Bifurcation and stability of elastic membranes: theory and biological applications

Citation for published version (APA):

Citing this paper
Please note that where the full-text provided on Manchester Research Explorer is the Author Accepted Manuscript or Proof version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version.

General rights
Copyright and moral rights for the publications made accessible in the Research Explorer are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Takedown policy
If you believe that this document breaches copyright please refer to the University of Manchester’s Takedown Procedures [http://man.ac.uk/04Y6Bo] or contact uml.scholarlycommunications@manchester.ac.uk providing relevant details, so we can investigate your claim.
Bifurcation and stability of elastic membranes: theory and biological applications

S. P. Pearce
PhD Thesis
February 2010
Keele University
Contents

1 Introduction .................................................. 1
   1.1 Introduction .............................................. 1

2 Mathematical Preliminaries ............................... 6
   2.1 Continuum Mechanics .................................... 6
       2.1.1 Introduction ........................................ 6
       2.1.2 Bodies and Configurations ......................... 6
       2.1.3 Tensor Algebra ...................................... 7
       2.1.4 Deformation Gradient ............................. 10
       2.1.5 Conservation of mass .............................. 13
       2.1.6 Conservation of Momentum ....................... 14
       2.1.7 Equations of Motion .............................. 15
       2.1.8 Constitutive Models .............................. 16
       2.1.9 Elasticity .......................................... 18
       2.1.10 Isotropy .......................................... 19
       2.1.11 Conservation of Energy ......................... 20
   2.2 Strain-Energy Functions ................................. 21
       2.2.1 Varga Strain-Energy Function .................... 23
       2.2.2 Neo-Hookean Strain-Energy Function ............ 24
       2.2.3 Mooney-Rivlin Strain-Energy Function .......... 24
       2.2.4 Gent Strain-Energy Function ..................... 24
       2.2.5 General Separable Strain-Energy Function ...... 25
2.2.6 Fung Strain-Energy Function ............................................. 26
2.3 Membrane Elasticity ............................................................ 26
  2.3.1 Membrane-Like Shells ................................................... 27
  2.3.2 Simple Membranes ...................................................... 27
  2.3.3 Generalised Membranes ............................................... 28
  2.3.4 Wrinkling ................................................................. 28
2.4 Linear Elasticity ............................................................... 29
  2.4.1 Introduction .............................................................. 29
  2.4.2 Linearisation ............................................................ 29
  2.4.3 Hooke’s Law ............................................................ 30
  2.4.4 Equilibrium equations ............................................... 33
2.5 Compound Matrix Method .................................................. 34
  2.5.1 Introduction .............................................................. 34
  2.5.2 Determinant Based Method ........................................... 35
  2.5.3 Compound Matrix Method ............................................ 37
2.6 Legendre Functions and Spherical Harmonics ............................ 40
  2.6.1 Introduction .............................................................. 40
  2.6.2 Spherical Harmonics .................................................. 40
  2.6.3 Legendre Polynomials ................................................ 41
  2.6.4 Orthogonality and Integral Formulae ............................... 43
  2.6.5 Function Expansion Theorem ....................................... 44
  2.6.6 Associated Legendre Functions .................................... 45

3 Non-Uniform Inflation of a Cylindrical Membrane ......................... 47
  3.1 Introduction .............................................................. 47
  3.2 Literature Review .......................................................... 48
    3.2.1 Experimental Studies .............................................. 48
    3.2.2 Analytical Studies ................................................ 50
  3.3 Inflation of a Cylindrical Tube ........................................... 52
    3.3.1 Governing Equations ............................................... 52

ii
### 3.3.2 Conditions at Infinity

#### 3.3.3 Bifurcation Condition

#### 3.3.4 Connection with the Pressure-Volume Curve

### 3.4 Localised Solutions

#### 3.4.1 Near-Critical Solutions

#### 3.4.2 Phase Plane Analysis

### 3.5 Amplitude-Stretch Diagrams

### 3.6 Kinked Solution

### 3.7 Numerical Solutions

### 3.8 Conclusion

### 4 Stability of a Bifurcated Cylindrical Membrane

#### 4.1 Introduction

#### 4.2 Stability of the Uniform State

#### 4.3 Stability of the Weakly-Nonlinear Solution

##### 4.3.1 Evolution Equation

##### 4.3.2 Compound Matrix Method

##### 4.3.3 Results

#### 4.4 Stability of the General Bifurcated State

##### 4.4.1 Introduction

##### 4.4.2 Governing Equations

##### 4.4.3 Compound Matrix Method

##### 4.4.4 Results

#### 4.5 Energy Minimisation

#### 4.6 Conclusion

### 5 Comparison of Strain-Energy Functions

#### 5.1 Introduction

#### 5.2 Varga Strain-Energy Function

##### 5.2.1 Bifurcation Condition
5.2.2 Amplitude-Stretch Diagram ........................................... 103
5.2.3 Stability ................................................................. 105
5.3 Neo-Hookean Strain-Energy Function ................................. 105
  5.3.1 Bifurcation Condition ................................................. 105
  5.3.2 Amplitude-Stretch Diagram ......................................... 106
5.4 Gent Strain-Energy Function ........................................... 107
  5.4.1 Bifurcation Condition ............................................... 107
  5.4.2 Amplitude-Stretch Diagram ....................................... 108
  5.4.3 Stability ............................................................. 109
5.5 Ogden Strain-Energy Function ....................................... 110
  5.5.1 Amplitude-Stretch Diagrams ...................................... 110
  5.5.2 Stability ............................................................. 113
5.6 Fung Strain-Energy Function ......................................... 113
5.7 Artery Modelling ......................................................... 114
5.8 Conclusion ............................................................... 115

6 Inflation of a Spherical Membrane .................................. 116
  6.1 Introduction ............................................................ 116
  6.2 Configuration .......................................................... 117
    6.2.1 Integral of the Governing Equations .......................... 119
    6.2.2 Spherically Symmetric Solution ............................... 120
    6.2.3 Asymmetric Solutions ......................................... 120
  6.3 Bifurcation Condition ............................................... 122
  6.4 Numerical Solutions .................................................. 125
  6.5 Biological Cells ....................................................... 128
    6.5.1 Constant Area Constraint .................................... 128
  6.6 Conclusion ............................................................. 131

7 Inflation of a General Axisymmetric Shell ......................... 133
  7.1 Introduction ............................................................ 133
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.2</td>
<td>Deformation of Axisymmetric Shells</td>
<td>134</td>
</tr>
<tr>
<td>7.2.1</td>
<td>Derivation</td>
<td>134</td>
</tr>
<tr>
<td>7.3</td>
<td>Equilibrium Equations</td>
<td>138</td>
</tr>
<tr>
<td>7.3.1</td>
<td>Evaluating the Equilibrium Equations</td>
<td>140</td>
</tr>
<tr>
<td>7.4</td>
<td>Invariants</td>
<td>141</td>
</tr>
<tr>
<td>7.4.1</td>
<td>Membrane Theory</td>
<td>142</td>
</tr>
<tr>
<td>7.5</td>
<td>Specialisation</td>
<td>143</td>
</tr>
<tr>
<td>7.5.1</td>
<td>Shell Theory</td>
<td>143</td>
</tr>
<tr>
<td>7.5.2</td>
<td>Biological Lipid Bilayers</td>
<td>145</td>
</tr>
<tr>
<td>7.6</td>
<td>Conclusions and Future Work</td>
<td>146</td>
</tr>
</tbody>
</table>

| 8       | Optical Tweezers                           | 147  |
| 8.1     | Introduction                               | 147  |
| 8.2     | Governing Equations                        | 149  |
| 8.2.1   | Conversion to Spherical Coordinates        | 149  |
| 8.2.2   | Displacements and Stresses                 | 150  |
| 8.2.3   | Flow Around a Rigid Sphere                 | 152  |
| 8.3     | Deformation of a Solid Elastic Sphere      | 154  |
| 8.3.1   | Results                                    | 156  |
| 8.4     | Deformation of a Hollow Elastic Sphere      | 159  |
| 8.4.1   | Results                                    | 161  |
| 8.5     | Conclusion                                 | 166  |
List of Figures

1.1.1 Schema of a typical cell ................................................. 3
3.2.1 Selected experimental tube configurations during inflation ................. 49
3.3.1 Axisymmetric deformation of a thin-walled elastic tube .................. 53
3.3.2 Pressure against $r_\infty$ for the Ogden strain-energy function with $z_\infty = 1$ . 58
3.3.3 Relation (3.3.18) showing the connection between $z_\infty$ and $r_\infty$ for a tube with closed ends .................................................. 60
3.4.1 Sketch of a function $F$ which permits bulging solitary wave solutions . . . 65
3.4.2 Plot of the bifurcation condition $\omega(r_\infty)$ for the Gent strain-energy function with $J_m = 97.2$ with closed ends. .......................... 68
3.4.3 Plots of $F(r)$ and phase planes for the Gent strain-energy function .......... 69
3.5.1 Dependence of $r_0 - r_\infty$ on $r_\infty$ for the closed Gent tube, $J_m = 97.2$ . . 70
3.6.1 Pressure as a function of volume for a closed Gent tube with $J_m = 30$ .... 74
3.7.1 Profile of $r(Z)$ as $r_\infty$ is changed for the closed Gent tube with $J_m = 97.2$ . 77
3.7.2 Deformed configuration as $r_\infty$ is changed for the closed Gent tube with $J_m = 97.2$. ................................................................. 77
3.7.3 Deformed configuration as $r_\infty$ is changed for the closed Gent tube with $J_m = 97.2$. ................................................................. 78
4.3.1 Evans function for the weakly nonlinear solution .......................... 90
4.3.2 Normalised eigenfunction of the weakly nonlinear solution .............. 90
4.4.1 Comparison of the calculated values of $\alpha$ with the relation (4.4.7) ....... 95
4.4.2 Plot of $\alpha$ for the closed Gent strain-energy function with $J_m = 97.2$ .... 96
4.5.1 $\rho(0)$ for the bulged section of the Gent strain-energy function. ........ 99
4.5.2 $\rho(Z)$ for the Gent strain-energy function ........................................... 99

5.2.1 Bifurcation condition $\omega(r_\infty)$ for the Varga strain-energy function .... 102
5.2.2 Potential bulge/neck amplitude $r_0 - r_\infty$ for the Varga strain-energy function 104
5.2.3 Deformed configurations a closed Varga tube ........................................ 104
5.2.4 Eigenvalues of the fully nonlinear bifurcated state for the Varga strain-
energy function ................................................................. 105
5.3.1 Bifurcation condition $\omega(r_\infty)$ for the neo-Hookean strain-energy function . 106
5.3.2 Potential bulge/neck amplitude $r_0 - r_\infty$ for the neo-Hookean strain-energy
function ................................................................. 106
5.4.1 Bifurcation condition for the Gent strain-energy function ......................... 107
5.4.2 Maximum $z_\infty$ at which bifurcation exists for the Gent strain-energy function 108
5.4.3 Bifurcation condition for the Gent strain-energy function ......................... 109
5.4.4 Plot of $\alpha$ for the Gent strain-energy function with $z_\infty = 1$ and $J_m = 97.2$ . 110
5.5.1 Bifurcation condition for the Ogden strain-energy function ....................... 111
5.5.2 Dependence of $r_0$ against $r_\infty$ for the open Ogden tube with $z_\infty = 1$ ...... 111
5.5.3 Dependence of $r_0$ against $r_\infty$ for the open Ogden tube with $z_\infty = 3.5$ .... 112
5.5.4 Dependence of $r_0$ against $r_\infty$ for the closed Ogden tube ....................... 112
5.5.5 Plot of $\alpha$ for the open tube with the Ogden strain-energy function ............. 113
5.6.1 Dependence of $r_0$ against $r_\infty$ for the closed Fung tube ....................... 114

6.3.1 $r(S)$ for the first 20 mode numbers, perturbation amplitudes normalised . 125
6.4.1 Numerical calculation of mode one bifurcation for the Varga strain-energy
function ................................................................. 127
6.4.2 Numerical calculation of mode two bifurcation for the Varga strain-energy
function ................................................................. 127

8.3.1 Sphere showing where the force is applied ............................................. 157
8.3.2 Solid Sphere: Effect of varying $\eta$ ..................................................... 158
8.3.3 Solid Sphere: Effect of varying $\nu$ ..................................................... 158
8.3.4 Solid Sphere: Effect of varying $p_0$ ..................................................... 159
8.3.5 Solid Sphere: Effect of varying $\delta$ .............................................. 159
8.4.1 Hollow Sphere: Effect of varying $\eta, \epsilon = 0.1$ ................................. 162
8.4.2 Hollow Sphere: Effect of varying $\eta, \epsilon = 0.05$ ................................. 163
8.4.3 Hollow Sphere: Effect of varying $\delta$ .............................................. 163
8.4.4 Hollow Sphere: Effect of varying $\nu$ ................................................. 164
8.4.5 Hollow Sphere: Effect of varying $p_0$ ................................................. 164
8.4.6 Hollow Sphere: Effect of varying $\epsilon$ .............................................. 165
Acknowledgements

As I stand at the end of this journey I would like to thank all those who have been involved in ensuring I reached my destination. I thank God for His illuminating power which has shown me the way to tread. I thank my parents, whose love and support helped show me to the start of the road, and have been a continuing support since my childhood. I thank my guide, Prof. Yibin Fu, who has always discerned a path to tread and who is equally skilled at both mathematics and badminton. I thank Andy, Charis, Mike, Rob, Steve, and Will for helping me to take plenty of rest by regularly providing distraction, and ensuring that the journey has been an extremely enjoyable one. Special thanks go to Rob, for all those useful discussions, occasional arguments and puzzles we shared. And of course, thank you to my long-suffering wife Hannah, who has been my companion throughout this adventure, and who deserves this more than me. Without her love, patience and motivational skills I would never have reached the goal. In addition, thank you to all the other friends who have made my life better through their friendship throughout my life so far, whose names are too many to count.

I’d also like to thank the people I have worked with, both in the research institute for Science and Technology in Medicine: Alicia El Haj, Isaac Liu, Hu Zhang and Jon Glossop, and everyone in the Mathematics department, particularly Shailesh Naire, John Chapman and David Bedford. And thank you to Dot, Janet, Madeleine and Sue, for those morning chats.

Special thanks go to the BBSRC and EPSRC for the funding they provided.
Abstract

Elastic membranes are commonly found in biological and engineering contexts. In this thesis we first consider the deformation of hyperelastic, isotropic, incompressible cylindrical membranes subject to an internal pressure, using the tools of nonlinear elasticity. A condition for the bifurcation from a uniform inflation state into a localised bulged or necked non-uniform state is found analytically for a general strain-energy function. The condition required for the existence of a kinked solution in which multiple uniform states are connected by non-uniform transition regions is also derived. It is found that in several common strain-energy functions a bulged solution may exist, along with a kinked solution for certain strain-energy functions. The stability of the bifurcated state is calculated using an eigenvalue method, showing that the bulged or necked solutions are always unstable, while the kinked solution appears to be stable.

A similar process is repeated for a spherical membrane, where the bifurcation from an initially spherical inflation into a non-spherically symmetric solution is considered. After this the inflation of a general axisymmetric shell by an internal pressure is considered, and it is shown that an integral of the governing equations always exists, which has not been utilised in the literature.

Finally, we conclude by considering the deformation of a spherical cell held in a fluid flow by optical tweezers, using linear elasticity. We initially model the cell as a solid sphere, followed by a hollow sphere, and show the results of varying the parameters involved.
Chapter 1

Introduction

1.1 Introduction

Membranes are found in many biological contexts, where the word is used to define a thin layer of tissue which delimits the boundary between two spaces (Humphrey 2003). For example, animal and plant cells (eukaryotes) are surrounded by the cell surface membrane, the plasmalemma, which separates the contents of cells from their environment and acts as a partially permeable barrier controlling exchange with the external environment (Soper et al. 1990). At a larger scale, blood vessels and the lens of the eye may be considered as biological membranes (Humphrey 1998).

In mechanics, such a solid structure is denoted a shell, rather than a membrane (Libai and Simmonds 1998). An additional restriction is given on the definition of a membrane, which is that a membrane is a thin shell considered to have negligible resistance to bending, where bending moments and transverse shears are considered insignificant compared to in-plane loadings (Libai and Simmonds 1998). Humphrey (1998) showed that many biological membranes may be modelled as mechanical membranes in this way, and gives a very informative review of the techniques with which to treat elastic biological membranes. Throughout this text the mechanical definition of membrane will be used.
Here we shall model membranes as elastic, that is as solids which deform instantaneously in response to an applied load, and return to their original shape after the load is removed. This is an approximation to the reality for biological applications, where time-dependent (viscoelastic) or permanent (plastic) deformations often occur. As in all mathematical modelling we hope that investigating this initial case will be instructive, despite its limitations.

Biological cells exist in a complex environment, where they respond to mechanical, biological and environmental signals to change their form and function (Bao and Suresh 2003). We are therefore interested in the stresses and strains which the cells experience, as this is known to be biologically relevant for various different cells, such as endothelial cells (Frangos et al. 1988, McCormick et al. 2001), and more generally by Chicurel et al. (1998).

Barthes-Biesel (1980) define a microcapsule to be a thin elastic membrane surrounding a Newtonian incompressible fluid, which can potentially support large deformations. They both occur naturally, such as cells and eggs, and are synthesised for various industrial applications such as the food, cosmetic and biomedical industries (Chang and Olbricht 1993). The main difficulty involved in modelling the behaviour of the microcapsule is the coupling between the fluid problem and the elasticity problem. The deformation of such initially spherical capsules in shear flows has been studied by Barthes-Biesel and Rallison (1981), Ramanujan and Pozrikidis (1998), Walter et al. (2001) and Finken and Seifert (2006) amongst others. Modelling red blood cells, with their complex geometry but biological significance, as such a microcapsule has been undertaken, as in the studies by Skotheim and Secomb (2007) and Abkarian et al. (2007). Assuming that a constant pressure is exerted on the inside face of the membrane, it is possible to simplify the problem by not explicitly modelling the fluid.

It is worth noting that this concept of the mechanical response of a cell coming entirely from the cell membrane is an extremely simplistic model of the underlying biological structure (Ingber 2003). In particular, the internal cytoskeleton, composed of intercon-
 connected microtubules, actin filaments and intermediate filaments, is considered to have a large influence on the cell’s ability to resist shape distortion (Ingber 2003).

Figure 1.1.1: Schema of a typical cell, showing the cell membrane and its various receptors, the cytoplasm and the nucleus (Humphrey 2003, p7)

is a schematic diagram of a typical cell reproduced from Humphrey (2003), showing important cellular structures and illustrating how the cell has a complex internal structure.

At a larger scale, blood vessels may be considered as elastic tubes, inside which a moving fluid exerts a force upon the inner walls. In arteries the walls are not relatively thin; they exhibit strongly anisotropic responses due to the presence of embedded collagen fibres (Holzapfel et al. 2000) and are viscoelastic (Fung 1993), amongst other modelling complications. Aneurysms are localised bulges which may form in a blood vessel, commonly the aortic artery. An aneurysm changes the flow and strength of the artery and rupturing is usually fatal (Watton et al. 2004). Therefore, the study of the causes of such a medical emergency is an important one, which is receiving much attention in the literature.

The elastin, which is one of the main components of artery walls, may be considered as an isotropic elastic material. Several strain energy functions for modelling this material are compared by Watton et al. (2009), with the conclusion that it behaves similarly to a neo-Hookean material at physiological stretches. This neo-Hookean behaviour is also considered for the elastin in Holzapfel et al. (2000).
In this thesis, we start by considering the inflation of a cylindrical, isotropic, incompressible, elastic membrane subject to an internal pressure, looking in particular for solutions which are localised in their non-uniformity along the length of the tube. We find an expression for the critical value of the pressure at which such a non-uniform solution may exist, which is given in terms of a general strain-energy function and is therefore applicable to a wide range of materials. Depending on the form of the strain-energy function, there may also exist what we call a ‘kinked’ solution, where uniform sections of different radii are connected by non-uniform transition regions. This solution is connected to the existence of a Maxwell line in the pressure-volume diagrams, in the same way as in other solid mechanics problems such as phase transitions in bars (Ericksen 1975) and propagating necks in metals (Hutchinson and Neale 1983). We also find an analytic expression for the shape of the bulge or neck close to the bifurcation point, as well as showing its existence further from the bifurcation point.

Although bifurcated states may exist, they are not necessarily observable if they are unstable with respect to small perturbations. We therefore follow the derivation of the bifurcation condition with a stability analysis of the bulged or necked solution close to the bifurcation point, in which we conclude that this weakly nonlinear solution is unstable. We then investigate the stability of the general bulged or necked state, using both the techniques of spectral analysis and energy minimisation. We conclude that the bifurcated state is definitely unstable and we suggest that the kinked solution is probably stable.

After our investigation of the inflation of a cylindrical membrane, we turn our attention to the case of an initially spherical membrane instead. Here, instead of localised solutions we find non-axisymmetric modal perturbations on the basic shape. We re-derive the bifurcation condition given by Haughton and Ogden (1978b) and Chen and Healey (1991), using a different method which exploits the existence of an integral of the governing equations.

We then consider the inflation of a general axisymmetric shell by including contributions
to the energy of the deformation from bending as well as from stretching, using the two-
dimensional membrane theory. We show that the integral of the governing equations
used in both the previous studies must exist for a shell of any undeformed shape, not
just a cylindrical or spherical membrane.

Finally, we look at the deformation of a spherical cell which is held immobile in a fluid
flow in optical tweezers, where the cell is restrained by the force produced by a focused
laser (Henon et al. 1999). We model this as part of a EPSRC/BBSRC grant on stem cells.
Baesu et al. (2004) discuss how important the non-destructive testing of such cells is,
and how optical tweezers are a useful method in such testing. Lim et al. (2006) com-
pare various mechanical models, and discuss how the experimental technique which is
being studied should influence the model which is used. In particular, if a low-force,
small-deformation technique such as optical tweezers is used then modelling the cell as
a linear elastic solid is viable. We therefore first take the cell to be a solid elastic body
and then a thin elastic shell, using only linear elasticity. As linear elasticity is only valid
for small deformations, we are interested in the value of the parameters at which the dis-
placements become large and the linear approximation is no longer valid, along with the
stresses induced on the cell up to this point.

Much of the content of Chapters 3, 4 and 5 has been published in two journal articles,
Fu et al. (2008) and Pearce and Fu (2010), although the material is presented here in a
self-contained manner.
Chapter 2

Mathematical Preliminaries

2.1 Continuum Mechanics

2.1.1 Introduction

In this section we outline the theory of continuum mechanics, before specialising to deformations of solid materials, specifically those which may be described as elastic, which will then be used throughout the remainder of the thesis. In particular, the majority of the chapters will focus on the theory of large deformations, where the linear approximation is not valid. The formulation here is standard in the literature, for more details see, Ogden (1997), Holzapfel (2000) and Bertram (2008) as well as the individual citations in the text.

2.1.2 Bodies and Configurations

A body $B$ is a set of material points, or particles, each of which can be mapped to a point in Euclidean space and is uniquely represented by a position vector at each time $t$. The distribution of the points comprising the body in Euclidean space at a specific time is called a configuration of the body.
We define $B_r$, the reference configuration, to be a stress-free configuration of the body prior to any deformation. Associated with each point in $B_r$ is a position vector $X$, relative to an origin $O$ in a coordinate system with basis vectors $(E_i)$, where we use bold letters to denote a vector or tensor. We shall also use undeformed to refer to the reference configuration.

Additionally, we define $B_t$ as the current configuration at time $t > 0$, and let $x$ be the position vector of a point in the body relative to $O$ in a coordinate system with basis vectors $(e_i)$. Again, we also denote this the deformed configuration. As $B_r$ and $B_t$ are different configurations of the body $B$, there must exist a bijective mapping from $B_r$ to $B_t$, which we denote by $\chi$. Therefore, we have $x = \chi(X,t)$, where the explicit dependence on $t$ is shown. Capital letters will be used wherever possible to denote quantities in the reference configuration, while lower case versions of the same letters are used to represent the corresponding quantities in the current configuration.

We shall require the two coordinate bases, $(E_i)$ and $(e_i)$, to be orthogonal but not necessarily Cartesian. It is possible to use non-orthogonal coordinate bases in continuum mechanics, for example see Eringen (1962), although it is not required in this thesis and therefore will not be discussed here. In general these two coordinate systems may be different, though it is often convenient in applications for them to be identical, and this will be the case here.

### 2.1.3 Tensor Algebra

This section introduces several results in tensor algebra which are required for this thesis, but is in no way complete. The following books give an excellent introduction to the use and theory of tensors, Ogden (1997), Spencer (1980), Chadwick (1999) and Bertram (2008).

First we introduce the concept of the tensor product, denoted by $\otimes$, in that for any two vectors $u$ and $v$ there exists a tensor product defined by its action on an arbitrary vector
\[(u \otimes v) \cdot a = (a \cdot v) u \quad (2.1.1)\]

In component form, if the two vectors have expressions relative to the basis \((e_i)\), \(u = u_i e_i\), \(v = v_i e_i\), the tensor product may be written as

\[u \otimes v = u_i v_j e_i \otimes e_j, \quad (2.1.2)\]

where in (2.1.2), and elsewhere throughout this work unless stated otherwise, the summation convention is assumed. That is, whenever a Latin sub- or superscript is repeated in an expression it is taken to imply a sum over that index between 1 and 3. The identity tensor, \(I\), may be written as,

\[I = e_i \otimes e_i. \quad (2.1.3)\]

For all coordinate systems we have the following expression for the total differential, using the chain rule,

\[dx = \frac{\partial x}{\partial s_i} ds_i \equiv g_i ds_i, \quad (2.1.4)\]

where \(s_i\) are any set of coordinates and \(g_i\) are metric tensors associated with this coordinate system. We now specify the values of these functions with respect to several common coordinate systems. In Cartesian coordinates we find,

\[s_1 = x_1, \ s_2 = x_2, \ s_3 = x_3, \ g_1 = e_x, \ g_2 = e_y, \ g_3 = e_z, \quad (2.1.5)\]

whereas for cylindrical polar coordinates,

\[s_1 = r, \ s_2 = \theta, \ s_3 = z, \ g_1 = e_r, \ g_2 = re_\theta, \ g_3 = e_z, \quad (2.1.6)\]

and for spherical polar coordinates,

\[s_1 = r, \ s_2 = \theta, \ s_3 = \phi, \ g_1 = e_r, \ g_2 = re_\theta, \ g_3 = r \sin \theta e_\phi. \quad (2.1.7)\]
We then also define a dual metric basis, which is a set of vectors \( g^i \) such that the following orthogonality relation holds,

\[
g^i \cdot g_j = \delta_{ij}, \tag{2.1.8}
\]

which as we have assumed that the \( g_i \) are an orthogonal set gives,

\[
g^i = g_i / (|g_i|^2). \tag{2.1.9}
\]

The gradient of a vector, which gives a tensor, may then be defined as,

\[
\text{grad } u = \frac{\partial u}{\partial s^i} \otimes g^i. \tag{2.1.10}
\]

We also introduce the divergence of a tensor, which results in a vector, and is given by

\[
\text{div } T = g^i \cdot \frac{\partial T}{\partial s^i}. \tag{2.1.11}
\]

In particular, we will use cylindrical coordinates throughout this thesis, and therefore we give an explicit result for (2.1.10) by writing \( u = u_r e_r + u_\theta e_\theta + u_z e_z \), and using (2.1.6),

\[
\text{grad } u = \frac{\partial u}{\partial r} \otimes e_r + \frac{\partial u}{\partial \theta} \otimes \frac{e_\theta}{r} + \frac{\partial u}{\partial z} \otimes e_z
\]

\[
= u_{r,r} e_r \otimes e_r + u_{r,\theta} e_\theta \otimes e_r + u_{z,r} e_z \otimes e_r +
\]

\[
+ \left( \frac{u_{r,\theta}}{r} - \frac{u_{\theta,\theta}}{r} \right) e_r \otimes e_\theta + \left( \frac{u_{\theta,\theta}}{r} + \frac{u_r}{r} \right) e_\theta \otimes e_\theta + \frac{u_{z,\theta}}{r} e_z \otimes e_\theta
\]

\[
+ u_{r,z} e_r \otimes e_z + u_{\theta,z} e_\theta \otimes e_z + u_{z,z} e_z \otimes e_z \tag{2.1.12}
\]

where a comma denotes differentiation with respect to that variable. In (2.1.12) the formulae for the differentiation of the unit vectors in cylindrical coordinates, \( \frac{\partial e_r}{\partial \theta} = e_\theta, \frac{\partial e_\theta}{\partial \theta} = -e_r \), have been used. Similarly, if a second order tensor \( T \) is given by \( T = T_{ij} e_i \otimes e_j \) where \((1, 2, 3) = (r, \theta, z)\), the specialisation of (2.1.11) for cylindrical coordinates is given
by,

\[
\text{div } \mathbf{T} = \left( T_{rr,r} + \frac{T_{r\theta,\theta}}{r} + T_{zr,z} + \frac{T_{rr} - T_{\theta\theta}}{r} \right) \mathbf{e}_r \\
+ \left( T_{r\theta,r} + \frac{T_{\theta\theta,\theta}}{r} + T_{z\theta,z} + \frac{T_{\theta r} + T_{r\theta}}{r} \right) \mathbf{e}_\theta \\
+ \left( T_{rz,r} + \frac{T_{\theta z,\theta}}{r} + T_{zz,z} + \frac{T_{rz}}{r} \right) \mathbf{e}_z
\]  

(2.1.13)

(2.1.14)

(2.1.15)

We also introduce the notations Grad and Div as being the gradient and divergence defined in the same way but with respect to the undeformed configuration, with an appropriate change in the \( s_i \) from \( x_i \) to \( X_i \), and from \( e_i \) in (2.1.4) to \( E_i \).

### 2.1.4 Deformation Gradient

We are interested in the deformation from the given reference configuration to a current configuration, without taking into account any intermediate steps. Indeed, we are chiefly interested in problems with no explicit time dependence and are therefore interested in quasi-static deformations where we look at states in which the equilibrium equations are satisfied.

With this in mind, as we are interested in how the particles of the body move from a point \( X \) in the reference undeformed configuration to a point \( x \) in the current deformed configuration, we define the deformation gradient tensor \( \mathbf{F} \) as the gradient of \( x \) with respect to \( X \),

\[
\mathbf{F} = \text{Grad } x = \frac{\partial x}{\partial X_j} \otimes E_j
\]  

(2.1.16)

where the repeated index represents the sum over \( j = 1, 2, 3 \) as mentioned previously. Equation (2.1.16) allows us to determine how a neighbourhood of \( X \) is locally deformed, and we can naturally write the following equation for the deformation of line elements between the two configurations given by,

\[
dx = \mathbf{F}dX
\]  

(2.1.17)
where \( dX \) and \( dx \) are infinitesimal line elements in the reference and current configurations respectively. The deformation gradient therefore enables us to transform line elements between the two configurations. An important conclusion from (2.1.17) is that \( F \) must be non-singular, else there exists some non-zero line elements \( X \) which vanish under the deformation, which is physically unacceptable.

Taking three vectors in the reference configuration, \( X^{(1)}, X^{(2)}, X^{(3)} \), we define an infinitesimal volume element in the reference configuration as,

\[
dV = [dX^{(1)}, dX^{(2)}, dX^{(3)}],
\]

and we may use basic tensor theory (Spencer 1980) to deduce that

\[
dv = JdV,
\]

where \( J = \det F > 0 \) is the local volume change between the two configurations and \( dv \) is the corresponding parallelepiped in the deformed configuration. Clearly, \( J \) is required to be strictly positive to ensure that volumes remain positive, and therefore physical, after deformation. A material is said to be incompressible if the volume does not change under any deformation, leading to the condition \( J \equiv 1 \) for all \( X \).

As \( F \) is a non-singular tensor with a positive determinant, the polar decomposition theorem (Chadwick 1999, Antman 2005) enables us to write it uniquely as,

\[
F = RU = VR,
\]

where \( R \) is a proper orthogonal tensor and \( U, V \) are positive definite symmetric tensors. This decomposition (2.1.20) represents a stretch \( U \), followed by a rotation \( R \), or a rotation \( R \) followed by a stretch \( V \). It may be shown that the two rotations here are identical, see for example Ogden (1997).

As \( U \) is a positive definite symmetric tensor, it may be written in a diagonal form with
respect to a particular basis as,
\[ U = \sum_{i=1}^{3} \lambda_i u_i^i \otimes u_i^i, \quad (2.1.21) \]

where the axes \((u_i^i)\) are denoted as the principal axes of stretch, and the \(\lambda_i\) are known as the principal stretches. In this decomposition, \(\lambda_i\) are the eigenvalues of \(U\) and \((u_i^i)\) are the corresponding eigenvectors. A similar decomposition may be used on \(V\), resulting in the same principal stretches but rotated axes, given by \(v_i = Ru_i^i\).

We define the right Cauchy-Green deformation tensor, \(C\), as
\[ C = F^T F = U^2 = \sum_{i=1}^{3} \lambda_i^2 u_i^i \otimes u_i^i, \quad (2.1.22) \]

where a superscript \(T\) denotes the transpose of a tensor and (2.1.21) has been used to express \(C\) as diagonal with respect to the basis \((u_i^i)\). \(C\) is clearly symmetric as \((F^T F)^T = F^T (F^T)^T = F^T F\). \(C\) is also positive definite as, for a non-zero arbitrary vector \(a \in \mathbb{R}^3\),
\[ a \cdot Ca = a \cdot F^T Fa = Fa \cdot Fa = |Fa|^2 > 0. \]

We also define the Green strain tensor, \(E\), as
\[ E = \frac{1}{2} (F^T F - I), \quad (2.1.23) \]

where \(I\) is the identity tensor.

The displacement of a particle is defined as \(u = x - X\), and therefore \(x = X + u\). We also note the following relation defining the displacement gradient \(\text{Grad } u\),
\[ F = \text{Grad } x = I + \text{Grad } u. \quad (2.1.24) \]

The velocity of a point \(x\) is given by
\[ \dot{v} \equiv \dot{x} = \frac{\partial}{\partial t} x (X, t), \quad (2.1.25) \]

where a superimposed dot represents differentiation with respect to \(t\), and this expresses
the partial derivative with respect to $t$ at a fixed $X$. We also define a velocity gradient tensor $L$ as,

$$L = \text{grad} \, v.$$  \hfill (2.1.26)

We immediately note the connection, $\text{Grad} \, v = (\text{grad} \, v) \, F = LF$, where the first equation comes from (2.1.16). Similarly, we can write $\text{Grad} \, v = \frac{\partial}{\partial t} \text{Grad} \, x = \dot{F}$, provided that the partial derivatives are suitably continuous which we will assume here. Therefore we find the following relation,

$$\dot{F} = LF.$$  \hfill (2.1.27)

We also note the relation

$$\text{tr} \, L = \text{tr} \, (\text{grad} \, v) = \text{div} \, v,$$  \hfill (2.1.28)

which follows from the definition (2.1.26).

### 2.1.5 Conservation of mass

It is assumed that there exists a scalar field, $\rho$, which is the mass density of the material of the body defined in the current configuration $B_t$. Let $R_t$ be an arbitrary region in the current configuration, then as the body deforms the mass of the material in this region must not change. Therefore we require the following equation representing the conservation of mass to hold,

$$\frac{d}{dt} \int_{R_t} \rho \, dv = 0.$$  \hfill (2.1.29)

By converting the integral over $R_t$ into an integral over the equivalent region in the reference configuration, $R_r$, we may take the derivative with respect to time inside the integral as $R_r$ does not depend on $t$. Therefore (2.1.29) becomes,

$$\frac{d}{dt} \int_{R_t} \rho \, dv = \int_{R_r} \frac{d}{dt} (\rho J) \, dV = \int_{R_t} \left( \dot{\rho} + \rho \frac{\dot{J}}{J} \right) \, dv = 0.$$  \hfill (2.1.30)

As the region $R_t$ is arbitrary, the equation $J \dot{\rho} + \rho \dot{J} = 0$ must hold throughout the configuration, where it has been assumed that this function is continuous. We can therefore
integrate and write $\rho J = \rho_r$, where $\rho_r$ is the density of the material in the undeformed configuration.

