Optimal Stopping for Exponential Lévy Models with Weighted Discounting

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Abstract

This paper considers an optimal stopping problem with weighted discounting, and the state process is modelled by a general exponential Lévy process. Due to the time inconsistency, we provide a new martingale method based verification theorem for the equilibrium stopping strategies. As an application, we generalize an investment problem with non-exponential discounting studied by Grenadier and Wang (2007) and Ebert et al. (2020) to Lévy models. Closed-form equilibrium stopping strategies are derived, which are closely related to the running maximum of the state process. The impacts of discounting preferences on the equilibrium stopping strategies are examined analytically.

Keywords: Weighted discounting, Optimal stopping, Time inconsistency, Lévy processes.

Mathematics Subject Classification (2020): 60G40, 60G51, 91B06

1 Introduction

The dilemma of what discount rate to use and how discount rates affect investment decision is a long-existing problem in economics which can be traced back to Ramsey (1928). The standard framework assumes a decision maker (DM) has a constant discount rate and then it leads to the classical exponential discounting form. However, there are cases that exponential discounting is unable to appropriately characterize how DMs discount future investment payoff.

First, it is common that investment decisions are made by a group of people with different discount rates. Regardless of a household or company, investment decisions are typically based on different opinions from a group of members, most likely, with diversified time preferences. Indeed, Weitzman (2001) conducted a survey from over 2000 economists (including 52 Nobel prize laureates) on opinions of discount rate values, and the result shows a widely dispersed discount rate distribution. Second, even if an investment decision is to be made by a single DM, she may have uncertainty about what discount rate to use, especially for some long-term investments or projects that the DM is not very experienced with. Third, there is strong empirical evidence that individuals' time preferences are present-biased such as hyperbolic discounting (Harvey (1986)) and generalized hyperbolic discounting (Loewenstein and Prelec (1992)).

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Recently, Ebert et al. (2020) proposed *weighted discounting functions* of the form

$$h(t) = \int_0^\infty e^{-rt} F(\mathrm{d}r),$$

where F, referred to as the *weighting distribution*, is a cumulative distribution function concentrated on $[0, \infty)$. As elaborated in Ebert et al. (2020), a weighted discount function h provides a flexible way to attach weights to different discount rates, and more importantly, it provides a unified form of discounting to resolve all three issues mentioned above, i.e., group diversity in discount rates, parameter uncertainty of the discount rate, and present-biased time preferences. To generally analyze the impact of discounting on investment decision, Ebert et al. (2020) consider the following optimal stopping problem related to real option, i.e.,

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[h(\tau) G(X_\tau)], \tag{1}$$

where we denote by $\mathcal{T}$ the set of stopping times $\tau \geq 0$, h a weighted discounting function, G a payoff function, and X a state process representing the project value of the real option.

The technical issue of the optimal stopping problem (1) is that it becomes time inconsistent, except the standard exponential discounting case, namely, $h(t) = e^{-rt}$. Time-inconsistent dynamic optimization has emerged to be one of the most important topics in the recent studies of stochastic control theory. One way to tackle this issue is to treat a time-inconsistent problem as an intra-personal game, and this leads to time-consistent equilibrium solutions. Thanks to the seminal works of Basak and Chabakauri (2010) and Bjork and Murgoci (2010), the time-inconsistent portfolio optimization type of problems have been well studied by this equilibrium approach. Mathematically, equilibrium strategies can be solved from a system of extended Hamilton-Jacobi-Bellman (HJB) equations.

However, unlike portfolio optimization problems, how to find equilibrium solutions to time-inconsistent optimal stopping problems is much less developed. As a matter of fact, even the definition of equilibrium stopping strategies has not been unified. In a finite horizon, it is natural to use backward induction argument as in O'Donoghue and Rabin (1999), but it is not clear for infinite-horizon problems like Problem (1). Huang and Nguyen-Huu (2018) define equilibrium stopping strategies via fixed-point iterations. In this setting, immediately stopping is always an equilibrium. In another way, Ebert et al. (2020) and Tan et al. (2021) treat stopping as a binary control and then define equilibrium stopping strategies using a comparison with its local deviations. This definition is more in line with Bjork and Murgoci (2010) and numerous following papers for the portfolio optimization problems. Hence, in this paper we will adopt the latter definition.

In this paper, we also study equilibrium stopping strategies for the weighted optimal stopping problem (1). Our contribution is twofold. First, we develop a *martingale method* to solve equilibrium stopping strategies for the time-inconsistent optimal stopping problem (1), while the previous works on this topic are all based on the variational inequalities method (e.g., Grenadier and Wang (2007), Hsiaw (2013) Ebert et al. (2020), and Tan et al. (2021)). To the best of our knowledge, this is also the first work using martingale method for time-inconsistent optimal stopping problems. The advantage of the martingale method is that it does not need to impose strong regularity assumptions on the optimal value function beforehand, which can be hard to verify unless the solutions are in very explicit form. Therefore, we believe the martingale method can shed some light on more general time-inconsistent optimal stopping problems or those with more general underlying state processes.

The second contribution is that we derive closed-form equilibrium stopping strategies for the real option problem (1) with payoff function $G(x) = x - I$ for some constant $I > 0$, where the state process X follows an exponential Lévy model, namely,

$$X_t = X_0 e^{Y_t},$$

in which $Y = \{Y_t : t \geq 0\}$ is a general Lévy process. Under the standard exponential discounting case, this is a classical irreversible investment problem of Brennan and Schwartz (1985) and McDonald and Siegel (1986). More recently, Grenadier and Wang (2007) and Ebert et al. (2020) study the problem with hyperbolic discounting and general weighted discounting, respectively, but the state process X follows a geometric Brownian motion model. Our results generalize Grenadier and Wang (2007) and Ebert et al. (2020) to a general exponential Lévy model via a different approach as aforementioned. We find that the equilibrium stopping threshold is closely related the *running maximum* of the Lévy process Y. Moreover, we find that the investment will be delayed in case of greater group diversity, greater discounting uncertainty, or more present-biased time preferences. This finding is consistent with Ebert et al. (2020).

There is a considerable volume of literature on the applications of Lévy processes in financial models due to the empirical evidence of jumps in various asset prices; see, e.g., Cont and Tankov (2003) for a complete review. But it is worth noting that, even in the time-consistent case, optimal control problems with Lévy models typically tackle with those with only one side of jumps (namely, spectrally positive/negative) for the sake of explicit solutions; see, e.g., Palmowski et al. (2020), López et al. (2021), and Palmowski et al. (2021) for some recent works. In this paper, we obtain an explicit form of the equilibrium stopping threshold for a general class of Lévy models with both sides of jumps under some mild conditions.

We shall also mention that there are other types of time-inconsistent optimal stopping problems in addition to the non-exponential discounting problems. For instance, Christensen and Lindensjö (2018) consider a reward function depending on the initial state, Bayraktar et al. (2019) consider a discrete-time infinite horizon mean-variance stopping problem, and Huang et al. (2020) consider an optimal stopping problem with probability distortion. Any one of these extensions (initial-state dependence, variance objective, and probability distortion) yields time inconsistency.

The remainder of the paper is organized as follows. Section 2 presents some preliminaries for Lévy processes. Section 3 formulates the optimal stopping problem with weighted discounting and defines the equilibrium stopping rules. Section 4 presents some general results including a verification theorem for equilibrium stopping rule based on the martingale method. Section 5 considers a specific investment problem with non-exponential discounting under the exponential Lévy model. Equilibrium stopping strategies are found in closed form and the impact of the decision maker's discounting preference is analyzed. Finally, section 6 provides some specific examples of Lévy processes where the equilibrium stopping thresholds are given explicitly. All proofs have been relegated to the Appendix.

2 Preliminaries

Let $Y = \{Y_t, t \geq 0\}$ be a Lévy process. That is, an almost surely càdlàg process that has independent and stationary increments such that $Y_0 = 0$. We take it to be defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by Y which is naturally enlarged (see Definition 1.3.38 of Bichteler (2002)).

From the stationary and independent increments property, the law of Y is characterized by the distribution of Y_1. We hence define the characteristic exponent of Y, $\Psi(\theta) := -\log(\mathbb{E}[e^{i\theta Y_1}])$, $\theta \in \mathbb{R}$. The Lévy–Khintchine formula guarantees the existence of constants, $\mu \in \mathbb{R}$, $\sigma \geq 0$ and a measure Π concentrated in $\mathbb{R} \setminus \{0\}$ with the property that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$ (called the Lévy measure) such that for any $\theta \in \mathbb{R}$,

$$\Psi(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y \mathbb{I}_{\{|y|<1\}})\Pi(\mathrm{d}y).$$

If $\mathbb{E}[e^{\beta Y_1}] < \infty$ for some $\beta \in \mathbb{R}$, we can define the Laplace exponent

$$\psi(\beta) = \log(\mathbb{E}[e^{\beta Y_1}]) = \mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{\mathbb{R}} (e^{\beta y} - 1 - \beta y \mathbb{I}_{\{|y|<1\}})\Pi(\mathrm{d}y).$$

Moreover, from the Lévy–Itô decomposition, we can write

$$Y_t = \sigma B_t + \mu t + \int_{[0,t]} \int_{(-\infty,-1)\cup(1,\infty)} y N(\mathrm{d}s \times \mathrm{d}y) + \int_{[0,t]} \int_{(-1,1)} y(N(\mathrm{d}s \times \mathrm{d}y) - \mathrm{d}s\Pi(\mathrm{d}y)), \quad (2)$$

where B is a standard Brownian motion and N is an independent Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}$ with intensity $\mathrm{d}t \times \Pi(\mathrm{d}x)$.

We state now some properties and facts about Lévy processes. The reader can refer, for example, to Bertoin (1998), Sato (1999) and Kyprianou (2014) for more details. The paths of Y are of *bounded variation* if and only if $\sigma = 0$ and $\int_{(-1,1)} |y|\Pi(\mathrm{d}y) < \infty$, in which case we can write

$$Y_t = \delta t + \int_{[0,t]} \int_{\mathbb{R}} y N(\mathrm{d}s \times \mathrm{d}y),$$

where $\delta = \mu - \int_{(-1,1)} y\Pi(\mathrm{d}y)$ is the drift.

We denote the supremum of Y at time $t \geq 0$ by

$$\overline{Y}_t = \sup_{0\leq s\leq t} Y_s. \quad (3)$$

Let $\mathbf{e}_r$ be an independent exponential time independent of Y with mean $1/r$. In particular, $\mathbf{e}_0 = \infty$. It is shown in Lemma 1 of Mordecki (2002a) that for any $r \geq 0$,

$$\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}] < \infty \quad \text{if and only if} \quad \psi(1) < r, \quad (4)$$

where we recall that $\psi(1) = \log(\mathbb{E}[e^{Y_1}])$. For any $y \in \mathbb{R}$, we define the stopping time

$$\tau_y^+ = \inf\{t > 0 : Y_t > y\}.$$

For any $r > 0$, $\beta \leq 0$ and $y \in \mathbb{R}$, we have the following identity (see Exercise 6.7 in Kyprianou (2014)),

$$\mathbb{E}\left[e^{\beta\overline{Y}_{\mathbf{e}_r}} \mathbb{I}_{\{\overline{Y}_{\mathbf{e}_r} > y\}}\right] = \mathbb{E}\left[e^{\beta\overline{Y}_{\mathbf{e}_r}}\right] \mathbb{E}\left[e^{-r\tau_y^+ + \beta Y_{\tau_y^+}} \mathbb{I}_{\{\tau_y^+ < \infty\}}\right], \quad (5)$$

which is also true for $\beta \geq 0$ provided that $\mathbb{E}\left[e^{\beta\overline{Y}_{\mathbf{e}_r}}\right] < \infty$.

Given $Y_0 = 0$, we say that the point 0 is *regular* for $(0, \infty)$ if

$$\mathbb{P}(\tau_0^+ = 0) = 1.$$

Otherwise, if the probability above is 0, we say that 0 is irregular for $(0, \infty)$. By Theorem 6.5 of Kyprianou (2014), we have that 0 is regular for $(0, \infty)$ if and only if one of the following three situation occurs: (1) Y has unbounded variation; (2) Y has bounded variation and positive drift;

(3) Y is not a compound Poisson process, has bounded variation, zero drift, and its Lévy measure satisfies

$$\int_0^1 \frac{x\Pi(\mathrm{d}x)}{\int_0^x \Pi(-\infty,-y)\mathrm{d}y} = \infty.$$

Moreover, from the identity $\mathbb{P}(\overline{Y}_{\mathbf{e}_r} = 0) = \mathbb{P}(\tau_0^+ > \mathbf{e}_r)$, we have that 0 is regular for $(0,\infty)$ if and only if $\mathbb{P}(\overline{Y}_{\mathbf{e}_r} = 0) = 0$ for any $r \geq 0$.

We say that Y *creeps upwards* if

$$\mathbb{P}(Y_{\tau_y^+} = y, \tau_y^+ < \infty) > 0,$$

for some (and then for all) $y > 0$. By Theorem 7.11 of Kyprianou (2014), we have that Y creeps upwards if and only if one of the following three situation occurs: (1) Y has bounded variation and positive drift; (2) Y has a Gaussian component; (3) Y has unbounded variation, no Gaussian component and its Lévy measure satisfies

$$\int_0^1 \frac{x\Pi(x,\infty)}{\int_0^x \Pi(-1,-y)\mathrm{d}y}\mathrm{d}x < \infty.$$

From the above characterizations, it is seen that if Y creeps upwards, then we have 0 is regular for $(0,\infty)$. Note that more results on Lévy processes that creep upwards will be given in Appendix A.1, which will be used later in the paper.

3 Problem formulation

Let $h : [0,\infty) \mapsto (0,1]$ be a *weighted discount function*, that is, a strictly decreasing function with $h(0) = 1$ such that can be written as

$$h(t) = \int_0^\infty e^{-rt} F(\mathrm{d}r), \tag{6}$$

where F, referred to as the *weighting distribution*, is a cumulative distribution function concentrated on $[0,\infty)$. As discussed in Ebert et al. (2020), the weighted discount function (6) provides a flexible way to attach weights to different discount rates. It can be used to resolve both *inter-personal* disagreement and *intra-personal* disagreement. Note that, by Bernstein's theorem, a function h is a weighted discount function if and only if it is continuous on $[0,\infty)$, infinitely differentiable on $(0,\infty)$ and it is completely monotone, that is,

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} h(t) \geq 0,$$

for any $n \geq 1$ and $t > 0$. More discussions on weighted discount functions can be found in Ebert et al. (2020).

