# GENERIC LOCAL UNIQUENESS AND STABILITY IN POLARIZATION TOMOGRAPHY

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Abstract. The problem of polarization tomography is considered on a Riemannian manifold. This problem comes from the physical problem of recovering the anisotropic part of the dielectric permittivity tensor of a quasi-isotropic medium from polarization measurements made around the boundary, but is more general. In greater than three dimensions local uniqueness and stability are established for generic background metrics, and near generic tensor fields through the study of a related linear inverse problem. The same results are established on a natural subspace of tensor fields in dimension three.

## 1. Introduction

We will study *polarization tomography* with the addition of phase information as formulated in [15], and also studied in [12]. This is an inverse problem defined on a Riemannian manifold of dimension at least 3 with boundary. In the case of dimension 3, the problem corresponds with the physical problem of recovering the anisotropic part of a quasi-isotropic medium from polarization measurements made at the boundary (see [15]). The physical problem of recovering electrical properties of a medium based on polarization measurements has also been studied extensively in the optics literature. In particular much work has been done in the context of the photoelastic effect, where one well known technique of inversion is *integrated photoelasticity* ([1], [2], [3], [8], [7]).

We will proceed to describe the problem of polarization tomography, but in order to do this we must first introduce some notations and definitions. Let $(M, g)$ be a compact connected Riemannian manifold with strictly convex boundary such that no geodesic has infinite length. Such a manifold will be called *compact non-trapping* (CNT). Given a CNT manifold, we will denote the sets of unit vectors, inward pointing unit vectors on $\partial M$, and outward pointing unit vectors on $\partial M$ respectively as $\Omega M$, $\partial_-\Omega M$, and $\partial_+\Omega M$. Let $\tau_1^1(M)$ denote the smooth sections of the bundle $(T_1^1)^{\mathbb{C}}M$ of complexified $(1, 1)$ tensors over $M$. For any $v \in TM$, we will write $\pi(v)$ for the natural projection of $v$ onto $M$, and define $\pi_v : T^{\mathbb{C}}_{\pi(v)}M \to T^{\mathbb{C}}_{\pi(v)}M$ to be the orthogonal projection onto the subspace perpendicular to $v$. Given $f \in (T_1^1)^{\mathbb{C}}M$, we define $(P_v\, f) = \pi_v \circ f \circ \pi_v$ where we are considering $f$ as an automorphism of $T^{\mathbb{C}}_{\pi(v)}M$. For any $v \in TM$ we denote the maximally extended geodesic with initial data $v$ by $\gamma_v$.

Now, take any $f \in \tau_1^1(M)$, $\xi \in \partial_-\Omega M$, and $\eta_0 \in T^{\mathbb{C}}_{\pi(\xi)}M$, and consider the initial value problem

$$\frac{D\eta}{\mathrm{d}s}(s) = (P_{\dot\gamma_\xi(s)}\, f)\, \eta(s) \quad , \quad \eta(0) = \eta_0 \tag{1}$$

for a vector field along $\gamma_\xi$. Here the derivative on the left is covariant differentiation along $\gamma_\xi$. Since $(M, g)$ is CNT, there must be some unique time $l(\xi) > 0$ when $\gamma_\xi$ intersects $\partial M$. We may now state the problem of polarization tomography with phase information.

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**Question 1.** *Given the solution $\eta(l(\xi))$ of (1) for every $\xi \in \partial_-\Omega M$ and initial vector $\eta_0 \in T^{\mathbb{C}}_{\pi(\xi)}M$, is it possible to recover the tensor field $f$?*

A positive answer to this question was given in [12] under assumptions on the curvature of $(M, g)$, which include the Euclidean case, in dimension at least 4 and for tensor fields $f$ sufficiently small. In dimension 3, Novikov and Sharafutdinov ([12]) also established the same result assuming additionally that $f$ is in a certain natural subspace of $\tau^1_1(M)$.

We may reformulate this problem by introducing the fundamental matrix for (1) along all unit speed geodesics. This fundamental solution takes the form of a $(1, 1)$ semi-basic tensor field over the unit sphere bundle $\Omega M$. In general, an $(m, n)$ semi-basic tensor field on $TM$ is a complex tensor field which in any set of natural coordinates takes the form

$$U(x, \xi) = u^{i_1 \dots i_m}_{j_1 \dots j_n}(x, \xi)\frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_m}} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_n}.$$

This definition is independent of the natural coordinate system chosen. Note that here we are writing the tangent vector $\xi$ as $(x, \xi)$, where $x = \pi(\xi)$, in order to stress the base point over which $\xi$ is a tangent vector. We will sometimes refer to tangent (and cotangent) vectors in this way throughout the remainder of this work. For any embedded submanifold $\mathcal{G} \subset TM$, the set $B^m_n(\mathcal{G})$ is the vector bundle of $(m, n)$ semi-basic tensors restricted to $\mathcal{G}$, and $\beta^m_n(\mathcal{G})$ is the set of $(m, n)$ semi-basic tensor fields over $\mathcal{G}$ (ie. the sections of $B^m_n(\mathcal{G})$). Semi-basic tensor fields of order $(1, 1)$ provide automorphisms on the spaces $T^{\mathbb{C}}_x M$ by the formula

$$U(x, \xi)\left(v^j \frac{\partial}{\partial x^j}\right) = v^j u(x, \xi)^i_j \frac{\partial}{\partial x^i}.$$

It may be checked that this mapping is independent of the coordinate system chosen. Because of this fact, $(1, 1)$ semi-basic tensors at a given point $(x, \xi) \in TM$ may be identified with $(1, 1)$ tensors over $M$ at $x$. Thus, we may consider $(1, 1)$ semi-basic tensor fields as $(1, 1)$ tensor fields that are also permitted to depend on the fiber variable.

The fundamental matrix for (1) is the field $U \in \beta^1_1(\Omega M \setminus T\partial M)$ which satisfies the equation

$$HU(x, \xi) = (P_\xi f)U(x, \xi) \quad , \quad U|_{\partial_-\Omega M} = E \tag{2}$$

where $E$ is the identity, and $H$ is the differentiation with respect to the geodesic flow (see [15]). The solutions of (2) and (1) are related by

$$\eta(s) = U(\gamma_\xi(s), \dot{\gamma}_\xi(s))\, \mathcal{I}^{\gamma_\xi}_{0,s}\, \eta_0 \tag{3}$$

where $\mathcal{I}^{\gamma_\xi}_{0,s}$ denotes the parallel translation along $\gamma_\xi$. From (3) we see that the data of the inverse problem are given precisely by $U|_{\partial_+\Omega M}$, and this leads us to the following reformulation of question 1.

**Question 2.** *Can we recover the tensor field $f$ from $U|_{\partial_+\Omega M}$ where $U$ is the solution of (2)?*

We call $U|_{\partial_+\Omega M}$ the *polarization data* corresponding to $f$.

Our main result applies to so called *simple* manifolds. A CNT manifold is called simple if the exponential map at every point is a diffeomorphism. The main result in the case of dimension at least 4 is the following theorem.

**Theorem 1.** *Assume that $(M, g)$ is a real-analytic simple manifold of dimension at least 4 with real-analytic metric $g$. If $\hat{f} \in \tau^1_1(M)$ is real-analytic, then there exists an $\epsilon > 0$ such that whenever $g' \in S_2(M)$ is another metric on $M$ and $f_1$, $f_2 \in \tau^1_1(M)$ are such that*

$$\|g - g'\|_{C^5 S_2(M)} < \epsilon, \quad \textit{and} \quad \|\hat{f} - f_i\|_{C^3 \tau^1_1(M)} < \epsilon \quad \textit{for } i = 1 \textit{ and } 2, \tag{4}$$

*if the polarization data of $f_1$ and $f_2$ with respect to the metric $g'$ are the same then $f_1 = f_2$. Furthermore, there is a stability estimate for such $f_1$ and $f_2$*

$$\|f_1 - f_2\|_{L^2\tau_1^1(M)} \leq C\|U_1' - U_2'\|_{H^1\beta_1^1((\partial_+\Omega)'M)} \tag{5}$$

*for some constant $C > 0$, and where $U_1'$ and $U_2'$ are the polarization data of $f_1$ and $f_2$ respectively taken with respect to $g'$.*

As we will discuss below in section 2, in the case of dimension 3 the polarization data does not uniquely determine $f$. However, when we restrict $f$ to an appropriate subspace we still have a similar result to Theorem 1 in dimension 3 (Theorem 5), but we must delay the statement.

The inverse problem of recovering $f$ from its polarization data is nonlinear. Our main tool in the proof of Theorems 1 and 5 will be an integral identity that in some sense linearizes the problem. The identity is the same as that used in [12] integrated along geodesics, and was previously used in [9] to show that the full jet of $f$ can be recovered at points in $\partial M$ from the polarization data. Let $f_1$ and $f_2 \in \tau_1^1(M)$, $\xi \in \partial_-\Omega M$, and choose any $\eta$, $\zeta \in T^{\mathbb{C}}_{\pi(\xi)}M$. Also let $U_1$ and $U_2$ be the solutions of (2) for $f_1$ and $f_2$ respectively. Then by equation (5.8) in [12] we have

$$\frac{\partial}{\partial s}\Big\langle \mathcal{I}^{\gamma_\xi}_{s,0}\big(U_2(\dot\gamma_\xi(s))^{-1}U_1(\dot\gamma_\xi(s)) - E\big)\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\eta,\,\zeta\Big\rangle_{g(x)} = \Big\langle \mathcal{I}^{\gamma_\xi}_{s,0}\,U_2^{-1}(\dot\gamma_\xi(s))\,\big[P_{\dot\gamma_\xi(s)}(f_1 - f_2)\big](\gamma_\xi(s))\,U_1(\dot\gamma_\xi(s))\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\eta,\,\zeta\Big\rangle_{g(x)}.$$

Now choose any $0 \leq t_1 < t_2 \leq l(\xi)$ and integrate the previous formula to get

$$\begin{aligned}&\Big\langle \Big(\mathcal{I}^{\gamma_\xi}_{t_2,0}\;U_2^{-1}(\dot\gamma_\xi(t_2))\,U_1(\dot\gamma_\xi(t_2))\,\mathcal{I}^{\gamma_\xi}_{0,t_2} - \mathcal{I}^{\gamma_\xi}_{t_1,0}\,U_2^{-1}(\dot\gamma_\xi(t_1))\,U_1(\dot\gamma_\xi(t_1))\,\mathcal{I}^{\gamma_\xi}_{0,t_1}\Big)\,\eta,\zeta\Big\rangle_{g(x)}\\ &\quad= \int_{t_1}^{t_2}\Big\langle U_2^{-1}(\dot\gamma_\xi(s))\,\Big[P_{\dot\gamma_\xi(s)}\,(f_1 - f_2)\Big]\,(\gamma_\xi(s))\,U_1(\dot\gamma_\xi(s))\;\mathcal{I}^{\gamma_\xi}_{0,s}\,\eta,\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\zeta\Big\rangle_{g(\gamma_\xi(s))}\,\mathrm{d}s.\end{aligned} \tag{6}$$

This is the main identity from which our results will follow. Note that if $0 = t_1$ and $l(\xi) = t_2$, then (6) becomes

$$\begin{aligned}&\Big\langle \big(U_2^{-1}(\dot\gamma_\xi(l(\xi)))\,U_1(\dot\gamma_\xi(l(\xi))) - E\big)\,\mathcal{I}^{\gamma_\xi}_{0,l}\,\eta,\,\mathcal{I}^{\gamma_\xi}_{0,l}\,\zeta\Big\rangle_{g(y)}\\ &\quad= \int_0^{l(\xi)}\Big\langle U_2^{-1}(\dot\gamma_\xi(s))\;\Big[P_{\dot\gamma_\xi(s)}(f_1 - f_2)\Big]\,(\gamma_\xi(s))\,U_1(\dot\gamma_\xi(s))\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\eta,\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\zeta\Big\rangle_{g(\gamma_\xi(s))}\mathrm{d}s.\end{aligned} \tag{7}$$

When $f_1$ and $f_2$ have the same polarization data the left hand side of (7) is equal to zero. We will view the right hand side of (7), with a fixed pair of semi-basic tensor fields $U_1$ and $U_2$, as a sort of X-ray transform acting on $f_1 - f_2$ that maps elements of $\tau_1^1(M)$ to sections of $B_2(\partial_-\Omega M)$. We will refer to this transformation as $I_{U_1,U_2}$. It is defined by

$$I_{U_1,U_2}[f](\eta,\zeta) = \int_0^{l(\xi)}\Big\langle U_2^{-1}(\dot\gamma_\xi(s))\;\Big[P_{\dot\gamma_\xi(s)}(f)\Big]\,(\gamma_\xi(s))\,U_1(\dot\gamma_\xi(s))\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\eta,\,\mathcal{I}^{\gamma_\xi}_{0,s}\,\zeta\Big\rangle_{g(\gamma_\xi(s))}\mathrm{d}s. \tag{8}$$

The inversion of the map $f \mapsto I_{U_1,U_2}[f]$ is a linear problem, and we will study it in section 3. Our main tool in analyzing this linear problem is the associated *normal operator*. A modification of the proof of [15, Theorem 4.2.1] shows that $I_{U_1,U_2} : L^2\tau_1^1(M, |\mathrm{d}v_g|) \to L^2\beta_2(\partial_-\Omega M, |\mathrm{d}V_{\partial_-\Omega M}|)$ continuously, and so we may define the $L^2$ transpose $I^*_{U_1,U_2} : L^2\beta_2(\partial_-\Omega M, |\mathrm{d}V_{\partial_-\Omega M}|) \to L^2\tau_1^1(M, |\mathrm{d}v_g|)$. Here $|\mathrm{d}v_g|$ is the Riemannian density, and

$$|\mathrm{d}V_{\partial_-\Omega M}| = |\langle\xi,\nu\rangle_g\, i_{\frac{\partial}{\partial r}}(i_\nu(\mathrm{d}V))|$$

where $\nu$ is the outer unit normal, $\frac{\partial}{\partial r}$ is the unit radial vector field in $TM \setminus \{0\}$, $\mathrm{d}V$ is the natural $2n$ form on $TM$, and $i_\xi$ denotes interior multiplication by $\xi$. From now on we will write $L^2\tau_1^1(M)$, and $L^2\beta_2(\partial_-\Omega M)$ for these $L^2$ spaces without reference to the respective densities used in their definitions. The normal operator associated to $I_{U_1,U_2}$ is

$$\mathcal{N}_{U_1,U_2} = I^*_{U_1,U_2} \circ I_{U_1,U_2} : L^2\tau_1^1(M) \to L^2\tau_1^1(M). \tag{9}$$

In order to state our main results for the linear problem we must also introduce an extension of $M$. Indeed, let $M_1$ be a new manifold with boundary such that $M \Subset M_1^{int}$. This is always possible by taking a collar neighborhood of $\partial M$ that is diffeomorphic to $\partial M \times [0,\epsilon)$, and then taking $M_1$ to be $M \cup (\partial M \times [-\epsilon, 0))$ where the charts for $M_1$ across $\partial M$ are defined in the obvious way. Furthermore, since $g$ is smooth up to $\partial M$, $g$ can be extended smoothly to a metric on $M_1$ that agrees with $g$ on $M$ (see [14]). If we assume that $M$ is a simple manifold, then we may assume that with the extended metric $M_1$ is also a simple manifold. We can extend any $f \in L^2\tau_1^1(M)$ to an element of $L^2\tau_1^1(M_1)$ that agrees with $f$ on $M$, and is equal to zero $M_1 \setminus M$. In this way we identify $L^2\tau_1^1(M)$ with the subspace of $L^2\tau_1^1(M_1)$ consisting of elements having support contained in $M$.

For the linear problem we have the following two results in dimension at least 4. In the statement of Theorem 2 we use the notation $\Omega_a^b M_1 = \{v \in TM \mid a < \|v\|_g < b\}$ for an annulus.

**Theorem 2.** *Suppose that $(M,g)$ is simple and has dimension at least 4. If $U_1$, $U_2 \in \beta_1^1(TM \setminus \{0\})$ are everywhere invertible, then the kernel of $I_{U_1,U_2}$ acting on $L^2\tau_1^1(M)$ is at most finite dimensional and contains only elements of $\tau_1^1(M)$ that are zero to infinite order on $\partial M$ (ie. all elements of the kernel are smooth and vanish to infinite order on $\partial M$). Furthermore, if $I_{U_1,U_2}$ is injective, then there is a stability estimate*

$$\|f\|_{L^2\tau_1^1(M)} \leq C \|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)}. \tag{10}$$

*The constant $C$ can be chosen so that there exists $\epsilon > 0$ such that the estimate (10) remains valid if $U_1$, $U_2$, and $g$ are replaced by $U_1'$, $U_2' \in C^3\beta_1^1(TM \setminus \{0\})$, and $g' \in C^4S_2M_1$ with $\|U_1 - U_1'\|_{C^3\beta_1^1(\Omega_a^b M_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta_1^1(\Omega_a^b M_1)} < \epsilon$, and $\|g - g'\|_{C^4S_2M_1} < \epsilon$ assuming that the unit sphere bundles with respect to both $g$ and $g'$ are contained in $\Omega_a^b M_1$.*

**Theorem 3.** *If $(M,g)$ is a real analytic simple manifold of dimension at least 4, and $U_1$, $U_2$ are real analytic, then $I_{U_1,U_2}$ is injective.*

Taken together these two theorems imply that $I_{U_1,U_2}$ is injective for generic simple metrics, and generic $U_1$ and $U_2$. Similar results hold in the case of three dimensions when we add some extra hypotheses for $f$ which will be explained below in section 2.

The next task is to examine the non-uniqueness that occurs in the 3 dimensional case, and present the corresponding results in that case. This will be done in section 2. Section 3 then contains the proofs of our results for the linear problem, and then finally in section 4 we apply these results to the nonlinear problem to prove Theorems 1 and 5.

The results in this paper are based upon the doctoral dissertation of the author which was completed at the University of Washington under the supervision of Gunther Uhlmann.

## 2. The three dimensional case

In dimension 3 there is a natural non-uniqueness to the inverse problem which we will now describe. This non-uniqueness was originally found in [12], and the formulation that we use here can also be found in [9]. We make the additional assumption that $(M,g)$ is oriented in which case we may define the Hodge star operator $* : \Lambda^k T^*M \to \Lambda^{3-k}T^*M$ from $k$ forms to $3-k$ forms on

$M$ (see for example [13]), and also the adjoint of the exterior derivative $\delta = *d*$ where $d$ is the exterior derivative. We have the following theorem.

**Theorem 4.** *If a CNT manifold $M^3$ is orientable and $h_1,\ h_2 \in C^\infty(M)$ are such that $h_1|_{\partial M} = h_2|_{\partial M}$, then the polarization data for $f_i = (\delta(h_i\,\mathrm{d}v_g))^{\#}$ $(i = 1, 2)$ are the same. The $\#$ indicates that the first index of $\delta(h\,\mathrm{d}v_g)$ has been raised, and $\mathrm{d}v_g$ is the Riemannian volume form.*

For a proof, see [12] or [9].

In view of the previous theorem we make the following definition. A tensor field $f \in \tau_1^1(M)$ will be called *coexact* if $f = (\delta(h\,\mathrm{d}v_g))^{\#}$ for some $h \in C^\infty(M)$. We now consider how to determine a subspace of $\tau_1^1(M)$ that is complementary to the space of coexact tensor fields.