Using the tensor identity, $\frac{\partial}{\partial t} (\det F) = \det F \text{tr} \left( F^{-1} \dot{F} \right)$ (Chadwick 1999) as well as (2.1.26) and (2.1.27), we find

$$\dot{J} = J \text{tr} L = J \text{div} \mathbf{v}. \quad (2.1.31)$$

Using (2.1.31), we therefore find the conservation of mass (2.1.30) becomes,

$$\dot{\rho} + \rho \text{div} \mathbf{v} = 0 \quad (2.1.32)$$

which must hold throughout the body. If the density is constant, which is true for an incompressible material as $\dot{J} = 0$, then (2.1.32) reduces to $\text{div} \mathbf{v} = 0$.

### 2.1.6 Conservation of Momentum

We consider a small region of the current configuration, on whose faces the forces from the remainder of the body act. These forces are known as tractions, which are represented by vectors which depend on the position in the configuration, along with the normal to the excised surface $\mathbf{n}$, given by $\mathbf{t}(\mathbf{x}, \mathbf{n})$. This description of the tractions is known as Cauchy’s stress principle, and is regarded as an axiom (Ogden 1997).

Therefore the forces which act on the surface of the arbitrary region in the current configuration $\partial R_t$ are given by $\int_{\partial R_t} \mathbf{t}(\mathbf{x}, \mathbf{n}) da$. If there are forces acting throughout the body, such as gravity, then these body forces give a contribution to the force in the current configuration which is given by $\int_{R_t} \rho \mathbf{b} dv$, where $\mathbf{b}$ is the body force per unit mass.

The linear momentum of the material in the region $R_t$ is given by mass multiplied by velocity, i.e. $\int_{R_t} \rho \mathbf{v} dv$. Therefore, using Newton’s second law, in the form Force = $\frac{d}{dt}$ (Momentum), along with the previous considerations, the following equation must hold,

$$\int_{\partial R_t} \mathbf{t}(\mathbf{x}, \mathbf{n}) da + \int_{R_t} \rho \mathbf{b} dv = \frac{d}{dt} \int_{R_t} \rho \mathbf{v} dv. \quad (2.1.33)$$
The right hand side of (2.1.33) may be written, by converting from $R_t$ to $R_r$, taking the derivative inside and converting back to $R_t$, as

$$\frac{d}{dt} \int_{R_t} \rho v \, dv = \int_{R_t} \frac{d}{dt} (\rho v J dV) = \int_{R_t} \frac{(\rho \dot{v} J - \rho J \dot{v})}{J} \, dv = \int_{R_t} \rho \dot{v} \, dv,$$

(2.1.34)

where the conservation of mass has been used in the last equality and $a = \dot{v}$. Therefore (2.1.33) may be written,

$$\int_{\partial R_t} t(x, n) \, da = \int_{R_t} \rho (a - b) \, dv.$$

(2.1.35)

### 2.1.7 Equations of Motion

We now state without proof Cauchy’s Theorem, as given in Ogden (1997) or Holzapfel (2000), which states that provided that the traction vector $t(x, n)$ is continuous in $x$ then it depends linearly on $n$, and therefore there exists a second-order tensor $\sigma$ independent of $n$ such that,

$$t(x, n) = \sigma^T(x, t) \cdot n$$

(2.1.36)

where $\sigma$ is expected to be a function of $x$ and $t$. The tensor $\sigma$ is called the Cauchy stress tensor. The components of $\sigma$, $\sigma_{ij}$, are the stresses acting in the direction $j$ on the $i$ plane, and therefore the components $\sigma_{ii}$ are normal stresses while the components $\sigma_{ij}, i \neq j$, are shear stresses.

The equation of linear momentum, (2.1.35), therefore becomes,

$$\int_{\partial R_t} \sigma^T \cdot n \, da = \int_{R_t} \rho (a - b) \, dv.$$

(2.1.37)

Using the divergence theorem, a standard result in vector calculus, the left hand side of (2.1.37) may be written as,

$$\int_{\partial R_t} \sigma^T \cdot n \, da = \int_{R_t} \text{div} \sigma \, dv,$$

(2.1.38)
and therefore, as $R_t$ is an arbitrary region, we find the following equation of motion,

$$\text{div} \, \sigma = \rho (a - b). \quad (2.1.39)$$

A similar procedure as that which produced this equation of motion may be used to derive an equation for the angular momentum, which reduces to

$$\sigma = \sigma^T, \quad (2.1.40)$$

implying that $\sigma$ must be a symmetric tensor. For details of this see Spencer (1980) or Ogden (1997). Throughout this thesis no body forces will be considered, and therefore we set $b = 0$. If the material is in equilibrium then $a = 0$, and (2.1.39) becomes the equilibrium equations,

$$\text{div} \, \sigma = 0. \quad (2.1.41)$$

We also wish to define the nominal stress tensor, $S$, which is a measure of the stress in the reference configuration. This is related to $\sigma$ by,

$$S = JF^{-1} \sigma, \quad (2.1.42)$$

and therefore we may write (2.1.41) as,

$$\text{Div} \, S = 0 \quad (2.1.43)$$

### 2.1.8 Constitutive Models

Equations (2.1.32), (2.1.39) and (2.1.40) give 7 relations between 13 unknowns, $(\rho, v, \sigma)$, or 4 relations between 10 unknowns after taking $\sigma$ to be symmetric. The remaining six relations need to be determined from the constitutive equations representing the properties of the material making up the body. So far these governing equations are valid for any continuum of material, whether solid or fluid; elastic, viscoelastic or plastic. The speci-
fication of the remaining six relations must be chosen in an appropriate way depending on the material of the body under consideration. These relations come from considering how the stresses in the material are related to the strains, from experimental and physical observations. The relation for a linear elastic material was first given by Hooke in 1678 as ‘As the extension, so the force’, stating that the stress is proportional to the strain, which we may generalise to multiple dimensions and write as,

\[
\text{Stress} \propto \text{Strain}, \tag{2.1.44}
\]

where the proportionality constant implied by (2.1.44) is an elastic modulus. The precise form of this relationship will be discussed later. A Newtonian fluid, named after Isaac Newton, is a fluid whose relation between stress and strain is given by,

\[
\text{Stress} \propto \frac{d\text{Strain}}{dt}, \tag{2.1.45}
\]

where the stress in the fluid is proportional to the rate of strain rather than the strain itself. The proportionality constant implied by (2.1.45) is a viscosity, and represent the resistance of the fluid-fluid interactions. Many common liquids and gases are such Newtonian fluids.

Some materials behave like solids in some circumstances and like fluids in other circumstances. These viscoelastic materials have relationships which involve both the amount of, and rate of, stress and strain. These materials exhibit time-dependent behaviour, including creep and relaxation. For example, the Kelvin-Voigt viscoelastic model relates the stress to both the amount and rate of strain,

\[
\text{Stress} \propto \left(\text{Strain}, \frac{d\text{Strain}}{dt}\right). \tag{2.1.46}
\]

There are also other models for viscoelastic materials which include the rate of stress into the constitutive equations. In addition, materials may behave plastically, where the stress depends both on the history of the deformation as well as the current state.
2.1.9 Elasticity

We wish to specify the equations of continuum mechanics which have been derived so far to the case of elasticity. As discussed above, we therefore require the stress to be proportional to the strain, and we also require that the Cauchy stress depends only on the deformation gradient, without depending on the history or the path taken to reach the point $F$, by assuming,

$$\sigma = g(F, X), \quad (2.1.47)$$

for some symmetric tensor valued function $g$, with $g(I, X) = 0$ so that the reference configuration is stress-free. This assumption is known as Cauchy elasticity, the alternative of allowing $g$ to depend on higher-order gradients of $F$ is possible but not considered here.

We will consider homogeneous materials from hereon, and therefore assume that $\sigma$ does not depend on $X$ and will drop this argument from the definition of $g$.

In theory any tensor function $g$ would be a potential choice for defining a constitutive equation. However, choosing such a function at ‘random’, while mathematically possible, would not be physically meaningful due to violation of certain physical considerations. In particular, it is necessary that the function $g$ is objective. That is, the stresses in the body must be independent of rigid-body motion after deformation has occurred (Ogden 1997). The form of such a rigid-body motion is given by

$$x^* = Q(t)x + c(t), \quad (2.1.48)$$

where $Q$ is a proper orthogonal rotation tensor and $c$ is a constant translation vector. The corresponding deformation gradient, $F^*$, is given by an application of the chain rule as $F^* = QF$. Similarly, the stress tensor with respect to $x^*$ is given by $\sigma^* = Q\sigma Q^T$. Therefore a tensor-valued function $g$ is an objective function of $F$ if and only if,

$$g(F^*) = g(QF) = Qg(F)Q^T = \sigma^*, \quad (2.1.49)$$

for each $F$ and all rotations $Q$ (Ogden 1997).
2.1.10 Isotropy

A body is isotropic relative to $B_r$ if the response of any small section of material cut from $B_r$ is independent of its orientation in $B_r$. For example, rubber is isotropic, but rubber reinforced with metal strips in a specific direction is anisotropic. For isotropic materials we have the additional requirement that the response function must be invariant under rigid-body rotation prior to deformation, that is

$$g(FQ) = g(F), \quad (2.1.50)$$

for each $F$ and all rotations $Q$. On taking $Q = R^T$ in (2.1.50), we have

$$\sigma = g(F) = g(V), \quad (2.1.51)$$

and therefore $\sigma$ only depends on $F$ through $V$. Therefore, combining (2.1.49) and (2.1.51), we find that the response function must satisfy, for any rotation $Q$,

$$g(Q^TVQ) = Q^Tg(V)Q. \quad (2.1.52)$$

A tensor-valued function $g$ which satisfies (2.1.52) is an isotropic tensor function of $F$. The representation theorem, see for example Ogden (1997), states that any such second-order isotropic tensor function must have the following representation,

$$g(V) = \phi_0 I + \phi_1 V + \phi_2 V^2, \quad (2.1.53)$$

where the $\phi_i$ are functions of the three principal invariants of a three-dimensional second-order tensor, given by $\text{tr} V, \frac{1}{2} ((\text{tr} V)^2 - \text{tr} (V^2))$ and $\text{det} V$ (Spencer 1980). If a tensor is given in spectral form as in (2.1.21) and (2.1.22), then these invariants may be written as,
for $V$,

\begin{align}
i_1 &\equiv \text{tr} (V) = \lambda_1 + \lambda_2 + \lambda_3 \\
i_2 &\equiv \frac{1}{2} [i_1^2 - \text{tr} (V^2)] = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \\
i_3 &\equiv \det V = \lambda_1 \lambda_2 \lambda_3,
\end{align}

(2.1.54)

or for the tensor $C$,

\begin{align}
I_1 &\equiv \text{tr} (C) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2 &\equiv \frac{1}{2} [I_1^2 - \text{tr} (C^2)] = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \\
I_3 &\equiv \det C = \lambda_1^2 \lambda_2^2 \lambda_3^2,
\end{align}

(2.1.55)

where we have introduced the notations $I_k$ and $i_k$, which are used throughout this work to represent the invariants of $C$ and $V$ respectively. These two sets of invariants may be connected using $I_1 = i_1^2 - 2i_2, I_2 = i_2^2 - 2i_1 i_3, I_3 = i_3^2$.

### 2.1.11 Conservation of Energy

We now derive the law of conservation of energy, following the method presented in Ogden (1997). We take the dot product of the equation of motion, (2.1.39), with the velocity $v$, giving

\[(\text{div} \sigma) \cdot v + \rho (b \cdot v) = \dot{v} \cdot v,\]

(2.1.56)

where $\dot{v}$ is the acceleration as defined earlier. We then rewrite the first term as,

\[\text{div} (\sigma v) - \text{tr} (\sigma L) + \rho (b \cdot v) = \dot{v} \cdot v,\]

(2.1.57)

where $L$ is the velocity gradient defined in (2.1.26). Integrating (2.1.57) over the current configuration, $B_t$, and using the divergence theorem and the conservation of mass we
\[
\int_{B_t} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial B_t} \mathbf{t} \cdot \mathbf{v} \, da = \frac{d}{dt} \int_{B_t} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_{B_t} \text{tr} (\sigma \mathbf{L}) \, dv. \quad (2.1.58)
\]

In (2.1.58) the left hand side represents the rate of working of the forces applied to the body, and the first term on the right hand side is the kinetic energy. If the deformation is conservative and no energy is lost, then this remaining term must represent the stored elastic energy. Rewriting this term as,

\[
\int_{B_t} \text{tr} (\sigma \mathbf{L}) \, dv = \int_{B_r} J \text{tr} (\sigma \mathbf{L}) \, dV, \quad (2.1.59)
\]

we may then interpret \( J \text{tr} (\sigma \mathbf{L}) \) as the rate of increase of elastic energy per unit volume in \( B_r \) (Ogden 1997), which is the work done by the forces acting on the body.

### 2.2 Strain-Energy Functions

For an isotropic, homogenous material we now introduce a strain-energy function \( W(\mathbf{F}) \) which represents the stored elastic energy per unit undeformed volume of the material,

\[
\frac{\partial}{\partial t} W(\mathbf{F}) = J \text{tr} (\sigma \mathbf{L}). \quad (2.2.1)
\]

An elastic material satisfying (2.2.1) is said to be hyperelastic. We may then write,

\[
\frac{\partial}{\partial t} W(\mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}_{ij}} \frac{\partial \mathbf{F}_{ij}}{\partial t} \equiv \text{tr} \left( \frac{\partial W}{\partial \mathbf{F}} \mathbf{F} \right) = \text{tr} \left( \frac{\partial W}{\partial \mathbf{F}} \mathbf{L} \right) = \text{tr} \left( \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} \mathbf{L} \right), \quad (2.2.2)
\]

where we have used (2.1.27), and we define \( \frac{\partial W}{\partial \mathbf{F}} \) as the tensor whose components are given by the convention,

\[
\left( \frac{\partial W}{\partial \mathbf{F}} \right)_{ji} = \frac{\partial W}{\partial \mathbf{F}_{ij}}. \quad (2.2.3)
\]
Comparing (2.2.1) and (2.2.2) we find a relation between the strain-energy function and the Cauchy stress,

\[ J\sigma = F\frac{\partial W}{\partial F} \] (2.2.4)

It is convenient to regard \( W \) as being either a function of the three stretches, \( W(\lambda_1, \lambda_2, \lambda_3) \), or the three invariants defined in (2.1.55). It is the character of \( W \) which expresses the differences between various materials, and prescribes the remaining constitutive equations to complete the formulation of elastic deformations. Due to the isotropy of the material, the strain-energy function must be a symmetric function of the three principal stretches and hence

\[ W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_2, \lambda_3, \lambda_1). \] (2.2.5)

A further restriction on \( W \) is that there is no stored energy or stress in the reference configuration, \( B_r \), implying

\[ W(1, 1, 1) = 0, \quad \frac{\partial W(1, 1, 1)}{\partial \lambda_i} = 0, \quad i \in 1, 2, 3. \] (2.2.6)

With respect to the principal stretches for an isotropic material, (2.2.4) becomes,

\[ \sigma_i = J^{-1}\lambda_i \frac{\partial W}{\partial \lambda_i}, \] (2.2.7)

which are in the principal directions \( u^i \).

If the material is incompressible then \( J = \det F = \lambda_1\lambda_2\lambda_3 = 1 \), and therefore the derivatives of (2.1.16) and (2.2.7) are no longer independent. We therefore introduce a pseudo-strain-energy function, \( \tilde{W} = W(F) - p(\det F - 1) \), where \( p \) is a Lagrange multiplier. It is clear that once we apply the constraint, \( \tilde{W} \) is identical to \( W \), and (2.2.4) becomes,

\[ \sigma = F\frac{\partial}{\partial F} (W - p(\det F - 1)) = F\frac{\partial W}{\partial F} - pI, \] (2.2.8)

where we have used \( \frac{\partial \det F}{\partial F} = F^{-1} \), and have applied the constraint after differentiation. In doing \( \text{tr}(\sigma L) \), the work done by the forces on the body, is not affected by \( p \), and so the
constraint does no work as required.

In terms of the principal stretches, (2.2.8) becomes,

\[ \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad (2.2.9) \]

instead of (2.2.7).

The strain-energy function incorporates an elastic modulus, in units of Force/Area, which corresponds to the shear modulus in infinitesimal, or linear, deformations and will be denoted here by \( \mu \). We now present a variety of strain-energy functions for isotropic incompressible elastic solids which will be considered in later chapters. As was mentioned above, the space of such functions is extremely large, in that any function of the two invariants \( I_1, I_2 \) which satisfies (2.2.5) and (2.2.6) may be considered as a viable strain-energy function, although there are other physical considerations.

### 2.2.1 Varga Strain-Energy Function

A simple strain-energy function when expressed with respect to the stretches is the Varga strain-energy function. This function was described by Varga (1966) to model rubber for small but not infinitesimal stretches of the order of \( 1.3 \) say. As a function of the stretches it is given by

\[ W = 2\mu(i_1 - 3) = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3), \quad (2.2.10) \]

where \( \mu \) is the infinitesimal shear modulus. The Varga strain-energy function is useful for theoretical work due to its simple linear mathematical structure, but does not model many physical behaviours observed in elastic materials.
2.2.2 Neo-Hookean Strain-Energy Function

The neo-Hookean strain-energy function is a generalisation of Hooke’s Law to the case of nonlinear elasticity, that is that the extension is linearly proportional to the stress,

\[
W = \frac{1}{2} \mu(I_1 - 3) = \frac{1}{2} \mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3). \tag{2.2.11}
\]

The neo-Hookean model also has a simple mathematical structure while having much wider physical application, particularly to the modelling of rubber at small to moderate stretches of up to approximately 1.5. In particular, Müller and Strehlow (2004) show that (2.2.11) is inappropriate for biaxial stretching above this range. Along with the Varga strain-energy function, the neo-Hookean model is one of the most commonly used strain-energy functions for analytical work due to their simplicity.

2.2.3 Mooney-Rivlin Strain-Energy Function

The Mooney-Rivlin is a more generalised model than the neo-Hookean, incorporating a term involving \( I_2 \), and does not follow Hooke’s Law for large deformations unless \( \alpha = 0 \),

\[
W = \frac{\mu}{2} ((1 - \alpha)(I_1 - 3) + \alpha(I_2 - 3)). \tag{2.2.12}
\]

In this model there is a second parameter, \( \alpha \), which expresses the measure of nonlinearity of the material. This strain energy function is known to model rubber to a good precision (Müller and Strehlow 2004). When \( \alpha = 0 \) this simplifies to the neo-Hookean strain-energy function given in (2.2.11).

2.2.4 Gent Strain-Energy Function

The simplified Gent strain-energy function was first introduced by Gent (1996) to model rubbers which are strain-hardening. Its derivation is from considering the hydrocarbon
chains, which comprise the rubber at an atomistic level, as being initially tangled and unextended, which then straighten and become taut as the continuum is stretched. The model has a second parameter, \( J_m \), which represents the maximum value of \( J_1 = I_1 - 3 \) beyond which the hydrocarbon chains may not extend any further, and the strain-energy function is given by

\[
W = -\frac{1}{2} \mu J_m \log \left(1 - \frac{J_1}{J_m}\right). \tag{2.2.13}
\]

This Gent strain-energy function, along with a more complex version involving a term in \( I_2 \), have become popular models for a range of rubber and bio-materials. Its use has been discussed in several papers including Horgan and Saccomandi (2003, 2006) and Ogden et al. (2006) in the context of other so called ‘finite-chain’ models, where the molecules which make up the materials are initially compact and tangled until a specific expansion ratio. Gent (1996) suggests that \( J_m = 97.2 \) or 114 are typical values of the parameter \( J_m \) for rubber, whereas for human arteries Horgan and Saccomandi (2003) give a range for \( J_m \) of between 0.422 and 3.93. The maximum stretch in any direction is given from the definition of \( I_1 \) by approximately \( \sqrt{J_m + 3} \).

### 2.2.5 General Separable Strain-Energy Function

Ogden (1997) present the general form of a separable strain-energy function expanded in terms of the stretches as

\[
W^S = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right), \tag{2.2.14}
\]

with \( \mu_n, \alpha_n \in \mathbb{R} \) and \( \mu_n \alpha_n > 0, \sum_{n=1}^{N} \mu_n \alpha_n = 2 \mu, \) for some \( N \geq 1 \). The Varga and neo-Hookean models discussed already fall into this category of strain-energy functions with \( N = 1, \alpha_1 = 1, 2 \) respectively. Equation (2.2.14) is commonly known as the Ogden model after it was introduced in Ogden (1972), with a set of constants based on experimental measurements of rubber, given by \( \mu_1 = 1.491, \mu_2 = 0.003, \mu_3 = -0.023, \alpha_1 = 1.3, \alpha_2 = 5.0, \alpha_3 = -2.0 \). This set of constants has been used in many studies, and will be denoted...
the Ogden model throughout this thesis.

Kyriakides and Chang (1991) fit such a strain-energy function to some rubber tubes, finding $\mu_1 = 617, \mu_2 = 1.86, \mu_3 = -9.79, \alpha_1 = 1.3, \alpha_2 = 5.08, \alpha_3 = -2.00$, which when non-dimensionalised with respect to $\mu$ gives very similar values as the Ogden model.

2.2.6 Fung Strain-Energy Function

Fung (1993) introduced a strain-energy function designed for the modelling of biological soft tissues, which commonly feature strain-stiffening behaviour, such as healthy arterial tissue as,

$$W = \frac{\mu (e^{\Gamma(I_1-3)} - 1)}{\Gamma},$$

(2.2.15)

where $\Gamma$ is a positive parameter representing the degree of strain stiffening. In the limit as $\Gamma$ approaches zero, (2.2.15) reproduces the neo-Hookean strain-energy function (2.2.11).

2.3 Membrane Elasticity

An elastic shell may be defined as a three-dimensional elastic body with a dimension which is much thinner than the other two, and therefore $\varepsilon = H/R \ll 1$, where $H$ is a typical thickness in the thin direction and $R$ is a typical lengthscale in the remainder of the body (Libai and Simmonds 1998). This enables us to consider the deformation of only the mid-plane of the elastic body through this thinner direction. An axishell is such a shell which has symmetry around an axis. A membrane may then be defined as a shell which has negligible resistance to bending (Libai and Simmonds 1998, Humphrey 2003). There are three different classes of membrane theories which may be used, as detailed below.
2.3.1 Membrane-Like Shells

The first of these membrane theories may be called a ‘membrane-like shell’ (Libai and Simmonds 1998), which takes the shell theory of three-dimensional elasticity and then neglects the contribution to strain-energy from bending in the shell theory. Then, from the asymptotic expansion in the small-thickness variable, the ‘membrane assumption’ of no stress through the thickness of the membrane, i.e. $\sigma_3 = 0$, is a consequence in the asymptotic approximation (Humphrey 1998, Haughton and Ogden 1978a).

In this formulation the thickness is included in the derivation, as $\lambda_3$ is still included in the governing equations. These equations explicitly allow the membrane to get thinner via stretching in the other directions. The membrane may be considered as either compressible or incompressible, although the assumption of incompressibility is commonly used.

This is a common method of treating membranes, and a comprehensive derivation may be found in Haughton and Ogden (1978a), Libai and Simmonds (1998) and Steigmann (2007). In particular, Haughton and Ogden (1978a) give explicit expressions for the errors introduced by the approximation, showing that the stress through the membrane is also of order $H/R$, as do Kydoniefs (1969) and Erbay and Demiray (1995) for particular problems in cylindrical coordinates. A variational treatment of this problem has also been considered, for example see Le Dret and Raoult (1996) and Chen (1997).

2.3.2 Simple Membranes

The second class of membrane theories occurs from considering the mid-plane of the membrane as a two-dimensional sheet of elastic material embedded into a three-dimensional space, entirely neglecting thickness effects through the membrane (Li and Steigmann 1995). Nadler and Rubin (2009) call this a ‘simple membrane’, and a consequence of this reduction is that the stretch through the membrane $\lambda_3$, does not appear directly in the formulation and is therefore assumed to remain constant. The deformation gradient
becomes two-dimensional, and there are only two strain invariants and two principal stretches rather than the three given in the three-dimensional theory.

The consequence of this is that, for instance, the equivalent of the neo-Hookean strain-energy function would be \( W = \frac{\mu}{2} (I_1 - 2) = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 - 2) \), which is clearly different to \( \hat{W} = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2} \lambda_2^{-2} - 3) \). It is possible to define two-dimensional strain-energy functions which have no counterpart in the three-dimensional theory (Haughton 2001), as well as to use three-dimensional strain-energy functions in the two-dimensional theory. It has been shown that the simple membrane and membrane-like shell approaches give the same governing equations to first order in \( \epsilon \) (Naghdi 1972, Haughton 2001, Steigmann 2007), and it is the specification of the constitutive behaviour which varies between the two theories. Further details of simple membranes may be found in Humphrey (1998), Libai and Simmonds (1998), Steigmann (2007) and a particularly concise derivation in Steigmann (2009).

### 2.3.3 Generalised Membranes

There is a third approach, denoted a ‘generalised membrane’ by Nadler and Rubin (2009), which takes the ‘membrane-like shell’ and removes the curvature of the reference geometry by assuming the reference state is flat. This has the benefit of simplifying the governing equations, but is still applicable to three-dimensional strain-energy functions unlike the ‘simple membrane’. However, if the reference geometry is highly curved then this theory will not give the same results as either of the two previous membrane theories or the fully three-dimensional theory.

### 2.3.4 Wrinkling

In addition, when considering deformations of membranes using any of the above theories, it is important to ensure that the membrane remains in tension, i.e. the principal stresses remain positive throughout the deformations (Humphrey 1998). If compressive
stresses occur then the membrane may wrinkle, which is not included in the formulation considered here. In particular, Andra et al. (2000) show that equilibrium states with distributions of infinitesimal wrinkles are possible when bending stiffness is neglected due to loss of convexity of the strain-energy function. For more details about the connection between wrinkling and compressive stresses as well as introducing quasiconvexity to extend the theory to include this case see Pipkin (1986), Li and Steigmann (1995) and Finken and Seifert (2006).

Throughout this thesis the first approach, of taking the membrane approximation from the fully three-dimensional theory, will be used with the exception of Chapter 7 in which the simple membrane theory is used to make a comparison between the two theories.

2.4 Linear Elasticity

2.4.1 Introduction

We will now consider the case of linear elasticity, that is the specification of the theory to (infinitesimally) small deformations. In this we consider the case where the displacement \( u \) is small, and will we neglect terms which are of order \( O(u^2) \) or higher. We therefore neglect any terms involving \( u_i u_j \) for any \( i \) and \( j \), as well as products of derivatives of \( u \). This approximation is the source of the name linear elasticity, and does not imply that the geometry must be linear.

2.4.2 Linearisation

On taking this linearisation and using (2.1.24), \( F = I + \text{Grad} \, u \), the Green strain tensor \( E \) becomes,

\[
E = \frac{1}{2} (F^T F - I) = \frac{1}{2} \left( \text{Grad} \, u + (\text{Grad} \, u)^T \right) + O(u^2),
\]  

(2.4.1)
or in component form in Cartesian coordinates, where we use $\varepsilon$ to denote the components of the tensor $E$,

$$
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
$$

(2.4.2)

where we have neglected the higher order terms. It should be noted that in non-Cartesian coordinates there are extra terms corresponding to the derivatives of the basis vectors. In the following Cartesian coordinates are used when component forms of the tensors are given, although the tensor relations still hold for other coordinate systems, as part of the definitions of tensors.

The diagonal components of the strain tensor, $\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}$ (no sum), are the normal strains in the directions given by $i$, and the off-diagonal components are given by $\varepsilon_{ij} = \frac{1}{2} \gamma_{ij}$, $i \neq j$, where $\gamma_{ij}$ denote the changes in angle between the initially orthogonal coordinate axes (Lur’ev 1964).

### 2.4.3 Hooke’s Law

The generalisation of Hooke’s Law mentioned in Section 2.1.8, corroborated by experimental results, is given by the injective relation

$$
\sigma_{ij} = f_{ij} \left( \varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{32}, \varepsilon_{33} \right)
$$

(2.4.3)

between the stresses $\sigma_{ij}$ and the strains $\varepsilon_{ij}$ (Sokolnikoff 1956). This relation holds for elastic materials provided the stresses do not exceed the elastic limit of the material, after which the material will fail, either from the beginning of irreversible plastic deformations, breaking or rupture.

Expressing the functions $f_{ij}$ as a power series in the strains, $\varepsilon_{ij}$, we may write,

$$
\sigma_{ij} = A_{ij} + c_{ijkl} \varepsilon_{kl} + O \left( \varepsilon_{ij}^2 \right),
$$

(2.4.4)
where $A_{ij}$ and $c_{ijkl}$ are the components of constant valued tensors of order two and four respectively. In tensor form, (2.4.4) may be written,

$$\sigma = A + c : E + O\left(E^2\right),$$  \hspace{1cm} (2.4.5)

where the central dots represent tensor contraction and $A, c$ are the appropriate constant valued tensors.

We require that the stresses $\sigma_{ij}$ vanish when the strains vanish, on the assumption that the initial state is stress-free, and therefore we require that $A_{ij} = 0$ for all $i, j$. Neglecting the second order terms in $\varepsilon$, as per our assumption of small strains, will result in the following constitutive relation for linear elasticity,

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl},$$  \hspace{1cm} (2.4.6)

where the 81 coefficients $c_{ijkl}$ in general vary throughout the material. If the $c_{ijkl}$ are independent of the position of the point then the material is homogeneous and we will henceforth make this assumption (Barber 2004). As the tensor $\sigma$ must be symmetric to satisfy the equation of angular momentum, the coefficients $c_{ijkl}$ are symmetric with respect to the indices $i$ and $j$, so $c_{ijkl} = c_{jikl}$. We can also assume, without loss of generality, that the coefficients are therefore symmetric with respect to the indices $k$ and $l$. This may be proved by reducing $c_{ijkl}$ into a symmetric and an anti-symmetric part given by

$$c_{ijkl}^S = \frac{1}{2}\left(c_{ijkl} + c_{ijlk}\right), \quad c_{ijkl}^A = \frac{1}{2}\left(c_{ijkl} - c_{ijlk}\right),$$  \hspace{1cm} (2.4.7)

whence $c_{ijkl}^S$ is symmetric in $k$ and $l$ and $c_{ijkl}^A$ is antisymmetric in the same indices. Thus the coefficients can be expressed as $c_{ijkl} = c_{ijkl}^S + c_{ijkl}^A$, so (2.4.6) can be written as:

$$\sigma_{ij} = c_{ijkl}^S\varepsilon_{kl} + c_{ijkl}^A\varepsilon_{kl}.$$  \hspace{1cm} (2.4.8)

However, the second term in this expression vanishes as $c_{ijkl}^A = -c_{ijlk}^A$ by definition and
\( \varepsilon_{kl} = \varepsilon_{lk} \) from the symmetry of \( E \).

Thus we can regard the coefficients \( c_{ijkl} \) as being symmetric with respect to both \( i, j \) and \( k, l \), and hence there are 36 independent coefficients \( c_{ijkl} \). For notational convenience we introduce the following notation for the six independent components of the stresses and strains, (Landau and Lifshitz 1986)

\[
\begin{align*}
\sigma_{11} &= \sigma_1, \sigma_{22} = \sigma_2, \sigma_{33} = \sigma_3, \sigma_{23} = \sigma_4, \sigma_{31} = \sigma_5, \sigma_{12} = \sigma_6 \\
\varepsilon_{11} &= \varepsilon_1, \varepsilon_{22} = \varepsilon_2, \varepsilon_{33} = \varepsilon_3, 2\varepsilon_{23} = \varepsilon_4, 2\varepsilon_{31} = \varepsilon_5, 2\varepsilon_{12} = \varepsilon_6,
\end{align*}
\]

(2.4.9) (2.4.10)

which introduces a factor of two into the off-diagonal elements of the strain tensor. This enables us to write concisely the six equations resulting from (2.4.6) as

\[
\sigma_i = c_{ij} \varepsilon_j,
\]

(2.4.11)

where the \( c_{ij} \) are the 36 independent elastic moduli. The number of independent constants reduces to 21 in the case where there exists a strain-energy function \( W = \frac{1}{2} c_{ij} \varepsilon_i \varepsilon_j \) with the property \( \frac{\partial W}{\partial \varepsilon_i} = \sigma_i \). As the quadratic form of \( W \) is always symmetric it follows that \( c_{ij} = c_{ji} \) and hence the reduction to merely 21 constants. The existence of the strain-energy function \( W \) has been argued using the first and second laws of thermodynamics (Sokolnikoff 1956). For a completely anisotropic material these 21 constants are the required minimum, though material symmetries reduce the number of constants which are independent. Sokolnikoff (1956) point out that in the derivation of (2.4.11) the components of strain \( \varepsilon_{ij} \) use the undeformed, or Lagrangian, coordinates while the stress components \( \sigma_{ij} \) are functions of the deformed, or Eulerian, coordinates. If the displacements are small, which we have assumed here, then this distinction is not required, but for large deformations this is not the case. Hence this formulation is only valid when the displacement derivatives are small compared to one.

For the case where the material is isotropic the number of elastic constants reduces to merely two. Landau and Lifshitz (1986) show that the stress-strain relationship for
isotropic elastic bodies may be written,

\[ \sigma_{ik} = \lambda \varepsilon_{ll} \delta_{ik} + 2G \varepsilon_{ik}, \quad (2.4.12) \]

or, in tensor form,

\[ \sigma = \lambda \text{tr}(E) I + 2GE, \quad (2.4.13) \]

where \( \lambda \) is the first Lamé constant, \( G \) is the shear modulus of the material and \( \text{tr}(E) \) is the trace of the tensor \( E \). We use the alternative notation \( G \) here for the shear modulus in order to reserve \( \mu \) for when we use linear elasticity in Chapter 8. These two parameters fully characterise the elastic properties of any isotropic material, although we will replace the first Lamé constant by the Poisson’s ratio, \( \nu \), given by \( \nu = \lambda / (2(G + \lambda)) \). The shear modulus represents the resistance to shearing deformations and Poisson’s ratio governs the relative contraction in the other two directions when the material undergoes uniaxial extension. For most materials Poisson’s ratio lies between 0 and \( \frac{1}{2} \), with the value of \( \frac{1}{2} \) corresponding to an incompressible material. Negative values of the Poisson’s ratio may also occur for specific materials with a high internal void fraction such as foams, which are called auxetic.

### 2.4.4 Equilibrium equations

With (2.4.1) and (2.4.13) the displacements and stresses are coupled together. Along with (2.1.41) and appropriate boundary conditions we have all the required equations to determine the deformation of an elastic material under boundary conditions prescribing either stresses or displacements. Eliminating the stress tensor \( \sigma \) in the equilibrium equations (2.1.41) by using (2.4.13) and (2.4.1), we find the equation for equilibrium in terms of the displacements is given by:

\[ \frac{1}{1 - 2\nu} \text{grad} \text{ div} u + \nabla^2 u + \frac{b}{G} = 0. \quad (2.4.14) \]
Note that for an incompressible material, $\nu = 1/2$, and hence $\lambda \to \infty$, which reduces the equilibrium equation (2.4.14) to

$$\text{div } u = 0.$$  \hfill (2.4.15)

### 2.5 Compound Matrix Method

#### 2.5.1 Introduction

In this section we present a method for solving eigenvalue problems in differential equations where the boundary conditions are prescribed at two ends, called the Compound Matrix method. This method works well for differential equations which may contain singularities or are otherwise numerically stiff. We also introduce the Evans function, a complex analytic function whose zeros correspond to the eigenvalues (Evans 1975, Alexander et al. 1990). The use of the Evans function ensures that the value of the eigenvalue found is not dependent on the point at which the solutions are matched. For further details on the use and derivation of the compound matrix method see Ng and Reid (1985) and Haughton (2009).