Consider a state process $X_t = X_0 e^{Y_t}$ where Y is a Lévy process. Throughout the paper, we denote by $X_s^{t,x}$ (for $s \geq t$) the value of X_s given $X_t = x$, for $x > 0$. It is known that any Lévy process is also Markov process (see, e.g., Theorem 3.1 in Kyprianou (2014)). The Markov property implies that for any $t < s < v$, $x \in \mathbb{R}$, and any integrable function f,

$$\mathbb{E}[f(X_v^{t,x})|\mathcal{F}_s] = \mathbb{E}[f(X_v^{s,y})]\big|_{y = X_s^{t,x}}. \tag{7}$$

Here and thereafter, $\mathbb{E}[\cdot]|_{y=\cdot}$ means evaluating the expectation first and then substituting the value of y into the expression of the expectation.

Let $G : [0, \infty) \to \mathbb{R}_+$ be a measurable function. For any given $t \geq 0$ and $x > 0$, we consider the optimal stopping problem

$$\sup_{\tau \in \mathcal{T}_t} J(t, x; \tau), \tag{8}$$

where $\tau \in \mathcal{T}_t$ is the set of all stopping times τ of X such that $\tau \geq t$, and the reward function is given by

$$J(t, x; \tau) = \mathbb{E}[h(\tau - t)G(X_\tau^{t,x})]. \tag{9}$$

Let $r_0 := \inf\{r \geq 0 : F(r) > 0\}$. To ensure the well-posedness of the problem, we assume that, for all $t \geq 0$ and $x > 0$,

$$\mathbb{E}\left[\sup_{s \geq t} e^{-r_0 s} |G(X_s^{t,x})|\right] < \infty.$$

Denote $\tau^*(t, x)$ as the solution of the optimal stopping problem (8), that is,

$$\sup_{\tau \in \mathcal{T}_t} J(t, x; \tau) = J(t, x; \tau^*(t, x)).$$

Naturally an agent wants to solve the problem repeatly at each time $s \geq t$ (assuming not stopped yet), i.e.,

$$\sup_{\tau \in \mathcal{T}_s} J(s, X_s^{t,x}; \tau).$$

Let $\tau^*(s, X_s^{t,x})$ the solution to the optimal stopping problem above. The raises the question whether the two optimal stopping times $\tau^*(t, x)$ and $\tau^*(s, X_s^{t,x})$ are consistent with each other (when $\tau^*(t, x) > s$). The next definition, which is extracted from Huang and Nguyen-Huu (2018), formally introduces this concept.

Definition 3.1. *We say the optimal stopping problem* (8) *is time consistent if for any $t \geq 0$, $x > 0$, and $s > t$, we have that $\tau^*(t, x) = \tau^*(s, X_s^{t,x})$ on the event $\{\tau^*(t, x) > s\}$. Otherwise, we say that the problem* (8) *is time inconsistent.*

It is known from Huang and Nguyen-Huu (2018) that the solution of Problem (8) is time inconsistent, except the standard exponential discounting case, i.e., the weighted discount function $h(t) = e^{-rt}$ for some $r \geq 0$. Hence, with a general weighting distribution F, we follow the definition of *equilibrium stopping rules* proposed in Ebert et al. (2020) and Tan et al. (2021) to find time-consistent solutions to the optimal stopping problem (8). We first introduce the concept of stopping rules and the stopping times associated to them.

Definition 3.2. *A stopping rule is a measurable function $u : [0, \infty) \times \mathbb{R}_+ \mapsto \{0, 1\}$, where the value 0 indicates "continue" and the value 1 indicates "stop". For each stopping rule u, $t \geq 0$ and $x > 0$, we define a (t-after) stopping time,*

$$\tau_u^{t,x} = \inf\{s \geq t : u(s, X_s^{t,x}) = 1\}.$$

Next we define the equilibrium stopping rules for the optimal stopping problem (8).

Definition 3.3. *Let u^* be a stopping rule, $\varepsilon > 0$ and $a \in \{0,1\}$. For any $s \geq t \geq 0$ and $x > 0$, we define a stopping rule*

$$u^{\varepsilon,a}(s,x) = \begin{cases} u^*(s,x), & s \in [t+\varepsilon, \infty), \\ a, & s \in [t, t+\varepsilon]. \end{cases}$$

Here $u^{\varepsilon,a}$ is called the (ε, a)-deviation from u^*.

Then the stopping rule u^* is called an equilibrium stopping rule for Problem (8) if for any $t \geq 0$, $x > 0$ and any $a \in \{0,1\}$,

$$\liminf_{\varepsilon \to 0} \frac{J(t,x;\tau_{u^*}^{t,x}) - J(t,x;\tau_{u^{\varepsilon,a}}^{t,x})}{\varepsilon} \geq 0. \tag{10}$$

It is worth noting that Definition 3.3 is in line with the seminal works of Basak and Chabakauri (2010) and Björk et al. (2017) on the definition of optimal (in the equilibrium sense) control laws to time-inconsistent stochastic control problems. It is seen that, when $a = 1$,

$$\tau_{u^{\varepsilon,1}}^{t,x} = t, \tag{11}$$

which indicates an immediate stop and, when $a = 0$,

$$\tau_{u^{\varepsilon,0}}^{t,x} = \inf\{s \geq t : u^{\varepsilon,0}(s, X_s^{t,x}) = 1\} = \inf\{s \geq t+\varepsilon : u^*(s, X_s^{t,x}) = 1\}, \tag{12}$$

which indicates a strategy to continue in $[t, t+\varepsilon)$ and then follow the stopping rule u^* after $t+\varepsilon$. In other words, we have $\tau_{u^*}^{t,x} = \tau_{u^{\varepsilon,0}}^{t,x}$ in the event of $\{\tau_{u^*}^{t,x} \geq t+\varepsilon\}$ and $\tau_{u^*}^{t,x} \approx t = \tau_{u^{\varepsilon,1}}^{t,x}$ in the event of $\{\tau_{u^*}^{t,x} < t+\varepsilon\}$. Therefore, $\tau_{u^{\varepsilon,a}}^{t,x}$ can be indeed regarded as a *local deviation* to $\tau_{u^*}^{t,x}$. The condition (10) means that the equilibrium stopping time $\tau_{u^*}^{t,x}$ is *superior* to any of the two local-deviated strategies, that are $\tau_{u^{\varepsilon,1}}^{t,x}$ and $\tau_{u^{\varepsilon,0}}^{t,x}$. Due to the Markov property and (12), we have that for any integrable function f,

$$\mathbb{E}[f(\tau_{u^{\varepsilon,0}}^{t,x})|\mathcal{F}_{t+\varepsilon}] = \mathbb{E}[f(\tau_{u^*}^{t+\varepsilon,y})]\Big|_{y=X_{t+\varepsilon}^{t,x}}. \tag{13}$$

4 General results

Instead of using the HJB equations approach as in Ebert et al. (2020), we use the martingale approach to solve the equilibrium stopping rules for the time-inconsistent optimal stopping problem (8). Let u^* be a candidate of equilibrium stopping rule and $\tau_{u^*}^{t,x} = \inf\{s \geq t : u^*(s, X_s^{t,x}) = 1\}$ be the corresponding stopping time after $t \geq 0$. For any $s, t \geq 0$ and $x > 0$, we define the following two functions,

$$V(t,x) := J(t,x;\tau_{u^*}^{t,x}) = \mathbb{E}[h(\tau_{u^*}^{t,x} - t)G(X_{\tau_{u^*}^{t,x}}^{t,x})], \tag{14}$$

$$H(s,t,x) := \int_0^\infty \mathbb{E}[re^{-r(\tau_{u^*}^{t,x} - t + s)}G(X_{\tau_{u^*}^{t,x}}^{t,x})]F(\mathrm{d}r), \tag{15}$$

and the following stochastic process,

$$Z_s^{t,x} := V(t+s, X_s^{t,x}) - \int_0^s H(0, t+v, X_{t+v}^{t,x})\mathrm{d}v. \tag{16}$$

Next we present a verification theorem[1] (i.e., sufficient conditions) for equilibrium stopping rules based on the martingale method.

[1]Note that the proof of Theorem 4.1 only relies on the Markov property of X. In other words, Theorem 4.1 holds for general Markov processes.

Theorem 4.1. *Let u^* be a stopping rule, $V(t,x)$ and $H(s,t,x)$ as per equations (14) and (15) and $Z_s^{t,x}$ defined by (16). Then u^* is an equilibrium stopping rule to Problem (8) and V is the corresponding equilibrium value function if the following two conditions are satisfied,*

i) $V(t,x) \geq G(x)$ for all $t \geq 0$ and $x > 0$,

ii) For each $t \geq 0$ and $x > 0$, the stochastic process $Z^{t,x} = \{Z_s^{t,x}, s \geq 0\}$ is a supermartingale.

Since Problem (8) is in infinite horizon and the underlying process X is time homogeneous, it is natural to seek for *time invariant* stopping rules. That is, we say a stopping rule u^* is time invariant if $u(t,x) = u(0,x)$ for all $t \geq 0$ and $x > 0$. Due to time homogeneity of X, we have, for any $s,t \geq 0$ and $x > 0$,

$$V(t,x) = \mathbb{E}[h(\tau_{u^*}^{t,x} - t)G(X_{\tau_{u^*}^{t,x}}^{t,x})] = \mathbb{E}[h(\tau_{u^*}^{0,x})G(X_{\tau_{u^*}^{0,x}}^{0,x})] = V(0,x),$$

and

$$\begin{aligned} H(s,t,x) &= \int_0^\infty \mathbb{E}[re^{-r(\tau_{u^*}^{t,x} - t + s)}G(X_{\tau_{u^*}^{t,x}}^{t,x})]F(\mathrm{d}r) \\ &= \int_0^\infty \mathbb{E}[re^{-r(\tau_{u^*}^{0,x} + s)}G(X_{\tau_{u^*}^{0,x}}^{0,x})]F(\mathrm{d}r) \\ &= H(s,0,x). \end{aligned}$$

Hence, by denoting $V(x) \equiv V(t,x)$ and $H(s,x) \equiv H(s,0,x)$ for every $s \geq 0$ and $x > 0$, we have the following corollary.

Corollary 4.2. *Let u^* be a time-invariant stopping rule. Then u^* is an equilibrium stopping rule to Problem (8) if the fowlloing two conditions are satisfied,*

i) $V(x) \geq G(x)$ for all $x > 0$,

ii) For each $x > 0$, the process $V(X_s^{0,x}) - \int_0^s H(0, X_v^{0,x})\mathrm{d}v$ is a supermartingale.

Remark 4.3. *In particular, when it reduces to exponential discounting (i.e., $h(t) = e^{-rt}$), one can show that Corollary 4.2 coincides with the supermartingale characterization for the classical optimal stopping problem. For instance, the reader can refer to Lemma 11.1 in Kyprianou (2014) for Lévy models, or to the superharmonic characterization given in Peskir and Shiryaev (2006) for strong Markov processes, which is equivalent to the supermartingale property under some integrability conditions, see statement (2.2.8) and Theorem 2.7 therein.*

Note that Theorem 4.1 do not need to pre-assume high regularity of the value function V, which is a well-known merit of the martingale method. In case V does belong to the domain of the infinitesimal generator of X, the supermartingale property of $Z^{t,x}$ implies the variational inequality

$$\frac{\partial}{\partial t}V(t,x) + \mathcal{A}_X V(t,x) - H(0,t,x) \leq 0,$$

for all $t \geq 0$ and $x \in \mathbb{R}$, where $\mathcal{A}_X$ is the infinitesimal generator of X. If further $h(t) = e^{-rt}$, for some $r \geq 0$, we recover, from the conditions in Corollary 4.2, the classical HJB system

$$\max\{\mathcal{A}_X V(x) - rV(x), G(x) - V(x)\} = 0.$$

Upon the existence of equilibrium stopping rules, the next proposition provides the characterization using differential equations.

Proposition 4.4. *Suppose that $u^*(x)$ is a time-invariant equilibrium stopping rule and let $V(x) := \mathbb{E}[h(\tau_{u^*})G(X_{\tau_{u^*}})|X_0 = x]$ be the corresponding value function. Denote by $D := \{x \in \mathbb{R} : u^*(x) = 1\}$ the stopping region and $C := \mathbb{R} \setminus D$ the continuation region. Assume that G belong to the domain of the infinitesimal generator $\mathcal{A}_X$. Then:*

i) For any $x \in D$, we have

$$\mathcal{A}_X G(x) - \overline{r}G(x) \le 0,$$

where $\overline{r} := \int_0^\infty rF(\mathrm{d}r)$ denotes the average discount rate.

ii) Let x^* be an equilibrium stopping threshold, i.e., $x^* \in \partial C$ which is the boundary points of the set C, and assume that $(-\infty, x^*) \cap C \neq \emptyset$ (resp., $(x^*, \infty) \cap C \neq \emptyset$). Assume further that C is an open set and V is continuously differentiable in the closure of C. Then

$$V'(x^*-) \le G'(x^*) \qquad (\textit{resp., } V'(x^*+) \ge G'(x^*)).$$

Remark 4.5. *Suppose that u^*, V and G satisfy the conditions given in Proposition 4.4 above. Since we have $V(x) = G(x)$ and thus $V'(x) = G'(x)$ for any x in the interior of D, it is natural to use the smooth pasting condition to find a candidate for equilibrium stopping threshold, i.e.,*

$$V'(x^*) = G'(x^*).$$

However, as pointed out by a recent paper of Tan et al. (2021), the smooth fit condition may fail for some optimal stopping problems with weighted discounting.

5 Investment with weighted discounting for exponential Lévy models

In this section we apply the results developed in the previous sections to solve the standard irreversible investment problem of Brennan and Schwartz (1985) and McDonald and Siegel (1986) under a general exponential Lévy model with weighted discounting. This problem (with weighted discounting) has been studied in Ebert et al. (2020) in a geometric Brownian motion model, and we are devoted to generalize the results to Lévy asset models. The key finding is that the optimal stopping threshold is closely related to the *running maximum* of the Lévy process.

Consider the following problem of finding the optimal time to install an irreversible project:

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[h(\tau)(X_\tau^{0,x} - I)], \tag{17}$$

where we denote by X the present value process of the project, $I > 0$ a fixed cost at the inception of the project, h the weighted discount function of decision maker(s), and $\mathcal{T}$ the set of all stopping times $\tau \ge 0$ of X representing the admissible times to install the project. As such, (17) is a special case of Problem (8) with $G(x) = x - I$.