First we define the *tangential component* of $f \in \tau_1^1(M)$ to be the section of the vector bundle $i^*(T_1^1 M)$ over $\partial M$ given by $\mathbf{t}f = P_\nu f$, and the *normal component* of $f$ to be $\mathbf{n}f = f - \mathbf{t}f$. Here $i : \partial M \to M$ is the inclusion map, and $\nu$ is the outward pointing unit normal to $\partial M$. The tangential and normal parts of $(0, 2)$-tensors can also be defined in a similar way. Indeed, if $h \in \tau_2(M)$ is a $(0, 2)$-tensor and $\xi,\ \eta \in i^*(TM)$, then $\mathbf{t}h(\xi, \eta) = h(\pi_\nu(\xi), \pi_\nu(\eta))$ and $\mathbf{n}h = h - \mathbf{t}h$. With these definitions the following formulas hold for $f \in \tau_1^1(M)$.

$$(\mathbf{t}f)^\flat = \mathbf{t}f^\flat \quad \text{and} \quad (\mathbf{n}f)^\flat = \mathbf{n}f^\flat. \tag{11}$$

The *antisymmetric part* of $f$, $f^a$, is defined by $f^a = (f - f^t)/2$ where $f^t$ is the transpose of $f$ with respect to the non-sesquilinear inner product corresponding to $g$. In coordinates this is given by

$$(f^t)_i^j = g_{ik} f_s^k g^{sj}. \tag{12}$$

We will say that $f$ satisfies the *tangential boundary condition* if $\mathbf{t}f^a = 0$.

Now consider the Helmholtz decomposition (see [13]) which says that any 2-form $h \in \Lambda^2(M)$ can be uniquely written as the sum of a coexact form with zero normal part, and a closed form. We can identify antisymmetric tensors in $\tau_1^1(M)$ with elements of $\Lambda^2(M)$ through the metric, and so in fact for any $f \in \tau_1^1(M)$ we can decompose $f$ as

$$f = f^s + (\alpha)^\sharp + (*\mathrm{d}\,\beta)^\sharp \tag{13}$$

where $f^s = (f + f^t)/2$ is the symmetric part of $f$, $\alpha \in \Lambda^2(M)$ is closed, $\beta \in C^\infty(M)$, and $*\mathrm{d}\,\beta$ has zero normal component. This last property is equivalent to the property that $\beta$ is constant on the boundary. The decomposition is also unique up to possibly changing $\beta$ by a constant, and so if we add the requirement that $\beta|_{\partial M} = 0$, then the decomposition (13) is unique. Note that $(*\mathrm{d}\,\beta) = (\delta * \beta)$, and so according to theorem 4 we cannot expect to recover $(*\mathrm{d}\,\beta)^\sharp$ from the polarization data of $f$. However, we expect to be able to find at least the normal part of $f$ at the boundary since this part does not depend on $(*\mathrm{d}\,\beta)^{\#}$. Furthermore, we might expect to be able to recover $f$ fully when the coexact part of $f$, $(*\mathrm{d}\,\beta)^\sharp$, is zero. In light of this, we now record the condition on $f$ which will guarantee that $*\mathrm{d}\,\beta$ is zero. To this end, let $\mathrm{Alt} : \tau_2(M) \to \Lambda^2(M)$ denote the projection onto the alternating tensor fields, which is given in coordinates by

$$\mathrm{Alt}(f_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j) = \sum_{j>i} (f_{ij} - f_{ji})\,\mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Now define $\mathrm{d}_\beta : \tau_1^1(M) \to C^\infty(M)$ by

$$\mathrm{d}_\beta(f) = *\,\mathrm{d}\,\mathrm{Alt}(f^\flat). \tag{14}$$

Since $\alpha$ is closed, from (13) we see that

$$\Delta_g\,\beta = \mathrm{d}_\beta f. \tag{15}$$

If we also require that $\beta|_{\partial M} = 0$, then $*\mathrm{d}\,\beta$ will have normal part zero, and so $\beta$ is given by the solution of the Dirichlet problem corresponding to (15). By uniqueness of solutions to the Dirichlet problem it is therefore clear that $*\mathrm{d}\,\beta = 0$ if and only if $\mathrm{d}_\beta f = 0$.

Armed with these new definitions and the facts presented in the previous paragraph, we are now prepared to state the main results in the 3 dimensional case. For the nonlinear problem we have the following theorem.

**Theorem 5.** *Assume that $(M, g)$ is a real-analytic simple manifold of dimension 3 with real-analytic metric $g$, and let $\hat{f} \in \tau_1^1(M)$ be real analytic. Then there exists an $\epsilon > 0$ such that whenever $g' \in S_2(M)$ and $f_1$, $f_2 \in \tau_1^1(M)$ satisfy (4), $\mathrm{d}_\beta(f_1 - f_2) = 0$, $f_1 - f_2$ satisfies the tangential boundary condition, and $f_1$ and $f_2$ have the same polarization data with respect to $g'$, then $f_1 = f_2$. If we additionally restrict $f_1$ and $f_2$ to have support contained in some compact set $K \Subset M^{int}$, then we have the estimate (5) where the constant $C$ may depend on $K$.*

It is important to note here that the operator $\mathrm{d}_\beta$ and the tangential boundary condition are taken with respect to the reference metric $g$.

We write $L^2_\beta\tau_1^1(M)$ and $C^3_\beta\tau_1^1(M)$ to denote the kernel of $\mathrm{d}_\beta$ acting on $L^2\tau_1^1(M)$ and $C^3\tau_1^1(M)$ respectively. With this notation we also have the following results for the linear problem.

**Theorem 6.** *If $M$ has dimension 3, and $U_1$, $U_2 \in \beta_1^1(TM_1\backslash\{0\})$ are everywhere invertible, then the kernel of $I_{U_1,U_2}$ acting on $L^2_\beta\tau_1^1(M)$ is at most finite dimensional and consists entirely of functions that are smooth on $M^{int}$. If we additionally assume that $f \in C^3_\beta\tau_1^1(M)$ is in the kernel of $I_{U_1,U_2}$ and satisfies the tangential boundary condition, then $f$ is smooth up to $\partial M$ and vanishes to infinite order there. Furthermore, if $I_{U_1,U_2}$ is injective on any closed subspace $\mathcal{L} \subset L^2_\beta\tau_1^1(M)$, then the stability estimate (10) holds for $f \in \mathcal{L}$, and the same statements concerning perturbations of $U_1$, $U_2$, and $g$ as in Theorem 2 also hold.*

We also have the following analog of Theorem 3.

**Theorem 7.** *Assume the same hypotheses as Theorem 3 except that the dimension of $M$ is 3. Then $I_{U_1,U_2}$ is injective on the subspace of $C^3_\beta\tau_1^1(M)$ consisting of fields that satisfy the tangential boundary condition.*

We now continue to study the linear problem and prove the results already presented for that case.

## 3. The linear problem

In this section we will analyze the problem of inverting the linear map $f \mapsto I_{U_1,U_2}[f]$ defined by (8) when $U_1$ and $U_2$ are arbitrary invertible $(1,1)$ semi-basic tensor fields. We will assume that we are working on a simple manifold $(M, g)$, and that we have an extension of $(M_1, g)$ as described in section 1. By using normal coordinates centered at a point in the interior of $M$ we may then work in global coordinates. To begin our analysis we will write out (8) with respect to these coordinates. For $v \in \Omega M$ we write $R(v)_{ab} = g(\pi(v))_{dj}\,(\mathcal{I}^{\gamma_v}_{-l(v),0})^j_b\,(U_2^{-1})(v)^d_a$ and $Q(v)^m_k = (U_1)(v)^m_p\,(\mathcal{I}^{\gamma_v}_{-l(v),0})^p_k$. With this notation

$$((I_{U_1,U_2})[f])(\xi)_{kb} = \int_0^{l(\xi)} R(\dot\gamma_\xi(s))_{ab}\,[P_{\dot\gamma_\xi(s)}\,f](\gamma_\xi(s))^a_m\,Q(\dot\gamma_\xi(s))^m_k\,\mathrm{d}s. \tag{16}$$

To fully analyze this operator in coordinates we also need the expansion

$$[P_v\,f]^a_m = \left(\delta^a_r - \frac{v^a\,v_r}{|v|^2_g}\right)\,f^r_u\left(\delta^u_m - \frac{v^u\,v_m}{|v|^2_g}\right). \tag{17}$$

Using (17) we can expand the integrand in (16) in terms of the components $f^r_u$. From [15, Lemma 4.1.1], we have that $l(v) \in C^\infty(\Omega M \setminus T\partial M)$ (recall that this function gives the positive time at which $\gamma_v$ meets $\partial M$), and this implies that each of the components of $I_{U_1,U_2}[f]$ given by (16) is in $C^\infty(\partial_-\Omega M)$ if $f$ is smooth. By the convexity of $\partial M$, we can further see that $I_{U_1,U_2}$ maps compactly supported tensor fields $(\tau^1_1)_c(M)$ to compactly supported smooth $(0,2)$ semi-basic tensor fields over $\partial_-\Omega M$, $(\beta_2)_c(\partial_-\Omega M)$. Thus, using (16) we obtain that $I_{U_1,U_2} : (\tau^1_1)_c(M) \to (\beta_2)_c(\partial_-\Omega M)$ is continuous. In the next section we further explore the properties of $\mathcal{N}_{U_1,U_2}$.

### 3.1. **Ellipticity of $\mathcal{N}_{U_1,U_2}$.**

Our analysis of both the linear and nonlinear problems are ultimately based upon the fact that the normal operator $\mathcal{N}_{U_1,U_2}$ is a pseudodifferential operator ($\Psi$DO) of order $-1$. In the case of dimension at least 4, $\mathcal{N}_{U_1,U_2}$ is elliptic. In dimension 3 we must add another operator to create an elliptic system. These results are contained in the following two propositions. The proofs use techniques from [5] and [6].

**Proposition 1.** *If $(M,g)$ is a simple manifold, then the operator $\mathcal{N}_{U_1,U_2}$ defined by (9) is a $\Psi DO$ of order $-1$ on the sections of the vector bundle $(T^1_1)^{\mathbb{C}}M$. Furthermore, if the dimension of $M$ is at least 3, then $\mathcal{N}_{U_1,U_2}$ is elliptic.*

*Proof.* For $h$ and $f \in \tau^1_1(M)$, we have from the definition of $\mathcal{N}_{U_1,U_2}$

$$\begin{aligned}
\langle \mathcal{N}_{U_1,U_2}[f], h\rangle_{L^2\tau^1_1(M)} &= \langle I_{U_1,U_2}[f], I_{U_1,U_2}[h]\rangle_{L^2\beta_2(\partial_-\Omega M)} \\
&= \Bigg( \int_{\partial_-\Omega M} |\mathrm{d}V_{\partial_-\Omega M}|(\xi)\, g(\pi(\xi))^{bb'}\, g(\pi(\xi))^{kk'} \\
&\qquad \times \left( \int_0^{l(\xi)} R(\dot\gamma_\xi(s))_{ab}\, [P_{\dot\gamma_\xi(s)}\, f](\gamma_\xi(s))^a_m\, Q(\dot\gamma_\xi(s))^m_k\, \mathrm{d}s \right) \\
&\qquad \times \overline{\left( \int_0^{l(\xi)} R(\dot\gamma_\xi(t))_{a'b'}\, [P_{\dot\gamma_\xi(t)}\, h](\gamma_\xi(t))^{a'}_{m'}\, Q(\dot\gamma_\xi(t))^{m'}_{k'}\, \mathrm{d}t \right)} \Bigg) \\
&= \Bigg( \int_{\partial_-\Omega M} |\mathrm{d}V_{\partial_-\Omega M}|(\xi) \\
&\qquad \times \Bigg( \int_0^{l(\xi)} (\mathcal{I}^{\gamma_{\dot\gamma_\xi(s)}}_{0,-s})^{b'}_b\, (U_2^{-1})(\dot\gamma_\xi(s))^b_a\, [P_{\dot\gamma_\xi(s)}\, f](\gamma_\xi(s))^a_m \\
&\qquad\qquad \times (U_1)(\gamma_\xi(s))^m_p\, (\mathcal{I}^{\gamma_{\dot\gamma_\xi(s)}}_{0,-s})^{k'}_k\, g(\gamma_\xi(s))^{kp}\, \mathrm{d}s \Bigg) \\
&\qquad \times \overline{\left( \int_0^{l(\xi)} R(\dot\gamma_\xi(t))_{a'b'}\, [P_{\dot\gamma_\xi(t)}\, h](\gamma_\xi(t))^{a'}_{m'}\, Q(\dot\gamma_\xi(t))^{m'}_{k'}\, \mathrm{d}t \right)} \Bigg).
\end{aligned}$$

In this last equality we have used the following property of parallel translation in relation to the metric

$$g(\pi(v))^{bb'}\, (\mathcal{I}^{\gamma_v}_{0,s})^j_b = (\mathcal{I}^{\gamma_v}_{s,0})^{b'}_o\, (\mathcal{I}^{\gamma_v}_{0,s})^o_\rho\, g(\pi(v))^{\rho b}\, (\mathcal{I}^{\gamma_v}_{0,s})^j_b = (\mathcal{I}^{\gamma_v}_{s,0})^{b'}_b\, g(\gamma_v(s))^{bj}.$$

The previous calculation shows that $\langle \mathcal{N}_{U_1,U_2}[f], h\rangle_{L^2\tau_1^1(M)}$ equals the sum

$$\int_{\partial_-\Omega M} \left(\int_0^{l(\xi)} w_2(\dot\gamma_\xi(s))_r^{uk'b'}\, f(\gamma_\xi(s))_u^r \,\mathrm{d}s\right) \overline{\times \left(\int_0^{l(\xi)} w_1(\dot\gamma_\xi(t))^\epsilon_{\alpha k'b'}\, h(\gamma_\xi(t))^\alpha_\epsilon \,\mathrm{d}t\right)} |\mathrm{d}V_{\partial_-\Omega M}|(\xi). \tag{18}$$

For any $(x,v)\in\Omega M$ the weights $w_1(x,v)^\epsilon_{\alpha k'b'}$ and $w_2(x,v)^{uk'b'}_r$ are given by

$$w_1(v)^\epsilon_{\alpha k'b'} = g(\pi(v))_{d'j'}\,(\mathcal{I}^{\gamma_v}_{-l(-v),0})^{j'}_{b'}\,(U_2^{-1})(v)^{d'}_{a'}\,[P_v]^{a'\epsilon}_{m'\alpha}\,(U_1)(v)^{m'}_{p'}\,(\mathcal{I}^{\gamma_v}_{-l(-v),0})^{p'}_{k'} \tag{19}$$

and

$$w_2(v)^{uk'b'}_r = (\mathcal{I}^{\gamma_v}_{0,-l(-v)})^{b'}_b\,(U_2^{-1})(v)^b_a\,[P_v]^{au}_{mr}\,(U_1)(v)^m_p\,(\mathcal{I}^{\gamma_v}_{0,-l(-v)})^{k'}_k\, g(\pi(x))^{kp}. \tag{20}$$

The components of $[P_v]^{a'\epsilon}_{m'\alpha}$ can be calculated from (17), although as we will see below this is unnecessary for our purposes.

Now we work separately with each of the terms in the sum (18). Following [5] and [6] we split the $s$ integral into two parts. For the first we take the part where $s>t$, and second is where $s<t$. In the $s>t$ portion we make the change of variables $(\xi,s,t)\mapsto v=(s-t)\,\dot\gamma_\xi(t)\in\mathcal{F}$ where $\mathcal{F}\subset TM$ is the domain of the exponential map. In the $s<t$ portion we instead make the change $(\xi,s,t)\mapsto v=-s\,\dot\gamma_\xi(t)\in\mathcal{F}$. The result is a sum of two integrals over $\mathcal{F}\setminus\{0\}$. Using global coordinates $\{x^i\}$ on $M$, we may introduce corresponding coordinates $\{x^i,\omega^i\}$ on $\mathcal{F}\setminus\{0\}$ and write each of the terms of the sum in (18) as two iterated integrals. Indeed, with the notation $\mathcal{F}_x=\mathcal{F}\cap T_xM$ for each $x\in M$, (18) becomes

$$\begin{aligned}&\int_M \overline{h(x)^\alpha_\epsilon}\Bigg(\int_{\mathcal{F}_x\setminus\{0\}} w_2(\dot\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^{uk'b'}_r\,\overline{w_1(\omega/\|\omega\|_g)^\epsilon_{\alpha k'b'}}\\ &\qquad\times f(\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^r_u\sqrt{\det(g)}\,\frac{\mathrm{d}\omega}{\|\omega\|_g^{n-1}}\Bigg)\sqrt{\det(g)}\,\mathrm{d}x\\ &+\int_M \overline{h(x)^\alpha_\epsilon}\Bigg(\int_{\mathcal{F}_x\setminus\{0\}} w_2(-\dot\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^{uk'b'}_r\,\overline{w_1(-\omega/\|\omega\|_g)^\epsilon_{\alpha k'b'}}\\ &\qquad\times f(\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^r_u\sqrt{\det(g)}\,\frac{\mathrm{d}\omega}{\|\omega\|_g^{n-1}}\Bigg)\sqrt{\det(g)}\,\mathrm{d}x.\end{aligned}$$

This shows that $\mathcal{N}_{U_1,U_2}[f]$ is given by

$$\begin{aligned}\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\epsilon = g(x)^{\alpha\alpha'}\,g(x)_{\epsilon\epsilon'}\Bigg(&\int_{\mathcal{F}_x\setminus\{0\}} w_2(\dot\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^{uk'b'}_r\,\overline{w_1(\omega/\|\omega\|_g)^{\epsilon'}_{\alpha'k'b'}}\\ &\times f(\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^r_u\sqrt{\det(g)}\,\frac{\mathrm{d}\omega}{\|\omega\|_g^{n-1}}\\ &+\int_{\mathcal{F}_x\setminus\{0\}} w_2(-\dot\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^{uk'b'}_r\,\overline{w_1(-\omega/\|\omega\|_g)^{\epsilon'}_{\alpha'k'b'}}\\ &\times f(\gamma_{\omega/\|\omega\|_g}(\|\omega\|_g))^r_u\sqrt{\det(g)}\,\frac{\mathrm{d}\omega}{\|\omega\|_g^{n-1}}\Bigg).\end{aligned} \tag{21}$$

This formula holds for arbitrary CNT manifolds. It is in the next step that we use the assumption that $(M,g)$ is simple. We apply the analysis of [6] to each of the terms in the sum (21) separately. A similar method is used to analyze such a system of operators in [5]. From this we see that the integral operators on the right hand side of (21) form a system of ΨDOs of order $-1$, and therefore a ΨDO on the sections of the vector bundle $(T^1_1)^{\mathbb{C}}(M)$. Furthermore, a principal symbol for this system is given by

$$\sigma_{\mathcal{N}_{U_1,U_2},p}(\xi)^{u\alpha}_{r\epsilon} = \|\xi\|_g^{-1} g(\pi(\xi))^{\alpha\alpha'}\, g(\pi(\xi))_{\epsilon\epsilon'}\, 2\pi \int_{\Omega_{\pi(\xi)}M\cap\xi^\perp} w_2(\omega)_r^{uk'b'}\, \overline{w_1(\omega)^{\epsilon'}_{\alpha'k'b'}}\, \mathrm{d}\omega. \tag{22}$$

The set $\xi^\perp$ is the collection of vectors annihilated by the covector $\xi$. The principal symbol $\sigma_{\mathcal{N}_{U_1,U_2},p}(\xi)$ is a linear map from $(T^1_1)^{\mathbb{C}}_{\pi(\xi)}(M)$ to $(T^1_1)^{\mathbb{C}}_{\pi(\xi)}(M)$ for each $\xi \in T^*M\setminus\{0\}$, and the ellipticity of $\mathcal{N}_{U_1,U_2}$ at $\xi$ is equivalent to the injectivity of this map.