The method works on a set of $n$ first-order linear ordinary differential equations, with variable coefficients, which include a parameter $\zeta \in \mathbb{R}$, given by,

$$\frac{dy}{dZ} = A(Z; \zeta) \cdot y, \quad a \leq Z \leq b \hfill (2.5.1)$$

where $y(Z) = (y_1(Z), y_2(Z), \ldots, y_n(Z))$ is the set of $n$ dependent variables, $Z$ is the independent variable and $A(Z)$ is a $n \times n$ matrix determined by the coefficients in the original differential equations. In addition, one or both of the two endpoints $a, b$ may be infinite.

We assume that there are associated boundary conditions, $n$ in total, defined at one or
both ends of the range of $Z$ by,

$$B(Z; \zeta)y = 0 \text{ at } Z = a \quad \text{(2.5.2a)}$$

$$C(Z; \zeta)y = 0 \text{ at } Z = b, \quad \text{(2.5.2b)}$$

where $B$ and $C$ are $m_1 \times n$ and $m_2 \times n$ matrices respectively, where $m_1 + m_2 = n$, and are both known functions of $Z$ and $\zeta$. These boundary conditions (2.5.2) enable us to specify any combination of boundary conditions at the endpoints. The case which we will consider here is when $n$ is even and $m_1 = m_2 = m = n/2$, although the method is general and works for the case $m_1 \neq m_2$. We wish to find the values of the parameter $\zeta$ such that the system has a non-trivial solution which satisfies the boundary conditions, and this value of $\zeta$ will be called an eigenvalue of the differential equations. Naively, it is possible to guess a value of $\zeta$, along with the unspecified components at $Z = a$, and numerically integrate from $a$ to $b$, varying both $\zeta$ and the unspecified components at $Z = a$ to satisfy the boundary conditions specified at $b$. However, this is naturally inefficient and unsystematic. Instead, it would be common to use a determinant based method in order to solve the equations, which we will describe here, before explaining the compound matrix method.

### 2.5.2 Determinant Based Method

First, for a given value of $\zeta$, assuming $B$ has rank $m$ we must be able to find $m$ linearly independent solutions to (2.5.2a) which we denote $y^{(i)}_0, i = 1, 2, \ldots, m$. Using these vectors as our initial value at $Z = a$, we may integrate using (2.5.1) towards $Z = b$, leading to a set of $m$ independent solutions $y^{(i)}(Z), i = 1, 2, \ldots, m$. Therefore a general solution which satisfies the left boundary condition and the equation (2.5.1) is given by,

$$y = \sum_{i=1}^{m} k_i y^{(i)}(Z), \quad \text{(2.5.3)}$$

for some constants $k_i$. 
We then proceed the same way from the right hand endpoint, denoting by $y^{(i)}_0, i = m + 1, m + 2, \ldots, n$ the independent solutions to (2.5.2), and integrating towards $Z = a$, resulting in a second set of $m$ independent solutions $y^{(i)}(Z), i = m + 1, m + 2, \ldots, n$. Therefore the general solution satisfying the right boundary condition is,

$$y = \sum_{i=m+1}^{n} k_i y^{(i)}(Z), \quad (2.5.4)$$

for some further constants $k_i$. We then impose that these two solutions must match at some intermediate point $Z = d$. Therefore,

$$\sum_{i=1}^{m} k_i y^{(i)}(Z) = \sum_{i=m+1}^{n} k_i y^{(i)}(Z), \quad \text{at } Z = d, \quad (2.5.5)$$

or equivalently,

$$N(d, \zeta) c = 0, \quad (2.5.6)$$

where $c = (k_1, k_2, \ldots, k_n, -k_{n+1}, -k_{n+2}, \ldots, -k_n)^T$ and $N(Z, \zeta)$ is a matrix formed from concatenating the $n$ vectors, $y^{(i)}(Z)$

$$N(Z, \zeta) = [y^{(1)}, y^{(2)}, \ldots, y^{(m)}, y^{(m+1)}, y^{(m+2)}, \ldots, y^{(n)}]. \quad (2.5.7)$$

Therefore we need to iterate on $\zeta$ until the matching condition,

$$|N(d, \zeta)| = 0, \quad (2.5.8)$$

is satisfied. This matching condition is dependent on $d$ as well as the value of $\zeta$. It may be shown that the following condition,

$$D(\zeta) = e^{-\int_{a}^{d} \text{tr } A(s, \zeta) ds} |N(d, \zeta)| = 0, \quad (2.5.9)$$

is independent of the value of $d$. This may be seen using the following matrix property Chadwick (1999), $\frac{d\det\text{A}}{dx} = \text{tr } (\text{adj } \text{A}) \frac{dA}{dx}$ for any square matrix $\text{A}$, that this function is
in fact independent of the matching point $d$.

### 2.5.3 Compound Matrix Method

The previous determinant method may become stiff for some eigenvalue problems, due to the presence of exponentially growing solutions. We therefore describe the compound matrix method, a method which enables us to remove this exponential dependence. We use the two sets of linearly independent solutions, $(y^{(1)}, y^{(2)}, \ldots, y^{(m)})$ and $(y^{(m+1)}, y^{(m+2)}, \ldots, y^{(n)})$, defined as previously. We now define an $n \times m$ matrix $M^{-}(Z, \zeta)$ defined by concatenating the solutions formed from integrating from the left endpoint,

$$M^{-}(Z, \zeta) = [y^{(1)}(Z), y^{(2)}(Z), \ldots, y^{(m)}(Z)], \quad (2.5.10)$$

The determinant of $M^{-}$ has $\binom{n}{m}$ 2 by 2 minors, which we denote by $\phi_1, \phi_2, \ldots$, and when $m = 2$ we have,

$$\phi_1 = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} \equiv (1, 2), \quad \phi_2 = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_3^{(1)} & y_3^{(2)} \end{vmatrix} \equiv (1, 3), \quad (2.5.11)$$

and we then define the other minors in the same way by, $\phi_3 = (1, 4), \phi_4 = (2, 3), \phi_5 = (2, 4), \phi_6 = (3, 4)$. These $\phi$’s satisfy first-order differential equations, which may be found by differentiating inside the determinant as,

$$\phi_1' = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} = \sum_{i=1}^{4} A_{1i} y_i^{(1)} + \sum_{i=1}^{4} A_{1i} y_i^{(2)} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} \sum_{i=1}^{4} A_{2i} y_i^{(1)} + \sum_{i=1}^{4} A_{2i} y_i^{(2)}$$

$$= A_{11}\phi_1 - A_{13}\phi_4 - A_{14}\phi_5 + A_{22}\phi_1 + A_{23}\phi_2 + A_{24}\phi_3, \quad (2.5.12)$$
which therefore may be written as a matrix equation,

\[ \phi' = Q(Z; \zeta) \phi, \quad a \leq Z \leq b, \]  

(2.5.13)

where the \( \binom{n}{m} \) by \( \binom{n}{m} \) matrix \( Q \) is given by, in the \( m = 2 \) case, as

\[
Q = \begin{pmatrix}
A_{11} + A_{22} & A_{23} & A_{24} & -A_{13} & -A_{14} & 0 \\
A_{32} & A_{11} + A_{33} & A_{34} & A_{12} & 0 & -A_{14} \\
A_{42} & A_{43} & A_{11} + A_{44} & 0 & A_{12} & A_{13} \\
-A_{31} & A_{21} & 0 & A_{22} + A_{33} & A_{34} & -A_{24} \\
-A_{41} & 0 & A_{21} & A_{43} & A_{22} + A_{44} & A_{23} \\
0 & -A_{41} & A_{31} & -A_{42} & A_{32} & A_{33} + A_{44}
\end{pmatrix}
\]  

(2.5.14)

The boundary conditions in terms of \( \phi' \)'s come from evaluating \( M^- \) or \( M^+ \) at the boundaries. As an example, if one of the sets of boundary conditions is given by \( y_1(a) = y_3(a) = 0 \), then \( y(a) = (0, y_2, 0, y_4)^T \). The two independent solutions would then be, \( y_0^{(1)} = (0, 1, 0, 0)^T, y_0^{(2)} = (0, 0, 0, 1)^T \), and the \( \phi_i \) are the minors of the matrix \( M^- \) and therefore,

\[
M^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi_1(a) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0, \quad \phi_2(a) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \]  

(2.5.15)

and additionally, \( \phi_3(a) = 0, \phi_4(a) = 0, \phi_5(a) = 1, \phi_6(a) = 0 \). A similar process would enable us to find the \( \phi_i \) at \( Z = b \), given appropriate boundary conditions.

With these boundary conditions at \( Z = a \) we may then shoot from both sides to match in the middle at \( Z = d \) again. We denote by \( \phi^- \) the \( \phi \) which is formed from the minors of \( M^- \) and \( \phi^+ \) is formed from the minors of \( M^+ = [y^{(m+1)}, y^{(m+2)}, \ldots, y^{(n)}] \), where the other formulae are as before. When \( m = 2 \) the matching condition \( |N| \) may be expanded using
the Laplace expansion of the first two columns as,

\[
|N(Z; \zeta)| = |[y^1(Z), y^2(Z), y^3(Z), y^4(Z)]| = \begin{vmatrix}
  y_1^{(1)} & y_1^{(2)} & y_1^{(3)} & y_1^{(4)} \\
  y_2^{(1)} & y_2^{(2)} & y_2^{(3)} & y_2^{(4)} \\
  y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\
  y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \\
\end{vmatrix}
\]

\[= \begin{vmatrix}
  y_1^{(1)} & y_1^{(2)} & (\ldots) \ (\ldots) & y_1^{(3)} & y_1^{(4)} \\
  y_2^{(1)} & y_2^{(2)} & (\ldots) & y_2^{(3)} & y_2^{(4)} \\
  y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\
  y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \\
\end{vmatrix}
\]

\[+ \begin{vmatrix}
  y_1^{(1)} & y_1^{(2)} & (\ldots) & y_1^{(3)} & y_1^{(4)} \\
  y_2^{(1)} & y_2^{(2)} & y_2^{(3)} & y_2^{(4)} \\
  y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\
  y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \\
\end{vmatrix}
\]

\[+ \begin{vmatrix}
  y_2^{(1)} & y_2^{(2)} & (\ldots) & y_2^{(3)} & y_2^{(4)} \\
  y_1^{(1)} & y_1^{(2)} & y_1^{(3)} & y_1^{(4)} \\
  y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\
  y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \\
\end{vmatrix}
\]

\[= \begin{vmatrix}
  y_1^{(1)} & y_1^{(2)} & (\ldots) & y_1^{(3)} & y_1^{(4)} \\
  y_2^{(1)} & y_2^{(2)} & y_2^{(3)} & y_2^{(4)} \\
  y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\
  y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \\
\end{vmatrix}
\]

which may then be written as,

\[
|N(d, \zeta)| = \psi_1^+ \psi_6^- - \psi_2^+ \psi_5^- + \psi_3^+ \psi_4^- + \psi_4^+ \psi_3^- - \psi_5^+ \psi_2^- + \psi_6^+ \psi_1^-,
\]

\[\text{(2.5.17)}\]

with the Evans function as defined in (2.5.9) still applying as before.
2.6 Legendre Functions and Spherical Harmonics

2.6.1 Introduction

In this section we present a set of functions which are known as the Legendre polynomials, which arise as the solution to a specific differential equation involved in the solution of Laplace’s equation in spherical coordinates. These will be used in later chapters and therefore are presented here for reference.

2.6.2 Spherical Harmonics

A spherical harmonic \( V \) is a homogeneous function of degree \( n \) in \( x, y, z \) which satisfies Laplace’s (or the harmonic) equation \( \nabla^2 V = 0 \) (Hobson 1931). When expressed in spherical coordinates, \((r, \theta, \phi)\), a spherical harmonic of degree \( n \) takes the form \( r^n f_n(\theta, \phi) \), where \( f_n(\theta, \phi) \) is denoted a surface or zonal spherical harmonic of degree \( n \) (Hobson 1931). Here we derive the equations for Legendre polynomials, a type of spherical harmonic, as well as several useful properties between them. In spherical coordinates, Laplace’s equation may be expressed as,

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0. \tag{2.6.1}
\]

Using the method of separation of variables we assume the solution is given in the form of

\[
V = R(r)\Theta(\theta)\Phi(\phi), \tag{2.6.2}
\]

where \( R, \Theta \) and \( \Phi \) are functions only of \( r, \theta \) and \( \phi \) respectively. Substituting (2.6.2) into (2.6.1) and dividing through by \( R(r) \), we find that the only terms involving \( r \) come from the first term of equation (2.6.2), and in order for this term to balance the other terms it must be a constant. Hence \( \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) \) is equal to a constant, which we choose to be \( n(n + 1) \) to simplify the resulting expression. Solving this equation for \( R(r) \) using a trial
solution of the form $R = r^k$ for some $k$, we find

$$R(r) = Ar^n + Br^{-n-1},$$

(2.6.3)

where $A$ and $B$ are arbitrary constants. Therefore Laplace’s equation (2.6.1) may be written as

$$n(n+1) \sin^2 \theta + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0.$$

(2.6.4)

Here we can see that the function $\Phi$ satisfies the equation $\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2$ by the same argument as used with the $r$-dependence, where we have chosen the constant to be $-m^2$. The choice of the negative constant means that, for real $m$, the solutions are given by trigonometric functions instead of exponential functions. With this choice of constant we may solve for the $\Phi$ dependence, giving

$$\Phi(\phi) = C \cos(m\phi + D),$$

(2.6.5)

where $C$ and $D$ are two further arbitrary constants. The remaining equation for the $\theta$ dependence of Laplace’s equation is therefore given by,

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left( n(n+1) - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0.$$

(2.6.6)

Equation (2.6.6) may be rewritten by using the substitution $\cos \theta = \mu$, resulting in what is known as the associated Legendre differential equation:

$$\frac{d}{d\mu} \left( (1 - \mu^2) \frac{d\Theta}{d\mu} \right) + \left( n(n+1) - \frac{m^2}{1 - \mu^2} \right) \Theta = 0.$$

(2.6.7)

2.6.3 Legendre Polynomials

If there exists symmetry around an axis in the three-dimensional space, we may choose the coordinate system such that this symmetry axis is aligned with the $\phi = 0$ axis. Therefore we may remove the $\phi$ dependence of the function $V$, and set $m = 0$ in the equation
The resulting differential equation is known as Legendre’s differential equation (Hobson 1931),

\[
\frac{d}{d\mu} \left( (1 - \mu^2) \frac{d\Theta}{d\mu} \right) + n(n + 1)\Theta = 0,
\]

and it is shown by Farrell and Ross (1963) that this equation has solutions of the first kind given by

\[
\Theta(\mu) = C_0 \left[ 1 - \frac{n(n+1)}{2!} \mu^2 + \frac{n(n+1)(n-2)(n+3)}{4!} \mu^4 - \ldots \right] \\
+ C_1 \left[ \mu - \frac{(n-1)(n+2)}{3!} \mu^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} \mu^5 - \ldots \right],
\]

valid for \(|\mu| < 1\) with \(C_0, C_1\) as arbitrary constants. There are also solutions of a second kind which converge for \(|\mu| > 1\). When \(n\) is an integer one of the two sums given in (2.6.9) terminates at a finite length, and we therefore choose to set the constant in front of the non-terminating series to be zero, depending on whether \(n\) is even or odd, and choose the other arbitrary constant to be \(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}\), which ensures that the finite polynomial has the value one at \(\mu = 1\). These solutions to (2.6.8) with integer \(n\) are called Legendre polynomials and are denoted by \(P_n(\mu)\), where \(n\) is the degree of the polynomial. The first few Legendre polynomials are given by (MacRobert 1947),

\[
P_0(\mu) = 1, \ P_1(\mu) = \mu, \ P_2(\mu) = \frac{1}{2}(3\mu^2 - 1),
\]

\[
P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu), \ P_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3),
\]

and are found tabulated in various sources. The \(n\)th Legendre polynomial has maximum degree \(n\) and only contains powers of \(\mu\) which have the same polarity as \(n\). There also exists the following symmetry in the index \(n\),

\[
P_n(\mu) = P_{-n-1}(\mu),
\]
resulting from the invariance of (2.6.8) when \( n \) is replaced by \(-(n + 1)\). Therefore, when \( m = 0 \), the resulting solutions of (2.6.1) are given by

\[
V(r, \theta) = \sum_{n=0}^{\infty} \left( Ar^n + Br^{-n-1} \right) \left( E P_n(\mu) + F Q_n(\mu) \right),
\]

where \( A, B, E, F \) are arbitrary coefficients determined by initial and boundary conditions and \( Q_n(\mu) \) is the \( n \)th Legendre function of the second kind, which is not convergent for \(|\mu| < 1\) and hence \( F = 0 \) when \( \mu = \cos \theta \). Due to (2.6.12) and the same symmetry in the \( r \) component, we only require the sum to take the positive values of \( n \).

The following expression, known as Rodrigue’s formula, gives a closed form for the Legendre polynomials (Hobson 1931),

\[
P_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n}(\mu^2 - 1)^n,
\]

which is particularly useful for calculating integral relations involving the Legendre polynomials, and is used in the following derivations.

If we allow \( n \) to be a non-integer in the definition (2.6.9), we obtain what are known as the Legendre functions. These are still denoted \( P_n(\mu) \), but do not have a closed form representation, and are defined for all complex \( n \) in the form of a hypergeometric series, as indeed they may be when \( n \) is an integer.

### 2.6.4 Orthogonality and Integral Formulae

The Legendre polynomials form an orthogonal set of functions. This may be derived by using the Rodrigue’s formula (2.6.14), and the orthogonality property is given by,

\[
\int_{-1}^{1} P_l(\mu) P_n(\mu)d\mu = \frac{2}{2n + 1} \delta_{ln},
\]

where \( \delta_{ln} \) is the Kronecker delta which is one if \( l = n \) or zero if \( l \neq n \). An orthonormal set of polynomials can be created by including the factor \( \sqrt{\frac{2}{2n+1}} \) as an additional coefficient.
in the definition of $P_n(\mu)$, though this alters the value of the polynomials at unity as well as Rodrigue’s formulae and other relations.

The following integral relations are also true for $l < n$ and $l \leq n$ respectively, which may be intuitively seen, apart from the constants, by expressing $\mu P_l(\mu)$ in terms of a sum of Legendre polynomials up to degree $l + 1$ and using (2.6.15),

$$\int_{-1}^{1} \mu P_l(\mu)P_n(\mu)d\mu = \frac{2n}{4n^2 - 1}\delta_{(n-1)}. \quad (2.6.16)$$

$$\int_{-1}^{1} \mu^2 P_l(\mu)P_n(\mu)d\mu = \frac{4n^2 + 4n - 2}{(2n - 1)(2n + 1)(2n + 3)}\delta_n + \frac{2n(n - 1)}{(2n - 3)(2n - 1)(2n + 1)}\delta_{(n-2)}, \quad (2.6.17)$$

2.6.5 Function Expansion Theorem

As the Legendre polynomials (with integer values of $n$) are a complete orthogonal set, any piecewise continuous function may be expressed as a linear combination of the infinite set of Legendre polynomials using the following theorem (Whittaker and Watson 1927 cited by Farrell and Ross 1963):

If a function $f(x)$ is bounded on the interval $[-1, 1]$, and continuous except for a finite number of discontinuities then there exists a series of constant coefficients, $A_k$, such that the series

$$\sum_{k=0}^{\infty} A_k P_k(x), \quad (2.6.18)$$

converges to $f(x)$ on every point of continuity of $f(x)$ on $[-1, 1]$ and converges on every discontinuity. The coefficients $A_k$ in this series are determined by the relation,

$$A_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x)P_k(x)dx. \quad (2.6.19)$$

This relation enables any suitable function to be approximated with a finite number of
Legendre polynomials, and therefore the expression in (2.6.13) is a suitably general solution to the Laplace equation.

2.6.6 Associated Legendre Functions

We now return to (2.6.7) and consider the case when $m$ is a positive integer. We define the Associated Legendre polynomials as being solutions to (2.6.7) with such a positive integer $m$. For each value of $n$ there are $2m + 1$ independent solutions of the first kind of (2.6.7), given by the following relation (MacRobert 1947),

$$P_n^m(\mu) = (-1)^m (1 - \mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m}, \quad m = 0, 1, \ldots, n$$

(2.6.20)

for the positive values of $m$. The associated Legendre functions for negative integer $m$ may then be calculated using the relation,

$$P_n^{-m}(\mu) = (-1)^m \frac{(n - m)!}{(n + m)!} P_n^m(\mu).$$

(2.6.21)

It should be noted that the inclusion of the factor $(-1)^m$ is a convention which is not followed in all definitions of the associated Legendre polynomials. These associated Legendre polynomials are used when the function $V$ has dependence on the third spherical coordinate, $\phi$, for when the condition of symmetry around an axis is not imposed. They have a closed form similar to Rodrigue’s formula by combining (2.6.20) with (2.6.14) which holds for both positive and negative integer values of $m$ and is given by:

$$P_n^m(\mu) = \frac{1}{2^n n!} (1 - \mu^2)^{-m/2} \frac{d^{m+n}[\mu^2 - 1]^n}{d\mu^{m+n}}.$$  

(2.6.22)

Note that, for odd values of $m$, the associated Legendre polynomial is not a polynomial due to the presence of the square root in (2.6.20) and (2.6.22), despite what the name would suggest. However, they are still commonly denoted the associated Legendre polynomials for integer values of $n$ and $m$, with $|m| \leq n$. The first few such functions are
given by

\[ P_0^0(\mu) = 1, \]
\[ P_{-1}^0(\mu) = \frac{1}{2} \sqrt{1 - \mu^2}, \quad P_1^0(\mu) = \mu, \quad P_1^1(\mu) = -\sqrt{1 - \mu^2} \]  \hspace{1cm} (2.6.23, 2.6.24)

The associated Legendre polynomials are also orthogonal with respect to \( n \) over \([-1, 1]\), with the following orthogonality relation including the constants,

\[ \int_{-1}^{1} P_l^m(\mu) P^m_n(\mu) d\mu = \frac{2}{2n + 1} \frac{(n + m)!}{(n - m)!} \delta_{ln}. \]  \hspace{1cm} (2.6.25)

The surface harmonics mentioned in (2.6.2) are given by,

\[ Y_n^m(\theta, \phi) = P_n^m(\cos \theta)e^{im\phi}, \]  \hspace{1cm} (2.6.26)

and the general solution to Laplace’s equation when the axial symmetry is not imposed, (2.6.1), may be written as,

\[ V(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( Ar^n + Br^{-n-1} \right) \left( C \cos m\phi + D \sin m\phi \right) \left( EP_n^m(\mu) + FQ_n^m(\mu) \right), \]  \hspace{1cm} (2.6.27)

subject to suitable boundary conditions to determine the six arbitrary constants.
Chapter 3

Non-Uniform Inflation of a Cylindrical Membrane

3.1 Introduction

In this chapter we consider a cylindrical, hyperelastic, isotropic, incompressible membrane, subject to a constant internal pressure. This boundary value problem has been studied by many authors as a classical problem of solid mechanics. Adkins and Rivlin (1952), Green and Adkins (1960) and Kydoniefs (1969) all consider this, whilst requiring the tube to remain cylindrical under the deformation. A connected bifurcation analysis considers whether or not multiple configurations of the deformed tube can exist for a given set of boundary conditions and internal pressure. Of particular interest are locally bulging solutions, where there exists an enlarged section of the tube connected to uniformly inflated sections away from the local bulge, in a similar manner to soliton solutions in wave propagation. The biological motivation behind interest in these solutions is the study of aneurysms found in blood vessels; such a local bulge causes a lowering of the blood pressure, blood stagnation and potential rupture, with serious medical complications.
In this chapter we consider the conditions on the strain-energy function of the material of the tube required in order for such bulging or necking solutions to exist. Chapter 4 will then study the stability of such solutions. Throughout these two chapters the examples will be mostly given for the Gent strain-energy function (2.2.13). Chapter 5 will then compare a range of strain-energy functions.

### 3.2 Literature Review

#### 3.2.1 Experimental Studies

The inflation of a thin-walled rubber tube via control of the internal volume or pressure has been considered experimentally by a range of authors, often entwined with analytical or numerical results. In the experiments of Kyriakides and Chang (1990), which are pressure controlled using compressed air, the tube inflates cylindrically until a limiting pressure is reached, and these experiments are shown in Figure 3.2.1. After this limiting pressure is reached a bulge forms at one point of the tube with a decrease in the radius elsewhere. Further inflation causes the diameter to expand before reaching a maximum size, at which point the bulge begins to spread laterally along the tube at a constant propagation pressure. After the bulge has filled the tube the inflation continues in a uniform manner, with an increasing pressure. While the non-uniformity is localised in this way, the remainder of the tube remains uniform. In Kyriakides and Chang (1991), the deformation is volume controlled by filling the tube with water, resulting in similar behaviour as described for the pressure controlled case. These experiments are consistent with those described by Alexander (1971).

When an ordinary cylindrical rubber party balloon is inflated these features may be observed, particularly the state in which two uniform sections of different radii are joined by a non-uniform section. Due to comparisons with various phenomena observed in other problems which will be discussed later, this behaviour will be called the *kinked* state.
throughout this thesis, and is discussed in Section 3.6. The initial bulging is a transient feature in this case, which suggests that this region is unstable for rubber balloons.

Pamplona et al. (2006) investigate the inflation of pre-stressed rubber tubes, showing the bulging behaviour as discussed above, along with the corresponding decrease in pressure. Their results are for tubes which have very thin walls, with thickness ratio $H/R \ll 1$. In addition to their experimental results, they use finite element simulation to produce corresponding results for tubes modelled by a Mooney-Rivlin strain-energy function. However, they do not find the kinked solution, in either the experiments or their numerical simulations, which we will later show to be consistent with our results for the Mooney-Rivlin strain-energy function. Goncalves et al. (2008) detail, for a range of physical thick-walled specimens with $\frac{1}{4} \leq H/R \leq \frac{1}{3}$, the same onset of localised bulging as found for the thin walled case.
3.2.2 Analytical Studies

Alexander (1971) considered the uniform inflation of the tube, using both linear elasticity and a specific nonlinear strain-energy function which has not been used in further studies. They found the relationship between the pressure and the inflated radius for these two cases. However, they assumed that the localised bulging found experimentally was caused by material or geometric variations and not as a consequence of the non-monotonic pressure-volume curve. They found that the linear theory incorrectly predicts that the axial load has no dependence on this deformation, and show that the limiting pressure at which localised bulging occurs varies with the value of the axial load for their strain-energy function. Benedict et al. (1979) extend this work to include a non-zero axial force, and present a numerical scheme for finding the limiting pressure in this case, as well as the case of prescribed elongation. However, they do not consider what happens after the limiting pressure, including any description of the bulged state.

Pipkin (1968) shows, in an extremely concise way, that both of the equations of equilibrium for an axisymmetric cylindrical membrane may be integrated, provided the membrane is isotropic and a strain-energy function exists. This is a very useful result as it reduces the order of the differential equations by one.

Yin (1977) presents an analytical study in which he, using parallels with the inflation of spherical balloons, makes some assumptions on the form of the strain-energy function in order to consider the propagating solution found experimentally. Yin (1977) finds that such a solution has a lower energy than a uniform tube with the same internal pressure, and therefore it is the energy-minimising configuration, assuming the internal pressure is from an isothermic gas. Details are given of a method for determining the properties of the strain-energy function from experimental results for a particular material sample. The very large deformations which occur during the inflation process make it particularly attractive to determine the behaviour of the strain-energy function at high strain using this technique.
Haughton and Ogden (1979a) consider bifurcations in both the case where the bifurcated mode remains axisymmetric, and also prismatic modes where the tube deforms to a non-axisymmetric shape throughout the tube. Haughton and Ogden (1979a) present a bifurcation condition on the general separable strain-energy function of the type given in (2.2.14) for the \( n \)th prismatic mode to exist, that is at least one of the exponents \( \alpha \) has an absolute value less than unity. In the following sections it will be shown that this condition, when evaluated for bifurcation into the zeroth mode, gives the condition for axisymmetric bifurcated modes which are non-uniform in the axial direction, as we are interested in. However, in Haughton and Ogden (1979a) this condition was discarded as the linear eigenfunction for the zeroth mode predicts a constant, whereas in fact a non-constant function exists for the nonlinear equation. They do correctly state that the bifurcation, if it exists, occurs before the pressure maximum, for the case of an infinite tube with fixed axial stretch. They do not consider the case of closed ends, which shall be discussed here.

Chater and Hutchinson (1984) show that the propagating solution exists whenever the pressure-volume curve for uniform inflation has a Maxwell construction; that is a value of the pressure such that two distinct areas bounded between the curve and the straight line have equal area. The existence of such a Maxwell line is guaranteed when the pressure-volume curve has a local, but not global, maximum followed by a minimum. Chater and Hutchinson (1984) point out the similarities of this deformation with propagating buckles in metal tubes (Kyriakides and Babcock 1981), propagating necks in materials under tension (Hutchinson and Neale 1983), and phase transitions in bars (Ericksen 1975). They also state that a characteristic of such behaviour is the high pressure required to enter the bifurcated mode, followed by a much lower value required once the mode has been initiated. This characteristic is seen in the previously cited studies as well as in the experiments by Kyriakides and Chang (1990).

Kyriakides and Chang (1991) solve the differential equations for the membrane tube numerically for various strain-energy functions, and find good agreement with their experimental results, where the ratio of thickness to radius for the rubber tubes involved was
approximately one quarter.

Gent (2005) and Kanner and Horgan (2007) discuss how the form of the strain-energy function which models a closed tube affects whether a limit-point instability exists, after which localised bulges occur. They show how these instabilities exist at the maximum of the pressure-volume curve belonging to the inflated cylindrical tube.

Haughton (2001) derives the bifurcation condition and provides examples of the bulged configuration for the Varga strain-energy function in a tube with no axial stretch. Here we shall expand this for general strain-energy functions, closed or open ended tubes and any value of axial stretch.

3.3 Inflation of a Cylindrical Tube

3.3.1 Governing Equations

We develop a description of the elastic deformation of such a tube for a general isotropic, incompressible strain-energy function, which may be validated by the experimental results conducted on rubber detailed above. We are then interested in the response of non-rubberlike strain-energy functions, particularly those which are appropriate to biological materials.

The analysis considered here is for an incompressible, isotropic, hyperelastic membrane tube. We consider only axially symmetric deformations from the originally axially symmetric configuration. For prismatic deformations we refer to Haughton and Ogden (1979a). In the reference configuration the tube is assumed to have uniform radius $R$, uniform thickness $H$ and is described by coordinates $0 \leq \Theta \leq 2\pi, -\infty \leq Z \leq \infty$. The position vector $X$ is given by $X = Re_{\mathcal{R}}(\Theta) + Ze_{Z}$. When inflated by a constant internal pressure, $P$, it is assumed that the inflated configuration maintains axial symmetry and the radius may be constant or vary along the axial direction. We assume that in the undeformed tube the thickness is much smaller than the radius, $H \ll R$, and we only
consider the deformation of the midplane of the tube wall, using the membrane assumption detailed in Section 2.3. It has been shown in Haughton and Ogden (1979a) that the magnitude of the error when using these approximations is of the order $O(H/R)^2$, as opposed to the expected $O(H/R)$. This extends the validity of the solution for thicker walled tubes than would otherwise be the case.

We describe the axisymmetric deformed configuration $(r, \theta, z)$ by

$$r = r(Z), \quad \theta = \Theta, \quad z = z(Z),$$

(3.3.1)

where $Z$ and $z$ are the axial coordinates of a representative material particle before and after inflation respectively and $r$ is the inner radius after inflation; see Figure 3.3.1. The lack of dependence of (3.3.1) on $R$ comes from considering solely the deformation of the midplane, as discussed above. The thickness of the deformed configuration is then given by $h$. The deformed position vector $x$ is given by $x = re_R(\theta) + ze_Z$, noting that we have chosen to use the same coordinate basis in the two configurations.

The stretch in the meridional direction is given by the ratio of the circumference in the deformed configuration with that in the undeformed configuration, while the stretch in the latitudinal direction is given by the derivative of the arc length $s$ in the deformed configuration with respect to the arc length $S$ in the undeformed configuration, as can be seen in Figure 3.3.1. It should be noted that $S = Z$ in this case as the undeformed
configuration is a tube with constant radius $R$. From Figure 3.3.1 we can see that this can be written as:

$$\lambda_0 = \frac{2\pi r}{2\pi R}, \quad \lambda_z = \frac{ds}{dS} = \frac{ds}{dZ}, \quad \lambda_r = \frac{h}{H}. \quad (3.3.2)$$

Since the deformation is axially symmetric, the principal directions of stretch coincide with the lines of latitude, the meridian and the normal to the deformed surface. Thus, the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = (r^2 + z'^2)^{\frac{1}{2}}, \quad \lambda_3 = \frac{h}{H}, \quad (3.3.3)$$

where we use the indices $(1, 2, 3)$ for the latitudinal, meridional and normal directions respectively and the primes indicate differentiation with respect to $Z$. Clearly, these three directions are mutually orthogonal, and are therefore the principal directions of the deformation. Thus the principal Cauchy stresses in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i W_i - p, \quad i = 1, 2, 3, \quad (3.3.4)$$

where $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function, $W_i = \partial W / \partial \lambda_i$ and $p$ is the Lagrange multiplier associated with the constraint of incompressibility. Utilising the incompressibility constraint, $\lambda_1 \lambda_2 \lambda_3 = 1$, and the membrane assumption of no stress in the thickness direction, $\sigma_3 = 0$, equation (3.3.4) becomes

$$\sigma_i = \lambda_i \hat{W}_i, \quad i = 1, 2, \quad (3.3.5)$$

where $\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1} \lambda_2^{-1})$ and $\hat{W}_i = \partial \hat{W} / \partial \lambda_i$ (Ogden 1997).