Recall that we model X by an exponential Lévy process, i.e., $X_t^{0,x} = xe^{Y_t}$, where $Y = \{Y_t, t \ge 0\}$ is a Lévy process with triple (μ, σ, Π). If $\mathbb{E}[e^{\beta Y_1}] < \infty$ for some $\beta \in \mathbb{R}$, we can define the Laplace exponent. That is to say, by the Lévy–Khintchine formula,

$$\psi(\beta) = \log(\mathbb{E}[e^{\beta Y_1}]) = \mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{\mathbb{R}} (e^{\beta y} - 1 - \beta y \mathbb{I}_{\{|y|<1\}})\Pi(\mathrm{d}y).$$

As for comparison, we first consider the benchmark case with exponential discounting, i.e., $h(t) = e^{-rt}$. Then Problem (17) becomes

$$V^{(r)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(X_\tau^{0,x} - I)], \tag{18}$$

which is a standard time-consistent optimal stopping problem. The following proposition gives the solution to Problem (18). We omit its proof since it has been well-studied (see, e.g., Theorem 11.6.10 and Example 11.21 in Boyarchenko and Levendorskii (2007) on page 220).

Proposition 5.1. *Suppose that $\psi(1) < r$. Then $u^*(x) = \mathbb{I}_{\{x > x_r^*\}}$ is an optimal stopping rule for Problem (18), where the stopping threshold is given by*

$$x_r^* = \mathbb{E}[e^{\overline{Y}_{e_r}}]I. \tag{19}$$

The corresponding value function is given by, for any $x > 0$,

$$V^{(r)}(x) = x\frac{\mathbb{E}[e^{\overline{Y}_{e_r}}\mathbb{I}_{\{\overline{Y}_{e_r} > \log(x_r^*/x)\}}]}{\mathbb{E}[e^{\overline{Y}_{e_r}}]} - I \cdot \mathbb{E}[\mathbb{I}_{\{\overline{Y}_{e_r} > \log(x_r^*/x)\}}].$$

Note that the condition $\psi(1) < r$ in Proposition 5.1 is to ensure the well-posedness of Problem (18). By equation (4), it is known that $\psi(1) < r$ implies $\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}] < \infty$.

Remark 5.2. *If Y is a process that creeps upwards, the results of Proposition 5.1 follow immediately from Theorem 5.3 below by taking $h(t) = e^{-rt}$. But actually one can show it with weaker conditions as in Proposition 5.1 by adapting a similar proof as for Theorem 11.2 in Kyprianou (2014).*

We then consider a general weighting distribution F. Recall that $r_0 := \inf\{r \geq 0 : F(r) > 0\}$, $\overline{r} := \int_0^\infty rF(\mathrm{d}r)$, and f_r is the density function of $\overline{Y}_{\mathbf{e}_r}$. Next theorem presents the equilibrium stopping rule to Problem (17).

Theorem 5.3. *Suppose that $\overline{r} < \infty$, and Y is a Lévy process that creeps upwards and satisfies $\psi(1) < r_0$. We further assume that $|f_r'(0+)| < \infty$ if Y is of unbounded variation. Then $u^*(x) = \mathbb{I}_{\{x > x_F^*\}}$ is an equilibrium stopping rule for Problem (17), where the stopping threshold is given by*

$$x_F^* := \frac{\int_0^\infty f_r(0)F(\mathrm{d}r)}{\int_0^\infty \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{e_r}}]}F(\mathrm{d}r)}I. \tag{20}$$

The corresponding value function is given by, for any $x > 0$,

$$V(x) = \int_0^\infty \left[\frac{x}{\mathbb{E}[e^{\overline{Y}_{e_r}}]}\int_{\log(x_F^*/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(x_F^*/x)}^\infty f_r(y)\mathrm{d}y\right]F(\mathrm{d}r).$$

The proof of Theorem 5.3 is rather lengthy, so Appendix A.4 is dedicated entirely on that purpose. It is worth noting that the proof relies on some additional results of Lévy processes that creep upwards which are summarized in Appendix A.1. More specifically, Lemma A.2 ensures the existence of the density function f_r, and gives an expression of $f_r(0)$ in terms of the Laplace exponent of the ascending ladder process. Lemma A.5 shows that

$$\int_0^\infty \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}F(\mathrm{d}r) \leq \int_0^\infty f_r(0)F(\mathrm{d}r) < \infty.$$

As such, it is only optimal to install the project when the present value exceeds the fixed cost, i.e., $x_F^* \geq I$.

The next proposition gives some sufficient condition to ensure the condition $|f_r'(0+)| < \infty$ in Theorem 5.3.

Proposition 5.4. *We have* $|f_r'(0+)| < \infty$ *if* $\Pi(0,\infty) < \infty$.

Note that the condition given in Proposition 5.4 is not sharp. For instance, in the spectrally positive case, it is sufficient to have a Gaussian component for the finiteness of $f_r'(0)$ regardless the Lévy measure Π (see Section 6.2).

Remark 5.5. *From the proof of Theorem 5.3, we can see that the assumption that* Y *creeps awards plays a crucial role to ensure the value function* $V(x)$ *is differentiable for all* $x > 0$*. If* 0 *is irregular for* $(0,\infty)$*, which implies that* Y *does not creep upwards, one can modify the proofs slightly and show that* $u^*(x) = \mathbb{I}_{\{x > x_F^*\}}$ *is an equilibrium stopping rule for Problem (17), where the stopping threshold is given by*

$$x_F^* = \frac{\int_0^\infty \mathbb{P}(\overline{Y}_{\boldsymbol{e}_r} = 0) F(\mathrm{d}r)}{\int_0^\infty \frac{\mathbb{P}(\overline{Y}_{\boldsymbol{e}_r} = 0)}{\mathbb{E}[e^{\overline{Y}_{\boldsymbol{e}_r}}]} F(\mathrm{d}r)} I. \tag{21}$$

It is interesting to see that the two expressions of x_F^* *in (20) and (21) are consistent in the sense that the probability density function of* $\overline{Y}_{\boldsymbol{e}_r}$ *at* 0 *that is* $f_r(0)$ *becomes the mass point at* 0 *that is* $\mathbb{P}(\overline{Y}_{\boldsymbol{e}_r} = 0)$*. In this case, the value function* V *is continuous, but non-differentiable at* x_F^**.*

In the presence of non-exponential discounting, it is crucial to understand how the DM's time preference affects the optimal time to install the project. This is studied in Ebert et al. (2020) via the infinity-stochastic dominance under the geometric Brownian motion model.

Definition 5.6. *Let* F *and* G *be two weighting distributions of the discount rate. We say that* G *is less infinity-stochastically dominant than* F*, denoted as* $G \preceq_{\infty SD} F$*, if*

$$\int_0^\infty u(r) G(\mathrm{d}r) \le \int_0^\infty u(r) F(\mathrm{d}r),$$

for all strictly increasing integrable functions u *such that* $(-1)^{n+1} u^{(n)}(r) \ge 0$ *for all* $n \ge 1$ *and* $x > 0$.

It is well-known that a distribution F is more infinity-stochastically dominant means that it has a higher mean, or equal mean but lower variance, or equal mean and variance but higher skewness, etc. Theorem 2 of Ebert et al. (2020) shows that $G \preceq_{\infty SD} F$ if and only if their discount functions

$$h_F(t) \le h_G(t), \qquad \text{for all } t \ge 0. \tag{22}$$

The next proposition generalizes Proposition 7 in Ebert et al. (2020) to exponential Lévy models. Since the weighted discounting functions provides a unified approach to model group diversity of discounting, discounting uncertainty of an individual, and present-biased time preferences simultaneously, Proposition 5.7 implies the investment will be *postponed* (with a larger present value threshold to install the project) in case of *greater group diversity, greater discounting uncertainty, or more present-biased time preferences.*

Proposition 5.7. *Let* F *and* G *be two weighting distributions of the discount rate such that* $G \preceq_{\infty SD} F$*. Denote by* x_F^* *and* x_G^* *the equilibrium stopping thresholds defined in (20) associated to* F *and* G*, respectively. Then we have that* $x_G^* \ge x_F^*$*.*

Recall that $r_0 := \inf\{r \ge 0 : F(r) > 0\}$ and $\overline{r} := \int_0^\infty r F(\mathrm{d}r)$. By Jensen's inequality, we have that for any $t \ge 0$,

$$e^{-\overline{r}t} \le h_F(t) \le e^{-r_0 t}.$$

Hence, Proposition 5.7 yields the following upper and lower bounds of x_F^*,

$$x_{\overline{r}}^* \le x_F^* \le x_{r_0}^*,$$

where $x_{\overline{r}}^*$ and $x_{r_0}^*$ are the optimal stopping thresholds given in (19) for $r \equiv \overline{r}$ and $r \equiv r_0$, respectively.

6 Examples

In this section, we consider some special examples for the Lévy process Y including spectrally negative Lévy processes, spectrally positive Lévy processes, and Kou's model with double exponential jumps. For each model, the stopping threshold x_F^* is derived in closed form.

6.1 Spectrally negative Lévy process

Suppose that Y is an spectrally negative Lévy process, that is with no positive jumps (and sample paths are non-monotone). In this case, we have that Y creeps upwards. Moreover, for each $r \ge 0$, the random variable $\overline{Y}_{\mathbf{e}_r}$ has exponential distribution with parameter $\Phi(r)$ (see, e.g., equation (8.4) of Kyprianou (2014)), where

$$\Phi(r) = \sup\{\beta \ge 0 : \psi(\beta) = r\},$$

is the right inverse of ψ, and we recall that $\psi(\beta) = \log(\mathbb{E}[e^{\beta Y_1}])$. Hence, for any $y \ge 0$, $r \ge 0$, and $\Phi(r) > 1$ (equivalently, $\psi(1) < r$),

$$f_r(y) = \Phi(r)e^{-\Phi(r)y} \quad \text{and} \quad \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}] = \frac{\Phi(r)}{\Phi(r)-1}.$$

Therefore, by (20), the stopping threshold to Problem (17) is given by

$$x_F^* = \frac{\int_0^\infty \Phi(r)F(\mathrm{d}r)}{\int_0^\infty \Phi(r)F(\mathrm{d}r) - 1} I,$$

provided $\psi(1) < r_0$.

Note that spectrally negative Lévy model of Y includes the Brownian motion model as a special case (and then X is a geometric Brownian motion). In this case, we have $\psi(\beta) = \mu\beta + \frac{1}{2}\sigma^2\beta^2$, and then

$$\Phi(r) = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2}.$$

This recovers Proposition 5 of Ebert et al. (2020).

6.2 Spectrally positive Lévy processes

Suppose that Y is a spectrally positive Lévy process, that is, $Y = -Z$, where Z is a spectrally negative Lévy process with Laplace exponent denoted by ϕ. Let $W^{(r)}(x) : \mathbb{R} \to \mathbb{R}$ be the scale function of Z, that is, $W^{(r)}$ is the only function such that $W^{(r)}(x) = 0$ for any $x < 0$ and

$$\int_0^\infty e^{-\beta x} W^{(r)}(x)\mathrm{d}x = \frac{1}{\phi(\beta) - r},$$

for $\beta > \Phi(r)$, where Φ is the right inverse of the function ϕ.

From the Wiener–Hopf factorization (see, e.g., equation (8.4) of Kyprianou (2014)), we have that for any $0 \le \beta \le 1$ and $r \ge r_0$,

$$\mathbb{E}[e^{\beta \overline{Y}_{\mathbf{e}_r}}] = \mathbb{E}[e^{-\beta \underline{Z}_{\mathbf{e}_r}}] = \frac{r}{\Phi(r)} \frac{\Phi(r)+\beta}{r-\phi(-\beta)}, \tag{23}$$

provided $\phi(-1) = \psi(1) < r_0$. Applying Laplace inversion to (23) (or see equation (8.24) of Kyprianou (2014)) yields the law

$$\mathbb{P}(\overline{Y}_{\mathbf{e}_r} \in \mathrm{d}y) = \frac{r}{\Phi(r)} W^{(r)}(\mathrm{d}y) - rW^{(r)}(y)\mathrm{d}y. \tag{24}$$

To meet the conditions of Theorem 5.3, we assume that Y is of unbounded variation and the Gaussian component $\sigma > 0$. Then by (24) and Lemmas 3.1 and 3.2 of Kuznetsov et al. (2012b), we have that

$$f_r(0) = \frac{r}{\Phi(r)} \frac{2}{\sigma^2}. \tag{25}$$

By (20), (23), and (25), the stopping threshold to Problem (17) is thus given by

$$x_F^* = \frac{\int_0^\infty \frac{r}{\Phi(r)} F(\mathrm{d}r)}{\int_0^\infty \frac{r-\psi(1)}{\Phi(r)+1} F(\mathrm{d}r)} I.$$

6.3 Kou's model with double exponential jumps

Suppose that Y follows the Kou's model with double exponential jumps, i.e.,

$$Y_t = \mu t + \sigma B_t + \sum_{i=1}^{N_t} \xi_i, \quad t \ge 0,$$

where $\{B_t, t \ge 0\}$ is a standard Brownian motion, $\mu \in \mathbb{R}$, $\{N_t, t \ge 0\}$ is an independent Poisson process with parameter $\lambda > 0$, and $\{\xi_i\}_{i=1}^\infty$ are independent and identically distributed random variables with probability density function given by

$$g(y) = p\eta_1 e^{-\eta_1 y} \mathbb{I}_{\{y>0\}} + (1-p)\eta_2 e^{\eta_2 y} \mathbb{I}_{\{y<0\}},$$

where $0 \le p \le 1$ and $\eta_1, \eta_2 > 0$.

