Proceedings of the Institution of Mechanical Engineers, Part A: Journal of Power and Energy
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Proceedings of the Institution of Mechanical Engineers, Part A: Journal of Power and Energy 2012 226: 350 originally published online 9 March 2012
DOI: 10.1177/0957650912439158

Joint economic and physical constraints on nuclear power: how much uranium would be needed to decarbonize electricity supply?

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The manuscript was received on 10 June 2011 and was accepted after revision for publication on 19 December 2011.

DOI: 10.1177/0957650912439158

Abstract: A new uranium supply and usage analysis takes into account growth in physical demand for electricity, both before and after decarbonization. It also models the economic willingness of investors to build the required reactors as a profit-maximizing decision. The analysis computes the optimum path of investment and disinvestment in one or more uranium-based fuel cycles as dynamic profit maximization under physical constraints, and the resulting economic decisions are compared with physical decarbonization goals. Investor decisions prove to be insensitive to the electricity price (and hence to higher carbon tax) above a low threshold. Above that price threshold, their decisions on reactor investment are dominated by the total supply of uranium and the speed at which reactors can be built. In order to pay for the very large generating capacity needed, profit-maximizing investors seem to need of the order of 36 M tons of uranium reserves to use (some seven times the world's presently identified reserves). Even given this supply, they will limit their maximum investment in capacity if the build rate of reactors is too slow. If the world does choose to decarbonize, it will be necessary to manage uranium prospecting, and reactor building, purchase and operation as an interdependent system, and there will be a steep task of uranium exploration over the next decade or two.

Keywords: nuclear power, uranium supply, fast reactors, dynamic optimization, real options

1 INTRODUCTION

The sustainability of nuclear power raises two questions: 'how long can nuclear energy itself be sustained?' – both for the world in total, and for the UK, if different – and 'how big a contribution can nuclear energy make to targets for world sustainability, such as climate control?' – again for the world in total, and for the UK if different. The problems at world and UK level are linked, because Britain cannot follow an independent nuclear power policy on either reactors or fuel, even though the UK accounts for less than 3 per cent of world demand [**1**]. The UK has no independent manufacturing capacity, but it has a larger stock than many countries of plutonium, both separated and unseparated [**1**]. Plutonium is a potentially limiting scarce resource, and is one of the fuels required to start fast reactors (FRs). The other fuel requirement for starting FRs, namely previously 'burned' nuclear fuel, is not a scarce resource, and once started, an FR only requires a feed of ^{238}U, which has enough resources to last for centuries. However the UK's total plutonium stock could only start enough FRs to meet 20 per cent of the UK's 2010 demand for electricity [**1**]. Hence, decisions by the UK to decarbonize are potentially constrained by world decisions on uranium resources and usage, decarbonization, and plutonium availability.

In this article, quantitative statements about 'decarbonization' will be restricted to scenarios in which nuclear power alone provides the whole of the

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demand now conventionally met by electricity. It is realistic to suppose that there will, in addition, be supplies of carbon-free electricity from renewables like wind, tidal, and solar power. This article assumes that if nuclear energy is used to decarbonize all existing ('conventional') electricity demand, any further supplies of decarbonized electricity will be free to replace fossil fuels in powering some fraction of land transport, making use of energy storage such as vehicle batteries or fuel cells in order to decouple the intermittencies of supply and demand. This article does not quantitatively model the decarbonization of land transport, though renewable supply and total demand seem of similar order, but some level of this additional form of decarbonization is made possible by a scenario in which nuclear power alone decarbonizes all existing demands for electricity.

This article examines two limiting policy scenarios for significant future world usage of nuclear power: first 'business as usual' (BAU), in which nuclear power provides its present proportion of total electricity demand indefinitely, and second 'fast and sustained total decarbonization' (DECARB) in which nuclear power grows rapidly to match 100 per cent of electricity demand in 2050, then matches subsequent demand growth. These limits exclude policies in which nuclear energy meets a declining share of electricity demand, since these pose fewer practical problems than either BAU or DECARB. The long-term growth rate of electricity demand is assumed to be its average over recent decades of 1.9 per cent [**2**].

The purpose of modelling the DECARB scenario is that proposals for climate stabilization often call for near total decarbonization by 2050, and often assume that nuclear power is the only technology that can achieve rapid decarbonization without disruptive shocks to the world's economic, social, and technical systems [**3–6**]. The target of meeting all ongoing growth of future electricity demand after decarbonisition is chosen within DECARB for two reasons: (a) in a growing world economy, any share given to fossil fuels must eventually grow its absolute fossil emissions without limit and (b) the rapid growth of BRIC and other formerly poor economies is mostly in warm countries and at a time of global warming. This leads to large absolute new demands for cooling, which will tend to offset percentage reductions in other electricity use caused by technology change and/or price increases. However, even if a scenario for nuclear power growth can reach, and then indefinitely sustain, an absolute target of zero CO_2 tons emitted per year by electricity generation, if that scenario takes decades to reach decarbonization, the scenario may stabilize the world's CO_2 equilibrium at too high a level. If so, revised plans are needed, which could effectively displace any unwanted future *cumulative emission* of CO_2 by earlier and faster *cumulative consumption* of uranium.

This article will not directly model the world output of CO_2, but we note that in DECARB the absolute CO_2 emissions continue to grow for thirteen years.

To define these limiting policy scenarios more precisely for the purposes of this study, BAU assumes a world electricity capacity of 2400 GW in 2011 (model year 1) of which nuclear energy provides 370 GW, and both will grow at 1.9 per cent. Scenario DECARB assumes that nuclear energy must grow from 370 GW in 2011 to match the total demand of 5000 GW in 2050, by some time path, and must grow at 1.9 per cent thereafter.

This article aims to explore how fast either BAU and DECARB policies, using light water reactors (LWRs) would deplete the available uranium supply, and identify resulting problems, using methods which do not seem to have been applied elsewhere. The resulting cumulative demands for uranium are compared with various assumptions about the physical supply of uranium. Results explore what joint levels of nuclear electricity price (assumed to be varied by a carbon tax) uranium supply, generator prices, and maximum rates of generator supply, would lead profit-maximizing investors (private or social) to choose rapid and sustained decarbonization using once-through uranium burning (UB) followed if needed, by an orderly transition to FRs.

Even if the available supply of uranium is physically adequate for decarbonization, profit-maximizing investors may not choose to do so. Their objective to be optimized is their return over the complete trajectory of building and decommissioning nuclear reactors, for one or two fuel cycles, in the light of total physical demand and various cost and price coefficients, and their joint dependence on stocks of uranium and plutonium.[1] To solve the profit-maximizing investor's dynamic problem, this article approximates the problem in state space form by a partial differential equation (PDE) and optimizes the trajectory from every state point. This model is used to identify the most sensitive variables for policy use.

The structure of the article is as follows: after the main assumptions, section 3 computes cumulative physical uranium demand from the BAU and

[1]Note that this article does not address thorium fuel cycles, which is a topic for future work. Meanwhile, it is understood that thorium FR cycles (breeder reactors) require starting charges of plutonium, which must be derived from previous once through UB and to that extent resemble uranium FR cycles. Also, thorium in once-through thermal reactors also consume some enriched natural uranium as fuel, and to that extent may be called UB.

DECARB policy scenarios for electricity generation. It compares these demands with various estimates of the natural reserves of uranium, both physically and at various economic extraction costs. It also simulates in realistic detail, but without optimization, a sample UB building programme which strongly increases the output of UB generation, then hands over to the FR cycle. The specific assumptions made do not decarbonize completely, nor do they provide a smooth handover from UB to FR, and uranium consumption is large. Section 5 applies a dynamic optimization model to optimize the scale and speed of investment and disinvestment in UB and FR under various price incentives, constraints on the rate of reactor build, and perceptions of the total economically usable uranium reserves. It is found that in order to motivate rapid decarbonization by UB, followed by a smooth handover to FR, the reactor build rate must be fast enough to achieve decarbonization, and the economically available uranium stock must be large enough to allow profit-maximizing utilization of a large UB reactor fleet. The electricity price must be high enough to give investment in FR a positive net present value (NPV), but beyond this further price rises have trivial effects.

2 ASSUMPTIONS AND NOMENCLATURE

Published actual, forecast, and assumed data about nuclear reactors vary widely, especially in their economic coefficients. The assumptions used in this study broadly represent expert industry opinion and sources such as WNA [**7**] (which themselves offer widely varying actuals, let alone estimates, of the cost and performance parameters of future FRs). For some users our assumptions on cost, build, and efficiency variables may serve as targets, which if met will justify this article's conclusions. Other readers may want to treat future financial and efficiency parameters as random variables. If so, they should allow for the value of being able to escape some of the effects of adverse outcomes, using real options theory. A list of notation and typical values of the parameters used throughout the article is in Table 1.

3 PHYSICAL DEMAND

To allow intuitive calibration of the following estimates of cumulative uranium demand, the WNA [**7**] has recently estimated known uranium reserves at 5.5 M tons (extractable at costs up to $120 per lb, which is well above the typical spot price in recent decades). A further 6 M tons is expected but not known to exist in similar geologies yet to be explored.

3.1 BAU and DECARB scenarios

Method for BAU

Assume that the starting conditions of total world electricity demand are $D_0 = D_{t=0} = 2400$ GW, and world UB nuclear generation capacity $G_1(t=0) = 370$ GW. The standard U tonnage per GW year = 182. Also assume that UB generation capacity G_1 grows at 1.9 per cent (annual compounding used).

Method for DECARB

Assume the following conditions:

- total world electricity demand $D_0 = D_{t=0} = 2400$ GW;
- world nuclear generation capacity $G_1(t=0) = 370$ GW;
- standard U tonnage per GW year = 182;

Table 1 Constants and terms used in the models

Term	Description	Units	Value
V	NPValue	US$	–
t	Time	Years	–
G_1	Current reactor capacity of UB (assuming 90% load factor)	GWe	370
G_2	Current reactor capacity of FR (assuming 90% load factor)	GWe	0
Q_1	Stock of uranium	10^6 tons (M tons)	–
Q_2	Stock of plutonium	tons	0
U	Amount of uranium to produce 1 GWe per unit time	ton/(GWe year)	182
P_1	Amount of plutonium produced by 1 GWe UB reactor per unit time	ton/(GWe year)	0.01
P_2	Amount of plutonium required as initial fuel for 1 GWe FR reactor	ton/GWe	11.8
P_3	Amount of plutonium produced by 1 GWe FR reactor per unit time	ton/(GWe year)	0.05
r	Long-term discount rate	year^{-1}	0.08
θ_1	Operating costs for UB	US $/(GWe year)	174×10^6
θ_2	Operating costs for FR	US $/(GWe year)	244×10^6
S	Selling price of electricity	US $/(GWe year)	–
π_1^+	Cost per GWe of investment in UB	US $/GWe	1300×10^6
π_1^-	Cost per GWe of decommissioning in UB	US $/GWe	480×10^6
π_2^+	Cost per GWe of investment in FR	US $/GWe	1820×10^6
π_2^-	Cost per GWe of decommissioning in FR	US $/GWe	672×10^6
κ_1	Reciprocal of time lag in build rate of UB	year^{-1}	0.1
κ_2	Reciprocal of time lag in build rate of FR	year^{-1}	0.1

- exponential growth in reactor capacity at rate $g=$ 7 per cent/year;
- select g so that $G_1(t=40)=368\exp(40g)$;
- bound $G_1(t>40)=2400\exp(0.019t)$.