Considering the balance of forces in an infinitesimal section of the material formed by cutting the tube twice perpendicular to the $Z$-axis, we find that

$$r(\sigma_2 h)' + h r'(\sigma_2 - \sigma_1) = 0. \quad (3.3.6)$$
Similarly, by considering equilibrium in the normal direction of a small volume element,

\[ \frac{\sigma_2 (r''z' - r'z'')}{\lambda_2^2} - \frac{z' \sigma_1}{r \lambda_2} + \frac{P^*}{\lambda_3} = 0, \]  

(3.3.7)

where \( P^* = P/H \). We will discuss later how these two equations may be derived from the equilibrium equations given in (2.1.41). Using (3.3.6) to express \( \sigma_1 \) as \( \sigma_1 = \frac{\langle h' r \sigma_2 \rangle'}{h r'} \), (3.3.7) is a first order differential equation for \( \sigma_2 \) which may be solved via an integrating factor leading to,

\[ \frac{\sigma_2 z'}{\lambda_2^2} - \frac{1}{2} P^* \lambda_1^2 R = C_2, \]  

(3.3.8)

where \( C_2 \) is a constant. This equation could be derived immediately by considering the resultant from balancing forces acting through the tube in the \( Z \)-direction, and exists even when \( R \) or \( H \) are non-constant functions of \( Z \). However, it does require that the internal pressure \( P \) is not a function of \( Z \), and therefore it is not appropriate in the case of tubes which are partially filled with a liquid, such as those considered by Pamplona et al. (2001). It was shown by Pipkin (1968) that there exists a second integral of the axisymmetric elastic tube provided that a strain-energy function exists as in (3.3.5). In this case, with \( H \) and \( R \) both being constant, (3.3.6) becomes

\[ \lambda_2 (W_2)' = \frac{r'}{R} W_1. \]  

(3.3.9)

Then, using \( W' = \lambda_1^1 W_1 + \lambda_2^2 W_2 \) and \( \lambda_1^1 = r'/R \), this leads to,

\[ W' = (\lambda_2 W_2)', \]  

(3.3.10)

which may then be integrated to give the second integral described by Pipkin (1968). Our two integrals of the equilibrium equations may therefore be given by,

\[ \dot{W} - \lambda_2 \dot{W}_2 = C_1, \]  

(3.3.11)

\[ \frac{\dot{W}_2 z'}{\lambda_2} - \frac{1}{2} P^* \lambda_1^2 R = C_2, \]  

(3.3.12)
where \( C_1, C_2 \) are constants. These two integrals are the result of symmetries in the variational form of the current deformation, with the second integral only existing when \( R \) and \( H \) are both constant, as mentioned above. Equations (3.3.11) and (3.3.12) are equivalent to equations (3.27) and (3.29) of Haughton (2001) respectively, which are derived from a more elaborate thin-shell theory considering the deformation of the midplane of a shell with negligible thickness. For more details on the derivation of the above results, including the asymptotic estimates from neglecting the small thickness which are also relevant to these results, see Haughton and Ogden (1979a) and Haughton and Ogden (1979b).

Epstein and Johnston (2001) give an alternative, equivalent, self-contained derivation of the equations of motion using a force-balance technique, which in our notation may be written as,

\[
\begin{align*}
\left[ R \sigma_2 \frac{z'}{\lambda_2^2} \right]' - P^* r r' &= \rho \ddot{z}, \\
\left[ R \sigma_2 \frac{r'}{\lambda_2^2} \right]' - \frac{\sigma_1}{\lambda_1} + P^* r z' &= \rho \ddot{r},
\end{align*}
\]

(3.3.13)

where \( \rho \) is the material density and a superimposed dot denotes differentiation with respect to \( t \). It is possible to derive (3.3.13) via a perturbation expansion of the full equations of motion, (2.1.41), between two axisymmetric cylindrical coordinate systems \( (R, \theta, Z) \) and \( (r, \theta, z) \), and we will outline this here. Taking (2.1.41) as the starting point, which involves derivatives with respect to both \( R \) and \( Z \), we first non-dimensionalise with respect to the inner radius \( R_i \) and the infinitesimal shear modulus \( \mu \). We introduce a dimensionless thickness coordinate \( \eta \) through \( R = R_i + H \eta, 0 < \eta < 1 \), which we use instead of \( R \). Taking a regular perturbation expansion in \( \epsilon = H/R_i \) of the quantities \( r, z, P \) and \( \sigma \), we then find, to first order in \( \epsilon \), the equations given in (3.3.13), along with the fact that the pressure \( P \) must be of order \( \epsilon \). Full details of this procedure may be found in Erbay and Demiray (1995) for a neo-Hookean strain-energy function and Tuzel and Erbay (2004) for any strain-energy function. We also note that these two studies, along with Kydoniefs and Spencer (1987) which also carries out the above asymptotic expansion for a particular problem, include the possibility of a tangential traction acting on the inner surface of the deformed membrane as well as the normal pressure. In Chapter 4 we show
that the static version of (3.3.13) may be derived via an energy minimisation approach; the dynamic equations (3.3.13) are derived in Chen (1997) using such an energy minimisation approach.

Equation (3.3.13) can be shown to be equivalent to equations (3.3.6) and (3.3.7) when static deformations are considered, although they represent the force balances in the $z$ and $r$ directions rather than in the tangential and normal directions respectively, which are given by $(r'e_r + z'e_z)/\lambda_2$ and $(z'e_r - r'e_z)/\lambda_2$ respectively.

To non-dimensionalise the problem all the length variables are rescaled by the original radius $R$, or equivalently $R$ is set to be unity. Similarly, we divide the strain-energy function $\hat{W}$, and the pressure $P^*$, by the infinitesimal shear modulus $\mu$ to non-dimensionalise the stresses.

### 3.3.2 Conditions at Infinity

In this work we are interested in localised bulging or necking solutions in which the tube has a constant radius $r_\infty$ and constant axial stretch $z_\infty$ far away from the localised bulge or neck if one exists. Hence we impose the following conditions \textit{a priori},

$$\lim_{Z \to \infty} r(Z) = r_\infty R, \quad \lim_{Z \to \infty} z(Z) = z_\infty Z,$$

(3.3.14)

and hence

$$r' \to 0, \quad z' \to z_\infty, \quad \lambda_1 \to r_\infty, \quad \lambda_2 \to z_\infty \text{ as } Z \to \pm \infty.$$  

(3.3.15)

Essentially we are imposing a pre-stress on the material, associated with the pressure $P$ through $r_\infty$ and $z_\infty$, and then determining whether there exist other non-uniform states which satisfy the governing equations but decay to the same constant radius and axial stretch at infinity.

We can find the constants $C_1$ and $C_2$ by evaluating (3.3.11) and (3.3.12) in the limit $Z \to$
and we also find the pressure $P$ from (3.3.7) in terms of $r_{\infty}$ and $z_{\infty}$ as,

$$C_2 = \hat{W}_2^{(\infty)} - \frac{1}{2}P^* r_{\infty}^2, \quad P^* = \frac{\hat{W}_1^{(\infty)}}{r_{\infty} z_{\infty}}, \quad C_1 = \hat{W}_1^{(\infty)} - z_{\infty} \hat{W}_2^{(\infty)},$$

(3.3.16)

where the superscript $(\infty)$ signifies evaluation at $\lambda_1 = r_{\infty}, \lambda_2 = z_{\infty}$. Equation (3.3.16) allows us to use $r_{\infty}$ and $z_{\infty}$ as the controlling parameters of the deformation instead of the pressure $P$. This enables us to use a monotonic parameter $r_{\infty}$ instead of $P$; which is often non-monotonic for the strain-energy functions we will consider here. Figure 3.3.2 shows an example of the connection between $P^*$ and $r_{\infty}$ when $z_{\infty} = 1$ for the Ogden strain-energy function. The curve for the Gent model is monotonic when $z_{\infty} = 1$ for all $J_m$, but not in the case of closed ends, as discussed below. For an infinite tube with open ends the remote axial stretch $z_{\infty}$ represents a prestrain of the material prescribed by the load applied at the end of the tube and is therefore treated as constant. The force required to keep the remote axial stretch at $z_{\infty}$ is given by Kyriakides and Chang (1990) as,

$$F = 2\pi H \left( \hat{W}_2^{(\infty)} - \frac{r_{\infty}}{2z_{\infty}} \hat{W}_1^{(\infty)} \right) = 2\pi H C_2.$$  

(3.3.17)

The significance of the constant $C_2$ in (3.3.8) is therefore revealed to be the scaled axial force applied at the ends of the tube, which may be compressive, particularly for $z_{\infty} = 1$. 

---

**Figure 3.3.2: Pressure against $r_{\infty}$ for the Ogden strain-energy function with $z_{\infty} = 1$.**
Naturally this depends on the value of $r_\infty$ as well as the strain-energy function. For human arteries Learoyd and Taylor (1966) give the axial stretch as 1.28 to 1.67 for people under 35, and 1.14 to 1.32 for people over 35, while Humphrey et al. (2009) states that it is of the order $z_\infty \approx 1.5$ for various mammals.

For a tube with closed ends and no axial loading, where we assume the ends are suitably far away from the localised bulge or neck that we may still consider the tube to be infinite, we require that the force balance in the $Z$ direction is zero, from which we can derive the condition $C_2 = 0$. For closed ends with no axial loading, we find the following relation from (3.3.12),

$$r_\infty \tilde{W}_1(r_\infty, z_\infty) = 2z_\infty \tilde{W}_2(r_\infty, z_\infty),$$

(3.3.18)

which can be used to determine $z_\infty$ for any given $r_\infty$. The relation given in (3.3.18) may have multiple branches, that is multiple values of $z_\infty$ which satisfy (3.3.18) for a given $r_\infty$. In this case, we need to choose the branch which passes through the undeformed solution $r_\infty = 1, z_\infty = 1$, which is always a solution to (3.3.18) for any admissible strain-energy function $W$ which satisfying the conditions given in (2.2.6).

The closed ends relation (3.3.18) for the Varga and Gent strain-energy function given by (2.2.10) and (2.2.13) become respectively,

$$1 + r_\infty^2 z_\infty - 2r_\infty z_\infty^2 = 0,$$

$$1 + r_\infty^4 z_\infty^2 - 2r_\infty^2 z_\infty^4 = 0,$$

(3.3.19)

which may be solved explicitly for $z_\infty$ and are plotted in Figure 3.3.3. It should be noted that the condition for the Gent strain-energy function (2.2.13) is independent of $J_m$ and in fact corresponds to the relation for the neo-Hookean strain-energy function. This second relation also holds for any other strain-energy functions which are solely a function of $I_1$, such as the LSS and Fung strain-energy functions. In addition, the first relation also holds for any function which is solely a function of $i_1$ as given in (2.1.54). A similar relation can be found for a strain-energy function which only depends on $I_2$ or $i_2$ for example, although no such relation may be found for the Ogden strain-energy function given by (2.2.14) as it is expressed as a function of the stretches rather than a single invariant.
Therefore, for either the open or closed ends case, we take $r_\infty$ as the controlling parameter, with $P^*$ and $z_\infty$ being potentially dependent upon it. Throughout the rest of this thesis formulae which involve $z_\infty$ should be taken to potentially depend on $r_\infty$.

### 3.3.3 Bifurcation Condition

We now arbitrarily set the origin of our coordinate system to be such that any non-uniform localised solution, either bulging or necking, is centred upon $Z = 0$. In doing so we do not lose any generality as the equations (3.3.11) and (3.3.12) are invariant under the transformation $Z \to Z + c$. It is assumed that local imperfections in the material properties or geometry of the tube will cause the localised solution to occur in a particular location, although we do not try and model such imperfections here.

With $C_1$ and $C_2$ known, equations (3.3.11) and (3.3.12) are two coupled first-order nonlinear differential equations for $r(Z)$ and $z(Z)$. Given that our coordinate system has been
set arbitrarily, the axial symmetry of the entire deformation together with the symmetry in the end conditions (3.3.15), we focus only on solutions which are symmetric about the origin \( Z = 0 \). The required conditions are therefore,

\[
\begin{align*}
    r'(0) &= 0, \\
    z(0) &= 0.
\end{align*}
\]

(3.3.20)

To find the corresponding value \( r(0) \) for a given \( r_\infty \), if one exists, (3.3.12) and (3.3.11) are evaluated at \( Z = 0 \), obtaining respectively,

\[
\begin{align*}
    \hat{W}_2(0, z'_0) &= \frac{\hat{W}_1(\infty)}{2r_\infty z_\infty} (r_0^2 - r_\infty^2) - \hat{W}_2(\infty) = 0, \\
    \hat{W}(r_0, z'_0) - z'_0 \hat{W}_2(r_0, z'_0) - \hat{W}(\infty) + z_\infty \hat{W}_2(\infty) &= 0,
\end{align*}
\]

(3.3.21)

(3.3.22)

where \( r_0 = r(0), z'_0 = z'(0) \), and both are required to be positive for physical deformations. In theory these two equations can be solved to find \( r_0 \) and \( z'_0 \) as functions of \( r_\infty \), though for most strain-energy functions this process must be numerical. Obviously, \( r_0 = r_\infty \) and \( z'_0 = z_\infty \) is always a solution of (3.3.21) and (3.3.22) corresponding to the trivial uniform solution \( r(Z) \equiv r_\infty \). A necessary condition for a non-uniform bifurcated solution to exist is therefore that (3.3.21) and (3.3.22) have a solution other than \( r_0 = r_\infty, z'_0 = z_\infty \).

Suppose that after the trivial solution \( r_0 = r_\infty \) has been factorised out, the solution is given by

\[
r_0 = q(r_\infty),
\]

(3.3.23)

for some unknown function \( q \). Any bifurcation values are therefore given by the roots of \( r_\infty = q(r_\infty) \); the intersection of the trivial solution with the nontrivial solutions. It should be noted that the existence of such a non-trivial value of \( r_0 \) does not guarantee that it corresponds to a localised solution of \( r(Z) \), or even a finite solution. In particular, it must be verified that \( z'_0 > 0 \) to ensure that the solution is physical. Section 5.2 details this process for the Varga strain-energy function as an illustration of this method, and may be read as an example before continuing here.
In general, for more complex strain-energy functions than the Varga strain-energy function, equations (3.3.21) and (3.3.22) cannot be solved explicitly for $r_0$. Another method is required to find the function $q$ which is presented here. Implicitly, from one of the equations (3.3.21) or (3.3.22), $z'_0$ can be expressed as a function of $r_0$. Hence we assume (3.3.21) may be implicitly solved to find $z'_0 = k(r_0)$ and we differentiate (3.3.22) twice with respect to $r_0$, evaluating each derivative at $r_0 = r_\infty$ to find,

\[
\hat{W}^{(\infty)}_1 - \hat{W}^{(\infty)}_{12} - \hat{W}^{(\infty)}_{22} k'(r_\infty) = 0, \tag{3.3.24}
\]

\[
\hat{W}^{(\infty)}_1 - \hat{W}^{(\infty)}_{12} - (\hat{W}^{(\infty)}_{22} + \hat{W}^{(\infty)}_{22}) k'(r_\infty)^2 - 2\hat{W}^{(\infty)}_{12} k'(r_\infty) - \hat{W}^{(\infty)}_{22} k''(r_\infty) = 0. \tag{3.3.25}
\]

These two equations can then be solved for $k'(r_\infty)$ and $k''(r_\infty)$, which can then be substituted into the first two derivatives of (3.3.21). When evaluated at $r_0 = r_\infty$ and using (3.3.24), (3.3.21) and its first derivative vanish, showing that the trivial solution $r_0 = r_\infty$ is always a double root of the equations, for any strain-energy function. The second derivative vanishes only when

\[
\frac{r_\infty(\hat{W}^{(\infty)}_1 - z_\infty \hat{W}^{(\infty)}_{12})^2 + z_\infty^2 \hat{W}^{(\infty)}_{22} (\hat{W}^{(\infty)}_1 - r_\infty \hat{W}^{(\infty)}_{11})}{r_\infty z_\infty^2 \hat{W}^{(\infty)}_{22}} = 0, \tag{3.3.26}
\]

which corresponds to the bifurcation condition $r_\infty = q(r_\infty)$, although it does not allow us to find $q$ explicitly. Equation (3.3.26) can therefore be solved for a specific strain-energy function $\hat{W}$, along with the condition for $z_\infty$, giving the bifurcation values. Equation (3.3.26) is consistent with Haughton’s (2001) equation (3.50), although the only plus in said equation should be a minus. It was also found by Haughton and Ogden (1979b), however the significance of (3.3.26) as this bifurcation condition was not fully appreciated as it corresponded to a zero mode number in a linear eigenvalue expansion.

### 3.3.4 Connection with the Pressure-Volume Curve

We now show how the turning points of the uniform-inflation pressure-volume curve correspond to the bifurcation points for the case of a closed tube. Taking the first two
equations in (3.3.16),
\[ C_2 = \hat{W}_{2}^{(\infty)} - \frac{1}{2}P^*r_\infty^2, \quad P^* = \frac{\hat{W}_{1}^{(\infty)}}{r_\infty z_\infty} \]  \quad (3.3.27)
we then differentiate both equations in (3.3.27) with respect to the volume of the uniformly inflated tube \( v \), giving,
\[ \hat{W}_{11}^{(\infty)} \frac{\partial r_\infty}{\partial v} + \hat{W}_{12}^{(\infty)} \frac{\partial z_\infty}{\partial v} - \frac{\partial P^*}{\partial v} r_\infty z_\infty - P^* \frac{\partial r_\infty}{\partial v} z_\infty - P^* r_\infty \frac{\partial z_\infty}{\partial v} = 0 \]  \quad (3.3.28)
\[ \hat{W}_{12}^{(\infty)} \frac{\partial r_\infty}{\partial v} + \hat{W}_{22}^{(\infty)} \frac{\partial z_\infty}{\partial v} - \frac{1}{2} \frac{\partial P^*}{\partial v} r_\infty^2 - P^* r_\infty \frac{\partial r_\infty}{\partial v} = 0. \]  \quad (3.3.29)
Eliminating \( \frac{\partial z_\infty}{\partial v} \) between the two equations and solving for \( \frac{\partial P^*}{\partial v} \) we find,
\[ \frac{\partial P^*}{\partial v} \left( \hat{W}_{12}^{(\infty)} \frac{r_\infty^2}{\hat{W}_{22}^{(\infty)}} - r_\infty z_\infty - \frac{P^* r_\infty^3}{2\hat{W}_{22}^{(\infty)}} \right) = \frac{\partial r_\infty}{\partial v} \frac{(\hat{W}_{12}^{(\infty)} - P^* r_\infty)^2 + \hat{W}_{22}^{(\infty)} (\hat{W}_{11}^{(\infty)} - P^* z_\infty)}{\hat{W}_{22}^{(\infty)}}. \]  \quad (3.3.30)
Using (3.3.27) \(_2\) it may be easily seen that the right hand side of (3.3.30) contains the bifurcation condition (3.3.26) as,
\[ \frac{\partial P^*}{\partial v} \left( \hat{W}_{12}^{(\infty)} \frac{r_\infty^2}{\hat{W}_{22}^{(\infty)}} - r_\infty z_\infty - \frac{P^* r_\infty^3}{2\hat{W}_{22}^{(\infty)}} \right) = \frac{\partial r_\infty}{\partial v} \frac{r_\infty}{z_\infty} \frac{z_\infty \hat{W}_{12}^{(\infty)} - \hat{W}_{1}^{(\infty)}}{\hat{W}_{22}^{(\infty)}} + \frac{z_\infty^2 \hat{W}_{22}^{(\infty)}}{r_\infty z_\infty^2 \hat{W}_{22}^{(\infty)}} \left( r_\infty \hat{W}_{11}^{(\infty)} - \hat{W}_{1}^{(\infty)} \right). \]  \quad (3.3.31)
Therefore, the bifurcation points for a closed tube occur at the turning points of the pressure-volume curve. Physically, the first turning point of the pressure-volume curve must be a maximum, which ensures that the first bifurcation point is subcritical, as will be shown in Section 4.2.

For the case of an open ended tube, we may differentiate \( P \) with respect to \( r_\infty \), giving
\[ \frac{dP}{dr_\infty} = \frac{r_\infty \hat{W}_{11}^{(\infty)} - \hat{W}_{1}^{(\infty)}}{r_\infty^2}. \]  \quad (3.3.33)
Therefore the bifurcation condition (3.3.26) may be written,
\[ \omega (r_\infty) = \frac{(\hat{W}_1^{(\infty)} - z_\infty \hat{W}_2^{(\infty)})^2}{z_\infty \hat{W}_2^{(\infty)} \hat{W}_2^{(\infty)}} - \frac{2z_\infty^2}{r_\infty \hat{W}_2^{(\infty)}} dP = 0, \]  
(3.3.34)
and therefore if a bifurcation point exists, it must come after the maximum of the pressure-volume curve.

### 3.4 Localised Solutions

We rewrite the differential equations (3.3.11) and (3.3.12) in the following way, defining two new functions \( f \) and \( g \),

\[ f(r, \lambda_2) \equiv \hat{W} - \lambda_2 \hat{W}_2 - C_1 = 0, \]  
(3.4.1)
\[ g(r, \lambda_2) \equiv \frac{\lambda_2}{\hat{W}_2} (C_2 + \frac{P}{2} r^2) = z'. \]  
(3.4.2)

Equation (3.4.1) allows us to express \( \lambda_2 = K(r) \) as a function of \( r \) for a given \( r_\infty \), although this relation will be implicit for most strain-energy functions. Using the definition of \( \lambda_2 \) given in (3.3.3), we can write

\[ r'^2 = \lambda_2^2 - z'^2 \]  
(3.4.3)
\[ = K(r)^2 - g(r, K(r)) \]
\[ = F(r; r_\infty), \]

defining a new function \( F \). The behaviour of \( F \) governs the existence and shape of the non-trivial solution. From results in the theory of dynamical systems, detailed by Bhatnagar (1979) and used in Epstein and Johnston (2001), it is shown that bulging solitary waves can exist in systems described by \( w'(x)^2 = Q(w(x)) \), when the function \( Q \) has a double root at \( w = w_1 \), another root at \( w = w_2 \), where \( w_2 > w_1 \), and \( Q(w) > 0 \) for \( w \in (w_1, w_2) \). Corresponding conditions can be given for necking solitary waves by
reversing the signs of the two inequalities. As was discussed in Section 3.3.3, there is always a double root of \( F \) at \( r = r_\infty \). So for an isolated bulge to exist there must be another root \( r_1 \) of \( F \) such that \( r_1 > r_\infty \) and \( F(r) > 0, r \in (r_\infty, r_1) \). A necking solution occurs if there exists a second root \( r_1 \) where \( r_1 < r_\infty \) and \( F(r) < 0, r \in (r_1, r_\infty) \). Therefore the behaviour of \( F \) enables us to see if a bulging or necking solution exists. Expanding \( F \) in (3.4.4) around \( r_\infty \) gives,

\[
 w'^2 = \omega(r_\infty) w^2 + \gamma(r_\infty) w^3 + \mathcal{O}(w^4), \tag{3.4.4}
\]

where \( w = r - r_\infty \) and the fact that \( r_\infty \) is a double root has been used. The function \( \omega(r_\infty) \) is given by,

\[
 \omega(r_\infty) = \frac{r_\infty (\hat{W}_{11}^{(\infty)} - z_\infty \hat{W}_{12}^{(\infty)})^2 + z_\infty^2 \hat{W}_{22}^{(\infty)} (\hat{W}_{11}^{(\infty)} - r_\infty \hat{W}_{11}^{(\infty)})}{r_\infty z_\infty \hat{W}_{22}^{(\infty)}} = 0. \tag{3.4.5}
\]

The extra root coalesces with the double root at critical points \( r_{cr} \) of \( \omega \), where \( \omega(r_{cr}) = 0 \). Therefore, around \( r_\infty = r_{cr} \), solitary wave solutions may only exist on one side of \( r_{cr} \), not both. Comparison of (3.4.5) with (3.3.26) shows that (3.4.5) is the bifurcation condition for the deformation, associated with where the character of the differential equation in (3.4.4) changes. We also note that the denominator of (3.3.26) is important, as it may potentially become infinite and/or change its sign. Equations (3.3.26) and (3.4.5) differ by a factor of \( z_\infty \hat{W}_{22}^{(\infty)} \), which from strong ellipticity must be positive, and therefore these two bifurcation conditions give exactly the same bifurcation points.
3.4.1 Near-Critical Solutions

We now show that solitary wave solutions do exist on one side of \( r_{cr} \) without evaluating \( F \) by looking for near-critical solutions of (3.4.4), where \( r_\infty \) is close to \( r_{cr} \) and we define \( \epsilon = r_\infty - r_{cr} \) which may be positive or negative. Expanding around \( r_{cr} \) we obtain

\[
w'^2 = \omega'(r_{cr}) \epsilon w^2 + \gamma(r_{cr}) w^3 + O(w^4), \tag{3.4.6}\]

where \( O(\epsilon w^2) = O(w^3) \). Naturally \( w' \) may only change sign when the right hand side of (3.4.6) is zero, where \( w = \frac{-\omega'(r_{cr}) \epsilon}{\gamma(r_{cr})} \). On the assumption that \( w' \equiv r' \) does not change sign, then when \( r_0 > r_\infty \), a bulging solution exists with \( w' < 0 \) and \( w > 0 \); whereas when \( r_0 < r_\infty \), a necking solution exists with \( w' > 0 \) and \( w < 0 \). Thus in both cases the square root may be taken in (3.4.6) to give,

\[
w' = -w\sqrt{\omega'(r_{cr}) \epsilon + \gamma(r_{cr})w}. \tag{3.4.7}\]

This equation is naturally related to the following four equations:

\[
w'_1 = -aw_1\sqrt{1 + bw_1}, \quad w'_2 = -aw_2\sqrt{1 - bw_2},
\]

\[
w'_3 = -aw_3\sqrt{-1 + bw_3}, \quad w'_4 = -aw_4\sqrt{-1 - bw_4},
\]

where \( w_1, w_2, w_3, w_4 \) are functions of \( Z \) and \( a, b \) are positive constants given by

\[
a = \sqrt{\left| \omega'(r_{cr}) \epsilon \right|}, \quad b = \left| \gamma(r_{cr}) \right|/a^2.
\]
It can be shown that these four equations have the following solutions,

\[
\begin{align*}
  w_1 &= \begin{cases} 
  \frac{1}{b} \left[ \sinh \left( \frac{1}{2} a Z + A \right) \right]^{-2}, & \text{if } w_1 > 0 \\
  -\frac{1}{b} \left[ \cosh \left( \frac{1}{2} a Z + A \right) \right]^{-2}, & \text{if } w_1 < 0
  \end{cases} \\
  w_2 &= \begin{cases} 
  -\frac{1}{b} \left[ \sinh \left( \frac{1}{2} a Z + A \right) \right]^{-2}, & \text{if } w_2 < 0 \\
  \frac{1}{b} \left[ \cosh \left( \frac{1}{2} a Z + A \right) \right]^{-2}, & \text{if } w_2 > 0
  \end{cases} \\
  w_3 &= \frac{1}{b \cos \left( \frac{1}{2} a Z + A \right)^2}, \\
  w_4 &= -\frac{1}{b \cos \left( \frac{1}{2} a Z + A \right)^2},
\end{align*}
\]

where \( A \) is an integration constant which can be set to zero, since the origin of \( Z \) has been arbitrarily chosen. The relevance of these equations to (3.4.7) is clearly given by,

\[
\begin{align*}
  w &= \begin{cases} 
  w_1 & \text{if } \omega'(r_{cr})(r_{\infty} - r_{cr}) > 0, \gamma(r_{cr}) > 0; \\
  w_2 & \text{if } \omega'(r_{cr})(r_{\infty} - r_{cr}) > 0, \gamma(r_{cr}) < 0; \\
  w_3 & \text{if } \omega'(r_{cr})(r_{\infty} - r_{cr}) < 0, \gamma(r_{cr}) > 0; \\
  w_4 & \text{if } \omega'(r_{cr})(r_{\infty} - r_{cr}) < 0, \gamma(r_{cr}) < 0.
  \end{cases}
\end{align*}
\] (3.4.8)

By inspection, \( w_3, w_4, w_1 > 0 \) and \( w_2 < 0 \) can be dismissed as unphysical solutions since they all blow up at finite values of \( Z \). Applying these results to (3.4.7), it is seen that near-critical solutions are only possible if

\[
\omega'(r_{cr}) \epsilon = \omega'(r_{cr})(r_{\infty} - r_{cr}) > 0.
\] (3.4.9)

Hence a super-critical solution is possible only if \( \omega'(r_{cr}) > 0 \), that is only if the curve of \( \omega(r_{\infty}) \) at the bifurcation point has a positive slope. Similarly, sub-critical solutions only exist if \( \omega'(r_{cr}) < 0 \). A super-critical solution corresponds to a bulging or aneurysm solution if \( q'(r_{cr}) > 0 \) and to a necking solution if \( q'(r_{cr}) < 0 \), where \( q(r_{cr}) \) is defined by (3.3.23). Therefore, in the near-critical case, the solutions are given by

\[
\begin{align*}
  w &= \frac{\omega'(r_{cr})(r_{\infty} - r_{cr})}{\gamma(r_{cr})} \text{sech}^2 \left( \frac{\sqrt{\omega'(r_{cr})(r_{\infty} - r_{cr})Z}}{2} \right),
\end{align*}
\] (3.4.10)
and such solitary waves exist only on the side of \( r_{cr} \) in which (3.4.9) is satisfied. Figure

![Figure 3.4.2: Plot of the bifurcation condition \( \omega(r_\infty) \) for the Gent strain-energy function with \( J_m = 97.2 \) with closed ends.](image)

3.4.2 shows \( \omega(r_\infty) \) for the Gent strain-energy function, which indicates that bulged solutions exist to the left of the first bifurcation point \( r_{cr} \) and necking solutions exist to the right of the second bifurcation point \( r_{cr2} \).

### 3.4.2 Phase Plane Analysis

Plotting the phase plane of \( r' \) against \( r \), we find that there exist fixed points, where \( r' = 0 \), whenever \( F \) has a root. From \( r'' = F'(r)/2 \), we can see that there exist centres in the phase plane at the maxima of \( F \) and saddles at the minima of \( F \). The solitary wave solution corresponds to the homoclinic orbit connecting the saddle back to itself, this homoclinic orbit can be seen in Figure 3.4.3(c). There are three mechanisms by which this homoclinic orbit appears, or disappears, as \( r_\infty \) is varied. The first such mechanism is that as \( r_\infty \to r_{cr} \) the third root \( r_1 \) coalesces to form a triple root, as explained above. The second mechanism is that the third root coalesces with a further root to become another double root, as shown in Figure 3.4.3(b). In this case there exists a heteroclinic orbit connecting the two saddles, as can be seen in Figure 3.4.3(d), corresponding to two
Figure 3.4.3: Plots for the closed Gent strain-energy function with $J_m = 97.2$: (a) $F(r)$ for the bulged solution with $r_\infty = 1.5$, (b) Phase plane for the bulged solution with $r_\infty = 1.5$, (c) $F(r)$ for the kinked solution with $r_\infty = 1.1694$ (d) Phase plane for the kinked solution with $r_\infty = 1.1694$

uniform sections of cylinder connected by a non-uniform transition region, which is the kinked state described in Section 3.1. Further discussion of the kinked solution will be found in section 3.6. The third mechanism is that $F$ has an asymptote before the third root $r_1$, and therefore no homoclinic orbit exists as the orbits in the phase plane tend to infinity at the asymptote. This occurs for the Varga strain-energy function and the Gent strain-energy function with open ends after the second bifurcation point, and is discussed more in the comparison of various strain-energy functions in Chapter 5.

### 3.5 Amplitude-Stretch Diagrams

Returning to the evaluation of the equilibrium equations at $Z = 0$, we may numerically solve (3.3.21) and (3.3.22) to find $r_0$ and $z'_0$ for a given $r_\infty$ and $z_\infty$ as has been previously stated. Figure 3.5.1 shows an example of how the maximum amplitude of the non-trivial state, $r_0 - r_\infty$, varies with $r_\infty$ for the Gent strain-energy function, and give important
Figure 3.5.1: Dependence of $r_0 - r_\infty$ on $r_\infty$ for the closed Gent tube, $J_m = 97.2$. Only the solid lines in (a) correspond to localised solutions, and the non-localised solutions are not plotted in (b).

insights to the bifurcation process. This plot corresponds to the non-trivial solution of (3.3.21) and (3.3.22), combined with (3.3.18), and shows the amplitude of the bulging or necking solutions. The existence of the solutions on one side of each critical point is guaranteed by the shape of $F$ in Figure 3.4.3 and also the shape of $\omega(r_\infty)$ in Figure 3.4.2. The solid lines in Figure 3.5.1(a) correspond to where localised solutions do exist, given by where the phase planes discussed in Section 3.4.2 have homoclinic orbits and are verified numerically by the method detailed later in Section 3.7. Dashed lines represent values of $r_0$ and $z'_0$ which satisfy (3.3.21) and (3.3.22) but do not satisfy the conditions imposed in (3.3.16) specifying localised solutions. The localised solutions can cease to exist through the methods discussed in Section 3.4.2 where the homoclinic orbit no longer exists, and are not limited to turning points of the $r_0 - r_\infty$ curve, although in this case that is where the localised solutions disappear.

Figure 3.5.1(a) shows that, for the closed Gent strain-energy function with $J_m = 97.2$, there exist two bifurcation points, the first of which corresponds to a bulged solution and the second to a necking solution, as verified by the plot of $\omega(r_\infty)$ given in Figure 3.4.2. There also exist two points, marked $A$ and $B$, where the gradient of $r_0 - r_\infty$ becomes infinite. These two points, which will be denoted $r_k$, are shown in section 3.6 to
correspond to kinked solutions where several uniform sections of two different sizes are connected by non-uniform transition regions. For the bulging solution, as we vary $r_\infty$ from the bifurcation point along the solid curve, the radius at the centre of the bulge will increase monotonically until we reach the turning point A, where the bulge flattens out at its centre, stops growing in radius and then starts to propagate outwards in both directions and becomes a kinked solution. A similar interpretation may be given to the solid curve which terminates at the second turning point B, although this represents a kinked solution arising from a necking solution at a higher value of $r_\infty$. Here the values of $r_0$, $z'_0$ and $r_\infty$, $z_\infty$ are swapped, corresponding to the same kinked solution but with the sizes of the uniform sections reversed. This may be seen in Figure 3.5.1(b) where it is clear that A and B are reflected around the line $r_0 = r_\infty$.

Therefore the entire inflation process for the Gent strain-energy function may be described using Figure 3.5.1(b). The initially stress-free unpressurised state is given by point C, corresponding to $r_\infty = 1$. Then uniform inflation follows the straight line $r_0 = r_\infty$ until the first bifurcation point D is reached. As inflation is continued past this point a bulge occurs, whose growth is described by the path DA. At point A the bulge is at its maximum radius and begins to propagate in both directions. For a finite tube, a uniform state is eventually achieved as the kink spreads to fill the entire tube, at point E. After this point further inflation will be uniform, although bounded due to the limit of $J_m$ in the Gent strain-energy function. If the tube is deflated after point E has been reached then the deflation will be uniform until the second bifurcation point F is reached, after which the necked solution occurs, which is described by FB. At point B the kinked solution is recovered but with the sizes of the sections reversed. This will begin to propagate in both directions until the kinks reach the ends and point G corresponds to the resulting uniform state. Further inflation or deflation will proceed in the same manner as the tube is considered elastic, and therefore no plastic effects are considered.
3.6 Kinked Solution

The location of a kinked solution where several uniform sections of differing sizes exist may be found analytically, if such a value of \( r_\infty \) exists. For a uniform state to exist at \( Z = 0 \) we require, in addition to \( r'(0) = 0 \) as prescribed already, \( r''(0) = 0 \). Subject to this requirement, the equilibrium equation (3.3.7) becomes, using (3.3.5) and (3.3.16),

\[
\begin{align*}
    r_0 z_0' \hat{W}_1(r_\infty, z_\infty) &= r_\infty z_\infty \hat{W}_1(r_0, z_0') . \\
    (3.6.1)
\end{align*}
\]

Therefore, combining (3.6.1) with (3.3.21), (3.3.22) and either the closed end condition (3.3.18) or a specified \( z_\infty \), we have a set of four equations for four unknowns, \( r_0, z_0', r_\infty, z_\infty \).

If a solution to these equations exists where \( r_0 \neq r_\infty \), then this is the kinked solution, with radius \( r_0 \) at the origin and \( r_\infty \) at infinity.