We now aim to prove that $\mathcal{N}_{U_1,U_2}$ is elliptic by showing that $\sigma_{\mathcal{N}_{U_1,U_2},p}(\xi)$ is injective at every point $\xi \in T^*M\setminus\{0\}$. Indeed, suppose that $f \in (T^1_1)^{\mathbb{C}}_{\pi(\xi)}M$ is not zero. We must show that this implies that $\sigma_{\mathcal{N}_{U_1,U_1},p}(\xi)[f]$ is not zero. By (22) we have

$$\langle \sigma_{\mathcal{N}_{U_1,U_1},p}(\xi)[f], f\rangle_{(T^1_1)^{\mathbb{C}}_{\pi(\xi)}M} = \frac{2\pi}{\|\xi\|_g}\int_{\Omega_{\pi(\xi)}M\cap\xi^\perp} w_2(\omega)_r^{uk'b'}\,\overline{w_1(\omega)^{\epsilon}_{\alpha k'b'}}\, f^r_u\, \overline{f^\alpha_\epsilon}\,\mathrm{d}\omega.$$

Applying (19), and (20) we can rewrite the integrand in this formula invariantly as

$$\|U_2^{-1}(\omega)\circ[P_\omega f]\circ U_1(\omega)\|^2_{(T^1_1)^{\mathbb{C}}_{\pi(\xi)}M}. \tag{23}$$

Since this quantity is always non-negative, to show that $\sigma_{\mathcal{N}_{U_1,U_1},p}(\xi)[f]$ is not zero it is sufficient to prove that $U_2^{-1}(\omega)\circ[P_\omega f]\circ U_1(\omega)$ is not zero at some point $\omega \in \Omega_{\pi(\xi)}M\cap\xi^\perp$. We note here that all of the steps in the proof until now apply equally well in dimension 3. It is in this final step that we must require dimension at least 4. Since $f$ is not zero, there exists some $v \in T_xM$ such that $f(v)\neq 0$. When the dimension is greater than 3, it is always possible to find a vector $\omega_v \in \Omega_{\pi(\xi)}M\cap\xi^\perp$ that is simultaneously perpendicular to $v$ and $f(v)$. Given such an $\omega_v$ we have

$$\langle U_2^{-1}(\omega_v)\circ[P_{\omega_v}f]\circ U_1(\omega_v)(U_1^{-1}(\omega_v)v),\, U_2^*(\omega_v)f(v)\rangle_{g(\pi(\xi))} = \|f(v)\|^2_{g(\pi(\xi))} > 0.$$

Therefore $U_2^{-1}(\omega_v)\circ[P_{\omega_v}f]\circ U_1(\omega_v)\neq 0$, and so $\mathcal{N}_{U_1,U_2}$ is elliptic. □

Next we will prove the related result for the 3 dimensional case. In 3 dimensions the operator $\mathcal{N}_{U_1,U_2}$ is not in general elliptic. However we can produce an elliptic system of operators by adding a second ΨDO. Indeed, let $\Lambda^2$ be a ΨDO on $M$ which is a parametrix for the positive Laplace-Beltrami operator $-\Delta_g$. Then we have the following result in 3 dimensions.

**Proposition 2.** *If $(M,g)$ is simple and the dimension of $M$ is 3, then the system of ΨDOs $(\mathcal{N}_{U_1,U_2}, \Lambda^2\,\mathrm{d}_\beta)^T$ from sections of $(T^1_1)^{\mathbb{C}}(M)$ to sections of $(T^1_1)^{\mathbb{C}}(M)\bigoplus\Lambda^0(M)$ is elliptic.*

*Proof.* We will first calculate the operator $\mathrm{d}_\beta : \tau^1_1(M)\to\Lambda^0(M)$ in coordinates and then determine its principal symbol. Taking any $f\in\tau^1_1(M)$ we have

$$\begin{aligned}\mathrm{d}_\beta(f^j_i\,\mathrm{d}x^i\otimes\tfrac{\partial}{\partial x^j}) &= *\mathrm{d}\left(\sum_{i<j}(g_{ik}\,f^k_j - g_{jk}\,f^k_i)\,\mathrm{d}x^i\wedge\mathrm{d}x^j\right)\\ &= \tfrac{1}{\sqrt{\det(g)}}\Big(\tfrac{\partial(g_{1k}\,f^k_2)}{\partial x^3} - \tfrac{\partial(g_{2k}\,f^k_1)}{\partial x^3} - \tfrac{\partial(g_{1k}\,f^k_3)}{\partial x^2} + \tfrac{\partial(g_{3k}\,f^k_1)}{\partial x^2}\\ &\qquad + \tfrac{\partial(g_{2k}\,f^k_3)}{\partial x^1} - \tfrac{\partial(g_{3k}\,f^k_2)}{\partial x^1}\Big)\end{aligned}$$

From this we can see that the principal symbol $\sigma_{\mathrm{d}_\beta,p}(\xi)$ is given by

$$\sigma_{\mathrm{d}_\beta,p}(\xi)[f] = \frac{i}{\sqrt{\det(g)}}(\xi_1(g_{2k}\, f^k_3 - g_{3k}\, f^k_2) - \xi_2(g_{1k}\, f^k_3 - g_{3k}\, f^k_1) + \xi_3(g_{1k}\, f^k_2 - g_{2k}\, f^k_1)).$$

Since composition of ΨDOs corresponds to multiplication at the level of principal symbols and the principal symbol of $\Lambda^2$ is $\|\xi\|_g^{-2}$, this implies that the principal symbol $\sigma_{\Lambda^2\,\mathrm{d}_\beta,p}$ of $\Lambda^2\,\mathrm{d}_\beta$ is given by

$$\sigma_{\Lambda^2\,\mathrm{d}_\beta,p}(\xi)[f] = \frac{i}{\|\xi\|_g^{-2}\sqrt{\det(g)}}(\xi_1(g_{2k}\, f^k_3 - g_{3k}\, f^k_2) - \xi_2(g_{1k}\, f^k_3 - g_{3k}\, f^k_1) + \xi_3(g_{1k}\, f^k_2 - g_{2k}\, f^k_1)). \tag{24}$$

To complete the proof we must show that for any $\xi \in T^*M \setminus \{0\}$, the map

$$(\sigma_{\mathcal{N}_{U_1,U_2},p}(\xi), \sigma_{\Lambda^2\,\mathrm{d}_\beta,p}(\xi))^T : (T^1_1)^{\mathbb{C}}_{\pi(\xi)}M \to (T^1_1)^{\mathbb{C}}_{\pi(\xi)}M \bigoplus \Lambda^0_{\pi(\xi)}(M) \tag{25}$$

is injective.

Indeed, let us suppose that $f \in (T^1_1)^{\mathbb{C}}_{\pi(\xi)}M$ is such that $(\sigma_{\mathcal{N}_{U_1,U_2},p}, \sigma_{\Lambda^2\,\mathrm{d}_\beta,p})^T[f] = 0$. We must show that this implies $f = 0$. Most of the proof of proposition 1 still applies to this case. In fact, following through that proof we see that it is sufficient to show that if $f \neq 0$ and $\sigma_{\Lambda^2\,\mathrm{d}_\beta}(\xi)[f] = 0$, then for any $\xi \in T^*M\setminus\{0\}$ there exists $\omega \in \Omega_{\pi(\xi)}M \cap \xi^\perp$ such that $U_2^{-1}(\omega)\circ[P_\omega f]\circ U_1(\omega)$ is not zero. Indeed, suppose that $f$ satisfies these hypotheses and split $f$ into symmetric and anti-symmetric parts by writing

$$f = f^s + f^a$$

where $f^s = (f + f^t)/2$ and $f^a = (f - f^t)/2$. Recall that $f^t$ is the transpose of $f$ given by (12). Since $f \neq 0$, one of $f^s$ or $f^a$ must be non-zero, and so we break the proof into two cases.
*Case 1: $f^s$ is not zero.* In this case there must be a real vector $v \in T_{\pi(\xi)}M$ such that $\langle f^s(v), v\rangle_{g(\pi(\xi))} \neq 0$. Take $\omega_v \in \Omega_{\pi(\xi)}M \cap \xi^\perp$ such that $\omega_v$ is perpendicular to $v$. Then since $v$ is real we have

$$\langle U_2^{-1}(\omega_v)\circ[P_{\omega_v} f]\circ U_1(\omega_v)(U_1^{-1}(\omega_v)v), U_2^*(\omega_v)v\rangle_{g(\pi(\xi))} = \langle f(v), v\rangle_{g(\pi(\xi))} = \langle f^s(v), v\rangle_{g(\pi(\xi))} \neq 0.$$

Therefore $U_2^{-1}(\omega_v)\circ[P_{\omega_v} f]\circ U_1(\omega_v) \neq 0$, and the proof is complete in this case.
*Case 2: $f^s$ is zero, but $f^a$ is not zero.* Since this entire calculation takes place in a single fiber over $M$, we assume that we are working with respect to an orthonormal frame and so $g_{ij} = \delta_{ij}$. By assumption $f$ is represented with respect to this frame by an anti-symmetric matrix

$$f = \begin{pmatrix} 0 & f^1_2 & f^1_3 \\ -f^1_2 & 0 & f^2_3 \\ -f^1_3 & -f^2_3 & 0 \end{pmatrix}.$$

If we also write $v = (v^1, v^2, v^3)^T$ in this frame, then it is easy to check that

$$f(v) = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times \begin{pmatrix} f^2_3 \\ -f^1_3 \\ f^1_2 \end{pmatrix} \tag{26}$$

where $\times$ denotes the Euclidean cross product. Also, in this orthonormal coordinate frame (24) becomes

$$\sigma_{\Lambda\,\mathrm{d}_\beta,p}(\xi)[f] = \frac{2i}{\|\xi\|^{-2}}(\xi_1 f^2_3 - \xi_2 f^1_3 + \xi_3 f^1_2).$$

Therefore, if we take $\omega = (f^2_3, -f^1_3, f^1_2)^T/|(f^2_3, -f^1_3, f^1_2)|$, then $\sigma_{\Lambda^2\,\mathrm{d}_\beta}(\xi)[f] = 0$ implies that $\omega\cdot\xi = 0$ and so $\omega \in \Omega_{\pi(\xi)} \cap \xi^\perp$. Finally, if we take $v$ to be nonzero and perpendicular to both $\omega$ and $\xi^\sharp$ (this is $\xi$ with the index raised), then by (26) $f(v)$ must be nonzero, parallel to $\xi^\sharp$, and perpendicular to $\omega$. Therefore

$$\langle U_2^{-1}(\omega)\circ[P_\omega f]\circ U_1(\omega)(U_1^{-1}(\omega)v), U_2^*(\omega)\,\xi^\sharp\rangle_{g(x)}\neq 0.$$

This shows that the map (25) is injective which completes the proof. □

Theorem 4 shows that in some sense tensor fields of the form $(*\mathrm{d}\beta)^\sharp$ ,where $\beta\in\Lambda^0(M)$ vanishes on $\partial M$, are in the "kernel of the nonlinear operator" associated to the inverse problem. The following lemma says that these tensor fields are also the kernel of $I_{U_1,U_2}$, but only to first order. The lemma will be required for the proof of Theorem 6

**Lemma 1.** *In dimension* 3*, the* $\Psi DO$ $\mathcal{N}_{U_1,U_1}\circ(\sharp\circ*\mathrm{d})$ *from* $\Lambda^0(M)$ *to sections of* $T_1^1(M)$ *is of order* $-1$*. Here* $\sharp$ *raises the first index.*

*Remark* 1. Note that $\sharp\circ*\mathrm{d}$ is a ΨDO of order 1 while $\mathcal{N}_{U_1,U_2}$ is of order $-1$, and so we would naively expect $\mathcal{N}_{U_1,U_2}\circ(\sharp\circ*\mathrm{d})$ to be of order 0.

*Proof.* By the calculus of ΨDOs, it is sufficient to show that for every $\xi\in T^*M\setminus\{0\}$, $\sigma_{\mathcal{N}_{U_1,U_2},p}(\xi)\circ\sigma_{\sharp\circ*\mathrm{d},p}(\xi)=0$. Based on (23), it is sufficient to show that for any non-zero complex number $\tilde\beta$ and any $\omega\in\Omega_{\pi(\xi)}M\cap\xi^\perp$

$$P_\omega[\sigma_{\sharp\circ*\mathrm{d},p}(\xi)\,\tilde\beta]=0.$$

Let us now proceed to calculate $\sigma_{\sharp\circ*\mathrm{d},p}$. Assume that we have an oriented coordinate system $(x^1,x^2,x^3)$ whose coordinate vectors are orthonormal at the point $\pi(\xi)$, $\xi=\mathrm{d}x^1|_{\pi(\xi)}$, and $\omega=\frac{\partial}{\partial x^2}|_{\pi(\xi)}$. Then if $\beta\in\Lambda^0(M)$, in these coordinates we have

$$\sharp\circ*\mathrm{d}(\beta)=\frac{\partial\beta}{\partial x^1}\,\partial_{x^2}\wedge\mathrm{d}x^3-\frac{\partial\beta}{\partial x^2}\,\partial_{x^1}\wedge\mathrm{d}x^3+\frac{\partial\beta}{\partial x^3}\,\partial_{x^1}\wedge\mathrm{d}x^2.$$

Therefore, if $\tilde\beta=\beta(x)$ then

$$\sigma_{\sharp\circ*\mathrm{d},p}(\xi)\,\tilde\beta=i\,\tilde\beta\,\partial_{x^2}\wedge\mathrm{d}x^3.$$

Similar to (26), if $v\in T_{\pi(\xi)}M$ is represented as the vector $(v^1,v^2,v^3)^T$ in these coordinates, then

$$\sigma_{\sharp\circ*\mathrm{d},p}(x,\xi)\,\tilde\beta(v)=i\,\tilde\beta\,\begin{pmatrix}v^1\\v^2\\v^3\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=i\,\tilde\beta\,\begin{pmatrix}0\\v^3\\-v^2\end{pmatrix}.$$

This implies (recall that $\omega=\frac{\partial}{\partial x^2}|_{\pi(\xi)}$)

$$P_\omega[\sigma_{\sharp\circ*\mathrm{d},p}(x,\xi)\,\tilde\beta](v)=i\,\tilde\beta\,\pi_\omega\begin{pmatrix}0\\v_3\\0\end{pmatrix}=0.$$

This completes the proof of the lemma. □

In the next section we use propositions 1 and 2 to prove Theorems 2 and 6.

3.2. **Stability for the linear problem.** We begin with the proof of Theorem 2, which consists essentially of two parts. First we apply a parametrix for $\mathcal{N}_{U_1,U_2}$ to show that the inversion of $\mathcal{N}_{U_1,U_2}$ on $L^2\tau_1^1(M)$ can be reduced to a Fredholm problem. Together with a bit of functional analysis this proves the first portion of the theorem. Then we study how $\mathcal{N}_{U_1,U_2}$ changes under perturbations of $U_1$, $U_2$, and $g$ to prove the second part.

*Proof of Theorem 2.* For this proof we will require an intermediate manifold $M_{1/2}$ such that $M \Subset M_{1/2}^{int} \Subset M_1^{int}$. If $M_1$ is constructed as described in section 1, then $M_{1/2}$ can be taken to be $M \cup (\partial M \times [-\epsilon/2, 0))$. Now, take a cut-off function $\phi \in C_c^\infty(M_1)$ such that $\phi = 1$ on $M_{1/2}$. By Theorem 1 the ΨDO $\mathcal{N}_{U_1,U_2}$ is elliptic of order $-1$, and therefore there exists a parametrix $\mathcal{A}$ for $\mathcal{N}_{U_1,U_2}$ which is a ΨDO of order 1. This means that, if we denote multiplication by $\phi$ as $\phi^m$, for any $f \in L^2\tau_1^1(M)$ we have

$$\mathcal{A} \circ (\phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f] = f + \mathcal{K}[f] \quad \text{on } M_{1/2}^{int} \tag{27}$$

where $\mathcal{K} : \mathcal{E}'\tau_1^1(M_1) \to \tau_1^1(M_1)$ is a properly supported smoothing operator. The addition of the cut-off functions in this formula is required for the composition of the two ΨDOs to be well-defined. Rearranging this last formula slightly, using the continuity properties of $\mathcal{A}$ and $\mathcal{K}$, and using the fact that $\phi f = f$, we obtain the estimate

$$\|f\|_{L^2\tau_1^1(M)} = \|f\|_{L^2\tau_1^1(M_1)} \leq C(\|\mathcal{N}_{U_1,U_1}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-s}\tau_1^1(M_1)}) \tag{28}$$

for any $s > 0$. For $f \in L^2\tau_1^1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$ this estimate becomes

$$\|f\|_{L^2\tau_1^1(M_1)} \leq C\|f\|_{H^{-s}\tau_1^1(M_1)}.$$

Since the inclusion map $L^2\tau_1^1(M_1) \hookrightarrow H^{-s}\tau_1^1(M_1)$ is compact when $s > 0$ this shows that the identity map restricted to $f \in L^2\tau_1^1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$ must be compact and so $L^2\tau_1^1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$ must be finite dimensional. From the definition of $\mathcal{N}_{U_1,U_2}$ it is clear that $L^2\tau_1^1(M) \cap \ker(\mathcal{N}_{U_1,U_2}) = L^2\tau_1^1(M) \cap \ker(I_{U_1,U_2})$, and so as claimed the kernel of $I_{U_1,U_2}$ acting on $L^2\tau_1^1(M)$ is finite dimensional. Further, if $f \in L^2\tau_1^1(M) \cap \ker(\mathcal{N}_{U_1,U_2})$, then by (27) $f = -\mathcal{K}[f]$ on $M_{1/2}^{int}$, and in particular $f \in \tau_1^1(M_{1/2})$. Since $f = 0$ on $M_{1/2} \setminus M$, this proves result that if $I_{U_1,U_2}[f] = 0$, then $f$ vanishes to infinite order on $\partial M$.

The stability estimate (10) follows from (28) and the following lemma which is similar to [20, Prop. V.3.1]. See also [17, Lemma 2]. A proof of this precise formulation can be found in [10].

**Lemma 2.** *If $X$, $Y$, and $Z$ are all Banach spaces, $A : X \to Y$ is a continuous and injective linear operator, $K : X \to Z$ is a compact linear operator, and we have the estimate*

$$\|x\|_X \leq C(\|Ax\|_Y + \|Kx\|_Z) \quad \forall x \in X, \tag{29}$$

*then in fact we have*

$$\|x\|_X \leq \tilde{C}\|Ax\|_Y \quad \forall x \in X.$$

Take $X = L^2\tau_1^1(M)$, $Y = H^1\tau_1^1(M_1)$, $Z = H^{-s}\tau_1^1(M_1)$, $A = \mathcal{N}_{U_1,U_2}$, and $K$ the inclusion map from $L^2\tau_1^1(M)$ to $H^{-s}\tau_1^1(M_1)$. Then (28) gives (29), and so if $I_{U_1,U_2}$, and therefore $\mathcal{N}_{U_1,U_2}$, is injective (10) is proved.

