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A Many-Body Treatment of Hamiltonian Lattice Gauge Theory

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The Hamiltonian SU(N) lattice gauge theory is studied in terms of angular variables. We show that the one-plaquette problem can be mapped onto a fermion problem for arbitrary N. The low-lying energy spectrum is investigated numerically.

1. INTRODUCTION

The two key aspects of the strong interaction, as described by QCD, are colour confinement and spontaneous breaking of chiral symmetry. The first gives rise to composite hadrons, which cannot break up into constituent quarks. The latter manifests itself in the absence from the physical spectrum of the chiral partners of hadrons, which would otherwise be expected on the basis of the symmetry of the QCD Lagrangian. There are two avenues towards a description of hadrons in terms of QCD. The main approach is based on the Euclidean-time path integral. From the ensuing Monte Carlo lattice calculations there are clear indications that QCD is the proper description of hadrons. Although no unexpected physics has yet arisen, there remain many open questions, such as the existence of exotic particles and glueballs. Recent experiments indicate the presence of some exotic particles (and see the article by K.K. Seth, in these proceedings).

Secondly, there is the Hamiltonian approach, where the key element is the availability of the vacuum wave functional, but in which gauge symmetry is not immediately evident. The availability of the wave functional is a big advantage, and allows us readily to study such typical quantum phenomena as collective excitations and order parameters. The price to pay for more insight into the quantum-mechanical nature of the problem, however, is the elaborate formulation, which has led people to pursue the Euclidean avenue much further than its Hamiltonian counterpart. Although it is hard to compete with numerical results from the Monte Carlo calculations at the moment, this is, however, not our present intention.

Rather, in this paper we discuss some aspects of a Hamiltonian lattice approach, and its results for the lowest, non-trivial approximation to the colour vacuum wave functional; the one-loop, or one-plaquette, problem [1]. One of the major problems of Hamiltonian lattice field theory is to keep control of the large number of degrees of freedom in the wave functional, once a proper set of variables is determined. We propose to use a many-body technique, the coupled-cluster method [2]. However, even within that method some important choices about the wave functional representation have to be made. Often, every added degree of freedom in a finite approximation contributes as much to the wave functional as any other, and the finite approximations do not converge. The problem of the large number of degrees of freedom needed for the wave functional is not just a problem

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of computer power. Indeed, approaches which are successful for SU(2) gauge theory or for two spatial dimensions become intangible for the physical case of SU(3) gauge theory in three spatial dimensions. Therefore, a careful study of low-order approximations should be a valuable first step for the investigation of a method that can later be pursued to high orders [2].

The gauge is fixed by using the maximal tree gauge [3] which is specific for the lattice version of gauge theory. It does not suffer from the typical problems of gauge fixing in the continuum. Consequently, we introduce angular variables such that the SU(N) one-loop problem can be mapped onto an N-fermion problem. We determine exact solutions for both the weak- and the strong-coupling limits. The eigenstates in the strong-coupling limit are the group characters [4] of the corresponding group, and in the weak-coupling limit the harmonic approximation to the fermion problem leads to the degeneracies of the equidistant energy levels. In terms of the group characters the eigenstates in the strong-coupling limit, the ground state and low-lying states can be determined for arbitrary coupling constant, since the Hamiltonian reduces to a linear equation in terms of these group characters. Another advantage of this approach is the cancellation of correlations between any two, spatially distinct trace variables, which suggests that in this formulation the spatial correlations might be weak and successive approximations might converge.

2. THEORY

Given the SU(N) Lagrangian the Hamiltonian does not directly follow from the Legendre transform. Instead, the gauge freedom leads to constrained dynamics, since the equation of motion which follows from the variation with respect to A_0^a does not contain a time derivative, and is therefore not dynamical. Thus, the use of the temporal gauge $A_0^a = 0$ is prerequisite. The residual gauge symmetry involves all time-independent local gauge transformations. These gauge freedoms are thence not fixed [5], and generally prevent one from establishing a proper set of variables in which the wave functional can be expressed. In a strong-coupling expansion it is not necessary to establish the wave functional. However, relations among different terms in the expansion, leading to non-trivial, non-vanishing contributions, are not straightforwardly determined for the case of SU(3).

We fix the gauge fully by separating all the links on the spatial lattice into two sets; one set connects all sites uniquely, and is referred to as the maximal tree [3], while the other set is its complement. The links on the maximal tree change under local gauge transformations, and are set to unity in the gauge fixing procedure. The other links are connected with a path over the maximal tree to the origin to form a contour from and to the origin and therefore transform only under gauge transformations of the origin, which is the global gauge transformation.

If the contours are combined in such a way that they form plaquette [5], we can consider a wave functional as a function only of the plaquette variables,

$$\mathrm{Tr}\,[U^p] = \mathrm{Tr}\left[\left(V^\dagger U V\right)^p\right] = \sum_{j=1}^{N} e^{ip\phi_j} \;; \qquad p \in [1, 2, \cdots, N-1] \;, \tag{1}$$

where V is an arbitrary unitary transformation of the product of links around a plaquette $U \in \mathrm{SU}(N)$. Therefore, the trace depends only on the eigenvalues $\exp\{i\phi_j\}$ of U. The

angular variables ϕ_j satisfy the constraint $\sum_{j=1}^{N}\phi_j = 0$. The group characters [4] $\chi_\lambda(\phi)$ of SU(N) can be represented as

$$\chi_{\lambda_1\lambda_2\cdots\lambda_{N-1}} = \frac{\epsilon(\lambda_1\lambda_2\cdots\lambda_N)}{\epsilon((N-1)(N-2)\cdots 210)}\ ; \qquad \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_N = 0\ , \tag{2}$$