To show that the points where kinked solutions exist correspond to those marked \( A \) and \( B \) in Figure 3.5.1, we first differentiate (3.3.21) and (3.3.22) with respect to \( r_\infty \), viewing \( z_0' \) as a function of \( r_\infty \) and \( r_0, r_\infty, z_\infty \) as functions of \( r_\infty \), giving respectively,

\[
\begin{align*}
    \frac{\partial r_0}{\partial r_\infty} (W_{12} - P^*r_0) + \frac{\partial z_0'}{\partial r_\infty} W_{22} + P^*r_\infty + \frac{\partial P^*}{\partial r_\infty} - W_{12}'(\infty) - \frac{\partial z_\infty}{\partial r_\infty} W_{22}'(\infty) &= 0 \\
    (3.6.2)
\end{align*}
\]

By taking the limit \( \partial r_0/\partial r_\infty \rightarrow \infty \) in the resulting equations, corresponding to the tangency observed in Figure 3.5.1, we obtain, using the chain rule on \( \partial z_0'/\partial r_\infty \),

\[
\begin{align*}
    \hat{W}_{12}(r_0, z_0') - \frac{r_0 \hat{W}_1'(\infty)}{r_\infty z_\infty} + \frac{\partial z_0'}{\partial r_0} \hat{W}_{22}(r_0, z_0') &= 0 , \\
    (3.6.4)
\end{align*}
\]

\[
\begin{align*}
    \hat{W}_1(r_0, z_0') - z_0' \hat{W}_{12}(r_0, z_0') + z_0' \frac{\partial z_0'}{\partial r_0} \hat{W}_{22}(r_0, z_0') &= 0 . \\
    (3.6.5)
\end{align*}
\]

On eliminating \( \partial z_0'/\partial r_0 \) between the two equations above, we reproduce the same condition as given by (3.6.1) by assuming \( r''(0) = 0 \),

\[
\begin{align*}
    \hat{W}_1(r_0, z_0') - \frac{r_0 z_0'}{r_\infty z_\infty} \frac{\hat{W}_1'(\infty)}{z_\infty} &= 0 . \\
    (3.6.6)
\end{align*}
\]

72
Therefore, at values of $r_{\infty}$ which satisfy (3.6.1), there exist two possible constant values of $r(Z)$ which satisfy the equilibrium equations; $r_{\infty}$ and $r_0$. Therefore any combination of uniform sections of radius $r_{\infty}$ and $r_0$ connected by non-uniform transition regions is a solution of the equilibrium equations.

We now proceed to show that these two uniform states satisfy the Maxwell equal-area rule which occurs in other related kink-band problems (Ericksen 1975, Chater and Hutchinson 1984).

The Maxwell equal-area rule is that value of the pressure $P_k$ such that the two areas bounded by the pressure function $P(v)$ and the line $P = P_k$ are equal giving,

$$\int_{v_1}^{v_2} P(v)dv = P_k(v_2 - v_1),$$

(3.6.7)

where $v$ is the volume of the tube. In the deformation considered here, the volume measure $v$ is given by,

$$v = r_{\infty}^2 z_{\infty},$$

(3.6.8)

which for uniform inflation is the volume change per unit volume in the undeformed configuration. With the additional use of (3.3.18) if required, we may view $r_{\infty}$ and $z_{\infty}$ both as functions of $v$. The two values of $v$ generated by (3.6.7) are the volumes corresponding to the two uniform sections of the kinked solution.

Figure 3.6.1 shows the pressure-volume curve for a typical closed Gent strain-energy function, along with the line $P_k$ which satisfies the equal-area rule. This pressure-volume curve is typical for rubber-like materials, as discussed in Carroll (1987) and (Gent 2005), but the pressure-volume curve is not required to be non-monotonic for the kinked solution to exist, as discussed below.

We now demonstrate that the equal-area condition (3.6.7) gives the same condition for the kinked solutions as (3.6.1). We define the strain-energy depending solely on the volume
Figure 3.6.1: Pressure as a function of volume for a closed Gent tube with \( J_m = 30 \) as \( \tilde{W}(v) = \hat{W}(r_\infty(v), z_\infty(v)) \). It can then be shown that, using the chain rule

\[
P^* = \frac{1}{r_\infty z_\infty} \frac{\partial \hat{W}(v)}{\partial r_\infty} = \frac{1}{r_\infty z_\infty} \frac{d\hat{W}(v)}{dv} \frac{\partial v}{\partial r_\infty} = 2 \frac{d\hat{W}(v)}{dv}.
\]

(3.6.9)

The Maxwell equal-area rule (3.6.7), evaluated at two separate uniform states then becomes using (3.6.9),

\[
P_k(v_2 - v_1) = 2 \left( \tilde{W}(v_2) - \tilde{W}(v_1) \right),
\]

(3.6.10)

where

\[
v_1 = r_\infty^2 z_\infty, \quad v_2 = r_0^2 z_0', \quad P_k = P^*_{|v=v_1} = P^*_{|v=v_2},
\]

(3.6.11)

with \((r_\infty, z_\infty)\) and \((r_0, z_0')\) being the two uniform states connected by the Maxwell line. It remains to show that the \((r_0, z_0', r_\infty, z_\infty)\) defined in this way also satisfy the kinked condition (3.6.1). We note that (3.6.11) may be written as the pressure in either of the two states,

\[
P_k = \frac{\hat{W}_1(r_\infty, z_\infty)}{r_\infty z_\infty}.
\]

(3.6.12)

We may now use the equilibrium equation (3.3.12) to rewrite the terms involving \( \tilde{W} = \hat{W} \)
on the right hand side of (3.6.10) in terms of \( \hat{W}_2 \) and \( C_2 \), which along with (3.6.12) gives,

\[
\frac{W_1(r_\infty, z_\infty)}{r_\infty z_\infty} (r_0^2 z_0' - r_\infty^2 z_\infty) = 2 (z_0' W_2(r_0, z_0') - z_\infty W_2(r_\infty, z_\infty)).
\]

(3.6.13)

We can then use the other equilibrium equation (3.3.11) to express \( \hat{W}_2 \) in terms of \( \hat{W}_1 \) and \( C_1 \), which then reduces (3.6.13) to the kinked equation (3.6.1).

For the case of the Gent strain-energy function with \( z_\infty = 1 \), \( P \) is a monotonic function of \( v \) and thus no Maxwell line exists. However, the condition given by (3.6.6) still has a solution which corresponds to the kinked solution, for example when \( J_m = 97.2 \), \( r_\infty = 1.17643 \), \( r_0 = 5.81723 \), \( z_0' = 4.09994 \) is a solution to (3.6.6). In this case, the two pressures given by (3.6.11) evaluated at the two pairs \((r_\infty, 1)\) and \((r_0, z_0')\) are equal, and therefore we conclude that the condition (3.6.1) is the requirement for the kink to exist, not the existence of the Maxwell line.

### 3.7 Numerical Solutions

We now look at directly solving the equilibrium equations given in (3.3.11) and (3.3.12) numerically, subject to the localised behaviour specified in (3.3.15). In doing so we can find, for the regions where it exists, the solitary wave solution or the kinked solution for a given \( r_\infty, z_\infty \) and strain-energy function. Initially we may take the first order differential equation derived in (3.4.4) and rewrite it as,

\[
\int_{r(Z)}^{r_0} \frac{dr}{\sqrt{F(r; r_\infty)}} = \int_{Z}^{0} dZ,
\]

(3.7.1)

where we take the negative sign for regions of bulging solutions as \( r'(Z) < 0 \), and the positive sign for necking solutions. Evaluating (3.7.1) at successive values of \( Z \) is a very accurate but computationally expensive method of finding the bifurcated solution for \( r(Z) \), which then can be used to find \( z(Z) \).
Kyriakides and Chang (1991) give a set of differential equations which may be used to find \( r(Z) \) and \( z(Z) \), without using the conservation laws (3.3.11) and (3.3.12). This is an equivalent set of differential equations to (3.3.6) and (3.3.7), but using \( \lambda_1, \lambda_2 \) and \( \phi \) as the independent coordinates, where \( \phi \) is the angle between the \( Z \)-axis and the membrane, defined in Figure 3.3.1:

\[
\begin{align*}
\lambda'_1 &= \lambda_2 \sin \phi, \\
\lambda'_2 &= \frac{\hat{W}_1 - \lambda_2 \hat{W}_{12}}{W_{22}} \sin \phi, \\
\theta' &= \frac{\hat{W}_1}{W_2} \cos \phi - \frac{\hat{W}_1^{(\infty)}}{r_{\infty} z_{\infty} W_2} \lambda_1 \lambda_2.
\end{align*}
\] (3.7.2)

Equations (3.7.2) are solved subject to the initial conditions \( \lambda_1(0) = r_0, \lambda_2(0) = z'_0, \phi(0) = 0 \), the last of these coming from the fact that the deformed tube is flat at \( Z = 0 \) as we impose \( r'(0) = 0 \) in the symmetry conditions. Using (3.7.2) is computationally more efficient than using (3.7.1) for solving the equilibrium equation and gives the same results. Consequently, it was the numerical scheme used throughout the work presented here, although comparison with the integral (3.7.1) was made for selected cases to ensure the results are accurate. Back substitution of the computed results into the equilibrium equations was also used to verify the accuracy of the solutions.

Figure 3.7.1 shows typical profiles of \( r(Z) \) for the solitary wave solution with varying \( r_\infty \) for the Gent strain-energy function with \( J_m = 97.2 \). This shows how the bulge increases while the far-radius decreases, as expected from the amplitude-stretch diagram given in Figure 3.5.1. In addition, it shows how the solution stops growing radially, flattens and propagates down the tube as the kinked solution is approached. However, in Figure 3.7.1 the radius is plotted against the position in the undeformed axial variable \( Z \). In order to visualise the tube in the deformed configuration it is necessary to plot \( r(Z) \) against \( z(Z) \), as shown in Figure 3.7.2, compared to the undeformed configuration where \(-10 < Z < 10\). This second plot shows how the axial stretch \( z'(Z) \) is involved in the deformation, and it is greater than unity for the bulged configuration. Figure 3.7.3 shows...
how the deformed tube expands and approaches the kinked solution as $r_\infty \to r_k$.

![Figure 3.7.1: Profiles of $r(Z)$ for a closed Gent tube with $J_m = 97.2$.](image1)

![Figure 3.7.2: Deformed configurations for a closed Gent tube with $J_m = 97.2$.](image2)

### 3.8 Conclusion

This chapter has detailed the conditions required for bulging or necking solutions to exist, and shown how the amplitude-stretch diagram discussed in (3.5) imparts a great amount of information about the process. The bifurcation condition, $\omega(r_\infty) = 0$, may be calculated easily for any given strain-energy function by using (3.4.5) and the side
Figure 3.7.3: Deformed configurations for the closed Gent tube with $J_m = 97.2$ as the kink is approached.

on which the solitary wave solution exists close to the critical points is given immediately from the sign of $\omega'(r_{cr})$. A calculation of the amplitude-stretch diagram, which involves numerically solving two, or three, algebraic but generally transcendental equations whilst varying $r_\infty$, then gives most of the information about the existence of solutions throughout the deformation. Numerical integration, or examining the phase planes at a few select points, then enables us, with the various results detailed in this chapter, to see which points bulged, necked, or kinked solutions exist.

Additionally, the pressure-volume curve for the uniform tube, given by (3.3.16) and (3.6.8), which again is easy to calculate for any given strain-energy function, gives immediate information as to the existence of solutions. If the pressure has a local maximum then localised solutions must exist, due to the connection of the bifurcation condition with the pressure maximum, which will then be subcritical due to the results in Chapter 4. If this pressure is a local, but not global, maximum, then a Maxwell line must exist and therefore the kinked solution must be present. The existence of a local minimum implies that another bifurcation point exists, which must be a supercritical solution due to the continuity of $\omega(r_\infty)$. Further maxima or minima in the pressure-volume curve would lead to more sub- or supercritical solutions in turn, along with associated kinked solutions between them, although some kind of localised strain-softening/stiffening or a high axial stretch would likely be required in the strain-energy function to produce such
a pressure-volume relationship.

It is not currently clear why the Maxwell line is not required for a kinked solution to exist, as in the Gent tube where the pressure-volume curve for the uniform tube is monotonically increasing. This requires further investigation, particularly in the case of open-ended tubes.
Chapter 4

Stability of a Bifurcated Cylindrical Membrane

4.1 Introduction

Chapter 3 showed the existence of bifurcated states of the cylindrical tube, and this chapter considers the stability of these bifurcated states with respect to axisymmetric perturbations. Initially, we discuss the stability of the uniformly inflated tube which is shown to be initially stable at \( r_\infty = 1 \) and change stability at the bifurcation points \( r_{cr} \). We derive the dispersion relation for the inflated tube and use this to construct a suitable perturbation expansion in order to derive an evolution equation which the weakly nonlinear equation is related to. Inserting into this evolution equation a time-dependent perturbation we find an eigenvalue problem, which we solve using the Evans function as part of the compound matrix method. In this way we show that the weakly nonlinear solution is always unstable, for both bulging or necking cases and any strain-energy function. After this we proceed by considering the spectral stability of the fully nonlinear bulged or necked solution using the same method, and show numerically that it is unstable. An alternative stability test using an energy method is used to investigate the bifurcated
state by considering the second variation of the energy in the tube. Finally the chapter is concluded by a discussion of the stability of the kinked solution.

4.2 Stability of the Uniform State

Chen (1997) studied the stability of the uniformly inflated tube and provides the following necessary conditions for local stability,

\begin{align}
\frac{r_1^2}{\hat{W}_{11}} - r_2 \hat{W}_1 &\geq 0, \quad \hat{W}_2 \geq 0 \quad \hat{W}_{22} \geq 0, \\
\frac{z_1^2}{\hat{W}_{22}}(r_1^2 \hat{W}_{11} - r_2 \hat{W}_1) - (r_\infty z_\infty \hat{W}_{12} - r_\infty \hat{W}_1)^2 &\geq 0,
\end{align}

(4.2.1)

where, as we are considering a uniform inflation state, all the values of \( \hat{W} \) are evaluated at \( \lambda_1 = r_\infty, \lambda_2 = z_\infty \). In addition, the strict inequality version of (4.2.1) gives sufficient conditions. Provided the membrane is in tension, as required to prevent wrinkling, then \( \sigma_2 \geq 0 \) and therefore we must have \( \hat{W}_2 \geq 0 \). The third inequality in (4.2.1) is satisfied as a consequence of strong ellipticity of the strain-energy function; which all the functions discussed here satisfy. With these two conditions holding the remaining two inequalities in (4.2.1) reduce to merely \( \omega(r_\infty) \geq 0 \). Therefore the uniform tube changes stability at the bifurcation points \( r_{cr} \), as often occurs in bifurcation problems. It should be noted that the undeformed tube, \( r_\infty = z_\infty = 1 \), is only stable provided that \( \omega(1) > 0 \), implying that the initial bifurcation point is subcritical, which will be seen in the plots of \( \omega \) for various strain-energy functions in Chapter 5. Any strain-energy function which does not satisfy (4.2.1) must therefore be unphysical as the undeformed tube would then be unstable. However, the function \( \gamma(r_\infty) \) which determines whether the solution is bulging or necking depends on the strain-energy function as opposed to having a simple determining factor.
4.3 Stability of the Weakly-Nonlinear Solution

We now consider stability of the weakly nonlinear bulging or necking solution found in Section 3, that is the solution which is valid in the region where $r_\infty$ is close to $r_{cr}$. This will then be used as a starting point for the fully nonlinear stability analysis to be presented in Section 4.4.

We recall equation (3.4.4) governing the shape of the tube,

$$w'^2 = \omega(r_\infty)w^2 + \gamma(r_\infty)w^3 + O(w^4),$$  \hspace{1cm} (4.3.1)

where $w = r - r_\infty$. Differentiating (4.3.1) with respect to $Z$ and expanding around $r_{cr}$ where $\epsilon = r_{cr} - r_\infty$, we find

$$w'' = \omega'(r_{cr})\epsilon w + \frac{3}{2}\gamma(r_{cr})w^2 + O(w^3),$$  \hspace{1cm} (4.3.2)

where $w = r - r_{cr}$ and $\epsilon = r_\infty - r_{cr}$. Equivalently (4.3.2) may be written as:

$$\frac{d^2V}{d\xi^2} = V - V^2,$$  \hspace{1cm} (4.3.3)

via the use of the transformation

$$w = -\frac{2\epsilon \omega'(r_{cr})}{3\gamma(r_{cr})}V(\xi), \quad \xi = \sqrt{\epsilon \omega'(r_{cr})}Z.$$  \hspace{1cm} (4.3.4)

In (4.3.4) it is assumed that $\epsilon \omega'(r_{cr}) > 0$, the necessary condition for the existence of localised bulging or necking solutions (3.4.9), holds and therefore $\xi$ must be real. The nonlinear equation (4.3.3) has an exact solitary-type solution given by

$$V(\xi) = V_0(\xi) \equiv \frac{3}{2}\text{sech}^2\left(\frac{\xi}{2}\right).$$  \hspace{1cm} (4.3.5)

This static aneurysm solution is in fact a ‘fixed’ point of an evolution equation when time dependence is included. For convenience we now consider the necking case where $\epsilon$ and
are both positive. The same analysis applies for the bulging case where \( \epsilon \) and \( \omega'(r_{cr}) \) are negative; absolute values for each variable must be taken when deriving the evolution equation.

### 4.3.1 Evolution Equation

We now derive an evolution equation for which equation (4.3.3) is the static equivalent, using the ideas from Fu (2001). In order to find the correct scalings in the perturbation analysis which will follow, we first obtain a dispersion relation for the tube. We therefore perturb the governing equations (3.3.13) around the undeformed configuration, by letting

\[
\begin{align*}
    r(Z, t) &= r_\infty + w(Z, t), \\
    z'(Z, t) &= z_\infty + u(Z, t),
\end{align*}
\]

(4.3.6)

and then linearising the governing equations (3.3.13) around \( u = w = 0 \). We then seek a travelling wave solution with wave number \( \hat{k} \) and wave speed \( \hat{c} \),

\[
(w, u) = (\tilde{w}, \tilde{u}) e^{i(k(Z-\hat{c}t))},
\]

(4.3.7)

and then, on setting the determinant of the coefficients of \( \tilde{w}, \tilde{u} \) to be zero, we find the following dispersion relation connecting the wave number and wave speed,

\[
\hat{k}^2 (\rho \hat{c}^2)^2 - (\rho \hat{c}^2) \left( \hat{k}^2 \left( \frac{\hat{W}_2^{(\infty)}}{z_\infty} + \hat{W}_2^{(\infty)} \right) + \hat{W}_1^{(\infty)} \right) + \frac{\hat{W}_2^{(\infty)}}{z_\infty} \left( \hat{k}^2 - \omega(r_\infty) \right) = 0
\]

(4.3.8)

For \( 0 < \epsilon \ll 1 \), the pre-stressed membrane tube will support travelling waves with small wave number and wave speed; as almost all elastic bodies do. From the dispersion relation, if the wave number is of order \( \epsilon \) then the wave speed must be of order \( \sqrt{\epsilon} \). It can also be deduced that the radial perturbation is of the order \( \sqrt{\epsilon} \) multiplying the order of axial perturbation, and this is detailed in Fu and Il’ichev (2010). Thus, we may define a far-distance variable \( \xi \) as in (4.3.4), a slow-time variable \( \tau \) through

\[
\tau = \epsilon t,
\]

(4.3.9)
and look for a perturbation solution of the form

\[ r = r_{cr} + \epsilon \lambda_1 + \epsilon \{ w_1(\xi, \tau) + \epsilon w_2(\xi, \tau) + \cdots \}, \tag{4.3.10} \]

\[ z = (z_{cr} + \epsilon \lambda_2) Z + \sqrt{\epsilon} \{ u_1(\xi, \tau) + \epsilon u_2(\xi, \tau) + \cdots \}, \tag{4.3.11} \]

where \( w_1, w_2, u_1, u_2, \lambda_1, \lambda_2 \) are all order one functions and \( z_{cr} = z_{\infty}(r_{cr}) \) is the critical value of \( z_{\infty} \). We also note that the internal pressure is given by (3.3.16), so that when \( r_{\infty} \) and \( z_{\infty} \) are perturbed as \( \epsilon = r_{\infty} - r_{cr} \), \( P^* \) expands as

\[ P^* = P_0 + \epsilon P_1 + \cdots, \tag{4.3.12} \]

where again \( P_0 \) and \( P_1 \) are order one functions. We now substitute (4.3.10), (4.3.11) and (4.3.12) into the equations of motion, (3.3.13), and equate the powers of \( \epsilon \). We find at first order in \( \epsilon \), after integrating one equation, two equations for \( w_1(\xi) \) and \( u_1(\xi) \), which may be rewritten as,

\[ L \begin{bmatrix} w_1 \\ \sqrt{\omega'(r_{cr})} u_{1\xi} \end{bmatrix} = 0, \quad \text{where} \quad L = \begin{bmatrix} -\hat{W}_1/z_{cr} + \hat{W}_{12} & \hat{W}_{22} \\ z_{cr}(\hat{W}_1 - r_{cr}\hat{W}_{11}) & r_{cr}(\hat{W}_1 - z_{cr}\hat{W}_{12}) \end{bmatrix}, \tag{4.3.13} \]

where \( \hat{W}_1, \hat{W}_2, \hat{W}_{12}, \hat{W}_{22} \) are all evaluated at \( r = r_{cr}, \ z' = z_{cr}, \) and \( u_{1\xi} \) denotes \( \partial u_1/\partial \xi \). Clearly, \( \det L = \omega(r_{cr})r_{cr}\hat{W}_2\hat{W}_{22} \). Thus, as we expected, \( \omega(r_{cr}) = 0 \) ensures that the matrix equation (4.3.13) has a non-trivial solution for \( w_1 \) and \( u_1 \). Proceeding to the next order, we find that \( w_2 \) and \( u_2 \) satisfy the inhomogeneous system

\[ L \begin{bmatrix} w_2 \\ \sqrt{\omega'(r_{cr})} u_{2\xi} \end{bmatrix} = b, \tag{4.3.14} \]

where the vector \( b \) only contains \( w_1 \) and its derivatives. As the matrix \( L \) is singular as discussed above, we now need to impose a solvability condition in order to be able to find a solution to (4.3.14). We do this by taking the dot product of (4.3.14) with the left eigenvector of \( L \), resulting in a zero left hand side of the resulting equation from the
definition of the left eigenvector. We then obtain an evolution equation from the right hand side of (4.3.14), which is of the form,

$$\frac{\partial^2 V}{\partial \xi^2} - c_1 \frac{\partial^2 V}{\partial \tau^2} = c_2 \frac{\partial^4 V}{\partial \xi^4} + c_3 \frac{\partial^2 (V^2)}{\partial \xi^2},$$

(4.3.15)

where $c_1, c_2, c_3$ are known constants, and $V$ is given by (4.3.4). Although the expressions for the constants $c_1, c_2, c_3$ are available from the above perturbation procedure, we may obtain their expressions more simply as follows.

First, from the fact that when $V$ is assumed to be independent of $\tau$, (4.3.15) must reduce to the static amplitude equation (4.3.3), we deduce that $c_2 = c_3 = 1$. To determine the remaining constant $c_1$, we linearise (4.3.15) and then look for a travelling wave solution of the form

$$V = e^{iK(\xi - v\tau)} = \exp\left(iK\sqrt{\epsilon \omega'(r_{cr})}(Z - \sqrt{\epsilon \omega'(r_{cr})}vt)\right),$$

(4.3.16)

resulting in the following linearised approximation to the dispersion relation,

$$v^2 = \frac{1 + K^2}{c_1}.$$  

(4.3.17)

From (4.3.16) we see that the connection between $K$ and $v$ and the actual wave number $\hat{k}$ and speed $\hat{c}$ given in (4.3.7) is,

$$\hat{k} = K \sqrt{\epsilon \omega'(r_{cr})}, \quad \hat{c} = v \sqrt{\frac{\epsilon}{\omega'(r_{cr})}}.$$  

(4.3.18)

Inserting (4.3.18) into the linearised dispersion relation (4.3.17) we find,

$$\hat{c}^2 = \frac{\epsilon}{c_1 \omega'(r_{cr})} + \frac{\hat{k}^2}{c_1 \omega'(r_{cr})^2} = \frac{r_{\infty} - r_{cr}}{c_1 \omega'(r_{cr})} + \frac{\hat{k}^2}{c_1 \omega'(r_{cr})^2}.$$  

(4.3.19)

Solving the exact dispersion relation (4.3.8) for $\hat{c}$, we obtain,

$$\frac{\rho \hat{c}^2}{\mu} = q(r_{\infty}) + O(\hat{k}^2) = q'(r_{cr})(r_{\infty} - r_{cr}) + O(\hat{k}^2, \epsilon^2),$$  

(4.3.20)
where the function $q$ is given by,

$$
q(r_\infty) = -\frac{r_\infty W_2^{(\infty)} W_{22}^{(\infty)}}{z_\infty (W_1^{(\infty)} - r_\infty W_{11}^{(\infty)})} \omega(r_\infty),
$$

(4.3.21)

and we have used the Taylor series around $r_{cr}$, utilising the fact that $\omega(r_{cr}) = 0$ by definition of $r_{cr}$. The value of the last coefficient in the evolution equation $c_1$ is then obtained by comparing (4.3.19) with (4.3.20), which gives,

$$
c_1 = \frac{\rho}{\mu q'(r_{cr}) \omega'(r_{cr})} = -\frac{z_{cr} W_1(r_{cr}, z_{cr}) - r_{\infty} W_{11}(r_{cr}, z_{cr})}{r_{cr} \mu W_2(r_{cr}, z_{cr}) W_{22}(r_{cr}, z_{cr})} \omega'(r_{cr}),
$$

(4.3.22)

where we have again used $\omega(r_{cr}) = 0$. Therefore our evolution equation is given by,

$$
\frac{\partial^2 V}{\partial \xi^2} - c_1 \frac{\partial^2 V}{\partial \tau^2} = \frac{\partial^4 V}{\partial \xi^4} + \frac{\partial^2 V}{\partial \xi^2},
$$

(4.3.23)

with $c_1$ given by (4.3.22). We note that when evaluated, $c_1$ is non-negative for values of $(r_\infty, r_0)$ which correspond to the bounded solutions. To study the stability of our near-critical solution $V_0(\xi)$ given by (4.3.5), we substitute the following ansatz,

$$
V(\xi, \tau) = V_0(\xi) + B(\xi)e^{\sigma \tau}
$$

(4.3.24)

into (4.3.15) and linearise to obtain

$$
\frac{d^4 B}{d\xi^4} - \frac{d^2 B}{d\xi^2} + \frac{2}{d\xi^2} \frac{d^2 (V_0 B)}{d\xi^2} + c_1 \sigma^2 B = 0,
$$

(4.3.25)

which is to be solved subject to the following decaying conditions ensuring the boundary conditions (3.3.16) remain satisfied,

$$
B \to 0 \quad \text{as} \quad \xi \to \pm \infty.
$$

(4.3.26)

The determination of the value(s) of $\sigma$ which satisfy (4.3.25) is therefore an eigenvalue problem. The static solution $V_0(\xi)$ is said to be linearly unstable or spectrally unstable if,
for some fixed complex $\sigma$ with $\text{Re} (\sigma) > 0$, there exists a solution of (4.3.25) which decays exponentially as $\xi \to \pm \infty$.

### 4.3.2 Compound Matrix Method

We therefore solve this eigenvalue problem by computing the Evans function for the compound matrix method as discussed in Section 2.5. We follow the procedure explained in Afendikov and Bridges (2001) in which the eigenvalue problem also involves a fourth-order differential equation. We rewrite the system (4.3.25) as a matrix system of first order differential equations as,

$$y' = A(\xi)y,$$

where $y = (B(\xi), B'(\xi), B''(\xi), B'''(\xi))^T$, and the matrix $A$ is given by,

$$A(\xi; \zeta) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\zeta - 2V''_0(\xi) & -4V'_0(\xi) & 1 - 2V_0(\xi) & 0
\end{pmatrix},$$

(4.3.28)

where $\zeta = c_1\sigma^2$. Evaluating (4.3.28) at $\xi \to \infty$, we find,

$$A_\infty = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\zeta & 0 & 1 & 0
\end{pmatrix},$$

(4.3.29)

which has two pairs of eigenvalues, given by $\pm k_1, \pm k_2$, where $k_1 = \sqrt{\frac{1}{2}(1 - \sqrt{1 - 4\zeta})}$, $k_2 = \sqrt{\frac{1}{2}(1 + \sqrt{1 - 4\zeta})}$, which will be denoted as matrix eigenvalues to distinguish them from the eigenvalues of the eigenproblem under consideration. Associated with each of these matrix eigenvalues is an eigenvector, and we denote the two eigenvectors associated with $k_1, k_2$ as $a^+_1, a^+_2$ respectively, and those associated with $-k_1, -k_2$ as $a^-_1, a^-_2$. 

87
respectively. As \( \xi \to \infty \), any decaying solution of (4.3.27) will be a linear combination of \( a_1^+ e^{-k_1 \xi} \) and \( a_2^+ e^{-k_2 \xi} \) and the nomenclature becomes clear. Similarly, as \( \xi \to -\infty \) decaying solutions will be a linear combination of \( a_1^- e^{k_1 \xi} \) and \( a_2^- e^{k_2 \xi} \).

For a fixed \( \zeta \) we may integrate from \(-L\) using \( a_1^- \) with \( k_1 \), resulting in a solution \( y_1^- (\xi) \), where \( L \) is a large number used instead of infinity for the numerical results. Similarly, we can find a second solution \( y_2^- (\xi) \) from using \( a_2^- \), and two solutions which come from integrating from \( L \), \( y_1^+ (\xi) \) and \( y_2^+ (\xi) \). Naively, we could then attempt to find a value of \( \zeta \) such that the solutions intersect in a non-trivial way, namely that the determinant formed by \( (y_1^- (\xi) y_2^- (\xi) y_1^+ (\xi) y_2^+ (\xi)) \) is zero at a specific \( \xi \).

However, we use the compound matrix method as discussed in Section 2.5.3, which ensures that the differential equations are non-stiff. The disadvantage of the compound matrix method is that it converts a system of order \( 2n \) into a system of order \( n(n+1) \) and is hence impractical for large \( n \), but this is an acceptable compromise here where \( n = 2 \) and therefore we change from fourth order to sixth order.

We define two new matrices \( M^- (\xi) \) and \( M^+ (\xi) \) by concatenating the two appropriate solutions, as discussed in Section 2.5.3, along with the two sets of minors \( \phi_i^- \) and \( \phi_i^+ \). We therefore reach a sixth order differential system, where (4.3.27) becomes

\[
\phi' = Q(\xi)\phi, \tag{4.3.30}
\]

where \( Q \) is given by (2.5.14).

The two sets of initial conditions for the \( \phi_i \) are given by the corresponding minors of \( M^- (-L) \) and \( M^+ (L) \), and then (4.3.30) is integrated from both sides for a given \( \zeta \). However, to remove the exponential growth which the solutions must obey in this case we write \( \phi(\xi) = \psi(\xi)e^{K\xi} \), where \( K = k_1 + k_2 \) for integration from \(-L\) and \( K = -(k_1 + k_2) \) for the integration from \( L \), and therefore (4.3.30) becomes

\[
\psi' = (Q - KI)\psi, \tag{4.3.31}
\]
where \( I \) is the six by six identity matrix. This step ensures that it is possible to find the eigenvalue without the exponentially large terms influencing the numerical computations.

This procedure breaks down at points where the matrix eigenvalues are repeated, that is \( k_1 = k_2 \), where there is only one ordinary eigenvector associated with each side and therefore the minors of \( Y^-(-L) \) and \( Y^+(L) \) are all zero. At this point we need to replace \( a_2^+, a_2^- \) by the corresponding generalised eigenvectors

\[
\begin{align*}
a_2^+ &= \lim_{k_2 \to k_1} \frac{a_2^+ - a_1^+}{k_2 - k_1}, \quad \text{and} \quad a_2^- &= \lim_{k_2 \to k_1} \frac{a_2^- - a_1^-}{k_2 - k_1},
\end{align*}
\] (4.3.32)

respectively.

To accommodate this isolated case, we may replace the determinant \(|N|\) in (2.5.17) by

\[
|N^*| = \left| \begin{array}{c}
y_1^-(d), \quad y_2^-(d) - y_1^-(d) \\
y_1^+(d), \quad y_2^+(d) - y_1^+(d)
\end{array} \right|,
\] (4.3.33)

which may be seen to be, using elementary properties of determinants, as

\[
|N| = |N^*|(k_2 - k_1)^2,
\]

and therefore the roots will be the same except \(|N^*|\) will not have a root where the matrix eigenvalues are repeated.

In this example, \( k_1 = k_2 \) at \( \zeta = \frac{1}{4} \) and we need to use the process discussed above to remove this point to create a modified Evans function. Therefore, we use \(|N(\zeta, d)|\) in the form

\[
|N(\zeta, d)| = \frac{1}{1 - 2\sqrt{\zeta}} \det[y_1^-(d), \ y_2^-(d), \ y_1^+(d), \ y_2^+(d)],
\] (4.3.34)

with \( D(\zeta) \) still defined as in 2.5.9.

### 4.3.3 Results

Doing this we find that there is just one positive solution of \( D(\zeta) = 0 \), which is given by \( \zeta_1 = \frac{3}{10} \), as shown in Figure 4.3.1. The normalised eigenfunction for this eigenvalue is given by Figure 4.3.2. To find the eigenfunction of (4.3.25) we try an expansion of the
Figure 4.3.1: Evans function for the weakly nonlinear solution, dashed line is the unmodified Evans function.

Figure 4.3.2: Normalised eigenfunction of the nontrivial weakly nonlinear solution corresponding to the eigenvalue $\zeta_1 = \frac{3}{16}$ form,

$$B(\xi) = b(\xi) \left( g_0 + g_1 \text{sech}(\xi/2) + g_2 \text{sech}^2(\xi/2) + g_3 \text{sech}^3(\xi/2) \right), \quad (4.3.35)$$

for some function $b$, where the form of this expression has been assumed from the $\text{sech}(\xi/2)$ dependence in $V_0(\xi)$. For $b(\xi) = 1$ the resulting system is only non-trivial for $\zeta = \frac{3}{16}$ and
may be simplified to

$$B(\xi) = B_0 \left( 2 \text{sech}^3 \left( \frac{\xi}{2} \right) - \text{sech} \left( \frac{\xi}{2} \right) \right) ,$$

(4.3.36)

where $B_0$ is an arbitrary amplitude. Therefore the analytic expression for the eigenfunction corresponding to $\zeta_1$ is given by (4.3.36). For $b(\xi) = \tanh(\xi/2)$, the eigenvalue at zero has an eigenfunction given by,

$$B(\xi) = B_1 V_0'(\xi) = -\frac{3B_1}{4} \tanh \left( \frac{\xi}{2} \right) \text{sech} \left( \frac{\xi}{2} \right) ,$$

(4.3.37)

where $B_1$ is an arbitrary amplitude. Therefore, as the above analysis is for a general strain-energy function and either the bulging or necking near-critical solution, we state that all such near-critical solutions are unstable with respect to axisymmetric perturbations.

### 4.4 Stability of the General Bifurcated State

#### 4.4.1 Introduction

In this section we consider the stability of the fully nonlinear bifurcated solution with respect to spectral perturbations of the form $a(Z)e^{\eta t}$, in the same way as in the section 4.3 for the weakly nonlinear solution. We insert these perturbations into the equations of motion and linearise, resulting in a differential equation for the unknown function $a(Z)$ which depends on $\eta$. This equation will then be solved subject to the boundary condition that $a(Z)$ decays as $Z \to \infty$.  

