Supplemental Materials for "Semiparametric Mean-Covariance Regression Analysis for Longitudinal Data"

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We address two issues in this supplemental file: 1. robustness of the semiparametric mean-covaraince model when errors deviate from the normality assumption; 2. additional proofs of Lemma 2 and Theorem 2.

1 Robustness

This section is used to assess the robustness of the proposed method if the error deviates from the normality assumption. To this end, we follow the simulation setup in Ye and Pan (2006) by using a normal mixture model and the choice of $\mathbf{W}_i$ as in the paper.

Suppose that observations for subject i are sampled from

$$\pi N_{n_i}(\boldsymbol{\mu}_i(1+\tau), \boldsymbol{\Sigma}_i) + (1-\pi) N_{n_i}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

where $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are formed using the parameters specified in Case 1 in Study 1, π is the mixing weight and τ is the mean-shift parameter. In order to see how the proposed approach behaves under different levels of mixtures, we choose $\pi = 0.25, 0.5$ and $\tau = 1/10, 1/5, 1/3$. Below we give the performance of the proposed estimator in these six different combinations of mixture, each with 1000 replications.

For the Normal mixture distribution, the expectation and variance are $\tilde{\boldsymbol{\mu}}_i = \boldsymbol{\mu}_i(1+\pi\tau)$, and $\tilde{\boldsymbol{\Sigma}}_i = \boldsymbol{\Sigma}_i + \pi(1-\pi)\tau^2\boldsymbol{\mu}_i\boldsymbol{\mu}_i'$. Because we have a mean shift in a mixture model, it is more appropriate to compare the estimated $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}_i$ to $\tilde{\boldsymbol{\beta}} = (1+\pi\tau)\boldsymbol{\beta}$ and $\tilde{\boldsymbol{\Sigma}}_i$. We define the relative errors by $\text{err}(\hat{\boldsymbol{\beta}})=\|\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}}\|/\|\tilde{\boldsymbol{\beta}}\|$ and $\text{err}(\hat{\boldsymbol{\Sigma}}_i)=\|\hat{\boldsymbol{\Sigma}}_i - \tilde{\boldsymbol{\Sigma}}_i\|/\|\tilde{\boldsymbol{\Sigma}}_i\|$, where $\|\cdot\|$ denotes the Euclidean norm. Table 1 gives the averages of relative errors for each combination of π and τ, showing that the resulting estimators are close to the true values in all cases.

Table 1: Simulation study. Average of relative errors $\text{err}(\hat{\boldsymbol{\beta}})$ and $\text{err}(\hat{\boldsymbol{\Sigma}})=\sum_{i=1}^{m}\text{err}(\hat{\boldsymbol{\Sigma}}_i)/m$ for 1000 random samples from the Normal Mixture distribution, with AR(1) structure and $\boldsymbol{\delta} = 0.2$ being specified for the covariances of ϵ_i^2.

(π, τ)	$\text{err}(\hat{\boldsymbol{\beta}})$	$\text{err}(\hat{\boldsymbol{\Sigma}})$
$(0.25, 1/10)$	0.0468	0.0737
$(0.25, 1/5)$	0.0583	0.0756
$(0.25, 1/3)$	0.0846	0.0846
$(0.50, 1/10)$	0.0638	0.0740
$(0.50, 1/5)$	0.0532	0.0773
$(0.50, 1/3)$	0.0879	0.0915

We conclude that in mild violation of the model, the proposed choice of $\mathbf{W}_i$ works satisfactorily.

2 Additional Proofs

Proof of Lemma 2. First, we show that $\|\hat{\boldsymbol{\zeta}}_1 - \tilde{\boldsymbol{\zeta}}_1\| = o_p(1)$. Similar to the arguments of the proof of Theorem 1, we have that

$$\sup_{\|\boldsymbol{\zeta}_1\| \le L, \|\boldsymbol{\zeta}_2\| \le k_n^{1/2}} \|\boldsymbol{\Psi}_1(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) - \boldsymbol{\Phi}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)\| = o_p(1), \quad \|\hat{\boldsymbol{\zeta}}_1\| = O_p(1).$$