We have now finished the first part of the proof referred to at the beginning of this section, and will move on to the second portion. For this part of the proof we rely on Theorem 8 which is stated at the end of this section. This theorem is used in the proof of both the present theorem and Theorem 6, and so we delay its statement. In the present case, if $U_1'$, $U_2'$, and $g'$ satisfy the hypotheses for any $\epsilon > 0$ sufficiently small, then Theorem 8 gives

$$\|\mathcal{N}_{U_1,U_2} - \mathcal{N}_{U_1',U_2'}\|_{L^2\tau_1^1(M) \to H^1\tau_1^1(M_1)} \leq C'\epsilon$$

for some new constant $C'$. If $\epsilon$ is taken to be less than $1/(2CC')$ where $C$ is the constant from (10) then we have

$$\begin{aligned}\|f\|_{L^2\tau^1_1(M)} &\leq C\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)}\\ &\leq C\|\mathcal{N}_{U_1',U_2'}[f]\|_{H^1\tau^1_1(M_1)} + C\|(\mathcal{N}_{U_1,U_2}-\mathcal{N}_{U_1',U_2'})[f]\|_{H^1\tau^1_1(M_1)}\\ &\leq C\|\mathcal{N}_{U_1',U_2'}[f]\|_{H^1\tau^1_1(M_1)} + \tfrac{1}{2}\|f\|_{L^2\tau^1_1(M)}.\end{aligned}$$

Therefore

$$\|f\|_{L^2\tau^1_1(M)} \leq 2C\|\mathcal{N}_{U_1',U_2'}[f]\|_{H^1\tau^1_1(M_1)}$$

and so the proof is complete.

□

We now proceed to the proof of Theorem 6.

*Proof of Theorem 6* Take $M_{1/2}$ and $\phi$ to be defined as in the proof of Theorem 2. By proposition 2 the system of ΨDOs $(\mathcal{N}_{U_1,U_2}, \Lambda^2\,\mathrm{d}_\beta)^T$ is elliptic, and so there exists a parametrix for this system which we will denote by $(A, B)$ where $A$ is a ΨDO from sections of $(T^1_1)^{\mathbb{C}}M_1$ to sections of $(T^1_1)^{\mathbb{C}}M_1$, and $B$ is a ΨDO from sections of $M_1\times\mathbb{C}$ to sections of $(T^1_1)^{\mathbb{C}}M_1$. For any $f\in L^2\tau^1_1(M_1)$, the analog of (27) in this case is

$$(A\circ\phi^m\circ\mathcal{N}_{U_1,U_2}\circ\phi^m)[f] + (B\circ\phi^m\circ\Lambda^2\,\mathrm{d}_\beta\circ\phi^m)[f] = f + \mathcal{K}[f] \quad \text{on } M^{int}_{1/2} \tag{30}$$

where as before $\mathcal{K}:\mathcal{E}'\tau^1_1(M_1)\to\tau^1_1(M_1)$ is a properly supported smoothing operator.

Now let us suppose that $f\in L^2_\beta\tau^1_1(M)$. Unfortunately this does not mean that $f\in L^2_\beta\tau^1_1(M_1)$ since derivatives of $f$ extended as zero to $M_1$ will in general be singular on $\partial M$. To overcome this problem we use the decomposition (13) on $M_1$. Indeed, take $\beta\in H^1_0(M_1)$ to be the solution of the Dirichlet problem

$$\Delta_g\beta = \mathrm{d}_\beta f \qquad \beta|_{\partial M_1} = 0. \tag{31}$$

We will denote the operator which takes $\mathrm{d}_\beta f$ to the function $\beta\in H^1_0(M_1)$ by $\Delta_g^{-1}$, and so with this notation $\beta = \Delta_g^{-1}\,\mathrm{d}_\beta\, f$. Now define $f^\beta = f - (*\mathrm{d}\beta)^\#$, and so $\mathrm{d}_\beta f^\beta = 0$ on $M_1^{int}$. Note also that $\beta$ is harmonic on $M^{int}$ and $M_1\setminus M$ since $\mathrm{d}_\beta f = 0$ on both of those sets, but that $\beta$ may be singular on $\partial M$.

Next we apply (30) to $f^\beta$. This yields

$$(A\circ\phi^m\circ\mathcal{N}_{U_1,U_2}\circ\phi^m)[f^\beta] + (B\circ\phi^m\circ\Lambda^2\circ *)[(\mathrm{d}\,\phi)\wedge\mathrm{Alt}(f^\beta)^\flat] = f^\beta + \mathcal{K}[f^\beta] \quad \text{on } M^{int}_{1/2}. \tag{32}$$

Note that the second term on the left hand side of (32) is a properly supported ΨDO of order $-1$ applied to $f^\beta$, and so we may rewrite (32) as

$$(A\circ\phi^m\circ\mathcal{N}_{U_1,U_2}\circ\phi^m)[f^\beta] = f^\beta + \mathcal{K}_1[f^\beta] \quad \text{on } M^{int}_{1/2}. \tag{33}$$

where $\mathcal{K}_1 = \mathcal{K} - (\phi^m\circ B\circ\phi^m\circ\Lambda^2\circ *(\mathrm{d}\phi)^\wedge\circ\mathrm{Alt}\circ\flat)$ is a properly supported ΨDO of order $-1$.

Now recall that $\Lambda^2$ is a parametrix for $-\Delta_g$ on $M_1$, and therefore $(\Delta_g^{-1}+\Lambda^2)\circ\mathrm{d}_\beta = \tilde{\mathcal{K}} : \mathcal{E}'\tau^1_1(M_1)\to\Lambda^0(M_1)$ is a smoothing operator. Using this fact we may write

$$f^\beta = f - (*\mathrm{d}\,\tilde{\mathcal{K}}\,[f])^\# + (*\mathrm{d}(\Lambda^2\circ\mathrm{d}_\beta)[f])^\#.$$

Plugging this into (33), using lemma 1, and making use of the fact that $[\phi^m, *\mathrm{d}]$ is a ΨDO of order 0, we have

$$(A\circ\phi^m\circ\mathcal{N}_{U_1,U_2}\circ\phi^m)[f] = f^\beta + \mathcal{K}_2[f] \quad \text{on } M^{int}_{1/2} \tag{34}$$

where $\mathcal{K}_2$ is a new operator with the same properties as $\mathcal{K}_1$.

From the last equation we have the following estimate which is similar to (28)

$$\|f^\beta\|_{L^2\tau^1_1(M_1)} \leq C(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)} + \|f\|_{H^{-1}\tau^1_1(M_1)}). \tag{35}$$

If we were able to replace $f^\beta$ by $f$ on the left hand side of this estimate, then the remainder of the proof would follow as in the proof of Theorem 2. We will now proceed to show that it is possible to make this replacement. To start note that since $f = 0$ on $M^{int}_{1/2} \setminus M$, from (34) we have

$$(*\mathrm{d}\beta)^\# = -(A \circ \phi^m \circ \mathcal{N}_{U_1,U_2} \circ \phi^m)[f] + \mathcal{K}_2[f] \quad \text{on } M^{int}_{1/2} \setminus M. \tag{36}$$

We will use this fact along with the definition of $\beta$, (31), to estimate $\|\beta\|_{H^1(M)}$. Directly from (36), we have the estimate

$$\|\mathrm{d}\beta\|_{L^2\tau_1(M_{1/2}\setminus M^{int})} = \|(*\mathrm{d}\beta)^\#\|_{L^2\tau^1_1(M_{1/2}\setminus M^{int})} \leq C\left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)} + \|f\|_{H^{-1}\tau^1_1(M_1)}\right). \tag{37}$$

Next we derive a Poincaré type inequality for $\beta$ in order to estimate its norm in $H^1(M_{1/2} \setminus M^{int})$. From the way we have defined $M_{1/2}$, $M_{1/2} \setminus M^{int}$ is diffeomorphic to $\partial M \times [0, \epsilon/2]$ and $(x,t) \in \partial M \times [0, \epsilon/2]$ provide boundary normal coordinates on $M_{1/2} \setminus M^{int}$ where $\{t = 0\} = \partial M$. Using the covector field $t\,\mathrm{d}t$ defined in these coordinates on $M_{1/2} \setminus M^{int}$ together with Stokes' theorem we have

$$\begin{aligned}
\|\beta\|^2_{L^2(M_{1/2}\setminus M^{int})} &= \int_{M_{1/2}\setminus M^{int}} |\beta|^2 \,\mathrm{d}v_g \\
&= -\int_{M_{1/2}\setminus M^{int}} \mathrm{d}(|\beta|^2) \wedge *(t\,\mathrm{d}t) + \int_{\partial(M_{1/2}\setminus M^{int})} |\beta|^2 \, *(t\,\mathrm{d}t) \\
&= -\int_{M_{1/2}\setminus M^{int}} 2\,\mathrm{Re}(\overline{\beta}\,\mathrm{d}\beta) \wedge *(t\,\mathrm{d}t) + \frac{\epsilon}{2}\int_{\partial M_{1/2}} |\beta|^2 \,\mathrm{d}v_{\tilde{g}} \\
&\leq C\left(\|\beta\|_{L^2(M_{1/2}\setminus M)}\,\|\mathrm{d}\beta\|_{L^2\tau_1(M_{1/2}\setminus M)} + \|\beta\|^2_{L^2(\partial M_{1/2})}\right) \\
&\leq C\left(\tfrac{1}{2C}\|\beta\|^2_{L^2(M_{1/2}\setminus M)} + \tfrac{C}{2}\|\mathrm{d}\beta\|^2_{L^2\tau_1(M_{1/2}\setminus M)} + \|\beta\|^2_{L^2(\partial M_{1/2})}\right).
\end{aligned}$$

Now, using the operators defined above we have that $\beta = \tilde{\mathcal{K}}[f] - (\Lambda^2 \circ \mathrm{d}_\beta)[f]$. If we take a function $\psi \in C^\infty_c(M_{1/2})$ that is equal to 1 on $M$, and $\tilde{\psi} \in C^\infty_c(M_1 \setminus \mathrm{supp}(\psi))$ that is equal to 1 on a neighborhood of $\partial M_{1/2}$, then on some neighborhood $V$ of $\partial M_{1/2}$ we will have

$$\beta|_V = (\tilde{\psi}^m \circ \tilde{\mathcal{K}} \circ \psi^m - \tilde{\psi}^m \circ \Lambda^2 \circ \mathrm{d}_\beta \circ \psi^m)[f].$$

The operator involved in the previous equation now has a properly supported $C^\infty$ kernel, and so maps $H^{-1}\tau^1_1(M_1) \to H^1(M_1)$ continuously. Combining this with the trace theorem we have

$$\|\beta\|_{L^2(\partial M_{1/2})} \leq C\,\|f\|_{H^{-1}(M_1)}.$$

Together with the calculation from above this shows that

$$\|\beta\|_{H^1(M^{int}_{1/2}\setminus M)} \leq C\left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)} + \|f\|_{H^{-1}\tau^1_1(M_1)}\right).$$

Once more by the trace theorem

$$\|\beta\|_{H^{1/2}(\partial M)} \leq C\left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)} + \|f\|_{H^{-1}\tau^1_1(M_1)}\right).$$

Since $\beta$ satisfies $\Delta_g\beta = \mathrm{d}_\beta f = 0$ on $M^{int}$, standard estimates prove that

$$\|\beta\|_{H^1(M)} \leq C\,\|\beta\|_{H^{1/2}(\partial M)} \leq C\left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau^1_1(M_1)} + \|f\|_{H^{-1}\tau^1_1(M_1)}\right). \tag{38}$$

Finally, since $f = f^\beta + (*\mathrm{d}\,\beta)^\#$ from (35) and (38) we obtain

$$\|f\|_{L^2\tau_1^1(M)} \leq C\left(\|f^\beta\|_{L^2\tau_1^1(M)} + \|\beta\|_{H^1(M)}\right) \leq C\left(\|\mathcal{N}_{U_1,U_2}[f]\|_{H^1\tau_1^1(M_1)} + \|f\|_{H^{-1}\tau_1^1(M_1)}\right).$$

The fact that the kernel of $I_{U_1,U_2}$ acting on $L^2_\beta\tau_1^1(M)$ is finite dimensional now follows as in the proof of Theorem 2. Also, by the pseudolocal property of ΨDOs, (32) implies that $f = (*\mathrm{d}\,\beta)^\#$ modulo a smooth function on the interior of $M$. Since $\beta$ is harmonic on $M^{int}$ it is smooth there, and so this implies that $f$ is smooth on $M^{int}$.

Now assume that $f \in C^3_\beta\tau_1^1(M)$ is in the kernel of $I_{U_1,U_2}$ and satisfies the tangential boundary condition. Then using the method of proof from [9, Theorem 4] we can show that $f$ vanishes to first order on $\partial M$, and thus $f \in C^1\tau_1^1(M_1)$. Since $\mathrm{d}_\beta(f) = 0$ on $M$ and $M_1 \setminus M$, this implies that $\mathrm{d}_\beta(f) = 0$ on all of $M_1$. From this we see that $f^\beta = f$, and so from (32) we conclude that $f \in \tau_1^1(M_1)$, which finally implies that $f$ must vanish to infinite order on $\partial M$.

If $\mathcal{L}$ is a closed subspace of $L^2_\beta\tau_1^1(M)$, then the stability estimate (10) follows just as in the proof of Theorem 2 by applying lemma 2 with $X = \mathcal{L}$, and then using Theorem 8. This completes the proof of Theorem 6.

□

To end the section we will prove a stability result used in the proofs of Theorem 2 and Theorem 6.

**Theorem 8.** *Let $M$, $M_1$, $U_1$, $U_2$, and $g$ be all as in either Theorem 2 or Theorem 6, and let $a$ and $b \in \mathbb{R}$ with $0 < a < 1 < b$. If $U_1'$, $U_2' \in C^3\beta_1^1(TM_1 \setminus \{0\})$, and $g' \in C^4S_2M$ with $\|U_1 - U_1'\|_{C^3\beta_1^1(\Omega_a^bM_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta_1^1(\Omega_a^bM_1)} < \epsilon$, $\|g - g'\|_{C^4S_2M} < \epsilon$ for $\epsilon$ sufficiently small, and the unit spheres with respect to both $g$ and $g'$ are both contained in $\Omega_a^bM_1$, then $(M, g')$ is still a simple manifold (in the sense that the exponential map is a $C^3$ diffeomorphism at every point), and*

$$\|\mathcal{N}_{U_1,U_2} - \mathcal{N}'_{U_1',U_2'}\|_{L^2\tau_1^1(M)\to H^1\tau_1^1(M_1)} \leq C'\epsilon$$

*for some constant $C'$ which depends only on $U_1$, $U_2$, and $g$.*

*Proof.* To prove this theorem we will carefully compare the kernels of the two operators $\mathcal{N}_{U_1,U_2}$ and $\mathcal{N}_{U_1',U_2'}$ in the global coordinates on $M_1$. The same method is also applied in [6] and [10] to similar problems. Let us assume that $U_1$, $U_2$, and $g$ are as in the statement, and that $\|U_1 - U_1'\|_{C^3\beta_1^1(\Omega_a^bM_1)} < \epsilon$, $\|U_2 - U_2'\|_{C^3\beta_1^1(\Omega_a^bM_1)} < \epsilon$, and $\|g - g'\|_{C^4S_2M} < \epsilon$ for some $\epsilon > 0$.

For each $x \in M_1$, let $\mathcal{F}_x^t$ denote the maximal domain of the map from $\mathbb{R} \times (\Omega_a^b)_xM_1$ to $M_1$ given by $(t,\omega) \mapsto \gamma_\omega(t\omega)$ where $\gamma_\omega$ is maximally extended geodesic with initial data $\omega$ for the metric $g$. Then we begin by considering the map $F_x : \mathcal{F}_x^t \to \mathbb{R} \times \mathbb{S}^{n-1}$ defined by

$$F_x(t,\omega) = \left( t\left|\int_0^1 \dot\gamma_\omega(rt)\,\mathrm{d}r\right|, \frac{\displaystyle\int_0^1 \dot\gamma_\omega(rt)\,\mathrm{d}r}{\left|\displaystyle\int_0^1 \dot\gamma_\omega(rt)\,\mathrm{d}r\right|}\right) = (\rho,\theta). \tag{39}$$

We also define an analogous map with respect to $g'$ and denote it by $F_x'$. In fact, throughout this proof primed maps and sets will refer to those defined with respect to $g'$, and unprimed maps and sets are defined in the same way with respect to $g$. It is clear that $F_x$ and $F_x'$ are $C^3$ functions on their entire respective domains, and also depend in a $C^3$ manner on $x$. By possibly extending $g$ and $g'$ continuously in the $C^4$ norm beyond $M_1$, we may assume that $F_x$ and $F_x'$ are both defined on the same domain.

As a first step we will estimate $\|F_x(t,\omega)-F'_x(t,\omega)\|_{C^3_{x,t,\omega}}$. To accomplish this we use the following result from [4].

**Lemma 3.** *Let $x$ and $\tilde{x}$ solve the ODE systems*

$$x' = G(t,x), \quad \tilde{x}' = \tilde{G}(t,\tilde{x}),$$

*where $G$, $\tilde{G}$ are continuous functions from $[0,T]\times U$ to a Banach space $\mathcal{B}$, where $U \subset \mathcal{B}$ is open. Let $G$ be Lipschitz w.r.t. $x$ with a Lipschitz constant $k > 0$. Assume that*

$$\|G(t,x) - \tilde{G}(t,x)\| \leq \delta, \quad \forall t \in [0,T],\ \forall x \in U,$$

*and that $x(t)$, $\tilde{x}(t)$ stay in $U$ for $0 \leq t \leq T$. Then for $0 \leq t \leq T$*

$$\|x(t) - \tilde{x}(t)\| \leq e^{kt}\|x(0) - \tilde{x}(0)\| + \frac{\delta}{k}\left(e^{kt} - 1\right).$$

We first apply this lemma to the exponential map by recalling that the geodesics $\gamma_\omega(t)$ and $\gamma'_\omega(t)$ satisfy respectively the initial value problems

$$\begin{cases} \dot{\gamma}_\omega(t)^j = g(\gamma_\omega(t))^{jk}\,\xi_\omega(t)_k \\ \dot{\xi}_\omega(t)_k = -\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}(\gamma_\omega(t))\,\xi_\omega(t)_i\,\xi_\omega(t)_j \\ \gamma_\omega(0)^j = x^j, \quad \xi_\omega(0)_k = g(x)_{kj}\,\omega^j, \end{cases} \text{ and } \begin{cases} \dot{\gamma}'_\omega(t)^j = g'(\gamma'_\omega(t))^{jk}\,\xi'_\omega(t)_k \\ \dot{\xi}'_\omega(t)_k = -\frac{1}{2}\frac{\partial g'^{ij}}{\partial x^k}(\gamma'_\omega(t))\,\xi'_\omega(t)_i\,\xi'_\omega(t)_j \\ \gamma'_\omega(0)^j = x^j, \quad \xi'_\omega(0)_k = g'(x)_{kj}\,\omega^j. \end{cases}$$

By differentiating these systems with respect to the initial conditions we may obtain similar systems for the derivatives of $\gamma_\omega(t)$ and $\gamma'_\omega(t)$. Applying lemma 3 to these systems and using the fact that $\|g - g'\|_{C^4 S_2(M)} < \epsilon$ we obtain

$$\|\gamma_{x,\omega}(t) - \gamma'_{x,\omega}(t)\|_{C^3_{x,t,\omega}} + \|\dot{\gamma}_{x,\omega}(t) - \dot{\gamma}'_{x,\omega}(t)\|_{C^3_{x,t,\omega}} < C\epsilon \tag{40}$$

for some constant $C > 0$ depending only on $g$ and $M_1$. This estimate shows that if $\epsilon$ is taken small enough then $(M_1, g')$ is still simple (ie. the exponential maps at every point are $C^3$ diffeomorphisms). Using the expressions from (39) for $F_x(t,\omega)$ and $F'_x(t,\omega)$ in terms of $\gamma_{x,\omega}(t)$ and $\gamma'_{x,\omega}(t)$ together with the estimate (40) gives

$$\|F_x(t,\omega) - F'_x(t,\omega)\|_{C^3_{x,t,\omega}} < C\epsilon \tag{41}$$

for a new constant $C > 0$ which still only depends on $g$ and $M_1$.

