Towards ℓ_2-stability of discrete-time reset control systems via dissipativity theory

Joaquín Carrasco[a], Eva M. Navarro-López[b]

[a] *Control Systems Centre, School of Electrical and Electronic Engineering, The University of Manchester, Sackville Street Building, Manchester M13 9PL, UK*
[b] *School of Computer Science, The University of Manchester, Oxford Road, Kilburn Building, Manchester M13 9PL, UK*

Abstract

This paper proposes conditions on input-output stability of discrete-time reset systems by using some key dissipativity properties. In the continuous-time setting, dissipativity of the base linear system is preserved under reset actions if the storage function is decreasing at reset times. Indeed, when the reset system is full reset, the dissipativity of the base linear system ensures the dissipativity of the reset system. However, in the discrete-time setting, this condition on the storage function is not enough to ensure the dissipativity of the base linear system. We define some dissipativity properties of discrete-time reset systems and give an appropriate definition of the reset system in order to preserve the $(Q, S, R)-$dissipativity of the base linear system under reset actions. As a result, ℓ_2-stability of feedback interconnected dissipative control reset systems is obtained.

Keywords: Discrete-time reset control systems, Dissipativity, ℓ_2-stability.

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1. INTRODUCTION

Reset systems were proposed more than 50 years ago [14]. Horowitz and co-authors [22, 23] were interested in the properties of reset systems due to interesting phase properties. The Clegg integrator – that is, an integrator which is set to zero whenever its input crosses zero – has a describing function

Email addresses: `joaquin.carrascogomez@manchester.ac.uk` (Joaquín Carrasco), `eva.navarro@cs.man.ac.uk` (Eva M. Navarro-López)

given by

$$CI(j\omega) = \frac{1.62}{j\omega} e^{j52^\circ}. \tag{1}$$

Hence, an improvement of about 52 degrees in phase lag can be reached compared to the linear integrator, which has a phase lag of -90 degrees [1]. Horowitz's contributions to reset control system are shown in [13]. During the last decade, several works have been devoted to reset control system and they have been summarised in [1]. Furthermore, under some conditions, reset actions make systems dissipate energy as shown in [9, 10, 34].

The first stability condition proposed for reset control systems is of Lyapunov type and is given for a family of Lyapunov functions which are decreasing at the reset times [7]. This condition is known as the H_β condition. Alternatively, Baños et al. [2] give a condition based on discrete-time Lyapunov functions which is not restricted to stable base linear systems. Moreover, Nešić et al [33] propose an alternative definition for reset systems where the system can only evolve within a subset of the state space, giving less restrictive results.

Dissipative and passive systems exhibit highly desirable properties, namely the ones related to stability and representation properties, which may simplify the system analysis and control design. Passivity theory has been applied in [12] to achieve an input-output stability result. In the same manner, [16] derives passivity conditions for the reset systems defined by Nešić et al [33]. Additionally, in [6], the H_β condition is re-interpreted as a passivity result.

Recently, the discrete version of reset control systems has been proposed [4, 5]. An intermediate version is presented in [18], where the reset actions on a linear continuous-time system are triggered by a discrete law. If continuous-time reset systems show complex behavior due to their discrete-type and continuous-time evolutions, the discrete-time counterpart may be simpler since both dynamics – system dynamics and reset actions – are of discrete type. Therefore, discrete-time reset control systems can be understood as discrete-time switching systems where the switching law depends on the inputs.

Undoubtedly, dissipativity and passivity theory has attracted less attention in the discrete-time than in the continuous-time setting. Even though input-output stability conditions can be stated in a general Hilbert space – that is, they are valid for $\mathcal{L}_2$ and ℓ_2 – dissipativity conditions in the state space realization are very different in the continuous-time and discrete-time settings; in the same way that Lyapunov-type stability conditions or Riccati equation are different. For example, passive discrete systems require a direct input-output link [11, 28, 29, 30, 31, 35]. For the case of linear systems this means that $D \neq 0$. There is a growing effort to remove this direct input-output link restriction for either the linear or the nonlinear case [27, 36]. Unlike discrete-time systems, continuous-time systems which are passive have a relative degree of one. Therefore, in general, the passivity property is not preserved under sampling.