From Kou and Wang (2003), it is known that for any $r > 0$,

$$\mathbb{E}[e^{-r\tau_y^+}] = \frac{\eta_1 - \beta_1(r)}{\eta_1} \frac{\beta_2(r)}{\beta_2(r)-\beta_1(r)} e^{-\beta_1(r)y} + \frac{\beta_2(r)-\eta_1}{\eta_1} \frac{\beta_1(r)}{\beta_2(r)-\beta_1(r)} e^{-\beta_2(r)y},$$

where $0 < \beta_1(r) < \eta_1 < \beta_2(r) < \infty$ are the only two positive roots of the equation

$$\psi(\beta) := \mu\beta + \frac{1}{2}\sigma^2\beta^2 + \lambda \left(\frac{p\eta_1}{\eta_1 - \beta} + \frac{(1-p)\eta_2}{\eta_2 + \beta} - 1 \right) = r.$$

Since $\mathbb{P}(\overline{Y}_{\mathbf{e}_r} > y) = \mathbb{P}(\tau_y^+ < \mathbf{e}_r) = \mathbb{E}[e^{-r\tau_y^+}]$, it follows that the probability density function of $\overline{Y}_{\mathbf{e}_r}$ is given by

$$f_r(y) = \frac{\beta_1(r)\beta_2(r)}{\beta_2(r)-\beta_1(r)} \left[\frac{\eta_1 - \beta_1(r)}{\eta_1} e^{-\beta_1(r)y} + \frac{\beta_2(r)-\eta_1}{\eta_1} e^{-\beta_2(r)y} \right],$$

and the moment generating function of $\overline{Y}_{\mathbf{e}_r}$ is given by

$$\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}] = \frac{\beta_1(r)\beta_2(r)(\eta_1 - 1)}{\eta_1(\beta_2(r) - 1)(\beta_1(r) - 1)},$$

which is finite provided $1 < \beta_1(r) < \beta_2(r)$. It follows that

$$f_r(0) = \frac{\beta_1(r)\beta_2(r)}{\beta_2(r) - \beta_1(r)} \frac{\beta_2(r) - \beta_1(r)}{\eta_1} = \frac{\beta_1(r)\beta_2(r)}{\eta_1},$$

and

$$\frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} = \frac{(\beta_2(r) - 1)(\beta_1(r) - 1)}{(\eta_1 - 1)}.$$

Therefore, by (20), the stopping threshold to Problem (17) is given by

$$x_F^* = \frac{\eta_1 - 1}{\eta_1} \frac{\int_0^\infty \beta_1(r)\beta_2(r)F(\mathrm{d}r)}{\int_0^\infty (\beta_2(r) - 1)(\beta_1(r) - 1)F(\mathrm{d}r)}.$$

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A Appendix

A.1 Lévy processes creeping upwards

From the Wiener–Hopf factorization (see, e.g. Theorem 6.15 of Kyprianou (2014)), we have that for any $\beta \geq 0$,

$$\mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_r}}] = \frac{\kappa(r, 0)}{\kappa(r, \beta)}. \tag{26}$$

Here κ is the Laplace exponent of the ascending ladder process $(L^{-1}, H) = \{(L_t^{-1}, H_t), t \geq 0\}$, that is defined as, for any $r, \beta \geq 0$,

$$\kappa(r, \beta) = -\log\left(\mathbb{E}(e^{-rL_1^{-1} - \beta H_1}\mathbb{I}_{\{1 < L_\infty\}})\right), \tag{27}$$

where $L = \{L_t, t \geq 0\}$ is the local time at the maximum, L_t^{-1} is the inverse local time at time $t \geq 0$ defined by

$$L_t^{-1} = \begin{cases} \inf\{s > 0 : L_s > t\}, & t < L_\infty, \\ \infty, & \text{otherwise}, \end{cases}$$

and

$$H_t = \begin{cases} Y_{L_t^{-1}}, & t < L_\infty, \\ \infty, & \text{otherwise}, \end{cases}$$

is the ladder height process. The inverse local time L^{-1} corresponds to the real times at which new maxima are reached, and the ladder height process H corresponds to the set of new maxima.

For any $r, \beta \geq 0$, we have the representation

$$\kappa(r, \beta) = q + \phi(r, \beta), \tag{28}$$

where $q \geq 0$ and ϕ is the Laplace exponent of a bivariate subordinator, say,

$$\phi(r, \beta) = ra + \beta b + \int_{(0,\infty)^2} (1 - e^{-rx-\beta y})\Lambda(\mathrm{d}x, \mathrm{d}y), \tag{29}$$

with $a, b \geq 0$ and Λ is a measure on $(0, \infty)^2$ satisfying

$$\int_{(0,\infty)^2} \left(1 \wedge \sqrt{x^2 + y^2}\right) \Lambda(\mathrm{d}x, \mathrm{d}y) < \infty.$$

Moreover the random variable $\overline{Y}_{\mathbf{e}_r}$ is equal in law to $\mathbb{H}_{\mathbf{e}_\chi}$, where $\mathbf{e}_\chi$ is an independent exponential random variable with parameter $\kappa(r, 0)$ and $\{\mathbb{H}_t, t \geq 0\}$ is a subordinator with Laplace exponent given by

$$\beta \mapsto \beta b + \int_{(0,\infty)^2} (1 - e^{-\beta y})e^{-rx}\Lambda(\mathrm{d}x, \mathrm{d}y) = \kappa(r, \beta) - \kappa(r, 0). \tag{30}$$

Note that from (27), we easily deduce that the function κ is non-decreasing in each argument. Moreover from (28) and (29) we can see that

$$(-1)^{n+1} \frac{\partial^n}{\partial r^n} \kappa(r, \beta) \geq 0, \tag{31}$$

for all $n \geq 1$, $r > 0$, and $\beta \geq 0$.

Lemma A.1. *Suppose that Y creeps upwards. If $\mathbb{E}[e^{\beta \overline{Y}_{e_r}}] < \infty$ for some $\beta > 0$ and $r \geq 0$, one can extend (26) to be*

$$\mathbb{E}[e^{\beta \overline{Y}_{e_r}}] = \frac{\kappa(r, 0)}{\kappa(r, -\beta)}, \tag{32}$$

where $\kappa(r, -\beta)$ can still be defined as in (27).

Proof. First, we note that $\kappa(r, 0) < \infty$. Indeed, since Y creeps upwards we have that the random variable L_∞ is a stricly positive random variable. Moreover, we have that L_∞ is exponentially distributed with parameter $\kappa(0, 0) \geq 0$ (see Theorem 6.9 of Kyprianou (2014)). Hence, we have that that the term inside of the expectation in (27) is stricly positive $\mathbb{P}$ a.s. implying that $\kappa(r, 0) < \infty$ as claimed.

Since $\overline{Y}_{\mathbf{e}_r}$ is equal in law to $\mathbb{H}_{\mathbf{e}_\chi}$ we have that

$$\int_0^\infty \chi e^{-\chi t} \mathbb{E}[e^{\beta \mathbb{H}_t}]\mathrm{d}t = \mathbb{E}[e^{\beta \mathbb{H}_\chi}] = \mathbb{E}[e^{\beta \overline{Y}_{\mathbf{e}_r}}] < \infty.$$

This implies that $\mathbb{E}[e^{\beta \mathbb{H}_t}] < \infty$ for each $t \geq 0$. In particular, from the Lévy–Khintchine formula and (30) we have that

$$-\log(\mathbb{E}[e^{\beta \mathbb{H}_1}]) = -\beta b + \int_{(0,\infty)^2} (1 - e^{\beta y}) e^{-rx} \Lambda(\mathrm{d}x, \mathrm{d}y) = \kappa(r, -\beta) - \kappa(r, 0).$$

Then we obtain that

$$\mathbb{E}[e^{\beta \overline{Y}_{\mathbf{e}_r}}] = \mathbb{E}[e^{\beta \mathbb{H}_\chi}] = \int_0^\infty \chi e^{-\chi t} \mathbb{E}[e^{\beta \mathbb{H}_t}] \mathrm{d}t = \frac{\chi}{\kappa(r, -\beta) - \kappa(r, 0) + \chi} = \frac{\kappa(r, 0)}{\kappa(r, -\beta)}.$$

□

The following lemma gives some useful properties for the density function of $\overline{Y}_{\mathbf{e}_r}$.

Lemma A.2. *Suppose that Y is a Lévy process that creeps upwards. For any $r \geq 0$ fixed, $\overline{Y}_{\mathbf{e}_r}$ has a bounded, strictly positive and continuously differentiable density f_r such that $r \mapsto f_r'(y) := \frac{\mathrm{d}}{\mathrm{d}y} f_r(y)$ is continuous on $[0, \infty)$. In particular, we have that*

$$f_r(0) = \frac{\kappa(r, 0)}{b} \quad \text{and} \quad f_r'(0+) = -\frac{\kappa(r, 0)}{b^2} \left(\int_{(0,\infty)^2} e^{-rx} \Lambda(\mathrm{d}x, \mathrm{d}y) + \kappa(r, 0) \right), \tag{33}$$

for all $r \geq 0$, where $b > 0$ and Λ are the drift and the Lévy measure, respectively, of the subordinator associated to the ladder height process H given in (29).

Proof. Let $r \geq 0$ fixed. From the Wiener–Hopf theory we know that the random variable $\overline{Y}_{\mathbf{e}_r}$ has the same law as $\mathbb{H}_{\mathbf{e}_\chi}$, where $\{\mathbb{H}_t, t \geq 0\}$ is a subordinator with Laplace exponent given in (30) and $\mathbf{e}_\chi$ is an independent exponential random variable with parameter $\chi = \kappa(r, 0)$. From Theorem VI.4 in Bertoin (1998) on page 175 we have that $b > 0$ if and only if Y creeps upwards. Moreover, from Corollary II.20 and Theorem II.16 in Bertoin (1998) we have that $\mathbb{H}_{\mathbf{e}\chi}$ (and hence $\overline{Y}_{\mathbf{e}_r}$) is absolutely continuous and has a bounded stricly positive density, say $u^{(\chi)}(y)$, which is equal to $f_r(y)$. Moreover, since the measure Λ has no atoms (see discussion after Lemma 7.10 in Kyprianou (2014)) we deduce from Corollary 3 in Döring and Savov (2011) that $u^{(\chi)}(y)$ is differentiable. Moreover, we have the series representation

$$(u^{(\chi)})'(y) = \sum_{n=1}^\infty \frac{(-1)^n}{b^{n+1}} (\overline{\Lambda} + \chi)^{*n}(y),$$

where $\overline{\Lambda}(y) = \Lambda([0, \infty), [y, \infty))$ for $y \geq 0$. Note that due to the fact that Λ does not have atoms, we also deduce that $y \mapsto (u^{(\chi)})'(y)$ is continuous. Furthermore, since $r \mapsto \chi = \kappa(r, 0)$ is continuous, we have that the mapping $r \mapsto f_r'(y)$ is also continuous, for each $y \geq 0$.

Further, by the Wiener–Hopf factorization (26), we deduce that

$$f_r(0) = \lim_{\beta \to \infty} \beta \mathbb{E}[e^{-\beta \overline{Y}_{\mathbf{e}_r}}] = \lim_{\beta \to \infty} \beta \frac{\kappa(r, 0)}{\kappa(r, \beta)} = \frac{\kappa(r, 0)}{b},$$

where the last equality follows from equations (28) and (29). Moreover, using integration by parts we obtain that for any $\beta \geq 0$,

$$\int_0^\infty e^{-\beta y} f_r'(y) \mathrm{d}y = \beta \mathbb{E}[e^{-\beta \overline{Y}_{\mathbf{e}_r}}] - f_r(0) = \beta \frac{\kappa(r, 0)}{\kappa(r, \beta)} - \frac{\kappa(r, 0)}{b} = \frac{\kappa(r, 0)}{b} \frac{\beta b - \kappa(r, \beta)}{\kappa(r, \beta)}.$$

Then from equations (30), (28), and (29), we conclude that

$$
\begin{aligned}
f_r'(0+) &= \lim_{\beta\to\infty} \beta \int_0^\infty e^{-\beta y} f_r'(y)\mathrm{d}y \\
&= \lim_{\beta\to\infty} \beta \frac{\kappa(r,0)}{b} \frac{\beta b - \kappa(r,\beta) + \kappa(r,0) - \kappa(r,0)}{\kappa(r,\beta)} \\
&= -\frac{\kappa(r,0)}{b} \left(\int_{(0,\infty)^2} e^{-rx} \Lambda(\mathrm{d}x,\mathrm{d}y) + \kappa(r,0) \right) \lim_{\beta\to\infty} \frac{\beta}{\kappa(r,\beta)} \\
&= -\frac{\kappa(r,0)}{b^2} \left(\int_{(0,\infty)^2} e^{-rx} \Lambda(\mathrm{d}x,\mathrm{d}y) + \kappa(r,0) \right).
\end{aligned}
$$

□

Remark A.3. *Note that, even though the lemma above includes a large family of Lévy processes, the result is not sharp. For instance, one can consider a spectrally positive process of bounded variation with negative drift which does not creep upwards. In this case, the cumulative distribution function of* $\overline{Y}_{\mathbf{e}_r}$ *can be written in terms of the scale functions and has an absolutely continuous density when* Π *has no atoms, see Section 6.2.*

Some examples in the literature for which a closed-form expression for f_r *exist include: spectrally negative Lévy processes (for which* $\overline{Y}_{\mathbf{e}_r}$ *has exponential distribution), spectrally positive Lévy processes (for which the distribution of* $\overline{Y}_{\mathbf{e}_r}$ *has a density when* $\sigma > 0$ *or* Π *has no atoms and is described in terms of the scale functions), Lévy processes with phase type positive jumps for which the distribution of* $\overline{Y}_{\mathbf{e}_r}$ *is of phase type (see Mordecki (2002b)), and the class of meromorphic Lévy processes (see Kuznetsov et al. (2012a)) which have jumps of a (possibly infinite) mixture of exponentials and then* $\overline{Y}_{\mathbf{e}_r}$ *has an atom at zero and* f_r *is also a mixture of exponentials.*

The next lemma describes the behaviour in terms of the variable r for some functions that will be used in the proof of Theorem 5.3.

Lemma A.4. *Suppose that* Y *is a Lévy process that creeps upwards.*

i) For any fixed $\beta \geq 0$*, the mappings* $r \mapsto f_r(0)$ *and* $r \mapsto \mathbb{E}[e^{-\beta \overline{Y}_{\mathbf{e}_r}}]$ *are non-decreasing.*

ii) For any $y \geq 0$*, we have that* $f_r(y) \leq f_r(0)$ *and the mapping* $r \mapsto \frac{f_r(y)}{f_r(0)}$ *is non-increasing.*

iii) Assume that $\mathbb{E}[e^{\beta \overline{Y}_{\mathbf{e}_{r_0}}}] < \infty$ *for some* $r_0 \geq 0$ *and* $\beta > 0$*. Then, for any* $z > 1$*, the mapping*

$$
r \mapsto z \frac{f_r(0)}{\mathbb{E}[e^{\beta \overline{Y}_{\mathbf{e}_r}}]} - f_r(0)
$$

is non-decreasing on $[r_0, \infty)$.