Next we introduce a map $F_x^{-1}$ defined by

$$F_x^{-1}(\rho,\theta) = \left(\operatorname{sign}(\rho)\left|\exp_x^{-1}(\rho\theta + x)\right|_g, \operatorname{sign}(\rho)\frac{\exp_x^{-1}(\rho\theta + x)}{\left|\exp_x^{-1}(\rho\theta + x)\right|_g}\right) \tag{42}$$

and the corresponding map $(F'_x)^{-1}$ for $g'$. It is not difficult to check that $F_x^{-1}$ and $(F'_x)^{-1}$ are right inverses for $F_x$ and $F'_x$ respectively, and two-sided inverses of $F_x|_{\Omega_x M_1}$ and $F'_x|_{\Omega'_x M_1}$.

We will now use (41) to estimate $\|F_x^{-1}(\rho,\theta) - (F')_x^{-1}(\rho,\theta)\|_{C^3_{x,\theta,\rho}}$. Indeed, working in some appropriate set of local coordinates for $\theta \in \mathbb{S}^{n-1}$, we have

$$(F_x^{-1}\circ F'_x - F_x^{-1}\circ F_x)(t,\omega) = \left(\int_0^1 DF_x^{-1}(sF'_x(t,\omega) + (1-s)F_x(t,\omega))\,\mathrm{d}s\right)\cdot(F'_x(t,\omega) - F_x(t,\omega)).$$

Taking derivatives of this last equation we see that $\|F_x^{-1}\circ F'_x - F_x^{-1}\circ F_x\|_{C^3_{x,t,\omega}}$ can be bounded in terms of (41), and $\|DF_x^{-1}\|_{C^3_{x,\rho,\theta}}$. Next note that

$$F_x^{-1} - (F'_x)^{-1} = (F_x^{-1}\circ F'_x - F_x^{-1}\circ F_x)\circ(F'_x)^{-1} + (F_x^{-1}\circ F_x - \mathrm{Id})\circ(F'_x)^{-1}, \tag{43}$$

and observe that $(F_x^{-1} \circ F_x)(t,\omega) = (t|\omega_g|, \omega/|\omega|_g)$. From the hypothesis $\|g - g'\|_{C^4 S_2 M} < \epsilon$, we may therefore conclude that $\|(F_x^{-1} \circ F_x - \mathrm{Id})\big|_{\Omega' M_1}\|_{C^3_{x,t,\omega}} < C\epsilon$. Finally, by (41), $\|F'_x\|_{C^3_{x,t,\omega}}$ and $\|(DF'_x)^{-1}\|_{C^2_{x,\rho,\theta}}$ are uniformly bounded if $\epsilon$ is sufficiently small. Thus, from (43) we conclude that

$$\|F_x^{-1}(\rho,\theta) - (F'_x)^{-1}(\rho,\theta)\|_{C^3_{x,\rho,\theta}} < C\epsilon \tag{44}$$

for a new constant $C > 0$.

With estimate (44) in hand, we will now continue to derive a formula for the kernels of $\mathcal{N}_{U_1,U_2}$ and $\mathcal{N}'_{U'_1,U'_2}$. Beginning from (21) first define

$$\begin{aligned} A_x(v)^{u\alpha}_{r\nu} =& g^{\alpha\alpha'}(x)\, g_{\nu\nu'}(x)(w_2(\dot\gamma_{v/\|v\|_g}(\|v\|_g))^{uk'b'}_r\, \overline{w_1(v/\|v\|_g)}^{\nu'}_{\alpha' k' b'} \\ &+ w_2(-\dot\gamma_{v/\|v\|_g}(\|v\|_g))^{uk'b'}_r\, \overline{w_1(-v/\|v\|_g)}^{\nu'}_{\alpha' k' b'}), \end{aligned} \tag{45}$$

and let $A'_x(v)$ be defined in the same way with $g$ replaced by $g'$, $w_1$ replaced by $w'_1$, and $w_2$ replaced by $w'_2$. Let $f \in \tau_1^1(M)$ be extended as zero to $M_1$. Then with this notation we have for $x \in M_1^{int}$

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\mathcal{F}_x\setminus\{0\}} A_x(v)^{u\alpha}_{r\nu}\, f(\exp_x(v))^r_u \sqrt{\det(g)}\, \frac{\mathrm{d}v}{\|v\|_g^{n-1}}.$$

Switching to polar coordinates $(t,\omega)$ on $\mathcal{F}_x M_1 \setminus \{0\}$ gives

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\Omega_x M_1} \int_0^{l(\omega)} A_x(t\omega)^{u\alpha}_{r\nu}\, f(\exp_x(t\omega))^r_u\, \mathrm{d}t\, \mathrm{d}\omega. \tag{46}$$

A similar formula also holds for $\mathcal{N}'_{U'_1,U'_2}[f]$. Next we introduce a cut-off function $\chi \in C_c^\infty(M_1^{int})$ that equals 1 on $M$. Since $f$ vanishes on $M_1 \setminus M$, we may multiply the integrand in (46) by $\chi(\exp_x(t\omega))$ and this does not change $\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu$. Finally, we change variables by the map $F_x^{-1}$ in (46) to get

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\mathbb{S}^{n-1}} \int_0^\infty \chi(x+\rho\theta) A_x((F_x^{-1})_t(\rho,\theta)\, (F_x^{-1})_\omega(\rho,\theta))^{u\alpha}_{r\nu} f(x+\rho\theta)^r_u \left|\frac{\partial F_x^{-1}}{\partial(\rho,\theta)}\right| \mathrm{d}\rho\, \mathrm{d}\theta. \tag{47}$$

Let us now define

$$\tilde A(x,\rho,\theta)^{u\alpha}_{r\nu} = \chi(x+\rho\theta) A_x((F_x^{-1})_t(\rho,\theta)\, (F_x^{-1})_\omega(\rho,\theta))^{u\alpha}_{r\nu} \left|\frac{\partial F_x^{-1}}{\partial(\rho,\theta)}\right| \tag{48}$$

and rewrite (47) in terms of this function as

$$\mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu = \int_{\mathbb{S}^{n-1}} \int_0^\infty \tilde A(x,\rho,\theta)^{u\alpha}_{r\nu}\, f(x+\rho\theta)^r_u\, \mathrm{d}\rho\, \mathrm{d}\theta. \tag{49}$$

We may check that $\tilde A(x,\rho,\theta)^{u\alpha}_{r\nu}$ is in fact smooth up $\rho = 0$, and so we take the linear approximation to $\tilde A(x,\rho,\theta)^{u\alpha}_{r\nu}$ near $\rho = 0$

$$\tilde A(x,\rho,\theta)^{u\alpha}_{r\nu} = \tilde A(x,0,\omega)^{u\alpha}_{r\nu} + \rho\, R(x,\rho,\theta)^{u\alpha}_{r\nu},$$

and plug this into (49). At the same time we also change the integration from polar to Cartesian coordinates ($y = x + \rho\theta$) to get

$$\begin{aligned} \mathcal{N}_{U_1,U_2}[f](x)^\alpha_\nu =& \int_{\mathbb{R}^n} \tilde A\left(x, 0, \frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu} f(y)^r_u \frac{\mathrm{d}y}{|y-x|^{n-1}} \\ &+ \int_{\mathbb{R}^n} R\left(x, |y-x|, \frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu} f(y)^r_u\, \frac{\mathrm{d}y}{|y-x|^{n-2}}. \end{aligned} \tag{50}$$

As before, we also have a similar formula for $\mathcal{N}'_{U_1',U_2'}[f]$.

Using (50), we now compare $\mathcal{N}_{U_1,U_2}$ and $\mathcal{N}'_{U_1',U_2'}$. Indeed, we have

$$\begin{aligned}\left(\mathcal{N}_{U_1,U_2}-\mathcal{N}'_{U_1',U_2'}\right)[f](x)^\alpha_\nu = \\ \int_{\mathbb{R}^n}\left(\tilde{A}\left(x,0,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu}-\tilde{A}'\left(x,0,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu}\right)f^r_u(y)\frac{\mathrm{d}y}{|y-x|^{n-1}} \\ +\int_{\mathbb{R}^n}\left(R\left(x,|y-x|,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu}-R'\left(x,|y-x|,\frac{y-x}{|y-x|}\right)^{u\alpha}_{r\nu}\right)f(y)^r_u\,\frac{\mathrm{d}y}{|y-x|^{n-2}}.\end{aligned}\tag{51}$$

For the moment assume that

$$\left\|\tilde{A}\left(x,0,\omega\right)^{u\alpha}_{r\nu}-\tilde{A}'\left(x,0,\omega\right)^{u\alpha}_{r\nu}\right\|_{C^1(\mathbb{R}^n_x\times\mathbb{S}^{n-1}_\omega)}<C\epsilon \tag{52}$$

and

$$\left\|R\left(x,\rho,\omega\right)^{u\alpha}_{r\nu}-R'\left(x,\rho,\omega\right)^{u\alpha}_{r\nu}\right\|_{C^1(\mathbb{R}^n_x\times\mathbb{R}_\rho\times\mathbb{S}^{n-1}_\omega)}<C\epsilon \tag{53}$$

where $C>0$ is some new constant. Since $|y-x|^{n-1}$ and $|y-x|^{n-2}$ are integrable singularities in each variable individually, we may apply [21, Proposition A.5.1] together with the above estimates and (51) to conclude that

$$\left\|\mathcal{N}_{U_1,U_2}-\mathcal{N}'_{U_1',U_2'}\right\|_{L^2\tau^1_1(M)\to L^2\tau^1_1(M_1)}<C\epsilon.$$

It remains to estimate the $L^2$ norms of the derivatives of the components of $(\mathcal{N}_{U_1,U_2}-\mathcal{N}_{U_1',U_2'})[f]$. Indeed, we may simply differentiate with respect to $x$ under the second integral in (51), and after doing this we have a new integral operator applied to $f$ whose kernel is still integrable in each of the variables. Furthermore, these integrals can be uniformly bounded using (53), and so the desired estimates follow once again from [21, Proposition A.5.1]. Estimating the derivatives of the first integral in (51) poses a problem since when we differentiate that kernel with respect to $x$ the result is no longer integrable. However, since $\tilde{A}(x,0,\omega)$ and $\tilde{A}'(x,0,\omega)$ are even with respect to $\omega$, we may still apply the Calderón-Zygmund Theorem to estimate the singular integral which results from differentiating the kernel, and by [11, Theorem XI.11.1] this is the derivative of the integral. This argument combined with (52) shows that the derivatives of the components of the first integral are bounded by $C\epsilon\|f\|_{L^2\tau^1_1(M)}$ in $L^2(M_1)$, and so this completes the proof assuming (52) and (53).

All that remains now is to prove the estimates (52) and (53). Since

$$R(x,\rho,\theta)^{u\alpha}_{r\nu}=\int_0^1\frac{\partial\tilde{A}}{\partial\rho}(x,\rho s,\omega)^{u\alpha}_{r\nu}\,\mathrm{d}s,$$

to prove (53) it is sufficient to show that

$$\left\|\tilde{A}(x,\rho,\theta)^{u\alpha}_{r\nu}-\tilde{A}'(x,\rho,\theta)^{u\alpha}_{r\nu}\right\|_{C^2(\mathbb{R}^n_x\times\mathbb{R}_\rho\times\mathbb{S}^{n-1}_\theta)}<C\epsilon. \tag{54}$$

Recall that $\tilde{A}$ and $\tilde{A}'$ are defined by (48), and note that

$$\begin{aligned}A_x\big((F_x^{-1})_t(\rho,\theta)\,(F_x^{-1})_\omega(\rho,\theta)\big)^{u\alpha}_{r\nu}-A'_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}= \\ \left(A_x\left((F_x^{-1})_t(\rho,\theta)\,(F_x^{-1})_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}-A_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}\right) \\ +\left(A_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}-A'_x\left((F_x^{-1})'_t(\rho,\theta)\,(F_x^{-1})'_\omega(\rho,\theta)\right)^{u\alpha}_{r\nu}\right).\end{aligned}$$

Therefore, using also (44) and the fact that all derivatives of $A_x(t\omega)$ are bounded (by (56) below), we see that in order to prove (54) it is sufficient to show that

$$\|A_x(t\omega)^{u\alpha}_{r\nu} - A'_x(t\omega)^{u\alpha}_{r\nu}\|_{C^2_{x,t,\omega}}. \tag{55}$$

Returning to the definition (45) of $A_x(t\omega)$ and $A'_x(t\omega)$ we see that

$$\begin{aligned} A_x(t\omega)^{u\alpha}_{r\nu} &= g(x)^{\alpha\alpha'}\, g(x)_{\nu\nu'} \\ &\quad \left( w_2(\dot\gamma_\omega(t))^{uk'b'}_r\, \overline{w_1((\omega))}^{\nu'}_{\alpha' k' b'} + w_2(-\dot\gamma_\omega(t))^{uk'b'}_r\, \overline{w_1((-\omega))}^{\nu'}_{\alpha' k' b'} \right). \end{aligned} \tag{56}$$

Using the definitions of $w_1$ and $w_2$, which are respectively (19) and (20), we can write out a more explicit version of (56). This is

$$\begin{aligned} A_x(t\omega)^{u\alpha}_{r\nu} = g(x)^{\alpha\alpha'}\, g(x)_{\nu\nu'}\, g(x)_{d'j'}\, g(\gamma_\omega(t))^{kp} \left(\mathcal{I}^{\gamma_\omega}_{t,0}\right)^{j'}_{b} \left(\mathcal{I}^{\gamma_\omega}_{t,0}\right)^{p'}_{k} \Big( \left(U_2^{-1}(\omega)\right)^{d'}_{a'} \\ \left(\overline{U_2^{-1}(\dot\gamma_\omega(t))}\right)^{b}_{a} [P_\omega]^{a'\nu'}_{m'\alpha'}\, [P_{\dot\gamma_\omega(t)}]^{au}_{mr}\, \left(U_1(\omega)\right)^{m'}_{p'} \left(\overline{U_1(\dot\gamma_\omega(t))}\right)^{m}_{p} - \left(U_2^{-1}(-\omega)\right)^{d'}_{a'} \\ \left(\overline{U_2^{-1}(-\dot\gamma_\omega(t))}\right)^{b}_{a} [P_{-\omega}]^{a'\nu'}_{m'\alpha}\, [P_{-\dot\gamma_\omega(t)}]^{au}_{mr}\, \left(U_1(-\omega)\right)^{m'}_{p'} \left(\overline{U_1(-\dot\gamma_\omega(t))}\right)^{m}_{p} \Big). \end{aligned}$$

A corresponding formula holds for $A'_x(t\omega)$, and we wish to estimate the difference of the two. To do this, it is sufficient to estimate the differences of the corresponding terms in the two formulas for $A_x(t\omega)$ and $A'_x(t\omega)$. If we note that the projections can be written in terms of $g$ and $g'$, we see that all of these differences are bounded in the $C^2$ norm by $C\epsilon$ by a combination of the hypotheses and (40), except for the difference in the parallel translation terms. To bound this last difference we note that for any vector $\eta^b$, $\left(\mathcal{I}^{\gamma_\omega}_{0,t}\right)^{j'}_{b} \eta^b$ satisfies the system of ODEs

$$\left(\frac{\partial \mathcal{I}^{\gamma_\omega}_{0,t}}{\partial t}\right)^{j'}_{b} \eta^b = \Gamma(\gamma_\omega(t))^{j'}_{kl}\, \dot\gamma_\omega(t)^k \left(\mathcal{I}^{\gamma_\omega}_{0,t}\right)^{l}_{b} \eta^b \quad \text{and} \quad \left(\mathcal{I}^{\gamma_\omega}_{0,0}\right)^{j'}_{b} \eta^b = \eta^b$$

where $\Gamma^{j'}_{kl}$ are the Christoffel symbols of the metric $g$. The same formula holds for the parallel translation with respect to the $g'$ metric when the Christoffel symbols and geodesics are those of the $g'$ metric. Therefore, by lemma 3, (40), the hypothesis that $\|g - g'\|_{C^4 S^2 M_1} < \epsilon$, and the definition of the Christoffel symbols,

$$\left\| \left( \left(\mathcal{I}^{\gamma_\omega}_{0,t}\right)^{j'}_{b} - \left(\mathcal{I}'^{\gamma'_\omega}_{0,t}\right)^{j'}_{b} \right) \eta^b \right\|_{C^3_{x,t,\omega}} < C\epsilon. \tag{57}$$

Since this holds for any vector $\eta^b$, and $\mathcal{I}^{\gamma_\omega}_{0,t} = \left(\mathcal{I}^{\gamma_{x,\omega}}_{t,0}\right)^{-1}$, this implies the needed estimate on the difference of the parallel translation factors, and so completes the proof. □

3.3. **Injectivity for the linear problem.** The results of the previous section establish that the set of $U_1$, $U_2$, and $g$ for which $I_{U_1,U_2}$ is injective is open in the $C^4$ topology when the dimension is greater than 3. In dimension 3 the same is true if $I_{U_1,U_2}$ is restricted to $L^2_\beta \tau^1_1(M)$. We would like to first know that this set is also nonempty, but in fact Theorems 3 and 7 give much more than this. They say that $I_{U_1,U_2}$ is injective for any real analytic $U_1$, $U_2$, and $g$.

Our proofs of these two theorems will use analytic microlocal analysis, and as a primary reference on this topic we use [16]. A different approach to analytic microlocal analysis is also given in [22]. Since we are using analytic methods, for this section we must assume that $M$ is a real analytic manifold (ie. that the transition maps are all real analytic). Our notation for the analytic wave front set of $f \in \mathcal{D}'\tau_1^1(M)$ will be $\mathrm{WF_a}(f)$. The main step in both proofs is the following lemma.

**Lemma 4.** *Suppose that* $(M,g)$, $U_1$, *and* $U_2$ *are as in Theorem 3 or 7, and* $\xi_0 \in T^*M^{int} \setminus \{0\}$. *In dimension greater than* 3 *let* $f \in L^2_c\tau_1^1(M)$, *and in dimension* 3 *assume that* $f \in (L^2_\beta)_c\tau_1^1(M)$. *If there is an open subset* $V \subset \Omega M$ *such that* $V \cap \xi_0^\perp \neq \emptyset$, *and on the set of unit speed geodesics whose tangent vectors pass through* $V$ $I_{U_1,U_2}[f]$ *is zero, then* $\xi_0 \notin \mathrm{WF_a}(f)$.

*Remark* 2. This result is actually more general than required for the proof of Theorem 3. Using this lemma we could show injectivity for the map $I_{U_1,U_2}$ composed with restriction to a smaller set than all of $\partial_-\Omega M$.

*Remark* 3. The method of proof used here was developed in [6] and [18], and is also used in [10]. Because many of the steps in the current proof are identical to the corresponding steps in those papers, we will mainly emphasize the portions that are different and only summarize those that are the same.