Within the last two decades, key results on dissipative nonlinear discrete-time systems have been proposed, which include the linear case and can be divided into two main groups. First, characterisation of passivity, dissipativity, and feedback passivity and dissipativity by using zero dynamics and relative degree properties [11, 27, 28, 31, 30, 35]. Second, preservation of dissipativity and stability properties under sampling [26, 24, 25, 32].

In comparison to the continuous-time case, there are very few results of the study of dissipativity-related properties in discrete-time switched systems, we highlight the work [8]. Recently, the results of Navarro-López given in [28, 31, 30] have been further expanded to discrete-time periodically controlled systems [39] and nonlinear discrete-time switched systems [38].

The aim of the present work is to extend the results given in [12] for continuous-time reset systems to discrete-time reset systems, for a class of dissipative systems. In particular (Q, S, R)-dissipative systems are studied. We will consider dissipativity properties of a switched system with two dynamics. Afterwards, this dissipativity result will apply to reset systems. As a consequence, the ℓ_2-stability of a reset system can be achieved applying dissipative stability results to discrete reset control systems. Unlike most of the results in the reset literature, it is worth noting that this result is not limited to linear plants or linear base systems.

2. PRELIMINARIES AND PROBLEM SETUP

2.1. Spaces and dissipativity definitions

We will use the standard ℓ_2 notation for the space of all sequences $x = \{x(0), x(1), x(2), ...\}$ of real numbers such that $\sum_{k=0}^{\infty} x(k)^2 < \infty$. ℓ_2^m is the space of all sequences $x = \{x(0), x(1), x(2), ...\}$ with $x(k) \in \mathbb{R}^m$ such that $\sum_{k=0}^{\infty} x^\top(k)x(k) < \infty$. A truncation of the sequence x at the instant t is

Table 1: Notation

Symbol	Meaning
$\mathbb{R}$	Set of real number
$\mathbb{R}^n$	Set of n-dimensional vectors
$\mathbb{R}^{n\times m}$	Set of matrices n by m
$\mathbb{S}^n$	Set of symmetric matrices n by n
ℓ_e	Space of sequences of elements in $\mathbb{R}$
ℓ_e^n	Space of sequences of elements in $\mathbb{R}^n$
ℓ_2	Space of square-summable sequences of elements in $\mathbb{R}$
ℓ_2^n	Space of norm square-summable sequences of elements in $\mathbb{R}^n$
x	Sequence
x_k	Truncate sequence at time k
$x(\cdot)$	An element of the sequence x
$\langle f, g\rangle$	Inner product: $\sum_0^\infty f(k)^* g(k)$
$\Vert f\Vert_2$	ℓ_2–norm: $\sqrt{\langle f, f\rangle}$
$\mathcal{M}$	Logic map $\mathcal{M} : \ell_e \times \ell_e \mapsto \{0,1\}$

given by

$$x_t(k) = \begin{cases} x(k) & \forall k \leq t, \\ 0 & \forall k > t. \end{cases} \tag{2}$$

Then, x belongs to the space ℓ_e if $x_t \in \ell_2$ for all $t \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of natural numbers including 0. In fact, the discrete-time counterpart of the extended space, ℓ_e, is the space of all possible real-valued sequences.

Different definitions of reset systems can be proposed for the discrete-time case. In order to reach general conditions, we investigate dissipative conditions on a discrete-time switched dynamical system with two dynamics as follows

$$\Sigma_s \ : \ \left\{ \begin{array}{lcll} x(k+1) & = & A_1 x(k) + B_1 u(k), & x(\cdot) \in \mathbb{R}^n, u(\cdot) \in \mathbb{R}^m, \mathcal{M}(x_k, u_k) = 0, \\ x(k+1) & = & A_2 x(k) + B_2 u(k), & \mathcal{M}(x_k, u_k) = 1, \\ y(k) & = & Cx(k) + Du(k) & y(\cdot) \in \mathbb{R}^l, \end{array} \right. \tag{3}$$

where A_1, B_1, A_2, B_2, C, and D have appropriate dimensions and $\mathcal{M}$: $\ell_2 \times \ell_2 \mapsto \{0,1\}$ is a logic law. Truncated sequences are used in the reset law to guarantee casuality. For instance, the zero-crossing reset law would be generated by

$$\mathcal{M}_{zc}(x_k, u_k) = 0 \text{ if } u(k)u(k-1) \geq 0,$$

and

$$\mathcal{M}_{zc}(x_k, u_k) = 1 \text{ if } u(k)u(k-1) < 0.$$

Note that this definition may easily be extended to more different dynamics by using a more complicated reset law and different Cs and Ds.