Proof. i) From (33) and the fact that κ is non-decreasing in each argument, we know that $f_r(0)$ is non-decreasing in r. It is also obvious to see $r \mapsto \mathbb{E}[e^{-\beta \overline{Y}_{\mathbf{e}_r}}]$ is non-decreasing as $\overline{Y}_t$ is a non-decreasing process and the mean of $\mathbf{e}_r$ is $1/r$.

ii) Let

$$
T_y = \inf\{t > 0 : \mathbb{H}_t > y\}.
$$

By Exercise 5.7 iii) of Kyprianou (2014), we have that for any $y \geq 0$,

$$\mathbb{E}[e^{-\chi T_y}\mathbb{I}_{\{\mathbb{H}_{T_y}=y\}}] = bu^{(\chi)}(y),$$

where $u^{(\chi)}(y)$ is the density of the potential measure

$$U^{(\chi)}(\mathrm{d}y) = \int_0^\infty e^{-\chi t}\mathbb{P}(\mathbb{H}_t \in \mathrm{d}y)\mathrm{d}t = \frac{\mathbb{P}(\mathbb{H}_{\mathbf{e}_\chi} \in \mathrm{d}y)}{\chi} = \frac{\mathbb{P}(\overline{Y}_{\mathbf{e}_r} \in \mathrm{d}y)}{\kappa(r,0)}.$$

Hence, for any $y \geq 0$,

$$f_r(y) = \frac{\kappa(r,0)}{b}\mathbb{E}[e^{-\kappa(r,0)T_y}\mathbb{I}_{\{\mathbb{H}_{T_y}=y\}}],$$

and, by (33),

$$\frac{f_r(y)}{f_r(0)} = \mathbb{E}[e^{-\kappa(r,0)T_y}\mathbb{I}_{\{\mathbb{H}_{T_y}=y\}}].$$

Therefore, $f_r(y)/f_r(0)$ is non-increasing in r.

iii) Note that $\mathbb{E}[e^{\beta\overline{Y}_{\mathbf{e}_{r_0}}}] < \infty$ implies $\mathbb{E}[e^{\beta\overline{Y}_{\mathbf{e}_r}}] < \infty$ for all $r \geq r_0$ as $\beta > 0$. By equations (32) and (33), we have that for any $r \geq r_0$ and $z > 1$,

$$\begin{aligned}
&z\frac{f_r(0)}{\mathbb{E}[e^{\beta\overline{Y}_{\mathbf{e}_r}}]} - f_r(0)\\
&= z\frac{\kappa(r,-\beta)}{b} - \frac{\kappa(r,0)}{b}\\
&= \frac{1}{b}\left[z\left(ra - \beta b + \int_{(0,\infty)^2}(1-e^{-rx+\beta y})\Lambda(\mathrm{d}x,\mathrm{d}y)\right) - ra - \int_{(0,\infty)^2}(1-e^{-rx})\Lambda(\mathrm{d}x,\mathrm{d}y)\right]\\
&= \frac{1}{b}\left[(z-1)ra - z\beta b + \int_{(0,\infty)^2}\left(z - 1 - e^{-rx}\left(ze^{\beta y}-1\right)\right)\Lambda(\mathrm{d}x,\mathrm{d}y)\right],
\end{aligned}$$

which is obviously non-decreasing in r as $z > 1$. □

Next lemma ensures the integrals in (20) are finite and it is only optimal to install the project when the present value exceeds the fixed cost, i.e., $x_F^* \geq I$.

Lemma A.5. *Let Y be any Lévy process that creeps upwards. Suppose that $\overline{r} < \infty$. Then we have that*

$$\int_0^\infty \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{e_r}}]}F(\mathrm{d}r) \leq \int_0^\infty f_r(0)F(\mathrm{d}r) < \infty.$$

Proof. Take any $\delta > 0$. We have that for any $\beta \geq 0$,

$$\begin{aligned}
\int_0^\infty \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_r}}]F(\mathrm{d}r) &= \int_0^\delta \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_r}}]F(\mathrm{d}r) + \int_\delta^\infty \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_r}}]F(\mathrm{d}r)\\
&\leq \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_\delta}}] + \int_\delta^\infty\int_0^\infty re^{-rt}\mathbb{E}[e^{-\beta\overline{Y}_t}]\mathrm{d}tF(\mathrm{d}r)\\
&\leq \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_\delta}}] + \int_\delta^\infty\int_0^\infty re^{-\delta t}\mathbb{E}[e^{-\beta\overline{Y}_t}]\mathrm{d}tF(\mathrm{d}r)\\
&\leq \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_\delta}}] + \frac{\overline{r}}{\delta}\mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_\delta}}],
\end{aligned}$$

where we used that the mapping $r \mapsto \mathbb{E}[e^{-\beta \overline{Y}_{\mathbf{e}_r}}]$ is non-decreasing (see part i) of Lemma A.4). By Fatou's lemma, we deduce that

$$\begin{aligned}\int_0^\infty f_r(0)F(\mathrm{d}r) &= \int_0^\infty \lim_{\beta\to\infty} \beta\mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_r}}]F(\mathrm{d}r)\\ &\le \lim_{\beta\to\infty} \beta \int_0^\infty \mathbb{E}[e^{-\beta\overline{Y}_{\mathbf{e}_r}}]F(\mathrm{d}r)\\ &\le f_\delta(0) + \frac{\overline{r}}{\delta} f_\delta(0).\end{aligned}$$

The finiteness of $\int_0^\infty f_r(0)F(\mathrm{d}r)$ then follows from (33). Lastly, by (32), (33), and that κ is non-decreasing in both arguments, it follows that

$$\int_0^\infty \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}F(\mathrm{d}r) = \int_0^\infty \frac{\kappa(r,-1)}{b}F(\mathrm{d}r) \le \int_0^\infty \frac{\kappa(r,0)}{b}F(\mathrm{d}r) = \int_0^\infty f_r(0)F(\mathrm{d}r) < \infty.$$

$\square$

A.2 Proof of Theorem 4.1

For any $t \ge 0$ and $x \in \mathbb{R}$, by (11) and the condition $V(t,x) = J(t,x;\tau_{u^*}^{t,x}) \ge G(x)$, it follows that (10) holds true when $a = 1$. It remains to show that (10) also holds for $a = 0$.

For any $t \ge 0$, $x \in \mathbb{R}$, and $\varepsilon > 0$, recall that $\tau_{u^{\varepsilon},0}^{t,x} = \inf\{s \ge t+\varepsilon : u^*(s, X_s^{t,x}) = 1\} := w$ as in (12), by (9), we have

$$\begin{aligned}J(t,x;\tau_{u^\varepsilon,0}^{t,x}) &= \mathbb{E}[h\left(\tau_{u^\varepsilon,0}^{t,x} - t\right) G(X_{\tau_{u^\varepsilon,0}^{t,x}}^{t,x})]\\ &= \mathbb{E}[h(w-t)G(X_w^{t,x})]\\ &= \mathbb{E}[h(w-t-\varepsilon)G(X_w^{t,x})] + \mathbb{E}\left[\left(h(w-t) - h(w-t-\varepsilon)\right)G(X_w^{t,x})\right]. \end{aligned} \tag{34}$$

By the Markov property of X (Eq. (7)), (13) and (14), we have

$$\begin{aligned}\mathbb{E}[h(w-t-\varepsilon)G(X_w^{t,x})|\mathcal{F}_{t+\varepsilon}] &= \mathbb{E}[h(\tau_{u^\varepsilon,0}^{t,x} - t - \varepsilon)G(X_{\tau_{u^\varepsilon,0}^{t,x}}^{t,x})|\mathcal{F}_{t+\varepsilon}]\\ &= \mathbb{E}[h(\tau_{u^*}^{t+\varepsilon,y} - t - \varepsilon)G(X_{\tau_{u^*}^{t+\varepsilon,y}}^{t+\varepsilon,y})]\Big|_{y=X_{t+\varepsilon}^{t,x}} \qquad (35)\\ &= J(t+\varepsilon, y; \tau_{u^*}^{t+\varepsilon, X_{t+\varepsilon}^{t,x}})\\ &= V(t+\varepsilon, X_{t+\varepsilon}^{t,x}).\end{aligned}$$

By the tower property, the first expectation in (34) becomes

$$\mathbb{E}\left[h(w-t-\varepsilon)G(X_w^{t,x})\right] = \mathbb{E}\left[\mathbb{E}[h(w-t-\varepsilon)G(X_w^{t,x})|\mathcal{F}_{t+\varepsilon}]\right] = \mathbb{E}[V(t+\varepsilon, X_{t+\varepsilon}^{t,x})]. \tag{36}$$

For the second expectation in (34), by Fubini's theorem and the Markov property of X,

$$
\begin{aligned}
& \mathbb{E}\left[(h(w-t)-h(w-t-\varepsilon))\, G(X_w^{t,x})\right] \\
= & \ \mathbb{E}\left[\int_0^\infty \left(e^{-r(w-t)} - e^{-r(w-t-\varepsilon)}\right) G(X_w^{t,x}) F(\mathrm{d}r)\right] \\
= & \ \mathbb{E}\left[\int_0^\infty e^{-r(w-t-\varepsilon)}(e^{-r\varepsilon}-1) G(X_w^{t,x}) F(\mathrm{d}r)\right] \\
= & \ -\int_0^\varepsilon \mathbb{E}\left[\int_0^\infty r e^{-r(w-t-\varepsilon+s)} G(X_w^{t,x}) F(\mathrm{d}r)\right] \mathrm{d}s \\
= & \ -\int_0^\varepsilon \mathbb{E}\left[\int_0^\infty \mathbb{E}[r e^{-r(w-t-\varepsilon+s)} G(X_w^{t,x}) | \mathcal{F}_{t+\varepsilon}] F(\mathrm{d}r)\right] \mathrm{d}s \\
= & \ -\int_0^\varepsilon \mathbb{E}\left[\int_0^\infty \left(\mathbb{E}[r e^{-r(\tau_{u^*}^{t+\varepsilon,y}-t-\varepsilon+s)} G(X^{t,x}_{\tau_{u^*}^{t+\varepsilon,y}})]\Big|_{y=X_{t+\varepsilon}^{t,x}} \right) F(\mathrm{d}r)\right] \mathrm{d}s \\
= & \ -\int_0^\varepsilon \mathbb{E}[H(s,t+\varepsilon,X_{t+\varepsilon}^{t,x})] \mathrm{d}s, \qquad (37)
\end{aligned}
$$

where the second last equality is by (35) and the last equality is from the definition of function H given in (15). Substitute (36) and (37) into (34) yields

$$
J(t,x;\tau_{u^{\varepsilon,0}}^{t,x}) = \mathbb{E}[V(t+\varepsilon, X_{t+\varepsilon}^{t,x})] - \int_0^\varepsilon \mathbb{E}[H(s,t+\varepsilon,X_{t+\varepsilon}^{t,x})] \mathrm{d}s. \qquad (38)
$$

On the other hand, by the condition that $\{Z_s^{t,x}, s \geq 0\}$, defined in (16), is a supermartingale, we have

$$
\mathbb{E}[V(t+\varepsilon, X_{t+\varepsilon}^{t,x})] - \mathbb{E}\left[\int_0^\varepsilon H(0,t+v,X_{t+v}^{t,x}) \mathrm{d}v\right] = \mathbb{E}[Z_\varepsilon^{t,x}] \leq Z_0^{t,x} = V(t,x).
$$

Dividing by ε and taking the limit yields

$$
\liminf_{\varepsilon \downarrow 0} \frac{V(t,x) - \mathbb{E}[V(t+\varepsilon, X_{t+\varepsilon}^{t,x})]}{\varepsilon} \geq -H(0,t,x).
$$

Therefore we have that for any $t > 0$ and $x \in \mathbb{R}$, by (38) and the last inequality,

$$
\begin{aligned}
& \liminf_{\varepsilon \downarrow 0} \frac{J(t,x;\tau_{u^*}^{t,x}) - J(t,x;\tau_{u^{\varepsilon,0}}^{t,x})}{\varepsilon} \\
= & \liminf_{\varepsilon \downarrow 0} \frac{V(t,x) - \mathbb{E}[V(t+\varepsilon, X_{t+\varepsilon}^{t,x})] + \int_0^\varepsilon \mathbb{E}[H(s,t+\varepsilon,X_{t+\varepsilon}^{t,x})]\mathrm{d}s}{\varepsilon} \\
\geq & -H(0,t,x) + H(0,t,x) \\
= & \ 0.
\end{aligned}
$$

This completes the proof.

A.3 Proof of Proposition 4.4

i) Since u^* is a time-invariant equilibrium stopping rule, for any $x \in D$, by (10) with $a = 0$ and (38), we have that

$$\begin{aligned}
0 &\le \liminf_{\varepsilon\to 0} \frac{J(t,x;\tau^t_{u^*}) - J(t,x;\tau^t_{u^{\varepsilon,0}})}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0} \frac{V(x) - \mathbb{E}[V(X^{0,x}_\varepsilon)] + \int_0^\varepsilon \mathbb{E}[H(s,X^{0,x}_\varepsilon)]\mathrm{d}s}{\varepsilon} \\
&\le \liminf_{\varepsilon\to 0} \frac{G(x) - \mathbb{E}[G(X^{0,x}_\varepsilon)]}{\varepsilon} + H(0,x) \\
&= -\mathcal{A}_X G(x) + \overline{r}G(x),
\end{aligned}$$

where the second inequality is by $V(x) = G(x)$ for $x \in D$ and $V(x) \ge G(x)$ for all $x > 0$ (by (10) with $a = 1$), while the last equality is due to, for $x \in D$,

$$H(0,x) = \int_0^\infty \mathbb{E}[re^{-r\tau^{0,x}_{u^*}} G(X^{0,x}_{\tau^{0,x}_{u^*}})]F(\mathrm{d}r) = G(x)\int_0^\infty rF(\mathrm{d}r) = \overline{r}G(x).$$

ii) Let x^* be an equilibrium stopping threshold such that $(-\infty, x^*) \cap C \neq \emptyset$. We prove $V'(x^*-) \le G'(x^*)$ by contradiction. Suppose that

$$V'(x^*-) > G'(x^*).$$

By the assumption that V is continuously differentiable in the closure of the continuation region, there exists some neighborhood $(x^* - \delta, x^*)$ for some $\delta > 0$ such that $V'(x) > G'(x)$ for any $x \in (x^* - \delta, x^*)$. By further $V(x^*) = G(x^*)$ as $x^* \in D$, we deduce that

$$V(x) < G(x), \quad \text{for any } x \in (x^* - \delta, x^*).$$

This contradicts to the optimality of x^* as in (10). The case when $(x^*, \infty) \cap C$ is analogous and thus omitted.