91
4.4.2 Governing Equations

We therefore insert into the equations of motion (3.3.13) perturbations of the general form,

$$r(Z,t) = \bar{r}(Z) + w(Z,t), \quad z(Z,t) = \bar{z}(Z) + u(Z,t), \quad (4.4.1)$$

where $\bar{r}$ and $\bar{z}$ are the static non-trivial solutions found from the integration of the integrated governing equations (3.3.11) and (3.3.12), as discussed in Chapter 3. Note that $\bar{r}$ and $\bar{z}$ both depend on $r_\infty, z_\infty$ and the form of $\hat{W}$. Inserting (4.4.1) into (3.3.13) and linearising around the static state we find:

$$\left[ \frac{\bar{W}_2w' + \frac{\bar{r}'}{\bar{\lambda}_2^2}(\bar{\lambda}_2\bar{W}_{22} - \bar{W}_2)(\bar{r}'w' + \bar{z}'u') + wz'\bar{W}_{12}}{\bar{\lambda}_2} \right]' - P^*(\bar{r}w' + w\bar{r}') = \rho\ddot{u}, \quad (4.4.2)$$

$$\left[ \frac{\bar{W}_2w' + \frac{\bar{r}'}{\bar{\lambda}_2^2}(\bar{\lambda}_2\bar{W}_{22} - \bar{W}_2)(\bar{r}'w' + \bar{z}'u') + \bar{r}'r_1\bar{W}_{12}}{\bar{\lambda}_2} \right]' - \frac{\bar{W}_{12}(\bar{r}'w' + \bar{z}'u')}{{\bar{\lambda}_2}} - w\bar{W}_{11} - P^*(\bar{r}u' + w\bar{z}') = \rho\ddot{w}, \quad (4.4.3)$$

where $\bar{\lambda}_2 = \sqrt{\bar{r}^2 + \bar{z}^2}$ and a bar indicates evaluation at the known values of $r = \bar{r}, z = \bar{z}$. It should be noted that $P^*$ in (4.4.2) and (4.4.3) is also a function of $r_\infty$ as given by (3.3.16).

We then assume that the perturbations have an exponential time dependence given by,

$$w(Z,t) = \tilde{w}(Z)e^{\eta t}, \quad u(Z,t) = \tilde{u}(Z)e^{\eta t}, \quad (4.4.4)$$

introducing $\eta$ as the eigenvalue. It is clear that if $\eta$ has a positive real part then the amplitude of the perturbation will grow exponentially in time, and hence the static solution is unstable.

It can be seen that (4.4.2) and (4.4.3), after use of (4.4.4), is a system of two coupled linear second order ordinary differential equations in $\tilde{w}(Z)$ and $\tilde{u}(Z)$, and the dependence on $\eta$ is entirely through the combination $\alpha = \rho \eta^2$, which defines $\alpha$. The determination of
values of $\alpha$ for which this system has non-zero solutions is hence an eigenvalue problem. We repeat the use of the compound matrix method as in section 4.3 to find such eigenvalues, if they exist, for the given initial state.

### 4.4.3 Compound Matrix Method

We rewrite the system (4.4.2), (4.4.3) in terms of $y = (\tilde{u}(Z), \tilde{u}'(Z), \tilde{w}(Z), \tilde{w}'(Z))^T$, which is the same form as that used in (4.3.27) but with $\xi$ replaced by $Z$. The matrix $A(Z)$ in (4.3.27) will be a function of $Z$ via the fully nonlinear static solution $\bar{r}, \bar{z}$, and is also therefore dependent of the value of $r_{\infty}$. Rearranging (4.4.2) and (4.4.3) to isolate $\tilde{u}''(Z)$ and $\tilde{w}''(Z)$, leads to the following structure for $A(Z)$,

$$
A(Z) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} \\
0 & 0 & 0 & 1 \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{pmatrix},
$$

(4.4.5)

where the non-zero components $A_{ij}$ can be found explicitly but are extremely long and not shown here. From the conditions governing the decay of the underlying state as $Z \to \pm \infty$, we know $\bar{r}(Z) \to r_{\infty}$, $\bar{z}'(Z) \to z_{\infty}$ and the matrix $A_{\infty} = \lim_{Z \to \infty} A(Z)$ becomes,

$$
A_{\infty} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{\alpha}{W_1^{(\infty)} + z_{\infty} W_2^{(\infty)}} & 0 & 0 & \frac{W_1^{(\infty)} - z_{\infty} W_2^{(\infty)}}{z_{\infty} W_2^{(\infty)}} \\
0 & 0 & 0 & 1 \\
0 & -\frac{W_1^{(\infty)} + z_{\infty} W_2^{(\infty)}}{W_2^{(\infty)}} & -\frac{z_{\infty} W_1^{(\infty)} + z_{\infty} r_{\infty} W_2^{(\infty)}}{r_{\infty} W_2^{(\infty)}} & 0
\end{pmatrix},
$$

(4.4.6)

where $r_{\infty}$, $z_{\infty}$ and $\bar{W}$ are known and involved in the determination of the underlying state. For the localised bulged or necked solutions considered here, the matrix eigenvalues of $A_{\infty}$ are given by two pairs, $\pm k_1, \pm k_2$, for $\alpha > 0$. There may exist, depending on $r_{\infty}$ and $\bar{W}$, two values of $\alpha$ where $k_1 = k_2$ given by $\alpha^1, \alpha^2$. Therefore the matrix eigenvalues $k_1$ and $k_2$ are real for $\alpha \in [0, \alpha^1]$, complex for $\alpha \in [\alpha^1, \alpha^2]$ and real again for $\alpha \in [\alpha^2, \infty)$. 

93
The complex region \([\alpha^1, \alpha^2]\) occurs for \(r_\infty\) close to \(r_{cr}\), as \(r_\infty\) moves away from \(r_{cr}\), \(\alpha^1\) and \(\alpha^2\) coalesce and the matrix eigenvalues \(k_1\) and \(k_2\) become real for all \(\alpha\). These two isolated cases can be removed, in the same way as \(\frac{1}{4}\) was in the weakly nonlinear case, after finding their values numerically from the matrix eigenvalues of \(A_\infty\) for a given \(r_\infty\) and \(\tilde{W}\). Alternatively, the unmodified compound matrix method may be used with an additional \textit{a posteriori} verification that any roots found do not correspond to where the matrix eigenvalues are repeated.

### 4.4.4 Results

We can therefore proceed with the compound matrix method to find the eigenvalue for each \(r_\infty\). Close to the critical point, \(r_{cr}\), we must recover the near-critical results discussed in Section 3.4.1. The connection between the weakly nonlinear eigenvalue \(\sigma\) and the fully nonlinear eigenvalue \(\eta\) is given by \(\eta = \epsilon \sigma\). Hence in the limit as \(r_\infty \to r_{cr}\), using \(\alpha = \rho \eta^2\) and \(\eta = c_1 \sigma^2\), we require the connection,

\[
\alpha = \frac{\rho \epsilon^2 \zeta_1}{c_1},
\]

where \(\zeta_1\) is the real eigenvalue of the weakly nonlinear Evans function given in Figure 4.3.1, that is \(\zeta_1 = \frac{3}{16}\). This connection also provides an approximate value of \(\alpha^1\) where \(k_1 = k_2\), if \(\zeta_1\) is replaced by \(\frac{1}{4}\), the counterpart of \(\alpha^1\) in the weakly nonlinear case. The second point where the matrix eigenvalues become repeated, \(\alpha^2\), corresponds to a repeated eigenvalue at ‘infinity’ in the weak situation, as the matrix eigenvalues tend towards zero as \(Z\) gets very large, and therefore this connection may not be expressed by (4.4.7).

The eigenvalues found this way using the fully nonlinear solution agree precisely with those given by the use of (4.4.7) in the limit as \(r_\infty \to r_{cr}\), for both open and closed tubes described by the various strain-energy functions considered here. Figure 4.4.1 shows how the proper eigenvalue for the Gent strain-energy function with closed ends is proportional to \((r_\infty - r_{cr})^2\) near \(r_{cr}\), with the coefficient given by \(\frac{\rho \zeta_1}{c_1}\). For comparison, the
values of $\alpha^1$ and $\alpha^2$ are 0.000045 and 3.770042 when $r_\infty = 1.595$, illustrating how $\alpha^2$ corresponds to the repeated matrix eigenvalues at infinity in the weakly nonlinear case, as it is five orders of magnitude larger than $\alpha^1$.

![Graph showing comparison of calculated values of $\alpha$ for the closed Gent strain-energy function with $J_m = 97.2$ near to the critical point $r_\infty = 1.59676$ (dots), with the relation (4.4.7) (dashed).]

It is found that as $r_\infty$ moves further away from the critical point $r_{cr}$ the two points $\alpha^1, \alpha^2$ coalesce and the matrix eigenvalues $k_1, k_2$ are real for all $\alpha > 0$. Figure 4.4.2 shows that there exists a positive eigenvalue throughout the range $(r_k, r_{cr})$ for the closed Gent strain-energy function with $J_m = 97.2$. It is seen that as the kinked solution is approached at $r_k = 1.1694$ the value of the eigenvalue rapidly approaches zero. Chapter 5 will show more examples of the calculated eigenvalues, showing that the bulged and necked solutions are unstable throughout the ranges in which they exist for both open and closed tubes.

### 4.5 Energy Minimisation

In this section, we test the stability of the fully nonlinear static solution by determining whether it is an energy minimiser with respect to variations which decay to zero as $Z \to$
±\infty. If the static solution is not an energy minimiser within this class of perturbations, which satisfy the same boundary conditions, then it will not be stable when subjected to small perturbations from the static solution; and therefore will not exist physically.

The total potential energy stored in the membrane tube is given by,

\[ E = \int_{-L}^{L} W(\lambda_1, \lambda_2)2\pi RHdZ - \int_{-L}^{L} P\pi r^2dz, \]  

where the first term is the strain-energy density function, which is defined with respect to the undeformed configuration, multiplied by the cross sectional area of the tube, then integrated over the length of the tube. The second term is the contribution from the pressure, acting on the internal area of the deformed tube.

We then convert the second integral to the undeformed configuration and rewrite (4.5.1) as

\[ E^* = \int_{-L}^{L} (2RW - P^*r^2z')dz, \]  

where \( E^* = E/(\pi H) \). To facilitate the use of calculus of variations we assume that all the dependent variables depend on a time-like variable \( \tau \) as well as the original variable \( Z \),
and we use
\[ r|_{\tau=0} = \frac{\partial r}{\partial \tau}|_{\tau=0} \tag{4.5.3} \]
to denote the variation in \( r \), and a superimposed dot has the same meaning for any other quantity. On differentiating (4.5.2) with respect to \( \tau \), we obtain
\[
\dot{E} = \int_{-L}^{L} \left\{ 2R \left( W_1 \dot{\lambda} + W_2 \dot{\lambda} \right) - P^* \left( 2rz' \dot{r} + r^2 \dot{z}' \right) \right\} dZ,
\]
\[
= \int_{-L}^{L} \left\{ 2W_1 \dot{r} + \frac{2RW_z}{\lambda_2} (r' \dot{r}' + z' \dot{z}') - 2Prz' \dot{r} - Pr^2 \dot{z}' \right\} dZ, \tag{4.5.4}
\]
\[
= \int_{-L}^{L} \left\{ \left[ 2W_1 - \left( \frac{2RW_z}{\lambda_2} \right)' \right] \dot{r} + \left( \frac{2RW_z}{\lambda_2} \right) \dot{r}' \right\} \dot{r}'
+ \left[ \frac{2RW_z}{\lambda_2} - P^* r^2 \right] \dot{z}' \right\} \dot{z}' dZ.
\]
Evaluating (4.5.4) at \( \tau = 0 \) and making use of the fact that the variations \( \dot{r} \) and \( \dot{z}' \) are arbitrary along with the boundary conditions, we obtain the following equilibrium equations,
\[
\left( \frac{2RW_z}{\lambda_2} \right)' - 2P^* r \dot{r}' = 0, \quad 2W_1 - \left( \frac{2RW_z}{\lambda_2} \right)' - 2P^* z' = 0, \tag{4.5.5}
\]
which may be seen to be the static version of the equations of motion, (3.3.13).

On differentiating (4.5.4) again with respect to \( \tau \) and then evaluating at \( \tau = 0 \) we obtain, after some manipulation, the second variation in \( E \),
\[
\ddot{E}|_{\tau=0} = 2 \int_{-L}^{L} (g_1 w'^2 + g_2 u'^2 + 2g_3 w' u + g_4 u'^2 + 2g_5 w' u') dZ, \tag{4.5.6}
\]
where we have introduced \( w = \dot{r}, u = \dot{z}' \) to simplify the expression, along with the following coefficients,
\[
g_1 = \frac{R}{\lambda_2^3} (\lambda_2 W_{22} r'^2 + W_{22} z'^2), \quad g_2 = \frac{W_{11}}{R} - \left( \frac{W_{12}}{\lambda_2} \right)' - P^* z',
\]
\[
g_3 = \frac{W_{12} z'}{\lambda_2} - P^* r, \quad g_4 = \frac{R}{\lambda_2^3} (\lambda_2 W_{22} z'^2 + W_{22} r'^2), \quad g_5 = \frac{R(\lambda_2 W_{22} - W_2)}{\lambda_2^3} r' z'. \tag{4.5.7}
\]
The terms in the integrand of the second variation (4.5.6) may now be rewritten in the
following way, as

$$
\tilde{E}|_{\tau=0} = 2 \int_{-L}^{L} \left\{ \left( \sqrt{a_1} w' + \frac{g_5}{\sqrt{g_1}} u \right)^2 + g_2 w^2 + 2g_3 wu + \left( g_4 - \frac{g_5^2}{g_1} \right) u^2 \right\} dZ
$$

$$
= 2 \int_{-L}^{L} \left\{ \left( \frac{g_1}{\sqrt{g_1}} w' + \frac{g_5}{\sqrt{g_1}} u \right)^2 + v \cdot G(Z) v^T \right\} dZ,
$$

(4.5.8)

where $v = (w, u)$, and

$$
G(Z) = \begin{pmatrix}
g_2 & g_3 \\
g_3 & g_4 - \frac{g_5^2}{g_1}
\end{pmatrix}.
$$

(4.5.9)

It can be immediately deduced that a sufficient condition for the positive semi-definiteness of the second variation of the potential energy is that $G$ is positive semi-definite for all $Z$, where a matrix $G$ is positive semi-definite if $x^T G x \geq 0$ for all non-zero vectors $x$. It may also be shown, using proof by contradiction with an appropriate choice of variations $(w, u)$, that this condition is also necessary (Y.B. Fu, personal communication, 6th April 2009). We thus conclude that if $G$ is positive semi-definite for all $Z$ for a given $\hat{r}, \hat{z}$, then the corresponding equilibrium solution is an energy minimiser.

An equivalent statement to $G$ being positive semi-definite is that both of the eigenvalues of $G$ are non-negative. We may therefore test the stability just from considering the eigenvalues of $G$, over the range of $Z$. We let $\rho(Z)$ be the smallest eigenvalue of $G$, at a given $Z$. For many bulged or necked solutions, it is only necessary to evaluate the eigenvalues of $G$ at $Z = 0$, as one of the eigenvalues is negative as shown in Figure 4.5.1.

As the critical point $r_{cr} = 1.59$ in Figure 4.5.1 is approached, $\rho(0)$ approaches zero with $\rho(Z) > 0$ for the remaining range of $Z$. As the kinked solution $r_k = 1.1674$ is approached, $\rho(0)$ becomes positive, although there is a range of $Z$ during which $\rho(Z)$ is still negative as may be seen in Figure 4.5.2. This is expected as around the origin the uniform state associated with the kink at $Z = 0$ is being approached which is stable as discussed above, and it is therefore only the nonuniform transition region which may be unstable. As the kink is approached further, this small range of $Z$ in which $\rho(Z) < 0$ moves away from
zero. Very similar results are also found for necking solutions, and for the other strain-energy functions under consideration here.

\[ \rho(0) \]

![Graph of \( \rho(0) \) for the bulged section of the Gent strain-energy function.]

\[ \rho(Z) \]

![Graph of \( \rho(Z) \) for the Gent strain-energy function near the kinked solution, \( r_\infty = 1.17 \) and 1.16941]

Figure 4.5.1: \( \rho(0) \) for the bulged section of the Gent strain-energy function.

Figure 4.5.2: \( \rho(Z) \) for the Gent strain-energy function near the kinked solution, \( r_\infty = 1.17 \) and 1.16941
4.6 Conclusion

Both the spectral stability and energy methods have shown that the solitary wave type solutions, in either the bulged or necked case, are unstable with respect to axisymmetric perturbations when the internal pressure is controlled. The analysis suggests that the kinked type solution is stable as the eigenvalue tends towards zero as the kinked solution is approached in the spectral stability test. What can be said for the kinked case is that, as the first bifurcation must be subcritical and the second supercritical, the two uniform states which the kinked solution consists of must both be stable; as may be seen from the projections onto the $r_0 = r_\infty$ line of the points $E$ and $G$ in Figure 3.5.1(b). Therefore it is just the transition region which may cause instability, as seen in the energy method above.

Experience with rubber modelling balloons suggests that when inflated the bulged state is unstable but the kinked state is stable and observable, indeed this was a motivating example for the considerations here. The experiments by Alexander (1971) and Kyriakides and Chang (1990, 1991), also appear to suggest that the kinked state is stable as it is observed. We therefore tentatively suggest that the kinked solution is stable, although we are unable to prove this to be true.
Chapter 5

Comparison of Strain-Energy Functions

5.1 Introduction

In this chapter we use the results from Chapters 3 and 4 to determine the inflation of the cylindrical membrane for various strain-energy functions with both closed and open ends, including varying the axial pre-stretch $z_\infty$ in the case of open ends. The bifurcation condition $\omega(r_\infty)$ is calculated, as well as the amplitude stretch diagram discussed in Section 3.5, for the different strain-energy functions. In addition, we show the eigenvalues found via the stability test detailed in Chapter 4.

5.2 Varga Strain-Energy Function

5.2.1 Bifurcation Condition

Initially we will derive results for the Varga strain-energy function given by (2.2.10). The simple form of this strain-energy function allows us to solve (3.3.21) and (3.3.22) explicitly to find $r_0$ and $z_0'$. Equation (3.3.22) can be solved to express $z_0'$ in terms of $r_0$, and then
may be solved to find \( r_0 \),

\[
(r_0 - r_\infty)^2 \left( r_0 - \frac{2(1 + r_\infty^2 z_\infty)}{r_\infty^3 z_\infty^2} \right) = 0. \tag{5.2.1}
\]

The first solution \( r_0 = r_\infty \) is expected and corresponds to the trivial solution, which occurs as a double root as discussed in Section 3.3.3. The second solution, which is associated with a potential non-uniform state, exists for all values of \( r_\infty \) including \( r_\infty = 1 \). The corresponding values for \( z'_0 \) connected to the non-trivial solution are given by,

\[
z'_0 = \frac{r_\infty^6 z_\infty^4}{(r_\infty^2 z_\infty + 1)(r_\infty^4 z_\infty^2 - 2)}. \tag{5.2.2}
\]

However, it can be seen that \( z'_0 \) becomes infinite when \( r_\infty^4 z_\infty^2 = 2 \), and hence the potential non-uniform state ceases to be physically realistic at some value \( r^* \), where \( r^* = \left(4(\sqrt{2} - 1)\right)^\frac{1}{3} \approx 1.183 \) for closed tubes and \( r^* = \left(\frac{2}{z_\infty^2}\right)^{1/4} \) for open tubes. When \( z_\infty = 1, r^* \approx 1.189 \) and the two values of \( r^* \) are very similar for these two cases. The bifurcation condition is shown in Figure 5.2.1, and is given by

\[
\omega(r_\infty) = \frac{z_\infty^2 (r_\infty^4 z_\infty^2 - 2r_\infty^2 z_\infty^2 - 2)}{2r_\infty^2 (r_\infty^2 z_\infty^2 - 1)}, \tag{5.2.3}
\]

with the closed ends condition as given in (3.3.19), that is \( 1 + r_\infty^2 z_\infty^2 - 2r_\infty^2 z_\infty^2 = 0 \).
For open ended tubes, the bifurcation point is therefore found by solving the quartic in the numerator of (5.2.3), giving $r_{cr} = \sqrt{\frac{1 + \sqrt{3}}{z_\infty}}$. For the closed tube we need to solve the two equations simultaneously, giving $r_\infty = 2^{2/3}$, $z_\infty = 2^{-4/3}(1 + \sqrt{3})$, which is close to the result for $z_\infty = 1$ as seen in Figure 5.2.1. Obviously, if the remote axial stretch $z_\infty$ is greater than $1 + \sqrt{3}$ then the critical point comes before unity, indicating that a large axial stretch is associated with a reduction in the far-radius. In the case where $z_\infty < 1$, the denominator of (5.2.3) can become infinite and hence $\omega(r_\infty)$ tend to zero for some $r_\infty > 1$, but we require that $z_\infty > 1$ as the membrane is in tension rather than compression. Therefore, for open ends $r^*$ is related to the critical point $r_{cr}$ by,

$$r^* = \frac{\sqrt{2}}{\sqrt{1 + \sqrt{3}}}r_{cr} \approx 0.7194r_{cr}, \quad (5.2.4)$$

implying that, for all values of the remote axial stretch, there is a range of $r_\infty$ for which a bulged tube may exist.

### 5.2.2 Amplitude-Stretch Diagram

The amplitude-stretch diagrams for a variety of $z_\infty$ along with the case of closed ends, are given in Figure 5.2.2, which shows how the amplitude of the potential bifurcated solution behaves as $r_\infty$ is changed. Figure 5.2.2 corresponds to the non-trivial solution of (5.2.1), and shows the amplitude of the bulged solutions. The existence of the bulged solutions indicated in Figure 5.2.2 is shown by the negative gradient of $\omega(r_{cr})$ in Figure 5.2.1, for the various $z_\infty$. The abrupt ends of the top curves represent $r^*$, where $z'_0$ becomes infinite and show that the values of $r_0$ at which this occurs are finite. Past this point the curves continue in the same manner but with negative $z'_0$, and are therefore not shown after this point. The case of closed ends is quantitatively similar to that of the open ends, with the effect of the variation in $z_\infty$ clearly shown, as it varies between the curves for $z_\infty = 1$ and $z_\infty = 1.5$. As $r^*$ is approached $z'_0$ tends to infinity, leading to a deformed configuration which flattens at $Z = 0$, in a similar way to the kinked solutions, as shown in Figure 5.2.3.
Figure 5.2.2: Potential bulge/neck amplitude \( r_0 - r_\infty \) for the Varga strain-energy function.

However, this is a separate mechanism than that which leads to the kinked solutions detailed in Section 3.6 as those solutions do not become infinite.

In addition, equation (3.4.4) may be written analytically as,

\[
 r'^2 = \frac{4z_\infty^2(\psi^2r - 4r_\infty^2z_\infty^2)^2 - r(r + r_\infty^2 + r_\infty^4z_\infty - r_\infty^2z_\infty - 2r_\infty^3z_\infty^2)^2)}{\psi^2r^2(\psi^2r - 4r_\infty^2z_\infty^2)^2}, \tag{5.2.5}
\]

where \( \psi = 2 + r_\infty z_\infty (r_\infty - r) \). Equation (5.2.5) may be solved subject to the second initial condition in (5.2.1), to obtain a bounded non-uniform solution for \( r(Z) \) for all values of
$r_\infty$ in $[r^*, r_\infty]$. This is therefore a simple first order differential equation for the deformed configuration.

5.2.3 Stability

Considering the stability of the Varga strain-energy function using the methods described in Chapter 4, we find that there is a single positive unstable eigenvalue for all values of $r_\infty$ between $r^*$ and $r_{cr}$, the value of which increases as $r_\infty \to r^*$. Figures 5.2.4 shows this behaviour both for the case of $z_\infty = 1$ and the closed tube.

![Figure 5.2.4: The eigenvalues of the fully nonlinear bifurcated state for the Varga strain-energy function, open ends with $z_\infty = 1$ and closed ends respectively.](image)

5.3 Neo-Hookean Strain-Energy Function

5.3.1 Bifurcation Condition

We now turn to the neo-Hookean strain-energy function given by (2.2.11). In this case, and all the following strain-energy functions, there is no longer a simple expression for $r_0$ and $z'_0$ as for the Varga strain-energy function, and so we use (3.4.5) to find the bifurcation condition as,

$$\omega(r_\infty) = \frac{2r_\infty^2 ( -3 - 6r_\infty^4 z_\infty^2 - 4r_\infty^2 z_\infty^4 + r_\infty^8 z_\infty^4 )}{r_\infty^2 ( -3 + 2r_\infty^2 z_\infty^4 + r_\infty^4 z_\infty^8 )} \quad (5.3.1)$$

with the closed ends condition given by (3.3.19)$_2$, that is $1 + r_\infty^4 z_\infty^2 - 2r_\infty^2 z_\infty^4 = 0$. In Figure
5.3.1 The bifurcation condition (5.3.1) is plotted for a range of \( z_\infty \) along with the closed ends case. In the limit as \( z_\infty \to \infty \) the bifurcation point \( r_{cr} \) approaches \( 2^{1/3} \) from above. Thus there exists a critical point at all values of \( z_\infty > 0 \), in the same way as for the Varga strain-energy function, but in this case it is always greater than unity.

5.3.2 Amplitude-Stretch Diagram

In addition, there exists a limit point \( r^* = (z_\infty)^{1/3} \) similarly to the Varga strain-energy function.
function, at which \( r_0 \) and \( z'_0 \) both become infinite, which may be seen in the amplitude-stretch diagram shown in Figure 5.3.2. This corresponds to a singularity in that the tube keeps expanding indefinitely, presumably followed by rupture. The case of closed ends has the singularity at \( r^* = 1 \) and the bifurcation point at \( r_{cr} = (17 + 3\sqrt{21})^{1/6}/2 \).

### 5.4 Gent Strain-Energy Function

#### 5.4.1 Bifurcation Condition

We now consider the Gent strain-energy function given by (2.2.13), which was used as the prevailing example throughout the derivations given in Chapter 3. The parameter \( J_m \), which represents the maximum stretch which the molecular chains comprising the material can extend to as detailed in Section 2.2.4, will be shown as 30 or 97.2 in these examples, although very similar qualitative behaviour is found across the range of \( J_m \), with the diagram being compressed for smaller \( J_m \). The bifurcation condition (3.4.5) is given by a sixth order polynomial in \( r^2_\infty \), which must be solved numerically. Figure 5.4.1

![Figure 5.4.1: Bifurcation condition for the Gent strain-energy function with \( J_m = 97.2 \)](image-url)

Figure 5.4.1: Bifurcation condition for the Gent strain-energy function with \( J_m = 97.2 \)
shows the plot of $\omega(r_\infty)$ for varying $z_\infty$, for $J_m = 97.2$ with a range of values of $z_\infty$ and the closed end case. It is seen that there are two bifurcation points, the first of which is subcritical and the second is supercritical, as expected. Therefore bulging and necking solutions may both exist. The case of closed ends substantially changes the distance between the two critical points of $\omega(r_\infty)$, a feature which will be seen to have a significant effect in the amplitude-stretch diagrams. At $z_\infty \approx 3.6198$ for $J_m = 97.2$, the two critical points coalesce and disappear. As $J_m$ decreases, the maximum value of the remote axial stretch for which localised solutions exist decreases, as shown in Figure 5.4.2.

![Figure 5.4.2: Maximum remote axial stretch $z_\infty$ for critical points to exist for the Gent strain-energy function as a function of $J_m$.](image)

5.4.2 Amplitude-Stretch Diagram

This bifurcation behaviour is consistent for a wide range of values of $J_m$, which can be seen in the profile of $\omega(r_\infty)$ for $J_m = 30$ in Figure 5.4.3. Figure 5.4.2 shows the maximum values of $z_\infty$ as a function of $J_m$, beyond which no critical points exist. For $z_\infty = 1$ this maximum value is $J_m^* = 11.070$ as shown in Figure 5.4.2, whereas for closed ends it is $J_m^* = 18.231$. As was mentioned previously, for realistic arteries $z_\infty \approx 1.5$, for which
\begin{align*}
J^*_m &= 19.472. \text{ In the limit as } J_m \to \infty, \text{ recovering the neo-Hookean strain-energy function, the second critical point moves towards infinity like } \sqrt{J_m}, \text{ as does the maximum value of } z_\infty \text{ at which solitary wave solutions exist. In this case only the first bifurcation point remains, which recovers the results for the neo-Hookean strain-energy function given in Section 5.3.}

In the preceding chapter Figure 3.5.1 showed the behaviour of the amplitude of the bulging and necking solutions, as well as their existence, for \( J_m = 97.2 \). For other values of \( J_m \) the same behaviour is found, as expected from the discussions regarding the bifurcation points above.

\subsection{5.4.3 Stability}

As was shown in Figure 4.4.2 in Chapter 4, the bulged or necked solutions are unstable when \( J_m = 97.2 \), although as the kinked solution is approached the eigenvalue tends towards zero. Figure 5.4.4 shows the behaviour for the case where \( z_\infty = 1 \). It should be noted that it becomes difficult to confidently find the eigenvalues as the kinked state is
Figure 5.4.4: Plot of $\alpha$ for the Gent strain-energy function with $z_\infty = 1$ and $J_m = 97.2$ approached due to numerical errors, and hence in Figure 5.4.4 these points have not been plotted.

5.5 Ogden Strain-Energy Function

5.5.1 Amplitude-Stretch Diagrams

Section 2.2.5 introduced the general separable strain-energy function defined in (2.2.14), along with a set of constants which were first given in Ogden (1972) for rubber. We shall now show the behaviour of such a tube modelled by this strain-energy function. In this case the bifurcation condition must be solved numerically. Figure 5.5.1 shows how there are between one and three bifurcation points, depending on the value of $z_\infty$ and whether the tube is open or closed.

The amplitude stretch diagram for the case of open ends with $z_\infty = 1$ is plotted in Figure 5.5.2. In this case there is just one bifurcation point, and as $r_\infty \to \infty$, $r_0$ behaves asymptotically as $1.23r_\infty^{1.48} - 0.39r_\infty^{0.52}$. Hence Figure 5.5.2 provides a complete picture of the bifurcation diagram with one bulging region and one kinked solution for this case. The kinked solution can therefore exist despite a second bifurcation point not existing, but
only in the case of an open tube. As $z_\infty$ is increased the same behaviour is found until $z_\infty \approx 3.1$, at which another pair of bifurcation points appears via the dip in the top curve seen in Figure 5.5.2 which gets increasingly pronounced until it crosses the axis. After this, as $z_\infty$ increases the behaviour stays the same, with three bifurcation points and a kinked solution, even as $z_\infty \to \infty$. Figure 5.5.3 shows the case of $z_\infty = 3.5$. In the portion of the curve between the second and third bifurcation points, observable necked solutions exist. For case of the closed ends there exists two bifurcation points and the form of the amplitude stretch diagram is given by Figure 5.5.4 which looks very similar to the one shown in Figure 3.5.1 for the Gent strain-energy function, including the two kinked
Figure 5.5.3: Dependence of $r_0$ against $r_\infty$ for the open Ogden tube with $z_\infty = 3.5$, dashed lines represent non-observable solutions.

Figure 5.5.4: Dependence of $r_0$ against $r_\infty$ for the closed Ogden tube. Only the solid lines in (a) correspond to localised solutions, and the non-localised solutions are not plotted in (b).

solutions. The labels $A – G$ on 5.5.4 correspond exactly to those discussed in Section 3.5 for the Gent strain-energy function, and the behaviour is the same. The behaviour for this Ogden strain-energy function is therefore dependent on the boundary conditions, changing whether there exists one, two or three critical points.
5.5.2 Stability

![Graph showing the plot of $\alpha$ for the open tube with the Ogden strain-energy function.](image)

Figure 5.5.5: Plot of $\alpha$ for the open tube with the Ogden strain-energy function.

Again, using the methods in Section 4.4, we show that there exists a positive eigenvalue of the spectral stability test and the bulged solution is therefore unstable, as given in Figure 5.5.5, which looks very similar to that in Figure 4.4.2 for the Gent strain-energy function with $J_m = 97.2$.

5.6 Fung Strain-Energy Function

We now consider the Fung strain-energy function described by (2.2.15) which includes a parameter $\Gamma > 0$ representing the amount of strain stiffening of the material. For a tube with open ends if $\Gamma > \Gamma_o = 0.166$ then no bifurcation points exist for any $z_\infty \geq 1$. For values of $\Gamma$ less than $\Gamma_o$, then a single bifurcation point exists for values of $z_\infty$ close to one, with very similar behaviour as the neo-Hookean strain-energy function. This is expected, as when $\Gamma = 0$ the neo-Hookean model is recovered. As $\Gamma$ increases towards $\Gamma_c$, the bifurcation point $r_{cr}$ moves towards infinity. When a closed end tube is considered, for small $\Gamma < \Gamma_c = 0.065$ there exists two bifurcation points and two kinked solutions
in exactly the same way as for the Gent and Ogden strain-energy functions, as shown in Figure 5.6.1.

5.7 Artery Modelling

Horgan and Saccomandi (2003) use the Gent strain-energy function to model arteries, giving values of $J_m$ in the range between 0.422 and 3.93 for healthy human arteries, corresponding to a maximum stretch ratio between 1.4 and 1.8, which is considerably smaller than that for rubber. These values of $J_m$ are less than $J_m^*$ for $z_\infty = 1.5$, implying there are no bifurcation points for values of $J_m$ appropriate to healthy artery walls. Similarly, Horgan and Saccomandi (2003) use the Fung strain-energy function to model arteries with values of $\Gamma$ between 1.067 and 5.547. For an open tube with $z_\infty = 1.5$, bifurcation points are only found for $\Gamma < 0.075$. Above this value of $\Gamma$ no bifurcation points exist, again showing that healthy artery walls are not subject to aneurysms when described using this formulation.

Various assumptions have been made here which may not be appropriate for arteries. For instance, the arterial wall comprises of three layers of tissue which have different
properties (Humphrey 2003). In addition, there are collagen fibres inside the arterial wall which, due to their relative stiffness compared to the elastic tissues, induce preferred directions of stretch into the artery walls (Holzapfel et al. 2000). Hence arteries are anisotropic in their deformations and require an anisotropic strain-energy function, as discussed in Horgan and Saccomandi (2003) for example. Similarly, Fung (1993) gives evidence of arterial walls being viscoelastic in their response to stress and strain. Another issue discussed by Alastrué et al. (2007) is that the arteries are not unstressed in the reference configuration, so the existence of this pre-stress may also be a factor in the existence of aneurysms. This problem is also considered by Gee et al. (2009) for finite-element simulations of arteries. In order to include such a pre-stress the wall thickness has to be assumed to be non-trivial, consequently the membrane assumption can not be used and another approach is required. Studies on thick-walled elastic tubes include Haughton and Ogden (1979b) and Zhu et al. (2008). In addition to these considerations, aneurysms also combine elements of growth and remodelling over time and therefore these factors need to be incorporated.

5.8 Conclusion

It has been shown that, for a range of strain-energy functions, bulging or necking solutions may exist. As expected and discussed in Section 4.2, the first bifurcation point, if it exists, has been shown to always be subcritical due to the stability of the undeformed tube, followed by a supercritical bifurcation point if a second exists. The comparison of several strain-energy functions shows how the existence and behaviour of non-uniform solutions varies depending on the choice of strain-energy function and the conditions imposed at infinity. In particular, when the tube has closed ends a closed curve may occur in the $r_0 - r_{\infty}$ plane, connecting two kinked solutions and two bifurcation points, with the interpretation given in Section 3.6.
Chapter 6

Inflation of a Spherical Membrane

6.1 Introduction

In this section we take the foundations of the boundary value problem studied for the isotropic, incompressible, homogenous elastic cylindrical membrane and apply them to the case of an initially spherical membrane. It is assumed that there is an internal pressure uniformly inflating the sphere, which is the controlling parameter for the deformation, and we again aim to find non-uniform solutions to this deformation.