Discrete-time switched (hybrid) systems have been studied in [8]. Nevertheless the system in (3) is slightly different since the evolution of the system not only depends on the current value of $x(k)$ and $u(k)$, but also on their past values. We use Byrnes and Lin's definition [11] of dissipative discrete-time systems for systems of the form (3). Consider a real-valued function $s(u(\cdot), y(\cdot))$ such that $s : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$, which is called the supply function.

Definition 2.1. *A dynamical system Σ is said to be dissipative with respect to $s(u(k), y(k))$ if there exists a non-negative function $V : \mathbb{R}^n \to \mathbb{R}$, with $V(0) = 0$, called the storage function, such that for all $(u(k), y(k)) \in \mathbb{R}^m \times \mathbb{R}^n$ and all $k \in \mathbb{N}_0$,*

$$V(x(k+1)) - V(x(k)) \leq s(u(k), y(k)).$$

In order to develop dissipative properties of system (3), our starting point will be to use dissipative properties of the following discrete linear time-invariant (LTI) system:

$$\Sigma_l \ : \ \begin{cases} x(k+1) &= Ax(k) + Bu(k), \qquad x(\cdot) \in \mathbb{R}^n, u(\cdot) \in \mathbb{R}^m, \\ y(k) &= Cx(k) + Du(k), \qquad y(\cdot) \in \mathbb{R}^l, \end{cases} \tag{4}$$

where A, B, C, and D are matrices with appropriate dimensions.

For linear systems, quadratic supply functions in the input and the output are typically considered. System (4) is said to be (Q, S, R)-dissipative if the system is dissipative with respect to $s(u(\cdot), y(\cdot)) = y(\cdot)^\top Q y(\cdot) + 2y(\cdot)^\top S u(\cdot) + u(\cdot)^\top R u(\cdot)$, where Q, S and R are appropriately dimensioned matrices with both Q and R symmetric. Using the classical result for the Generalised Discrete Positive Real Lemma (Lemma C.4.2 in [17]), a Linear Matrix Inequality (LMI) can be generated to test (Q, S, R)-dissipativity for linear systems.

Corollary 2.2. *Let (A,B,C,D) be a minimal state space representation of Σ_l. Then, Σ_l is (Q,S,R)-dissipative if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} A^\top PA - P - C^\top QC & A^\top PB - C^\top QD - C^\top S \\ \star & B^\top PB - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0. \tag{5}$$

■

Remark 2.3. *When Q and R are zero (that is, the passivity case with $s(y(\cdot), u(\cdot)) = y(\cdot)^T u(\cdot)$) $D \neq 0$ is a necessary condition. See [36] for an alternative definition of passivity in order to avoid this condition.*

2.2. Reset systems

By following [4, 5], and using the zero-crossing reset law $\mathcal{M}_{zc}$, we will consider the following SISO reset system:

$$\Sigma_r \;:\; \left\{ \begin{array}{lcll} x(k+1) & = & Ax(k) + Bu(k), & \mathcal{M}_{zc}(x_k, u_k) = 0, \\ x(k+1) & = & A_\rho x(k), & \mathcal{M}_{zc}(x_k, u_k) = 1, \\ y(k) & = & Cx(k) + Du(k), & \end{array} \right. \tag{6}$$

where $x(\cdot) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}^{1\times n}$. The reset matrix $A_\rho \in \mathbb{R}^{n\times n}$ is typically a diagonal matrix with zeros in the reset states and ones in the non-reset states.

By equivalence with the linear-case, the first dynamics in (6) is referred to as linear evolution, whereas the second dynamics in (6) is referred to as reset evolution[1]. Nevertheless, the definition of discrete-time reset systems has an essential difference with the definition for continuous-time reset systems in [12]. In the continuous-time setting, the set of reset times is a set of measure zero in $\mathbb{R}$, and this result is used in [12] to demonstrate the passivity of the continuous-time reset system. Roughly speaking, the set of reset times can be ignored in the integration of the supply function since the supplied energy to the reset system is zero during the reset action:

$$\lim_{\epsilon\to 0} \int_{t_k}^{t_k+\epsilon} s(x(t), y(t))dt = 0. \tag{7}$$

Consequently, by ensuring that the storage function is decreasing at the reset action, the dissipativity of the linear base system is preserved by reset actions.