where $\epsilon(\lambda)$ is the determinant of the matrix with entries $\{\exp\{i\lambda_k\phi_j\}\}_{kj}$. The determinant in the denominator is a Vandermonde determinant in terms of the eigenvalues: $\Delta \equiv \epsilon((N-1)(N-2)\cdots 210) = \prod[\exp(i\phi_k) - \exp(i\phi_j)]$, and allows us to map the one-plaquette problem onto a fermion problem, by absorbing Δ into the wave function. Thus, in the electric part of the Hamiltonian, we write

$$\sum_{a=1}^{N^2-1} E^a E^a \ \rightarrow\ \Delta\left(\sum_{a=1}^{N^2-1} E^a E^a\right)\frac{1}{\Delta} = D_N - \frac{N(N^2-1)}{24}\ , \tag{3}$$

$$\Psi'_\lambda \ \rightarrow\ \Delta\Psi_\lambda\ , \tag{4}$$

where the differential operator

$$D_N = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial\phi_i^2} + \frac{1}{2}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\partial}{\partial\phi_i}\right)^2 \tag{5}$$

acts on the antisymmetric wave function Ψ'. The magnetic part of the Hamiltonian $H_M = \mathrm{Tr}[U] + \mathrm{Tr}[U^\dagger]$, reduces to a multiplication [4] of group characters

$$[\chi_1(\phi) + \chi_1(-\phi)]\epsilon_\lambda = \sum_{\lambda'=\lambda+1}\epsilon_{\lambda'} + \sum_{\lambda'=\lambda-1}\epsilon_{\lambda'}\ , \tag{6}$$

where $\lambda \pm 1$ denotes all possible partitions where one of the $\lambda_i \rightarrow \lambda_i \pm 1$ $(1 \leq i \leq N)$ still fulfilling the requirement of Eq. (2). This result follows from the theory of multiplication of group characters, or directly from Eq. (2). Note, that for SU(N) any partition $\lambda_1 > \cdots > \lambda_N$ is equivalent to the partition $\lambda_1 - \lambda_N > \lambda_2 - \lambda_N > \cdots \lambda_{N-1} - \lambda_N > 0$ due to the constraint $\det U = 1$.

The one-plaquette problem can be related to the full lattice problem by the simple ansatz wave functional $|\Pi\rangle$:

$$\langle\{\phi_{\alpha i}\}|\Pi\rangle = \prod_{\text{plaquettes }\alpha} F(\phi_{\alpha 1}, \phi_{\alpha 2}, \cdots, \phi_{\alpha N})\ , \tag{7}$$

with F an arbitrary function, where the correlations between different plaquettes in $|\Pi\rangle$ vanish due to the special properties of the electric interaction of SU(N) gauge theory.

As a constrained N-fermion problem the SU(N) one-plaquette Hamiltonian reads

$$H = \frac{Lg^2}{2}D_N + \frac{1}{g^2 N}\sum_{i=1}^{N}(1 - \cos\phi_i) \tag{8}$$

where L is the length of the loop, which in case of the one plaquette equals 4, and g is the coupling constant [5]. The constraint $\det U = 1$ reduces to a constraint on the centre-of-mass motion, $\sum\phi_i = 0$. In the weak-coupling limit we use the harmonic approximation of the N-fermion problem. The centre-of-mass motion factorizes, which allows for the implementation of the constraint $\det U = 1$, and the degeneracies ν_n of the nth level of the equidistant spectrum are given by $\nu_n = P_N(n) - P_N(n-1)$, where $P_i(j)$ is the number of ways j can be represented as the sum of i distinct integers.

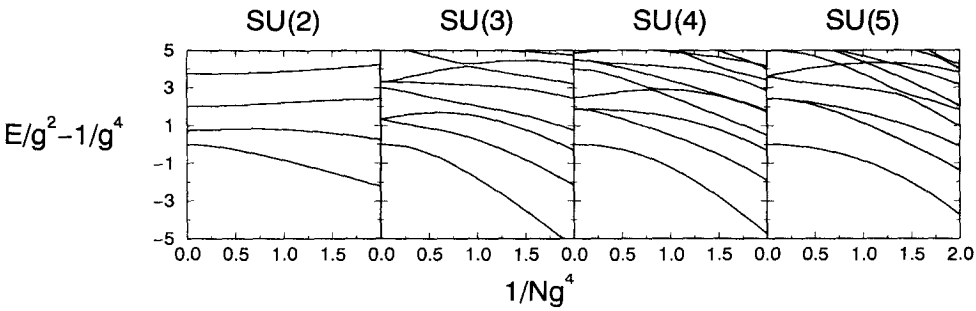

Figure 1. The spectra for SU(2), SU(3), SU(4), and SU(5), after subtraction of the linear term g^{-4}.

3. RESULTS AND CONCLUSIONS

For the low-lying spectra in Figure 1 we solve the linear eigenvalue equations numerically. An accuracy of 6 decimal places in the energy for the 10 lowest states over the whole range of coupling constants is achieved by using about 100-200 eigenstates of the electric operator, where the space is truncated at a particular $\lambda_1 \leq \lambda_{\max}$, where the number of states is approximately given by $\binom{\lambda_{\max}}{N-1}$. For SU(2), the spectrum is given by the odd characteristic values of the Mathieu equation. For $N = 3, 4$, and 5, the spectra are much richer. They include both crossings, indicating the presence of a symmetry, and avoided crossings. These features are being investigated.

The extension of this approach to include spatial correlations is also under investigation. The simplicity of the angular variables formulation is promising for more elaborate wave functionals [2]. We hope that the one-plaquette wave functional can serve as a first step in describing a spatially correlated wave functional.

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