In accordance with Chapter 3, we only consider deformations which are axisymmetric around the $Z$ axis, and we continue to use cylindrical polar coordinates instead of introducing spherical polar coordinates in order to facilitate this. This derivation follows the configuration introduced in Haughton (1980), although relabelling the 1 and 2 directions to coincide with those used in Chapter 3. In particular we are interested in axisymmetric modes superimposed upon the spherically symmetric solution. Physical applications of this mathematical deformation include the inflation of spherical rubber balloons, osmotic swelling of animal cells and imbibition (initial water uptake) of germinating plant seeds.

Gent (2005) and Kanner and Horgan (2007) discuss how the spherical inflation has a limit-point instability in the same way as the cylindrical tube, which occurs at the maximum of the pressure-volume diagram of the radially symmetric inflated sphere.
This bifurcation into a non-spherically symmetric solution has been considered by Shield (1972), Needleman (1976, 1977), Haughton and Ogden (1978b), Haughton (1980) and Chen and Healey (1991), all of which are relevant to the discussion here.

### 6.2 Configuration

The reference configuration is given by cylindrical coordinates, \((R, \Phi, Z)\), where the mid-plane coordinate \(R\) and the axial coordinate \(Z\) are both solely functions of the arc length \(S\). For a closed membrane with total arc length \(L\), we require \(R(0) = R(L) = Z'(0) = Z'(L) = 0\), and \(R(S) > 0, \ S \in (0, L)\), and the undeformed thickness is given as \(H(S)\).

For the case where the reference configuration is a spherical membrane with constant thickness, \(R(S) = \sin S, Z(S) = (1 - \cos S), H(S) = H\), where we have already non-dimensionalised with respect to the undeformed radius. Note that we take the arc length \(S\) to be increasing in the clockwise direction, instead of the usual counter-clockwise measure for angles, which coincides with the method used in Chapter 3 by ensuring that \(z'(S)\) will be positive. We also normalise with respect to the total arc length \(L\), and therefore \(S\) will just be an angle, though measured in a clockwise direction.

We also consider a deformed configuration given by cylindrical coordinates, \((r(S), \phi, z(S))\), where \(\Phi = \phi\) due to the assumed symmetry around the mid-plane. The requirements for closure of the deformed membrane after non-dimensionalisation are

\[
    r(0) = 0, \quad r(\pi) = 0, \quad z'(0) = 0, \quad z'(\pi) = 0. \tag{6.2.1}
\]

To characterise the deformation we use the three principal stretches, \(\lambda_1, \lambda_2, \lambda_3\), in the circumferential, meridional, and radial directions respectively. These are defined identically to those in Chapter 3, by

\[
    \lambda_1 = \frac{r}{R}, \quad \lambda_2 = \frac{ds}{dS} = \sqrt{r'^2 + z'^2}, \quad \lambda_3 = \frac{h}{H}. \tag{6.2.2}
\]
The material is assumed to be incompressible and hence \( \lambda_1 \lambda_2 \lambda_3 = 1 \). The only difference which should be noted is that \( R \) is now a function of \( S \) rather than a constant. Note that Haughton (1980), as do many other authors, reverses the labelling of the first two principal stretches, but we choose to be consistent with Chapter 3.

At the two poles of the membrane, both of the circumferential and meridional stretches become,

\[
\lim_{S \to 0} \lambda_1(S) = r'(0), \quad \lim_{S \to 0} \lambda_2(S) = r'(0),
\]

and similarly around \( S = \pi \), where it is noted that \( r'(0) \geq 0, r'(\pi) \leq 0 \) as the deformed radius must be increasing away from the poles.

The equilibrium equations, written in terms of the principal stretches, \( \lambda_i \) and the principal Cauchy stresses, \( \sigma_i \), are given by (Haughton 1980),

\[
(h r \sigma_2)' - h r' \sigma_1 = 0 \tag{6.2.4}
\]

\[
\frac{z' \sigma_1}{r \lambda_2} + \frac{\sigma_2 (r' z'' - r'' z')}{\lambda_2^3} - \frac{P^*}{\lambda_3} = 0, \tag{6.2.5}
\]

where a superimposed prime denotes differentiation with respect to \( S \), and \( P^* = P/H \).

We note that (6.2.4) and (6.2.5) reduce to the same equilibrium equations as for the cylindrical membrane discussed in Chapter 3, but for a non-constant \( R \). In addition, by the same method as described in Chapter 3, they may be derived using a perturbation expansion from the equilibrium equation (2.1.41) by expanding in the thickness coordinate through the membrane.

By introducing the principal curvatures of the membrane as

\[
\kappa_1 = \frac{z'}{r \lambda_2}, \quad \kappa_2 = \frac{(r' z'' - r'' z')}{\lambda_2^3} = \frac{1}{r'} \left( \frac{z'}{\lambda_2^2} \right)' = -\frac{1}{z'} \left( \frac{r'}{\lambda_2} \right)', \tag{6.2.6}
\]

equation (6.2.5) may be written as,

\[
\kappa_1 \sigma_1 + \kappa_2 \sigma_2 - P^* \lambda_1 \lambda_2 = 0. \tag{6.2.7}
\]
If the principal curvatures are rewritten in terms of the deformed arclength $s$, then
\[
\kappa_1 = \frac{1}{r} \frac{dz}{ds}, \quad \kappa_2 = \frac{1}{ds} \frac{d^2 z}{ds^2} = -\frac{1}{ds} \frac{d^2 r}{ds^2}. \tag{6.2.8}
\]

### 6.2.1 Integral of the Governing Equations

The two principal curvatures satisfy Codazzi’s equation (Pozrikidis 2003), which may be written as,
\[
r' \kappa_2 = (r \kappa_1)' \quad \text{or} \quad \frac{dr}{ds} \kappa_2 = \frac{d(r \kappa_1)}{ds}. \tag{6.2.9}
\]

Solving (6.2.4) for $\sigma_1$ and substituting into (6.2.5), we can then solve the resulting differential equation for $\sigma_2$ via the integrating factor method to find an integral of the governing equations. This integral corresponds to the fact that the force resultant in a cross section of the $z$-axis is a constant, in the same way as the cylindrical case in (3.3.12), and is given by,
\[
\frac{2p^2 H \sigma_2 \kappa_1}{\lambda_1 \lambda_2} - P r^2 = C_1, \tag{6.2.10}
\]

where $C_1$ is a constant to be determined. Considering how (6.2.10) behaves as $S \to 0$ or $\pi$, with (6.2.3), gives the result that $C_1 = 0$. It is important to note that the second integral which was found for the cylindrical case does not exist for the spherical case, as $R$ is no longer a constant. With $C_1 = 0$, (6.2.10) becomes,
\[
\sigma_2 \kappa_1 = \frac{P^* \lambda_1 \lambda_2}{2}, \tag{6.2.11}
\]

which we shall use along with one of the two original governing equations.

In the same way as for the cylindrical case and as explained in Chapter 2, we introduce a strain-energy function $W(\lambda_1, \lambda_2, \lambda_3)$, such that
\[
\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i = 1, 2, 3, \tag{6.2.12}
\]

where $p$ is the Lagrange multiplier associated with incompressibility. We then use the
membrane assumption of no stress through the thickness of the membrane, $\sigma_3 = 0$. This allows us to eliminate $p$ and leads to a new relation,

$$\sigma_i = \lambda_i \frac{\partial \hat{W}}{\partial \lambda_i}, \quad i = 1, 2,$$

(6.2.13)

in which we have introduced $\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1})$.

### 6.2.2 Spherically Symmetric Solution

Initially we wish to consider the case where the internal pressure $P$ inflates the undeformed sphere into another radially symmetric sphere with a larger radius. In this case, the two principal stretches must be equal, with $\lambda_1 = \lambda_2 = \lambda$. This is always a solution of (6.2.4) and (6.2.5) (or (6.2.10)) for any value of the pressure $P$. In this case we must have $r(S) = \lambda \sin S$ and $z'(S) = \lambda \sin S$, enabling us to find the following expression relating the internal pressure $P$ with the inflation stretch $\lambda$,

$$P = \frac{H\tilde{W}(\lambda)}{\lambda^2},$$

(6.2.14)

where $\tilde{W}(\lambda) = \hat{W}(\lambda, \lambda)$, and the subscript $\lambda$ represents differentiation with respect to $\lambda$. Equation (6.2.14) allows us to use the inflation stretch $\lambda$ instead of the pressure $P$, which again ensures that the bifurcation parameter is single valued and monotonic, in the same way as $r_\infty$ was used in Chapter 3. It should be noted from (6.2.14) that the pressure is of the order $H$, in the same way as the cylindrical case.

### 6.2.3 Asymmetric Solutions

We would now like to superimpose a small, non radially symmetric perturbation on top of the spherical solution, given by

$$r(S) = \lambda R(S) + \epsilon y(S), \quad z'(S) = \lambda R(S) + \epsilon w(S),$$

(6.2.15)
where \(0 < \epsilon \ll 1\) and \(y, w\) are order one functions. Using (6.2.14) along with (6.2.5) and (6.2.10) we have a second order differential system. To complete the system the boundary conditions \(y(0) = 0, y(\pi) = 0, w(0) = 0, w(\pi) = 0\) are required to ensure that the membrane remains closed throughout the deformation.

Substituting (6.2.15) into the first equilibrium equation (6.2.5) and the integral (6.2.10), and expanding both asymptotically as functions of \(\epsilon\), we recover (6.2.14) to zeroth order as expected, and at first order we find:

\[
\sin(2S)y'(S)(W_2 - \lambda W_{22}) - 2w(S)(W_2 \cos^2 S + \lambda W_{22} \sin^2 S)
\]
\[
+ y(S)(4W_2 - 2\lambda W_{12}) = 0 \quad (6.2.16)
\]

\[
\csc S y(S) \left( \lambda (W_{22} + W_{12}) - 2W_2 \right) + w(S) \left( (\csc S - 4\sin S) W_2 + \lambda \sin S (W_{22} + W_{12}) \right)
\]
\[
+ \cos S \left( w'(S)W_2 + \lambda y'(S)(W_{22} + W_{12}) \right) - \left( 4 \cos S y'(S) + \sin S y''(S) \right) W_2 = 0
\]
\[
\text{(6.2.17)}
\]

where \(W_2 = \frac{\partial W(\lambda, \lambda)}{\partial \lambda_2} = W_1, W_{12} = \frac{\partial^2 W(\lambda, \lambda)}{\partial \lambda_1 \partial \lambda_2}, W_{22} = \frac{\partial^2 W(\lambda, \lambda)}{\partial \lambda_2^2} = W_{11}\) and we have used (6.2.14) to express \(P\) as a function of \(\lambda\), as well as the symmetry of \(W\) with respect to the two stretches. We introduce the shortened notation \(\beta_0 = \lambda W_2, \beta_1 = \lambda^2 W_{22}, \alpha_1 = \lambda^2 W_{12}\), which was used in Fu and Il’ichev (2010).

Equation (6.2.16) can be used to find \(w(S)\) in terms of \(y(S)\). It should be noted that it is the integrated equation (6.2.10) which enables us to write \(w(S)\) in this way and the value of such a conservation law is therefore stressed. Using (6.2.16) we may therefore write,

\[
w(S) = \frac{\sin(2S)y'(S)(\beta_0 - \beta_1) + y(S)(4\beta_0 - 2\alpha_1)}{2(\beta_0 \cos^2 S + \beta_1 \sin^2 S)}.
\]
\[
\text{(6.2.18)}
\]

Using this relation, along with the substitution \(\mu = \cos S\), (6.2.17) may be used to derive a single second order differential equation for \(y(\mu)\),

\[
(1 - \mu^2)(\mu^2(\beta_0 - \beta_1) + \beta_1) y''(\mu) - 2\mu \beta_0 y'(\mu) + \chi(\lambda, \mu) y(\mu) = 0.
\]
\[
\text{(6.2.19)}
\]
where

\[
\chi(\lambda, \mu) = \frac{1}{\beta_0 \beta_1} \left( 2\mu^2 \beta_0^3 - \frac{(6 - 8\mu^2 + \mu^4)}{\mu^2 - 1} \beta_0^3 \beta_1 - 2\mu^2 \beta_0 \beta_1^2 + (1 + \mu^2) \beta_1^3 \\
- \alpha_1 \beta_0 (3\mu^2 \beta_0 + (5 - 3\mu^2) \beta_1) + \alpha_1^2 (\mu^2 \beta_0 - (-1 + \mu^2) \beta_1) \right) \tag{6.2.20}
\]

Equation (6.2.19) is a linear second order differential equation with variable coefficients, which is singular at the three points \( \mu = \pm 1, \infty \). The boundary conditions become \( y(-1) = y(1) = 0 \), and are to be applied at the singular endpoints of the differential equation. Writing \( y(\mu) = \sqrt{1 - \mu^2} Q(\mu) \), (6.2.19) becomes:

\[
(\mu^2 \beta_0 - (\mu^2 - 1) \beta_1) Q''(\mu) - 2\mu \beta_1 \left( 1 - \frac{\beta_0 (\mu^2 + 1)}{\beta_1 (\mu^2 - 1)} \right) Q'(\mu) \\
- \frac{(\alpha_1 - 2\beta_0 + \beta_1) (-\mu^2 \beta_0^2 - 3\beta_0 \beta_1 + (\mu^2 - 1) \beta_1^2 + \alpha_1 (\mu^2 (\beta_0 - \beta_1) + \beta_1))}{(\mu^2 - 1) \beta_0 \beta_1} Q(\mu) = 0, \tag{6.2.21}
\]

The exact solution to either of the differential equations (6.2.19) and (6.2.21) is not available in closed form, although it may be solved via series solutions around the singularities or numerical integration, given a strain-energy function.

### 6.3 Bifurcation Condition

Haughton and Ogden (1978b) use spherical coordinates throughout their paper, and consider axisymmetric modes in which the stretches in the \( e_\rho \) and \( e_\theta \) directions are expanded as Legendre polynomials and derivatives of Legendre polynomials. The corresponding modal perturbation \( Q(\mu) \) in our notation is \( Q(\mu) = A P_n(\mu) + B \mu P_n'(\mu) \). Inserting this ansatz into (6.2.19) and utilising the definition of the Legendre polynomials to simplify the higher order derivatives as discussed in Section 2.6, we find the following equation:

\[
(AG_{11}(\lambda, \mu) + BG_{12}(\lambda, \mu)) P_n(\mu) + (AG_{21}(\lambda, \mu) + BG_{22}(\lambda, \mu)) P_n'(\mu) = 0, \tag{6.3.1}
\]
where $G_{11}, G_{12}, G_{21}$ and $G_{22}$ are functions of the elastic modulii and $\mu$, given by

\[
G_{11} = -(-1 + \mu^2)(-(\alpha_1 - 2\beta_0 + \beta_1)((\alpha_1 - 3\beta_0 - \beta_1)\beta_1 \\
+ (\alpha_1 - \beta_0 - \beta_1)(\beta_0 - \beta_1)\mu^2) + \beta_0\beta_1(\beta_1 + \beta_0\mu^2 - \beta_1\mu^2) n(n + 1)),
\]

(6.3.2a)

\[
G_{12} = -2n(n + 1)(\mu^2 - 1)\beta_0\beta_1,
\]

(6.3.2b)

\[
G_{21} = -2\mu(\mu^2 - 1)\beta_0^2\beta_1,
\]

(6.3.2c)

\[
G_{22} = \mu G_{11} - 2\mu(\mu^2 - 1)\beta_0\beta_1(\beta_0 - 2\beta_1).
\]

(6.3.2d)

As the Legendre polynomials are linearly independent of their derivatives, we require that $AG_{11}(\lambda, \mu) + BG_{12}(\lambda, \mu) = 0$ and $AG_{21}(\lambda, \mu) + BG_{22}(\lambda, \mu) = 0$ must both be true in order for (6.3.1) to be satisfied for all $\mu$. Rewriting (6.3.1) as a matrix equation for the two constants $A$ and $B$, we have

\[
\begin{pmatrix}
G_{11}(\lambda, \mu) & G_{12}(\lambda, \mu) \\
G_{21}(\lambda, \mu) & G_{22}(\lambda, \mu)
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

(6.3.3)

In order for a non-trivial set of $A, B$ to exist for any given $n$ we require the determinant of the matrix $G = (G_{ij})$ to vanish, which gives the following equation,

\[
\det(M) = (\beta_1^2 - \alpha_1^2 + 3\alpha_1\beta_0 - 2\beta_0^2 - \beta_0\beta_1 + n(n + 1)\beta_0\beta_1) \zeta(\lambda, \mu) = 0,
\]

(6.3.4)

where $\zeta(\lambda, \mu)$ is an extremely complicated function of both the elastic modulii and $\mu$. The first bracket of (6.3.4) does not depend on $\mu$, and setting it to be zero therefore gives a sufficient condition for a mode of order $n$ to exist, depending only on $\lambda$ and the strain-energy function $\hat{W}$, and not on $\mu$. Hence the condition for bifurcation from the spherical solution into the $n$th mode is given by:

\[
-\alpha_1^2 + 3\alpha_1\beta_0 - 2\beta_0^2 + (n^2 + n - 1)\beta_0\beta_1 + \beta_1^2 = 0,
\]

(6.3.5)
or equivalently,

\[ \lambda W_2 \left( (n^2 + n - 1)W_{22} + 3W_{12} \right) + \lambda^2 (W_{22}^2 - W_{12}^2) - 2W_{22}^2 = 0. \]  

(6.3.6)

The bifurcation condition given by (6.3.5) is identical to that given by Haughton and Ogden (1978b) in their equation (52), utilising the connections \( \Sigma_1 = \beta_0, \Sigma_2 = \beta_1, \Sigma_3 = \beta_0 + \beta_1 - \alpha_1 \), where \( \Sigma_1, \Sigma_2, \Sigma_3 \) are those defined in Haughton and Ogden (1978b). It was also found by Chen and Healey (1991) to be the bifurcation condition, using a different method to that used here.

It is important to note that the condition (6.3.5) only predicts the bifurcation into a mode of order \( n \) from the spherical solution, whereas it is possible that the \( n \)th mode could occur earlier as a bifurcation from the \( (n - 1) \)th mode. This possibility is not considered here or in Haughton and Ogden (1978b). The value of \( \lambda \) such that the relation given in (6.3.5) is satisfied for a given \( n \) is the inflation ratio \( \lambda_n \) such that the \( n \)th order mode exists. Haughton and Ogden (1978b) show that if there are a finite number of modes possible for a specific strain-energy function, then the \( \lambda_n \) appear in pairs such that \( \lambda_1 < \lambda_2 < \ldots < \lambda_N < \lambda^*_N < \ldots < \lambda^*_2 < \lambda^*_1 \), for all \( n \geq N \). In this expression \( \lambda^*_n \) is the value of \( \lambda \) where the reverse bifurcation occurs and the \( n \)th order mode may no longer exist.

The Varga strain-energy function given by (2.2.10) has a single \( \lambda_n \) which satisfies (6.3.5) for all \( n \in \mathbb{N} \), and hence modes of all orders exist for a material described by the Varga strain-energy function. This behaviour for the Varga strain-energy function is incorrectly disregarded by Haughton and Ogden.

Figure 6.3.1 plots \( r(S) \) for the Varga strain-energy function, showing the first twenty bifurcated modes. It should be noted that the amplitude of the deformation is not given at first order in \( \epsilon \) and hence has been arbitrarily scaled in figure 6.3.1. To calculate the amplitude it is necessary to go to second order in \( \epsilon \), which will be necessary to do numerically, and has not been done here.

The neo-Hookean and Gent strain-energy functions do not satisfy (6.3.5) for any integer \( n \), and hence there are no non-trivial axisymmetric modes for the inflated sphere,
Figure 6.3.1: $r(S)$ for the first 20 mode numbers, perturbation amplitudes normalised for any $J_m > 0$ for the Gent strain-energy function. The Ogden strain-energy function experiences a mode 1 bifurcation and reversal at $\lambda = 1.77833$ and $\lambda = 2.51371$, but no others, as also shown by Haughton and Ogden (1978b). Therefore for the values of $\lambda \in [1.77833, 2.51371]$ a non-uniform solution is possible, as has been shown by Haughton (1980).

### 6.4 Numerical Solutions

We now solve the governing equations numerically. To do this, we use the following two equilibrium equations (6.2.5) which combined are third order in $r$ and $z$,

\[
(h r \sigma_2)' - h r' \sigma_1 = 0, \tag{6.4.1}
\]

\[
\frac{z' \sigma_1}{r \lambda_2} + \sigma_2 \frac{(r' z'' - r'' z')}{\lambda_2^2} - \frac{P^*}{\lambda_3} = 0,
\]

along with the boundary conditions $r(0) = 0, \ r(\pi) = 0, \ z'(0) = 0$. We initially try to solve (6.4.1) using a shooting method, ‘guessing’ a value of $r'(0)$ to give three conditions
at zero, and iterate on the value of $r'(0)$ until the third boundary condition is satisfied.

However, as the equations are singular at the point where we are trying to apply the boundary conditions, it is difficult to implement a numerical scheme starting at $S = 0$. We therefore choose to apply the boundary conditions at a point close to zero, given by $S = \delta$, where $\delta \ll 1$. This is viable as the singularity is only an artifact of the cylindrical coordinate system which we are applying, and is not essential to the problem under consideration. Expanding the functions around $S = 0$ we find,

\begin{align}
  r(\delta) &= r(0) + r'(0)\delta + \mathcal{O}(\delta^2) \approx r'(0)\delta, \\
  r'(\delta) &= r'(0) + r''(0)\delta + \mathcal{O}(\delta^2) \approx r'(0), \\
  z'(\delta) &= z'(0) + z''(0)\delta + \mathcal{O}(\delta^2) \approx z''(0)\delta,
\end{align}

where we have neglected terms of order $\delta^2$ and higher and used the boundary conditions specified above. We note that $r''(0) = 0$ due to the closure requirement at $S = 0$. Evaluating (6.4.1) as $S \to 0$, we find $z''(0) = \frac{\mathcal{P}^* r'(0)^3}{2 \hat{W}_2(r'(0), r''(0))}$, where we have used the symmetry of the two arguments of $\hat{W}$. Therefore we may now integrate our differential equations with these boundary conditions, using the shooting method described previously, with just $r'(0)$ as the only parameter to shoot on. Figure 6.4.1 shows an example of the calculated profile of $r(S)$ for the Varga strain-energy function after the mode one bifurcation, while Figure 6.4.2 shows the mode two bifurcation. In Figure 6.4.1, the value of $\lambda$ at that point is equal to $\lambda_1$, where the onset of the mode one bifurcation occurs. This means that the post-bifurcation path must have turning points the curve must track back on itself, as at the bifurcation point itself the non-uniform solution must not exist. Similarly with Figure 6.4.2, which is at $\lambda = 1.8874 < \lambda_3 = 1.9767$. 

126
Figure 6.4.1: Numerical calculation of mode one bifurcation for the Varga strain-energy function at $\lambda = \lambda_1 = 1.5874$, dashed lines represent undeformed shape.

Figure 6.4.2: Numerical calculation of the mode two bifurcation for the Varga strain-energy function at $\lambda = 1.8874$, dashed lines represent undeformed shape.
6.5 Biological Cells

Skalak et al. (1973) consider the deformation of red blood cells which have been ‘sphered’ by the addition of a hypotonic solution, so that they change from their natural, dumb-bell shape to a sphere. They report the results of experimental testing on red blood cells, which give (linear) elastic moduli ranging from $7.2 \times 10^5 - 3.0 \times 10^8$ dyn/cm$^2$ in different methods of testing. Skalak et al. conclude that the reason behind the wide variation in measured values is that the membrane has a high intrinsic resistance to deformations which change the surface area of the cell, and hence nonlinear elasticity is required to model the membrane accurately. This area-conserving property has been used as a formal constraint on the elastic deformations by later authors, such as Liu et al. (2006).

Skalak et al. (1973) introduce a two-dimensional membrane strain-energy function, where the theory does not explicitly involve the thickness, given by

$$ W = \frac{B}{4} (\frac{1}{2} K_1^2 + K_1 - K_2) + \frac{C}{8} K_2^2, \quad (6.5.1) $$

where $B, C$ are constants, $K_1 = \lambda_1 + \lambda_2 - 2$ is an invariant measuring the shearing component and $K_2 = \lambda_1^2 \lambda_2^2 - 1$ is an invariant which becomes zero when the areal strain is zero, regardless of the magnitude of the stretches. Skalak et al. (1973) suggest that constant $C$ is at least three orders of magnitude larger than $B$ in (6.5.1) in order to qualitatively model experimental data with the high resistance to area altering deformations.

### 6.5.1 Constant Area Constraint

If we impose this area conservation of cells as a constraint on the material properties in the sense of finite elasticity in addition to the standard incompressibility constraint, we require that

$$ \lambda_1 \lambda_2 = 1, \quad \lambda_3 = 1. \quad (6.5.2) $$
The addition of this second constraint requires us to introduce another Lagrange multiplier as well as the one associated with incompressibility. Starting with the second equilibrium equation (6.2.5) and the conservation law (6.2.10), we introduce the constrained stresses appropriately as

\[ \sigma_i = \lambda_i \frac{\partial \hat{W}(\lambda_1, \lambda_2)}{\partial \lambda_i} - q, \quad i = 1, 2 \]  

(6.5.3)

where \( q \) is the Lagrange multiplier associated with the constrained area and we set \( \lambda_1 \lambda_2 = 1 \) after differentiating.

We now perform the same analysis as for the proceeding case, looking for deformed configurations which are not radially symmetric. It is clear that the spherical expansion state which is found in the unconstrained case can not exist except for \( \lambda = \lambda_1 = \lambda_2 = 1 \), so we expand from the undeformed configuration instead of the spherically inflated configuration. Substituting (6.5.3) into (6.2.5) and (6.2.10) and using (6.5.2), we find

\[ \frac{2rH(W_2 - q)z'}{\lambda_1 \lambda_2^2} - P^* r^2 = 0, \]  

(6.5.4)

\[ \frac{z'(W_1 - q)}{r \lambda_2} + (W_2 - q) \left( \frac{r' z'' - r'' z'}{\lambda_2^2} \right) - P^* = 0, \]  

(6.5.5)

where \( W_i = \frac{\partial \hat{W}}{\partial \lambda_i} |_{\lambda_2 = \lambda_1^{-1}} \).

After eliminating \( \lambda_2 \) and \( \lambda_3 \) by using (6.5.2), equation (6.5.4) can be rearranged to find \( q \) is given by,

\[ q = \frac{\lambda_1^3 P^* r}{2 z'} - \lambda_1 W_1. \]  

(6.5.6)

Hence, (6.5.5) becomes, using (6.5.6),

\[ -\lambda_1^2 P^* + \frac{2z'}{r}(W_1 - W_2) + P^* r \left( -r'' + \frac{r' z''}{z'} \right) = 0. \]  

(6.5.7)

Expanding \( r \) and \( z' \) around the undeformed state in the same manner as for the unconstrained case, but with \( \lambda = 1, r(S) = \sin S + \epsilon y(S), z'(S) = \sin S + \epsilon w(S) \), we find (6.5.7)
becomes at first order in $\epsilon$,

$$(-1 + 4\gamma) \csc S y(S) + \cos S (-\cot Sw(S) + w'(S) + y'(S)) - \sin Sy''(S) = 0, \quad (6.5.8)$$

where $\gamma = 2G/P^*$ and $2G = W_{11}(1, 1) - W_{12}(1, 1)$ is the shear modulus of the material. We note that in this case, in contrast to the unconstrained case considered in the previous section, there is no equation which determines $P$ at the zeroth order which would be a counterpart of (6.2.14).

The second differential equation which we require can be derived from considering the first order in $\epsilon$ of $\lambda_2 = \sqrt{r'^2 + z'^2} = \lambda_1^{-1}$ and is given by

$$\sin Sw(S) + \csc Sy(S) + \cos Sy'(S) = 0,$$  

$(6.5.9)$

which may be solved for $w(S)$. Substituting (6.5.9) into (6.5.8) we find a single differential equation for $y(S)$, given by

$$(-(1 + 2\gamma) + 2(\gamma - 1) \cos(2S)) y(S) + \sin S(-\cos Sy'(S) + \sin Sy''(S)) = 0. \quad (6.5.10)$$

Using the transformation $\mu = \cos S$ the differential equation can be written as an associated Legendre function in the following way,

$$(1 - \mu^2)Y''(\mu) - 2\mu Y'(\mu) + \left(4\gamma - \frac{4\mu^2}{1 - \mu^2}\right) Y(\mu) = 0,$$  

$(6.5.11)$

where $Y(\mu) = \sin Sy(\cos S)$, and the relation of (6.5.11) to the associated Legendre equation is given by $l(l + 1) = 4\gamma, m^2 = 4$. Therefore the solutions of (6.5.11) are given by

$$y(S) = A \sin S P_{\frac{1}{2}\sqrt{-1 + \sqrt{1 + 16\gamma}}}'(\cos S) + B \sin S Q_{\frac{1}{2}\sqrt{-1 + \sqrt{1 + 16\gamma}}}'(\cos S). \quad (6.5.12)$$

The required boundary conditions for closure of the membrane are $y(0) = y(\pi) = 0$. The first of these is always satisfied by (6.5.12) provided $B = 0$, for any $\gamma$. In order to satisfy the second boundary condition we have the solvability condition that $\frac{1}{2}(-1 +
\[ \sqrt{17 + 16\gamma} = k, \text{ where } k \in \mathbb{Z}, \] to reduce the solution to a Legendre polynomial, else the solution is infinite at \( S = \pi \). Rearranging this equation for \( \gamma \) we find

\[ \gamma = \frac{1}{4}(k^2 + k - 4), \quad k \in \mathbb{Z}. \quad (6.5.13) \]

However, using the definition of \( \gamma \), we can rewrite (6.5.13) for \( P \),

\[ P = \frac{-4HG}{k^2 + k - 4}. \quad (6.5.14) \]

In the derivation of the governing equations it is a requirement that the pressure \( P \) is positive to ensure that the membrane is in tension and not compression. When one of the principal stresses becomes negative then the membrane will form wrinkles as discussed in Section 2.3.4, and a relaxed strain-energy function is required to describe this occurrence satisfactorily (Pipkin 1986). Hence, as the shear modulus and the thickness \( H \) are all positive, we require that the integer \( k \) satisfies \( k^2 + k - 4 < 0 \) to give a positive pressure from (6.5.14). This is only satisfied if \( k = -2, -1, 0, 1 \). So admissible solutions of the differential equation are given by

\[ y(S) = A \sin SP^n_k(\cos S), \quad (6.5.15) \]

where \( k = -2, -1, 0, 1 \). However, from the properties of the Legendre polynomials \( P^n_m(x) \), if \( -(m + 1) < n < m \) then the polynomial is everywhere zero. Therefore we conclude that there can be no non-trivial solutions to the inflation of an incompressible spherical membrane when the area is rigidly constrained in this manner.

### 6.6 Conclusion

In this chapter we have rederived the bifurcation condition given in Haughton and Ogden (1979a) and Chen and Healey (1991), utilising the integral arising from the conservation law.
When \( n = 0 \), the bifurcation condition (6.3.5) becomes \( \frac{dP}{d\lambda} = 0 \), and this must occur before the \( n = 1 \) mode, as discussed in Haughton and Ogden (1979a). In Chapter 3 the bifurcation condition which was found for the cylindrical tube corresponded to the zeroth mode in the linear bifurcation analysis, leading to a weakly nonlinear bifurcation analysis. This leads us to speculate that there may be a similar localised solution in the spherical case considered here, although we have not investigated this possibility and it remains as future work.

With respect to the constraint of area preservation on the surface of the membrane, we found that if the constraint is rigidly applied then no non-trivial solutions may exist. However, if the inflation of a shell was considered, the inclusion of the bending moments may allow the existence of non-trivial solutions, but this has not been considered here.
Chapter 7

Inflation of a General Axisymmetric Shell

7.1 Introduction

In this chapter we consider the axisymmetric inflation of a thin axishell via an internal pressure, rather than the cylindrical and spherical membranes considered in the previous chapters. Referring back to the definitions introduced in Section 2.3, we recall that an axishell is a thin axisymmetric shell whose response to deformation includes a contribution from bending in addition to the stretching response found in membranes. Initially we specialise the governing equations for a static deformation of a two-dimensional elastic surface deformed via an internal pressure, including the bending moments. We follow this by showing that, for an arbitrary axishell, a conservation law integral exists generalising the first such integral found in Chapter 3 for the cylindrical membrane and in Chapter 6 for the spherical membrane. After this we compare with the governing equations given in Chapter 6 for the spherical membrane, and the literature on the inflation of initially spherical axishells, showing that the governing equations derived here for the two-dimensional theory are identical with those for the three-dimensional theory, with an appropriate change in the definition of the strain-energy function. We conclude with
a discussion of biological fluid-like lipid bilayers, which are commonly modelled in the literature as elastic surfaces which only have contributions to the energy from bending, not stretching.

We shall use the theory of Steigmann and Ogden (1999) to describe the deformation of an axisymmetric two-dimensional elastic surface embedded into three-dimensional space, subject to axisymmetric deformations. In particular, we allow the energy of the material to depend on both the stretching of the elastic elements and the curvatures developing within the shell. This is an extension of the theories discussed previously, and is a thin-shell theory as opposed to a membrane theory. In addition, it extends the theory of nonlinear elasticity from Cauchy elasticity as the response of the material no longer depends only on the value of the deformation gradient \( F \), but also on the curvatures in the deformed configuration. These curvature terms are functions of the gradient of the deformation gradient, \( \nabla F \), but the energy is not allowed to depend generally on \( \nabla F \).

The notation and derivations here correspond to those used in Steigmann and Ogden (1999), who present a theory of elastic surface-substrate interactions wherein they consider the deformation of a thin coat of substrate bonded to a material, which may have different material properties. We shall use this theory without the solid body, to just describe the deformation of the thin elastic sheet. A variational treatment of this derivation may be seen in Baesu et al. (2004), although we work directly from Steigmann and Ogden (1999). We also note that Steigmann (2001) reiterates much of this theory, with additional details and explanations.

### 7.2 Deformation of Axisymmetric Shells

#### 7.2.1 Derivation

We consider a position function \( Y \) on the shell which is parameterised locally by two coordinates \( \theta^\alpha \), where \( \alpha \) and all future Greek sub- and superscripts take the values 1 or 2.
Thus $Y$ is the two-dimensional equivalent of $X$ in Chapter 2. In the same way as noted in Chapter 2, capital and small letters shall be used to denote the same quantities in the reference and current configurations respectively wherever possible.

We define the reference configuration with respect to cylindrical coordinates, $(R, \Theta, Z)$, and it is parameterised by

$$\theta^1 = A\Theta, \theta^2 = S,$$  \hspace{1cm} (7.2.1)

where $\Theta$ is the angle around the axis of symmetry, $S$ is the arc length around the sphere and $A$ is a lengthscale. The basis of three dimensional space which we will use is $(e_r, e_\theta, e_z)$ as unit vectors in the $R, \Theta$ and $Z$ directions respectively. We remain in cylindrical coordinates for convenience due to the axisymmetry, and choose to use the same basis for both the reference and current configurations. Therefore $Y$ is given by,

$$Y(\theta^1, \theta^2) = R(S)e_r(\Theta) + Z(S)e_z. \hspace{1cm} (7.2.2)$$

The two reference configurations which we are particularly interested in here are the cylindrical case, corresponding to the case considered in Chapter 3, and the spherical case. For the cylindrical case,

$$R(S) = A, \quad Z(S) = S,$$  \hspace{1cm} (7.2.3)

whereas for the spherical case,

$$R(S) = A\sin(S/A), \quad Z(S) = A(1 - \cos(S/A)),$$  \hspace{1cm} (7.2.4)

where we have chosen to define the coordinate system such that $Z'(S)$ is positive.