Nevertheless, when the reset system is defined in discrete time as (6), then the reset action will be performed within the subset $\{k_1^r, k_2^r, \dots\}$, where k_i^r means the i^{th} reset action, and this set has a nonzero measure in $\mathbb{N}$. As a consequence, in general, some energy will be supplied to or extracted from the system at reset times. This means that in order to define dissipativity in discrete-time reset systems, ensuring a decreasing storage function is not

[1]If $u(-1)$ is considered zero, then the first evolution is trivially a reset evolution; by choosing $u(-1) = u(0)$, this trivial evolution is avoided.

enough, because the linear evolution of the system can produce a new state with less energy than the reset evolution.

The second dynamics in (6) does not depend on the input $u(k)$. Therefore, the system evolution at reset times is not directly related to the supply function, which depends on the input. From a dissipativity point of view, then reset evolutions are hard to bound energetically. Taking this into account, we propose the following definition for a discrete-time reset system:

$$\Sigma_r \ : \ \begin{cases} x(k+1) & = & Ax(k) + Bu(k), & \mathcal{M}_{zc}(x_k, u_k) = 0, \\ x(k+1) & = & A(A_\rho x(k)) + Bu(k), & \mathcal{M}_{zc}(x_k, u_k) = 1, \\ y(k) & = & Cx(k) + Du(k). \end{cases} \tag{8}$$

System (8) has the advantage that both evolutions depend on the input. When the reset law holds, a reset action makes the state $x(k)$ be replaced by $A_\rho x(k)$. Then a linear evolution is performed as if the previous state is $A_\rho x(k)$ instead of $x(k)$. Both actions result in the second equation in (8).

2.3. Reset law

The results presented in this paper are general, independently on the election of the reset law as in [12]. We assume that a reset law, $\mathcal{M}(x_k, u_k) : \ell_e \times \ell_e \mapsto \{0, 1\}$, is given in such a way that the evolution of the system is perfectly characterised for any $k \geq 0$. That is, there is some logic law involving $\{x(0), x(1), \ldots, x(k), 0, \ldots\}$ and $\{u(0), u(1), \ldots, u(k), 0, \ldots\}$ in such a way that if the logic law is true then the reset action is triggered, and the evolution of the dynamical system is governed by the second equation in (8). Whereas if this logic law is false (equal to zero), then the evolution of the system is governed by the first equation (8).

In summary, the reset system given by (8) will be defined by five matrices (A, B, C, D, A_ρ) as follows:

$$\Sigma_R \ : \ \begin{cases} x(k+1) & = & Ax(k) + Bu(k), & \mathcal{M}(x_k, u_k) = 0, \\ x(k+1) & = & A(A_\rho x(k)) + Bu(k), & \mathcal{M}(x_k, u_k) = 1, \\ y(k) & = & Cx(k) + Du(k), \end{cases} \tag{9}$$

where the reset law $\mathcal{M}$ will be considered arbitrary. It is useful to reach dissipativity conditions which are independent on the reset law as in [12]. This has been used in [37], where modifications in the reset law allow an improvement on the performance without degradation on passivity properties of the system.

3. MAIN RESULTS

In the following, sufficient conditions for the (Q, S, R)-dissipativity of a system of the form (3) are given. Roughly speaking, since reset actions can be triggered at any point in the state space and any present value of the input, we ensure the (Q, S, R)-dissipativity of system (3) by ensuring that the two dynamics appearing in system (3) are (Q, S, R)-dissipative for the same storage function.