A.4 Proof of Theorem 5.3

Recall that we have defined

$$\tau^+_y = \inf\{t \ge 0 : Y_t > y\},$$

for any $y \in \mathbb{R}$. For $x > 0$ and $z > 0$, we further define the first passage times of X,

$$\sigma^+_{x,z} = \inf\{t \ge 0 : X^{0,x}_t > z\}.$$

Note that in this case, since $G(x) = x - I$, we have for any $x > 0$ that,

$$\mathbb{E}\left[\sup_{t\ge 0} e^{-r_0 t}|G(X^{0,x}_t)|\right] \le x\mathbb{E}\left[e^{\sup_{t\ge 0}(Y_t - r_0 t)}\right] + I < \infty,$$

where we used condition (4) and that $\{Y_t - r_0 t, t \ge 0\}$ is a Lévy process with Laplace exponent $\psi(\beta) - r_0\beta$ and then satisfies $\psi(1) - r_0 \le 0$, by assumption.

In order to prove Theorem 5.3, by Corollary 4.2, it remains to show that

i) $V(x) \geq G(x) = x - I$ for any $x > 0$.

ii) For any $x > 0$, the process $V(X_s^{0,x}) - \int_0^s H(0, X_v^{0,x})\mathrm{d}v$ for $s \geq 0$ is a supermartingale, where H is given by

$$H(s,x) = \int_0^\infty \mathbb{E}[re^{-r(\sigma^+_{x,x^*_F}+s)}(X^{0,x}_{\sigma^+_{x,x^*_F}} - I)]F(\mathrm{d}r).$$

Since the proof of the above two statements is rather lengthy, it will be split in a series of lemmas.

For any fixed $z > 0$ and $r \geq r_0$, we define the functions

$$V_z^{(r)}(x) = \mathbb{E}[e^{-r\sigma^+_{x,z}}(X^{0,x}_{x,\sigma^+_z} - I)], \quad x > 0, \tag{39}$$

and

$$V_z(x) = \int_0^\infty V_z^{(r)}(x)F(\mathrm{d}r) = \mathbb{E}[h(\sigma^+_{x,z})(X^{0,x}_{\sigma^+_{x,z}} - I)], \quad x > 0. \tag{40}$$

In particular, the equilibrium value function $V(x) = V_{x^*_F}(x)$.

Lemma A.6. *Suppose that Y is a Lévy process that creeps upwards and $\psi(1) < r_0$, we have*

i) For any $x, z > 0$ and $r \geq r_0$,

$$V_z^{(r)}(x) = x\frac{\mathbb{E}[e^{\overline{Y}_{e_r}}\mathbb{I}_{\{\overline{Y}_{e_r} > \log(z/x)\}}]}{\mathbb{E}[e^{\overline{Y}_{e_r}}]} - I\mathbb{E}[\mathbb{I}_{\{\overline{Y}_{e_r} > \log(z/x)\}}], \tag{41}$$

and

$$V_z(x) = \int_0^\infty \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{e_r}}]} \int_{\log(z/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(z/x)}^\infty f_r(y)\mathrm{d}y \right) F(\mathrm{d}r). \tag{42}$$

ii) For any $x, z > 0$ and $r \geq r_0$,

$$-I \leq V_z(x) \leq V_z^{(r_0)}(x) \leq x. \tag{43}$$

iii) $V_z'(z-) = 1$ if and only if $z = x^*_F$.

iv) For any $x < z \leq x^*_F$,

$$V_z'(x) \leq 1,$$

v) V is differentiable on $(0, \infty)$ with $V'(x^*_F) = 1$. Moreover, for all $x > 0$,

$$V'(x) \leq 1 \quad \text{and} \quad V(x) \geq x - I.$$

Proof. i) Recall that $X_t^{0,x} = xe^{Y_t}$. For any $x, z > 0$

$$\begin{aligned}
V_z^{(r)}(x) &= \mathbb{E}[e^{-r\sigma^+_{x,z}}(X^{0,x}_{\sigma^+_{x,z}} - I)] \\
&= \mathbb{E}[e^{-r\tau^+_{\log(z/x)}}(xe^{Y_{\tau^+_{\log(z/x)}}} - I)] \\
&= x\mathbb{E}[e^{-r\tau^+_{\log(z/x)}}e^{Y_{\tau^+_{\log(z/x)}}}] - I\mathbb{E}[e^{-r\tau^+_{\log(z/x)}}] \\
&= x\frac{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}\mathbb{I}_{\{\overline{Y}_{\mathbf{e}_r} > \log(z/x)\}}]}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} - I\mathbb{E}[\mathbb{I}_{\{\overline{Y}_{\mathbf{e}_r} > \log(z/x)\}}].
\end{aligned}$$

where the last equality is due to equation (5). Moreover, by Fubini's theorem, we get that

$$\begin{aligned}
V_z(x) &= \int_0^\infty V_z^{(r)}(x) F(\mathrm{d}r) \\
&= \int_0^\infty \left(x \frac{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}} \mathbb{I}_{\{\overline{Y}_{\mathbf{e}_r} > \log(z/x)\}}]}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} - I \mathbb{E}[\mathbb{I}_{\{\overline{Y}_{\mathbf{e}_r} > \log(z/x)\}}] \right) F(\mathrm{d}r) \\
&= \int_0^\infty \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \int_{\log(z/x)}^\infty e^y f_r(y) \mathrm{d}y - I \int_{\log(z/x)}^\infty f_r(y) \mathrm{d}y \right) F(\mathrm{d}r),
\end{aligned}$$

where f_r is the density of $\overline{Y}_{\mathbf{e}_r}$.

ii) From (39), (40), and the definition of r_0, it is clear that $-I \le V_z(x) \le V_z^{(r_0)}(x)$ for any $x > 0$. Moreover, $V_z^{(r_0)}(x) \le x$ is immediately from (41).

iii) By (42) and taking into account Lemma A.4 ii), one obtains that, for any $x < z$,

$$V_z'(x) = \int_0^\infty \left(\frac{1}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \int_{\log(z/x)}^\infty e^y f_r(y) \mathrm{d}y + \frac{z}{x \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} f_r(\log(z/x)) - \frac{I}{x} f_r(\log(z/x)) \right) F(\mathrm{d}r). \quad (44)$$

Since V_z is smooth for all $x \in (0, z]$, we have

$$V_z'(z-) = \lim_{x \uparrow z} V_z'(x) = 1 + \int_0^\infty \left(\frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} - \frac{I}{z} f_r(0) \right) F(\mathrm{d}r).$$

Thus, $V_z'(z-) = 1$ if and only if

$$z = \frac{\int_0^\infty f_r(0) F(\mathrm{d}r)}{\int_0^\infty \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} F(\mathrm{d}r)} I = x_F^*.$$

iv) For $x < z \le x_F^*$, it follows from (44) that

$$\begin{aligned}
V_z'(x) &\le 1 + \frac{I}{x} \int_0^\infty \left(\frac{z}{I \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} f_r(\log(z/x)) - f_r(\log(z/x)) \right) F(\mathrm{d}r) \\
&\le 1 + \frac{I}{x} \int_0^\infty \left(\frac{x_F^*}{I \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} f_r(\log(z/x)) - f_r(\log(z/x)) \right) F(\mathrm{d}r) \\
&= 1 - \frac{I}{x} \int_0^\infty \frac{f_r(\log(z/x))}{f_r(0)} \left(f_r(0) - \frac{x_F^*}{I} \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \right) F(\mathrm{d}r).
\end{aligned}$$

From Lemma A.4, we know that $r \mapsto \frac{f_r(\log(z/x))}{f_r(0)}$ and $r \mapsto f_r(0) - \frac{x_F^*}{I} \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}$ are both non-increasing mappings. As such, one can treat r as a random variable with distribution function $F(\mathrm{d}r)$, and then the two random variables $\frac{f_r(\log(z/x))}{f_r(0)}$ and $f_r(0) - \frac{x_F^*}{I} \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}$ are comonotonic. This implies their covariance is nonnegative, i.e.,

$$\begin{aligned}
&\int_0^\infty \frac{f_r(\log(z/x))}{f_r(0)} \left(f_r(0) - \frac{x_F^*}{I} \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \right) F(\mathrm{d}r) \\
&\le \left(\int_0^\infty \frac{f_r(\log(z/x))}{f_r(0)} F(\mathrm{d}r) \right) \left(\int_0^\infty \left(f_r(0) - \frac{x_F^*}{I} \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \right) F(\mathrm{d}r) \right).
\end{aligned}$$

It follows that, for any $x < z \le x_F^*$,

$$V_z'(x) \le 1 - \frac{I}{x}\left(\int_0^\infty \frac{f_r(\log(z/x))}{f_r(0)}F(\mathrm{d}r)\right)\left(\int_0^\infty \left(f_r(0) - \frac{x_F^*}{I}\frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}\right)F(\mathrm{d}r)\right) = 1,$$

where the last equality follows from the expression of x_F^*.

v) From part iii) and the fact that $V(x) = x - I$ for all $x \ge x_F^*$, we immediately have the differentiability of $V(x)$ at the point $x = x_F^*$ with $V'(x_F^*) = 1$. Moreover, by letting $z = x_F^*$ in part iv), one obtains that $V'(x) \le 1$ for all $x < x_F^*$ and $V'(x) = 1$ for all $x \ge x_F^*$. This implies that $V(x) - (x - I)$ is a non-increasing function in $x > 0$ and thus, for any $x < x_F^*$,

$$V(x) - (x - I) \ge V(x_F^*) - (x_F^* - I) = 0.$$

We then conclude that $V(x) \ge x - I$ for all $x > 0$, as $V(x) = x - I$ for $x \ge x_F^*$. □

Since we have shown $V(x) \ge x - I$ for all $x > 0$ in Lemma A.6, it remains to show that for any $x > 0$, the process $V(X_s^{0,x}) - \int_0^s H(0, X_v^{0,x})\mathrm{d}v$ for $s \ge 0$ is a supermartingale.

For any $z > 0$, we define the function

$$H_z(s,x) := \int_0^\infty re^{-rs}V_z^{(r)}(x)F(\mathrm{d}r) = \int_0^\infty \mathbb{E}[re^{-r(\sigma_{x,z}^+ + s)}(X_{\sigma_{x,z}^+}^{0,x} - I)]F(\mathrm{d}r),$$

with $s \ge 0$ and $x > 0$. From equation (41), one can express H_z as

$$H_z(s,x) = \int_0^\infty re^{-rs}\left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}\int_{\log(z/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(z/x)}^\infty f_r(y)\mathrm{d}y\right)F(\mathrm{d}r).$$

Note that for any $s \ge 0$ and $x, z > 0$, by (43),

$$-\overline{r}I \le H_z(s,x) \le \overline{r}V_z^{(r_0)}(x) < \overline{r}x. \tag{45}$$

Then we can define the stochastic process $\{Z_t^{(z)}, t \ge 0\}$, where

$$Z_t^{(z)} = V_z(X_t^{0,x}) - \int_0^t H_z(0, X_s^{0,x})\mathrm{d}s.$$

Note that we suppress the variable $x > 0$ in the above definition for ease of notation.

Lemma A.7. *For any $x, z > 0$, the stochastic process $\{Z_{t\wedge\sigma_{x,z}^+}^{(z)}, t \ge 0\}$ is a martingale.*

Proof. The integrability of $Z_t^{(z)}$ follows from equations (43) and (45) and the fact that

$$\mathbb{E}[X_t^{0,x}] = \mathbb{E}[xe^{Y_t}] < \infty,$$

where the last inequality is due to $\psi(1) < \infty$. For any $t > 0$,

$$\mathbb{E}\left[\int_0^{\sigma_{x,z}^+} H_z(0, X_s^{0,x})\mathrm{d}s\middle|\mathcal{F}_t\right] = \int_0^{t\wedge\sigma_{x,z}^+} H_z(0, X_s^{0,x})\mathrm{d}s + \mathbb{I}_{\{\sigma_{x,z}^+ > t\}}\mathbb{E}\left[\int_t^{\sigma_{x,z}^+} H_z(0, X_s^{0,x})\mathrm{d}s\middle|\mathcal{F}_t\right].$$

Due to the Markov property and the definition of H_z, one can simplify the second term on the right-hand side of the above equality as

$$\begin{aligned}\mathbb{E}\left[\int_t^{\sigma^+_{x,z}} H_z(0,X_s^{0,x})\mathrm{d}s\middle|\mathcal{F}_t\right] &= \int_t^\infty\int_0^\infty \mathbb{E}\left[1_{\{s<\sigma^+_{x,z}\}}\mathbb{E}\left[re^{-r\sigma^+_{x,z}}G(X^{0,X_s^{0,x}}_{\sigma^+_{x,z}})\right]\middle|\mathcal{F}_t\right]F(\mathrm{d}r)\mathrm{d}s\\ &= \int_t^\infty\int_0^\infty \mathbb{E}\left[\mathbb{E}\left[1_{\{s<\sigma^+_{x,z}\}}re^{-r(\sigma^+_{x,z}-s)}G(X^{0,x}_{\sigma^+_{x,z}})\middle|\mathcal{F}_s\right]\middle|\mathcal{F}_t\right]F(\mathrm{d}r)\mathrm{d}s\\ &= \mathbb{E}\left[\int_0^\infty (1-e^{-r(\sigma^+_{x,z}-t)})G(X^{0,x}_{\sigma^+_{x,z}})F(\mathrm{d}r)\middle|\mathcal{F}_t\right]\\ &= \mathbb{E}[(1-h(\sigma^+_{x,z}-t))G(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t]\\ &= \mathbb{E}[G(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t]-V_z(X_t^{0,x}),\end{aligned}$$

where we recall $G(x)=x-I$. It follows that

$$\mathbb{E}\left[\int_0^{\sigma^+_{x,z}} H_z(0,X_s^{0,x})\mathrm{d}s\middle|\mathcal{F}_t\right] = \int_0^{t\wedge\sigma^+_{x,z}} H_z(0,X_s^{0,x})\mathrm{d}s + \mathbb{I}_{\{\sigma^+_{x,z}>t\}}\left(\mathbb{E}[G(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t]-V_z(X_t^{0,x})\right).$$

On the other hand, since $V_z(x)=G(x)$, for all $x\geq z$, and $X^{0,x}_{\sigma^+_{x,z}}\geq x$, we have

$$\begin{aligned}\mathbb{E}[V_z(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t] &= \mathbb{I}_{\{\sigma^+_{x,z}>t\}}\mathbb{E}[V_z(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t] + V_z(X^{0,x}_{\sigma^+_{x,z}})\mathbb{I}_{\{t>\sigma^+_{x,z}\}}\\ &= \mathbb{I}_{\{\sigma^+_{x,z}>t\}}\mathbb{E}[G(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t] + V_z(X^{0,x}_{\sigma^+_{x,z}})\mathbb{I}_{\{t>\sigma^+_{x,z}\}}\\ &= \mathbb{I}_{\{\sigma^+_{x,z}>t\}}\left(\mathbb{E}[G(X^{0,x}_{\sigma^+_{x,z}})|\mathcal{F}_t]-V_z(X_t^{0,x})\right)+V_z(X^{0,x}_{t\wedge\sigma^+_{x,z}}).\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[V_z(X^{0,x}_{\sigma^+_{x,z}})-\int_0^{\sigma^+_{x,z}} H_z(0,X_s^{0,x})\mathrm{d}s\middle|\mathcal{F}_t\right] = V_z(X^{0,x}_{t\wedge\sigma^+_{x,z}})-\int_0^{t\wedge\sigma^+_{x,z}} H_z(0,X_s^{0,x})\mathrm{d}s,$$

or equivalently, $\mathbb{E}[Z^{(z)}_{\sigma^+_{x,z}}|\mathcal{F}_t]=Z^{(z)}_{t\wedge\sigma^+_{x,z}}$, which implies $\{Z^{(z)}_{t\wedge\sigma^+_{x,z}},t\geq 0\}$ is a martingale. □

Note that in the following Lemmas A.8 – A.10, we impose an additional assumption that the weighting distribution F has bounded support. It will be removed at the end of this subsection by Lemma A.11.

