of K-Taylor morphism, there is a unique morphism from T^* to every K-Taylor morphism.

Conversely, suppose L, F are differential K^*-algebras with structure maps η_L, η_F, respectively. Then, $\eta_L \circ T^*$ and $\eta_F \circ T^*$ define K-Taylor morphisms T, S for L, F, respectively. Let $\theta : L \to F$ be any differential K^*-algebra homomorphism. Let A be any differential ring, and let $\varphi : A \to K$ be a ring homomorphism. Then, by definition, $\theta \circ T_\varphi = \theta \circ \eta_L \circ T^*_\varphi$. Since θ is a differential K^*-algebra homomorphism, it preserves structure maps, thus $\theta \circ \eta_L = \eta_F$. Therefore we conclude that $\theta \circ T_\varphi = S_\varphi$, and θ is a morphism of K-Taylor morphisms $T \to S$, as required. □

By the same argument, for any arbitrary differential ring K, the category of K-Taylor morphisms is isomorphic to the category of differential K^*-algebras.

This partially answers a question regarding whether the ring of differentially algebraic power series admits a Taylor morphism:

Corollary 3.7.18. *Let (K, δ) be a di erential $\mathbb{Q}$-algebra, and let $K[[t]]_{\mathrm{alg}}$ denote the di erential subring of $(K[[t]], \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ consisting of the elements which are di erentially algebraic over K. If $K[[t]]_{\mathrm{alg}} \neq K[[t]]$, then $(K[[t]]_{\mathrm{alg}}, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ does not admit a K-Taylor morphism.*

Proof. Suppose otherwise. Then, there is a an embedding $\theta : K[[t]] \to K[[t]]_{\mathrm{alg}}$ over K. As there is an element $a \in K[[t]] \setminus K[[t]]_{\mathrm{alg}}$, there is an element

$b = \theta(a) \in K[[t]]_{\mathrm{alg}}$ differentially transcendental over K, a contradiction. □

Remark. It is not known in general, when K is not a constant field, whether $K[[t]]$ contains a differentially transcendental element over K.

3.8 A Note on Multiple Derivations

In this final section, we make minor modifications to our framework to work with differential fields with multiple commuting derivations. In this section, a ***di erential ring/field/algebra*** means a ring/field/algebra equipped with a set of n commuting derivations, and a ***di erential homomorphism*** is a homomorphism which respects all derivations.

In this new setting, we define a K-Taylor morphism exactly as in Definition 3.2.2, where all differential rings are now in n commuting derivations. The twisted Taylor morphism for $K((\mathbf{t}))$, where $\mathbf{t} = (t_1, ..., t_n)$, constructed in [32] is an example. We restate Lemma 3.1.6 for this case.

Lemma 3.8.1. *Let K be a di erential field, large as a field. Let L be a di erential field extension of K, admitting a twisted K-Taylor morphism T^*, and suppose that K is existentially closed in L as a di erential field. Let A be a di erentially finitely generated K-algebra, suppose that A has a K-rational point $\varphi : A \to K$. Then, A has a di erential K-rational point.*

Proof. By applying T^* to φ, we obtain a differential ring homomorphism $T^*_\varphi : A \to L$. By existential closure of K in L and 2.1(ii) of [32], we have that A has a differential point $A \to K$. □

Corollary 3.8.2. *Let K be a di erential field, large as a field. Suppose there is a di erential field extension L of K such that K is existentially closed in L, and L admits a twisted K-Taylor morphism. Then, K is di erentially large.*

Proof. By the previous lemma and 4.3(iv) of [32]. □

Theorem 3.8.3. *Let K be a di erential field that is large as a field. Then, K is di erentially large if and only if there exists an elementary extension L of K such that L admits a twisted K-Taylor morphism.*

Proof. First suppose that K is differentially large. Then, by 4.3(ii) of [32], K is existentially closed in $K((\mathbf{t}))$. Therefore, $K((\mathbf{t}))$ embeds in an elementary extension L of K. Further, $K((\mathbf{t}))$ admits a twisted K-Taylor morphism, so taking the composition with the inclusion, we obtain a twisted K-Taylor morphism for L.

The converse is given by the previous corollary. □

3.9 An Adjunction

We recall from [26] that the functor $H : \text{Ring} \to \text{DRing}$ which sends a ring to its ring of Hurwitz series is the right adjoint to the forgetful functor $U : \text{DRing} \to \text{Ring}$. It was noted by Tomašić that the twisted Taylor morphism could potentially be realised as the natural bijection of hom-sets in a certain coslice category.

Let $K = (K, \delta)$ be a differential ring. Let K/DRing and UK/Ring denote the coslice categories of DRing and Ring under K and UK, respectively. These categories are isomorphic to the categories of differential K-algebras, and non-differential K-algebras, respectively.

We define the induced functor $H_K : UK/\text{Ring} \to K/\text{DRing}$ as follows: Let A be a UK-algebra, with structure map σ_A. Define $H_K(A) = H(K)$ as a differential ring, equipped with the differential K-algebra structure map given by the composition

$$K \xrightarrow{\eta_K} HUK \xrightarrow{H\sigma_A} A$$

where η is the unit of the adjunction $(U \dashv H)$. Concretely, $\eta_K = H_{\text{id}}$, where id is the identity (ring) homomorphism $(K, \delta) \to K$. H_K acts by H on morphisms.

This is right adjoint to the induced forgetful functor $U : K/\text{DRing} \to UK/\text{Ring}$. From this, we see that for any differential K-algebra A, there is a natural bijection of hom-sets

$$UK/\text{Ring}(UA, UK) \to K/\text{DRing}(A, H_K(UK)).$$

This is concretely given by the map sending a UK-algebra homomorphism $\varphi : UA \to UK$ to the composition

$$A \xrightarrow{\eta_A} HUA \xrightarrow{H\varphi} H(K) = H_K(UK)$$

which we see is equal to the Hurwitz morphism of φ, $H_\varphi : A \to H(K)$.

From this, we observe that the Hurwitz morphism (restricted to differential K-algebras) is a K-Taylor morphism for $(H(K), \partial_K)$, where $H(K)$ is equipped with the differential K-algebra structure map H_{id}.

In the case where K is a differential $\mathbb{Q}$-algebra, this is equal to the classical Taylor morphism for $(K[[t]], \frac{\mathrm{d}}{\mathrm{d}t})$. By applying the twisting map from [32, Corollary 3.3], we can recover the twisted Taylor morphism.

4 | Differential Henselianity

In this chapter, we discuss the class of henselian valued fields equipped with 'generic derivations', which we will call *di erentially henselian*. These are a special case of a topological field with a generic derivation as studied by Cubides Kovacsics and Point in [12] and were also introduced by Guzy and Point in [19] as the class of henselian valued fields satisfying the axiom scheme (DL). These were also studied by Guzy in [18] as the models of a generalised uniform companion for henselian valued fields.

Throughout this chapter, we adopt the following convention:

Definition 4.0.1. A *valued-di erential field* (K, v, δ) is a field K of characteristic 0 equipped with a valuation v and a derivation δ. No interaction is prescribed between v and δ.

4.1 Existing Work

In this section, we give a brief summary of the main results of various papers by Cubides Kovacsics, Guzy and Point which are relevant to the topic of differentially henselian fields. In [19] and [12], Cubides Kovacsics, Guzy and Point study classes of topological differential fields with generic derivations, of which differentially henselian fields will be a special case. In the paper [18], Guzy generalises the work of Tressl in [46] on the uniform companion for large fields to the henselian context. Analogously to the case of differentially large fields (cf. Proposition 2.3.19), the models of Guzy's generalised uniform companion will coincide precisely with the class of differentially henselian fields.

Following the setup of [12], we let $\mathcal{L}_{\text{ring}} = (+, -, \cdot, 0, 1)$, $\mathcal{L}_{\text{field}} = \mathcal{L}_{\text{ring}} \cup \{^{-1}\}$, and $\mathcal{L}$ is a (possibly multi-sorted) language extending $\mathcal{L}_{\text{field}}$. We denote by Ω a (possibly empty) set of constant symbols, and let $\mathcal{L}_{\text{field}}^{\Omega} = \mathcal{L}_{\text{field}} \cup \Omega$.

Let $\mathcal{K}$ be an $\mathcal{L}$-structure. Then, we denote the field sort $\mathbf{F}(\mathcal{K})$ of $\mathcal{K}$ by K. Other sorts are known as *auxiliary sorts*.

Definition 4.1.1 ([12, Definition 1.1.2])**.** An *$\mathcal{L}$-definable field topology* τ (on $\mathcal{K}$) is a field topology τ on K such that there is an $\mathcal{L}$-formula $\chi_\tau(x, z)$, where x is a $\mathbf{F}$-variable, such that

$$\{\chi_\tau(K, \bar{a}) : \bar{a} \in \mathcal{K}^{\bar{z}}\}$$

is a basis of neighbourhoods of 0.

That is, we require the field topology τ to be uniformly definable by a single $\mathcal{L}$-formula. The work of Cubides Kovacics and Point relies mainly on the assumption that the theory of the topological field is an ***open $\mathcal{L}$-theory of topological fields***:

Definition 4.1.2 ([12, Definition 1.2.1])**.** An ***$\mathcal{L}$-theory of topological fields*** is an $\mathcal{L}$-theory T such that any model $\mathcal{K} \models T$ satisfies the following conditions:

1. The field sort K of $\mathcal{K}$ is a field of characteristic 0.
2. The restriction of $\mathcal{L}$ to the sort $\mathbf{F}$ is a relational extension of $\mathcal{L}^{\Omega}_{\text{field}}$.
3. For any $\mathbf{F}$-valued term $t(\bar{x}, \bar{z})$ with $\bar{x}$ a tuple of $\mathbf{F}$-variables and $\bar{z}$ a tuple of auxiliary sort variables, there is an $\mathbf{F}$-valued term $\tilde{t}(\bar{x})$ such that
$$\mathcal{K} \models \forall \bar{z} \forall \bar{x} (t(\bar{x}, \bar{z}) = \tilde{t}(\bar{x})).$$
4. K has an $\mathcal{L}$-definable field topology.

If T also satisfies the following condition, we call it an ***open $\mathcal{L}$-theory of topological fields***:

5. For any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{z})$ with $\bar{x}$ a tuple of $\mathbf{F}$-variables and $\bar{z}$ a tuple of auxiliary sort variables, there is a finite set H, and for each $h \in H$, an $\mathcal{L}$-formula $\psi_h(\bar{z})$ and a finite set I_h, and for each $h \in H$ and $i \in I_h$, an $\mathcal{L}$-formula $\theta_{ih}(\bar{x}, \bar{z})$, and finite set J_{ih}, and for each $h \in H, i \in I_h, j \in J_{ih}$, a nonzero polynomial $P_{ijh} \in \mathbb{Q}(\Omega)[x]$, such that $\varphi(\bar{x}, \bar{z})$ is equivalent modulo T to:
$$\bigvee_{h \in H} \left(\psi_h(z) \to \left(\bigvee_{i \in I_h} \bigwedge_{j \in J_{ih}} P_{ijh}(\bar{x}) = 0 \wedge \theta_{ih}(\bar{x}, \bar{z}) \right) \right)$$
and for any model $\mathcal{K} \models T$ and every $\bar{a} \in \mathcal{K}^{\bar{z}}$, $\theta_{ih}(\mathcal{K}, \bar{a})$ is an open set.

Examples 4.1.3 ([12, Examples 1.2.5])**.**

1. Let $\mathcal{L} = \mathcal{L}_{\text{vf}}$, and T be an $\mathcal{L}$-theory of a henselian valued field of characteristic 0. Then, T is an open $\mathcal{L}$-theory of topological fields, where the topology is the valuation topology. This follows from quantifier elimination in the RV language.
2. Let $\mathcal{L} = \mathcal{L}_{\text{ring}} \cup \{<\}$ and let $T = \text{RCF}$. Then, T is an open $\mathcal{L}$-theory of topological fields, with topology given by the order topology. This follows by quantifier elimination of RCF in $\mathcal{L}$.

Remark. Let T be an open $\mathcal{L}$-theory of topological fields. Then its Morleyisation T_{Mor} is an open $\mathcal{L}_{\text{Mor}}$-theory of topological fields. This is since $\mathcal{L}_{\text{Mor}}$ is a relational expansion of $\mathcal{L}$ by predicates definable in $\mathcal{L}$, and thus automatically satisfies (1)-(4). For (5), we simply observe that every $\mathcal{L}_{\mathrm{Mor}}$-formula is equivalent modulo T_{Mor} to a $\mathcal{L}$-formula, which satisfies condition (5).

For the same reason, every expansion of an open $\mathcal{L}$-theory of topological fields by $\mathcal{L}$-definable relations is also an open theory of topological fields.

We let $\mathcal{L}_\delta$ denote the language $\mathcal{L} \cup \{\delta\}$, where δ is a unary function symbol (in the field sort) which will be interpreted as a derivation. Let T be an $\mathcal{L}$-theory of topological fields. Denote by T_δ the $\mathcal{L}_\delta$-theory consisting of T and the axiom stating that δ is a derivation.

Definition 4.1.4 ([12, Definition 2.2.2])**.** Let T be an $\mathcal{L}$-theory of topological fields, and let $\mathcal{K}$ be a model of T_δ. We say that $\mathcal{K}$ satisfies the axiom scheme (DL) if, for every differential polynomial $f(x) \in K\{x\}$ of order n,

$$\forall \bar{z}(\exists \bar{y}(f_{\mathrm{alg}}(\bar{y}) = 0 \wedge s(f)_{\mathrm{alg}}(\bar{y}) \neq 0) \rightarrow \exists x(f(x) = 0 \wedge s(f)(x) \neq 0 \wedge \chi_\tau(\mathrm{Jet}_n(x) - \bar{y}, \bar{z})))$$

holds in $\mathcal{K}$. The scheme (DL) is clearly axiomatisable in the language $\mathcal{L}_\delta$ by quantifying over the coefficients of f. Denote by T_δ^* the $\mathcal{L}_\delta$-theory $T_\delta \cup (\mathrm{DL})$.

Remark. We can think of the scheme (DL) as stating the following property of the topological differential field K: for every differential polyonomial $f(x) \in K\{x\}$ (in one variable) of order n, if there is a tuple $\bar{a} \in K^{n+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$ (i.e. $\bar{a}$ is a simple root of f_{alg}), then, for any open neighbourhood B of $\bar{a}$, there is a differential root b of f such that $\mathrm{Jet}_n(b) \in B$, and $s(f)(b) \neq 0$.

We assume for the rest of this section that T is complete and T_δ^* is a consistent theory. General consistency results can be found in [12, Section 2.3]. In particular, where T is the theory of a henselian valued field or a real closed field, T_δ^* is consistent.

Example 4.1.5. Let $T = \mathrm{RCF}$. Then, T_δ^* is complete and is precisely the theory CODF of closed ordered differential fields.

The main relative quantifier elimination result in [12] is the following:

Theorem 4.1.6 ([12, Theorem 2.4.2])**.** *Let T be an open $\mathcal{L}$-theory of topological fields. If T eliminates field sort quantifiers then T_δ^* also eliminates field sort quantifiers.*

Remark. By a previous remark, the Morleyisation T_{Mor} of an open $\mathcal{L}$-theory of topological fields T is an open $\mathcal{L}_{\text{Mor}}$-theory of topological fields. Thus, by the above theorem, for any open $\mathcal{L}$-theory of topological fields T, $(T_{\text{Mor}})^*_\delta$ eliminates field sort quantifiers in the language $\mathcal{L}^\delta_{\text{Mor}}$.

From the above relative quantifier elimination result and by passing to the Morleyisation, Cubides Kovacsics and Point prove the following relative completeness theorem:

Corollary 4.1.7 ([12, Corollary 2.4.7]). *If T is a complete open $\mathcal{L}$-theory of topological fields, then the theory T^*_δ is complete.*

With a similar argument, Cubides Kovacsics and Point also show a number of results which transfer various model-theoretic properties from the topological field to the topological differential field. Again, we make the assumption that T is a complete open $\mathcal{L}$-theory of topological fields.

Theorem 4.1.8 ([12, Theorem A.0.3]). *The theory T^*_δ is NIP if and only if T is NIP.*

Theorem 4.1.9 ([12, Theorem A.0.5]). *The theory T^*_δ is distal if and only if T is distal.*

Theorem 4.1.10 ([12, Theorem A.0.6]). *The theory T^*_δ eliminates $\exists^\infty$ in the field sort.*

We also recall an Ax-Kochen/Ershov type result by Guzy and Point for existential closure of henselian fields with generic derivations satisfying the axiom scheme (DL).

Theorem 4.1.11 ([19, Theorem 8.3]). *Let (K, v, δ) be a valued-di erential field such that (K, v) is henselian and (K, v, δ) satisfies (DL). Let (L, w, ∂) be a valued-di erential field extension of (K, v, δ) such that:*

- $w(C_L) = wL$,
- $Kv \preccurlyeq_\exists Lw$
- $vK \preccurlyeq_\exists wL$

Then (K, v, δ) is existentially closed in (L, w, ∂) as valued-di erential fields.

In [18], Guzy shows that there is a analogue of Tressl's uniform companion for the theory of henselian valued fields with K commuting derivations. We denote by $\mathcal{L}_{\text{vf}}$ the language of valued fields, i.e. the language of rings expanded by a binary predicate $\mid$ interpreted as valuation divisibility. The language of valued-differential fields (with k commuting derivations) $\mathcal{L}^k_{\mathrm{vd}}$ is the language of valued fields $\mathcal{L}_{\mathrm{vf}}$ expanded by a k unary function symbols $\delta_1, ..., \delta_k$ interpreted as commuting derivations. We write $\bar{\delta}$ for the tuple of derivations $(\delta_1, ..., \delta_k)$. For the remainder of this section, a ***valued-di erential field*** means a valued field (K, v) equipped with k commuting derivations $\delta_1, ..., \delta_k$.

Guzy constructs the $\mathcal{L}^k_{\mathrm{vd}}$-theory (UC'_k) which carries many of the same properties for henselian valued fields as (UC) has for large fields. As the construction is fairly intricate, we do not reproduce it here and instead direct the reader to [18, Definition 2.7]. We list some of the main results below:

Theorem 4.1.12 ([18, Theorem 3.14], cf. [46, Main Theorem 6.2(II)])**.** *Let $(K, v, \bar{\delta})$ be a valued-di erential field, henselian as a pure valued field. There is a valued-di erential field extension $(L, w, \bar{\partial})$ of $(K, v, \bar{\delta})$ such that $(L, w, \bar{\partial}) \models (\mathrm{UC}')$, and $(K, v) \preccurlyeq (L, w)$.*

Following Tressl and Guzy, we will adopt the following notational convention:

Notation. For $\mathcal{L}$ an arbitrary language, and M, N arbitrary $\mathcal{L}$-structures with A a common subset of M and N, we write

$$M \equiv\rangle_{\exists,A} N$$

if every existential $\mathcal{L}(A)$-formula which holds in M also holds in N. We also write $M \equiv_{\exists,A} N$ if $M \equiv\rangle_{\exists,A} N$ and $N \equiv\rangle_{\exists,A} M$. We observe that $M \equiv_{\exists,A} N$ if and only if M and N have the same universal $\mathcal{L}(A)$-theory.

Theorem 4.1.13 ([18, Theorem 3.14], cf. [46, Theorem 3.3])**.** *Let $(K, v, \bar{\delta})$, $(L, w, \bar{\partial})$ be valued-di erential fields, and A be a common valued-di erential subfield. Assume that*

1. *As pure valued fields, $(K, v) \equiv_{\exists,A} (L, w)$;*
2. *$(L, w, \bar{\partial})$ is a model of (UC'_k).*

Then,

$$(K, v, \bar{\delta}) \equiv\rangle_{\exists,A} (L, w, \bar{\partial})$$

as valued-di erential fields.

We will show a version of this result for the class of differentially henselian fields in Section 4.4 as Corollary 4.4.7. An immediate corollary of this is the analogous version of [46, Main Theorem 6.2(I)]:

Corollary 4.1.14. *Let* $(K, v, \bar{\delta})$, (L, w, ∂) *be models of* (UC'_k)*, and suppose that* A *is a common valued-di erential subfield. Then, if* (K, v) *and* (L, w) *have the same universal theory over* A *as pure valued fields, i.e.* $(K, v) \equiv_{\exists,A} (L, w)$*, then* $(K, v, \bar{\delta})$ *and* $(L, w, \bar{\partial})$ *also have the same universal theory over* A *as valued-di erential fields.*

We finally remark that, as with (UC), the theory (UC'_k) is also inductive:

Lemma 4.1.15. *Let* $(K_i, v_i, \bar{\delta}_i)i \in I$ *be a chain of valued-di erential fields which are models of* (UC'_k)*. Then,* $(K, v, \bar{\delta}) = \bigcup_{i \in I}(K_i, v_i, \bar{\delta}_i)$ *is also a model of* (UC'_k)*. That is,* (UC'_k) *is an inductive theory.*

Proof. Fix a J-algebraically prepared system $\{f_1, ..., f_l; Q_1, ..., Q_{n-d}\}$ over K with respect to the tuples $\bar{a}, \bar{a}'$ in K, and fix $\gamma \in vK$. It is easy to see that if $\{f_1, ..., f_l; Q_1, ..., Q_{n-d}\}$ is a J-algebraically prepared system over K with respect to the tuples $\bar{a}, \bar{a}'$, then it is also a J-algebraically prepared system in any member K_i of the chain containing all the coefficients and $\bar{a}, \bar{a}'$. Take i such that $\gamma \in v_i K_i$. As $(K_i, v_i, \bar{\delta}_i) \models (\mathrm{UC}'_k)$, a solution of the desired form exists in $(K_i, v_i, \bar{\delta}_i)$, thus a solution exists in the union of the chain. □

4.2 Basic Properties

In this section, we discuss the basic properties of differentially henselian fields, including properties of the constant subfield, various equivalent axiomatisations, and their relationship with the class of differentially large fields. We begin by fixing a choice of language:

Definition 4.2.1. The *language of valued-di erential fields*, denoted $\mathcal{L}_{\mathrm{vd}}$, consists of the language of valued fields $\mathcal{L}_{\mathrm{vf}}$ (i.e. the language of rings along with a binary relation symbol | for valuation divisibility), along with a unary function symbol δ which will be interpreted as the derivation.