We are now able to deduce the cumulative U needed to sustain 1.9 per cent growth of a decarbonized electricity generation system for 100 years from 2050 (this is 1.67 reactor lives from 2050, assuming a reactor life of 60 years). It is noted (for later) that potential Pu production is 1 per cent of cumulative U use.

Discussion

The cumulative usage of uranium is presented in Table 2 for both BAU and DECARB scenarios. Examining the entries for the BAU scenario, it can be seen that there is only enough known uranium for 54 years of usage at current growth rates, which is less than a typical reactor life of 60 years. Many other commentators express the known stock of approximately 5 M tons as '76 years of supply at current usage rates'.

Table 2 also states that to decarbonize the world electricity supply by 2050 requires 12.4 M tons of U, which exceeds the world's total 'suspected but not yet found' reserves of easily extracted uranium [**7**]. To decarbonize by 2050 and then run the decarbonized system for 100 years after that (less than two reactor lifetimes) takes in total over 20 times the world's 'suspected but not yet found' reserves of 11 M tons of easy to extract uranium.

Clearly, a world policy decision to decarbonize by UB nuclear energy places severe demands on the 'known and suspected' supplies of easy to extract uranium. If stocks of U are indeed of the order of 11 M tons, UB might achieve deacarbonization by 2050, but cannot sustain it after that. However, this case requires exponential growth of reactor capacity; therefore, the last and largest cohorts of reactors, which are built just before 2050 could operate for lives of only a year or so. This would be a crashingly costly way to avoid only a few tons of CO_2, for either a profit-maximizing private investor or for an economically rational state-owned enterprise. Section 4.2 computes a profit-maximising approach to reactor capacity phasing.

Table 2 The cumulative U usage calculated in future years

Year	BAU (M tons)	DECARB (M tons)
2039		5.5
2050	3.7	12.4
2064	5.5	26.5
2100	14.6	85.4
2150	42.7	272.2

Flexing the uranium quantity constraint assumed by the WNA must involve flexing the upper cost constraint of $120 per ton, assumed in past WNA estimates. Hence, physical uranium supply cannot be divorced from its economics.

The next section considers these challenges in two stages. First, a highly optimistic recent forecast [**8**] of the long-term physical availability and cost of uranium is re-evaluated. This leads to a possible need for a handover to a successor fuel cycle, and a physically feasible handover from UB to FR is modelled in some detail.

4 PHYSICAL AND ECONOMIC MODELLING OF SUPPLY

A recent MIT study [**8**] predicts that over a cumulative output of 50 M tons (roughly 700 years at present consumption rates), the mining cost per ton of uranium will increase by a factor of only 1.4. A confidence interval is given around this estimate showing a 15 per cent probability that the cost increase factor will exceed 3, and a 15 per cent probability that it falls below 0.6 which is a 40 per cent fall from today's unit cost. The data source is a 1978 planet-wide census of uranium quantities and concentrations [**9**]. The present study joins MIT in recommending an updated census, adding more detail and confidence intervals.

The data source reports worldwide total tons T of uranium in various geological formations and features (including ocean water and fresh water), each having its own mean concentration C of uranium (units: parts per million). The data for the geological features are arranged in descending order of concentration C on a log scale (spanning eight orders of magnitude of C) and the total tonnage T of uranium in each formation is plotted, also on a log scale (spanning 10 orders of magnitude of T). The plotted logT(logC) points fall close to an inverted parabola (with gaps and outliers). This is the expected shape of the log likelihood function of T when T is normally distributed across the log of C.

The MIT model fits a smooth parabola through the plotted logC and logT points, and then approximates this parabola as a straight line in the region of $T=10^7$. It derives from this an elasticity function or differential equation (DE) to relate the marginal percentage increase in tonnage obtained by a marginal decrease in yield. This DE is solved numerically to get absolute tonnage/yield data. Similar information can be read directly from the original plot. The results assume that both learning and scale benefits apply, and the model assumes a wide range of possible values for the joint effects of these.

There are some questions about this method.

1. The use of both scale and learning coefficients: it is well known that during exponential growth either a learning or a scale model can fit the data well. To use both is double counting. A pure scale effect seems preferable for long-term work, since it is known that pure learning always exists locally, but it tends to a local asymptote, after which further improvements await the next long-term change in scale and/or technology, the latter often including a scale benefit [**10**]. In addition, U mining is a small fraction of the world's open cast mining; hence, effects within U alone may not change the world's cumulative learning or scale in mining by a significant amount.
2. It seems unrealistic to assume widely different scale (or learning) effects as equally likely in simulations. Learning and scale benefits tend to be smallest for simple processes, with high power usage and whose costs are incurred throughout the volume of the work unit (e.g. the cost of fuel flowing through an engine) rather than when costs are incurred in constructing the outer surface of a work unit which is essentially a container (e.g such as a car, ship, or engine). Steel rolling is a good example of the former, and ore extraction and mining seem to resemble it. The well-known 'two thirds power law' of scale applies to constructing a container, idealized as a sphere, whose surface strength need not rise with its mass, but mining is an operation on the whole of the mass of ore.
3. Because mining is performed on tons of ore, the effect of (for example) halving the ore concentration C is to double the tons of ore to be extracted per ton of uranium, but this also doubles the work of milling (or other concentration process) needed per ton of ore. The latter produces a doubly explosive increase in milling cost per ton of U, which eventually dominates the cost of mining. The work load of extraction and milling more tons of ore, per ton of U, increases faster than log linear scale economies reduce the cost per ton of ore from processing more ore in total.

The following method corrects for some of the above optimism by omitting learning as a benefit independent of scale, and by adding the explosive growth of milling for poorer grade ores. It continues to assume optimistic parameters for scale economies. It also continues to assume that future geological formations will not pose extra mining difficulties (e.g. from rock hardness) and that the lower grade ore deposits will have sizes and shapes that permit continued economies of scale in mining.

Assumptions
Ore yield (concentration) for (most) uranium deposits presently being worked is $Y_{M,0} = 0.7$ per cent; milling or other refinement processes raise the ore concentration to $Y_a = 5$ per cent; All these assumptions are compatible with the plotted data from reference [**1**] for 'Vein deposits'. Also, the plotted total U tonnage of vein deposits is close to 11 M tons, which is the WNA estimate of suspected but not identified uranium. It is optimistically assumed that extraction has scale economies based on pure surface effects (giving net extraction a scale power $s_E = 0.67$ which gives 20 per cent cost saving per scale doubling); that milling is energy intensive (giving net milling a scale power of $s_M = 0.85$ which gives 10 per cent cost saving per scale doubling); assume that geological formations not yet mined (sandstones) have identical rock hardness to vein deposits, therefore an identical energy cost of milling and extraction per ton of rock; assume that extraction costs and milling costs are roughly equal in present mines; that the cost of mined uranium is of the order of \$20/lb or £27/kg, not the upper limit of \$120/lb assumed in WNA data, or the market price at the time of writing of over \$60/lb (which applies to few transactions and may not endure).

For a detailed description of the method and results see Appendix 2. The main results are that within the cumulative output range 20–90 M tons uranium costs are multiplied by at least 3.3 and within the range 90–190 M tons they grow explosively.

4.1 A detailed scenario for uranium supply and use in once-through reactors and FRs

One natural successor fuel cycle to UB is the 'FR'. Over the long run (centuries), the FR cycle can be completely independent of outside fuel inputs apart from ^{238}U, and since the FR cycle seems capable of growing its own Pu stock at 5 per cent it can indefinitely match the historical demand growth of 1.9 per cent. Since the more expensive FR cycle is unlikely to be introduced without some shortage of uranium supply, a handover from UB to FR is likely to be constrained by economic pressures as well as physical fuel dynamics.

The detail at which nuclear systems are modelled
Modern UB reactors, such as the two pressurised water reactor designs (PWR) now being considered for new nuclear build in the UK, use fuel containing low enriched uranium (LEU) where the content of ^{235}U has been increased from the 0.71 per cent in natural uranium, generally to around 4.5–5 per cent. The process is such that only 10–15 per cent of the mined uranium makes up the fuel, with 85–90

per cent ending up as *enrichment tails* at around 0.2–0.3 per cent ^{235}U, which is stored or disposed of.

Though PWR reactors produce some power from plutonium, which is itself generated during irradiation of the fuel, they mainly produce power from fission of the ^{235}U by thermal neutrons, and are termed thermal reactors. They are relatively heavy users of uranium, typically consuming around 2.4 te LEU per TWh of electricity generated.

The spent fuel discharged from the UB reactor contains plutonium, typically at around 0.8–1.4 per cent. This can be used as part of the initial fuel for FR which uses fission by fast neutrons, and can burn the ^{238}U which remains in the spent UB fuel, and in the stored enrichment tails. This improves uranium utilization, typically by a factor of 50–60, increasing the energy potential of the world's uranium inventories.

Another feature of FRs is that they can produce more plutonium than they consume, taking 10–15 years to double the amount of the plutonium attainable. An indefinitely expanding programme of FRs can then be supported using plutonium from reprocessed fuel from the initial FR, both to refuel the initial reactor itself and to start up new reactors. The FR system essentially burns ^{238}U, derived from UB thermal reactor spent fuel and/or from stored enrichment tails, and this can be cycled through the reactor many times, isolating after each cycle the plutonium produced.

Though the physics and the technology for FRs have been demonstrated, they are not currently economically competitive, but the technology is complete enough to make valid physical assumptions for use in future scenarios.

To illustrate the factors at work, a UK-only scenario for UB construction is first considered, which is later grossed up to world level. The construction programme builds 50 GWe of modern PWRs, with the first reactor start-up in 2018 and the final reactor start up in 2037. The initial programme is bounded to a single generation of LWRs, with reactor and fuel assumptions typical of the Generation 3+ PWRs currently being evaluated by the UK nuclear regulators in the Generic Design Assessment [**11**].