*Proof.* Let $\xi_0$ and $V$ be as in the statement of the theorem. Now take any $v \in V \cap \xi_0^\perp$. Then by the hypothesis there exists a $v' \in \partial_-\Omega M$ such that the tangent vector of $\gamma_{v'}$ passes through $v$, and for every $w$ in a neighborhood of $v$, $I_{U_1,U_2}[f](\gamma_w) = 0$. Following the method of [6], [18], and [10], we work in a set of coordinates centered at $x_0 = \pi(\xi_0)$ and defined in a neighborhood $U$ of $\gamma_{v'} \cap \mathrm{supp}(f)$. Using the complex method of stationary phase from [16] we are able to show that for some $\epsilon > 0$ sufficiently small, constant $C$ sufficiently large, phase function $\Phi$, and symbol $P$ described below we have

$$\left| \int_{\{|x-y|<\epsilon|\}} e^{\frac{i}{h}\Phi(x,y,\zeta)}\, P(x,y,\zeta;\, h)_{akb}^m\, f(x)_m^a \,\mathrm{d}x \right| = \mathcal{O}(e^{-\frac{C}{h}}) \tag{58}$$

for every $\zeta$ in a complex neighborhood of $\xi_0$ as $h \to 0^+$ (many steps have been omitted here, see [6], [18], or [10]). The phase function $\Phi(x,y,\zeta)$ satisfies

$$\Phi(x,x,\zeta) = 0, \quad -\Phi_y(x,x,\zeta) = \Phi_x(x,x,\zeta) = \zeta, \tag{59}$$

and

$$\mathrm{Im}(\Phi(x,y,\zeta)) > C'\,|x-y|^2 \tag{60}$$

for a positive constant $C'$. The functions $P(x,y,\eta;\, h)_{akb}^m$ are an array of classic analytic symbols of order 0 (see [16]) with principal symbol

$$\sigma_P(0,0,\xi_0)_{akb}^m = R(v')_{sb}(P_{v'})_{at}^{ms}\, Q(v')_k^t. \tag{61}$$

Recall that $R_{sb}$ and $Q_k^t$ are defined above (16) at the beginning of section 3, while the components of $(P_{v'})_{at}^{ms}$ may be calculated from (17).

Now we take a basis $\{v_j\}_{j=1}^{n-1}$ for $\xi_0^\perp$ contained in $\xi_0^\perp \cap V$, which is possible since $V$ is open. Furthermore, by choosing $\{v_j\}_{j=1}^{n-1}$ sufficiently close together we can be sure that $(v_i+v_j)/\|v_i+v_j\| \in \xi_0^\perp \cap V$ for every $i$ and $j$, and we add these $(n-1)(n-2)/2$ extra vectors to $\{v_j\}_{j=1}^{n-1}$ to obtain a set of vectors $\{v_j\}_{j=1}^{n(n-1)/2}$. Doing this we have from (58) a system of equations

$$\left| \int_{\{|x-y|<\epsilon/C|\}} e^{\frac{i}{h}\Phi_j(x,y,\eta)}\, \tilde{P}_j(x,y,\eta;\, h)_{akb}^m\, f(x)_m^a \,\mathrm{d}x \right| = \mathcal{O}(e^{-\frac{C_j}{h}}) \tag{62}$$

where $\Phi_j(x, y, \eta)$ are all phase functions satisfying (59) and (60), and $\tilde{P}_j(x, y, \eta; h)^m_{akb}$ are classical analytic symbols of order 0 with principal symbols satisfying, according to (61),

$$\sigma_{\tilde{P}_j}(0, 0, \xi_0)^m_{akb} = R(v_j)_{sb}(P_{v_j})^{ms}_{at}\, Q(v_j)^t_k.$$

Suppose that $f \in (T^1_1)^{\mathbb{C}}_{x_0} M$ satisfies $\sigma_{\tilde{P}_j}(0, 0, \xi_0)^m_{akb} f^a_m = 0$ for every $j$. If we recall the definitions of $R$ and $Q$, and use the fact that $U_1$ and $U_2$ are invertible, we see that this implies $P_{v_j} f = 0$ for every $j$. We will now show that in dimension at least 4 this implies that $f = 0$. Indeed, let us express the components of $f^\flat$ with respect to the basis $\{v_1, \dots, v_{n-1}, \xi_0^\sharp\}$ as $f_{ij}$. Then we have for all $1 \le j,\, k,\, l \le n-1$ the following equations

$$0 = \big((P_{v_j} f)\big)^\flat (v_k,\, v_l) = f_{kl} - \langle v_j, v_k\rangle_{g(x_0)} f_{jl} - \langle v_j, v_l\rangle_{g(x_0)} f_{kj} + \langle v_j, v_l\rangle_{g(x_0)}\langle v_j, v_k\rangle_{g(x_0)} f_{jj}, \qquad (63)$$

$$0 = \big((P_{v_j} f)\big)^\flat (\xi_0^\sharp,\, v_l) = f_{nl} - \langle v_j, v_l\rangle_{g(x_0)} f_{nj},$$

$$0 = \big((P_{v_j} f)\big)^\flat (v_k,\, \xi_0^\sharp) = f_{kn} - \langle v_j, v_k\rangle_{g(x_0)} f_{jn},$$

and

$$0 = \big((P_{v_j} f)\big)^\flat (\xi_0^\sharp,\, \xi_0^\sharp) = f_{nn}.$$

The last three of these equations together with the fact that $-1 < \langle v_j, v_l\rangle_{g(x_0)} < 1$ imply that $f_{nj}$ and $f_{jn} = 0$ for all $j$. Thus it only remains to show that $f_{kl} = 0$ for $1 \le k,\, l \le n-1$. From the conditions that $P_{v_j} f = 0$ for $j = n$ to $n(n-1)/2$, we obtain for all $1 \le k,\, l \le n-1$ that

$$0 = \Big(\big((P_{v_k+v_l} f)\big)^\flat v_k,\, v_l\Big) = \frac{1}{4}\langle f(v_k - v_l),\, v_l - v_k\rangle_{g(x_0)} = \frac{1}{4}(f_{kl} + f_{lk} - f_{kk} - f_{ll}). \qquad (64)$$

These equations together with (63) form a system of linear equations for the components $f_{jk}$ of $f^\flat$. We will now show that the only solution is $f_{jk} = 0$ for all $1 \le j, k \le n-1$.

First we deal with the symmetric part of $f^\flat$. Note that when we set $k = l$ in (63), we get

$$0 = f_{kk} + \langle v_j, v_k\rangle^2_{g(x_0)} f_{jj} - \langle v_j, v_k\rangle_{g(x_0)}(f_{jk} + f_{kj}). \qquad (65)$$

By subtracting the corresponding equation with $j$ and $k$ switched we get

$$(1 - \langle v_j, v_k\rangle^2_{g(x_0)}) f_{kk} = (1 - \langle v_j, v_k\rangle^2_{g(x_0)}) f_{jj} \Rightarrow f_{jj} = f_{kk}$$

since $|\langle v_j, v_k\rangle_{g(x_0)}| < 1$. Thus all the diagonal entries of $f$ are equal. Therefore (64) shows that $f_{jj} = \frac{1}{2}(f_{kl} + f_{lk})$ for any $j$, $k$, and $l$. Plugging this back into (65) then shows that

$$0 = (1 + \langle v_j, v_k\rangle^2_{g(x_0)} - 2\langle v_j, v_k\rangle_{g(x_0)}) f_{jj} = (1 - \langle v_j, v_k\rangle_{g(x_0)})^2 f_{jj},$$

and since $\langle v_j, v_k\rangle_{g(x_0)} < 1$, this implies that the symmetric part of $f$ is zero.

Now we show that the antisymmetric part of $f = 0$. Suppose that the $i$, $j$, and $k$ in (63) are all distinct (note that this portion of the proof fails in dimension 3 because in that case there are only two vectors, $v_1$ and $v_2$, in the basis for $\xi_0^\perp$). Then we may cyclically permute the three indices in (63) and subtract the resulting equations pairwise to obtain

$$\begin{aligned}
(f_{kl} - f_{lk}) - \langle v_j, v_k\rangle_{g(x_0)}(f_{jl} - f_{lj}) - \langle v_j, v_l\rangle_{g(x_0)}(f_{kj} - f_{jk}) &= 0,\\
-\langle v_j, v_l\rangle_{g(x_0)}(f_{kl} - f_{lk}) + \langle v_l, v_k\rangle_{g(x_0)}(f_{jl} - f_{lj}) + (f_{kj} - f_{jk}) &= 0,\\
\langle v_j, v_k\rangle_{g(x_0)}(f_{kl} - f_{lk}) - (f_{jl} - f_{lj}) - \langle v_l, v_k\rangle_{g(x_0)}(f_{kj} - f_{jk}) &= 0.
\end{aligned}$$

If we consider this as a system of equations for $f_{kl} - f_{lk}$, $f_{jl} - f_{lj}$, and $f_{jk} - f_{kj}$, and then compute the determinant of the coefficient matrix we get

$$1 - (\langle v_l, v_k\rangle^2_{g(x_0)} + \langle v_j, v_k\rangle^2_{g(x_0)} + \langle v_j, v_l\rangle^2_{g(x_0)}) + 2\langle v_j, v_k\rangle_{g(x_0)}\langle v_j, v_l\rangle_{g(x_0)}\langle v_l, v_k\rangle_{g(x_0)}.$$

The fact that $v_j$, $v_k$, and $v_l$ are linearly independent implies that this quantity is not zero. Therefore the antisymmetric part of $f$ is zero, and we have completed the proof that $f = 0$. This shows that our system of equations (62) provides an elliptic system near $(0, 0, \xi_0)$ in dimension greater than 3, in the sense that the principal symbol at $(0, 0, \xi_0)$ admits a left inverse. In order to have an elliptic system in dimension 3, we must add another equation corresponding to the condition that $\mathrm{d}_\beta(f) = 0$. We will now proceed to introduce this extra equation.

Let $U$ be a neighborhood of $x_0$ on which we have a set of coordinates centered at $x_0$, and let $\chi_0 \in C_c^\infty(U)$ be a smooth cut-off function that is equal to 1 on a neighborhood of $x_0 = 0 \in U$ . Then, since $\mathrm{d}_\beta(f) = 0$ we have

$$h\, e^{\frac{i}{h}\Phi_1(x,y,\zeta)}\, \chi_0(x)\, \mathrm{d}_\beta(f)(x) = 0$$

where $\Phi_1$ is one of the phase functions from (62). By integrating this equality in the $x$ variable we obtain

$$\int h\, e^{\frac{i}{h}\Phi_1(x,y,\zeta)}\, \chi_0(x)\, \mathrm{d}_\beta(f)(x)\mathrm{d}x = 0,$$

and then integration by parts yields

$$\begin{aligned}\int e^{\frac{i}{h}\Phi_1(x,y,\zeta)} \Bigg( i\frac{\chi_0}{\sqrt{\det(g)}} \Big[ \frac{\partial \Phi_1}{\partial x^3}(g_{1k}f_2^k - g_{2k}f_1^k) + \frac{\partial \Phi_1}{\partial x^2}(g_{3k}f_1^k - g_{1k}f_3^k) \\ + \frac{\partial \Phi_1}{\partial x^1}(g_{2k}f_3^k - g_{3k}f_2^k)\Big] + h\, D(x)_a^m f_m^a(x) \Bigg)\mathrm{d}x = 0\end{aligned} \tag{66}$$

where $D(x)_a^m$ is calculated from derivatives of $\chi_0$ and the metric $g$. We consolidate the factors in the integrand into one classical analytic symbol and thus rewrite the previous formula as

$$\int e^{\frac{i}{h}\Phi_1(x,y,\zeta)} \chi_0(x) \tilde{D}(x, y, \zeta; h)_m^a f(x)_a^m \mathrm{d}x = 0$$

where the array of principal symbols of $\tilde{D}$ at $(0, 0, \xi_0)$ is given by

$$\sigma_{\tilde{D}}(0, 0, \xi_0)[f] = \frac{i}{\sqrt{\det(g)}} \left( g_{2k}f_1^k - g_{1k}f_2^k \right) \tag{67}$$

where we are once again expressing the components of $f$ and $g$ with respect to the basis $\{v_1, v_2, \xi_0^\sharp\}$. To show that the addition of the extra equation creates an elliptic system we must show that, for $f \in (T_1^1)_{x_0}^{\mathbb{C}}$, $\sigma_{\tilde{D}}(0, 0, \xi_0)[f] = 0$ and $P_{v_j} f = 0$ for each $j$ as above implies that $f = 0$. The argument to show that the symmetric part of $f$ is zero still holds in this case, so we only need to show that the antisymmetric part is zero. Using the previous notation this means that we must show $f_{21} - f_{12} = 0$, but by (67) this is exactly equivalent to $\sigma_{\tilde{D}}(0, 0, \xi_0)[f] = 0$. Therefore the system provided by the extra equation in dimension 3 is elliptic in the same sense as before.

To finish the proof we now must generalize a result from [16] to the case of systems of operators. This generalization has already been done in [18] and [10], and applied to different systems of operators. I will repeat the arguments given there and apply them in the present situation. We first combine the systems of operators derived so far in the proof as

$$\int_{|x-y|<C} e^{\frac{i}{h}\Phi_j(x,\beta)} \mathbf{A}(x, \beta; h)_{mkbj}^a\, f(x)_a^m \mathrm{d}x = \mathcal{O}(e^{-\frac{C}{h}}) \tag{68}$$

where $C$ is a new constant, and following [16] we write $\beta = (y, \zeta)$. $\mathbf{A}(x, \beta; h)_{mkbj}^a$ is an array of classic analytic symbol of order 0 made up entirely of the symbols $\tilde{P}_j$ from (62) in the case of dimension greater than 3. In dimension 3 the symbol $\tilde{D}$ from (66) is also added. The key feature of $\mathbf{A}(x,\beta;h)^a_{mkbj}$ is that its principal symbol is injective as a map from $(T^1_1)_{x_0}M$ to $\bigoplus_{j=1}^{n(n-1)/2}(B_2)_{v'_j}(\partial_-\Omega M)$ in the case of greater than 3 dimensions and to $\bigoplus_{j=1}^{n(n-1)/2}(B_2)_{v'_j}(\partial_-\Omega M)\times \mathbb{C}$ in the case of three dimensions. Thus these maps have left inverses.

Now, following [16], [18], and [10] we define a system of ΨDO's in the complex domain

$$\mathrm{Op}(\mathbf{A})[f](y)_{kbj} = \iint e^{\frac{i}{h}\left(\Phi_j(y,\beta)-\overline{\Phi_j(x,\beta)}\right)}\mathbf{A}(x,\beta;h)^a_{mkbj}\, f(x)^m_a \mathrm{d}x\,\mathrm{d}\beta. \tag{69}$$

These operators have different phase functions $\Phi_j$, but using the trick of Kuranishi we may make an appropriate series of changes of variables to change them all to the same phase function $\Phi$ without changing the principal symbols. Therefore we may construct a parametrix for $\mathrm{Op}(\mathbf{A})$ and use this parametrix to express $\mathbf{Id}\, e^{\frac{i}{h}\Phi}$ (where $\mathbf{Id} : (T^1_1)_{x_0}M \to (T^1_1)_{x_0}M$ is the identity map) as a superposition of the $\mathbf{A}^a_{mkbj}\, e^{\frac{i}{h}\Phi}$ modulo an exponentially decreasing function. Following now the same argument as is given for proposition 6.2 in [16], but with matrix valued symbols, we have

$$\int_{|x-y|<C} e^{\frac{i}{h}\Phi(x,\beta)}\mathbf{Id}[f](x)\,\mathrm{d}x = \mathcal{O}(e^{-C/h}),$$

possibly with a new constant $C$, for every $\beta=(y,\zeta)$ in a neighborhood of $(0,\xi_0)$. This proves that $(x_0,\xi_0)$ is not in $\mathrm{WF_a}(f)$. □

*Proof of Theorems 3 and 7* First we consider Theorem 3 where the dimension is greater than 3. Assume that the hypotheses are all satisfied and $f\in L^2\tau^1_1(M)$ is in the kernel of $I_{U_1,U_2}$. Let $M_1$ be as in Theorem 2. Then, by Theorem 2, when we extend $f$ as zero on $M_1\setminus M$, the resulting function is smooth on all of $M_1$, and still in the kernel of $I_{U_1,U_2}$ acting now on $L^2\tau^1_1(M_1)$. Now by lemma 4 applied on $M_1$ the analytic wavefront set of $f$ is empty, and therefore $f$ is analytic. Since $f$ vanishes on $M_1\setminus M$ this implies that $f=0$, and therefore proves that $I_{U_1,U_2}$ is injective.

Now let us turn to Theorem 7 in which there are 3 dimensions. Once again assume that the hypotheses are all met, and that $f\in C^3_\beta\tau^1_1(M)$ is in the kernel of $I_{U_1,U_2}$, and that $f$ satisfies the tangential boundary condition. Let $M_1$ be as in Theorem 6. Then by Theorem 6 when we extend $f$ as zero on $M_1\setminus M$, the resulting tensor field is smooth on all of $M_1$, and satisfies $\mathrm{d}_\beta(f)=0$ on $M_1$. Therefore lemma 4 implies that the analytic wavefront set of $f$ is empty, and just as above this implies that $f=0$. This completes the proof.

□

## 4. The nonlinear problem.

We will now return to the nonlinear problem of recovering a tensor field $f\in\tau^1_1(M)$ from its polarization data $(U|_{\partial_+\Omega M}$. Our main goal will be to prove Theorems 1 and 5, and to accomplish this we will use the stability and injectivity results for the linear problem established in section 3. Our method here mostly fits into a general approach to the linearization of nonlinear inverse problems presented in [19].

In order to apply the results of section 3 we must first deal with the issue of extending the semi-basic tensor fields $U_1$ and $U_2$ given respectively by (2) for some $f_1$ and $f_2\in\tau^1_1(M)$ to the larger manifold $M_1$. Note that $U_1$ and $U_2$ are only defined from (2) on $\Omega M\setminus T\partial M$, and so it is not clear that $U_1$ and $U_2$ can be extended to semi-basic fields on the larger manifold $\Omega M_1$. In fact it is not possible in general to make such an extension since the derivatives of $U_1$ and $U_2$ may be unbounded near $T\partial M$. We will avoid this issue by replacing $U_1$ and $U_2$ with another pair of semi-basic tensor fields $\tilde{U}_1$ and $\tilde{U}_2$ such that (6) still holds. These fields are obtained by solving (2) on a larger manifold. A second related issue arises since we would like to establish uniqueness results for generic metrics $g'$ obtained by perturbations near an analytic metric $g$. When the metric is changed the set $\Omega M$ also changes, and so we actually need to consider $U_1$ and $U_2 \in \beta^1_1(TM_1 \setminus \{0\})$ as in Theorems 2 and 6. Section 4.1 deals with these issues of extending $U_1$ and $U_2$. In Section 4.2 we prove the main theorems.

4.1. **Extending $U_1$ and $U_2$.** Suppose that $(M, g)$ is a simple manifold. Also suppose that we have an extension $M_1$ of $M$ as described in section 1 and let $M_2$ be an extension of $M_1$ accomplished in the same way as the extension from $M$ to $M_1$. In particular we are assuming that $(M_2, g)$ is still a simple manifold, and that $M \Subset M_1 \Subset M_2$. The metric $g$ will be the reference around which we will perturb in order to obtain results for "generic" metrics.