Definition 4.2.2. We say that a valued-differential field (K, v, δ) is *di erentially henselian* if (K, v) is nontrivially henselian, and satisfies the following axiom scheme:

> Let $f \in K\{x\}$ be a differential polynomial of order n, and $\bar{a} \in K^{n+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Then, for any $\gamma \in vK$, there is some $b \in K$ such that $f(b) = 0$ and $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$.

We denote the $\mathcal{L}_{\mathrm{vd}}$-theory of a differentially henselian field by DH.

Remark. In this sense, a 'differentially henselian field' is precisely a valued-differential field which is henselian as a pure valued field and satisfies the axiom scheme (DL).

Remark. There is a notion of 'differential henselianity' introduced by Aschenbrenner, van den Dries and van der Hoeven in [2] in the setting of transseries fields and valued-differential fields with small derivation. This notion is unrelated to the one discussed in this thesis.

Intuitively, the above axiom scheme states that if a differential polynomial has a simple *algebraic* root $\bar{a}$, then it has a differential root b which is arbitrarily close to $\bar{a}$ with respect to the valuation topology.

We begin by exhibiting some of the basic properties of the constant subfield of differentially henselian fields.

Proposition 4.2.3. *Let* (K, v, δ) *be a di erentially henselian field, and let* $(C_K, w, 0)$ *denote its constant subfield, considered as a valued-di erential subfield of* K. *Then, as pure valued fields:*

(i) C_K *is dense in* K *with respect to the valuation topology.*

(ii) $C_K w = Kv$ *and* $wC_K = vK$.

(iii) (C_K, w) *is a henselian valued field.*

(iv) If K *is equicharacteristic 0 or of mixed characteristic and unramified, then* $(C_K, w) \prec (K, v)$ *as pure valued fields.*

Proof. (i) It suffices to show that for every ball around a point $a \in K$ of radius $\gamma \in vK$, there exists an element $c \in C_K \cap B_\gamma(a)$. Consider the differential polynomial $f(x) = x'$, and the point $(a, 0) \in K^2$. Observe that $f_{\text{alg}}(a, 0) = 0$, and $s(f)_{\text{alg}} = 1$, thus we may apply axiom (2) and obtain $c \in K$ such that $c' = 0$, and $c \in B_\gamma(a)$, as required.

(ii) This follows immediately from (i).

(iii) Let $f \in \mathcal{O}_w[x]$ be a polynomial and $a \in \mathcal{O}_w$ such that $w(f(a)) > 0$ and $w(f'(a)) = 0$. Then, as K is henselian and $\mathcal{O}_w \subseteq \mathcal{O}_v$, there is $b \in \mathcal{O}_v$ such that $f(b) = 0$ and $v(a - b) > 0$. Since b is algebraic over the constant subfield, b is itself constant, thus $b \in \mathcal{O}_w = \mathcal{O}_v \cap C_K$ and (C_K, w) is henselian, as required

(iv) Since $(C_K, w) \subseteq (K, v)$ are henselian and equicharacteristic 0 or unramified mixed characteristic, and $C_K w = Kv$ and $wC_K = vK$, we may apply the elementary substructure version of the appropriate AKE theorem (Theorems 2.5.28, 2.5.29) to conclude that $(C_K, w) \prec (K, v)$ as pure valued fields. □

We now show that we can simplify our axiomatisation to only reference the valuation ring.

Proposition 4.2.4. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. Then, (K, v, δ) is di erentially henselian if and only if it satisfies the following axiom scheme:*

For every di erential polynomial $f(x) \in K\{x\}$ of order n, and $\bar{a} \in K^{n+1}$ such that $f_{\text{alg}}(\bar{a}) = 0$ and $s(f)_{\text{alg}}(\bar{a}) \neq 0$, there exists $b \in K$ such that $f(b) = 0$ and $\delta^i(b) - a_i \in \mathcal{O}_v$ for each i.

Proof. Clearly, if (K, v, δ) is differentially henselian, it satisfies the above axiom scheme. It remains to show the reverse direction. Let $f(x)$ be a differential polynomial of order n, and $\bar{a} \in K^{n+1}$ such that $f_{\text{alg}}(\bar{a}) = 0$ and $s(f)_{\text{alg}}(\bar{a}) \neq 0$. Let $\gamma \in vK$. Take $c \in C_K$ such that $v(c) > \gamma$.

Let $g(x)$ be the differential polynomial defined by $g(x) = f(cx)$. Observe that the tuple $(a_0/c, ..., a_n/c)$ is a simple algebraic root of g. Thus, by the above axiom scheme, there exists $b \in K$ such that $g(b) = 0$ and for each i, $\delta_i(b) - a_i/c \in \mathcal{O}_v$. Set $d = cb$. By construction, d is a root of f, and we have that $\frac{\delta_i(d) - a_i}{c} \in \mathcal{O}_v$. Further, $v(\delta_i(d) - a_i) \geqslant v(c) > \gamma$, as required. □

In fact it suffices to take polynomials with coefficients in the valuation ring:

Proposition 4.2.5. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. Then (K, v, δ) is di erentially henselian if and only if it satisfies the following axiom scheme:*

For every di erential polynomial $f(x) \in \mathcal{O}_v\{x\}$ of order n, and $n+1$-tuple $\bar{a}$ such that $f_{\text{alg}}(\bar{a}) = 0$ and $s(f)_{\text{alg}}(\bar{a}) \neq 0$, there exists $b \in K$ such that $f(b) = 0$ and $\delta^i(b) - a_i \in \mathcal{O}_v$ for each i.

Proof. The forward direction is clear from the previous proposition. For the backwards direction, multiply f with a nonzero element $c \in K$ with sufficiently large valuation such that all the coefficients of cf lie in $\mathcal{O}_K$. Then apply the above axiom scheme to obtain the desired solution. □

In all of the above, we can restrict our axiomatisation to consider only *irreducible* differential polynomials $f(x)$, as we can replace a reducible f with an irreducible factor on which $\bar{a}$ vanishes:

Lemma 4.2.6. *Let (K, δ) be a di erential field, and $f(x) \in K\{x\}$ be a differential polynomial of order n such that $f = gh$, where $g, h \in K\{x\}$ are not units. Let $\bar{a} \in K^{n+1}$ such that $f_{\text{alg}}(\bar{a}) = 0$ and $s(f)_{\text{alg}}(\bar{a}) \neq 0$. Then, either* *(i)* $g_{\mathrm{alg}}(\bar{a}) = 0, h_{\mathrm{alg}}(\bar{a}) \neq 0, s(g)_{\mathrm{alg}}(\bar{a}) \neq 0$ *and* $\mathrm{ord}(g) = n$*, or*

(ii) $h_{\mathrm{alg}}(\bar{a}) = 0, g_{\mathrm{alg}}(\bar{a}) \neq 0, s(h)_{\mathrm{alg}}(\bar{a}) \neq 0$ *and* $\mathrm{ord}(h) = n$*.*

Proof. As $f_{\mathrm{alg}}(\bar{a}) = 0$ and $f = gh$, at least one of $g_{\mathrm{alg}}(\bar{a})$ and $h_{\mathrm{alg}}(\bar{a})$ vanishes. Consider $s(f)_{\mathrm{alg}}(\bar{a}) = \frac{\partial f}{\partial x^{(n)}}(\bar{a})$. By the product rule,

$$s(f)_{\mathrm{alg}}(\bar{a}) = \left(\frac{\partial g_{\mathrm{alg}}}{\partial x^{(n)}}\right)(\bar{a}) h_{\mathrm{alg}}(\bar{a}) + g_{\mathrm{alg}}(\bar{a}) \left(\frac{\partial h_{\mathrm{alg}}}{\partial x^{(n)}}\right)(\bar{a}) \neq 0$$

In particular, it is not the case that both $g_{\mathrm{alg}}(\bar{a})$ and $h_{\mathrm{alg}}(\bar{a})$ vanish, otherwise $s(f)_{\mathrm{alg}}(\bar{a}) = 0$, a contradiction. Without loss, suppose that $g_{\mathrm{alg}}(\bar{a}) = 0$ and $h_{\mathrm{alg}}(\bar{a}) \neq 0$. If $\mathrm{ord}(g) < n$, then $\frac{\partial g_{\mathrm{alg}}}{\partial x^{(n)}} = 0$, and $s(f)_{\mathrm{alg}}(\bar{a}) = 0$, again a contradiction. Thus $\mathrm{ord}(g) = n$. Now,

$$s(f)_{\mathrm{alg}}(\bar{a}) = s(g)_{\mathrm{alg}}(\bar{a}) h_{\mathrm{alg}}(\bar{a}) + 0 \neq 0$$

and since $h_{\mathrm{alg}}(\bar{a}) \neq 0$, we obtain that $s(g)_{\mathrm{alg}}(\bar{a}) \neq 0$ also. The other case follows by symmetry. □

Lemma 4.2.7. *Let* (K, δ) *be a di erential field, and* $f(x) \in K\{x\}$ *be a differential polynomial of order* n*, and* $\bar{a} \in K^{n+1}$ *such that* $f_{\mathrm{alg}}(\bar{a}) = 0$ *and* $s(f)(\bar{a}) \neq 0$*. Then, there is a unique irreducible factor* g *of* f *such that* $g_{\mathrm{alg}}(\bar{a}) = 0$*. Further,* $\mathrm{ord}(g) = n$ *and* $s(g)_{\mathrm{alg}}(\bar{a}) \neq 0$*.*

Proof. As $K\{x\}$ is a unique factorisation domain, we may write f as the product of its irreducible factors. Applying Lemma 4.2.6 repeatedly to this factorisation, we observe that there is a unique irreducible factor g such that g vanishes on $\bar{a}$. Further, also by Lemma 4.2.6, such a factor necessarily has order n, and $s(g)_{\mathrm{alg}}(\bar{a}) \neq 0$. □

Now, it is clear that we can restrict our axiomatisation of differentially henselian fields to only irreducible polynomials:

Proposition 4.2.8. *Let* (K, v, δ) *be a valued-di erential field, henselian as a pure valued field. Then,* (K, v, δ) *is di erentially henselian if and only if it satisfies the following axiom scheme:*

For every irreducible di erential polynomial $f(x) \in K\{x\}$ *of order* n*, and* $\bar{a} \in K^{n+1}$ *such that* $f_{\mathrm{alg}}(\bar{a}) = 0$ *and* $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$*, for any* $\gamma \in vK$*, there is* $b \in K$ *such that* $f(b) = 0$ *and* $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$*.*

Proof. The forwards direction is trivial. For the reverse, let $f(x) \in K\{x\}$ be an arbitrary differential polynomial of order n and $\bar{a} \in K^{n+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Let $\gamma \in vK$.

By Lemma 4.2.7, there is a unique irreducible factor g of f such that $g_{\mathrm{alg}}(\bar{a}) = 0$, $s(g)_{\mathrm{alg}}(\bar{a}) \neq 0$ and $\mathrm{ord}(g) = n$. By the axiom scheme above, there is $b \in K$ such that $g(b) = 0$ and $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$. Then as g is a factor of f, we also have that $f(b) = 0$, as required. □

As we stated earlier, every differentially henselian field is differentially large. We can see this by applying the characterisation of differentially large fields from Proposition 2.3.17:

Proposition 4.2.9. *Let (K, v, δ) be a di erentially henselian field. Then, (K, δ) is di erentially large.*

Proof. Let $f, g \in K\{x\}$ with $n = \mathrm{ord}(f) > \mathrm{ord}(g)$, and $\bar{a} \in K^{n+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$, $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$ and $g_{\mathrm{alg}}(\bar{a}) \neq 0$. By the continuity of g, there is some $\gamma \in vK$ be such that $B_\gamma(\bar{a})$ does not intersect the solution set of $g_{\mathrm{alg}}(\bar{x}) = 0$. By differential henselianity of K, there is $b \in K$ such that $f(b) = 0$ and $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$. In particular, $g(b) \neq 0$. Thus, K satisfies the condition in Proposition 2.3.17, and is differentially large. □

We are able to obtain a partial converse to the above, which follows from work by Widawski relating the étale-open topology to differential largeness. We direct the reader to the article [24] by Johnson, Tran, Walsberg and Ye for the requisite background and definition of the étale-open topology which we do not reproduce here. The critical fact that we will need is the following consequence of Theorem B of [24]:

Theorem 4.2.10. *Let K be a henselian field that is not algebraically closed. Then, the étale-open topology coincides with the unique henselian valuation topology.*

We now apply the following theorem due to Widawski, which says that regular differential points of varieties of a certain type are dense in the set of regular points with respect to the étale-open topology:

Theorem 4.2.11 ([47])**.** *Let (K, δ) be a di erential field which is large as a pure field. Then, (K, δ) is di erentially large if and only if, for every irreducible closed variety $V \subseteq \mathbb{A}^{n+1}$ of dimension n, and not of the form $W \times \mathbb{A}$ for a subvariety W of $\mathbb{A}^n$, the set $(K^{n+1})_d \cap \mathrm{Reg}(V) \cap V(K)$ is dense* *in* $\mathrm{Reg}(V) \cap V(K)$ *with respect to the étale-open topology on* $V(K)$*, where* $(K^{n+1})_d := \{\mathrm{Jet}_n(a) : a \in K\}$*, and* $\mathrm{Reg}(V)$ *denotes the regular points of* V*.*

Combining these two results, we readily conclude the following:

Theorem 4.2.12. *Let* (K, v, δ) *be a valued-di erential field such that* K *is not algebraically closed,* (K, v) *is henselian, and* (K, δ) *is di erentially large. Then,* (K, v, δ) *is di erentially henselian.*

Proof. Let f be an irreducible differential polynomial of order n, and suppose we have $\bar{a} \in K^{n+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Let $V \subseteq \mathbb{A}^{n+1}$ denote the variety defined by the equation $f_{\mathrm{alg}} = 0$. Then, we have that $\bar{a}$ is a regular K-rational point of V. Let $\gamma \in vK$, and consider the open set $U = B_\gamma(\bar{a}) \cap V$. By Theorem 4.2.10, U is also open with respect to the étale-open topology. Now, by Theorem 4.2.11, as U contains a regular K-rational point, it contains a differential regular K-rational point $\mathrm{Jet}_n(b)$, as required. □

This result is optimal in the sense that it cannot be extended to the case of algebraically closed fields. We construct a counterexample:

Proposition 4.2.13. *Every di erentially closed field* $(K, \delta) \models \mathrm{DCF}_0$ *admits a nontrivial henselian valuation* v *such that* (K, v, δ) *is not di erentially henselian.*

Proof. Let (K, δ) be a differentially closed field, and denote by C_K the field of constants of K. Let $a \in K \setminus C_K$. As C_K is algebraically closed, a is transcendental over C_K, and $C_K[a]$ is isomorphic to the ring of polynomials $C_K[t]$. Let $\mathfrak{p}$ be the maximal ideal $aC_K[a]$ of $C_K[a]$.

By Chevalley's Extension Theorem (Theorem 2.4.17), there is a valuation ring $\mathcal{O}$ of K containing $C_K[a]$ such that $\mathfrak{m} \cap C_K[a] = \mathfrak{p}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. In particular, $\mathcal{O}$ is a nontrivial valuation ring of K containing C_K. Let v denote the valuation on K corresponding to the valuation ring $\mathcal{O}$. As K is algebraically closed, v is automatically a henselian valuation.

Now, we observe that $v(C_K^\times) = \{0\}$, as every nonzero constant is a unit of the valuation ring. Thus, there are no nonzero constants in the (open) ball $B_0(0) = \mathfrak{m}$, and C_K is not dense in K with respect to the valuation v. As the constants of any differentially henselian field are dense with respect to the valuation topology (Proposition 4.2.3(i)), we conclude that (K, v, δ) is not differentially henselian. □

Corollary 4.2.14. *The theory* $\mathrm{DCF}_0 \cup \mathrm{ACVF}_{(0,0)}$ *is not complete.*

Proof. By Corollary 4.3.9, there exist models of $\mathrm{DCF}_0 \cup \mathrm{ACVF}_{(0,0)}$ which are differentially henselian, and by Proposition 4.2.13, there are models which are not differentially henselian. □

Definition 4.2.15. A *di erentially closed valued field* is a valued-differential field which is a model of DCVF := ACVF ∪ DH. The theory of differentially closed valued fields of characteristic 0 with residue characteristic p, denoted $\mathrm{DCVF}_{(0,p)}$, is the $\mathcal{L}_{\mathrm{vd}}$-theory $\mathrm{ACVF}_{(0,p)} \cup \mathrm{DH}$.

Proposition 4.2.16. *The theory* $\mathrm{DCVF}_{(0,p)}$ *is complete, and eliminates quantifiers in the language* $\mathcal{L}_{\mathrm{vd}}$*.*

Proof. This is by completeness and quantifier elimination of $\mathrm{ACVF}_{(0,p)}$ in the language $\mathcal{L}_{\mathrm{vf}}$ (Theorem 2.5.25 and Corollary 2.5.26), which lifts to the differentially henselian extension by Theorem 4.1.6 and Corollary 4.1.7. □

We may also combine Theorem 4.2.12 with Corollary 4.1.7 to obtain the following:

Corollary 4.2.17. *Let* (K, v, δ) *be a valued-di erential field, not algebraically closed, such that* (K, δ) *is di erentially large, and* (K, v) *is henselian. Then,*

$$\mathrm{Th}(K, v) \cup \mathrm{Th}(K, \delta) \models \mathrm{Th}(K, v, \delta).$$

Proof. As (K, v, δ) is differentially large, not separably closed, and henselian, by Theorem 4.2.12, (K, v, δ) is differentially henselian. Thus, we may apply Corollary 4.1.7, to obtain that the theory $\mathrm{Th}(K, v, \delta)$ is determined by $\mathrm{Th}(K, v)$. □

We note in particular that the above axiomatisation specifies no interaction between the derivation and valuation.

4.3 Constructing Differentially Henselian Fields

In this section, we will exhibit various methods for constructing differentially henselian fields. These are adaptations of the methods used to construct differentially large fields as described in Section 2.3.4.

We begin by considering a construction via iterated power series (cf. Example 2.3.21). Let (K, v, δ) be a valued-differential field, henselian as a pure valued field. Let $(K_0, v_0, \delta_0) = (K, v, \delta)$, and for each $n < \omega$, let

$$(K_{n+1}, v_{n+1}, \delta_{n+1}) = (K_n((t_n^{\mathbb{Q}})), v_n \circ v_{t_n}, \hat{\delta}_n + \tfrac{\mathrm{d}}{\mathrm{d}t_n}).$$

Define

$$(K_\infty, v_\infty, \delta_\infty) = \bigcup_{n<\omega} (K_n, v_n, \delta_n)$$

to be the union of the chain that we have constructed.

We can explicitly describe the valuation v_∞ as follows: we let v_0 be the valuation $v : K^\times \to vK$, and suppose we have constructed $v_n : K_n \to vK \times \mathbb{Q}^n$, ordered reverse lexicographically. Then, let $a = \sum_{i \in \mathbb{Q}} a_i t_n^i$ be an element of $K_{n+1} = K_n((t_n^{\mathbb{Q}}))$. We then define

$$v_{n+1}(a) = (v_n(a_N), N) \in vK \times \mathbb{Q}^{n+1}$$

where $N = \min \operatorname{supp}(a)$. Ordered reverse lexicographically, this gives precisely the valuation v_{n+1} on K_{n+1}. Taking the union, we have that the valuation v_∞ is valued in $vK \times \mathbb{Q}^{<\omega}$, where $\mathbb{Q}^{<\omega}$ is considered as the subset of elements of $\mathbb{Q}^\omega$ with finite support, ordered reverse lexicographically.

Proposition 4.3.1. *The valued-di erential field* $(K_\infty, v_\infty, \delta_\infty)$ *is di erentially henselian.*

Proof. Firstly, (K_∞, v_∞) is henselian as it is a union of henselian valued fields. We show this by induction: for $n = 0$, $v_0 = v$ is henselian by assumption. Assume v_n is henselian. Then, $v_{n+1} = v_n \circ v_{t_n}$ by definition. v_n and v_{t_n} are both henselian, thus v_{n+1} is henselian.

Now, let $f \in K_\infty\{x\}$ be a differential polynomial of order k, and let $\bar{a} \in K_\infty^{k+1}$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Let $\gamma = (q, \sigma) \in vK \times \mathbb{Q}^{<\omega} = v_\infty K_\infty$. Let $N < \omega$ such that $f \in K_N\{x\}$, $\bar{a} \in K_N$, and $N > \max(\operatorname{supp}(\sigma))$.

Let T^* denote the standard twisted K_N-Taylor morphism for $K_N[[t]]$, considered as a K_N-subalgebra of $K_N((t^{\mathbb{Q}}))$. We drop the subscript N of t for readability. Let $A = K_N\{x\}/I(f)$. Define an algebraic K_N-algebra homomorphism $\varphi : A \to K_N$ as follows: for $i \leqslant k$, set $\varphi(x^{(i)}) = a_i$, and for $i > k$, define $\varphi(x^{(i)})$ recursively by taking derivatives of the relation $\varphi(f(x)) = 0$ and rearranging.