Thus, $\|\tilde{\boldsymbol{\zeta}}_1 - \hat{\boldsymbol{\zeta}}_1\| = o_p(1)$. Similarly, $\|\hat{\boldsymbol{\xi}}_1 - \tilde{\boldsymbol{\xi}}_1\| = o_p(1)$. To prove $\|\sqrt{m}(\hat{\boldsymbol{\gamma}}_m - \boldsymbol{\gamma}_0) - \tilde{\boldsymbol{\gamma}}\| = o_p(1)$, it suffices to prove this claim when all $\mathbf{D}_i$ are known to be $\mathbf{D}_{0i}$. Let $\tilde{\mathbf{B}}_m = \sum_{i=1}^m \mathbf{V}_i' \mathbf{D}_{0i}^{-1} \mathbf{V}_i / m$. By (4) in the main paper we have

$$\sqrt{m}(\hat{\boldsymbol{\gamma}}_m - \boldsymbol{\gamma}_0) = \tilde{\mathbf{B}}_m^{-1} \frac{1}{\sqrt{m}} \sum_{i=1}^m \mathbf{V}_i' \mathbf{D}_{0i}^{-1} (\mathbf{r}_i - \hat{\mathbf{r}}_i).$$

Obviously, $\tilde{\mathbf{B}}_m$ can be written as follows by Kronecker product,

$$\begin{aligned}
\tilde{\mathbf{B}}_m &= \frac{1}{m} \sum_{i=1}^m (\mathbf{r}_{0i}' \otimes \mathbf{I}_q + (\boldsymbol{\nabla\mu}_i)' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\mathbf{r}_{0i} \otimes \mathbf{I}_q + \boldsymbol{\nabla\mu}_i \otimes \mathbf{I}_q) \\
&= \frac{1}{m} \sum_{i=1}^m (\mathbf{r}_{0i}' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\mathbf{r}_{0i} \otimes \mathbf{I}_q) + \frac{1}{m} \sum_{i=1}^m (\mathbf{r}_{0i}' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\boldsymbol{\nabla\mu}_i \otimes \mathbf{I}_q) \\
&+ \frac{1}{m} \sum_{i=1}^m (\boldsymbol{\nabla\mu}_i)' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\mathbf{r}_{0i} \otimes \mathbf{I}_q) + \frac{1}{m} \sum_{i=1}^m ((\boldsymbol{\nabla\mu}_i)' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\boldsymbol{\nabla\mu}_i \otimes \mathbf{I}_q) \\
&=: \frac{1}{m} \sum_{i=1}^m (\mathbf{r}_{0i}' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\mathbf{r}_{0i} \otimes \mathbf{I}_q) + \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3,
\end{aligned}$$

where $\boldsymbol{\Omega}_i$ is a lower triangular matrix with 0's on its diagonal and the j row is $(\omega_{ij1}', \cdots, \omega_{ij(j-1)}', 0, \cdots, 0)$. Since for any $\mathbf{a} \in R^q$ satisfying $\|\mathbf{a}\| = 1$, by the proof of Theorem 1, we have

$$\begin{aligned}
&(E\mathbf{a}'\mathbf{J}_1\mathbf{a})^2 = \left\{ \frac{1}{m} E \sum_{i=1}^m \mathbf{a}' (\mathbf{r}_{0i}' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\boldsymbol{\nabla\mu}_i \otimes \mathbf{I}_q) a \right\}^2 \\
&= \frac{1}{m} \sum_{i=1}^m E\mathbf{a}' (\mathbf{r}_{0i}' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\mathbf{r}_{0i} \otimes \mathbf{I}_q) \mathbf{a} \cdot \frac{1}{m} \sum_{i=1}^m E\mathbf{a}' ((\boldsymbol{\nabla\mu}_i)' \otimes \mathbf{I}_q) \boldsymbol{\Omega}_i' \mathbf{D}_{0i}^{-1} \boldsymbol{\Omega}_i (\boldsymbol{\nabla\mu}_i \otimes \mathbf{I}_q) \mathbf{a} \\
&\le C \frac{1}{m} \sum_{i=1}^m \|\boldsymbol{\nabla\mu}_i\|^2 = o(n^{-(2s-1)/(2s+1)}).
\end{aligned}$$

That is $\mathbf{J}_1 \to 0$ in probability. Similarly we can show that $\mathbf{J}_2 \to 0$ and that $\mathbf{J}_3 \to 0$ in probability. Thus $\mathbf{B}_m - \frac{1}{m}\sum_{i=1}^m \mathbf{V}_i^{0\prime}\mathbf{D}_{0i}^{-1}\mathbf{V}_i^0 \to 0$ in probability. Similarly, we can prove that $\frac{1}{\sqrt{m}}\sum_{i=1}^m \mathbf{V}_i'\mathbf{D}_{0i}^{-1}(\mathbf{r}_i - \hat{\mathbf{r}}_i) - \frac{1}{\sqrt{m}}\tilde{\mathbf{S}}_2 \to 0$ in probability. The proof is completed by an application of Slutsky theorem.