Lemma A.8. *Under the conditions of Theorem 5.3 and further assuming that F has bounded support, we have that for each $z>0$, the function $V_z\in C^2(\mathbb{R}^+\setminus\{z\})$ with $V_z''(z-)<\infty$. In particular,*

$$V''(x_F^*-)\geq 0, \tag{46}$$

Proof. Recall from (42) that, for $0<x\leq z$,

$$V_z(x)=\int_0^\infty\left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathrm{e}_r}}]}\int_{\log(z/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(z/x)}^\infty f_r(y)\mathrm{d}y\right)F(\mathrm{d}r).$$

By the dominated convergence theorem, we have that for any $x < z$,

$$V_z''(x) = \int_0^\infty \left[\frac{z}{x^2 \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} f_r(\log(z/x)) - \left(\frac{z}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} - I \right) \frac{f_r'(\log(z/x)) + f_r(\log(z/x))}{x^2} \right] F(\mathrm{d}r)$$

$$= \frac{I}{x^2} \int_0^\infty \left[f_r(\log(z/x)) + f_r'(\log(z/x)) - \frac{z}{I \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} f_r'(\log(z/x)) \right] F(\mathrm{d}r),$$

where all the integrals above are finite since we are assuming that F has bounded support and since f_r' exists and is continuous as a function of r (see Lemma A.2). This together with $V_z(x) = x - I$ for all $x \geq z$ as well as $f_r \in C^1$, shown in Lemma A.2, yields that $V_z \in C^2(\mathbb{R}^+ \setminus \{z\})$. Moreover, by the assumption that $|f_r'(0+)| < \infty$,

$$V''(z-) = \lim_{x \uparrow z} V_z''(x) = \frac{I}{z^2} \int_0^\infty \left[f_r(0+) + f_r'(0+) - \frac{z}{I \mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} f_r'(0+) \right] F(\mathrm{d}r) < \infty.$$

When $z = x_F^*$, by part v) of Lemma A.6, we know that $V'(x) \leq 1$ for all $x < x_F^*$ and $V'(x_F^*) = 1$. As such,

$$V''(x_F^*-) \geq 0.$$

□

Recall that the infinitesimal generator of the Lévy process Y is given by

$$\mathcal{A}_Y f(y) = \mu f'(y) + \frac{1}{2}\sigma^2 f''(y) + \int_{\mathbb{R}} [f(y+u) - f(y) - f'(y) u \mathbb{I}_{\{|u|<1\}}] \Pi(\mathrm{d}u),$$

where $f \in C^2(\mathbb{R})$. By Itô's formula, it is seen that the infinitesimal generator of X is given by

$$\mathcal{A}_X f(x) = (\mu + \frac{1}{2}\sigma^2) x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) + \int_{\mathbb{R}} [f(xe^u) - f(x) - x f'(x) u \mathbb{I}_{\{|u|<1\}}] \Pi(\mathrm{d}u),$$

for any $f \in C^2(\mathbb{R}^+)$.

Since we have shown that $V_z \in C^2(\mathbb{R}^+ \setminus \{z\})$ under the assumption of Lemma A.8, the next lemma implies that $\mathcal{A}_X V_z(x)$ is well defined for all $x \in \mathbb{R}^+ \setminus \{z\}$.

Lemma A.9. *Under the assumptions of Theorem 5.3 and further assuming that F has bounded support, we have that for any $z > 0$ and $x \in \mathbb{R}^+ \setminus \{z\}$,*

$$\int_{\mathbb{R}} \left| V_z(xe^u) - V_z(x) - x V_z'(x) u \mathbb{I}_{\{|u|<1\}} \right| \Pi(\mathrm{d}u) < \infty.$$

Moreover, the above holds for all $x > 0$ if $z = x_F^*$.

Proof. We first consider the case that Y is of finite variation, which implies that

$$\int_{(-1,1)} |u| \Pi(\mathrm{d}u) < \infty. \tag{47}$$

In this case we do not need the assumption that F has bounded support. Since $|V_z'(x)| < \infty$ for all $x \in \mathbb{R}^+ \setminus \{z\}$, (see (44)) and $V_z(x) = x - I$ for $x > z$, it suffices to show that

$$\int_{\mathbb{R}} |V_z(xe^u) - V_z(x)| \, \Pi(\mathrm{d}u) < \infty.$$

Assume that $x > z$, and thus $\log(z/x) < 0$. By $V_z(x) = x - I$ for $x > z$, we have

$$\begin{aligned}\int_{\mathbb{R}} |V_z(xe^u) - V_z(x)|\,\Pi(\mathrm{d}u) &= \int_{[\log(z/x),\infty)} |xe^u - x|\,\Pi(\mathrm{d}u) + \int_{(-\infty,\log(z/x))} |V_z(xe^u) - x + I|\Pi(\mathrm{d}u) \\ &\le x\int_{\mathbb{R}} |e^u - 1|\Pi(\mathrm{d}u) + \left(\sup_{0<y<z} V_z(y) + x + I\right)\Pi(-\infty, \log(z/x)) \\ &< \infty,\end{aligned}$$

where in the last inequality we used the fact V_z is uniformly bounded in $(0, z]$, as it is continuous with $V_z(0+) = 0$ and $V_z(z) = z - I$, Π is a finite measure on sets away from zero, and that

$$\int_{\mathbb{R}} |e^u - 1|\Pi(\mathrm{d}u) \le \int_{|u|<1} e|u|\Pi(\mathrm{d}u) + \int_{|u|\ge 1} (e^u + 1)\Pi(\mathrm{d}u) < \infty.$$

Note that the last step is due to (47) and $\int_{|u|\ge 1} e^u\Pi(\mathrm{d}u) < \infty$, where the latter is by $\psi(1) < \infty$ and Theorem 3.6 of Kyprianou (2014). Similarly, when $x < z$, we have that

$$\begin{aligned}&\int_{\mathbb{R}} |V_z(xe^u) - V_z(x)|\,\Pi(\mathrm{d}u) \\ &= \int_{[\log(z/x),\infty)} |xe^u - I - V_z(x)|\,\Pi(\mathrm{d}u) + \int_{(-\infty,\log(z/x))} |V_z(xe^u) - V_z(x)|\,\Pi(\mathrm{d}u) \\ &\le x\int_{[\log(z/x),\infty)} e^u\Pi(\mathrm{d}u) + (I + V_z(x))\Pi([\log(z/x), \infty)) + \sup_{0<y<z} |V_z'(y)|x\int_{(-\infty,\log(z/x))} |e^u - 1|\Pi(\mathrm{d}u) \\ &< \infty,\end{aligned}$$

In particular, when $x = z = x_F^*$, we have

$$\begin{aligned}\int_{\mathbb{R}} |V(x_F^*e^u) - V(x_F^*)|\,\Pi(\mathrm{d}u) &= x_F^*\int_{[0,\infty)} |e^u - 1|\,\Pi(\mathrm{d}u) + \int_{(-\infty,0)} |V(x_F^*e^u) - V(x_F^*)|\,\Pi(\mathrm{d}u) \\ &\le x_F^*\int_{[0,\infty)} |e^u - 1|\,\Pi(\mathrm{d}u) + \sup_{0<y<z} |V'(y)|x_F^*\int_{(-\infty,0)} |e^u - 1|\Pi(\mathrm{d}u) \\ &< \infty.\end{aligned}$$

Next we consider the case that Y is of infinite variation. First, we assume that $x > z$. Then

$$\begin{aligned}&\int_{\mathbb{R}} \left|V_z(xe^u) - V_z(x) - xV_z'(x)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u) \\ &\quad = \int_{(-\infty,\log(z/x))} \left|V_z(xe^u) - V_z(x) - xV_z'(x)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u) \\ &\qquad + \int_{[\log(z/x),\infty)} \left|xe^u - x - xV_z'(x)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u) \\ &\quad \le \left[\sup_{0<y<z} |V_z(y) - V_z(x)| + x\right]\Pi((-\infty, \log(z/x)) + x\int_{[\log(z/x),\infty)} |e^u - 1 - u\mathbb{I}_{\{|u|<1\}}|\Pi(\mathrm{d}u) \\ &\quad < \infty,\end{aligned}$$

where in the last step we used that $V_z'(x) = 1$ since $x > z$ and the fact that

$$\int_{\mathbb{R}} |e^u - 1 - u\mathbb{I}_{\{|u|<1\}}|\Pi(\mathrm{d}u) \le \int_{|u|<1} eu^2\Pi(\mathrm{d}u) + \int_{|u|\ge 1} (e^u + 1)\Pi(\mathrm{d}u) < \infty, \tag{48}$$

by the integrablity condition of the Lévy measure that $\int_{\mathbb{R}}(1\wedge u^2)\Pi(\mathrm{d}u)<\infty$. We then assume $x<z$, and then we have

$$\begin{aligned}
&\int_{\mathbb{R}}\left|V_z(xe^u)-V_z(x)-xV_z'(x)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&=\int_{(-\infty,\log(z/x))}\left|V_z(xe^u)-V_z(x)-xV_z'(x)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&\quad+\int_{[\log(z/x),\infty)}\left|xe^u-I-V_z(x)-xV_z'(x)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&\le\int_{(-\infty,\log(z/x))\cap(-1,1)}\left|V_z(xe^u)-V_z(x)-xV_z'(x)u\right|\Pi(\mathrm{d}u)\\
&\quad+\int_{(-\infty,\log(z/x))\setminus(-1,1)}\left|V_z(xe^u)-V_z(x)\right|\Pi(\mathrm{d}u)\\
&\quad+x\int_{[\log(z/x),\infty)}e^u\Pi(\mathrm{d}u)+(I+V_z(x)+xV_z'(x))\Pi([\log(z/x),\infty))\\
&\le x^2\sup_{xe^{-1}<y<z}|V_z''(y)|\int_{(-\infty,\log(z/x))\cap(-1,1)}(e^u-1)^2\Pi(\mathrm{d}u)\\
&\quad+\sup_{0<y<z}|V_z(y)-V_z(x)|\,\Pi((-\infty,\log(z/x))\setminus(-1,1))\\
&\quad+x\int_{[\log(z/x),\infty)}e^u\Pi(\mathrm{d}u)+(I+V_z(x)+xV_z'(x))\Pi([\log(z/x),\infty))\\
&<\infty,
\end{aligned}$$

where in the last inequality we used that the finiteness of V_z'' (see Lemma A.8), and $(e^u-1)^2\le e^2u^2$ for $|u|<1$. In particular, when $x=z=x_F^*$, by $V'(x_F^*)=1$,

$$\begin{aligned}
&\int_{\mathbb{R}}\left|V(x_F^*e^u)-V(x_F^*)-x_F^*V'(x_F^*)u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&=\int_{(-\infty,-1)}|V(x_F^*e^u)-V(x_F^*)|\,\Pi(\mathrm{d}u)+\int_{(-1,0)}\left|V(x_F^*e^u)-V(x_F^*)-x_F^*V'(x_F^*)u\right|\Pi(\mathrm{d}u)\\
&\quad+x_F^*\int_{[0,\infty)}\left|e^u-1-u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&\le\Pi((-\infty,-1))\sup_{0<y<x_F^*}|V(y)-V(x_F^*)|+\int_{(-1,0)}\left|V(x_F^*e^u)-V(x_F^*)-x_F^*V'(x_F^*)(e^u-1)\right|\Pi(\mathrm{d}u)\\
&\quad+x_F^*V'(x_F^*)\int_{(-1,0)}|e^u-1-u|\,\Pi(\mathrm{d}u)+x_F^*\int_{[0,\infty)}\left|e^u-1-u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&\le\Pi((-\infty,-1))\sup_{0<y<x_F^*}|V(y)-V(x_F^*)|+(x_F^*)^2\sup_{x_F^*e^{-1}<y<x_F^*}|V''(y)|\int_{(-1,0)}(e^u-1)^2\Pi(\mathrm{d}u)\\
&\quad+x_F^*\int_{[-1,\infty)}\left|e^u-1-u\mathbb{I}_{\{|u|<1\}}\right|\Pi(\mathrm{d}u)\\
&<\infty,
\end{aligned}$$

where the last step is due to the finiteness of V'' (see Lemma A.8), $(e^u-1)^2\le u^2$ for $u\in(-1,0)$, and (48). □

The next lemma shows that the value function V satisfies the standard variation inequality. Recall from Lemma A.6 that V is a $C^1(0,\infty)$. However, V is not twice differentiable at the point $x = x_F^*$ and hence the infinitesimal generator $\mathcal{A}_X V(x)$ does not exist at $x = x_F^*$.

Lemma A.10. *Under the assumptions of Theorem 5.3 and further assuming that F has bounded support, we have that*

$$\mathcal{A}_X V(x) - H(0,x) \le 0,$$

for any $x \in \mathbb{R}^+ \setminus \{x_F^*\}$.