Consider the element $\alpha = T^*_\varphi(x) \in K_{N+1}$. We claim that, for $n \leqslant k$, the constant term of $\alpha^{(n)}$ is precisely a_n. Let $\alpha = \sum_i \alpha_i t^i$, and recall that the coefficients are given by the formula

$$\alpha_i = \frac{1}{i!}\sum_{j\leqslant i}(-1)^{i-j}\binom{i}{j}\partial^{i-j}(\varphi(\delta^j x)),$$

where ∂ and δ denote the derivations on K_N and A, respectively. Thus, for $i \leqslant j$, we have that

$$\alpha_i = \frac{1}{i!}\sum_{j\leqslant i}(-1)^{i-j}\binom{i}{j}\partial^{i-j}(a_j).$$

Now, we compute the constant term of $(\partial + \frac{\mathrm{d}}{\mathrm{d}t})^n(\sum_i \alpha_i t^i)$. Observe that this is precisely

$$\begin{aligned}\sum_{m=0}^{n} m!\binom{n}{m}\partial^{n-m}\alpha_m &= \sum_{m=0}^{n}\binom{n}{m}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\partial^{n-j}(a_j)\\ &= \sum_{m=0}^{n}\sum_{j=0}^{m}(-1)^{m-j}\frac{n!}{m!(n-m)!}\frac{m!}{j!(m-j)!}\partial^{n-j}(a_j)\end{aligned}$$

Collect the terms by the index j, and observe that the sum is equal to:

$$\sum_{j=0}^{n}\sum_{m=j}^{n}(-1)^{m-j}\frac{n!}{(n-m)!}\frac{1}{j!(m-j)!}\partial^{n-j}(a_j).$$

Setting $l = m - j$, we rewrite the sum as follows:

$$\begin{aligned}&\sum_{j=0}^{n}\sum_{l=0}^{n-j}(-1)^{l}\frac{n!}{(n-j-l)!j!l!}\partial^{n-j}(a_j)\\ &\quad= \sum_{j=0}^{n}\sum_{l=0}^{n-j}(-1)^{l}\frac{n!}{j!(n-j)!}\frac{(n-j)!}{l!(n-j-l)!}\partial^{n-j}(a_j)\\ &\quad= \sum_{j=0}^{n}\binom{n}{j}\sum_{l=0}^{n-j}(-1)^{l}\binom{n-j}{l}\partial^{n-j}(a_j).\end{aligned}$$

Since $\sum_{l=0}^{n-j}(-1)^l\binom{n-j}{l}$ is simply an alternating sum of binomial coefficients, it is 1 precisely when $n = j$, and 0 otherwise. We therefore conclude that the constant term of $\alpha^{(n)}$ is precisely a_n.

Thus, for $i \leqslant k$, $v_\infty(\alpha^{(i)} - a_i) > \gamma$, as $\alpha^{(i)} - a_i$ is a series in t_{n+1} with zero t_{n+1}-constant term, and $v(t_{n+1}) > \gamma$. Since T^*_φ is a differential K_N-algebra homomorphism, α is a solution to f with $\mathrm{Jet}_k(\alpha) \in B_\gamma(\bar{a})$, as required. □

Remark. As noted previously, if K is an algebraically closed, real closed or p-adically closed field, then so is $K((t^{\mathbb{Q}}))$. Thus, K_∞ as constructed in these cases gives natural concrete examples of models of algebraically closed, real closed and p-adically closed differentially henselian fields.

We note that, for the same reason as in the differentially large construction, it suffices to take $K((t))$ in place of $K((t^{\mathbb{Q}}))$. We therefore obtain:

Corollary 4.3.2. *Let* (K, v, δ) *be a valued-di erential field which is henselian as a pure valued field. Then, the union of the chain*

$$(K, v, \delta) \subseteq (K((t_0)), v \circ v_{t_0}, \delta + \tfrac{\mathrm{d}}{\mathrm{d}t_0}) \subseteq (K((t_0))((t_1)), v \circ v_{t_0} \circ v_{t_1}, \delta + \tfrac{\mathrm{d}}{\mathrm{d}t_0} + \tfrac{\mathrm{d}}{\mathrm{d}t_1}) \subseteq \ldots$$

is di erentially henselian.

We recall that a formulation of largeness for pure fields states that a field is large if and only if it is existentially closed in a henselian field (Corollary 2.3.5). We now use the above construction to present a generalisation to differential fields:

Proposition 4.3.3. *A di erential field* (K, δ) *is di erentially large if and only if there is a di erentially henselian field* (L, w, ∂) *with* $(L, \partial) \supseteq (K, \delta)$ *such that* (K, δ) *is existentially closed in* (L, ∂) *as a di erential field.*

Proof. For the backwards direction, suppose that (K, δ) and (L, w, ∂) are as above. We show that (K, δ) satisfies the condition in Proposition 2.3.17. Let $f, g \in K\{x\}$ be differential polynomials with $\mathrm{ord}(g) < \mathrm{ord}(f) = n$, and let $\bar{a} \in K$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$, $g(\bar{a}) \neq 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Considering f_{alg} and g_{alg} as polynomials in L, by continuity, there is $\gamma \in vL$ such that $B_\gamma(\bar{a})$ contains no root of g. Now, by differential henselianity of (L, w, ∂), there is $b \in L$ such that $f(b) = 0$, and $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$. In particular, we have that $g(b) \neq 0$. Finally, by existential closure of (K, δ) in (L, ∂), there exists $c \in K$ such that $f(c) = 0$ and $g(c) \neq 0$.

For the forwards implication, we first note that by 4.3(iii) of [32], (K, δ) is existentially closed in $(K_n, \delta_n) := (K((t_0))((t_1))\ldots((t_n)), \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t_1} + \ldots + \frac{\mathrm{d}}{\mathrm{d}t_n})$ for any $n < \omega$. Thus (K, δ) is existentially closed in the union $(L, \partial) = \bigcup_{n<\omega}(K_n, \delta_n)$. By Corollary 4.3.2, (L, w, ∂) is differentially henselian, as required. □

We will now show that any henselian valued field with sufficiently large transcendence degree admits a derivation which induces the structure of a differentially henselian field. This is an adaptation of the construction in Theorem 2.3.24 to the henselian case.

To simplify the notation, we adapt the notion of a *di erentially large problem* as in Definition 2.3.22 to the valued context:

Definition 4.3.4. Let (K, v, δ) be a valued-differential field. A *v-Singer problem over K of order n* is a triple $(f, \bar{c}, \gamma)$, where $f \in K\{x\}$ is a differential polynomial of order n, $\bar{c} \in K^{n+1}$ satisfies $f_{\mathrm{alg}}(\bar{c}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{c}) \neq 0$, and $\gamma \in vK$. We say that $(f, \bar{c}, \gamma)$ is *irreducible* if f is irreducible.

Let (L, w, ∂) be a valued-differential field extension of (K, v, δ). An element $a \in L$ is a *solution* to the v-Singer problem $(f, \bar{c}, \gamma)$ if $f(a) = 0$ and $\mathrm{Jet}_n(a) \in B_\gamma(\bar{c})$. We say that L *solves* $(f, \bar{c}, \gamma)$ if it contains a solution a to $(f, \bar{c}, \gamma)$.

From the definition of differential henselianity, it is clear that a valued-differential field is differentially henselian if and only if it is henselian and solves all v-Singer problems over itself. Further, by Proposition 4.2.8, it suffices to solve all irreducible v-Singer problems over itself. We begin by constructing a solution for a single v-Singer problem.

Lemma 4.3.5. *Let (K, v, δ) be a valued-di erential field, and let $(f, \bar{c}, \gamma)$ be an irreducible v-Singer problem over K of order n. Let (L, w) be a henselian valued field extending (K, v) as a pure valued field, such that $\mathrm{trdeg}(L/K) \geqslant n$. Then, there is a derivation on L extending the derivation on K such that there is a solution $a \in L$ to $(f, \bar{c}, \gamma)$ with $\mathrm{Jet}_{n-1}(a)$ algebraically independent over K.*

Proof. Since $f_{\mathrm{alg}}(\bar{c}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{c}) \neq 0$, and as $s(f)_{\mathrm{alg}}(\bar{c}) \in K^\times$, by the Implicit Function Theorem for Henselian Fields (Theorem 2.5.7), there is $\mu \in vK$ such that there is a unique continuous function $g : U = B_\mu(c_0, ..., c_{n-1}) \to L$ with $f_{\mathrm{alg}}(\bar{y}, g(y)) = 0$ for any $\bar{y} \in U$. By shrinking the ball if necessary, and by continuity of g, we may also assume that for any $\bar{y} \in U$, $(\bar{y}, g(y)) \in B_\gamma(\bar{c})$.

We claim that U contains a point of transcendence degree n over K. Since L/K is an extension of transcendence degree of at least n over K, there is a tuple $(b_0, ..., b_{n-1}) \in L^n$ algebraically independent over K. By replacing b_i with b_i^{-1} if necessary, we may also assume that $v(b_i) \geqslant 0$ for each i. Take $d \in K^\times$ with $v(d) > \mu$. Then, we claim that the tuple $(a_0, ..., a_{n-1}) := (db_0 + c_0, ..., db_{n-1} + c_{n-1})$ suffices.

If the a_i are not algebraically independent over K, then there is a nonzero polynomial $h(x_0, .., x_{n-1}) \in K[x_0, ..., x_{n-1}]$ such that $h(a_0, ..., a_{n-1}) = 0$. Then, defining

$$\tilde{h}(x_0, ..., x_{n-1}) = h(dx_0 + c_0, ..., dx_{n-1} + c_{n-1}),$$

and since $d, c_0, ..., c_{n-1} \in K$ with $d \neq 0$, we have that $\tilde{h} \in K[x_0, ..., x_{n-1}]$ is nonzero with $\tilde{h}(b_0, ..., b_{n-1}) = 0$, which contradicts the algebraic independence of the b_i over K. Further, $v(a_i - c_i) = v(db_i) = v(d) + v(b_i) \geqslant \mu$ as we have assumed that $v(b_i) \geqslant 0$ and $v(d) \geqslant \mu$. Thus $(a_0, ..., a_{n-1})$ is a point of transcendence degree n in U.

Setting $a_n = g(a_0, ..., a_{n-1})$ and defining $a_i' = a_{i+1}$ for $i < n$, we obtain a derivation on $K(a_0, ..., a_n)$. This derivation extends uniquely to its relative algebraic closure in L. We now extend this arbitrarily to a derivation ∂ on L.

Now observe that, by construction, $a = a_0$ is a solution to the v-Singer problem $(f, \bar{c}, \gamma)$ in (L, w, ∂). □

Note. We may construct the derivation ∂ on L such that (L, ∂) is a differentially algebraic extension of (K, δ): we observe that every element of the algebraic closure of $K[a_0, ..., a_n]$ in L is differentially algebraic over K. Extending $(a_0, ..., a_{n-1})$ to a transcendence basis B of L/K, and by setting $\partial(b) = 0$ for every $b \in B \setminus \{a_0, ..., a_{n-1}\}$, we obtain a derivation ∂ on L such that every element of L is differentially algebraic over (K, δ), as required.

We now show with an inductive construction that given a henselian valued field extension $(K, v) \subseteq (L, w)$ of sufficient transcendence degree, it is possible to construct a derivation on L such that every irreducible v-Singer problem over K has a solution in L.

Lemma 4.3.6. *Let (K, v, δ) be a valued-di erential field, and let (L, w) be a henselian valued field extension of (K, v) with $\operatorname{trdeg}(L/K) \geqslant |K|$. Then, there is a derivation ∂ on L extending δ such that (L, w, ∂) solves all irreducible v-Singer problems over K.*

Proof. Let $|K| = \kappa$, and enumerate all irreducible v-Singer problems over K by $(S_\alpha)_{\alpha<\kappa}$. This is possible, as the set of irreducible v-Singer problems is a subset of $K\{x\} \times K^{<\omega} \times vK$ which has cardinality κ. Let $B \subseteq L$ be a transcendence basis for L over K, and let $(B_\alpha)_{\alpha<\kappa}$ be a partition of B such that $|B_\alpha| \geqslant \aleph_0$ for each $\alpha < \kappa$.

For $\alpha < \kappa$, let L_α be the relative algebraic closure of $K\left(\bigcup_{\beta<\alpha} B_\beta\right)$. In particular, L_α is henselian for all $\alpha < \kappa$. Set $\partial_0 = \delta$, fix some $\alpha < \kappa$, and suppose we have constructed ∂_β for all $\beta < \alpha$ such that (L_β, δ_β) solves the v-Singer problem S_μ for every $\mu < \beta$.

For $\alpha = \xi + 1$, since S_ξ is a v-Singer problem over L_ξ of finite order, by Lemma 4.3.5, there is a derivation ∂_α extending ∂_ξ such that $(L_\alpha, \partial_\alpha)$ solves S_ξ. When α is a limit, let $\partial_\alpha = \bigcup_{\beta<\alpha} \partial_\beta$.

Take $\partial = \bigcup_{\alpha<\kappa} \partial_\alpha$, and observe that ∂ is a derivation on L such that (L, w, ∂) all v-Singer problems over K. □

Note. By applying the previous remark in every step, we may construct ∂ such that (L, ∂) is a differentially algebraic extension of (K, δ).

Theorem 4.3.7. *Let (K, v, δ) be a valued-di erential field, and let (L, w) be a henselian extension of (K, v) with $\operatorname{trdeg}(L/K) \geqslant |K|$. Then, there is a derivation ∂ on L extending δ such that (L, w, ∂) is di erentially henselian.*

Proof. We construct a derivation ∂ on L such that (L, w, ∂) solves all irreducible v-Singer problems over itself. Let $\kappa = \operatorname{trdeg}(L/K) \geqslant |K|$, and let B be a transcendence basis for L over K. Let $(B_n)_{n<\omega}$ be a partition of B such that $|B_n| = \kappa$ for each $n < \omega$. Let L_n be the relative algebraic closure of $K\left(\bigcup_{m<n} B_m\right)$ in L. In particular, L_n is henselian for every $n < \omega$.

By construction, $\operatorname{trdeg}(L_{n+1}/L_n) = \kappa$ for each $n < \omega$. We construct the derivation ∂ inductively. Let ∂_0 be the unique extension of δ to the relative algebraic closure of K in L. Suppose we have constructed ∂_n on L_n such that (L_n, ∂_n) solves all v-Singer problems over L_m for each $m < n$. Applying Lemma 4.3.6, we find a derivation ∂_{n+1} on L_{n+1} such that $(L_{n+1}, \partial_{n+1})$ solves all irreducible v-Singer problems over L_n.

Take $\partial = \bigcup_{n<\omega} \partial_n$. Since every (irreducible) v-Singer problem over L is an irreducible v-Singer problem over L_n for some $n < \omega$, and since (L, w, ∂) solves all irreducible v-Singer problems over L_n for every $n < \omega$, we have that (L, w, ∂) solves all irreducible v-Singer problems over itself, as required. □

Again by the previous remark, we may construct each derivation ∂_n such that $(L_n, \partial_n) \subseteq (L_{n+1}, \partial_{n+1})$ is differentially algebraic for every n. Thus, we obtain the following:

Corollary 4.3.8. *Let (K, v, δ) be a valued-di erential field, and let (L, w) be a henselian extension of (K, v) with $\operatorname{trdeg}(L/K) \geqslant |K|$. Then, there is a derivation ∂ on L extending δ such that (L, w, ∂) is di erentially henselian, and the extension of di erential fields $(L, \partial)/(K, \delta)$ is di erentially algebraic.*

We also obtain that every henselian valued field of infinite transcendence degree admits a derivation such that the resulting valued-differential field is differentially henselian.

Corollary 4.3.9. *Every henselian valued field (K, v) of infinite transcendence degree admits a derivation δ such that (K, v, δ) is di erentially henselian.*

Proof. Take $\mathbb{Q}$ with the trivial derivation and induced valuation as a valued-differential subfield of K, and apply Theorem 4.3.7. $\square$

We recall that given any differential field (K, δ) which is large as a pure field, there is a differential field extension (L, ∂) such that (L, ∂) is differentially large and $K \preccurlyeq L$ as pure fields (cf. Theorem 2.3.18). From the above, we can extract an analogous result for valued fields, and recover [12, Theorem 2.3.4] for the differentially henselian case.

Corollary 4.3.10. *Let (K, v, δ) be a valued-di erential field which is henselian as a pure valued field. There is a valued-di erential field extension (L, w, ∂) of (K, v, δ) such that (L, w, ∂) is di erentially henselian, and $(K, v) \preccurlyeq (L, w)$ as pure valued fields.*

Proof. Let (L, w) be an elementary extension of (K, v) such that $\mathrm{trdeg}(L/K) \geqslant |K|$, for example, taking any elementary extension with cardinality at least $|K|^+$ suffices. Then, apply Theorem 4.3.7 to obtain a derivation ∂ extending δ such that (L, w, ∂) is differentially henselian, as required. $\square$

4.4 Existential Lifting

In this section, we will prove an existential transfer theorem from pure valued fields to differentially henselian fields. This follows work by Tressl and Guzy for models of UC and UC$'$, respectively. From this work, we will also extract a powerful relative embedding theorem for differentially henselian fields, which will, in later sections, allow us to adapt many classical model-theoretic results from pure henselian fields to differentially henselian fields.

We will show that if (K, v, δ) and (L, w, ∂) are valued-differential fields, where (L, w, ∂) is differentially henselian, and A is a common valued-differential subfield, then, if $(K, v) \equiv\rangle_{\exists,A} (L, w)$ as pure valued fields, then we also have that $(K, v, \delta) \equiv\rangle_{\exists,A} (L, w, \partial)$ as valued differential fields (cf. Theorem 4.1.13).

Our proof strategy is as follows: replacing (L, w, ∂) with a sufficiently saturated elementary extension if necessary, we show that for any finite tuple $\bar{a}$ in K, and any realisation $\bar{b}$ of the quantifier-free $\mathcal{L}_{\mathrm{vf}}$-type of $\mathrm{Jet}(\bar{a})$, we can find a realisation $\bar{c}$ of the quantifier-free $\mathcal{L}_{\mathrm{vd}}$-type of $\bar{a}$ in L such that $\mathrm{Jet}(\bar{c})$ is arbitrarily close to $\bar{b}$. This implies the existential lifting condition above.

We recall that, in the language $\mathcal{L}_{\mathrm{vd}}$, every atomic formula with parameters in A is equivalent (modulo the theory of valued differential fields) to an atomic formula in of the form

- ‘$f(\bar{x}) = 0$’; or
- ‘$v(g(\bar{x})) \leqslant v(h(\bar{x}))$’,

where f, g, h are differential polynomials over A. For convenience, we will say that atomic formulae of these forms are ‘*algebraic*’ and ‘*valuation*’, respectively. As every atomic formula in the language $\mathcal{L}_{\mathrm{vd}}(A)$ is equivalent modulo the theory of valued differential fields to an algebraic or valuation atomic formula, we will assume that for every $\mathcal{L}_{\mathrm{vd}}(A)$-formula, every atomic subformula is of one of these two forms.

We begin with a preparatory lemma.

Lemma 4.4.1. *Let (K, v) be a valued field. Let $\varphi(\bar{x})$ be a consistent $\mathcal{L}_{\mathrm{vf}}(K)$ formula which is a boolean combination of valuation atomic formulae. Let $\bar{a} \in K$ be such that $K \models \varphi(\bar{a})$. Then, there exists $\gamma \in vK$ such that for any $\bar{b} \in B_\gamma(\bar{a})$, $K \models \varphi(\bar{b})$. That is, the set defined by $\varphi(\bar{x})$ is open in K^n with respect to the valuation topology.*

Proof. As polynomials are continuous with respect to the valuation topology, for every polynomial f appearing in φ, there is some $\gamma \in vK$ such that for all $\bar{b} \in B_\gamma(\bar{a})$, $v(f(\bar{b}) - f(\bar{a})) > v(f(\bar{a}))$. In particular, for any $\bar{b} \in B_\gamma(\bar{a})$, $v(f(\bar{a})) = v(f(\bar{b}))$. Fix $\gamma \in vK$ such that the above holds for all polynomials f appearing in φ. Now, for any atomic formula $\psi(\bar{x})$ of the form $v(f(\bar{x})) \leqslant v(g(\bar{x}))$ appearing in φ, and for any $b \in B_\gamma(\bar{a})$, $K \models \psi(\bar{a})$ if and only if $K \models \psi(\bar{b})$. As the truth of every atomic formula is preserved by replacing $\bar{a}$ with $\bar{b}$, we also have that $K \models \varphi(\bar{b})$. □

Remark. We can also present a topological version of this proof: we first note that the set $\{(x, y) \in K^2 : v(x) \leqslant v(y)\}$ is clopen in K^2, and for any polynomials $f(\bar{x}), g(\bar{x})$ over K, the map $K^n \to K^2 : \bar{x} \mapsto (f(\bar{x}), g(\bar{x}))$ is continuous. Thus, the preimage $\{\bar{x} \in K^n : v(f(\bar{x})) \leqslant v(g(\bar{x}))\}$ is also clopen. Since the set defined by $\varphi(\bar{x})$ is a finite boolean combination of clopen sets, it is also clopen (and hence open).

Lemma 4.4.2. *Let (K, v, δ) be a valued-di erential field, and A a di erential subfield of K. Let $\varphi(\bar{x})$ be a quantifier-free $\mathcal{L}_{\mathrm{vd}}(A)$-formula, and $\bar{a} \in K$ such that $K \models \varphi(\bar{a})$. There are $\mathcal{L}_{\mathrm{vd}}(A)$-formulae $\psi(\bar{x}), \chi(\bar{x})$ such that*

$$\models \forall \bar{x}((\psi(\bar{x}) \wedge \chi(\bar{x})) \to \varphi(\bar{x})),$$

and

$$K \models \psi(\bar{a}) \wedge \chi(\bar{a}),$$

and ψ, χ are conjunctions of algebraic and valuation atomic formulae and negations of algebraic and valuation atomic formulae, respectively.

Proof. We begin by enumerating the algebraic and valuation atomic formulae appearing in φ as $(\psi_i)_{i<n}$ and $(\chi_i)_{i<m}$, respectively. Let

$$\psi(\bar{x}) = \bigwedge_{i:K\models\psi_i(\bar{a})} \psi_i(\bar{x}) \wedge \bigwedge_{i:K\models\neg\psi_i(\bar{a})} \neg\psi_i(\bar{x}).$$

Similarly, set

$$\chi(\bar{x}) = \bigwedge_{i:K\models\chi_i(\bar{a})} \chi_i(\bar{x}) \wedge \bigwedge_{i:K\models\neg\chi_i(\bar{a})} \neg\chi_i(\bar{x}).$$

By construction, if $\bar{b}$ is a tuple of a $\mathcal{L}_{\mathrm{vd}}(A)$-structure M, such that $M \models \psi(\bar{b}) \wedge \chi(\bar{b})$, we have that $M \models \theta(\bar{b}) \iff K \models \theta(\bar{b})$ for every atomic formula θ appearing in φ. Thus, $M \models \varphi(\bar{b})$ and so we have that $\models \forall\bar{x}((\psi(\bar{x}) \wedge \chi(\bar{x})) \to \varphi(\bar{x}))$.