Proof of Theorem 2. By Lemma 2, we only need to show the asymptotic normality of $(\tilde{\boldsymbol{\xi}}_1', \tilde{\gamma}', \tilde{\zeta}_1')'/\sqrt{m}$. This is equivalent to the asymptotic normality of $(\tilde{\mathbf{S}}_1', \tilde{\mathbf{S}}_2', \tilde{\mathbf{S}}_3')'/\sqrt{m}$. Note that Conditions (A1), (A3), (A4) and (A6) imply that

$$E_0\left[\boldsymbol{\psi}'\left\{\mathbf{X}_i^{*\prime}\boldsymbol{\Delta}_{0i}\boldsymbol{\Sigma}_{0i}^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_{0i})\right\} + \boldsymbol{\omega}'\left\{\mathbf{V}_{0i}\mathbf{D}_{0i}^{-1}(\mathbf{r}_{0i} - \hat{\mathbf{r}}_{0i})\right\} + \boldsymbol{\varphi}'\left(\mathbf{Z}_i^{*\prime}\mathbf{D}_{0i}\mathbf{W}_{0i}^{-1}(\boldsymbol{\epsilon}_{0i}^2 - \boldsymbol{\sigma}_{0i}^2)\right\}\right]^3 < \kappa,$$

for any $\boldsymbol{\psi} \in \mathbb{R}^{p+K}$, $\boldsymbol{\omega} \in \mathbb{R}^q$ and $\boldsymbol{\varphi} \in \mathbb{R}^{d+K'}$, where κ is a constant independent of i. Furthermore, we have

$$\begin{aligned}\tfrac{1}{m}\textstyle\sum_{i=1}^m V\left[\boldsymbol{\psi}'\left\{\mathbf{X}_i^{*\prime}\boldsymbol{\Delta}_{0i}\Sigma_{0i}^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_{0i})\right\} + \boldsymbol{\omega}'\left\{\mathbf{V}_{0i}\mathbf{D}_{0i}^{-1}(\mathbf{r}_{0i} - \hat{\mathbf{r}}_{0i})\right\} + \boldsymbol{\varphi}'\left(\mathbf{Z}_i^{*\prime}\mathbf{D}_{0i}\mathbf{W}_{0i}^{-1}(\boldsymbol{\epsilon}_{0i}^2 - \boldsymbol{\sigma}_{0i}^2)\right\}\right] \\ = (\boldsymbol{\psi}', \boldsymbol{\omega}', \boldsymbol{\varphi}')\tfrac{1}{n}\boldsymbol{\Delta}_n(\boldsymbol{\psi}', \boldsymbol{\omega}', \boldsymbol{\varphi}')' \to (\boldsymbol{\psi}', \boldsymbol{\omega}', \boldsymbol{\varphi}')\boldsymbol{\Delta}(\boldsymbol{\psi}', \boldsymbol{\omega}', \boldsymbol{\varphi}')' > 0.\end{aligned}$$

Therefore the asymptotic normality of $(\tilde{\mathbf{S}}_1', \tilde{\mathbf{S}}_2', \tilde{\mathbf{S}}_3')'/\sqrt{m}$ is easily proved by multivariate Liapounov central limit theorem. Therefore,

$$\begin{aligned}\sqrt{m}\begin{pmatrix}\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}_0 \\ \hat{\boldsymbol{\gamma}}_m - \boldsymbol{\gamma}_0 \\ \hat{\boldsymbol{\lambda}}_m - \boldsymbol{\lambda}_0\end{pmatrix} &= \begin{pmatrix}(\mathbf{A}_m/m)^{-1} & 0 & 0 \\ 0 & (\mathbf{B}_m/m)^{-1} & 0 \\ 0 & 0 & (\mathbf{C}_m/m)^{-1}\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{S}}_1/\sqrt{m} \\ \tilde{\mathbf{S}}_2/\sqrt{m} \\ \tilde{\mathbf{S}}_3/\sqrt{m}\end{pmatrix} \\ \to N&\left\{\mathbf{0}, \begin{pmatrix}\boldsymbol{\delta}^{11} & 0 & 0 \\ 0 & \boldsymbol{\delta}^{22} & 0 \\ 0 & 0 & \boldsymbol{\delta}^{33}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\delta}^{11} & \boldsymbol{\delta}^{12} & \boldsymbol{\delta}^{13} \\ \boldsymbol{\delta}^{21} & \boldsymbol{\delta}^{22} & \boldsymbol{\delta}^{23} \\ \boldsymbol{\delta}^{31} & \boldsymbol{\delta}^{32} & \boldsymbol{\delta}^{33}\end{pmatrix}\begin{pmatrix}\boldsymbol{\delta}^{11} & 0 & 0 \\ 0 & \boldsymbol{\delta}^{22} & 0 \\ 0 & 0 & \boldsymbol{\delta}^{33}\end{pmatrix}^{-1}\right\}.\end{aligned}$$

in distribution as $m \to \infty$. The proof of Theorem 2 is completed.