Proposition 3.1. *System Σ_s in (3) with an arbitrary reset law $\mathcal{M}$ is dissipative with respect to a supply function $s(y(\cdot), u(\cdot)) = y(\cdot)^\top Qy(\cdot)+2y(\cdot)^\top Su(\cdot)+u(\cdot)^\top Ru(\cdot)$ if there exists $P > 0$ such that*

$$\begin{bmatrix} A_1^\top PA_1 - P - C^\top QC & A_1^\top PB_1 - C^\top QD - C^\top S \\ \star & B_1^\top PB_1 - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0, \quad (10)$$

and

$$\begin{bmatrix} A_2^\top PA_2 - P - C^\top QC & A_2^\top PB_2 - C^\top QD - C^\top S \\ \star & B_2^\top PB_2 - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0. \quad (11)$$

Proof. Let P be a symmetric matrix which satisfies (10) and (11). Then, let us consider $V(x(k)) = x(k)^\top Px(k)$ as storage function, in short $V(k) = V(x(k))$. Thus, if $x(k+1) = Ax(k)+Bu(k)$ by using (10), for all $(x(k), u(k)) \in \mathbb{R}^n \times \mathbb{R}^m$, and applying Corollary 2.2, we have for all $k \in \mathbb{N}_0$:

$$V(k+1)-V(k) \leq \begin{bmatrix} y(k) & u(k) \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \quad \forall (x(k), u(k)) \in \mathbb{R}^{n_p} \times \mathbb{R}^m. \quad (12)$$

On the other hand, if $x(k + 1) = A_2x(k) + B_2u(k)$, then applying Corollary 2.2, (11) ensures that for all $k \in \mathbb{N}_0$:

$$V(k+1)-V(k) \leq \begin{bmatrix} y(k) & u(k) \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \quad \forall (x(k), u(k)) \in \mathbb{R}^{n_p} \times \mathbb{R}^m. \quad (13)$$

Therefore, independently on which dynamics is governing the evolution of the system, for all $(x(k), u(k)) \in \mathbb{R}^{n_p} \times \mathbb{R}^m$ and for all $k \in \mathbb{N}_0$:

$$V(k+1)-V(k) \leq \begin{bmatrix} y(k) & u(k) \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \quad \forall (x(k), u(k)) \in \mathbb{R}^{n_p} \times \mathbb{R}^m, \quad (14)$$

and the discrete-time reset system with any reset law is (Q,S,R)-dissipative. ■

Remark 3.2. *It is worth noting that the reset action depends generally on the present value and past of the input, that is, the dynamics of the system is not defined by the values $(x(k), u(k))$. Hence, the passification of a system via reset actions would require further modifications in the definition of the reset system. Nevertheless, the flexibility in the reset law can be exploited by using the reset law as a tuning parameter [37].*

3.1. Partial reset

The partial reset situation is when $A_\rho \neq 0$. First of all, we will consider this case. Proposition 3.1 is applied to SISO reset systems as (9), where Q, S, and R are scalar, providing the following results.

Corollary 3.3. *A reset system of the form* (9) *with an arbitrary reset law* $\mathcal{M}$ *is* (Q, S, R)*-dissipative if there exists* $P > 0$ *such that*

$$\begin{bmatrix} A^\top PA - P - C^\top QC & A^\top PB - C^\top QD - C^\top S \\ \star & B^\top PB - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0, \quad (15)$$

and

$$\begin{bmatrix} (AA_\rho)^\top PAA_\rho - P - C^\top QC & (AA_\rho)^\top PB - C^\top QD - C^\top S \\ \star & B^\top PB - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0. \quad (16)$$

■

For the passivity case, that is, when $s(y(\cdot), u(\cdot)) = y(\cdot)u(\cdot)$ – which is equivalent to consider $Q = 0$, $S = 1/2$, $R = 0$ – we can obtain an LMI condition for passive discrete-time reset systems of the form (9), as the following corollary shows.

Corollary 3.4. *A reset system of the form* (9) *with an arbitrary reset law* $\mathcal{M}$ *is passive if there exists* $P > 0$ *such that*

$$\begin{bmatrix} A^\top PA - P & A^\top PB - 1/2C^\top \\ \star & B^\top PB - 1/2(D^\top + D) \end{bmatrix} \leq 0, \quad (17)$$

and

$$\begin{bmatrix} (AA_\rho)^\top PAA_\rho - P & (AA_\rho)^\top PB_2 - 1/2C^\top \\ \star & B^\top PB - 1/2(D^\top + D) \end{bmatrix} \leq 0. \quad (18)$$

■

Moreover, the limitation of (6) can be clearly stated now.