Further, it is clear by construction that $K \models \psi(\bar{a}) \wedge \chi(\bar{a})$, and that ψ and χ are of the form required. $\square$

Note. In the above lemma, the formula $\psi \wedge \chi$ is precisely the disjunct of the disjunctive normal form of φ which holds for $\bar{a}$ in K, where we separate the algebraic and valuation atomics into ψ and χ, respectively.

For convenience, we will call the formulae ψ and χ in the above lemma the *algebraic* and *valuation parts* of φ (with respect to $\bar{a}$), respectively.

Notation. For a valued-differential field (K, v, δ) and a set of parameters $A \subseteq K$, the quantifier-free $\mathcal{L}_{\mathrm{vf}}$ and $\mathcal{L}_{\mathrm{vd}}$-types of $\bar{a} \in K$ are denoted $\mathrm{qftp}_{\mathrm{vf}}(\bar{a}/A)$ and $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$, respectively.

We first consider the case where the tuple $\bar{a}$ is differentially algebraically independent over A.

Lemma 4.4.3. *Let (K, v, δ) be a valued-di erential field, (L, w, ∂) a di erential henselian field, and A a common valued-di erential subfield. Suppose that (L, w, ∂) is $|A|^+$-saturated, and let $\bar{a} = (a_i)_{i<n} \in K$ be di erentially algebraically independent over A. Let $\bar{c} = (c_{ij})_{i<n, j<\omega} \in L$ be a realisation of $\mathrm{qftp}_{\mathrm{vf}}(\mathrm{Jet}(\bar{a})/A)$ in L. Then, for any $\varepsilon \in wL$, there is $\bar{u} \in L$ realising $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$ and $\mathrm{Jet}(\bar{u}) \in B_\varepsilon(\bar{c})$.*

Proof. We show that the type $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$ along with '$\mathrm{Jet}(\bar{y}) \in B_\varepsilon(\bar{c})$' is finitely satisfiable in L. Let $\varphi(\bar{y}) \in \mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$, and let ψ and χ be the algebraic and valuation parts of φ, respectively. Let N be the highest order derivative of any variable appearing in φ. Since $\bar{a}$ is differentially algebraically independent over A, we may assume that ψ consists only of differential polynomial inequations. As polynomials are continuous with respect to the valuation topology, there is some $\mu \in wL$ such that for any differential polynomial inequation $g(\bar{y}) \neq 0$ appearing in ψ, and nN-tuple $\bar{u} \in B_\mu((c_{ij})_{i<n,j\leqslant N})$, we have that $g(\bar{u}) \neq 0$.

Further, by Lemma 4.4.1, there is some $\lambda \in wL$ such that for any nN-tuple $\bar{u} \in B_\lambda((c_{ij})_{i<n,j\leqslant N})$, $\psi(\bar{u})$ holds. Let $\varepsilon' = \max(\varepsilon, \mu, \lambda)$. By differential henselianity, there is $\bar{u} = (u_i)_{i<n} \in L$ such that $\mathrm{Jet}_N(v_i) \in B_{\varepsilon'}((c_{ij})_{j\leqslant N})$. Thus, we find that $L \models \varphi(\bar{v}) \wedge \mathrm{Jet}_N(\bar{v}) \in B_\varepsilon((c_{ij})_{i<n,j\leqslant N})$, as required. Now, apply saturation. □

Lemma 4.4.4. *Let (K, v, δ) be a valued-di erential field, (L, w, ∂) a di erentially henselian field, and A be a common valued-di erential subfield. Suppose that (L, w, ∂) is $|A|^+$-saturated. Let $\bar{a}b \in K$ be such that $\bar{a} = (a_i)_{i<n}$ is di erentially algebraically independent over A, and b is di erentially algebraic over $A\langle\bar{a}\rangle$. Suppose there is $\bar{c}\bar{d} = (c_{ij})_{i<n,j<\omega}(d_k)_{k<\omega} \in L$ realising $\mathrm{qftp}_{\mathrm{vf}}(\mathrm{Jet}(\bar{a})\,\mathrm{Jet}(b)/A)$. Then, for any $\gamma \in wL$ there is $\bar{\alpha}\beta \in L$ such that $\bar{\alpha}\beta \models \mathrm{qftp}_{\mathrm{vd}}(\bar{a}b/A)$ and $\mathrm{Jet}(\alpha)\,\mathrm{Jet}(\beta) \in B_\gamma(\bar{c}\bar{d})$.*

Proof. Let $f(x)$ be the differential minimal polynomial of b over $A\langle\bar{a}\rangle$. Let $\mathrm{ord}(f) = m$. By clearing denominators, we may write $f(x)$ as $F(\bar{a}, x)$, where $F(\bar{y}, x)$ is a differential polynomial over A in $n+1$ variables. Denote its separant (with respect to x) by $s(F)(\bar{y}, x)$.

We claim that $p(\bar{y}, x) \coloneqq \mathrm{qftp}_{\mathrm{vd}}(\bar{a}b/A)$ along with the partial type expressing that '$\mathrm{Jet}(\bar{y})\,\mathrm{Jet}(x) \in B_\gamma(\bar{c}\bar{d})$' is finitely satisfiable in L. Let $\varphi(\bar{y}, x)$ be some formula in p. Let ψ and χ denote the algebraic and valuation parts of φ, respectively.

For a tuple $\bar{t}u$ to satisfy $\psi(\bar{y}, x)$, it suffices for $\bar{t}$ to be differentially algebraically independent over A, $F(\bar{t}, u) = 0$ and for each differential polynomial inequation $g(\bar{y}, x) \neq 0$ appearing in ψ, $g(\bar{t}, u) \neq 0$. By continuity, there is some $\mu \in wL$ such that $B_\mu(\bar{c}d)$ contains no roots of g_{alg} for each of the g appearing above, and also does not contain any roots of $s(F)_{\mathrm{alg}}$. Further, by Lemma 4.4.1, there is some $\lambda \in wL$ such that for any $\bar{t}u \in L$ with $\mathrm{Jet}(\bar{t})\,\mathrm{Jet}(u) \in B_\lambda(\bar{c}\bar{d})$, $\chi(\bar{t}u)$ holds.

Let M be the highest derivative of x appearing in the formula φ. Since the differential algebraic relation $F(\bar{y}, x) = 0$ holds, we may express the derivatives $y^{(m+1)}, \ldots, y^{(M)}$ as continuous (in fact, rational) functions of $\bar{y}$ and $x, x', \ldots, x^{(m)}$.

Therefore, there is some $\zeta \in wL$ such that, for any $\bar{t}u \in L$ with $F(\bar{t}, u) = 0$, $s(F)(\bar{t}, u) \neq 0$, $\mathrm{Jet}(\bar{t}) \in B_\zeta(\bar{b})$ and $\mathrm{Jet}_m(u) \in B_\zeta(\bar{c}\restriction_m)$, we have that $\mathrm{Jet}_M(u) \in B_{\max(\lambda,\mu,\gamma)}(\bar{c}\restriction_M)$.

By the implicit function theorem for henselian fields (Theorem 2.5.7), there is some $\theta \in wL$, such that for any differentially algebraically independent tuple $\bar{t}$ over A with $\mathrm{Jet}(\bar{t}) \in B_\theta(\bar{c})$, there is a unique $\hat{d}_m \in L$ such that $F_{\mathrm{alg}}(\mathrm{Jet}(\bar{t}), d_0, ..., d_{m-1}, \hat{d}_m) = 0$, and $w(\hat{d}_m - d_m) > \max(\lambda, \mu, \zeta, \gamma) = \eta$.

Let $\varepsilon = \max(\lambda, \mu, \theta, \zeta, \gamma)$. By Lemma 4.4.3, there is a tuple $\bar{t} \in L$ differentially algebraically independent over A, realising $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$, and $\mathrm{Jet}(\bar{t}) \in B_\varepsilon(\bar{c})$. We find $\hat{d}_m \in B_\eta(d_m)$ such that $F_{\mathrm{alg}}(\mathrm{Jet}(\bar{t}), d_0, ..., d_{m-1}, \hat{d}_m) = 0$ and $s(F)_{\mathrm{alg}}(\mathrm{Jet}(\bar{t}), d_0, ..., d_{m-1}, \hat{d}_m) \neq 0$. Therefore, by differential henselianity, there is a $u \in L$ such that $F(\bar{t}, u) = 0$, and $\mathrm{Jet}_M(u) \in B_\eta(\bar{d}\restriction_M)$.

Thus, $L \models \varphi(\bar{t}, u)$, and the partial type p is finitely satisfiable in L. By saturation, we find $\bar{\alpha}\beta \in L$ realising p, as required. □

We will now drop the restriction on the form of the tuple we embed, and show that the valued-differential quantifier-free type of a finite tuple with a realisation of its valued-field quantifier-free type can be realised.

Proposition 4.4.5. *Let* (K, v, δ) *be a valued-di erential field,* (L, w, ∂) *be a di erentially henselian field, and* A *a common valued-di erential subfield. Suppose that* (L, w, ∂) *is* $|A|^+$*-saturated and that* $(K, v) \equiv\rangle_{\exists,A} (L, w)$ *as pure valued fields. Let* $\bar{a} = (a_i)_{i<n} \in K$ *be an arbitrary finite tuple. Suppose there is* $\bar{b} = (b_{ij})_{i<n,j<\omega}$ *realising* $\mathrm{qftp}_{\mathrm{vf}}(\mathrm{Jet}(\bar{a})/A)$*. Then, for any* $\gamma \in wL$*, there is* $\bar{c} \in L$ *realising* $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$ *such that* $\mathrm{Jet}(\bar{c}) \in B_\gamma(\bar{b})$*.*

Proof. We will show that the quantifier-free type $p(\bar{x}) = \mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$ along with the partial type stating that '$\mathrm{Jet}(\bar{x}) \in B_\gamma(\bar{b})$' is finitely satisfiable in (L, w, ∂), and use compactness.

First, we assume that the differential field extension $K \subseteq K\langle\bar{a}\rangle = F$ has nonzero differential transcendence degree. If not, possibly after replacing K with a suitable elementary extension, adjoin an arbitrary differentially transcendental element t to $\bar{a}$.

By Corollary 2.1.26, there are subtuples $\bar{a}_0$ and $\bar{a}_1$ partitioning $\bar{a}$ such that $\bar{a}_0$ is differentially algebraically independent over A, and the extension $A\langle\bar{a}_0\rangle \subseteq A\langle a_0\rangle\langle a_1\rangle = F$ is differentially algebraic. Partition the tuple $\bar{x}$ of variables similarly as $\bar{x}_0\bar{x}_1$ and similarly reindex $\bar{b} = (b_{ij})$ as $\bar{b}_0\bar{b}1 = (b_{0,i,j})(b_{1,i,j})$. That is, $\bar{b}_0\bar{b}_1$ realises $\mathrm{qftp}_{\mathrm{vf}}(\mathrm{Jet}(\bar{a}_0)\,\mathrm{Jet}(\bar{a}_1)/A)$.

As the derivative on $A\langle\bar{a}_0\rangle$ is nontrivial (in particular, there exists an element differentially transcendental over A), we may apply the differential primitive element theorem (Theorem 2.1.30) to obtain a single element $c \in F$ such that $A\langle\bar{a}_0\rangle\langle c\rangle = F$. In particular, writing $\bar{a}_1 = (a_{1,i})_i < m$, each derivative $a_{1,i}^{(j)}$ is expressible as $q_{i,j}(\bar{a}_0, c)$, where $q(\bar{x}_0, y)$ is a differential rational function over A (i.e. a ratio of differential polynomials over A with non-vanishing denominator).

As $\bar{b}$ is a realisation of $\mathrm{qftp}_{\mathrm{vf}}(\bar{a}/A)$, setting $\mathrm{Jet}(\bar{a}) \mapsto \bar{b}$ induces an embedding of valued fields over A. Let $\bar{d}$ be the image of $\mathrm{Jet}(c)$ under this embedding, i.e. $\bar{b}_0\bar{d}$ is a realisation of $\mathrm{qftp}_{\mathrm{vf}}(\mathrm{Jet}(a_0c))$ in L.

By continuity of the $q_{i,j}$, for any $N < \omega$, there is some $\delta \in wL$ such that for any realisation $\bar{\alpha}\beta$ of $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}_0c/A)$ in L with $\mathrm{Jet}(\bar{\alpha}\beta) \in B_\delta(\bar{b}_0\bar{d})$, we have that $w(q_{i,j}(\bar{\alpha}, \beta) - b_{1,i,j}) > \gamma$ for each $i < m, j < N$.

By Lemma 4.4.4, there is a realisation $\bar{\alpha}\beta$ of $\mathrm{qftp}(\bar{a_0}c)$ in L such that $\mathrm{Jet}(\bar{\alpha}\beta) \in B_{\max(\delta,\gamma)}(\bar{b}_0\bar{d})$. Let $\varphi : A\langle\bar{a}\rangle \to L$ be the valued-differential field embedding induced by setting $\bar{a}_0c \mapsto \bar{\alpha}\beta$. By construction, $\varphi(\mathrm{Jet}_N(\bar{a}_0\bar{a}_1)) \in B_\gamma(\bar{b}_0\bar{b}_1)$.

Thus, $\varphi(\bar{a}_0\bar{a}_1)$ is a realisation in (L, w, ∂) of $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}_0\bar{a}_1/A)$ as well as the formula stating '$\mathrm{Jet}_N(\bar{x}_0\bar{x}_1) \in B_\gamma(\bar{b}_0\restriction_N\bar{b}_1\restriction_N)$. We conclude therefore that the desired partial type is finitely satisfiable in (L, w, ∂). □

From this, we harvest a number of more applicable results regarding embeddings of valued-differential fields.

Proposition 4.4.6. *Let (K, v, δ) be a valued-di erential field, (L, w, ∂) be a di erentially henselian field, and A a common valued-di erential subfield. Let (L, w, ∂) be $|A|^+$-saturated. Suppose that $(K, v) \equiv\rangle_{\exists,A}(L, w)$ as pure valued fields. Then, every di erentially finitely generated extension $A\langle\bar{a}\rangle \subseteq K$ embeds in L over A as valued-di erential fields.*

Proof. As $(K, v) \equiv\rangle_{\exists,A}(L, w)$, and by saturation of (L, w), there is a realisation $\bar{b}$ of $\mathrm{qftp}_{\mathrm{vf}}(\mathrm{Jet}(\bar{a}/A)$ in L. By Proposition 4.4.5, there is a realisation $\bar{c}$ of $\mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$ in (L, w, ∂). In particular, setting $\bar{a} \mapsto \bar{c}$ induces an embedding of valued-differential fields $A\langle\bar{a}\rangle \to L$, as required. □

From the above embedding lemma, we obtain the desired existential lifting property of differentially henselian fields (cf. [18, Theorem 3.14])

Corollary 4.4.7. *Let (K, v, δ) be a valued-di erential field, (L, w, ∂) a di erentially henselian field, and A a common valued-di erential subfield. Suppose* *that* $(K, v) \equiv\rangle_{\exists,A}(L, w)$ *as pure valued fields. Then,* $(K, v, \delta) \equiv\rangle_{\exists,A}(L, w, \partial)$ *as valued-di erential fields.*

Proof. Let $\varphi(\bar{x})$ be an $\mathcal{L}_{\mathrm{vd}}(A)$-formula, and suppose that $(K, v, \delta) \models \exists\bar{x}\varphi(\bar{x})$. Let $\bar{a} \in K$ such that $K \models \varphi(\bar{a})$. Let (L^+, w^+, ∂^+) be an $|A|^+$-saturated elementary extension of (L, w, ∂). By Lemma 4.4.6, there is an embedding $f : A\langle\bar{a}\rangle \to L^+$ as valued-differential fields. Since φ is quantifier-free and f is an embedding of valued-differential fields, we have that $(L^+, w^+, \partial^+) \models \varphi(f(\bar{a}))$, and so $(L^+, w^+, \partial^+) \models \exists\bar{x}\varphi(\bar{x})$. Since $(L, w, \partial) \preccurlyeq (L^+, w^+, \partial^+)$, we also have that $(L, w, \partial) \models \exists\bar{x}\varphi(\bar{x})$ as required. □

Corollary 4.4.8. *Let* (K, v, δ) *be a valued-di erential field,* (L, w, ∂) *a di erentially henselian field, and* A *a common valued-di erential subfield. Suppose that* $(K, v) \equiv\rangle_{\exists,A}(L, w)$ *as pure valued fields, and* (L, w, ∂) *is* $\max(|A|^+, |K|)$*-saturated. Then,* (K, v, δ) *embeds in* (L, w, ∂) *over* A *as valued di erential fields.*

Proof. Let $|K| = \kappa$, and enumerate K as $\bar{a} = (a_\alpha)_{\alpha<\kappa}$. It suffices to find a realisation of the quantifier-free type $p(\bar{x}) = \mathrm{qftp}_{\mathrm{vd}}(\bar{a}/A)$. Since we have that $(K, v) \equiv\rangle_{\exists,A}(L, w)$ as pure valued fields, by Corollary 4.4.7, we also have that $(K, v, \delta) \equiv\rangle_{\exists,A}(L, w, \partial)$ as valued-differential fields. Thus, p is finitely satisfiable in L, and by saturation, we also have that p is realised in L. □

From these existential lifting results, we will now show that differentially henselian fields can be characterised in a similar way to differentially large fields, that is, in terms of an existential closure condition (cf. Definition 2.3.11).

Theorem 4.4.9. *Let* (K, v, δ) *be a valued-di erential field. Then,* (K, v, δ) *is di erentially henselian if and only if* (K, v) *is henselian, and for any valued-di erential field extension* (L, w, ∂) *of* (K, v, δ)*, if* (K, v) *is existentially closed in* (L, w) *as a valued field, then* (K, v, δ) *is existentially closed in* (L, w, ∂) *as a valued-di erential field.*

Proof. For the forwards direction, let (K, v, δ) be a differentially henselian field, and let (L, w, ∂) be a valued-differential field extension such that (K, v) is existentially closed in (L, w) as pure valued fields. Observe that $(K, v) \preccurlyeq_\exists (L, w)$ is equivalent to the condition $(L, w) \equiv\rangle_{\exists,K}(K, v)$. By Corollary 4.4.7, we also have that $(L, w, \partial) \equiv\rangle_{\exists,K}(K, v, \delta)$ as valued-differential fields, i.e. (K, v, δ) is existentially closed in (L, w, ∂).

For the backwards direction, suppose that (K, v, δ) is henselian as a pure valued field and, for every valued-differential field extension (L, w, ∂) in which

K is existentially closed as a pure valued field, we have that K is existentially closed in L as a valued-differential field.

Conversely, by Corollary 4.3.10, we let (L, w, ∂) be a differentially henselian field such that $(K, v) \preccurlyeq (L, w)$ as valued fields. As (L, w) is an elementary extension of (K, v), we have in particular that (K, v) is existentially closed in (L, w). Thus, by assumption, we have that (K, v, δ) is existentially closed in (L, w, ∂). Let f be a differential polynomial over K of order n, and let $\bar{a} \in K$ such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Let $\gamma \in vK$. As (L, w, ∂) is differentially henselian, there is $b \in L$ such that $f(b) = 0$ and $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$. By existential closure, we have that there is a $c \in K$ satisfying the same. Thus, (K, v, δ) is differentially henselian, as required. □

Using this characterisation of differential henselianity, we can re-prove the fact that every differentially henselian field is differentially large without appealing to the characterisation in Proposition 2.3.17:

Proof of Proposition 4.2.9. Suppose that (K, v, δ) is a differentially henselian field. Let (L, ∂) be a differential field extension of (K, δ) such that K is existentially closed in L as pure fields. Let (K^*, v^*, δ^*) be an elementary extension of (K, v, δ), sufficiently saturated such that L embeds in K^* as a pure field. Let w be the valuation induced on L under this embedding. Since (L, w) embeds in an elementary extension of (K, v), (K, v) is existentially closed in (L, w) as valued fields. Thus, by Theorem 4.4.9, (K, v, δ) is existentially closed in (L, w, ∂) as valued-differential fields. Taking reducts, we also have that (K, δ) is existentially closed in (L, ∂) as pure differential fields. Therefore (K, δ) is differentially large, as required. □

We will now harvest a useful relative embedding theorem:

Theorem 4.4.10 (Relative Embedding Theorem)**.** *Let* (K, v, δ) *be a valued-di erential field,* (L, w, ∂) *be a di erentially henselian field and* A *a common valued-di erential subfield. Suppose that there is an embedding* $\varphi : (K, v) \to (L, w)$ *as pure valued fields over the common subfield* A*. Suppose also that* (L, w, ∂) *is* $|K|^+$*-saturated. Then, for any* $\gamma \in wL$*, there is an embedding* $\psi : (K, v, \delta) \to (L, w, \partial)$ *of valued-di erential fields over* A *such that for all* $a \in K$*,* $\psi(a) \in B_\gamma(\varphi(a))$*.*

Proof. Enumerate K as $\bar{k}$ and consider the quantifier-free type $p(\bar{x})$ which contains $\mathrm{qftp}_{\mathrm{vd}}(\bar{k}/A)$ along with the formulae which express '$\bar{x} \in B_\gamma(\varphi(\bar{k})$'. This is finitely satisfiable by Proposition 4.4.5, as any finite subset of p involves only finitely many variables, and p restricted to these variables is realisable in L. Applying compactness, we find a realisation $\bar{l}$ of p in L, i.e. setting $\bar{k} \mapsto \bar{l}$ induces a differential embedding ψ of K into L such that for any $a \in K$, we have that $\psi(c) \in B_\gamma(\varphi(a))$. □

From this, we can reprove a relative completeness result for differentially henselian fields in the case where the theory of the underlying valued field is model complete. This is an analogue of [46, Theorem 7.1], and is a version of [18, Corollary 3.16] in the case of one derivation.

Theorem 4.4.11. *Let* $(K, v, \delta), (L, w, \partial)$ *be di erentially henselian. Suppose that* $(K, v) \equiv (L, w)$ *and are model complete as pure valued fields. Then* $(K, v, \delta) \equiv (L, w, \partial)$ *as valued-di erential fields.*

Proof. Let $(K_0, v_0, \delta_0) = (K, v, \delta)$ and $(L_0, w_0, \partial_0) = (L, w, \partial)$. By standard model theoretic results, there is an embedding $(K_0, v_0) \to (L_1, w_1)$, where (L_1, w_1, ∂_1) is a sufficiently saturated elementary extension of (L_0, w_0, ∂_0). By Theorem 4.4.10, taking the common differential subfield to be $\mathbb{Q}$, we find an embedding $(K_0, v_0, \delta_0) \to (L_1, w_1, \partial_1)$. Identifying (K_0, v_0, δ_0) with its image, and by model completeness, we have that $(K_0, v_0) \preccurlyeq (L_1, w_1)$.