Now let $f_1$ and $f_2 \in \tau^1_1(M)$. By [14] it is possible to define a linear and continuous extension mapping $E : \tau^1_1(M) \to (\tau^1_1)_c(M_2)$ so that all tensor fields in the range of $E$ have support contained within a given compact set $K$ such that $M \Subset K \Subset M_2$. We replace (2) with

$$HU(\xi) = [(P_\xi f)(x)]U(\xi) \quad \text{on } TM_2 \setminus \{\{0\} \cup T\partial M_2\}, \quad U|_{\partial_- TM_2} = E. \tag{70}$$

Here $\partial_- TM_2$ is the space of inward pointing tangent vectors not necessarily having unit length. In this case (3) holds where $\eta$ is still given by (1), but with $\xi \in \partial_- TM_2$ rather than $\partial_-\Omega M$.

Now define $\tilde{U}_1$ and $\tilde{U}_2 \in \beta^1_1(TM_2 \setminus \{\{0\} \cup T\partial M_2\})$ by solving (70) with $f$ replaced respectively by either $E[f_1]$ or $E[f_2]$ on $M_2$. Certainly $\tilde{U}_1$ and $\tilde{U}_2$ restrict to smooth semi-basic tensor fields in $\beta^1_1(TM_1 \setminus \{0\})$, and thus we will be able to apply the results from section 3 to the operator $I_{\tilde{U}_1,\tilde{U}_2}$.

The main task in this section is to prove that $\tilde{U}_1$ and $\tilde{U}_2$ have the properties given in the following lemma. We first recall the definition of the annulus $\Omega^b_a M_1 = \{(x, v) \in TM_1 \,|\, a < \|v\|_g < b\}$ where $0 < a < b$.

**Lemma 5.** *The tensor fields $\tilde{U}_1$ and $\tilde{U}_2$ defined above posses the following properties.*

- *For every $v \in TM_1 \setminus \{0\}$, $\tilde{U}_1(v), \tilde{U}_2(v) : T^{\mathbb{C}}_{\pi(v)}M_1 \to T^{\mathbb{C}}_{\pi(v)}M_1$ are invertible.* (71)
- *If $\gamma$ is any geodesic between points in $\partial M_1$ of length $l$, and $0 \le t_1 \le t_2 \le l$, then (6) holds where $M$ is replaced by $M_1$.* (72)
- *For fixed $f_1$ there exists an $\epsilon > 0$ such that if $\|f_1 - f_2\|_{C^3\tau^1_1(M)} < \epsilon$ then*
$$\|\tilde{U}_1 - \tilde{U}_2\|_{C^3\beta^1_1(\Omega^b_a M_1)} < C\|f_1 - f_2\|_{C^3\tau^1_1(M)} \tag{73}$$
*for some constant $C > 0$ which may depend on $a$, $b$, $g$, and $f_1$, but does not depend on $f_2$.*

*Proof.* The first statement (71) is an immediate consequence of the definition of $\tilde{U}_1$ and $\tilde{U}_2$. The second statement (71) follows from the derivation of the main identity in section 1. Now we turn to the proof of (73).

For any given $\xi \in \partial_-\Omega^b_a M_2$ and $\eta_0 \in T^{\mathbb{C}}_{\pi(\xi)}M_2$, let $\eta_i(s, \xi)$ ($i = 1$ or 2) be the solution of (1) on $M_2$ corresponding to $f_i$. Using global coordinates (1) becomes

$$\begin{aligned}\frac{\partial \eta^j_i}{\partial s}(s, \xi) &= \left([(P_{\dot\gamma_\xi(s)} f)(\gamma_\xi(s))]^j_k + \Gamma(\gamma_\xi(s))^j_{lk}\,\dot\gamma_\xi(s)^l\right)\eta^k_i(s, \xi)\\ &= G_i(s, \eta_i(s, \xi), \xi)^j\end{aligned} \tag{74}$$

and

$$\eta^j_i(0, \xi) = \eta^j_0.$$

Here the $\Gamma(x)^j_{lk}$ are the Christoffel symbols of the metric $g$. Now, in order to estimate $U_i(x,v)$ for any $(x,v) \in \Omega^b_a M_1$ we modify (3) to get

$$\eta^j_i \left( l(x,-v), \left( \mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)} \right)^b_a v^a \right) = \tilde{U}_i(x,v)^j_c \left( \mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0} \right)^c_d \eta^d_0. \tag{75}$$

Here $l(x,v)$ gives the positive endpoint of the maximally extended geodesic $\gamma_{x,v}$ in $M_2$. From this last equation we see that $\|\tilde{U}_1(x,v) - \tilde{U}_2(x,v)\|_{\beta^1_1(\Omega^b_a M_1)}$ may be bounded if we can bound the difference $\eta_1(s,\xi) - \eta_2(s,\xi)$ for every $\xi \in \partial_- \Omega^b_a M_2$, $s \in \mathbb{R}$ such that $\gamma_\xi(s) \in M_1$, and initial vectors $\eta_0$. To do this we use lemma 3 and the fact that

$$G_1(s,\eta,\xi)^j - G_2(s,\eta,\xi)^j = [(P_{\gamma_\xi(s)}(f_1 - f_2)(\gamma_\xi(s))]^j_k \, \eta^k.$$

If we assume a priori that $f_2$ is close to $f_1$, then $\eta_2(s,\xi)$ will be bounded, and so with the hypotheses this implies that $\|\eta_1(s,\xi) - \eta_2(s,\xi)\| < C\|f_1 - f_2\|_{C\tau^1_1(M)}$ uniformly for the required values of $s$ and $\xi$ where $C > 0$ does not depend on $f_2$. The norm on the left hand side of this estimate could be any norm on $\mathbb{C}^n$.

Next, differentiating (74) with respect to either $\xi$ or $s$ we may obtain ODEs satisfied by the derivatives of the left hand side of (75). Using this we can apply the same analysis as above to bound derivatives of $\tilde{U}_1(x,v) - \tilde{U}_2(x,v)$, except we also require bounds on the corresponding derivatives of $f_1 - f_2$. This proves the result. □

The final part of lemma 5 shows how the solution of (70) behaves when $f$ is perturbed, but we also want to see this behavior under perturbations of the metric $g$. Thus, we now suppose that $g' \in S_2M_2$ is another metric on $M_2$. If $g'$ is sufficiently close to $g$ in $C^4 S_2 M_2$, then by Theorem 8 $(M_2, g')$ is still a simple manifold, and we will always assume that $g'$ is such a metric. Let $f \in \tau^1_1(M)$ be one of the two tensor fields from above (either $f_1$ or $f_2$), and let $\tilde{U}$ also be the corresponding semi-basic tensor field defined from (70). If $\tilde{U}'$ is defined from (70) with $g$ replaced by $g'$, then we have the following lemma.

**Lemma 6.** *For a fixed metric $g$, there is an $\epsilon > 0$ such that whenever $\|g - g'\|_{C^5 S_2 M_2} < \epsilon$, for every $A > 0$ there is a constant $C$ such that*

$$\|\tilde{U} - \tilde{U}'\|_{C^3 \beta^1_1(\Omega^b_a M_1)} < C\|g - g'\|_{C^4 S_2 M_2}$$

*for every $f$ with $\|f\|_{C^3 \tau^1_1(M)} < A$.*

*Remark* 4. In fact we could only require that $\|g'\|_{C^5 S_2 M_2} < C'$ for a fixed constant $C'$ and $\|g - g'\|_{C^4 S_2 M_2} < \epsilon$.

*Proof.* As in the proof of (73) above, this result follows essentially from lemma 3. Throughout we will use primes to indicate objects corresponding to the metric $g'$, while unprimed objects will be those corresponding to $g$. As above we use (75) to estimate the difference $\tilde{U} - \tilde{U}'$. Indeed, working in global coordinates and using (75) we have

$$\begin{aligned} &\eta \left( l(x,-v), \left( \mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)} \right)^b_a v^a \right)^j - \eta' \left( l'(x,-v), \left( \mathcal{I'}^{\gamma'_{x,v}}_{0,-l'(x,-v)} \right)^b_a v^a \right)^j \\ &\quad + \tilde{U}'(x,v)^j_c \left( \mathcal{I'}^{\gamma'_{x,v}}_{-l'(x,-v),0} - \mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0} \right)^c_d \eta^d_0 \\ &\qquad = \left( \tilde{U}(x,v) - \tilde{U}'(x,v) \right)^j_c \left( \mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0} \right)^c_d \eta^d_0. \end{aligned} \tag{76}$$

We will estimate each of the lines in (76) separately, but in order to do this we first need an estimate of the difference $l(x,v) - l'(x,v)$.

We now proceed to establish this estimate. For every $s \in [0,1]$ we define the metric $g_s$ by

$$g_s = sg + (1-s)g' \tag{77}$$

so that $g_0 = g'$ and $g_1 = g$. When $\epsilon$ is small enough every $g_s$ will still be a simple metric on $M_2$. Now let $\exp^{\mathrm{s}}$ denote the exponential map corresponding to each $g_s$, and let $l^s$ be the corresponding positive function on $\Omega_a^b M_1$ defined by

$$\rho(\exp_x^s(l^s v)) = 0 \tag{78}$$

where $\rho$ is a defining function for $\partial M_2$. By the implicit function theorem, $l^s(x,v)$ is a smooth function of $s$, $x$, and $v$. Furthermore, we may calculate the derivative of $l^s(x,v)$ with respect to $s$ from (78). Indeed

$$\frac{\partial l^s}{\partial s}(x,v) = -\frac{1}{\mathrm{d}\rho\left(\dot\gamma^s_{x,v}(l^s(x,v))\right)}\,\mathrm{d}\rho\left(\frac{\partial(\exp_x^s)}{\partial s}(l^s(x,v)v)\right). \tag{79}$$

By examining the ODE defining the exponential map, we can bound the second term on the right hand side of (79) by $C\|g-g'\|_{C^1 S_2 M_2}$ for any $(x,v) \in \Omega_a^b M_1$ where the constant $C$ does not depend on $g'$. The first term on the right side of (79) can be also be bounded, using also the simplicity assumption, uniformly for any $(x,v) \in \Omega_a^b M_1$. Therefore since $l^1 = l$ and $l^0 = l'$, the mean value theorem shows that $\|l - l'\|_{C(\Omega_a^b M_1)} < C\|g-g'\|_{C^1 S_2 M_2}$. Differentiating (79) we can similarly show that $\|l-l'\|_{C^3(\Omega_a^b M_1)} < C\|g-g'\|_{C^4 S_2 M_2}$.

Now we return to estimating the left hand side of (76). By the argument used to establish (57) at the end of the proof of Theorem 8, we already have that

$$\left\|\left(\left(\mathcal{I}_{0,t}^{\gamma_{x,v}}\right)^j_b - \left(\mathcal{I}'^{\gamma'_{x,v}}_{0,t}\right)^j_b\right)\eta^b\right\|_{C^3_{x,t,v}} < C\|g-g'\|_{C^4 S_2 M_2}$$

where the norm on the left is over $(x,v) \in \Omega_a^b M_1$ and $t$ in the domain of both $\gamma_{x,v}$ and $\gamma'_{x.v}$. Since the determinants of the parallel translations are bounded below uniformly, and $\mathcal{I}_{t,0}^{\gamma_{x,v}} = (\mathcal{I}_{0,t}^{\gamma_{x,v}})^{-1}$, we also obtain

$$\left\|\left(\left(\mathcal{I}_{t,0}^{\gamma_{x,v}}\right)^j_b - \left(\mathcal{I}'^{\gamma'_{x,v}}_{t,0}\right)^j_b\right)\eta^b\right\|_{C^3_{x,t,v}} < C\|g-g'\|_{C^4 S_2 M_2}.$$

Additionally, using (74) and (75) as well as lemma 3, we have $\|\tilde U'(x,v)\|_{C^3\beta_1^1(\Omega_a^b M_1)} < C\|f\|_{C^3\tau_1^1(M_1)}$ (by for example comparing (74) for $f$ with (74) for the zero tensor field) were $C$ may depend on $g$ and $\epsilon$. Putting all this together and assuming that $f$ satisfies a bound as given in the hypothesis, this establishes the desired bound on the second line of (76).

Finally, we estimate the first line in (76) by using (74), and the previous bound on $l - l'$ established in this proof. Using (76) for every vector $\eta_0$, we thus obtain that for $(x,v) \in \Omega_a^b M_1$

$$\|\left(\tilde U(x,v) - \tilde U'(x,v)\right)^j_c \left(\mathcal{I}^{\gamma_{x,v}}_{-l(x,-v),0}\right)^c_d \eta_0^d\|_{C^3_{x,v}} < C\|g-g'\|_{C^4 S_2 M_2}.$$

Since $\mathcal{I}^{\gamma_{x,v}}_{0,-l(x,-v)}$ is uniformly bounded in $C^3_{x,v}$ for $(x,v) \in \Omega_a^b M_1$, this proves the result.

□

4.2. **Local invertibility of the nonlinear problem.** We are now prepared to return to the full inverse problem and provide the proof of Theorems 1 and 5.

*Proof of Theorems 1 and 5.* Let $(M,g)$, $g'$, $\hat{f}$, $f_1$, and $f_2$ be as in the statement of the theorems. Also let $\hat{U}$, $\tilde{\hat{U}}$, $U_1$, $\tilde{U}_1$, $U_2$ and $\tilde{U}_2$ denote the respective semi-basic tensor fields described in section 4.1 corresponding respectively to $\hat{f}$, $f_1$ and $f_2$. Further, the same objects defined with respect to $g'$ will be denoted with a prime. Now define $f \in L^2\tau_1^1(M_1)$ by setting

$$f(x) = \begin{cases} f_1(x) - f_2(x) & \text{if } x \in M \\ 0 & \text{if } x \in M_1 \setminus M. \end{cases} \tag{80}$$

In the 3 dimensional case note that $f \in L^2_\beta(M)$ since by assumption $\mathrm{d}_\beta(f_1 - f_2) = 0$ on the interior of $M$. We will consider the X-ray transform $I'_{\tilde{U}_1',\tilde{U}_2'}[f]$ on $M_1$. Our first task will be to show that $I'_{\tilde{U}_1',\tilde{U}_2'}[f] = 0$.

Let $\xi \in \partial_-\Omega' M_1$, and let $\gamma'_\xi$ denote the maximally extended geodesic for $g'$ in $M_1$ with initial data $\dot{\gamma}'_\xi(0) = \xi$ ($\dot{\gamma}'_\xi$ is the tangent vector to $\gamma'_\xi$). Suppose that $\gamma'_\xi$ has length $l'(\xi)$. If $\gamma'_\xi$ does not pass through the interior of $M$, then we easily see from the definition of $f$ that $I_{\tilde{U}_1',\tilde{U}_2'}[f](\xi) = 0$. Thus, suppose that $\gamma'_\xi$ does pass through the interior of $M$. Then since $\partial M$ is convex and $(M_1, g')$ is simple there must be unique times $t_1$ and $t_2$ with $0 \le t_1 \le t_2 \le l'(\xi)$ when $\gamma'_\xi$ enters and exits $M$ respectively. Then by (6), which holds according to (72), we have for any $\eta$ and $\zeta \in T^{\mathbb{C}}_{\pi(\xi)}(M_1)$

$$\begin{aligned}
&\left\langle \left( \mathcal{I}^{\gamma'_\xi}_{t_2,0} (\tilde{U}_2')^{-1}(\dot{\gamma}'_\xi(t_2))\, \tilde{U}_1'(\dot{\gamma}'_\xi(t_2))\, \mathcal{I}^{\gamma'_\xi}_{0,t_2} - \mathcal{I}^{\gamma'_\xi}_{t_1,0} (\tilde{U}_2')^{-1}(\dot{\gamma}'_\xi(t_1))\, \tilde{U}_1'(\dot{\gamma}'_\xi(t_1))\, \mathcal{I}^{\gamma'_\xi}_{0,t_1} \right)\eta, \zeta \right\rangle_{g'(\pi(\xi))} \\
&\quad = \int_{t_1}^{t_2} \left\langle (\tilde{U}_2')^{-1} \left[ P_{\dot{\gamma}'_\xi(s)} (f_1 - f_2) \right] (\gamma'_\xi(s))\, \tilde{U}_1'\, \mathcal{I}^{\gamma'_\xi}_{0,s}\, \eta, \mathcal{I}^{\gamma'_\xi}_{0,s}\, \zeta \right\rangle_{g'(\gamma'_\xi(s))} \mathrm{d}s \\
&\quad = \int_{0}^{l'(\xi)} \left\langle (\tilde{U}_2')^{-1} \left[ P_{\dot{\gamma}'_\xi(s)} (f_1 - f_2) \right] (\gamma'_\xi(s))\, \tilde{U}_1'\, \mathcal{I}^{\gamma'_\xi}_{0,s}\, \eta, \mathcal{I}^{\gamma'_\xi}_{0,s}\, \zeta \right\rangle_{g'(\gamma'_\xi(s))} \mathrm{d}s \\
&\quad = I'_{\tilde{U}_1',\tilde{U}_2'}[f](\xi)(\eta,\zeta).
\end{aligned} \tag{81}$$

We will show that when $f_1$ and $f_2$ have the same polarization data the first line of (81) is zero.

In order to do this, let us consider how $U_i'(\dot{\gamma}'_\xi(s))$ and $\tilde{U}_i'(\dot{\gamma}'_\xi(s))$ are related for $t_1 \le s \le t_2$. Here and in the rest of this paragraph $i$ may be either 1 or 2. Recall that $U_i'$ is given by solving (2) on $M$, while $\tilde{U}_i'$ is obtained by solving (70) on $M_2$ with $f$ replaced by $E[f]$. Take any $\eta \in T^{\mathbb{C}}_{\gamma'_\xi(t_1)}M$ and note that both $U_i'(\dot{\gamma}'_\xi(s))\, \mathcal{I}^{\gamma'_\xi}_{t_1,s}\, \eta$ and $\tilde{U}_i'(\dot{\gamma}'_\xi(s))\, \mathcal{I}^{\gamma'_\xi}_{0,s}\, \mathcal{I}^{\gamma'_\xi}_{t_1,0}\, (\tilde{U}_i')^{-1}(\dot{\gamma}'_\xi(t_1))\, \eta$ solve (1) where the initial data is taken at $t_1$ rather than 0. Therefore, by the uniqueness of solutions to (1) we obtain that

$$U_i'(\dot{\gamma}'_\xi(s))\, \mathcal{I}^{\gamma'_\xi}_{t_1,s} = \tilde{U}_i'(\dot{\gamma}'_\xi(s))\, \mathcal{I}^{\gamma'_\xi}_{0,s}\, \mathcal{I}^{\gamma'_\xi}_{t_1,0}\, (\tilde{U}_i')^{-1}(\dot{\gamma}'_\xi(t_1)) \tag{82}$$

for $t_1 \le s \le t_2$. If $f_1$ and $f_2$ have the same polarization data, then $U_1'(\dot{\gamma}'_\xi(t_2)) = U_2'(\dot{\gamma}'_\xi(t_2))$. Using (82) this can be written in terms of $\tilde{U}_1'$ and $\tilde{U}_2'$ as

$$\tilde{U}_1'(\dot{\gamma}'_\xi(t_2))\, \mathcal{I}^{\gamma'_\xi}_{t_1,t_2}\, (\tilde{U}_1')^{-1}(\dot{\gamma}'_\xi(t_1)) = \tilde{U}_2'(\dot{\gamma}'_\xi(t_2))\, \mathcal{I}^{\gamma'_\xi}_{t_1,t_2}\, (\tilde{U}_2')^{-1}(\dot{\gamma}'_\xi(t_1)). \tag{83}$$

Finally, using (83) to simplify the first line of (81) we obtain

$$\begin{aligned}
0 = \Big\langle \mathcal{I}^{\gamma'_\xi}_{t_2,0}\,(\tilde U_2')^{-1}(\dot\gamma'_\xi(t_2))\Big(\tilde U_1'(\dot\gamma'_\xi(t_2))\,\mathcal{I}^{\gamma'_\xi}_{t_1,t_2}(\tilde U_1')^{-1}(\dot\gamma'_\xi(t_1))\\
- \tilde U_2'(\dot\gamma'_\xi(t_2))\,\mathcal{I}^{\gamma'_\xi}_{t_1,t_2}(\tilde U_2')^{-1}(\dot\gamma'_\xi(t_1))\Big)\tilde U_1'(\xi(t_1))\,\mathcal{I}^{\gamma'_\xi}_{0,t_1}\,\eta,\zeta\Big\rangle_{g'(\pi(\xi))}\\
=\Big\langle \Big(\mathcal{I}^{\gamma'_\xi}_{t_2,0}(\tilde U_2')^{-1}(\dot\gamma'_\xi(t_2))\,\tilde U_1'(\dot\gamma'_\xi(t_2))\,\mathcal{I}^{\gamma'_\xi}_{0,t_2} - \mathcal{I}^{\gamma'_\xi}_{t_1,0}\,(\tilde U_2')^{-1}(\dot\gamma'_\xi(t_1))\,\tilde U_1'(\dot\gamma'_\xi(t_1))\,\mathcal{I}^{\gamma'_\xi}_{0,t_1}\Big)\,\eta,\zeta\Big\rangle_{g'(\pi(\xi))}\\
= I'_{\tilde U_1',\tilde U_2'}[f](\xi)(\eta,\zeta).
\end{aligned} \tag{84}$$

Therefore $I'_{\tilde U_1',\tilde U_2'}[f] = 0$.