The details of this UK-only scenario are given in the Table 4. This build rate is much higher than current UK projections and is higher also than the 'high' case of 38 GWe currently being looked at by the House of Lords [**12**], but it is similar to that in Level 4 in the UK Government's Pathways analysis up to 2035–2040 [**13**]. The UK policy documents which underlie this scenario do not consciously consider a potential handover from UB to FR, and they do not set targets for nuclear energy's share of ongoing total electricity demand and hence for its potential contribution to decarbonizing the UK electricity supply.

To give a simplistic examination of the effect of a worldwide build programme on the same scale, the UK programme is scaled up from the 1.9 per cent UK share of world electricity production in 2008 [**14**,**15**]. All existing and new reactors are assumed to be LWRs (PWRs and boiling water reactors) with similar fuel and uranium usage to the PWRs assumed for the UK. The present scenario's rescaling of UK growth to world scale agrees with the WNA high-growth scenario at world scale until around 2050 [**7**]. This nuclear capacity expansion scenario was used to calculate world uranium usage, electrical output, and the amount of plutonium contained in the spent fuel. The results obtained are presented in Table 5.

A key observation is that the consumption of uranium by 2040 (7.46 M te U) exceeds the current known economically recoverable reserves of 5.5 M tons. The uranium usage from the scenario is presented in Table 6.

Therefore, for our UK scenario to be deliverable on a world scale, considerably more uranium needs to be discovered and exploited in the next 30 years or so, to a total of nearly 30 M te U (some 5.5 times the currently known economically extractable reserves).

Table 3 Data for input into the reactor used in the model

Type	Capacity (MWth)	Efficiency (%)	Capacity (GWe)	U usage (t/y)	Fuel usage (teHM/y)
EPR	4500	36	1.458	276	29.6
AP1000	3400	32.85	1.005	209	22.4
Model reactor	4082	35	1.286	251	26.8

Source: authors' calculations using assumptions stated in text.

Table 4 UK Scenario used in the model

Year	2030	2050	2070	2090
UK new build reactors	22	35	35	13
UK reactors (GWe)	31.4	50.0	50.0	18.6
UK TWh/year from new build	400.4	637.0	637.0	236.6

Another factor is that the required reactor build rates are high. The programme requires a peak rate of reactor installation of some 137 GWe per annum (or about 85 EPRs in a year). This compares with a national rate of around 4.5 GWe per annum maintained in France through the 1980s [**16**].

This programme's carbon reduction rewards appear large, since it generates 1.33×10^6 TWh. Using the range of 7–22 gCO_2/kWh from the 2008 UK White Paper [**17**], the LWR world programme would save between 9.32×10^9 and 2.93×10^{10} te CO_2. This, however, is only around 1–3 years of the world's total energy sector emissions in 2008.

The next stage of the scenario examined is to use the plutonium contained in the LWR (UB) spent fuel to fuel FRs. The Gen IV GFR Case 6 design concept was chosen as an example FR. It has a 1080 MWe output and an assumed life of 50 years. Fuel loading is based on 90 GWd/t average discharge burnup and 45 per cent thermal efficiency. The total fuel inventory assumed for these reactors was 82.2 teHM, with a plutonium content of 15.5 per cent giving a required plutonium inventory of around 12.7 tePu (implying ≈11.8 tePu per GWe as the reactor start-up charge).

The scenario then assumes that all future spent fuel from the world's existing LWR reactors and the programme's new build is reprocessed, and the separated plutonium is used to fuel FRs. There will be a time lag between the spent LWR fuel being discharged from reactors and being available for reprocessing, and for illustrative purposes this is taken to be 10 years.

There is an assumption to be made on the availability date and subsequent build rate of fast reactors and the attendant fuel technology. Again, for illustrative purposes, build commencement has been taken 30 years after the first discharge of spent fuel from the new LWR reactors, so that FR operation starts from 2040. For simplicity, the world's existing stocks of plutonium, both separated and within stored spent fuel, have not been considered but they do not greatly change the cumulative Pu supply. The resulting FR programme and total world nuclear capacity are presented in Table 7.

We assume that the FRs are used to breed plutonium, supporting further increases in FR capacity. If the breeding rate is assumed to have doubling time of 15 years (corresponding to an annual increase of 4.73 per cent), the FR capacity development is as seen in Table 8.

Results of the scenario are shown in Fig. 1. Clearly, this scenario not only requires an integer multiple of 2009 known uranium stocks, but also provides a rather discontinuous world total output of nuclear energy. The level of CO_2 reduction is not game changing, because the scenario's 2099 output of 2500 GWe is less than half of total world electricity demand in 2099, which allows fossil fuel CO_2 emissions to double or treble from 2010 levels.

Table 5 Results obtained by scaling the UK growth to 1.9%

Year	2030	2050	2070	2090
UK new build reactors (GWe)	31.4	50.0	50.0	18.6
World GWe installed	1654	2632	2632	977
World existing GWe	254	105	0	0
World Total GWe	1908	2736	2632	977
World Total TWh	15 057	21 588	20 762	7 711
Cumulative world total U	2.95×10^6	1.23×10^7	2.17×10^7	2.91×10^7
Cumulative world Total Fuel	3.16×10^5	1.32×10^6	2.32×10^6	3.12×10^6
Cumulative total Pu in spent fuel	9026.3	37 712.6	66 330.1	89 093.2

Table 6 Uranium usage as given by the model

Year	2011	2030	2050	2070	2090
Cumulative world total U need	6.59×10^4	2.95×10^6	1.23×10^7	2.17×10^7	2.91×10^7
Proportion of currently identified reserves	0.01	0.54	2.24	3.94	5.29

Table 7 FR programme as given by model

Year	2011	2030	2050	2070	2090
World total LWR GWe	376	1 908	2 736	2 632	977
Cumulative separated Pu	–	938	9 593	21 914	33 687
Cumulative total GWe	–	0	91	867	1 908
Total world capacity GWe	376	1 908	2 828	3 498	2 885

Table 8 World capacity development

Year	2011	2030	2050	2070	2090
World total thermal GWe	376	1908	2736	2632	977
Cumulative total GWe	–	0	91	867	1908
Cumulative FR Gwe at 3.53%	–	0	95	905	1996
Total world nuclear capacity GWe	376	1908	2828	3498	2885

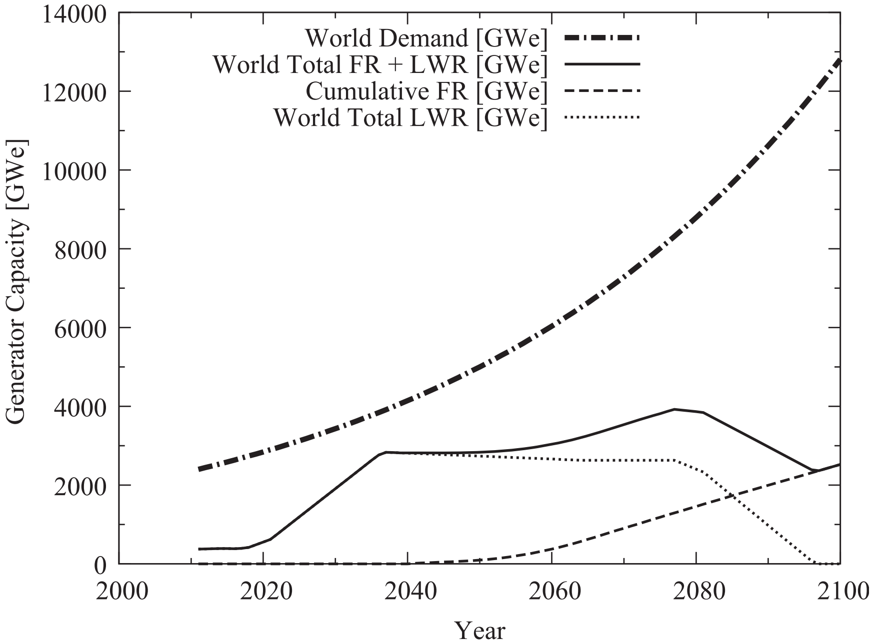

Fig. 1 World capacity from LWR and fast reactors

Discussion

The physical demand scenarios here show that within decades, large increases in nuclear electricity generation can exhaust an integer multiple of the world's known reserves of easily extracted uranium, yet this may not even briefly decarbonize the world's electricity supply. This problem could be ameliorated if large new discoveries can be made of easily extracted uranium (beyond the doubling of presently known U reserves, which the WNA already predicts) or if increasing shortages of U will justify the mining of presently uneconomic grades of U ore (for example, the enrichment tails of past U extraction). Both arguments are made in MIT [**8**], but the analysis earlier in this section suggests that the economically extractable supply of uranium cannot continue to grow exponentially, whereas the demand for electricity can grow exponentially for centuries, as can the supply of FR plutonium. Therefore, the sustainability of nuclear decarbonization must sooner or later depend on FR.

This completes the more technical analysis of the purely physical demand for uranium for decarbonization. The following section models (in less technical detail), the economic choices for profit-maximizing investors to invest in decarbonization.

5 ECONOMIC DEMAND

5.1 Optimal development of once-through uranium and its handover to FRs under varying levels of carbon tax

Microeconomic assumptions

The level of carbon tax required to trigger disinvestment in gas generation and rapid investment in nuclear energy is widely thought to be of the order of £100 per ton (∼\$150 per ton), which may cause a 40 per cent jump in the cost of gas-generated electricity. This sharp and permanent jump is outside the statistical variation of existing market prices. This should cause rapid investment in UB until nuclear supply meets the total conventional electricity demand. Beyond that level, excess electricity supply will not be sellable.

Assumptions
Define today's 'conventional electricity demand' as the world's total electricity demand at the time of writing (roughly 2400 GW) and assume that 'conventional electricity demand' will grow at its historic 1.9 per cent. Alternative assumptions are easy to implement. For simplicity, it is assumed that full decarbonization requires zero fossil contribution to the growth of conventional electricity demand (this assumes that additional electricity supplies from intermittent renewables may supply a further 40 per cent of conventional electricity demand, and that this can be absorbed by decarbonizing road transport using vehicle batteries or other storage to decouple the intermittencies of renewable supply and transport demand).

The model
Consider two kinds of nuclear reactor capacity, G_1 is the total number of world reactors for once-through UB, and G_2 the total number for FRs. Capacity in each is a continuous variable [units: GWe], an acceptable approximation at the world level. The once-through reactor capacity G_1 consumes a finite stock Q_1 of uranium at the rate UG_1, and in doing so produces plutonium at the rate P_1G_1. Then, the time evolution of the stock of U, Q_1 is governed by

$$\frac{dQ_1}{dt} = -UG_1(t)$$

Assume that the time and cost to extract the plutonium from used fuel is not a binding constraint, and that the cost of Pu extraction is included into the annual operating cost θ_2 for FRs (units: [$ per GW year]) so raising this cost above the annual operating cost θ_1 of the once-through reactors. The G_1 reactor capacity does not voluntarily use reprocessed fuel due to its high cost.