We now perform an inductive construction. Assume that we have constructed a chain of valued-differential fields

$$(K_0, v_0, \delta_0) \subseteq (L_1, w_1, \partial_1) \subseteq \ldots \subseteq (K_{n-1}, v_{n-1}, \delta_{n-1}) \subseteq L_n, w_n, \partial_n)$$

for some $n < \omega$, and assume that the following hold:

1. The subchain

$$(K_0, v_0, \delta_0) \preccurlyeq (K_1, v_1, \delta_1) \preccurlyeq \ldots \preccurlyeq (K_{n-1}, v_{n-1}, \delta_{n-1})$$

is an elementary chain of valued-differential fields.

2. The subchain

$$(L_0, w_0, \partial_0) \preccurlyeq (L_1, w_1, \partial_1) \preccurlyeq \ldots \preccurlyeq (L_n, w_n, \partial_n)$$

is also an elementary chain of valued-differential fields.

3. The reduct of the chain to the language of valued fields

$$(K_0, v_0) \preccurlyeq (L_1, w_1) \preccurlyeq \ldots \preccurlyeq (K_{n-1}, v_{n-1}) \preccurlyeq (L_n, w_n)$$

is an elementary chain of valued fields.

As $(K_{n-1}, v_{n-1}) \preccurlyeq (L_n, w_n)$, we have in particular that (K_{n-1}, v_{n-1}) is existentially closed in (L_n, w_n), thus (L_n, w_n) embeds in any sufficient saturated elementary extension of (K_{n-1}, v_{n-1}) over (K_{n-1}, v_{n-1}).

Let (K_n, v_n, δ_n) be such an elementary extension of $(K_{n-1}, v_{n-1}, \delta_{n-1})$. By Theorem 4.4.10, there is an embedding of (L_n, w_n, ∂_n) over $(K_{n-1}, v_{n-1}, \delta_{n-1})$ into (K_n, v_n, δ_n) as valued-differential fields. Identifying L_n with its image in K_n and applying model completeness, we have that $(L_n, w_n) \preccurlyeq (K_n, v_n)$ as valued fields.

Now, by a symmetric argument exchanging the roles of L and K, we find an elementary extension $(L_{n+1}, w_{n+1}, \partial_{n+1})$ of (L_n, w_n, ∂_n) such that (K_n, v_n, δ_n) embeds in $(L_{n+1}, w_{n+1}, \partial_{n+1})$ as valued-differential fields, and the embedding of valued fields $(K_n, v_n) \to (L_{n+1}, w_{n+1})$ is elementary.

Let

$$(F, u, \mathrm{d}) = \bigcup_{i<\omega}(K_i, v_i, \delta_i) = \bigcup_{i<\omega}(L_i, w_i, \partial_i).$$

Since (F, u, d) is the union of the elementary chain $(K_i, v_i, \delta_i)_{i<\omega}$, it is an elementary extension of (K, v, δ). Similarly, we also have that $(L, w, \partial) \preccurlyeq (F, u, \mathrm{d})$. Thus, $(K, v, \delta) \equiv (L, w, \partial)$, as required. □

We will now demonstrate that the class of differentially henselian fields coincides with the henselian models of Guzy's (UC'_k) when $k = 1$. The argument is essentially the same as for the differentially large case, i.e. for Proposition 2.3.19.

Proposition 4.4.12. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. Then, (K, v, δ) is di erentially henselian if and only if $(K, v, \delta) \models (\mathrm{UC}'_1)$.*

Proof. First suppose that (K, v, δ) is differentially henselian. Then, by Theorem 4.1.12, there is a valued-differential field extension (L, w, ∂) of (K, v, δ) such that $(L, w, \partial) \models (\mathrm{UC}'_1)$ and $(K, v) \preccurlyeq (L, w)$ as pure valued fields. Then, as (UC'_1) is inductive by Corollary 4.1.14, (K, v, δ) is also a model of (UC'_1).

Now, suppose that $(K, v, \delta) \models (\mathrm{UC}'_1)$. We verify the condition in Theorem 4.4.9. Let (L, w, ∂) be any valued-differential field extension of (K, v, δ) such that $(K, v) \preccurlyeq_\exists (L, w)$. By existential closure, there is an elementary extension (F, u) of (K, v) such that (L, w) embeds in (F, u) over (K, v). Extend the derivation ∂ arbitrarily to a derivation d on F. It suffices to show that (K, v, δ) is existentially closed in $(F, u\mathrm{d})$, so replace (L, w, ∂) with (F, u, d) (in particular, we may assume that $(K, v) \preccurlyeq (L, w)$).

By Theorem 4.1.12, there is a valued-differential field extension (L^*, w^*, ∂^*) of (L, w, ∂) such that $(L^*, w^*, \partial^*) \models (\mathrm{UC}_1')$ and $(L, w) \preccurlyeq (L^*, w^*)$. In particular, $(K, v) \preccurlyeq (L^*, w^*)$ as valued fields, and thus $(K, v) \preccurlyeq_\exists (L^*, w^*)$, i.e. (K, v) and (L, w) have the same universal theory over K as pure valued fields. Now, as both (K, v, δ) and (L^*, w^*, ∂^*) are models of (UC_1'), by Corollary 4.1.14, they have the same universal $\mathcal{L}_{\mathrm{vd}}$-theory over K. That is, (K, v, δ) is existentially closed in (L^*, w^*, ∂^*). This is sufficient, as $(K, v, \delta) \subseteq (L, w, \partial) \subseteq (L^*, w^*, \partial^*)$, so $(K, v, \delta) \preccurlyeq_\exists (L, w, \partial)$, as required. □

4.5 Ax-Kochen/Ershov Principles

In this section, we apply the various embedding results from Section 4.4 to prove Ax-Kochen/Ershov-type results for various classes of differentially henselian fields.

We recall from Section 2.5.2 that in certain 'tame' classes of henselian valued fields, the theory (and other model-theoretic properties) of the valued field can be understood completely in terms of the theories of the value group and residue field. We mainly consider the cases of equicharacteristic 0 and unramified mixed characteristic fields.

The Equicharacteristic 0 Case

We begin by proving a few versions (non-relative and relative) of this result for differentially henselian fields of equicharacteristic 0. We begin with the existential-closure version, which follows immediately from our existential lifting results from the previous section:

Theorem 4.5.1. *Let* $(K, v, \delta) \subseteq (L, w, \partial)$ *be an extension of di erentially henselian fields of equicharacteristic 0, and suppose that* $vK \preccurlyeq_\exists wL$ *and* $Kv \preccurlyeq_\exists Lw$ *as ordered abelian groups and pure fields, respectively. Then,* $(K, v, \delta) \preccurlyeq_\exists (L, w, \partial)$ *as valued-di erential fields.*

Proof. By the existential closure version of the Ax-Kochen/Ershov theorem for non-differential henselian valued fields of equicharacteristic 0 (Theorem 2.5.28(iii)), we have that $(K, v) \preccurlyeq_\exists (L, w)$ as valued fields. As (K, v, δ) is differentially henselian, we may apply Theorem 4.4.9 to conclude that $(K, v, \delta) \preccurlyeq_\exists (L, w, \partial)$ as valued-differential fields as well. □

This reproves a special case of Theorem 4.1.11. We now consider other formulations of the Ax-Kochen/Ershov principle. Let us begin with a definition:

Definition 4.5.2. Let $G \subseteq H$ be abelian groups. We say that G is *pure* in H if the quotient H/G is torsion-free, equivalently, for any positive integer n, and $h \in H$, if $nh \in G$, then $h \in G$.

Recall the following embedding result in the non-differential henselian setting:

Lemma 4.5.3 ([39, Lemma 4.6.2])**.** *Let $(K_1, v_1), (K_2, v_2)$ be henselian fields of equicharacteristic 0, with respective henselian subfields $(K_1', v_1'), (K_2', v_2')$. Let $\sigma' : (K_1', v_1') \to (K_2', v_2')$ be an isomorphism of valued fields, and let $\sigma_r' : K_1'v_1' \to K_2'v_2'$ and $\sigma_g' : v_1'K_1' \to v_2'K_2'$ be the induced isomorphisms of the residue field and value group, respectively. Suppose $v_1'K_1'$ is pure in v_1K_1. Then, if K_2 is $|K_1|^+$-saturated, and if σ_r' and σ_g' extend to embeddings σ_r and σ_g of K_1v_1 and v_1K_1 into K_2v_2 and v_2K_2, respectively, then σ' also extends to an embedding $\sigma : K_1 \to K_2$ of valued fields, inducing σ_r and σ_g.*

We will apply the above lemma along with the relative embedding theorem to show a similar result for differentially henselian fields.

Lemma 4.5.4. *Let (K_1, v_1, δ_1) and (K_2, v_2, δ_2) be valued-di erential fields of equicharacteristic 0, where $(K_1, .v_1)$ is henselian and (K_2, v_2, δ_2) is di erentially henselian. Let (K_1', v_1', δ_1') and (K_2', v_2', δ_2') be valued-di erential subfields of K_1 and K_2, respectively, henselian as valued fields. Let $\sigma' : K_1' \to K_2'$ be an isomorphism of valued-di erential fields, and let $\sigma_r' : K_1'v_1' \to K_2'v_2'$ and $\sigma_g' : v_1'K_1' \to v_2'K_2'$ be the induced isomorphisms of their residue fields and value groups, respectively. Suppose that $v_1'K_1'$ is pure in v_1K_1. Then, if (K_2, v_2, δ_2) is $|K_1|^+$-saturated, and if σ_r' and σ_g' extend to embeddings $\sigma_r : K_1v_1 \to K_2v_2$ and $\sigma_g : v_1K_1 \to v_2K_2$, respectively, then σ' extends to an embedding $\sigma : K_1 \to K_2$ of valued-di erential fields which induces σ_r and σ_g on the residue field and value group, respectively.*

Proof. By the non-differential version of the embedding lemma (Lemma 4.5.3), there is an embedding $\sigma_{\text{alg}} : (K_1, v_1) \to (K_2, v_2)$ of pure valued fields extending σ which induces σ_r and σ_g. As (K_2, v_2, δ_2) is $|K_1|^+$-saturated, there exists $\gamma \in v_2K_2$ such that $\gamma > \sigma_g(v_1K_1)$.

Now, by Theorem 4.4.10, there is a valued-differential field embedding $\sigma : (K_1, v_1, \delta_1) \to (K_2, v_2, \delta_2)$ with $\sigma\restriction_{K_1} = \sigma'$ and for every $a \in K_1$, $v_2(\sigma(a) - \sigma_{\text{alg}}(a)) > \gamma$. In particular, $v_2(\sigma(a) - \sigma_{\text{alg}}(a)) > v_2(\sigma_{\text{alg}}(a))$ for every a, thus $v_2(\sigma(a)) = v_2(\sigma_{\text{alg}}(a))$ for every a. Hence σ and σ_{alg} induce the same embedding of value groups, i.e. they both induce σ_g. Similarly, σ also induces σ_r on the residue field, as required. $\square$

Applying our differential embedding lemma, we now adapt Lemma 6.1 of [30] to the differential case to show that the class of equicharacteristic 0 differentially henselian fields is ***relatively subcomplete***.

Proposition 4.5.5. *The class of di erentially henselian fields of equicharacteristic 0 is relatively subcomplete. That is, if* (K, v, δ) *and* (L, w, ∂) *are differentially henselian fields of equicharacteristic 0, and if* (F, u, d) *is a common valued-di erential subfield such that* uF *is pure in* vK*, then if* $Kv \equiv_{Fu} Lw$ *and* $vK \equiv_{uF} wL$*, as pure fields and ordered abelian groups, respectively, then we also have* $(K, v, \delta) \equiv_F (L, w, \partial)$ *as valued-di erential fields.*

Proof. Let (K, v, δ) and (L, w, ∂) be differentially henselian fields with a common valued-differential subfield (F, u, d), and suppose that $Kv \equiv_{Fu} Lw$ and $vK \equiv_{uF} wL$. Further, suppose that uF is pure in vK.

As the henselisation of a valued field is an immediate extension, replacing F with its henselisation does not change our assumptions on the residue field and value group. Thus, we may assume that (F, u) is henselian.

We may also assume that (K, v, δ) and (L, w, ∂) are $|F|^+$-saturated, by replacing them with suitable elementary extensions of $(K, v, \delta)_F$ and $(L, w, \partial)_F$. Using the differential embedding lemma, we will construct chains $(K_i, v_i, \delta_i)_{i<\omega}$ and $(L_i, w_i, \partial_i)_{i<\omega}$ of valued-differential subfields of K and L, respectively, along with a chain of valued-differential isomorphisms $\sigma_i : K_i \to L_i$.

Let $(K_0, v_0, \delta_0) = (F, u, \mathrm{d}) = (L_0, w_0, \partial_0)$, and $\sigma_0 : K_0 \to L_0$ be the identity. We perform a back-and-forth construction as follows:

Suppose that we have constructed, for some $n < \omega$, the valued-differential fields $(K_{2n}, v_{2n}, \delta_{2n}), (L_{2n}, w_{2n}, \partial_{2n})$ and the isomorphism $\sigma_{2n} : K_{2n} \to L_{2n}$. We will assume that (K_{2n}, v_{2n}) and (L_{2n}, w_{2n}) are henselian as pure valued fields and of cardinality $|F|$, and the induced embeddings $\sigma^r_{2n} : K_{2n}v_{2n} \to L_{2n}w_{2n}$ and $\sigma^g_{2n} : v_{2n}K_{2n} \to w_{2n}L_{2n}$ of the residue field and value group, respectively, are elementary.

Let $(K_{2n+1}, v_{2n+1}, \delta_{2n+1})$ be an elementary substructure of $(K, v, \delta)_F$ of cardinality $|F|$, containing K_{2n}, such that the extensions $K_{2n}v_{2n} \subseteq K_{2n+1}v_{2n+1}$ and $v_{2n}K_{2n} \subseteq v_{2n+1}K_{2n+1}$ are elementary. Then, $(K_{2n+1}v_{2n+1})_{Fu} \equiv Lw_{Fu}$, and by saturation, there exists an elementary embedding $\sigma^r_{2n+1} : K_{2n+1}v_{2n+1} \to Lw$ extending the embedding σ^r_{2n}. Similarly, there is an elementary embedding $\sigma^g_{2n+1} : v_{2n+1}K_{2n+1} \to wL$ extending the embedding σ^g_{2n}. Further, as $v_{2n}K_{2n} \preccurlyeq v_{2n+1}K_{2n+1}$, we have that $v_{2n}K_{2n}$ is pure in $v_{2n+1}K_{2n+1}$.

We can now apply the differential embedding lemma (Lemma 4.5.4). This gives an embedding $\sigma_{2n+1} : (K_{2n+1}, v_{2n+1}, \delta_{2n+1}) \to (L, w, \partial)$ extending σ_{2n} and inducing σ^g_{2n+1} and σ^r_{2n+1}. Denote the image of σ_{2n+1} by $(L_{2n+1}, w_{2n+1}, \partial_{2n+1})$. By exchanging the roles of K and L, in the even steps we construct the embedding $(\sigma_{2n+2})^{-1} : L_{2n+2} \to K_{2n+2}$ in the same way.

We now observe that the chain $(K_{2n+1}, v_{2n+1}, \delta_{2n+1})_{n<\omega}$ is an elementary chain of elementary substructures of $(K, v, \delta)_F$. Similarly, we also have that $(L_{2n+2}, w_{2n+2}, \partial_{2n+2})_{n<\omega}$ is an elementary chain of elementary substructures of $(L, w, \partial)_F$. Thus, by taking $\hat{K} = \bigcup K_n$, $\hat{L} = \bigcup L_n$ and $\hat{\sigma} = \bigcup \sigma_n$, we obtain an isomorphism between elementary substructures of $(K, v, \delta)_F$ and $(L, w, \partial)_F$. Thus, we conclude that $(K, v, \delta) \equiv_F (L, w, \partial)$, as required. □

As a direct consequence, we obtain that

Corollary 4.5.6. *Let* (K, v, δ) *be a di erentially henselian field of equicharacteristic 0. Suppose that* vK *and* Kv *are model complete as an ordered abelian group and a pure field, respectively. Then,* (K, v, δ) *is model complete as a di erential-valued field.*

Proof. Let $(K, v, \delta) \subseteq (L, w, \partial)$ be differentially henselian fields of equicharacteristic 0, such that $(K, v, \delta) \equiv (L, w, \partial)$. Assume that $Kv \equiv Lw$ and $vK \equiv wL$ are model complete. Then, we have that $Kv \preccurlyeq Lw$ and $vK \preccurlyeq wL$. Equivalently, $Kv \equiv_{Kv} Lw$ and $vK \equiv_{vK} wL$. By relative subcompleteness, we have that $(K, v, \delta) \equiv_K (L, w, \partial)$ and thus $(K, v, \delta) \preccurlyeq (L, w, \partial)$. Thus (K, v, δ) is model complete. □

From relative subcompleteness, we also obtain a differential version of the 'relative model completeness' version of the Ax-Kochen/Ershov theorem in equicharacteristic 0.

Theorem 4.5.7. *Let* $(K, v, \delta) \subseteq (L, w, \partial)$ *be di erentially henselian fields of equicharacteristic 0, and suppose that* $Kv \preccurlyeq Lw$ *and* $vK \preccurlyeq wL$*. Then,* $(K, v, \delta) \preccurlyeq (L, w, \partial)$ *as valued-di erential fields.*

Proof. The conditions $Kv \preccurlyeq Lw$ and $vK \preccurlyeq wL$ are equivalent to $Kv \equiv_{Kv} Lw$ and $vK \equiv_{vK} wL$, respectively. Further, $vK \preccurlyeq wL$ implies that vK is pure in wL. Thus, by Proposition 4.5.5, we have that $(K, v, \delta) \equiv_K (L, w, \partial)$, i.e. $(K, v, \delta) \preccurlyeq (L, w, \partial)$. □

We are now able to show the 'elementary equivalence' version of the Ax-Kochen/Ershov theorem for differentially henselian fields of equicharacteristic 0. The back-and-forth construction construction in the proof below is largely similar to the one in the classical case.

Theorem 4.5.8. *Let* (K, v, δ) *and* (L, w, ∂) *be di erentially henselian fields of equicharacteristic 0 with* $Kv \equiv Lw$ *and* $vK \equiv wL$*. Then,* $(K, v, \delta) \equiv (L, w, \partial)$ *as valued-di erential fields.*

Proof. We first assume that (K, v, δ) and (L, w, ∂) are $\aleph_1$-saturated, by replacing them with elementary extensions if necessary. We will use the differential embedding lemma to perform a back-and-forth construction to produce isomorphic elementary substructures of (K, v, δ) and (L, w, ∂). We construct ascending chains $(K_i, v_i, \delta_i)_{i<\omega}$ and (L_i, w_i, ∂_i) of (K, v, δ) and $(L, w\partial)$, respectively, along with a chain of isomorphisms $\sigma_i : (K_i, v_i, \delta_i) \to (L_i, w_i, \partial_i)$.

Let (K_0, v_0, δ_0) and (L_0, w_0, ∂_0) be $\mathbb{Q}$ equipped with the trivial derivation and valuation, and let $\sigma_0 : K_0 \to L_0$ be the identity. Suppose we have constructed $(K_{2n}, v_{2n}, \delta_{2n})$, $(L_{2n}, w_{2n}, \partial_{2n})$ and the isomorphism σ_{2n}. Assume that K_{2n} and L_{2n} are countable, and let $\sigma^g_{2n} : v_{2n}K_{2n} \to w_{2n}L_{2n}$ and $\sigma^r_{2n} : K_{2n}v_{2n} \to L_{2n}w_{2n}$ be the induced isomorphisms of value groups and residue fields, respectively. Further, if $n \neq 0$, assume that $v_{2n}K_{2n}$, $w_{2n}L_{2n}$, $K_{2n}v_{2n}$ and $L_{2n}w_{2n}$ are elementary substructures of vK, wL, Kv and Lw, respectively.

Let $(K_{2n+1}, v_{2n+1}, \delta_{2n+1})$ be a countable elementary substructure of (K, v, δ) such that $v_{2n}K_{2n} \preccurlyeq v_{2n+1}K_{2n+1} \preccurlyeq vK$ and $K_{2n}v_{2n} \preccurlyeq K_{2n+1}v_{2n+1} \preccurlyeq Kv$. By elementary equivalence of the value groups and residue fields and $\aleph_1$-saturation of (L, w, ∂), there are elementary embeddings $\sigma^g_{2n+1} : v_{2n+1}K_{2n+1} \to wL$ and $\sigma^r_{2n+1} : K_{2n+1}v_{2n+1} \to Lw$ extending σ^g_{2n} and σ^r_{2n}.

In the $n = 0$ case, the trivial group is a pure subgroup of v_1K_1, and if $n > 0$, then $v_{2n}K_{2n} \preccurlyeq v_{2n+1}K_{2n+1}$ and therefore is pure. Hence we may apply the differential embedding lemma to obtain an embedding of valued-differential fields $\sigma_{2n+1} : (K_{2n+1}, v_{2n+1}, \delta_{2n+1}) \to (L, w, \partial)$ inducing σ^r_{2n+1} and σ^g_{2n+1}. Denote by $(L_{2n+1}, w_{2n+1}, \partial_{2n+1})$ the image of σ_{2n+1}.

Exchanging the roles of K and L, construct similarly a countable elementary substructure $(L_{2n+2}, w_{2n+2}, \partial_{2n+2})$ of (L, w, ∂) and an embedding $\sigma^{-1}_{2n+2} : (L_{2n+2}, w_{2n+2}, \partial_{2n+2}) \to (K, v, \delta)$ such that the induced embeddings of value groups and residue fields are elementary. Let $(K_{2n+2}, w_{2n+2}, \partial_{2n+2})$ be the image of σ_{2n+2}.