For the next paragraph we consider only the case of dimension greater than 3. In this case we will show that $I'_{\tilde U_1',\tilde U_2'}$ is injective. The first step is to apply (73) and lemma 6, which when taken together imply that for $\epsilon$ sufficiently small

$$\|\tilde{\hat U} - \tilde U_i'\|_{C^3\beta^1_1((\Omega^b_a)^{\mathbb{R}}M_1)} < C(\|\hat f - f_i\|_{C^3\tau^1_1(M)} + \|g-g'\|_{C^4S_2M_2}) \tag{85}$$

for a constant $C>0$ that does not depend on $f_i$ or $g'$. Now note that by Theorem 3 $I_{\tilde{\hat U},\tilde{\hat U}}$ is injective on $L^2\tau^1_1(M)$, since $\tilde{\hat U}$ is analytic, if the dimension is greater than 3. Therefore if $\epsilon$ is taken to be small enough, by Theorem 2 we have the stability estimate

$$\|f\|_{L^2\tau^1_1(M)} \leq C\|\mathcal{N}'_{\tilde U_1',\tilde U_2'}[f]\|_{H^1\tau^1_1(M_1)}. \tag{86}$$

By (84) this implies that $f=0$, or $f_1=f_2$ when the polarization data are the same. This proves the local injectivity part of the theorem in dimension greater than 3. We next prove local injectivity in dimension 3.

In the case of dimension 3, (84) and (85) from the previous paragraph still apply. The difficulty in the rest of the proof is that Theorem 3 only shows that $I_{\tilde{\hat U},\tilde{\hat U}}$ is injective on $C^3_\beta\tau^1_1(M)$, which is not a closed subspace of $L^2_\beta\tau^1_1(M)$ and so the stability estimate (10) may not hold. We avoid this difficulty however by noting that when $\mathrm{d}_\beta(f_1-f_2)=0$ on $M$ and $f_1-f_2$ satisfies the tangential boundary condition with respect to $g$, then by a slight variant of the main result of [9] $f_1-f_2$ vanishes to first order on $\partial M$. Therefore $f$ is actually in $L^2_\beta\tau^1_1(M_1)$ (with respect to $g$), and has support contained in $M$. Now we introduce an intermediate manifold $M_{1/2}$ such that $M \Subset M_{1/2} \Subset M_1$. The subspace $\mathcal{L}$ of $L^2_\beta\tau^1_1(M_{1/2})$ consisting of tensor fields having support contained in $M$ is a closed subspace of $L^2_\beta\tau^1_1(M_{1/2})$, and by Theorem 2 and Theorem 3 applied on the manifold $M_{1/2}$, $I'_{\tilde U',\tilde U'}$ is injective on $\mathcal{L}$. Therefore, by Theorem 6 (10) holds for tensor fields in $\mathcal{L}$, and so (86) also holds there if $\epsilon$ is small enough. Since $f$ is in $\mathcal{L}$, this proves that $f=0$, or $f_1=f_2$.

Now we move to the proof of the local stability estimate (5). As above, we first work in the case of dimension greater than 3. Assuming that $\epsilon$ is sufficiently small, most of the argument in the previous paragraph still holds, and we may still establish (86). By Theorem 9 this implies

$$\|f\|_{L^2\tau^1_1(M)} \leq C\|I'_{\tilde U_1',\tilde U_2'}[f]\|_{H^1\beta_2(\partial_-\Omega' M_1)} \tag{87}$$

We no longer have that the right hand side is zero, but we still have (81). Using (82) with $s = t_2$ we may rewrite (81) partially in terms of $U_1'$ and $U_2'$ as follows

$$\begin{aligned} I'_{\tilde{U}_1',\tilde{U}_2'}[f](\xi)(\eta,\zeta) = \Big\langle \mathcal{I}^{\gamma_\xi'}_{t_2,0}\,(\tilde{U}_2')^{-1}(\dot{\gamma}_\xi'(t_2)) \Big( \tilde{U}_1'(\dot{\gamma}_\xi'(t_2))\,\mathcal{I}^{\gamma_\xi'}_{t_1,t_2}\,(\tilde{U}_1')^{-1}(\dot{\gamma}_\xi'(t_1)) \\ -\tilde{U}_2'(\dot{\gamma}_\xi'(t_2))\,\mathcal{I}^{\gamma_\xi'}_{t_1,t_2}\,(\tilde{U}_2')^{-1}(\dot{\gamma}_\xi'(t_1)) \Big)\,\tilde{U}_1'(\dot{\gamma}_\xi'(t_1))\,\mathcal{I}^{\gamma_\xi'}_{0,t_1}\,\eta,\zeta \Big\rangle_{g(\pi(\xi))} \\ = \Big\langle \mathcal{I}^{\gamma_\xi'}_{t_2,0}\,(\tilde{U}_2')^{-1}(\dot{\gamma}_\xi'(t_2)) \big( U_1'(\dot{\gamma}_\xi'(t_2)) - U_2'(\dot{\gamma}_\xi'(t_2)) \big)\,\mathcal{I}^{\gamma_\xi'}_{t_1,t_2}\,\tilde{U}_1'(\dot{\gamma}_\xi'(t_1))\,\mathcal{I}^{\gamma_\xi'}_{0,t_1}\,\eta,\zeta \Big\rangle_{g(\pi(\xi))}. \end{aligned}$$

In the second equality we made use of (83). By (73) and the hypothesis that $\|\hat{f} - f_i\|_{C^3\tau_1^1(M)} < \epsilon$, for $\epsilon$ small enough the terms $\tilde{U}_2^{-1}(\dot{\gamma}_\xi'(t_2))$ and $\tilde{U}_1(\dot{\gamma}_\xi'(t_1))$ are bounded and have bounded derivatives, and so this last identity together with (87) shows that

$$\|f_1 - f_2\|_{L^2\tau_1^1(M)} = \|f\|_{L^2\tau_1^1(M)} \leq C\|U_1' - U_2'\|_{H^1\beta_1^1(\partial_+\Omega M)}.$$

This completes the proof of the stability estimate for dimension greater than 3.

In dimension 3 we once again have the problem that $I_{\tilde{\hat{U}},\tilde{\hat{U}}}$ is only injective on $C^3_\beta\tau_1^1(M)$, which is not a closed subspace of $L^2_\beta\tau_1^1(M)$. However, if we restrict to consider the space $\mathcal{L}$ of tensor fields in $L^2_\beta\tau_1^1(M)$ having support within a fixed compact set $K \Subset M$, then by Theorems 6 and 3 $I_{\tilde{\hat{U}},\tilde{\hat{U}}}$ is injective on $\mathcal{L}$. Therefore by Theorem 6 we have (86) for $f \in \mathcal{L}$, and the remainder of the proof follows as in the higher dimensional case. □

## Appendix A. Continuity of $I^*_{U_1,U_2}$ on $H^1$

This appendix is dedicated to the proof of the continuity result for $I^*_{U_1,U_2}$ used to establish (87) in the proof of Theorems 1 and 5. We must first introduce the weighted geodesic X-ray transform of a function $f$. Indeed, let $(M,g)$ be a compact non-trapping manifold, and let $w \in C^\infty(\Omega M)$. Then the X-ray transform $\mathbb{I}_w : L^2(M) \to L^2(\partial_-\Omega M)$ is defined by

$$\mathbb{I}_w[f](\xi) = \int_0^{l(\xi)} w(\dot{\gamma}_\xi(t))\,f(\gamma_\xi(t))\,\mathrm{d}t.$$

We also define the adjoint $\mathbb{I}_w^*$ which is automatically a continuous map from $L^2(M)$ to $L^2(\partial_-\Omega M)$. Following notation used in [5], for any function $h$ on $\partial_-\Omega M$ we define a function $h_\sharp$ on $\Omega M\backslash T\partial M$ by setting $h_\sharp = h$ on $\partial_-\Omega M$ and making $h$ constant on every orbit of the geodesic flow in $\Omega M \setminus T\partial M$. Note that $h_\sharp$ may be written as a composition of $h$ with a smooth function from $\Omega M \setminus T\partial M$ to $\partial_-\Omega M$. Indeed, if $p(v)$ is the inverse of the map $(s,\xi) \mapsto \dot{\gamma}_\xi(s)$, from the subset of $\mathbb{R}\times\partial_-\Omega M$ where $\dot{\gamma}_\xi(t)$ is defined to $\Omega M \setminus T\partial M$, then $h_\sharp(v) = h(\pi_{\partial_-\Omega M} \circ p(v))$. Here $\pi_{\partial_-\Omega M}$ is the projection from $\mathbb{R}\times\partial_-\Omega M$ to $\partial_-\Omega M$. Using Santaló's formula we may then show that for any $h \in C_c^\infty(\partial_-\Omega M)$

$$\mathbb{I}_w^*[h](x) = \int_{\Omega_x M} \overline{w(v)}\,h_\sharp(v)\;|\mathrm{d}V_{\Omega_x M}| \tag{88}$$

We now prove the following result about the continuity of $\mathbb{I}_w^*$ and $I^*_{U_1,U_2}$. Note that this theorem does not require that $(M,g)$ be simple.

**Theorem 9.** *The operator $\mathbb{I}_w^*[h](x)$ is continuous from $H^1_c(\partial_-\Omega M)$ to $H^1_{loc}\tau_1^1 M$, and the operator $I^*_{U_1,U_2}$ is continuous from $H^1_c\beta_2(\partial_-\Omega M)$ to $H^1_{loc}\tau_1^1 M$.*

*Proof.* First we consider only $\mathbb{I}_w^*$. Let $K' \Subset \partial_-\Omega M$ and $K \Subset M$ be compact sets, and let $h \in C_c^\infty(\partial_-\Omega M)$ have support contained in $K'$. Since we already know that $\mathbb{I}_w^*$ is continuous on $L_c^2(\partial_-\Omega M)$, it is sufficient to show that for any $\psi \in C_c^\infty(K)$, and any smooth vector field $X$ defined on $K$

$$\|X(\psi\,\mathbb{I}_w^*[h])\|_{H^1(M)} \le C\|h\|_{H^1(\partial_-\Omega M)}$$

where the constant $C$ may depend on $X$, $K'$, and $\psi$, but not on $h$. Let $\{\phi_{j'}\} \subset C_c^\infty(\partial_-\Omega M)$ and $\{\psi_j\} \subset C_c^\infty(M)$ be partitions of unity on $K'$ and $K$ respectively such that each of the cut-off functions has support contained within the domain of a coordinate chart. Then we have

$$\psi(x)\,\mathbb{I}_w^*[h](x) = \sum_{j,j'} \psi(x)\psi_j(x)\mathbb{I}_w^*[\phi_{j'}h](x) \tag{89}$$

Let us consider one of the terms in this sum. Indeed, we introduce coordinates $\{x^i\}$ on the support of $\psi_j$, and an orthonormal frame over $E_i$ on the same set with corresponding coordinates in the fiber labeled as $v^i$. Finally, we denote coordinates on the support of $\phi_{j'}$ as $\xi^i$. From (88) we then obtain

$$\psi(x^i)\,\psi_j(x^i)\,\mathbb{I}_w^*[\phi_{j'}h](x^i) = \psi(x^i)\,\psi_j(x^i)\,\int_{\mathbb{S}^{n-1}} \overline{w(x^i,v^i)}\,(\pi_{\partial_-\Omega M}\circ p)^*(\phi_{j'}\,h)(x^i,v^i))\,\mathrm{d}v$$

where the measure d$v$ is the Euclidean measure on $\mathbb{S}^{n-1}$. Setting $w_j(x^i,v^i) = \psi(x^i)\,\psi_j(x^i)\overline{w(x^i,v^i)}$ we have

$$\begin{aligned}\frac{\partial}{\partial x^l}\left(\psi(x^i)\,\psi_j(x^i)\,\mathbb{I}_w^*[\phi_{j'}h](x^i)\right) &= \int_{\mathbb{S}^{n-1}}\left(\frac{\partial}{\partial x^l}w_j(x^i,v^i)\right)(\pi_{\partial_-\Omega M}\circ p)^*(\phi_{j'}\,h)(x^i,v^i))\,\mathrm{d}v\\ &\quad+\int_{\mathbb{S}^{n-1}} w_j(x^i,v^i)\,\frac{\partial}{\partial x^l}(\pi_{\partial_-\Omega M}\circ p)^k(x^i,v^i)\\ &\quad\times\left((\pi_{\partial_-\Omega M}\circ p)^*\frac{\partial}{\partial \xi^k}(\phi_{j'}\,h)\right)(x^i,v^i))\,\mathrm{d}v\\ &= \mathbb{I}^*_{\frac{\partial}{\partial x^l}w_j}[\phi_{j'}h] + \sum_k \mathbb{I}^*_{w_j(x^i,v^i)\,\frac{\partial}{\partial x^l}(\pi_{\partial_-\Omega M}\circ p)^k(x^i,v^i)}[\frac{\partial}{\partial \xi^k}(\phi_{j'}h)].\end{aligned}$$

The $L^2$ continuity of $\mathbb{I}_w^*$ for arbitrary weights therefore implies that

$$\left\|\frac{\partial}{\partial x^l}\left(\psi(x^i)\,\psi_j(x^i)\,\mathbb{I}_w^*[\phi_{j'}h](x^i)\right)\right\|_{L^2(M)} \le C\|h\|_{H^1(\partial_-\Omega M)},$$

and together with (89) this shows that $\mathbb{I}_w^* : H_c^1(\partial_-\Omega M) \to H_{loc}^1(M)$ is continuous.

Now we continue to prove the continuity of $I^*_{U_1,U_2}$. As we will see, when we look in local coordinates this reduces to the continuity of a system of operators in the form $\mathbb{I}_w^*$ acting on components. Indeed, let $K'$ and $K$ be as above and let $F \in (\beta_2)_c\,\partial_-\Omega M$ with support contained within $K'$. Finally, let $\{\phi_{j'}\} \subset C_c^\infty(\partial_-\Omega M)$ and $\{\psi_j\} \subset C_c^\infty(M^{int})$ be finite partitions of unity on $K'$ and $K$, and $\psi \in C_c^\infty(M)$ have support contained in $K$ as before. Then

$$\psi I^*_{U_1,U_2}[F] = \sum_{j,j'} \psi\psi_j\, I^*_{U_1,U_2}[\phi_{j'}\,F],$$

and so it is sufficient to show that for each pair of $j$ and $j'$, $(I^*_{U_1,U_2})_{jj'} = (\psi\psi_j)^m \circ I^*_{U_1,U_2} \circ \phi^m_{j'}$ : $H_c^1\beta_2\partial_-\Omega M \to H_c^1\tau_1^1 M$ is continuous. Here $(\psi\psi_j)^m$ denotes multiplication by $\psi\psi_j$, and similarly for $\phi^m_{j'}$.

Next we derive a formula for the components of $(I^*_{U_1,U_2})_{jj'}[F]$ in the local coordinates. We have

$$\begin{aligned}
\langle (I^*_{U_1,U_2})_{jj'}[F],\, h\rangle_{L^2\tau^1_1(M)} &= \langle \phi_{j'}\, F, I_{U_1,U_2}[\psi\psi_j h]\rangle_{L^2\beta_2(\partial_-\Omega M)} \\
&= \int_{\partial_-\Omega M}\int_0^{l(\xi)} \phi_{j'}(\xi)\,\psi\psi_j(\gamma_\xi(t))\,F(\xi)_{bk}\,(\mathcal{I}^{\gamma_{\dot\gamma_\xi(t)}}_{0,-t})^b_{b'}\,\overline{(U_2^{-1})(\dot\gamma_\xi(t))^{b'}_{a'}} \\
&\qquad \times [P_{\dot\gamma_\xi(t)}\,\overline{h}](\gamma_\xi(t))^{a'}_{m'}\,\overline{(U_1)(\dot\gamma_\xi(t))^{m'}_{p'}}\,(\mathcal{I}^{\gamma_{\dot\gamma_\xi(t)}}_{0,-t})^k_{k'}\,g(\gamma_\xi(t))^{k'p'}\,\mathrm{d}t\,\left|\mathrm{d}V_{\partial_-\Omega M}(\xi)\right| \\
&= \sum_{b,k,\alpha,\epsilon}\langle \phi_{j'}\,F_{bk}, \mathbb{I}_{w^{bk\epsilon}_\alpha}[\psi\psi_j\, h^\alpha_\epsilon]\rangle_{L^2(\partial_-\Omega M)} \\
&= \sum_{b,k,\alpha,\epsilon}\langle \mathbb{I}^*_{w^{bk\epsilon}_\alpha}[\phi_{j'}\,F_{bk}], \psi\psi_j\, h^\alpha_\epsilon\rangle_{L^2(M)}
\end{aligned}$$

where

$$w^{bk\epsilon}_\alpha(v) = (\mathcal{I}^{\gamma_v}_{0,-l(-v)})^b_{b'}\,(U_2^{-1})(v)^{b'}_{a'}\,(P_v)^{a'\epsilon}_{m'\alpha}\,(U_1)(v)^{m'}_{p'}\,(\mathcal{I}^{\gamma_v}_{0,-l(-v)})^k_{k'}\,g(\pi(v))^{k'p'}.$$

From this calculation we see that in local coordinates

$$(I^*_{U_1,U_2})_{jj'}[F]^\alpha_\epsilon(x) = \sum_{b,k,\alpha',\epsilon'} g(x)^{\alpha\alpha'}\,g(x)_{\epsilon\epsilon'}\,\psi\psi^m_j(x)\,\mathbb{I}^*_{w^{bk\epsilon'}_{\alpha'}}[\phi^m_{j'}\,F_{bk}](x).$$

Therefore the continuity of $\mathbb{I}^*_{w^{bk\epsilon'}_{\alpha'}}$ on $H^1_c(\partial_-\Omega M)$ implies that $\|(I^*_{U_1,U_2})_{jj'}[F]^\alpha_\epsilon\|_{H^1(M)} \le C\|F\|_{H^1\beta_2(\partial_-\Omega M)}$ for a constant $C > 0$, which in turn implies the result. □

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