The capacity of either type of reactor can be changed, but only with a lagged effect. Approximate this by specifying a 'target level' H_1 [units: GWe] for capacity G_1, then the actual level of reactor capacity G_1 moves at a rate κ_1 proportional to the difference between G_1 and H_1. Hence, we have

$$\frac{dG_1}{dt} = \kappa_1(H_1 - G_1)$$

The rate of change of G_1 can be bounded to reflect a constrained rate of reactor production. At each instant, a cash capital cost π_1 must be paid at the rate that the actual capacity is changed, namely $\Delta G_1 = \kappa_1(H_1 - G_1)$. In general, there are different capital costs π_1^+ and π_1^- to reflect different capital costs of $\Delta G_1 > 0$, investment, and $\Delta G_1 < 0$, decommissioning. The resulting cost over the instant dt is therefore $\pi_1 \Delta G_1 dt$. At each instant in dt, the value of ΔG_1 is a control variable and may be chosen so as to maximize the value of the system. Since our main topic of analysis is the time evolution of the stocks of U and Pu, the replacement of the existed stock of reactors is not modelled directly. Instead, replacement cost of reactors is included into the running cost of current reactors. The annual operating cost of G_1 is $\theta_1 G_1$ [units: $].

Similarly, a 'target level' H_2 [units: GWe] is specified for capacity G_2. The actual level of reactor capacity G_2 moves towards H_2 at the speed κ_2, resulting in the equation

$$\frac{dG_2}{dt} = \kappa_2(H_2 - G_2)$$

Again the rate of change in G_2 is bounded, and the different capital costs π_2^+ and π_2^- reflect different capital costs of $\Delta G_2 > 0$, investment and $\Delta G_2 < 0$, decommissioning. The actual capital cost of the actual change in capacity achieved is $\pi_{G_2} \Delta G_2$. At any instant, ΔG_2 is a control variable. The annual operating cost of G_2 is $\theta_2 G_2$ [units: $]. Assume that all available reactor capacity will be used up to a maximum of total conventional electricity demand D (Units: GWe) since nuclear reactors always prefer to operate, and in all but a small region of the problem space they are insensitive to fuel cost.

The use of an FR generates surplus plutonium at the rate P_3G_2. However, to start off an FR requires a one off charge of plutonium at the rate $P_2\kappa_2(H_2 - G_2)$. Therefore, the evolution of the stock of Pu may be expressed as

$$\frac{dQ_2}{dt} = P_1G_1 - P_2\kappa_2(H_2 - G_2) + P_3G_2$$

Now define $V(Q_1, G_1, Q_2, G_2, t)$ as the NPV of all future nuclear generation by both UB and FR. The rate of positive income per GWe of capacity per year is defined as the selling price of electricity S, multiplied by the number of reactors, multiplied by the utilization factor (0.85 in the following results). Then, by standard assumptions in financial mathematics – no arbitrage and continuous hedging – and standard operations – applying Taylor series to V and approximating to first order – yields the following partial differential equation for V

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \kappa_1(H_1 - G_1)\frac{\partial V}{\partial G_1} - UG_1\frac{\partial V}{\partial Q_1} \\ &+ \kappa_2(H_2 - G_2)\frac{\partial V}{\partial G_2} + (P_1G_1 - P_2\kappa_2(H_2 - G_2) \\ &+ P_3G_2)\frac{\partial V}{\partial Q_2} - rV + f(S, G_1, G_2) = 0\end{aligned}$$

where

$$f(S, G_1, G_2) = S\min[G_1 + G_2, D(t)] - \theta_1 G_1 - \theta_2 G_2 - \pi_1 \Delta G_1 - \pi_2 \Delta G_2$$

and $D(t)$ is the current global demand. The function f defines the cash flows in the system, which are explained in more detail later in this section.

The aim here is to maximize the value of V over all possible paths in the state space by choosing the optimal values of investment or disinvestment at each and every point in the state space. This is therefore a classic 'differential game', and can be stated as a Hamilton–Jacobi–Bellman equation and can therefore be solved using the Bellman principle [**18**].

Normally, one might choose to include the parameter S, the selling price of electricity as a stochastic variable. However, it is quite obvious that the choice of S will dominate the problem – too small an S means nuclear investment will cease or (even lower) disinvestment begins. A large enough S means that one or both of G_1 and G_2 will be expanding at the maximum rate. Only over a small range of S will the solution be highly dependent on small changes in S. Outside this range of S, the value V is dominated by other terms in the equation. Therefore, the required leve of S (for any policy-relevant solution) is estimated outside the model and is supplied as a parameter.

5.2 A simplified model for UB reactors only

Here, the simplified model is considered, which merely optimizes the 'end game' of once-through generation $G = G_1$ where Q is finite and ignoring residual plutonium stock Q_2 as a by product of no value.

Select $H(G, Q, t)$ to optimize the NPV V of a particular set of assets (reactors G and uranium stock Q) at a particular time t. More mathematically, the V to be maximized is the function $V(t, G, Q)$ at every (t, G, Q) point, which is achieved by selecting the jointly best values of H at every (t, G, Q) point. The equation that governs V is a first-order PDE

$$\frac{\partial V}{\partial t} + \kappa(H - G)\frac{\partial V}{\partial G} - UG\frac{\partial V}{\partial Q} - rV + f(t, G, Q) \qquad (1)$$

Here, $H(t, G, Q)$ is the target for reactor output, f the cost function described below, and κ, U and r constants which are also described below. The cost function f is given by

$$f(t, G, Q) = SG - \theta G + \pi \Delta G \qquad (2)$$

where S is the selling price of electricity (assumed fixed), θ the rate of operating cost of running a gigawatt of reactor capacity per year, and $\pi\Delta G$ an asymmetric term representing the cost of commissioning or decommissioning reactor capacity. The parameters used here are stated in Table 1.

Equation (1) must be solved over a selected region in the (t, G, Q) state space. Obvious constraints are that both $G \geqslant 0$ and $Q \geqslant 0$. The world's present nuclear generating capacity is of the order 370 GW and it seems that scenarios where this grows beyond 4500 GW are unrealistic, so the maximum capacity is bounded by $G_{max} = 4500$. There is some obvious behaviour along the boundary given by $Q = 0$. Here, the NPV is completely determined by the cost of operating and decommissioning any existing reactors. To simplify things, it is assumed that the yearly cost of operating a reactor does not depend on whether it is burning uranium. It would be straightforward if needed to split operating costs into fixed costs and costs incurred by the actual burning process.

In the region close to $Q = 0$, note that at the start of a year there may not be enough uranium left to fuel the reactor for a whole year. To compensate for this, assume a sales income of $\frac{Q}{182G}S$ which will correct the year's revenue since uranium is burned at a rate of 182 tons per GW year. At the terminal time (here $t = 200$ years), assume that the only cash flow is the cost of owning any remaining reactors. This means $V(200, G, Q) = f(200, G, 0)$ which is an affine function of G. The above bounds on G, Q and t can be varied as needed to set an economically relevant solution region.

The rate of change in reactor capacity is given by

$$\frac{dG}{dt} = \kappa(H - G)$$

this rate is included in the dynamics of V given by equation (1). It is unrealistic to build or decommission reactors arbitrarily quickly, so these terms are bound by some functions of time. The following results assume either a linear or a quadratic growth rate in the number of reactors that can be built each year. Devising economically optimized growth rates for future reactor production is a subject for future work.

5.3 Results

5.3.1 NPV at initial time of investment

The model's full solution is a five-dimensional (5D) object (H, V, G, Q, t) of which plots may display only evolving 3D subspaces (surfaces) or 2D subspaces (line trajectories). Figure 2 shows the NPV surface V of the system at $t = 0$ over various starting values of G,

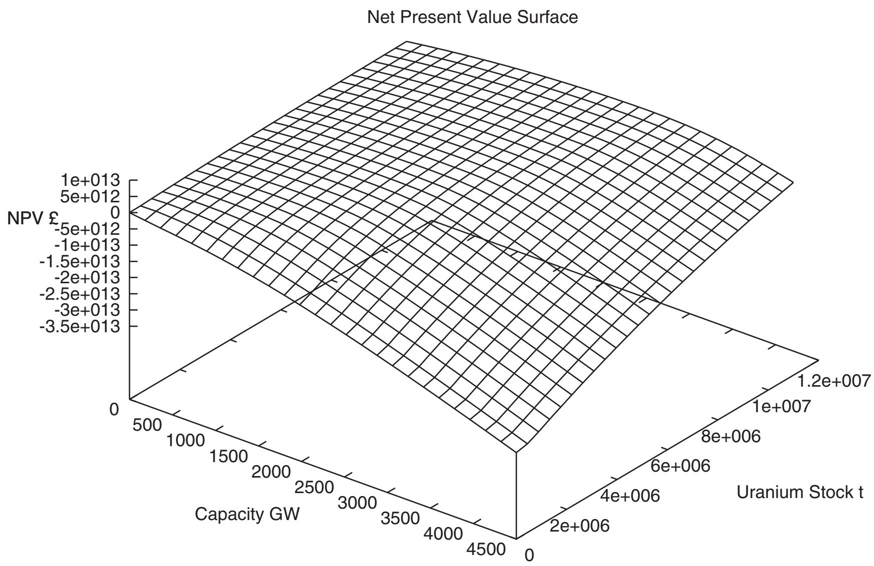

Fig. 2 The NPV of the system at time $t=0$ when $S=1.5\,\theta$ and $Q_{\max}=11\times10^6$ tons

Q assuming $S=1.5\,\theta$ and uranium stock $Q=11\times10^6$ tons. This surface was computed at 2500 points in (G, Q) and at each of 200 values of t, so its evolution through t can in principle be shown as a 3D movie. The value $V(G, Q, 0)$ is the expected value of starting the system at time $t=0$ with the given combination of V, Q (both endowed at zero cost) provided that the optimal policy $H(V, Q, t)$ is followed at every G, Q point reached for all t. A 'movie' could also be created for the optimal policy surface $H(G, Q, t)$.