Set $(\hat{K}, \hat{v}, \hat{\delta}) = \bigcup_{n<\omega}(K_n, v_n, \delta_n)$, $(\hat{L}, \hat{w}, \hat{\partial}) = \bigcup_{n<\omega}(L_n, w_n, \partial_n)$ and $\sigma = \bigcup_{n<\omega} \sigma_n$. Then, $(\hat{K}, \hat{v}, \hat{\delta})$ and $(\hat{L}, \hat{w}, \hat{\partial})$ are countable elementary substructures of (K, v, δ) and (L, w, ∂), respectively. Further, $\sigma : \hat{K} \to \hat{L}$ is, by construction, an isomorphism of valued-differential fields. Since (K, v, δ) and (L, w, ∂) have isomorphic elementary substructures, we conclude that $(K, v, \delta) \equiv (L, w, \partial)$ as required. □

Remark. The above theorem is also a consequence of Corollary 4.1.7 for differentially henselian fields of equicharacteristic 0.

The Unramified Mixed Characteristic Case

In this section, we consider Ax-Kochen/Ershov type results for differentially henselian fields in the unramified mixed characteristic case. To do this, we will adapt results due to Anscombe and Jahnke in [1], where the authors prove various embedding properties for Cohen rings, which are applicable to unramified mixed characteristic henselian valued fields. The AKE theorem for unramified differentially henselian fields then follows from an adaptation of the arguments in [1] and the application of the corresponding AKE principle for differentially henselian fields in the equicharacteristic 0 case.

We begin by recalling some of the background and terminology introduced in [1].

Recall that a ring A is ***local*** if it has a unique maximal ideal (usually denoted $\mathfrak{m}$). The ***residue field*** of a local ring A, usually denoted k is the quotient ring $A/\mathfrak{m}$, and the quotient map is called the ***residue map***, denoted $\mathrm{res} : A \to k$. The ***residue characteristic*** is the characteristic of k. We will think of a local ring A as the pair (A, k), where k is the residue field of A. A local ring has a natural topology called the ***local topology***, also known as the $\mathfrak{m}$-adic topology, which is generated by the base of neighbourhoods around 0 given by $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \ldots$.

Definition 4.5.9 ([1, Definition 2.3]). A ***pre-Cohen ring*** is a local ring (A, k) such that A is Noetherian and the maximal ideal $\mathfrak{m}$ is pA, where p is the residue characteristic.

In particular, pre-Cohen rings have residue characteristic p, and have either characteristic 0 or p^m for some m.

Lemma 4.5.10 ([1, Lemma 2.4]). *Let (A, k) be a pre-Cohen ring. The following are equivalent:*

(i) A has characteristic 0,

(ii) A is an integral domain,

(iii) A is a valuation ring.

Definition 4.5.11 ([1, Definition 2.5]). We say that a pre-Cohen ring (A, k) is ***strict*** if it satisfies any of the equivalent conditions of 4.5.10.

Definition 4.5.12 ([1, Definition 2.7]). A *Cohen ring* is a pre-Cohen ring (A, k) which is complete with respect to the local topology.

Example 4.5.13. The ring of p-adic integers $\mathbb{Z}_p$ is a strict Cohen ring.

The crucial results we will need are the following:

Theorem 4.5.14 ([1, Corollary 6.4 (Relative Embedding Theorem)]). *Let (A_1, k_1) and (A_2, k_2) be two Cohen rings, and let (A_0, k_0) be a common Cohen subring. Suppose that $\varphi_k : k_1 \to k_2$ is an embedding over k_0, and that both k_1/k_0 and $k_2/\varphi_k(k_1)$ are separable. Then, there is an embedding $\varphi : A_1 \to A_2$ inducing φ_k and fixing A_0 pointwise. Moreover, if φ_k is an isomorphism then φ is an isomorphism.*

Theorem 4.5.15 ([1, Corollary 6.6, Cohen Structure Theorem v.2]). *Let (A_1, k_1) and (A_2, k_2) be Cohen rings of the same characteristic, and let $\varphi_k : k_1 \to k_2$ be an isomorphism of residue fields. Then, there exists an isomorphism of Cohen rings $\varphi : A_1 \to A_2$ inducing φ_k.*

We now prove a differential version of Theorem 8.3 of [1], by adapting their proof with the differential version of the embedding theorem above.

Theorem 4.5.16. *Let k be a field of characteristic p, and Γ be an ordered abelian group with a minimal positive element. The $\mathcal{L}_{\mathrm{vd}}$-theory T of unramified di erentially henselian fields with residue field and value group elementarily equivalent to k and Γ, respectively, is complete.*

Proof. Let $(K_1, v_1, \delta_1), (K_2, v_2, \delta_2)$ be models of T. By the Keisler-Shelah Theorem, we replace (K_1, v_1, δ_1) and (K_2, v_2, δ_2) with suitable ultrapowers such that both (K_1, v_1, δ_1) and (K_2, v_2, δ_2) are $\aleph_1$-saturated, and that there are isomorphisms $\varphi_r : K_1v_1 \to K_2v_2$ and $\varphi_g : v_1K_1 \to v_2K_2$ of residue fields and value groups, respectively.

Let w_i denote the finest proper coarsening of the valuation v_i on K_i, given by the quotient of v_iK_i by the convex subgroup $\mathbb{Z}$ generated by their respective minimal positive elements. Denote the residue field of K_iw_i by k_i, and let $\bar{v}_i$ denote the valuation induced by v_i on k_i. Note in particular that $\mathrm{char}(k_i) = 0$, and that (K_i, w_i) are of equicharacteristic 0.

By $\aleph_1$-saturation, the $(k_i, \bar{v}_i)$ are spherically complete and hence the valuation rings $\mathcal{O}_{\bar{v}_i}$ are strict Cohen rings. By the Cohen Structure Theorem (Theorem 4.5.15), there is an isomorphism $\varphi : O_{\bar{v}_1} \to O_{\bar{v}_2}$ which induces the isomorphism φ_r of residue fields. Further, φ induces an isomorphism of the value groups $\bar{\varphi}_g : \bar{v}_1 k_1 \to \bar{v}_2 k_2$, as any isomorphism of ordered abelian groups sends the minimal positive element, and hence the convex subgroup generated by the minimal positive element, to the corresponding copy in the codomain.

Since (K_i, v_i, δ_i) are differentially henselian, their coarsenings (K_i, w_i, δ_i) remain differentially henselian, and are of equicharacteristic 0. Thus, we may apply Theorem 4.5.8 to obtain that $(K_1, w_1, \delta_1) \equiv (K_2, w_2, \delta_2)$ as valued-differential fields. Further, by [21, Corollary 2], we have that the valuations v_1, v_2 are 0-definable in K_1, K_2, respectively, in the language of rings. Thus, we conclude that $(K_1, v_1, \delta_1) \equiv (K_2, v_2, \delta_2)$ as valued-differential fields. □

In other words, we have the following Ax-Kochen/Ershov relative completeness theorem:

Theorem 4.5.17 (Relative Completeness). *Let (K, v, δ) and (L, w, ∂) both be unramified di erentially henselian fields. Then, $(K, v, \delta) \equiv (L, w, \partial)$ as valued-di erential fields if and only if $(K, v) \equiv (L, w)$ as pure valued fields. That is, the theory of an unramified di erentially henselian field is determined by the theory of the underlying valued field.*

The above recovers Theorem 4.1.7 in the case of unramified differentially henselian fields. We also want a relative version, where we have completeness over a common subfield. This is an adaptation of Theorem 9.2 of [1].

Theorem 4.5.18 (Relative Model Completeness). *Let $(K, v, \delta) \subseteq (L, w, \partial)$ be unramified di erentially henselian fields such that the induced embeddings $Kv \subseteq Lw$ and $vK \subseteq wL$ are elementary as pure fields and ordered abelian groups, respectively. Then, the inclusion $(K, v, \delta) \subseteq (L, w, \partial)$ is elementary as valued-di erential fields.*

Proof. We will show the following modified statement: let (K_1, v_1, δ_1) and (K_2, v_2, δ_2) be unramified differentially henselian fields with elementarily equivalent value group and residue field, and let (K_0, v_0, δ_0) be a common differentially henselian subfield of K_1, K_2 such that $K_0 v_0 \preccurlyeq K_1 v_1, K_2 v_2$ and $v_0 K_0 \preccurlyeq v_1 K_1, v_2 K_2$. Then, (K_1, v_1, δ_1) and (K_2, v_2, δ_2) are elementarily equivalent over (K_0, v_0).

By the Keisler-Shelah Theorem, we may replace each field with a suitable ultrapower such that $(K_0, v_0, \delta_0), (K_1, v_1, \delta_1)$ and (K_2, v_2, δ_2) are all $\aleph_1$-saturated, and there are isomorphisms $\varphi_r : K_1 v_1 \to K_2 v_2$ and $\varphi_g : v_1 K_1 \to v_2 K_2$ over $K_0 v_0$ and $v_0 K_0$, respectively. Let w_i denote the finest proper coarsening of v_i, i.e. the quotient by the convex subgroup generated by the minimum positive element, denote the reside field K_iw_i by k_i, and let $\bar{v}_i$ denote the valuation on k_i induced by v_i.

We observe that these coarsenings are compatible, i.e. (K_0, w_0, δ_0) is a common valued-differential subfield of (K_1, w_1, δ_1) and (K_2, w_2, δ_2), and $(K_0w_0, \bar{v}_0)$ is a common valued subfield of $(K_1w_1, \bar{v}_1)$ and $(K_2w_2, \bar{v}_2)$. Further, φ_g induces an isomorphism $\hat{\varphi}_g : w_1K_1 \to w_2K_2$ fixing w_0K_0, as φ_g restricts to an isomorphism of the convex subgroups generated by the minimum positive elements of v_1K_1 and v_2K_2.

By $\aleph_1$-saturation, the valuation rings of each $(K_iw_i, \bar{v}_i)$ are strict Cohen rings. Apply Theorem 4.5.14 to obtain an isomorphism $\varphi : (K_1w_1, \bar{v}_1) \to (K_2w_2, \bar{v}_2)$ over K_0w_0 inducing φ_r. Thus, we obtain that (K_1, w_1, δ_1) and (K_2, w_2, δ_2) are differentially henselian fields of equicharacteristic 0, with value groups and residue fields isomorphic over w_0K_0 and k_0, respectively.

Thus, by Lemma 4.5.5, we have that (K_1, w_1, δ_1) and (K_2, w_2, δ_2) are elementarily equivalent over K_0. Again by [21, Corollary 2], we have that v_i is 0-definable in K_i by the same formula in the language of rings, thus $(K_1, v_1) \equiv_{K_0} (K_2, v_2)$, as required.

Now, if $(K, v, \delta) \subseteq (L, w, \partial)$ are unramified differentially henselian fields with $Kv \preccurlyeq Lw$ and $vK \preccurlyeq wL$, we obtain from the above statement that $(K, v, \delta) \equiv_K (L, w, \partial)$ as valued-differential fields. Thus, (K, v, δ) is elementary in (L, w, ∂), as required. □

We also obtain the following relative model-completeness result:

Corollary 4.5.19. *Let (K, v, δ) be an unramified di erentially henselian field. Then, if Kv and vK are model complete as a pure field and ordered abelian group, respectively, then (K, v, δ) is model complete as a valued-di erential field.*

Now, we move to adapting the embedding lemma [1, Proposition 10.1] to differentially henselian fields. The classical version states the following:

Proposition 4.5.20 ([1, Proposition 10.1], Embedding Lemma). *Let (L_1, v_1) and (L_2, v_2) be extensions of (K, v), where all three are $\aleph_1$-saturated unramified henselian fields. Suppose that L_1v_1/Kv is separable, and v_1L_1/vK is torsion-free. Further, assume that (L_2, v_2) is $|L_1|^+$-saturated, and there are embeddings $\varphi_r : L_1v_1 \to L_2v_2$ and $\varphi_g : v_1L_1 \to v_2L_2$ over Kv and vK, respectively. Suppose that $L_2v_2/\varphi_r(L_1v_1)$ is separable. Then, there is an embedding $\varphi : (L_1, v_1) \to (L_2, v_2)$ over K inducing φ_r and φ_g on the residue field and value* *group, respectively. Further, if φ_r and φ_g are elementary embeddings, then φ is also an elementary embedding.*

We will prove the following differential version:

Proposition 4.5.21. *Let (L_1, v_1, δ_1) and (L_2, v_2, δ_2) be extensions of (K, v, δ), where all three are $\aleph_1$-saturated unramified di erentially henselian fields. Suppose that L_1v_1/Kv is separable, and v_1L_1/vK is torsion-free. Further, assume that (L_2, v_2, δ_2) is $|L_1|^+$-saturated, and there are embeddings $\varphi_r : L_1v_1 \to L_2v_2$ and $\varphi_g : v_1L_1 \to v_2L_2$ over Kv and vK, respectively. Suppose that $L_2v_2/\varphi_r(L_1v_1)$ is separable. Then, there is an embedding $\varphi : (L_1, v_1, \delta_1) \to (L_2, v_2, \delta_2)$ over K inducing φ_r and φ_g on the residue field and value group, respectively. Further, if φ_r and φ_g are elementary embeddings, then φ is also an elementary embedding.*

Proof. By the classical Embedding Lemma, there is an embedding $\psi : L_1 \to L_2$ over K as valued fields. Thus, by Theorem 4.4.10, there is a differential embedding $\varphi : L_1 \to L_2$ over K such that for any $a \in K$, $\varphi(a) \in B_\gamma(\psi(a))$, where $\gamma > v_2\varphi(L_1)$. In particular, φ also induces the embeddings φ_r and φ_g.

If φ_r and φ_g are elementary, then, identifying L_1 with its image in L_2 under φ, v_1L_1 and L_1v_1 are elementary substructures of v_2L_2 and L_2v_2, respectively. Thus, by Theorem 4.5.18, L_1 is an elementary substructure of L_2, i.e. φ is an elementary embedding. □

We now have everything in order to adapt [1, Theorem 10.2], an AKE principle for existential closure, to the differential case.

Theorem 4.5.22. *Let $(K, v, \delta) \subseteq (L, w, \partial)$ be an extension of unramified differentially henselian fields, such that Kv and Lw have the same finite degree of imperfection. If vK and Kv are existentially closed in wL and Lw, as an ordered abelian group and pure field, respectively, then (K, v, δ) is existentially closed in (L, w, ∂) as a valued-di erential field.*

The above follows from the proof of Theorem 10.2 in [1] without significant modification. We reproduce the proof below for the convenience of the reader.

Proof. By replacing with appropriate ultrapowers if necessary, we may assume that (K, v, δ) and (L, w, ∂) are $\aleph_1$-saturated. Let (K^*, v^*, δ^*) be a $|L|^+$-saturated elementary extension of (K, v, δ). As vK and Kv are existentially closed in wL and Lw, respectively, there are embeddings $\varphi_g : wL \to v^*K^*$ and $\varphi_r : Lw \to K^*v^*$. Let $B \subseteq Kv$ be a p-basis. As Kv is existentially closed in

Lw, B is also a p-basis of Lw. Since K^*v^* is an elementary extension of K, we also have that B is a p-basis of K^*v^*. In particular, $K^*v^*/\varphi_r(Lw)$ is separable. Further, since vK is existentially closed in wL, wL/vK is torsion-free. Finally, applying Proposition 4.5.21, we find an embedding $(L, w, \partial) \to (K^*, v^*, \delta^*)$ of valued-differential fields inducing φ_r and φ_g, as required. □

We will now make an adaptation to show a relative subcompleteness result for unramified differentially henselian fields.

Proposition 4.5.23. *Let (K, v_K, δ) and (L, v_L, ∂) be unramified di erentially henselian fields, and let (F, v_F, ∂_F) be a common valued-di erential subfield such that $v_F F$ is pure in $v_K K$. Then, if $(K, v_K) \equiv_F (L, v_L)$ as valued fields, then $(K, v_K, \partial_K) \equiv_F (L, v_L, \partial_L)$ as valued-di erential fields.*

Proof. Applying the Keisler-Shelah theorem, we may replace K, L, F with appropriate ultrapowers such that (K, v_K) and (L, v_L) are isomorphic as valued fields over (F, v_F).

We begin by considering the finest proper coarsenings of v_K, v_L, v_F, which we denote by w_K, w_L, w_F, respectively. This is the result of the quotient of their respective value group by the convex subgroup generated by the minimal positive element 1. We now observe that (F, w_F, ∂_F) is a common valued-differential subfield of (K, w_K, ∂_K) and (L, w_L, ∂_L). Further, (K, w_K, ∂_K) and (L, w_L, ∂_L) are differentially henselian fields of equicharacteristic 0.

We claim that $w_F F$ is a pure subgroup of $w_K K$. As $\mathbb{Z} = \langle 1 \rangle$ is a common convex subgroup, and as $v_F F$ is pure in $v_K K$, we have that $v_F F/\mathbb{Z} = w_F F$ remains pure in $v_K K/\mathbb{Z} = w_K K$. Now, we claim that $w_K K \equiv_{w_F F} w_L L$ and $K w_K \equiv_{F w_F} L w_L$ as ordered abelian groups and rings, respectively. In fact, by our assumption on isomorphism above, we have that $w_K K$ and $w_L L$ are isomorphic over $w_F F$ as ordered abelian groups, and $K w_K$ and $L w_L$ are isomorphic over $F w_F$ as rings, respectively.

Now, we may apply Lemma 4.5.5 to obtain that (K, w_K, ∂_K) and (L, w_L, ∂_L) are elementarily equivalent over F. Finally, as v_K and v_L are 0-definable (in the language of rings) with the same formula, we also conclude that $(K, v_K, \partial_K) \equiv_F (L, v_L, \partial_L)$, as required. □

4.6 Angular Components and Quantifier Elimination

In this section, we adapt relative quantifier elimination results for equicharacteristic 0 henselian valued fields in the Pas language to the differential context. We generalise results by Borrata ([10, Corollary 4.3.27]) from the context of closed ordered differential valued fields to differentially henselian fields. We begin by recalling a definition:

Definition 4.6.1. Let (K, v) be a valued field. An *angular component map* for (K, v) is a map $\mathrm{ac} : K \to Kv$ satisfying:

(1) $\mathrm{ac}(x) = 0$ if and only if $x = 0$;

(2) $\mathrm{ac}\restriction_{K^\times} : K^\times \to Kv^\times$ is a group homomorphism, and

(3) $\mathrm{ac}\restriction_{\mathcal{O}_v^\times} = \mathrm{res}\restriction_{\mathcal{O}_v^\times}$.

An ac-valued field is a valued field equipped with an angular component map. Similarly, an ac-valued differential field is an ac-valued field equipped with a derivation.

Example 4.6.2. Let K be a field of characteristic 0, and consider the field of formal Laurent series $K((t))$. For a nonzero $a \in K((t))$, we set $\mathrm{ac}(a) = a_{v(a)}$, i.e. $\mathrm{ac}(a)$ is the first non-zero coefficient in a.

Note. Not every henselian valued field admits an angular component map.

If we have a cross section $s : vK \to K^\times$, that is, s is a group homomorphism satisfying $v \circ s = \mathrm{id}_{vK}$, then we can define an angular component map by setting

$$\mathrm{ac}(x) = \begin{cases} \mathrm{res}(x/s(v(x))) & \text{for } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.6.3. The angular component map for $K((t))$ in the example above is given by the cross-section $s : \mathbb{Z} \to K((t))$ defined by $n \mapsto t^n$.

Lemma 4.6.4. *Let $(K, v) \subseteq (L, w)$ be an unramified extension of valued fields, i.e. $vK = wL$, and let ac be an angular component map for (K, v). Then, there is a unique angular component map on L extending ac.*

Proof. Let $b \in L^\times$. Since the extension is unramified, there is $a \in K^\times$ such that $v(a) = w(b)$. Then, $w(b/a) = 0$, and thus $b/a \in \mathcal{O}_w^\times$. If ac is an angular component map on (L, w), then we have that $\mathrm{ac}(b/a) = \mathrm{res}(b/a)$. By multiplicativity of ac, we have that $\mathrm{ac}(b) = \mathrm{ac}(b/a)\mathrm{ac}(a) = \mathrm{res}(b/a)\mathrm{ac}(a)$, which is uniquely determined by ac on K. This gives the unique extension of ac to L. □

We work in the three-sorted ***Denef-Pas language*** $\mathcal{L}_{\mathrm{Pas}}$, with a sort K for the valued field (the 'field sort'), a sort Γ for the value group, and a sort k for the residue field. We add the usual symbols for operations and relations on each of the sorts, i.e. the language of rings in the sort K, the language of ordered abelian groups in the sort Γ and the language of rings for the sort k. We also include symbols $v : K \to \Gamma$ for the valuation, and a symbol $\mathrm{ac} : K \to k$ for an angular component map. Note that we do not need to include the residue map as it is quantifier-free definable from ac as $\mathrm{ac} = \mathrm{res}$ on $\mathcal{O}^{\times}$, and we can set $\mathrm{res}(x) = 0$ otherwise.

We denote by $\mathcal{L}^{\delta}_{\mathrm{Pas}}$ the language $\mathcal{L}_{\mathrm{Pas}}$ augmented with a unary function symbol $\delta : K \to K$ in the field sort. That is, $\mathcal{L}^{\delta}_{\mathrm{Pas}}$ is the ***language of di erential*** ac***-valued fields***.

