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**THE $O(4)_{D+1}$ NONLINEAR SIGMA MODEL:
The Vacuum State and the Mass Gap**

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ABSTRACT

We study the ground state and low-lying excited states in the nonlinear sigma model with the coupled cluster method of quantum many-body theory.

1. Introduction

Although the Hamiltonian formulation of quantum field theory is more involved than its Euclidean equivalent, it has some fundamental advantages. These advantages start already at the level of the vacuum. Indeed, the *vacuum* is only clearly defined in the Hamiltonian formulation; it is the wave functional of the lowest-energy state. Given the vacuum, the particles are the low-lying excitations on this state, and therefore the properties of particles and their interactions are determined by the vacuum.¹ Even the absence of degrees of freedom, such as quarks, from the physical spectrum should almost automatically follow from the analysis of the vacuum.

Perturbations in single-quark degrees of freedom around the vacuum state should lead to high-energy states, probably a superposition of low-energy collective states. These perturbations can be considered the theoretical counterparts of deep-inelastic scattering, where an initial hit of a constituent leads to jets of particles in a “break-up” reaction. If the energy is high enough, the collective modes can move off freely.

Although the observables should follow easily when the vacuum state is determined, the practice is different. The vacuum problem is one of the hardest to solve. The number of non-perturbative methods for treating the vacuum state in the Hamiltonian formulation is restricted. Moreover, in QCD there are additional questions arising from the gauge invariance and the required gauge fixing.

The vacuum is invariant under space symmetries: rotations and translations. Therefore, the wave function can be restricted from the start. Usually it is restricted to mean-field states, where fields on each point act independently. However, in the true physical system correlations play an important role, not least to imprint the length scale of the collective modes which live on the vacuum.

If one wants to recover the global properties of a non-perturbative vacuum in an ab-initio calculation, one needs to retain as many as possible of the correlations allowed in the vacuum. For this purpose we investigate the possibility to apply the coupled-cluster method (CCM), a many-body technique, to nontrivial field theories. The CCM

has a number of properties, which, in principle, makes it very suitable for the study of the vacuum state and low-lying excitations. As a many-body technique it is designed to deal with a large number of degrees of freedom, which can be chosen to satisfy the symmetries of the vacuum. Its quasi-analytical nature allows for a linear response approximation (RPA) upon the vacuum state to recover the excitation spectrum.

For an initial investigation we study the nonlinear sigma model. It is known to exhibit symmetry breaking, beyond which the whole nature of the ground state changes. The model is used as an effective field theory for QCD from different perspectives. It is best known for the dynamics of pions, dominated by the chiral symmetry breaking expressed in the low-energy theorem.

2. The Nonlinear Sigma Model

The nonlinear sigma model consists of a single field ϕ_α with four components ϕ_1, ϕ_2, ϕ_3 , and ϕ_4 constrained by $\phi_\alpha^2 = 1$. The field is considered free and massless. The self-interaction among the components is the result of the constraint, which ties them together. The Lagrangian density

$$\mathcal{L} = \partial_\mu \phi_\alpha \partial^\mu \phi_\alpha \quad , \quad (1)$$

leads to the corresponding $O(4)$ spin-lattice Hamiltonian

$$H = \frac{1}{2} \sum_i \mathbf{I}_i^2 + \lambda \sum_{\langle i, j \rangle} (1 - \phi_{\alpha, i} \phi_{\alpha, j}) \quad , \quad (2)$$

where λ is proportional to $a^{2(D-1)}$ for a D -dimensional cubic lattice with spacing a ,² and \mathbf{I} is the angular momentum operator in four dimensions. The interaction term is summed over all nearest neighbour pairs of lattice sites $\langle i, j \rangle$.

If the model is thus regularised by considering only fields at lattice points it has two phases, although the case $D = 1$ is pathological since the coupling constant does not then depend on lattice spacing. As the coupling constant is small, the fields at each lattice point are virtually independent and the lowest-energy state consists of free rotors in the S -state at each site. If the coupling constant increases, the fields, which are vectors on a four-dimensional unit sphere, will align, and, in terms of the free-rotor spectrum, higher states will be filled. This behaviour is general for many systems: the kinetic and potential terms tending to drive the system into different phases.