Corollary 3.5. *A reset system of the form* (6) *with an arbitrary reset law* $\mathcal{M}$ *is not passive if the output explicitly depends on the non-reset states.*

Proof. If system definition (6) is used, then $A_2 = A_\rho$ and $B_2 = 0$. In addition, for the passivity case, we have $Q = 0$, $S = 1/2I$, and $R = 0$. Then condition (11) takes the following form:

$$\begin{bmatrix} A_\rho P A_\rho - P & 1/2C^\top \\ \star & -1/2(D^\top + D) \end{bmatrix} \leq 0, \tag{19}$$

and by using the Schur complements, we obtain that reset system (6) is passive if:

$$P - A_\rho P A_\rho - 1/2C^\top (D^\top + D)^{-1} C \geq 0. \tag{20}$$

In the case where A_ρ is a diagonal matrix of ones for the non-reset states and zero for the resetting states, then inequality (20) can be expressed in blocks as follows

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} - \begin{bmatrix} I_{\bar{r}} & 0 \\ 0 & 0_r \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \begin{bmatrix} I_{\bar{r}} & 0 \\ 0 & 0_r \end{bmatrix} - 1/2(D^\top + D)^{-1} \begin{bmatrix} C_{\bar{r}}^\top \\ C_r^\top \end{bmatrix} \begin{bmatrix} C_{\bar{r}} & C_r \end{bmatrix} \geq 0. \tag{21}$$

yielding

$$\begin{bmatrix} -1/2(D^\top + D)^{-1} C_{\bar{r}}^\top C_{\bar{r}} & \star \\ \star & \star \end{bmatrix} \geq 0. \tag{22}$$

Since $(D^\top + D) > 0$ and $C_{\bar{r}} \neq 0$ by assumption, the condition (19) cannot be satisfied. As a result, system (6) is not passive if the output explicitly depends on the non-reset states. ∎

3.2. Full reset

The results obtained in the previous section are applied here to the case of having $A_\rho = 0$. In this situation, the system is referred to as full reset system.

Corollary 3.6. *A reset system of the form* (9) *with an arbitrary reset law* $\mathcal{M}$ *is dissipative with respect to a supply function* $s(y(\cdot), u(\cdot)) = y(\cdot)Qy(\cdot) + 2y(\cdot)Su(\cdot) + u(\cdot)Ru(\cdot)$ *if there exists* $P > 0$ *such that*

$$\begin{bmatrix} A^\top PA - P - C^\top QC & A^\top PB - C^\top QD - C^\top S \\ \star & B^\top PB - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0, \tag{23}$$

and

$$\begin{bmatrix} -P - C^\top QC & -C^\top QD - C^\top S \\ \star & B^\top PB - R - D^\top S - S^\top D - D^\top QD \end{bmatrix} \leq 0. \tag{24}$$

■

For the passivity case, we can obtain an LMI condition for the passivity of full reset systems of the form 9, as the following corollary shows.

Corollary 3.7. *A full reset system of the form* (9) *with an arbitrary reset law* $\mathcal{M}$ *is passive if there exists* $P > 0$ *such that*

$$\begin{bmatrix} A^\top PA - P & A^\top PB - 1/2C^\top \\ \star & B^\top PB - 1/2(D^\top + D) \end{bmatrix} \leq 0, \tag{25}$$

and

$$1/2(D^\top + D) - B^\top PB - 1/4C^\top P^{-1}C > 0. \tag{26}$$

■

Remark 3.8. *Condition* (26) *is obtained by applying Schur complement.*

As a result, the passivity of a full reset system is not ensured by the passivity of the base linear system. This represents a clear difference between discrete-time reset systems and their continuous-time counterpart. In continuous-time reset systems, the passivity of the base linear system is enough to ensure the passivity of the full reset system.

3.3. ℓ_2-stability of a reset control system in discrete-time

Proposition 3.1 has been developed by using the state-space representation for dissipative systems [20, 40]. The input-output formalism of dissipative systems [21] can be also used. These results are generally given for continuous-time, e.g. [19]. Their extension to discrete-time is trivial, since they can be formulated in abstract Hilbert spaces for inputs and outputs, similarly as the passivity theorem is given in [15].