The relative quantifier elimination result in the classical setting is as follows:

Theorem 4.6.5 ([37]). *The theory of henselian* ac*-valued fields of equicharacteristic 0 eliminates K-quantifiers in the language $\mathcal{L}_{\mathrm{Pas}}$.*

We claim that the appropriate analogue holds in the differential case. From the proof of the classical theorem found in [20], we can extract the following lemma:

Lemma 4.6.6. *Let (K, v, ac_K) and (L, w, ac_L) be henselian* ac*-valued fields, considered as $\mathcal{L}_{\mathrm{Pas}}$-structures. Suppose that K is countable, and (L, w, ac) is $\aleph_1$-saturated. Let (A, u, ac_A) be an $\mathcal{L}_{\mathrm{Pas}}$-substructure of (K, v, ac_K), and let $f : (A, u, \mathrm{ac}_A) \to (L, w, \mathrm{ac}_L)$ be an $\mathcal{L}_{\mathrm{Pas}}$-embedding such that the induced maps f_g and f_r on the value groups and residue fields are elementary. Then, f extends to an $\mathcal{L}_{\mathrm{Pas}}$-embedding of (K, v, ac_K) into (L, w, ac_L).*

We require one more preparatory lemma before we proceed to the relative quantifier elimination result:

Lemma 4.6.7. *Let (K, v, ac_K) and (L, w, ac_L) be* ac*-valued fields. Let $\varphi : K \to L$ be an* ac*-valued field embedding, and $\psi : K \to L$ be an embedding of valued fields such that $w(\varphi(a) - \psi(a)) > w(\varphi(K))$ for any $a \in K$. Then, for any $b \in K$, $\mathrm{ac}_L(\varphi(b)) = \mathrm{ac}_L(\psi(b))$, and ψ is also an embedding of* ac*-valued fields.*

Proof. We begin by showing that, for $a \in L^\times$ and $b \in L$ such that $w(b) > w(a)$, we have that $\mathrm{ac}_L(a) = \mathrm{ac}_L(a+b)$. By multiplicativity, we have that:

$$\mathrm{ac}_L(a+b)(\mathrm{ac}_L(a^{-1})) = \mathrm{ac}_L(1 + a^{-1}b)$$

Since $w(a) < w(b)$, we have $w(a^{-1}b) = w(b) - w(a) > 0$, thus $w(1 + a^{-1}b) = 0)$ and $1 + a^{-1}b \in \mathcal{O}_w^\times$. Thus,

$$\mathrm{ac}_L(1 + a^{-1}b) = \mathrm{res}_w(1 + a^{-1}b) = 1$$

and so $\mathrm{ac}_L(a+b) = \mathrm{ac}_L(a) = \mathrm{ac}_L(a^{-1})^{-1}$ as required.

Now, let φ, ψ be as above. Observe that for any $b \in K$, $w(\varphi(b) - \psi(b)) > w(\varphi(b))$, thus by the above claim, $\mathrm{ac}_L(\psi(b)) = \mathrm{ac}_L(\varphi(b))$. Since φ is an ac-valued field embedding, so is ψ. □

We will now apply the above lemmas to adapt the proof of the classical Pas' theorem found in [20] to the differentially henselian case.

Theorem 4.6.8. *The $\mathcal{L}^\delta_{\mathrm{Pas}}$-theory of di erentially henselian* ac*-valued fields eliminates quantifiers in the field sort.*

Proof. Let $(K, v, \mathrm{ac}_K, \delta)$ and $(L, w, \mathrm{ac}_L, \partial)$ be differentially henselian ac-valued fields. We assume K is countable and $(L, w, \partial, \mathrm{ac}_L)$ is $\aleph_1$-saturated. Let $(A, u, \mathrm{ac}_A, \mathrm{d})$ be an $\mathcal{L}^\delta_{\mathrm{Pas}}$-substructure of K, and $f : A \to L$ be an $\mathcal{L}^\delta_{\mathrm{Pas}}$-embedding, such that the embeddings f_r and f_g of the residue field and value group sorts are elementary with respect to the languages of rings and ordered abelian groups, respectively. It now suffices to show that f extends to an $\mathcal{L}^\delta_{\mathrm{Pas}}$-embedding $K \to L$.

Now, by Lemma 4.6.6, there is an $\mathcal{L}_{\mathrm{Pas}}$-embedding $\varphi : K \to L$ extending f. By $\aleph_1$-saturation of L, there is some $\gamma \in wL$ with $\gamma > w(\varphi(K))$. Now, apply Theorem 4.4.10 to obtain a valued-differential field embedding $\psi : K \to L$ extending f with $w(\varphi(a) - \psi(a)) > \gamma$ for every $a \in K$. By Lemma 4.6.7, ψ also preserves ac, thus is an $\mathcal{L}^\delta_{\mathrm{Pas}}$-embedding, as required. □

4.7 Stable Embeddedness

In this section, we prove a stable-embeddedness result for the value group and residue field of a differentially henselian field in both the equicharacteristic 0 and unramified mixed characteristic cases, adapting analogous results for their pure valued field counterparts.

We recall the definition of stable-embeddedness:

Definition 4.7.1. Let M be an arbitrary first-order structure, and let $P \subseteq M^n$ be a definable set. We say that P is ***stably embedded*** if, for all formulae $\varphi(\bar{x}, \bar{y})$. and all $\bar{b} \in M^{|\bar{y}|}$, $\varphi(M^{n|\bar{x}|}, \bar{b}) \cap P^{|\bar{x}|}$ is P-definable.

To work with the value group and residue field as definable sets in the valued field, for the rest of this section, we consider a valued field K as a three-sorted structure (K, vK, Kv) equipped with the usual languages for each sort, along with functions for the valuation and residue map. Our result and proof are an adaptation of [1, Theorem 11.3] to the differential context, and by transitivity, also of [22, Lemma 3.1]

Theorem 4.7.2. *Let (K, v, δ) be an di erentially henselian field, either of equicharacteristic 0 or mixed characteristic and unramified. Then, the value group vK and residue field Kv are stably embedded as a pure ordered abelian group and pure field, respectively.*

Proof. Let (L, w, ∂) be a sufficiently saturated elementary extension ($|K|^+$-saturated suffices) of (K, v, δ), and let $a, b \in Lw$ have the same type over Kv in the language of rings. We will show that a and b have the same type over K.

Let (L_0, w_0, ∂_0) be an $\aleph_1$-saturated elementary substructure of (L, w, ∂) containing Ka. Since a and b have the same type over Kv, there is an elementary embedding $\varphi_r : L_0w_0 \to L_w$ over Kv with $\varphi_r(a) = b$. Let $\varphi_g : w_0L \to wL$ be the inclusion.

In the equicharacteristic 0 case, we observe that vK is pure in w_0L_0, as vK is elementary in w_0L_0. Thus we may apply Lemma 4.5.4 to find an embedding $\varphi : (L_0, w_0, \partial_0) \to (L, w, \partial)$ inducing φ_r and φ_g.

In the unramified mixed characteristic case, we apply observe that since φ_r is elementary, $Lw/\varphi_r(L_0w_0)$ is a separable extension. Similarly to the equicharacteristic 0 case, vK is pure in w_0L_0. Thus, we apply Proposition 4.5.21 to obtain an embedding $\varphi : (L_0, w_0, \partial_0) \to (L, w, \partial)$ inducing φ_r and φ_g.

By Theorems 4.5.7 and 4.5.18, respectively, as φ_r and φ_g are elementary embeddings, so is φ.

We finally conclude the following:

$$\mathrm{tp}_L(a/K) = \mathrm{tp}_{L_0}(a/K) = \mathrm{tp}_{\varphi(L_0)}(b/K) = \mathrm{tp}_L(b/K),$$

where $\mathrm{tp}_L(a/K)$ denotes the valued-differential field type of a over K in L, as required. This gives that the residue field vK is stably embedded. A similar argument replacing the conjugation of residue field elements with value group elements gives that the value group is also stably embedded. □

We now consider the case of the constant subfield. For a differentially henselian field (K, v, δ), it is not generally the case that the constant subfield (C_K, v) is stably embedded as a pure valued field in K. To see this, consider the following:

Example 4.7.3 ([11, Proposition 2.5])**.** Let $(K, v) \models \mathrm{ACVF}_{0,0}$. Let $(K, v) \prec (L, w)$ be an elementary extension such that K is dense in L. Let $a \in L \setminus K$, and consider the following definable set:

$$D = \{(y, z) \in K^2 : v(a - y) > v(a - z)\}.$$

Suppose that D is definable in with parameters in K. Then, the family of open balls

$$D_z = \{y \in K : (y, z) \in D\}$$

is definable in K. Observe that D_z is a nested family of nonempty balls with empty intersection. However, every model of ACVF is definably maximal, i.e. every definable family of nested non-empty balls has nonempty intersection (see [11, Definition 2.1]).

In particular, where (L, w, ∂) is a model of $\mathrm{DCVF}_{(0,0)}$, taking (K, v) to be its constant subfield equipped with the induced valuation, we have that $(K, v) \preccurlyeq (L, w)$ as pure valued fields, and K is dense in L with respect to the valuation topology. Applying the above, we find that D is not definable with parameters in K.

Thus, we expand the language to include sets definable in the pair of pure valued fields (K, C_K). Let $\mathcal{L}_{\mathrm{vf}}^P$ denote the language $\mathcal{L}_{\mathrm{vf}} \cup \{P\}$, where P is a unary predicate (later to be interpreted as a distinguished subfield). Let K be a differentially henselian field, and endow K with a $\mathcal{L}_{\mathrm{vf}}^P$-structure by interpreting $P(K) = C_K$.

Now, we will equip C_K with the structure induced from the pair (K, C_K). For every $\mathcal{L}_{\mathrm{vf}}^P(K)$-definable subset A of $(C_K)^n$, we let P_A be a new n-ary predicate. Define

$$\mathcal{L}_C = \mathcal{L}_{\mathrm{vf}} \cup \{P_A : A \subseteq (C_K)^n \text{ is } \mathcal{L}_{\mathrm{vf}}^P(K)\text{-definable}\}.$$

We equip C_K with an $\mathcal{L}_C$-structure in the natural way, by interpreting P_A as $A \subseteq (C_K)^n$ for each such predicate P_A.

Lemma 4.7.4. *Let $\mathcal{L}$ be a relational expansion of the language of rings, and let $\mathcal{L}^\delta = \mathcal{L} \cup \{\delta\}$. Let K be an $\mathcal{L}^\delta$-structure which is a di erential field. Suppose A is quantifier-free definable by an $\mathcal{L}^\delta(K)$-formula $\varphi(\bar{x})$. Then, $A \cap (C_K)^{\bar{x}} = B \cap (C_K)^{\bar{x}}$ for some quantifier-free $\mathcal{L}(K)$-definable set B.*

Proof. It is sufficient to prove this statement in the case where A is definable by an atomic $\mathcal{L}^\delta(K)$-formula and taking the relevant boolean combinations.

Every atomic $\mathcal{L}^\delta(K)$-formula $\varphi(\bar{x})$ is of the form $P((f_i(\bar{x}))_{i<n})$, where P is an n-ary relation symbol of $\mathcal{L}$, and $f_i(\bar{x})$ is a differential polynomial in $K\{\bar{x}\}$ for each $i < n$. Let $\bar{x} = (x_0, ..., x_m)$.

Writing $f_i = (f_i)_{\text{alg}}(x_0, x_0', ..., x_0^{(k)}, ..., x_m^{(k)})$, we set

$$g_i(x_0, ..., x_m) = (f_i)_{\text{alg}}(x_0, 0, ..., 0, x_1, 0, ..., 0, x_m, 0, ..., 0).$$

That is, g_i is the algebraic polynomial in $K[\bar{x}]$ obtained from $(f_i)_{\text{alg}}$ by setting $x_i^{(j)} = 0$ for all $j > 0$.

Let $\psi(\bar{x})$ be the $\mathcal{L}(K)$-formula $P(g_0(\bar{x}), ..., g_{n-1}(\bar{x}))$, and let $B = \psi(K^{\bar{x}})$. We claim that $B \cap (C_K)^m = A \cap (C_K)^m$. We observe that for any $\bar{a} \in (C_K)^m$, we have $f_i(\bar{a}) = g_i(\bar{a})$, since every element of $\bar{a}$ is a constant. Thus, we have that

$$K \models \forall \bar{x}((\bar{x} \in (C_K)^m) \to (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))).$$

Finally, we have that $A \cap (C_K)^m = B \cap (C_K)^m$, as required. □

Theorem 4.7.5. *Let (K, v, δ) be a di erentially henselian field. Then, C_K is stably embedded in K as an $\mathcal{L}_C$-structure.*

Proof. Let T be the $\mathcal{L}_{\text{vf}}$-theory of K, and let T^{Mor} be its Morleyisation with corresponding language $\mathcal{L}_{\text{Mor}}$. By Theorem 4.1.6, the theory $T^{\text{Mor}} \cup \text{DH}$ admits quantifier elimination in the language $\mathcal{L}_{\text{Mor}}(\delta)$. Since K (with its $\mathcal{L}_{\text{Mor}}(\delta)$-structure) is a model of $T^{\text{Mor}} \cup \text{DH}$, it eliminates quantifiers in the language $\mathcal{L}_{\text{Mor}}(\delta)$.

Let $\varphi(\bar{x})$ be an $\mathcal{L}_{\text{vd}}(K)$-formula, and let $A = \varphi(K^{\bar{x}}) \cap (C_K)^{\bar{x}}$. As $T^{\text{Mor}} \cup \text{DH}$ has quantifier elimination in $\mathcal{L}_{\text{Mor}}(\delta)$, we have that

$$T^{\text{Mor}} \cup \text{DH} \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

for some quantifier-free $\mathcal{L}_{\text{Mor}}(\delta, K)$-formula $\psi(\bar{x})$.

By Lemma 4.7.4, there is a quantifier-free $\mathcal{L}_{\text{Mor}}(K)$-formula $\chi(\bar{x})$ such that $\chi(K^{\bar{x}}) \cap (C_K)^{\bar{x}} = A$. As χ is a $\mathcal{L}_{\text{Mor}}(K)$-formula, it is equivalent modulo T^{Mor}

to an $\mathcal{L}_{\mathrm{vf}}(K)$-formula $\theta(\bar{x})$, possibly with quantifiers. Thus, A is $\mathcal{L}_{\mathrm{vf}}^{P}$-definable, and so by construction, A is also $\mathcal{L}_C$-definable in C_K. □

4.8 Equivalent Characterisations

In this section, we will show a number of equivalent characterisations of differentially henselian fields, in the style of Theorem 4.3 of [32]. We begin by making a modification to the notion of an (abstract) Taylor morphism for application to differentially henselian fields, namely that a 'valued' Taylor morphism should only move points infinitesimally.

Definition 4.8.1. Let $(K, v, \delta) \subseteq (L, w, \partial)$ be valued-differential fields. Let T be a (K, δ)-Taylor morphism for (L, ∂). We say that T is ***valued***, if, for any differential ring A, ring homomorphism $\varphi : A \to K$ and $a \in A$, we have that

$$w(\varphi(a) - T_\varphi(a)) > vK.$$

Alternatively, we will say that T is a ***valued*** (K, v, δ)***-Taylor morphism for*** (L, w, ∂).

Proposition 4.8.2. *Let* (K, v, δ) *be a valued-di erential field. Then, the standard twisted Taylor morphism* T^* *for* $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ *is valued.*

Proof. This follows by direct computation. Let (A, ∂) be a differential ring, let $\varphi : A \to K$ be a ring homomorphism, and let $a \in A$. Recall that $T^*_\varphi(a)$ is given by the series $\sum_i b_i t^i$, where the coefficients b_i are determined by the following formula:

$$b_i = \frac{1}{i!} \sum_{j \leqslant i} (-1)^{i-j} \binom{i}{j} \delta^{i-j}(\varphi(\partial^j(a))).$$

Observe that $b_0 = \varphi(a)$, and thus $T^*_\varphi(a) - \varphi(a) = \sum_{i \geqslant 1} b_i t^i$. Since $v_t(T^*_\varphi(a) - \varphi(a)) > 0$, we have that $(v \circ v_t)(T^*_\varphi(a) - \varphi(a)) > vK$, as required. □

Lemma 4.8.3. *Let* $(K, v, \delta) \subseteq (L, w, \partial)$ *be valued-di erential fields, where* (K, v) *is henselian as a pure valued field. Suppose that* (K, v, δ) *is existentially closed in* (L, w, ∂) *as valued-di erential fields, and that* (L, w, ∂) *admits a valued* (K, v, δ)*-Taylor morphism. Then* (K, v, δ) *is di erentially henselian.*

Proof. Let $f(x) \in K\{x\}$ be a differential polynomial of order n, and suppose that $\bar{a}$ is an algebraic root of f_{alg} with $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Let $\gamma \in vK$.

Let A be the differential ring $K\{x\}/I(f)$. Define the ring homomorphism $\varphi : A \to K$ by evaluation at $\bar{a}$ (as A is generated as a K-algebra by $x, .., x^{(n)}$).

Consider $T_\varphi^* : A \to L$. As T_φ is differential, $T_\varphi(x)$ is a differential root of f. Since T is valued, we have that $v(T_\varphi(x^{(i)}) - \varphi(x^{(i)})) > vK$, in particular, greater than γ. Hence,

$$(L, w, \partial) \models \exists x(f(x) = 0 \wedge \mathrm{Jet}_n(x) \in B_\gamma(\bar{a})).$$

By existential closure of (K, v, δ) in (L, w, ∂), we obtain $b \in K$ with $f(b) = 0$ and $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$ also. □

For a henselian valued field (K, v), we know that as K is large, K is existentially closed in $K((t))$ as a field. Let $v \circ v_t$ denote the composition of the t-adic valuation v_t on $K((t))$ with the valuation v on the residue field. Then, we have the following:

Proposition 4.8.4. *Let (K, v) be a henselian valued field. Then (K, v) is existentially closed in $(K((t)), v \circ v_t)$.*

Proof. This is a consequence of the proof of Theorem 5.14 of [30]. □

Theorem 4.8.5. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. Then, (K, v, δ) is di erentially henselian if and only if (K, v, δ) is existentially closed in $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$.*

Proof. For the forward direction, suppose (K, v, δ) is differentially henselian. Then, it is existentially closed in any valued-differential field extension in which it is existentially closed as a pure valued field, by Theorem 4.4.9. By Proposition 4.8.4 (K, v) is existentially closed in $(K((t)), v \circ v_t)$, we have that (K, v, δ) is existentially closed in $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$.

For the backwards direction, suppose that (K, v, δ) is existentially closed in $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ as a valued-differential field. By assumption, (K, v) is nontrivially henselian. By Proposition 4.8.2, $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ admits a valued (K, v, δ)-Taylor morphism. By Lemma 4.8.3, we have that (K, v, δ) is differentially henselian, as required. □

Iterating this argument yields a version for iterated power series extensions:

Theorem 4.8.6. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. Then, (K, v, δ) is di erentially henselian if and only if (K, v, δ) is existentially closed in $(L, w, \partial) = (K((t_0))...((t_{n-1})), v \circ v_{t_0} \circ ... \circ v_{t_n}, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t_0} + ... + \frac{\mathrm{d}}{\mathrm{d}t_n})$ for any $n < \omega$.*

Proof. By applying Proposition 4.8.4 repeatedly, we have that (K, v) is existentially closed in (L, w) by transitivity of existential closure. Thus, by Theorem 4.4.9, (K, v, δ) is existentially closed in (L, w, ∂).

Conversely, (L, w, ∂) admits a valued (K, v)-Taylor morphism, as we can take the inclusion of $(K((t_0)), v \circ v_{t_0}, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t_0})$ in L. We conclude by Lemma 4.8.3, (K, v, δ) is differentially henselian. □

We now generalise the characterisation of differential largeness in Theorem 2.3.16(iv), which says that a differential field is differentially large if and only if it is large as a pure field, and every differentially finitely generated K-algebra with a K-rational point also has a differential K-rational point.

Theorem 4.8.7. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. Then, (K, v, δ) is di erentially henselian if and only if for every di erentially finitely generated K-algebra A with a finite di erential generating set $(a_i)_{i<n}$, and a K-rational point $\varphi : A \to K$, there is, for any $\gamma \in vK$, a di erential K-rational point $\psi : A \to K$ such that $v(\varphi(a_i) - \psi(a_i)) > \gamma$ for each $i < n$.*

Proof. Suppose (K, v, δ) is differentially henselian. Let A be a differentially finitely generated K-algebra, $\bar{a} = (a_i)_{i<n}$ a finite differential generating set for A. Suppose A has a K-rational point $\varphi : A \to K$, and let $\gamma \in vK$.

By assumption, A is isomorphic to a differential K-algebra of the form $K\{x_0, ..., x_{n-1}\}/I$, where I is the kernel of the map π which evaluates x_i at a_i. Write $\bar{x} = (x_0, ..., x_{n-1})$.

Applying the standard twisted Taylor morphism to φ, we obtain a differential K-algebra homomorphism $T^*_\varphi : A \to K((t))$. By Proposition 4.8.2, for any $f \in A$, we have that $(v \circ v_t)(\varphi(f) - T^*_\varphi(f)) > vK$. So, in particular, $T^*_\varphi(\bar{a}) \in B_\gamma(\varphi(\bar{a}))$.

Let $\mathfrak{p}$ denote the preimage of $\ker(T^*_\varphi)$ under π in $K\{\bar{x}\}$. Clearly, $\mathfrak{p}$ is a differential prime ideal of $K\{\bar{x}\}$, thus by the Ritt-Raudenbush basis theorem, it is finitely generated as a differential radical ideal of $K\{\bar{x}\}$. Let $f_0(\bar{x}), ..., f_{k-1}(\bar{x})$ be such a generating set.

By construction, we now have that $T^*_\varphi(\bar{a})$ is a solution to the system of differential polynomials $f_0(\bar{x}) = ... = f_{k-1}(\bar{x}) = 0$. As (K, v, δ) is differentially henselian, it is existentially closed in $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ by Theorem 4.8.5.

Applying existential closure, there is some tuple $\bar{b} \in K$ with $f_i(\bar{b}) = 0$ for each $i < k$, and also $\bar{b} \in B_\gamma(\varphi(\bar{a})$. Equivalently, the differential prime ideal $\mathfrak{p}$ vanishes on $\bar{b}$, and so in particular, vanishes on I.

Thus, the map $K\{\bar{x}\} \to K$ defined by evaluation at $\bar{b}$ factors through $K\{\bar{x}\}/I = A$, thus A has a differential point $\psi : A \to K$ with $\psi(\bar{a}) = \bar{b}$. Further, $\psi(\bar{a}) = \bar{b} \in B_\gamma(\varphi(\bar{a}))$, as required. □

From the proof above, it is easy to see that we may reduce to the case where the differentially finitely generated K-algebra A is a domain. Another variation on this condition is that we may require the image of the differential generators $\bar{a}$ to be 'close up to order n' for some finite n. We state these more precisely as follows:

Proposition 4.8.8. *Let (K, v, δ) be a valued-di erential field, henselian as a pure valued field. The following are equivalent:*

(i) (K, v, δ) *is di erentially henselian.*

(ii) For any di erentially finitely generated K-algebra A with a finite differential generating set $\bar{a}$ and K-rational point $\varphi : A \to K$, for any $\gamma \in vK$, there is a di erential K-rational point $\psi : A \to K$ such that $\psi(\bar{a}) \in B_\gamma(\varphi(\bar{a}))$.