On the V surface displayed here, $V(0, 0, 0)=0$ because, given no uranium and no existing reactor, it is pointless to invest. Along $V(G, 0, 0)$, every reactor already owned at time $t=0$ reduces V because it cannot be fuelled and its decommissioning costs must be paid. Along the boundary $V(4500, Q, 0)$ where 4500 GW is the maximum reactor capacity modelled, every extra ton of uranium Q raises value by earning some income before decommissioning costs. Along most of $V(G, 1.2\times10^7, 0)$ value V is fairly flat. This is not because all generating capacities G are equally desirable, but because the optimal trajectories from this set of G values converge rapidly towards a single, globally optimal G, Q, t combination.

Note that at $t=0$, only one V point in the plotted G, Q domain is factual – hypothetically for example, assuming WNA known reserves, $V(370, 5.4)$. The V values at other presently non-factual G, Q points are calculated anyway as the model works backwards from all possible future states and decisions. Many of the non-factual points can become factual, for example if an accident sharply reduces G, or if new discoveries increase Q. In either case, $H(G, Q, t)$ at the new G, Q point is the required optimal control action, and V at the newly factual point on the $V(G, Q, t)$ surface is the expected value of following the new optimal trajectory. Comparing $V(G, Q)$ between the new and old G, Q points shows the net value gained or lost by the change in G, Q, t, provided this change is responded to by the newly optimal sequence of control actions.

5.3.2 Optimal investment trajectories through time

The foregoing Fig. 2 showed the NPV of all combinations of reactor capacity and U stock Q at a fixed time. To generate this, the optimization model must simultaneously compute, backwards in time, the complete 'movies' of $V(G, Q, t)$ and $H(G, Q, t)$ surfaces through t. These surfaces jointly define (and value) the optimal trajectory from every state point (G, Q, t).

It is often useful to work from a single G, Q, t state point, known to be factual, and to inspect the whole of the optimal trajectory from that point, plotting forward against time one or more of: the optimal control action H, capacity G, remaining reserves Q, and the remaining NPV V. This can be done from the realistic point ($G=370$, Q, $t=0$) assuming various values for the presently unknown Q. Any of the model's other fixed parameters, notably the selling price S, can be changed to produce a different trajectory.

Figure 3 has been optimized under a dynamic model of the lagged effects of control, in which the control variable H is a 'target' for capacity, towards which actual capacity G converges exponentially, subject to a maximum annual rate of reactor build

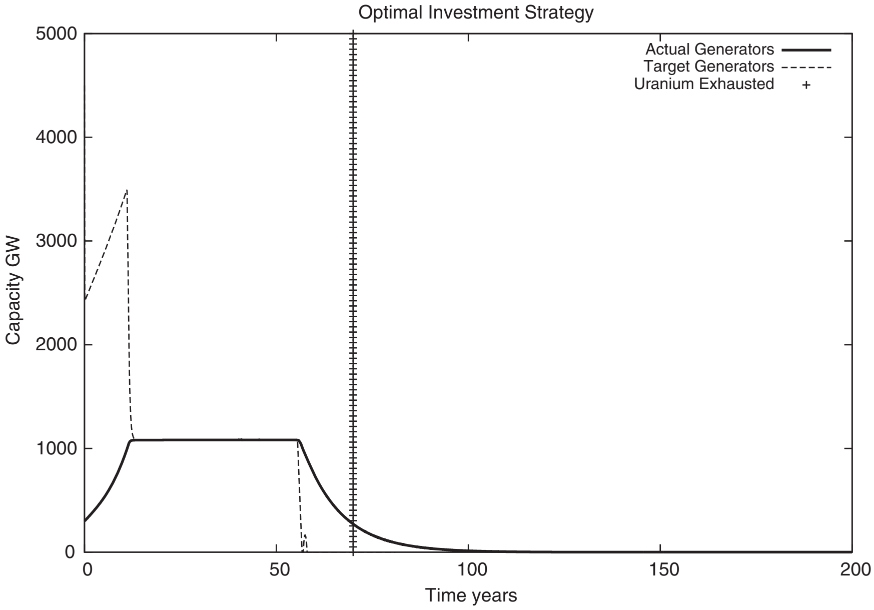

Fig. 3 Optimal target and actual investment strategy for $S = 1.7\,\theta$ and $Q_{max} = 11 \times 10^6$ tons

and of decommissioning. This figure also assumes a quadratically growing limit to the growth in number of reactors. The quadratic model was selected after a sensitivity analysis to the maximum build rate. There is no space to report sensitivity analysis in detail, but for a given U stock, a low maximum annual build rate of reactor capacity constrains the total amount of capacity that will ever be built. The intuition for this is that in order to repay an investment, the last reactor built should remain operating for some decades. Hence, one should ideally stop building reactors well before half of the fuel stock has been used. A slow uniform reactor build rate takes a longer time (and more important a larger cumulative usage of U) to reach any given number of reactors. However, it is not realistic to assume there can be an instantaneous jump to fast uniform build rates at any future time.

Accordingly, all the results presented here assume a growing maximum build rate. This model assumes that fresh facilities for manufacturing reactors are laid down at a quadratically growing annual rate (defined as $\kappa_{max}(t) = 0.75(t-2010)^2 + 40$), leading to a cubically growing cumulative output of reactors. The joint optimization of reactor manufacturing and reactor usage is left for future work.

Figure 3 assumes starting values of G, Q as (370, 11) and the sale price of electricity (per GW year) is fixed at a relatively low level $S =$ \$33.8/MWh (or £20.94/MWh). In all of Figs. 3 to 6, the fine line plots the currently optimal target capacity H, the heavy line shows the resulting optimized trajectory of active capacity G and the vertical line of crosses marks the time when the uranium stock Q reaches zero.

The qualitative features of Fig. 3 are found in all optimal solutions: the target capacity H first stays above actual capacity G, dragging it upwards at the maximum allowed rate. When it becomes optimal to stop growing, the target H falls rapidly but smoothly to coincide with actual G. Capacity is held constant for a while, then starts to fall (pulled down as H falls to zero) before Q is exhausted. Most of this figure's initial Q of 11 M tons of U (similar to WNA's [**7**] estimate of U reserves extractable at up to \$120 per ton) is used up within 70 years. The rate of decline of capacity G before and after exhaustion strikes an optimal balance between maximizing the NPV of income and minimizing the NPV of operating losses and decommissioning costs (both before and after exhaustion).

To show the effect of extremely high price, in Fig. 6, the electricity price is trebled from 1.7 times annual operating costs to 5 times (\$99/MWh or £61.58/MWh at 62 p/\$) leaving other parameters unchanged. At this price, profit-maximizing investors build 3300 GW of capacity, decarbonizing the expected demand of 34 years ahead for about 5 years. However, most reactor lives are inefficiently short, well below 30 years. These effects suggest that very high carbon prices (alone) are a practically inefficient way, as well as a politically intolerable one, to enforce decarbonization. Hence, solutions are sought at a lower price (lower carbon tax) but with a higher perceived uranium stock and, if needed, a faster allowed rate of reactor build.

Figure 4 raises the opening uranium stock Q to 30 M tons for the same low price as Fig. 3. The plotting conventions are unchanged. Comparing with Fig. 3, trebling the available stock (from 11 to 30 M tons) has induced investors to raise their peak capacity from 1100 to 2800, sustained for about 50 years. This requires the world to identify roughly six times the presently identified reserves of U within 6–10 years.

Figure 5 leaves the sale price of electricity at $S = \$33.8$/MWh (or £20.94/MWh) and further raises opening Q to 50 M tons. Peak capacity rises from 2800 to 4000 GW, and it is sustained for 50 years, between year 30 (when FR may become available) and year 80, before exhausting the reserves in year 93. This seems adequate for brief decarbonization but it requires the world to find nearly ten times the presently identified stock of U in the next 10 years (to meet a 10-year lead time on the first very large orders). Approximate trebling of the extraction cost of uranium, which such a high cumulative output is likely to cause, is not allowed for.

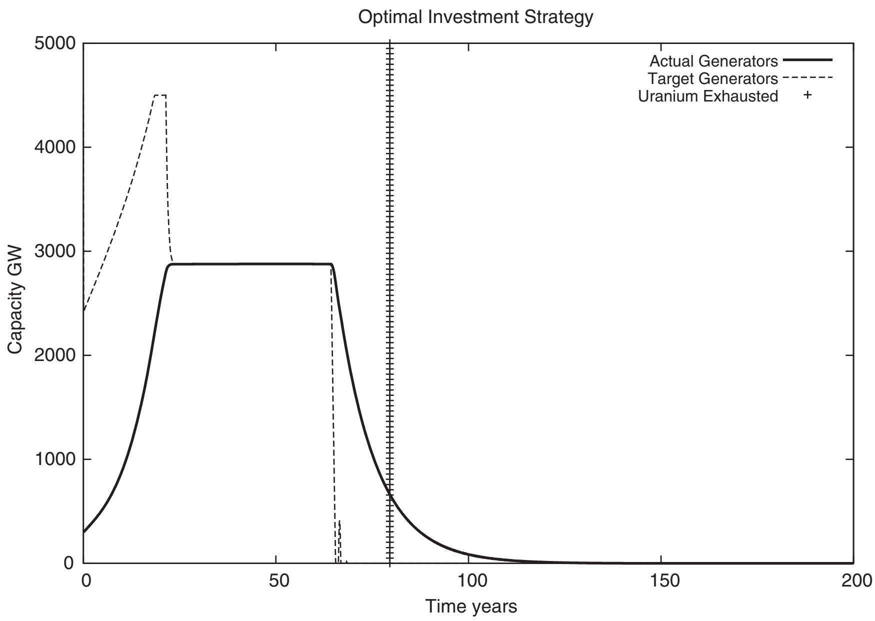

Fig. 4 Optimal target and actual investment strategy for $S = 1.7\,\theta$ and $Q_{max} = 30 \times 10^6$ tons