Let G be a nonlinear dynamical system given by

$$G \; : \; \begin{cases} x_G(k+1) \;=\; f(x_G(k), u_2(k)), & x_G(\cdot) \in \mathbb{R}^n_G, u_2(\cdot) \in \mathbb{R}^l, \\ y_2(k) \;=\; h(x_G(k), u_2(k)), & y_2(\cdot) \in \mathbb{R}^m. \end{cases} \tag{27}$$

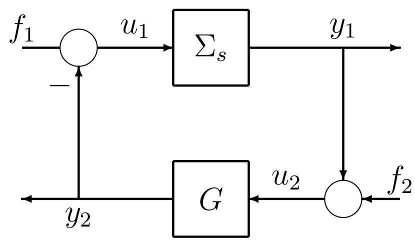

Figure 1: Reset control system. Negative feedback interconnection of systems (3) and (27).

Let us suppose that the plant G is dissipative with respect to the supply function $s(y_2(\cdot), u_2(\cdot)) = y_2(\cdot)^\top Q_G y_2(\cdot) + 2y_2(\cdot)^\top S_G u_2(\cdot) + u_2(\cdot)^\top R_G u_2(\cdot)$, where Q_G, S_G and R_G are appropriately dimensioned matrices with both Q_G and R_G symmetric.

By considering systems G and Σ_s as causal input-output mappings, i.e. $G : \ell_e^l \to \ell_e^m$ and $\Sigma_s : \ell_e^m \to \ell_e^l$, the negative feedback interconnection of both systems (see Fig. 2) is defined as follows:

$$\begin{cases} u_1 = f_1 - G u_2, \\ u_2 = f_2 + \Sigma_s u_1. \end{cases} \tag{28}$$

We assume that the system is well-posed, i.e. the map $(u_1, u_2) \to (f_1, f_2)$ defined by (28) has a causal inverse on ℓ_e^{l+m}. The feedback interconnection is said ℓ_2-stable if for any $(f_1, f_2) \in \ell_2^m \times \ell_2^l$ then $(u_1, u_2) \in \ell_2^m \times \ell_2^l$.

As commented, ℓ_2-stability of the feedback system can be obtained using classical result for the continuous-time.

Theorem 3.9. *Let $\Sigma_s : \ell_e^l \to \ell_e^m$ and $G : \ell_e^l \to \ell_e^m$ be causal operator dynamical systems such that the feedback interconnection defined by* (28) *is well-posed. Furthermore, let $Q, R_G \in \mathbb{S}^m$, $Q_G, R \in \mathbb{S}^l$, and $S, S_G^\top \in \mathbb{R}^{m \times l}$ be such that there exists a scalar $\sigma > 0$ such that*

$$\begin{bmatrix} Q_+ \sigma R_G & -S + \sigma S_G^\top \\ \star & R + \sigma Q_G \end{bmatrix} < 0. \tag{29}$$

If Σ_s is (Q,S,R)-dissipative, and G is (Q_G,S_G,R_G)-dissipative, then the feedback interconnection defined by (28) *is ℓ_2-stable.*

Proof. The proof is equivalent to the proof given in [19], where for any $u, y \in \ell_2^m$ the meaning of $\langle u, y \rangle$ should be considered as

$$\langle u, y \rangle = \sum_{k=0}^{\infty} u^\top(k) y(k). \tag{30}$$

In the same way, for any $u, y \in \ell_e^m$, then $\langle u, y \rangle_k = \langle u_k, y_k \rangle$ where the truncation of a signal at the instant k, u_k, is defined by (2). Since the proof only uses properties of the norm and inner product, these properties are also valid for the Hilbert space ℓ_2. ∎

4. Conclusion

This paper has extended the dissipativity results given in [12] for discrete-time reset systems. An adequate definition of discrete-time reset systems has been provided in order to perform the reset evolution in a virtual time. This definition allows general conditions for the dissipativity of discrete-time reset systems. It is worth to highlight that the dissipativity of a discrete-time full reset system is not ensured by the dissipativity of the base linear system. This is a significant difference between continuous-time and discrete-time reset systems. Moreover, stability of discrete-time reset control systems has been stated via dissipativity theory, whose stability theorem is equivalent for continuous-time and discrete-time systems since it is valid for any Hilbert space.

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