The mean-field results give a rough indication of the behaviour of the system. In this case one has to introduce a direction into the system and investigate whether the vacuum aligns with respect to this direction. Generally, the mean-field state is

$$\langle \{ \phi_{\alpha, j} \} | \Psi \rangle = \prod_i f(\phi_i \cdot \hat{e}) \quad , \quad (3)$$

where $\hat{e} = (0, 0, 0, 1)$ and f is an arbitrary function. However, the position of the phase transition and the global behaviour is already given with the simple linear trial function: $f(\phi \cdot \hat{e}) = \cos \alpha + \phi \cdot \hat{e} \sin \alpha$. Below the critical coupling $\lambda_c = 3/D$ the ground

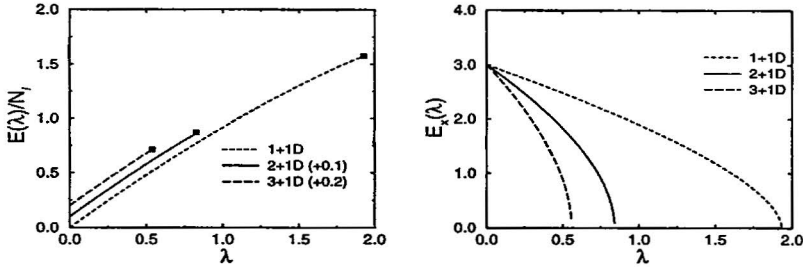


Fig. 1. The results using the functional NCCM: left, the ground-state energy per link in one, two, and three dimensions; and right, the corresponding lowest excitation energy. The solutions terminate as the excitation energy vanishes.

state is the constant state. However, above the critical coupling the ground state has a nontrivial solution,

$$f(x) = \sqrt{(\lambda D + 3)/(2\lambda D)} + \sqrt{(\lambda D - 3)/(2\lambda D)} x . \quad (4)$$

The exact mean-field result above the critical coupling is given by the first odd characteristic function ce_2 of the Mathieu equation. These results extend easily to the corresponding $O(N)$ model, yielding a critical coupling of $\lambda_c = 2(N - 1)/D\sqrt{N}$. The mean-field results predicts essentially the same behaviour for any dimension, although it is known that in the one-dimensional system the symmetry is never broken.

3. The Coupled Cluster Method

The CCM is a powerful technique for quantum many-body calculations. Here we will review only the essential features, and we refer to the literature³ for the details. There are three key features of the CCM: firstly, the exponentiated wave functional

$$|\Psi\rangle = e^{\hat{S}}|\Phi\rangle , \quad (5)$$

where $|\Phi\rangle$ is an uncorrelated model state, and \hat{S} is a linear combination of all multi-configurational creation correlation operators, which commute among themselves and annihilate the model bra state. Due to the exponentiation the independent correlations are automatically summed in the correct way. Secondly, the bra state is parametrised in a way which builds in a similarity transform,

$$\langle\tilde{\Psi}| = \langle\Phi|(1 + \tilde{\hat{S}})e^{-\hat{S}} = \langle\Phi|e^{\tilde{\hat{S}}}e^{-\hat{S}} , \quad (6)$$

where the first form is known as the *normal* CCM (NCCM), and the second as the *extended* CCM (ECCM).⁴ The third key feature of the CCM is the choice of operators: $\tilde{\hat{S}}$ and $\tilde{\hat{S}}$ contain only destruction operators; they are conjugate to the creation operators

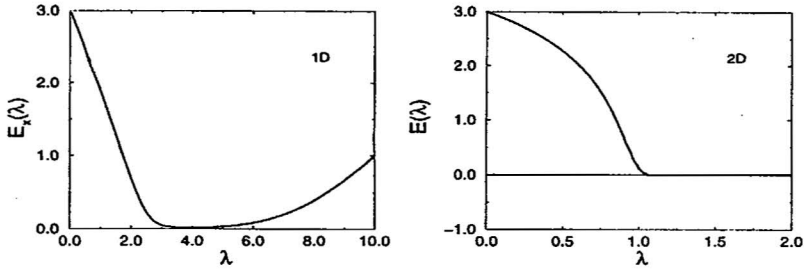


Fig. 2. The excitation energy, for operatorial ECCM, in one dimension (left) and two dimensions (right) (preliminary results).

in \hat{S} . The similarity transform then has the advantage that only correlations linked to the Hamiltonian are retained. The bi-variational equations in S and \hat{S} (or S and $\tilde{\Sigma}$) are therefore only of moderate complexity.

The NCCM SUB2 form, where only two-body correlations are retained, allows the use of *functions* as correlation operators

$$\hat{S} = \sum_{i,j} S_{i,j}(\phi_i \cdot \phi_j) , \quad (7)$$

which yield the results shown in Fig. 1.⁵ However, the NCCM has the same failure as the mean-field result; it finds a phase transition in one dimension where there should not be one. The NCCM also fails to track across the phase transition, which is the ultimate goal, as we want to be able to describe the collective low-lying modes in terms of local degrees of freedom.

The ECCM seems to fulfill above demands.⁵ Preliminary investigations show that we are indeed able to cross the phase transition and in one dimension the phase transition disappears, as we can see in Fig. 2, where the excitation energy remains massive in one dimension, but still vanishes, as expected, in two dimensions.

4. Conclusion and Outlook

The CCM allows an ab-initio study of the vacuum state and low-lying excitations in the nonlinear sigma model. However, to cross the phase transition requires the more elaborate *extended* CCM in an operatorial form. This work is in progress.

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