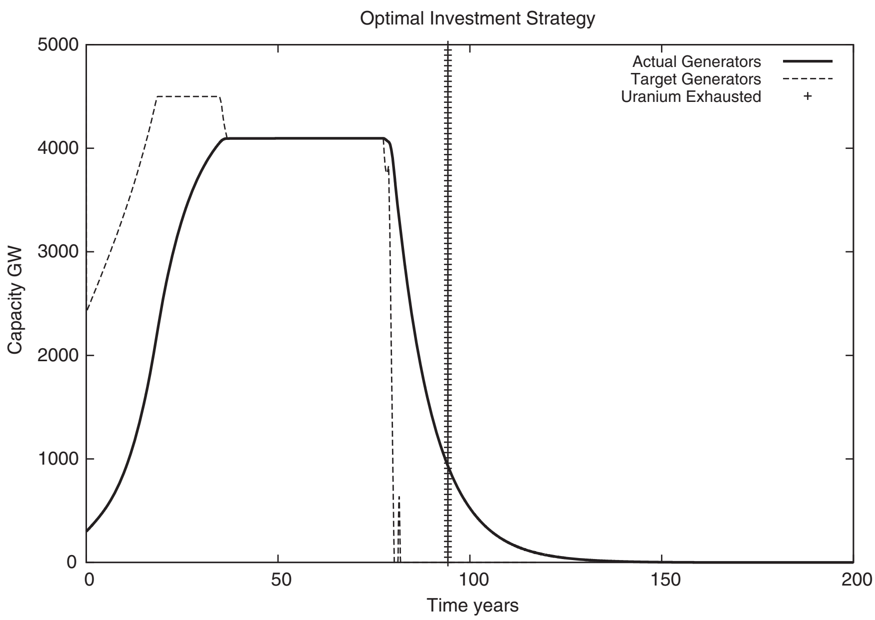

Fig. 5 Optimal target and actual investment strategy for $S = 1.7\,\theta$ and $Q_{max} = 50 \times 10^6$ tons

5.4. Hand over to FRs

For this article's last result, Fig. 7, the problem of Fig. 1 is revisited, applying its optimization method to a yet larger U supply at a high selling price of \$59.6/MWh or £37.5/Mwh (at 0.63 £/\$) although in fact G_2 can cover its capital costs at approximately \$45/MWh or £27.9 (at 62p/\$). A higher price is needed to motivate the successor FR technology, which is assumed to be 40 per cent more costly than UB. However, a 'nuclear waste tax' on UB, or a 'nuclear waste disposal credit' on FR would reduce the difference between the two. The value of U reserves of 35 M tons is maximized subject to a linearly growing limit on the annual rate of growth in reactors ($|\Delta G_1|$, $|\Delta G_2| < 15(t+1)$ for $2010 < t < 2030$

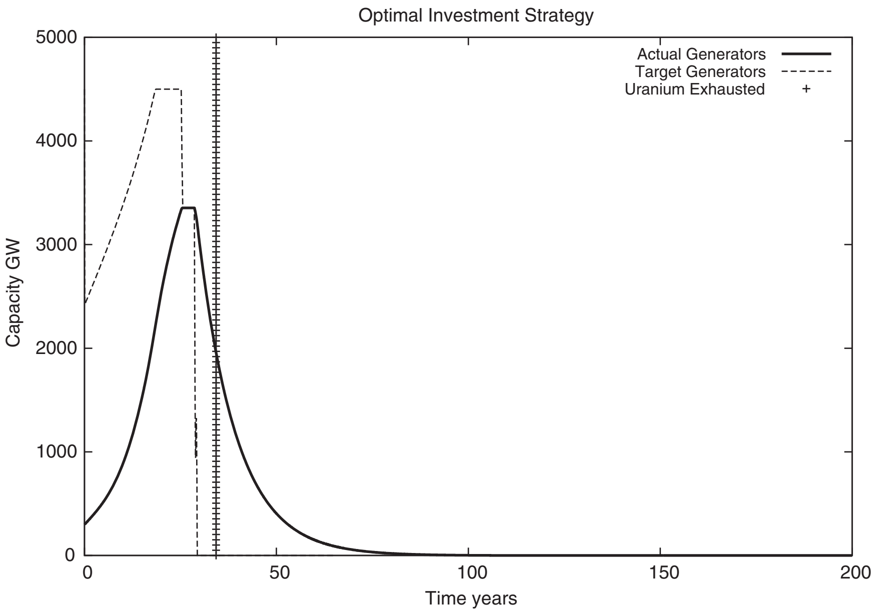

Fig. 6 Optimal target and actual investment strategy for $S = 5\,\theta$ and $Q_{\max} = 11 \times 10^6$ tons

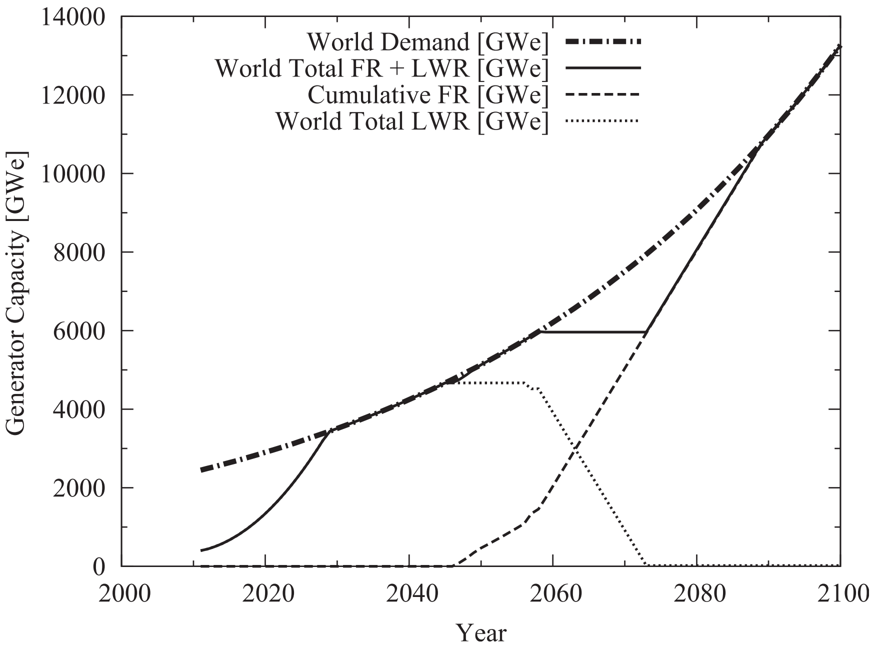

Fig. 7 Potential hand over from UB to FR reactors, after exhaustion of 35 M tons of U. The price of electricity of chosen as $S = 3\,\theta_1$

Table 9 Findings for miners in BAU scenario

Year	Reactors	Yearly U usage (tons)	Cumulative 21-year usage (tons)	Discoveries needed (tons)
1	360	62 640	1 598 195	91 272
5	388	67 538	1 755 904	98 408
10	426	74 203	1 823 282	108 119
20	515	89 570	2 328 700	130 510
41	764	132 990	3 393 104	193 777

Source: authors' calculations using assumptions stated in text.

Table 10 Findings for Reactors in BAU scenario

Year	Reactors	Yearly U usage (tons)	Cumulative U usage (tons)
1	360	62 640	62 640
5	388	67 538	325 330
10	426	74 203	682 763
20	515	89 570	1 506 924
50	905	160 532	5 152 198
71	1344	238 352	9 248 000

Source: authors' calculations using assumptions stated in text.

Table 11 Findings for miners in accelerated DECARB scenario

Year	Reactors	Yearly U usage (tons)	Cumulative 21-year usage (tons)	Discoveries needed (tons)
1	360	62 640	3 158 492	291 962
5	490	85 221	4 297 093	397 212
10	720	125 218	6 313 840	583 634
20	1554	270 336	12 431 390	888 957
41	5206	905 847	23 111 762	1 295 282

Source: authors' calculations using assumptions stated in text.

and $|\Delta G_1|, |\Delta G_2| < 300$ for $t \geqslant 2030$), and to an exponentially growing demand for electricity. This model uses the more costly FR heuristically, to 'plug any gaps' in supply after optimal use of the more profitable UB.

The solution achieves very rapid decarbonization, but there is a 15-year deficit in total supply between 2060 and 2085. It is hard to plug this by any plausible growth rate of FR, from its assumed commercial introduction in 2040. The supply deficit could be closed by a larger uranium supply, of the order of 40 M tons by 2080 (compare with the maximum 50 M tons modelled in MIT [**8**]). Alternatively, waste reprocessing, which is needed for FR anyway, might extend the life of UB. More work seems needed in this area.

Discussion and policy implications

Three main variables seem to dominate the path along which profit-motivated investors would expand UB capacity rapidly towards decarbonization (if at all), namely: electricity price, economic uranium reserves and the maximum build profile of reactors. Hence, if world governments decide to decarbonize using UB, their policy must rationally allow for the effects of all three variables on profit-maximizing behaviour, not only by the UB generation companies, but also, as found, by reactor manufacturers and uranium miners. Here, profit has only been explicitly maximized on behalf of UB generation companies, but similar issues face reactor manufacturers.

Price is a potential policy variable (for example by imposing a carbon tax). The results show that beyond a minimum price threshold, needed to trigger any investment in UB at all, the effects of further price increases are small, and are dominated by the size of the available uranium reserves (these must be large enough to give long enough operating lives to a fleet of reactors large enough to achieve decarbonization) and by the maximum available build rate of reactors (slower build rates not only delay decarbonization per se, but also consume more of a finite U reserve in doing so, thus encountering sooner the inability to give the next reactor a long enough life).

On reactor availability, these results have assumed that reactor capacity has a fixed price, whatever the time pattern of its delivery, and that manufacturers will be willing to supply reactors at a linearly increasing rate (quadratically increasing cumulative output). Clearly, a linearly expanding output rate requires a constant annual rate of investment in facilities, ideally terminating early enough to give to the last-built facilities a long life. Most of these results do not assume such early termination, and they may, therefore, overstate the ease of reaching decarbonization. Future work should clearly try to combine the dynamics of reactor supply and reactor use, and if possible jointly optimize them.

On uranium supply, these results show a minimum requirement of around 35 M tons to decarbonize even briefly. It is (trivially) essential to be able to foresee fuelling a new UB reactor throughout its life. These examples require very rapid investment in reactors, mostly within the next 25 years. Hence, if there is a 10-year lead time on building a reactor, and a 10- to 20-year lead time on finding and developing a new uranium mine, uranium prospecting faces a steep climb to identify within the next 10 years seven times the resources that have been discovered in the previous 60 years. This raises optimization decisions for mining companies themselves: they have other opportunities, are a small part of the nuclear system, and have a put option to leave the nuclear business without penalty (see Appendices 1 and 3).

6 CONCLUSIONS AND FUTURE WORK

If the world decides to invest in nuclear power, the practically important policies lie between BAU (nuclear power grows with background electricity demand at 1.9 per cent) and rapid decarbonization by 2050. This article's results on the physical dynamics and economics of uranium mining, reactor production and reactor purchase suggest the following.