Proof. First, for any $0 < x < x_F^*$, we deduce from Lemma A.7 that

$$\mathcal{A}_X V(x) - H(0,x) = 0,$$

which is equivalent to

$$(\mu + \frac{1}{2}\sigma^2)xV'(x) + \int_{\mathbb{R}} [V(xe^u) - V(x) - xV'(x)u\mathbb{I}_{\{|u|<1\}}]\Pi(\mathrm{d}u) - H(0,x) = -\frac{1}{2}\sigma^2 x^2 V''(x).$$

Moreover, by letting $x \uparrow x_F^*$ and using $V \in C^2(\mathbb{R}^+ \setminus \{x_F^*\})$ and $V \in C^1(\mathbb{R}^+)$ with $V'(x_F^*) = 1$, one obtains

$$(\mu + \frac{1}{2}\sigma^2)x_F^* + \int_{\mathbb{R}} [V(x_F^* e^u) - (x_F^* - I) - x_F^* u\mathbb{I}_{\{|u|<1\}}]\Pi(\mathrm{d}u) - H(0,x_F^*) = -\frac{1}{2}\sigma^2 x^2 V''(x_F^*-) \le 0, \tag{49}$$

where the last inequality follows from (46).

We then consider the case $x > x_F^*$. Since $V(x) = x - I$ and $H(0,x) = \overline{r}(x - I)$ for $x > x_F^*$, we have that

$$\mathcal{A}_X V(x) - H(0,x) = (\mu + \frac{1}{2}\sigma^2)x + \int_{\mathbb{R}} [V(xe^u) - (x - I) - xu\mathbb{I}_{\{|u|<1\}}]\Pi(\mathrm{d}u) - \overline{r}(x - I).$$

For any $x > x_F^*$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[\mathcal{A}_X V(x) - H(0,x)] &= \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} \left[V'(xe^u)e^u - 1 - u\mathbb{I}_{\{|u|<1\}}\right] \Pi(\mathrm{d}u) - \overline{r} \\
&\le \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} \left[e^u - 1 - u\mathbb{I}_{\{|u|<1\}}\right] \Pi(\mathrm{d}u) - \overline{r} \\
&= \psi(1) - \overline{r} \\
&< 0,
\end{aligned}$$

where in the first inequality we used that $V'(x) \le 1$ for all $x > 0$ (see part v) of Lemma A.6) and the last inequality is by $\psi(1) < r_0 \le \overline{r}$. Hence, we deduce that the function $x \mapsto \mathcal{A}_X V(x) - H(0,x)$ is non-increasing on (x_F^*, ∞). That means that for any $x_F^* < y \le x$,

$$\mathcal{A}_X V(x) - H(0,x) \le \mathcal{A}_X V(y) - H(0,y).$$

By taking $y \downarrow x_F^*$,

$$\begin{aligned}
\mathcal{A}_X V(x) - H(0,x) &\le \lim_{y \downarrow x_F^*} [\mathcal{A}_X V(y) - H(0,y)] \\
&= (\mu + \frac{1}{2}\sigma^2)x_F^* + \int_{\mathbb{R}} [V(x_F^* e^u) - (x_F^* - I) - x_F^* u\mathbb{I}_{\{u>-1\}}]\Pi(\mathrm{d}u) - H(0,x_F^*) \\
&\le 0,
\end{aligned}$$

where the last step is due to (49). □

The next lemma is devoted to prove that the process

$$Z_t := Z_t^{(x_F^*)} = V(X_t^{0,x}) - \int_0^t H(0, X_s^{0,x}) \mathrm{d}s$$

is supermartingale, and this will complete the proof of Theorem of 5.3.

Lemma A.11. *Under the assumptions of Theorem 5.3, we have that for each $X_0 = x > 0$, the process $\{Z_t, t \geq 0\}$ is a supermartingale.*

Proof. First we assume that F has bounded support. From Lemmas A.6 and A.8, we have that $V' \leq 1$ on $(0, \infty)$ and is twice differentiable on $(0, \infty) \setminus \{x_F^*\}$. For any $K > 0$, recall that

$$\sigma_{x,K}^+ = \inf\{t \geq 0 : X_t^{0,x} > K\}.$$

By applying Itô's formula (see, e.g., Theorem 71 of Protter (2005)), we obtain that, for any $x > 0$,

$$\begin{aligned}
V(X_{t\wedge\sigma_{x,K}^+}^{0,x}) &= V(xe^{Y_{t\wedge\sigma_{x,K}^+}}) \\
&= V(x) + \int_0^{t\wedge\sigma_{x,K}^+} V'(xe^{Y_{s-}})xe^{Y_{s-}} \mathrm{d}Y_s + \frac{1}{2}\sigma^2 \int_0^{t\wedge\sigma_{x,K}^+} [V''(xe^{Y_{s-}})(xe^{Y_{s-}})^2 + V'(xe^{Y_{s-}})xe^{Y_{s-}}] \mathrm{d}s \\
&\quad + \int_0^{t\wedge\sigma_{x,K}^+} \int_{\mathbb{R}} [V(xe^{Y_{s-}+u}) - V(xe^{Y_{s-}}) - uxe^{Y_{s-}}V'(xe^{Y_{s-}})] N(\mathrm{d}u, \mathrm{d}s) \\
&= V(x) + \int_0^{t\wedge\sigma_{x,K}^+} V'(X_{s-}^{0,x})X_{s-}^{0,x} \mathrm{d}Y_s + \frac{1}{2}\sigma^2 \int_0^{t\wedge\sigma_{x,K}^+} [V''(X_{s-}^{0,x})(X_{s-}^{0,x})^2 + V'(X_{s-}^{0,x})X_{s-}^{0,x}] \mathrm{d}s \\
&\quad + \int_0^{t\wedge\sigma_{x,K}^+} \int_{\mathbb{R}} [V(X_{s-}^{0,x}e^u) - V(X_{s-}^{0,x}) - uX_{s-}^{0,x}V'(X_{s-}^{0,x})] N(\mathrm{d}u, \mathrm{d}s) \\
&= V(x) + M_{t\wedge\sigma_{x,K}^+} + \int_0^{t\wedge\sigma_{x,K}^+} \mathcal{A}_X V(X_{s-}^{0,x}) \mathrm{d}s,
\end{aligned}$$

where $\{M_{t\wedge\sigma_{x,K}^+}, t \geq 0\}$ is a martingale, by Lemma A.9, V' is bounded, and $0 < X_s^{0,x} \leq K$ for all $s \leq \sigma_{x,K}^+$. Hence, we have that

$$\begin{aligned}
Z_{t\wedge\sigma_{x,K}^+} &= V(X_{t\wedge\sigma_{x,K}^+}^{0,x}) + \int_0^{t\wedge\sigma_{x,K}^+} H(0, X_s^{0,x}) \mathrm{d}s \\
&= V(x) + M_{t\wedge\sigma_{x,K}^+} + \int_0^{t\wedge\sigma_{x,K}^+} \left[\mathcal{A}_X V(X_{s-}^{0,x}) + H(0, X_s^{0,x})\right] \mathrm{d}s.
\end{aligned}$$

From Lemma A.10 we know that $\mathcal{A}_X V(x) - H(0, x) \leq 0$ for all $x \in \mathbb{R}^+ \setminus \{x_F^*\}$. Thus, for each $K > 0$, the process $\{Z_{t\wedge\sigma_{x,K}^+}, t \geq 0\}$ is a supermartingale. From equations (43) and (45), we know that for each $t \geq 0$ and $K > 0$,

$$-I \leq V(X_{t\wedge\sigma_{x,K}^+}^{0,x}) \leq X_{t\wedge\sigma_{x,K}^+}^{0,x} \leq xe^{\overline{Y}_t},$$

and

$$-\overline{r}It \leq \int_0^{t\wedge\sigma_{x,K}^+} H(0, X_s^{0,x}) \mathrm{d}s \leq \overline{r} \int_0^{t\wedge\sigma_{x,K}^+} X_s^{0,x} \mathrm{d}s \leq \overline{r}txe^{\overline{Y}_t}.$$

Since for every $t \geq 0$, the random variable $e^{\overline{Y}_t}$ has finite expectation as $\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}] = \int_0^\infty re^{-rt}\mathbb{E}[e^{\overline{Y}_t}]\mathrm{d}t < \infty$, by the dominated convergence theorem, by letting $K \to \infty$, we deduce that the process $\{Z_t, t \geq 0\}$ is a supermartingale when F has bounded support.

Now assume that F is an arbitrary distribution function with support on $\mathbb{R}^+$. For $N > 0$, consider the modified distribution function

$$F_N(r) = \begin{cases} F(r), & r < N, \\ 1, & r \geq N. \end{cases}$$

By equation (20), let

$$x_N^* = \frac{\int_0^\infty f_r(0)F_N(\mathrm{d}r)}{\int_0^\infty \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}F_N(\mathrm{d}r)}I = \frac{\int_0^N f_r(0)F(\mathrm{d}r) + f_N(0)(1-F(N))}{\int_0^N \frac{f_r(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]}F(\mathrm{d}r) + \frac{f_N(0)}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_N}}]}(1-F(N))}I.$$

be the equilibrium stopping threshold corresponding to F_N. By equation (42), and for each $x > 0$,

$$\begin{aligned} V(x;N) &:= \int_0^\infty \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \int_{\log(x_N^*/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(x_N^*/x)}^\infty f_r(y)\mathrm{d}y \right) F_N(\mathrm{d}r) \\ &= \int_0^N \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \int_{\log(x_N^*/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(x_N^*/x)}^\infty f_r(y)\mathrm{d}y \right) F(\mathrm{d}r) \\ &\quad + \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_N}}]} \int_{\log(x_N^*/x)}^\infty e^y f_N(y)\mathrm{d}y - I\int_{\log(x_N^*/x)}^\infty f_N(y)\mathrm{d}y \right) (1-F(N)), \end{aligned}$$

and

$$\begin{aligned} H(s,x;N) &:= \int_0^\infty re^{-rs} \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \int_{\log(x_N^*/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(x_N^*/x)}^\infty f_r(y)\mathrm{d}y \right) F_N(\mathrm{d}r) \\ &= \int_0^N re^{-rs} \left(\frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}]} \int_{\log(x_N^*/x)}^\infty e^y f_r(y)\mathrm{d}y - I\int_{\log(x_N^*/x)}^\infty f_r(y)\mathrm{d}y \right) F(\mathrm{d}r) \\ &\quad + \left(Ne^{-Ns} \frac{x}{\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_N}}]} \int_{\log(x_N^*/x)}^\infty e^y f_N(y)\mathrm{d}y - I\int_{\log(x_N^*/x)}^\infty f_N(y)\mathrm{d}y \right) (1-F(N)). \end{aligned}$$

From the last part, since F_N has bounded support, we have that, for each $N > 0$, the process

$$V(X_t^{0,x};N) + \int_0^t H(0, X_v^{0,x};N)\mathrm{d}v$$

is a supermartingale. That is, for each $x > 0$ and $s \geq 0$,

$$\mathbb{E}\left[V(X_{t+s}^{0,x};N) + \int_0^{t+s} H(0,X_v^{0,x};N)\mathrm{d}v \middle| \mathcal{F}_t \right] \leq V(X_t^{0,x};N) + \int_0^t H(0,X_v^{0,x};N)\mathrm{d}v.$$

By the dominated convergence theorem (see (43) and (45)), we deduce that

$$\lim_{N\to\infty} V(x;N) = V(x) \qquad \text{and} \qquad \lim_{N\to\infty} H(s,x;N) = H(s,x).$$

Therefore, we conclude that the process $\{Z_t, t \geq 0\}$ is a supermartingale. □

A.5 Proof of Proposition 5.4

From equation (33), we can see that $|f_r'(0+)|$ is finite if the Lévy measure of the ascending ladder height process, $\Pi_H(y,\infty) := \Lambda((0,\infty),(y,\infty))$, is a finite measure. From Theorem 7.8 of Kyprianou (2014), we know that

$$\Pi_H(y,\infty) = \int_{[0,\infty)} \widehat{U}(\mathrm{d}z)\Pi(z+y,\infty),$$

where $\widehat{U}(\mathrm{d}z) = \mathbb{E}[\int_0^\infty \mathbb{I}_{\{\widehat{H}_t \in \mathrm{d}z\}}\mathrm{d}t]$ and $\widehat{H}$ is the descending ladder height process. Hence, by noting that $\widehat{U}([0,x]) < \infty$ for all $x > 0$ (see Proposition III.1 in Bertoin (1998)), a sufficient condition for $|f_r'(0+)| < \infty$ is that $\Pi(0,\infty) < \infty$.

A.6 Proof of Proposition 5.7

By equations (20), (32), and (33), the equilibrium stopping threshold for $H \in \{F, G\}$ can be written as

$$x_H^* = \frac{\int_0^\infty \kappa(r,0)H(\mathrm{d}r)}{\int_0^\infty \kappa(r,-1)H(\mathrm{d}r)} I = \frac{\int_0^\infty \kappa(r,0)H(\mathrm{d}r)}{\int_0^\infty (\kappa(r,-1)-\kappa(r,0))H(\mathrm{d}r) + \int_0^\infty \kappa(r,0)H(\mathrm{d}r)} I.$$

Hence, since $\kappa(r,-1) > 0$ for all $r \geq 0$, by $\mathbb{E}[e^{\overline{Y}_{\mathbf{e}_r}}] < \infty$ and Lemma A.1), the inequality $x_G^* \geq x_F^*$ is equivalent to

$$\int_0^\infty \kappa(r,0)G(\mathrm{d}r)\left[\int_0^\infty (\kappa(r,0)-\kappa(r,-1))F(\mathrm{d}r)\right] \leq \int_0^\infty \kappa(r,0)F(\mathrm{d}r)\left[\int_0^\infty (\kappa(r,0)-\kappa(r,-1))G(\mathrm{d}r)\right].$$

Since $G \preceq_{\infty SD} F$, by Definition 5.6 and (31), we have that

$$\int_0^\infty \kappa(r,0)G(\mathrm{d}r) \leq \int_0^\infty \kappa(r,0)F(\mathrm{d}r).$$

On the other hand, from equation (30),

$$\kappa(r,0) - \kappa(r,-1) = b + \int_{(0,\infty)^2} (e^y - 1)e^{-rx}\Lambda(\mathrm{d}x,\mathrm{d}y) \geq 0.$$

By Fubini's theorem and (22), we obtain

$$\begin{aligned}\int_0^\infty (\kappa(r,0)-\kappa(r,-1))F(\mathrm{d}r) &= b + \int_{(0,\infty)^2} (e^y-1)h_F(x)\Lambda(\mathrm{d}x,\mathrm{d}y) \\ &\leq b + \int_{(0,\infty)^2} (e^y-1)h_G(x)\Lambda(\mathrm{d}x,\mathrm{d}y) \\ &= \int_0^\infty (\kappa(r,0)-\kappa(r,-1))G(\mathrm{d}r).\end{aligned}$$

This completes the proof.

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