(iii) For any di erentially finitely generated K-algebra A, which is a domain, with a finite di erential generating set $\bar{a}$ and K-rational point $\varphi : A \to K$, for any $\gamma \in vK$, there is a di erential K-rational point $\psi : A \to K$ such that $\psi(\bar{a}) \in B_\gamma(\varphi(\bar{a}))$.

(iv) For any di erentially finitely generated K-algebra A with a finite di erential generating set $\bar{a}$ and K-rational point $\varphi : A \to K$, for any $\gamma \in vK$ and $n < \omega$, there is a di erential K-rational point $\psi : A \to K$ such that $\psi(\mathrm{Jet}_n(\bar{a})) \in B_\gamma(\varphi(\mathrm{Jet}_n(\bar{a})))$.

Proof. The equivalence of (i) and (ii) is given by Theorem 4.8.7. The implications (ii) $\implies$ (iii) and (iv) $\implies$ (ii) are trivial. It remains to show (iii) $\implies$ (ii) and (ii) $\implies$ (iv). For (iii) $\implies$ (ii), as in the proof of Theorem 4.8.7, we may replace A with $K\{\bar{x}\}/\pi^{-1}(\ker(T^*_\varphi))$ and proceed. For (ii) $\implies$ (iv), we simply observe that $\mathrm{Jet}_n(\bar{a})$ is also a finite differential generating set for A, and apply (ii). □

An interesting corollary of this result is that isolated K-rational points of differentially finitely generated K-algebras must be differential:

Definition 4.8.9. Let (K, v, δ) be a valued-differential field, and let A be a differentially finitely generated K-algebra. We say that a K-rational point $\varphi : A \to K$ is *isolated* (with respect to the valuation topology) if there is a finite differential generating set $\bar{a} = (a_i)_{i<n}$ and $\gamma \in vK$ such that for any K-rational point $\psi : A \to K$ with $\psi(\bar{a}) \in B_\gamma(\varphi(\bar{a}))$, we have that $\varphi = \psi$.

Corollary 4.8.10. *Let (K, v, δ) be di erentially henselian, and let A be a di erentially finitely generated K-algebra. Let $\varphi : A \to K$ be an isolated K-rational point. Then φ is di erential.*

Proof. As φ is isolated, there is a finite differential generating set $\bar{a} = (a_i)_{i<n}$ of A and $\gamma \in vK$ such that there is no K-rational point $\psi : A \to K$ with $\psi \neq \psi$ and $\psi(\bar{a}) \in B_\gamma(\varphi(\bar{a})$.

Theorem 4.8.7, there is a differential K-rational point $\chi : K \to A$ with $\chi(\bar{a}) \in B_\gamma(\varphi(\bar{a}))$. By the above, we necessarily have that $\varphi = \chi$, thus φ is already differential. □

4.9 Differential Weil Descent on Valued Fields

In the paper [33], Léon Sánchez and Tressl use a differential version of the Weil descent to show that algebraic extensions of differentially large fields are themselves differentially large. In this section, we will adapt some of their machinery to prove a corresponding result for differentially henselian fields.

We first recall the corresponding result for differentially large fields:

Theorem 4.9.1 ([33, Theorem 6.1])**.** *Let (K, δ) be a di erentially large field, and let (L, ∂) be a di erential field extension where L/K is algebraic. Then, (L, ∂) is di erentially large.*

We are able to obtain a partial result in the case where the field is not real closed using previous results:

Proposition 4.9.2. *Let (K, v, δ) be a di erentially henselian field, where K is not real closed. Let (L, w, ∂) be an algebraic valued-di erential field extension of (K, v, δ). Then, (L, w, ∂) is di erentially henselian.*

Proof. It suffices to show that every subextension of finite degree is itself differentially henselian. Thus we may assume that L itself is an extension of finite degree of K.

Since (K, v) is henselian, the extension w of v to L is uniquely determined. In particular, (L, w) is also henselian. Further, as K is not real closed, L is not algebraically closed. Further, (L, ∂) is differentially large by Theorem 4.9.1. We then conclude by Theorem 4.2.12 that (L, w, ∂) is differentially henselian. □

We will reprove this later as Theorem 4.9.10 without restriction. To do this, we will require the use of the differential Weil descent. We begin by constructing the classical Weil descent, closely following the setup of [33]. For full details and algebraic technicalities, we direct the reader to the aforementioned paper.

Let K be a ring, and L be a K-algebra. Let

$$F : K\text{-}\textsc{Alg} \to L\text{-}\textsc{Alg}$$

be the 'extension of scalars' functor, given by $F(A) = A \otimes_K L$ on objects, and for a morphism $\varphi : A \to B$, $F(\varphi) = \varphi \otimes_K \mathrm{id}_L$. We assume that tensor products are taken over K, unless otherwise stated, and suppress the relevant subscripts. The Weil descent functor $W : L\text{-}\textsc{Alg} \to K\text{-}\textsc{Alg}$ we construct shall be the left adjoint of F.

We assume that L is a free and finitely generated K-module of dimension l over K. Fix a basis $b_1, ..., b_l$ of L as a K-module. For each i, define $\lambda_i : L \to K$ by

$$\lambda_i \left(\sum_j a_j b_j \right) = a_i,$$

i.e. $\lambda_i(x)$ is the ith coordinate of x with respect to the basis $b_1, ..., b_l$. For a K-algebra A, define $\lambda_i^A = \mathrm{id}_A \otimes \lambda_i : A \otimes L \to A \otimes K = A$.

Definition 4.9.3 ([33, Definition 2.3])**.** Let T be a set of indeterminates. Define a K-algebra

$$W(L[T]) = K[T]^{\otimes l}.$$

For $i = 1, ..., l$ and $t \in T$, write

$$t(i) = 1 \otimes ... \otimes 1 \otimes t \otimes 1 \otimes ... \otimes 1 \in K[T]^{\otimes l}$$

where the t occurs in the ith position. Define the L-algebra homomorphism $W_{L[T]} : L[T] \to F(W(L[T])) = K[T]^{\otimes l} \otimes L$ by setting for each $t \in T$

$$W_{L[T]}(t) = \sum_i (t(i) \otimes b_i).$$

Define $F_{K[T]} : W(F(K[T])) = K[T]^{\otimes l} \to K[T]$ by setting, for each $t \in T$ and $i = 1, ..., l$:

$$F_{K[T]}(t(i)) = \lambda_i(1)t.$$

We may choose $W(L[T])$ to be the Weil descent of $L[T]$, and $W_{L[T]}$ to be the unit of the adjunction at $L[T]$. We then obtain, for any K-algebra A

$$\tau = \tau(L[T], A) : \mathrm{Hom}_{A\text{-}\mathrm{ALG}}(K[T]^{\otimes l}, A) \to \mathrm{Hom}_{L\text{-}\mathrm{ALG}}(L[T], A \otimes L)$$
$$\varphi \mapsto F(\varphi) \circ W_{L[T]}$$

a bijective map. We can explicitly compute, for $\varphi : K[T]^{\otimes l} \to A$ and $t \in T$

$$\tau(\varphi)(t) = \sum_i \varphi(t(i)) \otimes b_i \in A \otimes L.$$

Conversely, if $\psi : L[T] \to A \otimes L$ is an L-algebra homomorphism, we obtain the corresponding K-algebra homomorphism $\varphi : K[T]^{\otimes l} \to A$ by setting, for each $t \in T$, $i = 1, ..., l$:

$$\varphi(t(i)) = \lambda_i^A(\psi(t)).$$

Now, we may construct the Weil descent of an arbitrary L-algebra B. Let $\pi_A : L[T] \to B$ be a surjective L-algebra homomorphism for some set T of indeterminates. Let I_B be the ideal generated in $W(L[T]) = K[T]^{\otimes l}$ by all elements of the form

$$\lambda_i^{W(L[T])}(W_{L[T]}(f))$$

where $i = 1, ..., l$, and $f \in \ker \pi_B$. Define

$$W(B) = W(L[T])/I_B,$$

and set

$$W(\pi_B) : W(L[T]) \to W(B)$$

to be the residue map.

Then, for any K-algebra A, the bijection $\tau(L[T], A)$ from above induces a bijection

$$\tau(B, A) : \mathrm{Hom}_{K\text{-}\mathrm{ALG}}(W(B), A) \to \mathrm{Hom}_{L\text{-}\mathrm{ALG}}(B, F(A))$$

such that the following square commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{K\text{-}\mathrm{ALG}}(W(B), A) & \xrightarrow{\tau(B,A)} & \mathrm{Hom}_{L\text{-}\mathrm{ALG}}(B, F(A)) \\ \downarrow{\scriptstyle -\circ W(\pi_B)} & & \downarrow{\scriptstyle -\circ \pi_B} \\ \mathrm{Hom}_{K\text{-}\mathrm{ALG}}(W(L[T]), A) & \xrightarrow{\tau(L[T],A)} & \mathrm{Hom}_{L\text{-}\mathrm{ALG}}(L[T], F(A)). \end{array}$$

From the commutativity of the above diagram, for $\varphi \in \mathrm{Hom}_{K\text{-}\mathrm{ALG}}(W(B), A)$

we obtain the following:

$$\tau(B,A)(\varphi) \circ \pi_B = \tau(L[T],A)(\varphi \circ W(\pi_B)) = ((\varphi \circ W(\pi_B)) \otimes \mathrm{id}_L) \circ W_{L[T]}.$$

Explicitly computing $W_B = \tau(B, W(B))(\mathrm{id}_{W(B)})$, for any $t \in T$, we obtain:

$$W_B(\pi_B(t)) = \sum_i W(\pi_B)(t(i)) \otimes b_i = \sum_i (t(i) + I_B) \otimes b_i.$$

We will now show that, where L/K is a finite algebraic extension of valued fields, B an L-algebra, and $\varphi, \psi : B \to L$ are L-rational points, then φ and ψ are 'close' if the corresponding K-rational points of the Weil descent $W(B)$ are also 'close'.

Notation. Let B be an L-algebra, and let $a \in B$. Let $\pi_B : L[T] \to B$ be a surjective L-algebra homomorphism, and assume there is $t \in T$ with $\pi_B(t) = a$. Write

$$a(i) = W(\pi_B)(t(i)) \in W(B).$$

Equivalently, $a(i) = \lambda_j^{W(B)}(W_B(a))$.

In particular, we should note that the definition of $a(i)$ has no dependence on the choice of indeterminates T or the surjective homomorphism π_B.

Theorem 4.9.4. *Let (K, v) be a valued field, let (L, w) be a finite algebraic extension of (K, v), and fix $b_1, ..., b_l$ a basis of L as a K-vector space. Let B be an L-algebra, let $a \in B$ and fix $\gamma \in wL$. Then, for any pair of K-algebra homomorphisms $\tilde{\varphi}, \tilde{\psi} : W(B) \to K$ satisfying the property that*

$$v(\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) > \gamma - \varepsilon,$$

for each $i = 1, ..., l$, where $\varepsilon = \min_i w(b_i)$, we have that

$$w(\varphi(a) - \psi(a)) > \gamma,$$

where $\varphi, \psi : B \to L$ denote the images of $\tilde{\varphi}$ and $\tilde{\psi}$ under $\tau(B, K)$, respectively.

Proof. We compute $\varphi(a)$ in terms of the $a(i)$. Let π_B be a surjective L-algebra homomorphism $L[T] \to B$, where there is $t \in T$ with $\pi_B(t) = a$. Then:

$$\begin{aligned}\varphi(a) &= (((\tilde{\varphi} \circ W(\pi_B)) \otimes \mathrm{id}_L) \circ W_{L[T]})(t) \\ &= ((\tilde{\varphi} \circ W(\pi_B)) \otimes \mathrm{id}_L)\left(\sum_i t(i) \otimes b_i\right) \\ &= \sum_i (\tilde{\varphi} \circ W(\pi_B))(t(i)) \otimes b_i \\ &= \sum_i \tilde{\varphi}(a(i)) \otimes b_i.\end{aligned}$$

Replacing $\tilde{\varphi}$ with $\tilde{\psi}$, we obtain a corresponding result for ψ. Now, we see that

$$\varphi(a) - \psi(a) = \sum_i (\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) \otimes b_i.$$

Taking w and applying the ultrametric inequality, we obtain

$$w(\varphi(a) - \psi(a)) \geqslant \min_i (w(\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) + w(b_i)).$$

Suppose that

$$v(\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) > \gamma - \varepsilon$$

holds for each i. Then,

$$\begin{aligned}w(\varphi(a) - \psi(a)) &\geqslant \min_i w(\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) + \min_i w(b_i) \\ &> \gamma - \varepsilon + \varepsilon = \gamma.\end{aligned}$$

This shows the desired inequality. □

We will now show that the converse also holds in a restricted setting, where we assume that the extension $(K, v) \in (L, w)$ admits a 'separated' basis. For full details on separated extensions, we refer the reader to [13].

Definition 4.9.5. Let $(K, v) \subseteq (L, w)$ be an extension of valued fields. We say that a finite set of elements $(b_i)_{i<n}$ is ***separated*** if, for any $(a_i)_{i<n} \in L^n$, we have that

$$w\left(\sum_i a_i b_i\right) = \min_i w(a_i b_i).$$

We say that the extension $(K, v) \subseteq (L, w)$ is ***separated*** if every finite dimensional K-vector subspace of L admits a separated basis.

Remark. Clearly, any separated set of elements is linearly independent.

Examples 4.9.6.

1. [13, Corollary 7] Every algebraic extension $(K, v) \subseteq (L, w)$ of henselian valued fields of equicharacteristic 0 is separated.
2. [13, Proof of Corollary 7] Any finite algebraic extension $(K, v) \subseteq (L, w)$ with

$$[L : K] = (vL : vK) \cdot [Lw : Kv]$$

is separated.

Proposition 4.9.7. *Let $(K, v) \subseteq (L, w)$ be a finite algebraic extension of valued fields. Let $b_1, ..., b_l$ be a separated basis of L as a K-vector space. Let B be an L-algebra, and let $a \in B$. Let $\varphi, \psi : B \to L$ be L-algebra homomorphisms, and denote their images under $\tau(B, K)^{-1}$ by $\tilde{\varphi}$ and $\tilde{\psi}$, respectively. Then, for each i,*

$$v(\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) \geqslant w(\varphi(t) - \psi(t)) - w(b_i).$$

Proof. We claim that

$$\tilde{\varphi}(a(i)) = \lambda_i(\varphi(a)).$$

To see this, we define $\tilde{\varphi}$ by setting $\tilde{\varphi}(a(i)) = \lambda_i(\varphi(a))$, and we apply $\tau(B, K)$:

$$\tau(B, K)(\tilde{\varphi})(a) = \sum_i \tilde{\varphi}(a(i)) \otimes b_i = \sum_i \lambda_i(\varphi(a)) \otimes b_i = \varphi(a).$$

Thus $\tilde{\varphi} = \tau(B, K)^{-1}(\varphi)$.

Now observe the following:

$$w(\varphi(a) - \psi(a)) = w\left(\sum_i (\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) \otimes b_i\right).$$

As the basis (b_i) is separated, we have that for any j,

$$\begin{aligned} w(\varphi(a) - \psi(a)) &= \min_i (w(\tilde{\varphi}(a(i)) - \tilde{\psi}(a(i))) + w(b_i)) \\ &\leqslant w(\tilde{\varphi}(a(j)) - \tilde{\psi}(a(j))) + w(b_j). \end{aligned}$$

Rearranging, we obtain the inequality

$$w(\tilde{\varphi}(a(j)) - \tilde{\psi}(a(j))) \geqslant w(\varphi(a) - \psi(a)) - w(b_j).$$

as required. □

We now recall the main result on the differential version of the Weil descent.

For a differential ring (K, δ), we denote the category of differential (K, δ)-algebras by (K, δ)-$\textsc{Alg}$.

Theorem 4.9.8 ([33, Theorem 3.4]). *Let (K, δ) be a di erential ring, and (L, ∂) be a di erential K-algebra, finitely generated and free as a K-module. Then:*

(i) The functor $F^{\text{diff}} : (K, \delta)\text{-}\textsc{Alg} \to (L, \partial)\text{-}\textsc{Alg}$ which sends a di erential (K, δ)-algebra (A, η) to $(A \otimes L, \eta \otimes \mathrm{id}_L + \mathrm{id}_A \otimes \partial)$ has a left adjoint $W^{\text{diff}} : (L, \partial)\text{-}\textsc{Alg} \to (K, \delta)\text{-}\textsc{Alg}$ known as the differential Weil descent *from (L, ∂) to (K, δ). The di erential Weil descent sends a di erential (B, d) algebra to $(W(B), \mathrm{d}^W)$, where d^W is uniquely determined in [33, Theorem 3.2].*

(ii) Let $(A, \eta) \in (K, \delta)\text{-}\textsc{Alg}$ and $(B, \mathrm{d}) \in (L, \partial)\text{-}\textsc{Alg}$. Then, the bijection

$$\tau(A, B) : \mathrm{Hom}_{K\text{-}\textsc{Alg}}(W(B), A) \to \mathrm{Hom}_{L\text{-}\textsc{Alg}}(B, F(A))$$

from the classical Weil descent restricts to a bijection

$$\mathrm{Hom}_{(K,\delta)\text{-}\textsc{Alg}}(W^{\text{diff}}(B, \mathrm{d}), (A, \eta)) \to \mathrm{Hom}_{(L,\partial)\text{-}\textsc{Alg}}((B, \mathrm{d}), F^{\text{diff}}(A, \eta)).$$

We apply this now to show that every finite algebraic extension of a differentially henselian field is again differentially henselian.

Proposition 4.9.9. *Let (K, v, δ) be a di erentially henselian field, and let (L, w, ∂) be a finite algebraic extension of (K, v, δ). Then, (L, w, ∂) is di erentially henselian.*

Proof. Let $f \in L\{x\}$ be a differential polynomial of order n, and let $\bar{a} \in L$ such that $f_{\text{alg}}(\bar{a}) = 0$ and $s(f)_{\text{alg}}(\bar{a}) \neq 0$. Fix γ in wL. Let (B, d) be the differential L-algebra $L\{x\}/I(f)$, and $\varphi : B \to L$ be the L-algebra homomorphism given by evaluation at $\bar{a}$.

Observe that B is generated as an L-algebra by $x, x', ..., x^{(n-1)}$. We take $\pi_B : L[t_0, ..., t_{n-1}]$ to be the morphism which sends t_i to $x^{(i)}$. Applying the Weil descent, and the bijection $\tau(B, K)$, we obtain a K-algebra homomorphism $\tilde{\varphi} : W(B) \to K$. We also have that $W(B)$ is finitely generated as a K-algebra by the $t_i(j)$, where $i = 0, ..., n-1$ and $j = 1, ..., l$. Let $\eta \in vK$ be an element satisfying the condition in Theorem 4.9.4.

As (K, v, δ) is differentially henselian, it is existentially closed in $(K((t)), v \circ v_t, \hat{\delta} + \frac{\mathrm{d}}{\mathrm{d}t})$ by Theorem 4.8.5, and as $(W(B), \mathrm{d}^W)$ is differentially finitely generated by the $t_i(j)$ via the surjective homomorphism $W(\pi_B) : W(L[T]) \to K$, we may apply Theorem 4.8.7 to obtain a differential K-algebra homomorphism $\tilde{\psi} : (W(B), \mathrm{d}^W) \to (K, \delta)$ such that for each i, j, $v(\tilde{\varphi}(W(\pi_B)(t_i(j))) - \tilde{\psi}(W(\pi_B)(t_i(j)))) > \eta$.

Write $\psi = \tau(B, K)(\tilde{\psi})$, and by Theorem 4.9.8, we have that $\psi : (B, \mathrm{d}) \to (L, \partial)$ is a differential L-algebra homomorphism. Now, by Theorem 4.9.4, we have that $w(\varphi(\pi_B(t_i)) - \psi(\pi_B(t_i))) > \gamma$ for each i.

Taking $b = \psi(\pi_B(t))$, we observe that $f(b) = 0$, and $b^{(i)} = \psi(\pi_B(t_i))$. Further, by the above, $w(a_i - b_i) > \gamma$ for each i. Thus, (L, w, ∂) is differentially henselian, as required. □

This extends simply to arbitrary algebraic extensions:

Theorem 4.9.10. *Let (K, v, δ) be a di erentially henselian field. Then every algebraic extension (L, w, ∂) of (K, v, δ) is di erentially henselian.*

Proof. By the previous proposition, every finite subextension is differentially henselian. Let $f \in L\{x\}$ be a differential polynomial of order n, let $\bar{a} \in L^{n+1}$ be such that $f_{\mathrm{alg}}(\bar{a}) = 0$ and $s(f)_{\mathrm{alg}}(\bar{a}) \neq 0$. Let $\gamma \in wL$. Let F be the extension of K generated by the coefficients of f and $\bar{a}$. Then, since F/K is finite, (F, u, d) (taking the appropriate restrictions of w and ∂) is differentially henselian. Let $\mu \in uF$ with $\mu > \gamma$. By differential henselianity, there is $b \in F$ with $f(b) = 0$ and $\mathrm{Jet}_n(b) \in B_\mu(\bar{a})$. In particular, $b \in L$ with $\mathrm{Jet}_n(b) \in B_\gamma(\bar{a})$. Thus (L, w, ∂) is also differentially henselian, as required. □

Finally, we observe that the algebraic closure of a differentially henselian field must be a model of DCVF:

Corollary 4.9.11. *Let (K, v, δ) be a di erentially henselian field of characteristic $(0, p)$, where p is a prime or 0. Then, its algebraic closure $(\overline{K}, \bar{v}, \bar{\delta})$ is a model of $\mathrm{DCVF}_{(0,p)}$.*

Proof. The algebraic closure $(\overline{K}, \bar{v})$ of (K, v) is a model of $\mathrm{ACVF}_{(0,p)}$, and $(\overline{K}, \bar{v}, \bar{\delta})$ is differentially henselian by Theorem 4.9.10. □

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