1. To decarbonize the world electricity supply by 2050 would consume the whole of the 11 M tons of uranium which the WNA suspects to exist at extraction costs up to \$120 per lb. Without fresh uranium discoveries, or suitable alternatives, a rapidly built reactor fleet would have no fuel for 2051.
2. Dynamic optimization results suggest that rational investors in nuclear reactions would need a much larger guaranteed world uranium supply, exceeding 30 M tons, in order to justify ordering and building a reactor fleet large enough to achieve decarbonization (but to sustain it for only a decade or two).
3. The economic results suggest that nuclear investors do not need, and will not respond to, prices much above the minimum required to achieve a positive investment return. The effective bound on the total capacity that investors are willing to buy is not set by price but by (a) the guaranteed uranium supply, as already noted in 2, above, and (b) the maximum rate at which suppliers can manufacture reactors.
4. Because uranium supply sets an economic (as opposed to a physical) bound on reactor investment, and because there is a 10- to 20-year lead time for finding and opening new uranium mines, decarbonization requires a huge uranium prospecting drive over the next 10 years, in order to raise identified reserves from 5.5 M tons to (order of at least) 30 M tons. Nuclear investors need to see these reserves before they can optimally place orders for enough reactors to achieve decarbonization (orders which have a 10-year lead time). The incentive to prospect is not symmetric between nuclear generating companies and uranium mining companies. If a uranium mining company has a newly opened mine, it has a 'put option' to sell the mine's existing uranium over a 20-year horizon and exit the uranium business with no further cost. In contrast, if a nuclear investor has a newly built reactor, this must be operated for 60 years to recover its cost, during which it depends on outside fuel supplies. The asymmetric motives and bargaining powers of uranium miners and reactor owners may lead some generating companies to acquire uranium mining companies, as China seems to be doing.
5. Because the rate of reactor supply also sets an economic bound on decarbonization, rapid decarbonization needs a crash programme of reactor building (which may act as a stimulant for the world economy). In this article's models, the reactor manufacturing companies are assumed to invest in reactor production facilities at a constant or an increasing rate, leading to a quadratic or cubic growth in the total reactor fleet. Future work needs to investigate how reactor manufacturers could actually maximize profit over a complete trajectory of investment and disinvestment in manufacture. This should be jointly optimised with the generating companies' trajectories for buying and operating these reactors.
6. Any delay in the development of the required nuclear fleet increases the total quantity of uranium required to decarbonize and the cumulative output of CO_2 before decarbonization.
7. Nuclear investors need protection from fossil competitors, led by CCGT gas, the cheapest and lowest in CO_2 emissions. The best taxation policy would be to introduce gradually a carbon tax on new fossil reactors, which is low enough to keep existing gas reactors in use, but high enough to prevent their replacement. In this way, the fossil component withers away over the life of a fossil reactor (typically 30 years). However, no existing fossil generator should be able to pass on the full tax, because the nuclear or other competitors which expand to replace the retiring fossil capacity should have costs below those of existing CCGT, after it is uplifted by a 40 per cent or so carbon tax (around £100 per ton) designed to make the replacement of gas capacity uneconomic.
8. If the world chooses to decarbonize by 'once-through' UB, sustained growth of world electricity demand at 1.9 per cent would take the cumulative uranium required to 227 M tons by 2150. This article's most optimistic forecast of long-term uranium supply and costs suggests that at a cumulative output of 50 M tons costs would have trebled, and between cumulative outputs of 90 and 190 M tons costs will be increasing rapidly beyond 10 times the present costs. In the worst case, if sandstone deposits prove costly to mine, or offer few opportunities for economies of scale, uranium costs may rise sharply after cumulative outputs of only 11 M tons. Hence, in the best case, a successor fuel cycle to UB may be needed within

100 years, and in the worst case in less than 50 years.

9. Possible successor fuel cycles include uranium-based FRs, the use of reprocessed uranium fuel, and/or reprocessed enrichment tails in thermal (once through) reactors, and thorium fuel cycles. All are subjects for future work. This study's results suggest that a smooth handover to any successor fuel cycle may not be easy to design, even in an economically sub-optimal way. It is noted that while FR cycles have 'extra costs', including fuel reprocessing, this fuel reprocessing eliminates much of the ongoing problem of nuclear waste storage. Hence, FR investors would be taking on a cost which present nuclear industry economics leave unassigned. This may give the FR cycle a strong appeal to countries with large nuclear waste legacies, and FR investors might campaign to receive a negative 'waste storage' tax.

FUNDING

This study was carried out as part of the SPRIng project, aimed at developing a decision-support framework for assessing the sustainability of nuclear power, led by the University of Manchester and carried out in collaboration with City and Southampton Universities. The authors acknowledge gratefully the Engineering and Physical Sciences Research Council and the Economic and Social Research Council for supporting this study. The views expressed in the article are those of the authors and not necessarily those of the SPRIng project.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the advice of Adisa Azapagic, two anonymous referees and Juliann Tilley.

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APPENDIX 1

Targets for the rate of prospecting needed to ease uranium constraints

Two contrasting scenarios for nuclear energy growth are examined and extra questions of each are asked.

1. What annual rate of new discoveries of uranium is needed to sustain each scenario's opening and/or closing growth trend indefinitely?
2. What existing stocks of uranium are needed either to reassure a miner that it can replace its mining capacity within a lead time of 20 years, or reassure a generation firm that it can operate a newly acquired 'once through' uranium reactor to the end of its physical life?

The first scenario (BAU) assumes that nuclear energy maintains its present 15 per cent share of world electricity production, growing indefinitely at 1.9 per cent. This allows CO_2 emissions to grow at 1.9 per cent. The second scenario (a slightly accelerated DECARB)assumes growth at 8 per cent compound for 33 years to achieve early decarbonization of 100 per cent of conventional electricity demand.

Mines and generation firms have different economic criteria. A mining company takes around 20 years to develop a new mine, so the firm must at a minimum have reserves to supply the next 20 years, all being developed at the correct rates, and it must discover new reserves each year to supply each fresh 20th year ahead.

In contrast, UB reactors are among the longest lasting of all capital assets – up to 60 years of operation, preceded by around 10 years of planning and construction. A long life can exploit an operating nuclear reactor's large positive cash flow and defer the costs of decommissioning. Hence investors in nuclear reactors need an adequate uranium supply over a 70-year lead time, both for their own reactor and all subsequent increases in nuclear capacity that may compete with their own new reactor for uranium.

Method

From initial UB capacity of $G_1 = 360$ GW, project the stated growth rate of capacity and deduce uranium needs $UG_1(t)$ for each future year. Sum required reserves in each of the next 20 years to detect total reserves over lead time from discovery to working mine. Take actual output in year $t+20$ as required rate of new ore discovery in year t.

Findings

Under the BAU capacity growth scenario (1.9 per cent pa) miners need in year t to discover 1.46 times as much as the uranium ore that they consume $UG_1(t)$ (this ratio is constant over time). The minimum reserve that the mines need at any time, in order to supply demand over the next 20 years, grows indefinitely, rising from 1.5 M tons in year 1, to 3.4 M tons in year 41 (this is always 25.5 years of the current demand).

Consider in contrast a UB generating company, also in BAU, which runs a 10-year reactor building project (over years 1 to 10) intended to operate for the 60 years from year 11 to year 71. The company would wish to see year 1 world uranium stocks large enough to fuel the new reactor for the whole of its life, along with any subsequently built reactors envisioned in the scenario. The needed stock is 9.4 M tons, or nearly twice the present stock of 5.5 M tons. Hence, every new nuclear generation investment must take on trust the discovery of significant future reserves.

The accelerated DECARB scenario likewise also has different impacts on mines and reactors. The uranium mining companies need a minimum starting stock of 3.5 M tons to supply the next 20 years (this could supply 86.8 years at the year 1 rate of usage). Then, during the 8 per cent growth phase, they must discover each year about 4.66 times as much as they consume. The required discovery rate per year tails off during the phase of slow growth after the dash to decarbonization, falling to 1.46 times the current output rate, just as in the scenario of perpetual growth at 1.9 per cent.

In the fast-growth scenario, a nuclear energy generating company needs to see very large uranium stocks in year 1 if it wants a guarantee that existing stocks can fuel all existing and likely new reactors up to a 71-year horizon, justifying the investment. This requires a starting stock of 52.8 M tons, nearly 10 times presently known world stocks (no table presented).

Clearly, neither miners nor reactors have an unconditional commitment to make this scenario be realized as planned. We have discussed in the main text what incentives are needed to make miners, nuclear reactors, and fossil reactors willingly take actions in line with the scenario.

Discussion

If the world starts at the time of writing a sudden dash for nuclear decarbonization at 8 per cent compound growth, present world reserves are enough to fuel the first 20 years of usage, as mines require. If during those 20 years the mines fail to find enough fresh economically extractable ore, the mines are not obliged to carry on mining. However, if for any reason, the owners of an operating nuclear reactor cannot charge enough to recover their operating costs, they have no option to terminate their losses or their liabilities to decommissioning expenses. If there is a dash to decarbonize, new investors in UB will have to take on trust the discovery over the next 50 years of nearly 10 times the world's presently known stock of easily extracted uranium.

If the actual rates of uranium discovery fall below this target, the fast nuclear growth strategy would become technically infeasible. In that case, rational nuclear investors should at some point cut back building plans, to allow all reactors already built to have profitably long operating lives. How soon investors would actually cut back their plans in this situation is uncertain, but in section 4.2, we give an optimizing model for such an end-game, assuming a reasonably accurate estimate of the reserves of economically usable uranium.

APPENDIX 2

Feasibility of finding large extra uranium supplies at acceptable cost

Method

Re-analyse the Deffeyes diagram as follows.

1. Identify the total tonnage T at each ore grade Y in successively poorer formations.
2. Compute per formation the tons T_O of ore extraction needed per ton of U where $T_O = 1/Y$. Compute tons of extraction per ton of U relative to extraction rate at assumed current ore grade $Y_{M,0} = 0.7$ per cent or 700 ppm as $Y_{M,0}/Y$. Compute scale effect of higher extraction rate per ton of ore: unit cost per ore ton extracted $= (1/Y)^{s_E}$.
3. Compute relative effect of ore extraction cost at lower yield, relative to extraction cost per te of U, taking advantage of scale improvements in extraction, namely

$$C_E = (Y_{E,0}/Y)^{s_E}$$

4. Similarly, per formation, compute the milling load per ton of ore (relative to present load at present higher yield) $= (Y_{M,0}/Y)$
5. Square this and apply the milling scale power s_M to give the resulting total relative milling cost per te of UC_M where

$$C_M = (Y_{M,0}/Y)^2 s_M$$

6. Assuming that the costs of extraction and milling are presently roughly equal, the net impact of any particular formation's yield Y on costs is $0.5C_E + 0.5C_M$.

Results

In DECARB a cumulative output of 90 M tons is needed to decarbonize by year 2050 and sustain growth to 2200. From Table 12, assuming current rates of uranium output, this would increase uranium mining costs by a factor in the range 5.4–27 (where the higher number imitates the five-fold cost increase at the top end of most WNA estimates). Similarly, the main text suggests that a larger cumulative output of 220 M tons is needed to decarbonize electricity supply by 2050 and sustain its growth at 1.8 per cent from 2050 to 2150 (less than two reactor lifetimes). It can be seen in Table 12 that output above this would drive mining costs up by a factor of 68 (or up to 343 if the typical WNA cost range of 5:1 is used).

The foregoing analysis allows for cost changes due to lower ore yield only, at an unchanged output of uranium. To allow for increases in the rate of uranium output, a final scale adjustment is needed. In DECARB, world gigawattage rises between 2010 and 2100 by the multiple 13 000/370, which is just over five doublings ($5 \log 2 \sim \log (13\,000/370)$). Assuming the same multiplier for the rate of uranium output and a 90 per cent scale curve (reflecting the growing dominance of milling), this reduces the cost multiplier for 90 M tons by a factor of $0.9^5.135 = 0.582$. The resulting expected cost increase per ton of U falls to (the 1:5 range) 3.15 to 15.7.

The largest cumulative U output in the MIT estimate [**8**] is 50 M tons, for which the cost multiplier is estimated at 1.4. The same output is reached in DECARB in model year 72, at an instantaneous output rate of 9132 GW. The resulting scale multiplier is $9132/370 = 24.7$ which is 4.6264 doublings.

Table 12 A reinterpretation of the MIT analysis

Formation	T	Cumulative T	Y	$\frac{1}{Y}$	$\frac{Y_{E,0}}{Y}$	C_M	$\frac{Y_a}{Y}$	$\frac{Y_{M,0}}{Y}$	$\frac{Y_{M,0}}{Y^2}$	C_M	$\frac{C_E+C_M}{2}$
Vein deposits	20	20	0.7	1.43	1	1	7.14	1	1	1	1
Fossil placers sandstones	70	90	0.2	5	3.5	2.40	25	3.5	12.25	8.4	5.4
Fossil placers sandstones	100	190	0.04	25	17.5	7.42	125	17.5	306.25	129.8	68.6
Volcanic deposits	100	290	0.02	50	35	12.05	250	35	1225	421.6	216.8

Source: authors' estimated readings of T and Y from [**1**] plus calculations using models and assumptions stated in the text. NB: The caption 'Fossil placers and sandstones' occurs twice identically in [**1**], as here, implying that otherwise similar geological deposits differ only in their uranium yield.

Raising the scale exponent of 0.9 to this power gives a cost reduction factor of 0.614, leading to a final multiplier of today's cost of $0.614 \times 5.4 = 3.3$.[†]

Conclusions

The costs of uranium start to rise noticeably in the range 20–90 M tons, and threaten to become prohibitive in the cumulative output range 90 190 M tons. Scenario DECARB passes 190 M tons in 2035. Both these limits occur within fossil placer and sandstone deposits, neither of which has yet been mined. There is a further trebling of U cost in the volcanic deposits formation, which raises cumulative U supplies over the range 190–1000 M tons.

These conclusions assume that the successive geological formations all have identical conditions for mining, e.g. from rock hardness, and that the low concentration ore deposits will be of sizes and shapes that permit larger scales of mine operation, to justify the scale economies assumed. A more complete geological study of these factors, including their range of statistical variation, seems very desirable.

APPENDIX 3

Joint optimization of mining and generating and prospecting

Rapid decarbonization by nuclear power would require generation capacity to rise 10-fold by 2050, from 370 to 3700 GW and beyond. Reactor expansion would need an even faster expansion in uranium mining (which is currently depressed while former military plutonium is being used up in MOX fuel for electricity reactors). Therefore, the optimization of a large joint expansion of generation and mining is briefly modeled. Following the Bellman principle of optimization, the normal chronological sequence of mining decisions, to prospect for ore, to develop an ore, discovery into a mine and to operate or expand the mine, is reserved.

Day-to-day mine operating decisions have been reported elsewhere [**19–23**] and here decisions to expand mines are considered. Nuclear reactors and uranium miners have different time horizons (their respective lead times of risk exposure to an existing asset base are 60 and 20 years). As long as ore is plentiful in principle, but temporarily in short supply, reactors can send strong price signals to miners, by bidding up the uranium price five-fold to around \$100 per lb. Historically mines have quickly responded to restore the supply, and with it the long-term price of around \$25 per lb. However, if easily extracted ores ever become short, the generating companies, whose horizons are longer term, might buy up uranium companies, whose exit possibilities are sooner, and manage mining and generation jointly. This appears to be the national policy of China.

The partial differential equation for joint decisions to expand or contract mines and reactors takes the following form

$$\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (M - UG_1)\frac{\partial V}{\partial Q_1} - rV + f(S, G_1, M) = 0 \qquad (3)$$

where $f(S,G_1,M) = S(G_1+M) - \theta_1 G_1 - \theta_M M - \pi_{G_1}\Delta G_1 - \pi_M \Delta M$). Here, G_1 is generation capacity in gigawatt, M mining capacity in tons of uranium per year, U uranium usage in tons per gigawatt year. The control variables are ΔG_1 and ΔM , which allow bounded rates of adjustment in G_1 and M. They can be optimized jointly by maximizing the total system's expected value V with respect to both of them at every point in the solution region. Control actions incur capital costs at the rate π_{G_1} for actual changes in G_1 and π_M for actual changes in M, both in units of money per unit of capacity. As in the main text, it is

[†]The largest cumulative U output in DECARB is 227 M te. The forecast cost multiplier yield used is 68.6. Repeating the above calculation to allow for the increase in U scale: Multiplier for U scale: $32\,841/370 = 88.8$ which is 6.47 doublings. Raising 0.9 to this power gives the U scale adjustment factor 0.506. Therefore, net multiple of today's U cost $= 68.6 \times 0.506 = 34.7$. This suggests that once-through UB becomes uneconomic somewhere in the range 90 to 190 M tons of cumulative U output, since fuel costs will rise from 3 per cent of other operating costs to roughly equal them. This is a severe economic burden.

conjectured that optimization can be iterative between the S, G_1, Q_1, t subspace and the S, M, Q_1, t subspace.

Because a dash to decarbonize by 2050 would exhaust all known reserves, plus the equally large unknown reserves, which are expected to exist but not yet identified, mines would need to take decisions at unprecedented rates on both prospecting, and mine development. The decision to prospect for uranium is a function of not only uranium price, but also the remaining stock of economically accessible ore and the expected physical demand for uranium.

Decisions around prospecting can themselves be split into a time sequence, namely waiting before deciding to prospect (value V_W) and actually prospecting (value V_P). Actual prospecting may be successful, in which case the outcome is a new reserve, increasing the uranium stock Q. The value of successful prospecting is the value V_O of an option to open a mine to exploit the extra Q (this can be calculated using an adjusted version of standard real options methods, in which a finite mine reserve is allowed for). Alternatively, prospecting may find nothing, in which case the value V_O of mining on the site is zero, and the costly process of prospecting has an expiry value of $V_P = 0$.

Optimization of these decisions proceeds in reversed time sequence. One first needs to value the option V_O to open a mine, if a reserve is found (it may not be optimal to exercise this option immediately). This value V_O (which can be related to $\frac{V}{M}$ in the above PDE for the larger system of mining plus generation) provides one of the possible future exit boundary values from the state of active prospecting V_P (the latter's value is also affected by its negative cash flows). The value V_W of the preceding option, to start actively prospecting, is that of a pure waiting game, with no cash flows in or out, until the selling price, ore quantity, and sales rates become suitable to spend on prospecting, which will acquire the value V_P. There may be a time constraint, in the form of a finite licence to explore and develop an area. In this case, V_W is a finite-life option, which is less valuable than 'perpetual' options, to prospect and to develop a mine, at any future time. The options become perpetual if the mining company owns all relevant mineral rights.

The value $V_P(S, Q_1, G, t)$ of prospecting in progress is governed by

$$\frac{\partial V_P}{\partial t} + rS\frac{\partial V_P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_P}{\partial S^2} + (M - UG_1)\frac{\partial V}{\partial Q_1} - rV + f(\epsilon_P) = 0. \tag{4}$$

Prospecting lasts a known finite time, and the value and probability of finding something can be represented by an 'up and in' option, whose positive payoff is the value V_O of an option to exploit a reserve of the size discovered. Its alternative payoff is zero. While prospecting, the option V_P incurs a continuous cost at the rate ϵ_P. This equation omits partial information (e.g. finding suitable geological formations, or uranium at low concentrations) before prospecting confirms the presence or absence of a useful find. The size and likelihood of the positive payoff may stochastically depend on the partial information, which can in principle be added to the PDE.

The preceding option, V_W, to wait before prospecting at all, generates no revenue or costs, so it is an American option (perpetual or finite as appropriate) which can be modelled by a homogeneous diffusion equation. Its value is set by a boundary condition, namely the value V_P of prospecting, which in turn is set by the expected value V_O of the resulting option to open a discovered mine. The option V_O is acquired (with probability less than one) by buying V_P. The NPV V_W of waiting for the right price conditions before starting to prospect is governed by the PDE

$$\frac{\partial V_W}{\partial t} + rS\frac{\partial V_W}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_W}{\partial S^2} - rV_W = 0 \tag{5}$$

with appropriate boundary conditions.

APPENDIX 4

Numerical method

The method of characteristics can be used to solve equation (1). In particular, writing the *total derivative* $\frac{\mathrm{d}}{\mathrm{d}t}$ of V along a characteristic curve as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \kappa(H - G)\frac{\partial V}{\partial G} - UG\frac{\partial V}{\partial Q}$$

we solve

$$\frac{\mathrm{d}V}{\mathrm{d}t} - rV + f = 0. \tag{6}$$

Writing this as a finite difference

$$\frac{V(t + dt, G^*, Q^*) - V(t, G, Q)}{dt} - rV(t + dt, G^*, Q^*) + f(t + dt, G^*) = 0$$

where (G^*, Q^*) is the image of the characteristic curve starting at the point (G, Q) in the state space.

Hence, the scheme used to back-solve our PDE (1) is given by

$$V(t, G, Q) = (1 - rdt)V(t + dt, G^*, Q^*) + dtf(t + dt, G^*) \quad (7)$$

The characteristic curves $\gamma(s) = (G(s), \ Q(s))$ solve $\dot{\gamma} = (\kappa(H - G), - UG)$; so, over small enough time steps, it is reasonable to take the linear approximations

$$G^* = G + \kappa(H - G)\mathrm{d}t \quad (8)$$

$$Q^* = Q - UG\mathrm{d}t \quad (9)$$

The bilinear interpolation can then be used to give the value of $V(t + dt, G^*, Q^*)$.