The shell deforms to a current configuration $y$ described by cylindrical coordinates, $(r, \Theta, z)$, where we have imposed the condition of axisymmetry. Therefore $y$ is given by,

$$y(\theta^1, \theta^2) = r(S)e_r(\Theta) + z(S)e_z. \hspace{1cm} (7.2.5)$$
These coordinates induce tangent vectors to the surface of the shell in the reference configuration,

\[ G_\alpha = Y_{,\alpha}, \quad g_\alpha = y_{,\alpha} \]  

(7.2.6)

where \(,\alpha\) represents differentiation with respect to \(\theta^\alpha\), and therefore \(,1 = \frac{1}{A} \frac{\partial}{\partial \theta^\alpha}, 2 = \frac{\partial}{\partial S}\). The tangent vectors in the reference configuration are then given by,

\[ G_1 = Y_{,1} = \frac{R(S)}{A} e_\theta, \quad G_2 = Y_{,2} = R'(S)e_r + Z'(S)e_z, \]  

(7.2.7)

where hereon a prime represents differentiation with respect to \(S\). We find the components of the metric tensors \((G_{\alpha\beta})\), \((G^{\alpha\beta})\) as well as the dual vectors \(G^{\alpha}\), using the following formulae,

\[ G^{11} = \frac{R^2}{A^2}, \quad G^{12} = G^{21} = 0, \quad G^{22} = R'^2 + Z'^2; \]

(7.2.8)

\[ G_1 = \frac{A}{R} e_\theta, \quad G_2 = \frac{R'}{\sqrt{R'^2 + Z'^2}} e_r + \frac{Z'}{\sqrt{R'^2 + Z'^2}} e_z; \]

(7.2.10)

\[ G = \text{det}(G_{\alpha\beta}) = \frac{R^2 (R'^2 + Z'^2)}{A^2}, \]

(7.2.11)

and thus the normal in the reference configuration is given by

\[ N = \frac{1}{\sqrt{G}} (G_1 \wedge G_2) = \frac{A}{R \sqrt{R'^2 + Z'^2}} \left( \frac{RZ'}{A} e_r - \frac{RR'}{A} e_z \right) \]

\[ = \frac{Z' e_r - R' e_z}{\sqrt{R'^2 + Z'^2}}. \]

(7.2.12)

The derivatives of the tangent vectors (7.2.7) are given by

\[ G_{1,1} = -\frac{R}{A^2} e_r, \quad G_{1,2} = G_{2,1} = \frac{R'}{A} e_\theta, \quad G_{2,2} = -R'' e_r + Z'' e_z. \]

(7.2.13)

The reference normal curvatures associated with the embedding of the surface into the
three-dimensional space, $B_{\alpha\beta} = N \cdot G_{\alpha,\beta}$, are given by

$$B_{11} = -\frac{RZ'}{A^2 \sqrt{R'^2 + Z'^2}}, \quad B_{12} = B_{21} = 0, \quad B_{22} = \frac{(R''Z' - R'Z'')}{\sqrt{R'^2 + Z'^2}}, \quad (7.2.14)$$

Similarly, in the deformed configuration we may define all the equivalent quantities, starting with the tangent vectors,

$$g_1 = y_1 = \frac{r(S)}{A} e_\theta, \quad g_2 = y_2 = r'(S)e_r + z'(S)e_z, \quad (7.2.15)$$

We find the components of the metric tensors $(g_{\alpha\beta})$, $(g^{\alpha\beta})$ as well as the dual vectors $g^\alpha$, all defined in the same way as for the undeformed configuration, as

$$g_{11} = \frac{r^2}{A^2}, \quad g_{12} = g_{21} = 0, \quad g_{22} = (r'^2 + z'^2), \quad (7.2.16)$$

$$g^{11} = \frac{A^2}{r^2}, \quad g^{12} = g^{21} = 0, \quad g^{22} = \frac{1}{(r'^2 + z'^2)}, \quad (7.2.17)$$

$$g^1 = \frac{A}{r} e_\theta, \quad g^2 = \frac{r'e_r + z'e_z}{(r'^2 + z'^2)}; \quad (7.2.18)$$

$$g = \det(g_{\alpha\beta}) = \frac{r^2(r'^2 + z'^2)}{A^2}, \quad J = g/G, \quad (7.2.19)$$

and thus the normal in the current configuration is,

$$n = \frac{1}{\sqrt{g}} (g_1 \wedge g_2) = \frac{1}{r \sqrt{r'^2 + z'^2}} (\frac{r'z'}{A} e_r - \frac{rr'}{A} e_z),$$

$$= \frac{z'e_r - r'e_z}{\sqrt{r'^2 + z'^2}}. \quad (7.2.20)$$

The derivatives of the tangent vectors are given by

$$g_{1,1} = -\frac{r}{A^2} e_r, \quad g_{1,2} = g_{2,1} = \frac{r'}{A} e_\theta, \quad g_{2,2} = r''e_r + z''e_z. \quad (7.2.21)$$

And the components of the embedded deformed curvature tensor are then given using $b_{\alpha\beta} = n \cdot g_{\alpha,\beta\nu}$

$$b_{11} = -\frac{rz'}{A^2 \sqrt{r'^2 + z'^2}}, \quad b_{12} = b_{21} = 0, \quad b_{22} = \frac{(r''z' - r'z'')}{\sqrt{r'^2 + z'^2}}. \quad (7.2.22)$$
The Christoffel symbols on the deformed surface are given by $\Gamma^\gamma_{\alpha\beta} = g^\gamma g_{\alpha,\beta}$ and become,

\[ \Gamma^1_{11} = 0, \quad \Gamma^2_{11} = \frac{-rr'}{A(r'^2 + z'^2)}, \quad (7.2.23) \]
\[ \Gamma^1_{12} = \frac{r'}{r}, \quad \Gamma^2_{12} = 0, \quad (7.2.24) \]
\[ \Gamma^1_{21} = \frac{r'}{r}, \quad \Gamma^2_{21} = 0, \quad (7.2.25) \]
\[ \Gamma^1_{22} = 0, \quad \Gamma^2_{22} = \frac{r'r'' + z'z''}{r'^2 + z'^2}. \quad (7.2.26) \]

Using the chain rule we may write,

\[ g_\alpha = FG_\alpha, \quad (7.2.27) \]

where $F$ is the surface deformation gradient, $F = g_\beta \otimes G^\beta$. Therefore we have the Cauchy Green surface deformation tensor $C$, given by

\[ C = F^T F = g_{\alpha\beta} G^\alpha \otimes G^\beta. \quad (7.2.28) \]

The relative curvature $\kappa = -b$, given with respect to the undeformed dual metric tensors is,

\[ \kappa = \kappa_{\alpha\beta} G^\alpha \otimes G^\beta. \quad (7.2.29) \]

### 7.3 Equilibrium Equations

We define $\dot{y}$ as an incremental variation in $y$, and henceforth a superimposed dot denotes an incremental variation in the relevant quantity. Assuming that the energy of the surface is given by a strain-energy function $U$, which is a function of both the deformation gradient $C$, and the curvature $\kappa$, Steigmann and Ogden (1999) show that

\[ \dot{U} = T^\alpha \cdot \dot{y}_\alpha + M^{\alpha\beta} \cdot \dot{y}_{\alpha\beta}, \quad (7.3.1) \]
where

\[
T^\alpha \equiv J(\sigma^{\alpha\beta}g_\beta + m^{\lambda\beta}\Gamma_{\lambda\beta}^\alpha n), \quad M^{\alpha\beta} \equiv -Jm^{\alpha\beta},
\]

\[
J\sigma^{\alpha\beta} = \frac{\partial U}{\partial g_{\alpha\beta}} + \frac{\partial U}{\partial g_{\beta\alpha}}, \quad Jm^{\alpha\beta} = \frac{\partial U}{\partial \kappa_{\alpha\beta}}.
\]

In (7.3.2), \(\sigma^{\alpha\beta}\) and \(m^{\alpha\beta}\) are the contravariant components of the stress and moment tensors in the deformed configuration respectively. These components may be related to the covariant components which are generally used by, \(\sigma_{\alpha\beta} = g_{\alpha\beta}\sigma^{\alpha\beta}\) and a similar formula holds for the moments (Epstein and Johnston 2001).

Steigmann and Ogden (1999) then derive how the increment of the energy of the static elastic surface may be written as,

\[
\oint_P \dot{U} \, dA = \oint_P G^{-1/2}[(G^{1/2}T^\alpha)_{,\alpha} - (G^{1/2}M^{\beta\alpha})_{,\beta\alpha}] \cdot \dot{y} \, dA,
\]

where a superimposed dot represents the increment. This leads to the following equilibrium equation,

\[
G^{-1/2}[(G^{1/2}T^\alpha)_{,\alpha} - (G^{1/2}M^{\beta\alpha})_{,\beta\alpha}] = S^T N,
\]

where \(S\) is the three-dimensional nominal stress tensor associated with the forces acting on the elastic surface as defined in (2.1.42). We note the connections between these three-dimensional, \(S^T = J\sigma F^{-T}\), and \(N = F^T n\).

This enables us to write the external forces from (7.3.5) as,

\[
S^T N = J\sigma F^{-T} F^T n = J\sigma n.
\]

If the external forces are given by a constant hydrostatic pressure \(P\) in the deformed configuration, then \(\sigma n = -P n\), and therefore using \(J = \sqrt{g}/\sqrt{G}\) we have,

\[
S^T N = -JP \frac{z' e_r}{\sqrt{r'^2 + z'^2}} + JP \frac{r' e_z}{\sqrt{r'^2 + z'^2}} = -P \frac{r' z'}{\sqrt{G}} e_r + P \frac{r' e_z}{\sqrt{G}}.
\]
### 7.3.1 Evaluating the Equilibrium Equations

We now non-dimensionalise with respect to $A$, which we do implicitly by setting $A = 1$ and considering all lengths, including $S$, to be rescaled with respect to $A$. Evaluating the equilibrium equation (7.3.5) gives,

$$
\frac{(J\sigma^{22} r' \sqrt{G})'}{\sqrt{G}} - \frac{J\sigma^{11} r}{A} - \frac{1}{\sqrt{G}} \left( \frac{Jm^{11} rr' z' \sqrt{G}}{A(r^2 + z'^2)^{3/2}} \right)' \\
- \frac{Jm^{11} z'}{(r^2 + z'^2)^{1/2}} + \frac{1}{\sqrt{G}} \left( \frac{(Jm^{22} z' \sqrt{G})'}{(r^2 + z'^2)^{1/2}} \right)' + P \frac{r z'}{\sqrt{G}} = 0,
$$

(7.3.8)
in the $e_r$ direction, and

$$
\frac{(J\sigma^{22} z' \sqrt{G})'}{\sqrt{G}} + \frac{1}{\sqrt{G}} \left( \frac{Jm^{11} rr^2 \sqrt{G}}{A(r^2 + z'^2)^{3/2}} \right)' - \frac{1}{\sqrt{G}} \left( \frac{(Jm^{22} z' \sqrt{G})'}{(r^2 + z'^2)^{1/2}} \right)' - P \frac{rr'}{\sqrt{G}} = 0,
$$

(7.3.9)
in the $e_z$ direction, where we recall that $\sqrt{G} = R(R^2 + Z'^2)^{1/2}/A$.

We now note that, after multiplying through by $\sqrt{G}$, (7.3.9) may now be integrated to give the resultant in the $Z$-direction as,

$$
J\sigma^{22} z' R\sqrt{R^2 + Z'^2} - \frac{Jm^{11} rr^2 R\sqrt{R^2 + Z'^2}}{(r^2 + z'^2)^{3/2}} + \frac{(Jm^{22} z' R\sqrt{R^2 + Z'^2})'}{(r^2 + z'^2)^{1/2}} - \frac{P* r^2}{2} = C_1,
$$

(7.3.10)

where $C_1$ is a constant of integration which will be determined by the boundary conditions of the case under consideration. The exploitation of this integral allowed us to derive the bifurcation conditions in the cylindrical and spherical cases in Chapters 3 and 6 respectively.

It is possible to combine these equations to give expressions in the directions normal and tangent to the deformed surface, that is $n$ and $(r' e_r + z' e_z)/\sqrt{r'^2 + z'^2}$ respectively. In the
the invariants of the deformation, hemitropic, and the energy must be invariant with respect to rotations around the axis of the axisymmetric configurations which are under consideration here are necessarily not immediately useful here. In the normal direction, we find

\[ \sqrt{r'^2 + z'^2} \left( \frac{\sigma_{22} \sqrt{G}}{\sqrt{r'^2 + z'^2}} \right)' - \frac{\sigma_{11} \sqrt{G} r'}{r} - \frac{m_{11} r' z' \sqrt{G}}{r^2 \sqrt{r'^2 + z'^2}} + \frac{m_{11} r' \sqrt{G}}{r(r'^2 + z'^2)^{3/2}} (z'' r' - r'' z') \]

\[ + \sqrt{r'^2 + z'^2} \left( \frac{m_{22} \sqrt{G}}{(r'^2 + z'^2)^2} \right)' + \left( \frac{m_{22} \sqrt{G}}{\sqrt{r'^2 + z'^2}} \right)' = 0. \]

We note that it is possible to combine the last two terms in (7.3.11) by using either of the two following identities, \((xy)' + x'y = 2x'y + xy' = (x^2 y)' / x = 2 \sqrt{y}(x \sqrt{y})'\), although they are not immediately useful here. In the normal direction, we find

\[ (z'' r' - r'' z') \left( \frac{\sigma_{22} \sqrt{G}}{(r'^2 + z'^2)^2} \right)' - \frac{\sigma_{11} \sqrt{G} z'}{r} - \frac{m_{11} z'^2 \sqrt{G}}{r^2 \sqrt{r'^2 + z'^2}} - \sqrt{r'^2 + z'^2} \left( \frac{m_{11} r' \sqrt{G}}{(r'^2 + z'^2)^2} \right)' \]

\[ + z' \left( \frac{1}{\sqrt{r'^2 + z'^2}} \left( \frac{m_{22} z' \sqrt{G}}{(r'^2 + z'^2)^2} \right)' \right)' + r' \left( \frac{1}{\sqrt{r'^2 + z'^2}} \left( \frac{m_{22} r' \sqrt{G}}{(r'^2 + z'^2)^2} \right)' \right)' + P'' r(r'^2 + z'^2) = 0 \]

(7.3.12)

### 7.4 Invariants

The axisymmetric configurations which are under consideration here are necessarily hemitropic, and the energy must be invariant with respect to rotations around the axis of symmetry. Therefore we use equation (6.20) from Steigmann and Ogden (1999) to find the invariants of the deformation,

\[ I_1 = \text{tr} C = g_{\alpha \beta} G^{\alpha \beta} = \frac{r^2}{R^2} + \frac{(r'^2 + z'^2)}{R^2 + Z'^2}, \]

(7.4.1a)

\[ I_2 = \det C = \frac{g}{G} = \frac{r^2(r'^2 + z'^2)}{R^2(R^2 + Z'^2)}, \]

(7.4.1b)

\[ I_3 = \text{tr} \kappa = \kappa_{\alpha \beta} G^{\alpha \beta} = \frac{r' z'}{R^2 \sqrt{r'^2 + z'^2}} + \frac{(r' z'' - r'' z')}{\sqrt{r'^2 + z'^2} (R^2 + Z'^2)}, \]

(7.4.1c)

\[ I_4 = \det \kappa = \frac{1}{G} \kappa_{11} \kappa_{22} = \frac{r' (r' z'' - r'' z')}{R^2 (r'^2 + z'^2) (R^2 + Z'^2)}, \]

(7.4.1d)

with another three invariants corresponding to coupling terms between the stresses and bending moments which are not considered here. From (7.4.1a) and (7.4.1b), we may see
that the principal stretches may be written as,

\[ \lambda_1 = \frac{r^2}{R^2}, \quad \lambda_2 = \frac{\sqrt{r'^2 + z'^2}}{\sqrt{R'^2 + Z'^2}}, \]  

(7.4.2)

corresponding with those used in the preceding chapters, although without the third stretch as the formulation here is two-dimensional. The stresses and moments are given by Steigmann and Ogden (1999),

\[ \frac{1}{2} J \sigma^{\alpha\beta} = \frac{\partial U}{\partial I_1} G^{\alpha\beta} + \frac{\partial U}{\partial I_2} \tilde{C}^{\alpha\beta}, \]  

(7.4.3)

\[ J m^{\alpha\beta} = 2 I_3 \frac{\partial U}{\partial I_3} G^{\alpha\beta} + \frac{\partial U}{\partial I_4} \tilde{\kappa}^{\alpha\beta}, \]  

(7.4.4)

where a tilde represents the adjugate of a tensor from \( \tilde{T} = (\text{tr}T)I - T \), although care must be taken to use the correct trace of the tensor, i.e. \( I_1 \) or \( I_3 \) for \( C \) or \( \kappa \) respectively, rather than just the sum of the two components. We may also use the formulae, \( \tilde{C}^{\alpha\beta} = I_2 g^{\alpha\beta}, \tilde{\kappa}^{\alpha\beta} = \mu^{\alpha\gamma} \mu^{\beta\lambda} \kappa_{\gamma\lambda}, \) where \( \mu^{\alpha\beta} = \epsilon^{\alpha\beta}/\sqrt{G} \), where \( \epsilon^{\alpha\beta} \) is the alternator symbol with \( \epsilon^{11} = \epsilon^{22} = 0, \epsilon^{12} = 1, \epsilon^{21} = -1. \)

\[ \tilde{C}^{11} = \frac{r'^2 + z'^2}{R^2(R'^2 + Z'^2)}, \quad \tilde{C}^{22} = \frac{r^2}{R^2(R^2 + Z^2)}, \]  

(7.4.5)

\[ \tilde{\kappa}^{11} = \frac{\kappa_{22}}{A^2 R^2(R^2 + Z^2)}, \quad \tilde{\kappa}^{22} = \frac{\kappa_{11}}{A^2 R^2(R^2 + Z^2)}. \]  

(7.4.6)

### 7.4.1 Membrane Theory

As discussed previously in Section 2.3, the membrane theory ignores the contribution to the energy from the curvature of the surface. Therefore, for a membrane we assume that \( U \) is a function of \( I_1 \) and \( I_2 \) only, and therefore \( m^{11} = m^{22} = 0 \). Using the covariant-contravariant connections, \( \sigma_{11} = r^2 \sigma^{11}, \sigma_{22} = (r'^2 + z'^2) \sigma^{22}, \) we find (7.3.8) and (7.3.9).
In the cylindrical case, $\sqrt{G} = 1$ as given in (7.2.3), we may immediately see (7.4.7) and (7.4.8) can be reduced to be the static form of the equations of motion in Chapter 3 (3.3.13), after using the connections $J\sigma_{11} = \sigma_1, J\sigma_{22} = \sigma_2$. This is therefore the connection between the two dimensional theory developed here, and the three dimensional theory used in Chapter 3. The two integrals used in Chapter 3 still exist here, with the first of them corresponding to (7.3.10).

For the spherical case, using (7.2.4), we find that (7.4.7) and (7.4.8) are equivalent to the equilibrium equations given in Chapter 6, (6.2.4) and (6.2.5). To see this, the equations (7.3.11) and (7.3.12), which are given in tangential and normal directions respectively, may be used. Again we require the connections $J\sigma_{11} = \sigma_1, J\sigma_{22} = \sigma_2$ between the two- and three- dimensional theories.

### 7.5 Specialisation

#### 7.5.1 Shell Theory

In the literature on deformations of spherical shells, the following three differential equations are introduced, such as by Pamplona and Calladine (1993), Pozrikiolis (2001), Blyth and Pozrikiolis (2004), Liu et al. (2006), Preston et al. (2008):

\begin{align}
\frac{d(rN_2)}{ds} - N_1 \frac{dr}{ds} - Qr \kappa_2 &= 0, \\
N_1 \kappa_1 + N_2 \kappa_2 + \frac{1}{r} \frac{d(Qr)}{ds} &= P, \\
\frac{d(rM_2)}{ds} - M_1 \frac{dr}{ds} + Qr &= 0,
\end{align}

(7.5.1a, 7.5.1b, 7.5.1c)
where \( s \) is the arc length in the deformed configuration, \( N_1, N_2, M_1, M_2 \) are principal stress resultants and bending moments respectively, \( Q \) is the transverse shear tension and the principal curvatures are given by,

\[
\kappa_1 = \frac{\kappa_{11}}{r^2}, \quad \kappa_2 = \frac{\kappa_{22}}{(r^2 + z'^2)}.
\]  
(7.5.2)

Equation (7.5.1a) and (7.5.1b) are in the tangential and normal directions respectively. In order to compare with our derived equations, we need the connections,

\[
N_i = h\sigma_{ii} = \frac{R}{r\sqrt{r'^2 + z'^2}}\sigma_{ii}, \quad M_i = h\mu_{ii} = \frac{R}{r\sqrt{r'^2 + z'^2}}\mu_{ii},
\]  
(7.5.3)

and the formula \( \frac{ds}{dS} = \sqrt{r'^2 + z'^2} \) as was discussed in Chapter 6. Using these formulae we may convert (7.3.11) and (7.3.12) into three equations as in (7.5.1), which are given by,

\[
\frac{d(rN_2)}{ds} - N_1\frac{dr}{ds} - Qr\kappa_2 - \frac{M_1}{r}\frac{dr}{ds} \frac{dz}{ds} + \frac{d(M_2r\kappa_2)}{ds} = 0,
\]  
(7.5.4a)

\[
N_1\kappa_1 + N_2\kappa_2 + \frac{1}{r}\frac{d(Qr)}{ds} + \frac{1}{r^2}M_1\frac{dz^2}{ds} + \kappa_2^2M_2 = P,
\]  
(7.5.4b)

\[
\frac{d(rM_2)}{ds} - M_1\frac{dr}{ds} + Qr = 0.
\]  
(7.5.4c)

In (7.5.4) there are additional terms to those in (7.5.1), indicated by the underbraces. These terms only involve the bending moments, and therefore when the membrane theory is considered the two derivations are equivalent. We also remark that it does not seem possible to include the additional terms in the definition of \( Q \), given by (7.5.4c), and retain the structure of the remaining equations. The equations in (7.5.1) only contain an integral when the membrane assumption is made, \( M_1 = M_2 = 0 \). This discrepancy requires further investigation, although it shall be left as future work here. In particular, the existence of the integral (7.3.10) would reduce the system of equations by one.
7.5.2 Biological Lipid Bilayers

Many biological membranes consist of lipid bilayers, where a double layer of long, relatively rigid molecules is orientated normal to the surface (Jenkins 1977). These molecules have hydrophilic heads and hydrophobic tails, and therefore molecular forces keep the membranes intact (McMahon and Gallop 2005). Such bilayers have fluid-like properties, and have often been modelled as continua which have negligible resistance to stretching but do resist changes to their curvature.

In order to model such lipid bilayers, Zhong-Can and Helfrich (1989) and Jian-Guo and Zhong-Can (1993) introduce a variational formulation in which they minimise the variations of the following integral,

\[
E = k_c \oint (\kappa_\phi + \kappa_s - c_0)^2 dA + \Delta p \int dV, \tag{7.5.5}
\]

where \(k_c\) is the bending rigidity, \(c_0\) is the spontaneous curvature in the undeformed state, and \(\Delta p\) is the pressure difference across the surface. The connection between the curvatures in (7.5.5) and those defined previously is given by,

\[
\kappa_\phi = \frac{\kappa_{11}}{r^2}, \quad \kappa_s = \frac{\kappa_{22}}{A^2(r'^2 + z'^2)}, \tag{7.5.6}
\]

where it should be noted that \(r^2 = g_{11}\) and \(A^2(r'^2 + z'^2) = g_{22}\), so \(\kappa_{11} g_{11} = \kappa_\phi, \kappa_{22} g_{22} = \kappa_s\).

Blyth and Pozrikidis (2004) review the literature on this variational formulation, and state a differential equation in terms of the curvature \(k_s\) which was found by Jian-Guo and Zhong-Can (1993). They discuss how several further authors have shown an integral of this equation exists, but there appears to be confusion in the literature as to the correct behaviour in certain limits. Blyth and Pozrikidis (2004) also discuss the alternative shell theory approach, but do not manage to make any connections between the two approaches.

We may therefore connect the integrand in (7.5.5) with our derivations above by noting that it may be written as \((\kappa_{11} \lambda_1^2 + \kappa_{22} \lambda_2^2 - c_0)^2\), which is precisely the third invariant \((I_3 - c_0)^2\).
as would be expected. We should therefore be able to compare the two approaches, in particular we would like to see how the integral given in (7.3.10) relates to that which was discussed in Blyth and Pozrikidis (2004), but we leave this as future work.

7.6 Conclusions and Future Work

In this chapter we have derived the equations for an axisymmetric shell which is subjected to a constant transmural pressure using two-dimensional elasticity, and compared with those used in the preceding chapters for membranes which were found using three-dimensional elasticity. We have shown an integral exists for a general axisymmetric shell and encourage its use whenever possible in order to reduce the order of the differential equations by one. We have found discrepancies between the equilibrium equations (7.3.8) - (7.3.9) and those given by Preston et al. (2008) and others. The equations given in Preston et al. (2008) are derived via force and moment balances, and it is possible that components have been overlooked in this derivation. The two approaches agree when the strain-energy only depends on the stretches, not the curvatures. Resolving this discrepancy has been left as future work which should be undertaken.

The modelling of biological lipid bilayers was briefly discussed and the connection between the theory given here and that discussed in Jian-Guo and Zhong-Can (1993) and others was shown, although this is also left as future work. In particular, the connection between the shell theory equations given by (7.3.8) - (7.3.9) with the equation resulting from the calculus of variations appearing in Jian-Guo and Zhong-Can (1993) should be investigated.
Chapter 8

Optical Tweezers

8.1 Introduction

The deformation of biological cells placed into a fluid flow is important in clinical medicine and the biological sciences. Cells which are inserted into the blood stream in one location may be deformed due to the stresses from the fluid flow before arriving at their intended destination. Such deformation could have major effects on the function of the cells, such as regulating the gene expressions of the cell. McCormick et al. (2001) show that many of the genes in endothelial cells are regulated by shear stresses, altering the expressed phenotypes, as also discussed in Frangos et al. (1988) and Bao and Suresh (2003). We are particularly interested here in stem cells, which are pluripotent cells with the potential to differentiate into various other types of cell (Deans and Moseley 2000). They are therefore considered useful in stem cell therapy treatments, where cells are inserted into the body as described above. Hence the knowledge of the stresses and deformation of a cell within a fluid flow is of practical interest. The internal structure of a cell is highly complex, comprising of many components with specialised functions (Humphrey 2003). It is assumed that, on a macroscopic scale, the internal structure is chiefly regulated by the fluid cytosol which is contained inside the cell and may be treated as a viscous Newtonian fluid. By considering the deformation of a single cell or microcapsule in a viscous Newtonian
fluid we wish to determine information about the behaviour of the cell when in blood as blood plasma has been found to behave like a Newtonian fluid via experiments using a viscometer (Fung 1993). New technology, such as optical tweezers, is improving the microscopic study of cells and their deformation (Lim et al. 2004, Titushkin and Cho 2006).

Optical tweezers are a recent advance in the manipulation of biological cells and artificial capsules (Henon et al. 1999, Fery and Weinkamer 2007). Zhang and Liu (2008) detail how focused lasers may be used to produce restoring forces on a microscopic object, enabling it to be kept immobile during an experiment.

In this technique, the microcapsule to be studied has one or more silicon beads bonded to its surface. Lim et al. (2004) use beads of diameter $4.12\mu$m attached to cells of diameter $7 - 8.5\mu$m. These beads are then trapped using a focused laser to produce forces directed towards the centre of focusing, before being either stretched in an ‘optical stretcher’ (Guck et al. 2001, Ananthakrishnan et al. 2005) or held stationary in an external flow (Henon et al. 1999, Foo et al. 2004). Using the beads as ‘handles’ (Henon et al. 1999) the microcapsule undergoes some kind of deformation until the restraining forces produced by the laser traps is not sufficient to hold the microcapsule in place. Typical forces for the laser trap used by Dao et al. (2003) are up to $600\text{pN}$. The beads are used to prevent heating of the microcapsule by the lasers potentially distorting the experiment though (Henon et al. 1999) present findings of no significant change on focusing lasers on red blood cells. We depart from the previous chapters by using linear elasticity rather than nonlinear elasticity to model this phenomenon, due to the small deformations seen in the experiments. Li et al. (2004) use a finite element package to investigate this deformation for the nonlinear Mooney-Rivlin strain energy function given by (2.2.12).
8.2 Governing Equations

We will take what is known as the Papkovich-Neuber form of the solution to the displacement equilibrium equations for linear elasticity, (2.4.14). This is given in vector form in Cartesian coordinates by Lur’e (1964) and Alexandrov and Pozharskii (2001) as

$$ \mathbf{u} = 4(1 - \nu)\mathbf{B} - \text{grad}(\mathbf{x} \cdot \mathbf{B} + B_0), $$

(8.2.1)

where $\mathbf{B}$ is a harmonic vector in Cartesian coordinates and $B_0$ is a harmonic scalar, i.e. $\nabla^2 \mathbf{B} = 0, \nabla^2 B_0 = 0$. It can be verified that (8.2.1) is a solution of (2.4.14) when $b = 0$ by direct substitution and manipulating the component form of the resulting expression utilising the harmonic properties of $\mathbf{B}$ and $B_0$. It follows from (8.2.1) that, as we are representing three independent variables $u_i$ by four new variables, we could disregard any one of these variables. However, keeping this dependent variable allows us to choose the most convenient form when solving particular boundary value problems (Lur’e 1964) and we therefore keep (8.2.1) overspecified for the moment.

8.2.1 Conversion to Spherical Coordinates

We now wish to take the Cartesian form of the Papkovich-Neuber solution given by (8.2.1) and convert it into spherical coordinates. Due to the harmonic nature of $\mathbf{B}$ and $B_0$ we have

$$ \nabla^2 B_x = 0, \nabla^2 B_y = 0, \nabla^2 B_z = 0, \nabla^2 B_0 = 0, $$

(8.2.2)

where $B_x, B_y$ and $B_z$ are the components of $\mathbf{B}$ in a Cartesian coordinate system $(x, y, z)$, and each satisfy the harmonic equation. In the case when there is central symmetry around an axis, we first convert to cylindrical coordinates $(r, \phi, Z)$, where $Z$ is the axis of symmetry, and the projections of the elements of $\mathbf{B}$ are given by (Lur’e 1964)

$$ B_x = B_r \cos \phi - B_\phi \sin \phi, \quad B_y = B_r \sin \phi + B_\phi \cos \phi. $$

(8.2.3)
As there is symmetry around the \( z \) axis, the components of \( \mathbf{B} \) can not depend on \( \phi \) and hence \( B_r, B_\phi \) are the coefficients of \( e^{i\phi} \) in the expressions \( \nabla^2 (B_r e^{i\phi}) = 0, \quad \nabla^2 (B_\phi e^{i\phi}) = 0. \) From (8.2.3) we have an expression for \( \mathbf{x} \cdot \mathbf{B} = xB_x + yB_y + zB_z = rB_r + zB_z. \)

Converting into spherical coordinates \((R, \theta, \phi)\), where the angle \( \theta \) is the angle from the \( z \)-axis and \( \phi \) is the same angle as in the cylindrical coordinates, we have \( r = R \sin \theta, z = R \cos \theta, \) and thus \( rB_r + zB_z = RB_R. \) Using all this, equation (8.2.1) becomes (Lur’e 1964):

\[
\begin{align*}
 u_R &= 4(1 - \nu)B_R - \frac{\partial}{\partial R} (RB_R + B_0), \quad (8.2.4) \\
 u_\theta &= 4(1 - \nu)B_\theta - \frac{1}{R} \frac{\partial}{\partial \theta} (RB_R + B_0), \quad (8.2.5)
\end{align*}
\]

where \( B_R, B_\theta \) are not harmonic potentials, though they are related to the harmonic function \( B_x \) and the function \( B_r \), which is the coefficient of \( e^{i\phi} \) in a harmonic potential as mentioned previously. Expressing the functions \( B_R, B_\theta \) as spherical harmonics and using the fact that one function of the functions may be chosen arbitrarily as discussed above, Lur’e (1964) shows that these functions can be expressed as

\[
\begin{align*}
 B_R &= - \sum_{n=0}^{\infty} A_n (n + 1) R^{n+1} P_n(\cos \theta), \\
 B_\theta &= \sum_{n=0}^{\infty} A_n R^{n+1} \frac{dP_n(\cos \theta)}{d\theta}, \quad (8.2.6) \\
 B_0 &= - \sum_{n=0}^{\infty} B_n R^n P_n(\cos \theta),
\end{align*}
\]

where \( A_n, B_n \) are new arbitrary coefficients replacing the three quantities \( B_R, B_\theta, B_0 \).

### 8.2.2 Displacements and Stresses

The spherical harmonics which are independent of \( \phi \) can be written as

\[
R^n P_n(\mu), \quad \frac{1}{R^{n+1}} P_n(\mu), \quad (8.2.7)
\]

as discussed in Section 2.6.
The first of these sets of functions remains finite provided \( R \) is bounded, along with its derivatives, while the second set remains finite provided \( R \) does not approach zero. Therefore, if the origin is included in the domain only the first set of functions is appropriate, and we shall derive the following expressions for this case. In Section 8.4 the second set of functions will be used.

Combining (8.2.6) with (8.2.5), the following expressions for the displacements of a symmetrically loaded linearly elastic sphere can be expanded as an infinite series of Legendre Polynomials given by

\[
\begin{align*}
    u_R &= \sum_{n=0}^{\infty} \left[ A_n(n+1)(n-2+4\nu)R^{n+1} + B_n n R^{n-1} \right] P_n(\mu), \\
    u_\theta &= \sum_{n=1}^{\infty} \left[ A_n (n+5-4\nu)R^{n+1} + B_n R^{n-1} \right] \frac{dP_n(\mu)}{d\theta},
\end{align*}
\]

(8.2.8)

where \( \mu = \cos \theta \). The formulae for the stresses are then derived as Lur’e (1964),

\[
\begin{align*}
    \frac{\sigma_R}{2G} &= \sum_{n=0}^{\infty} \left[ A_n (n+1)(n^2-n-2-2\nu)R^n + B_n n(n-1)R^{n-2} \right] P_n(\mu) \\
    \frac{\tau_{R\theta}}{2G} &= \sum_{n=1}^{\infty} \left[ A_n (n^2+2n-1+2\nu)R^n + B_n (n-1)R^{n-2} \right] \frac{dP_n(\mu)}{d\theta}.
\end{align*}
\]

(8.2.10)

(8.2.11)

For a given external loading to have a consistent solution of the linear elasticity equations, it is required that the resultant force and moment in each direction is zero (Howell et al. 2009). In particular, the force resultant, \( Z^* \), in the \( Z \)-direction around which the axisymmetry is assumed, is given by

\[
Z^* = \int_0^{2\pi} d\phi \int_0^\pi \left( \sigma_R \cos \theta - \tau_{R\theta} \sin \theta \right) R_0^2 \sin \theta d\theta
= 2\pi R_0^2 \int_{-1}^1 \left( \mu \sigma_R - \sin \theta \tau_{R\theta} \right) d\mu,
\]

(8.2.12)

and we therefore require that \( Z^* = 0 \) to ensure there is no resultant force in the \( Z \) direction. For a boundary value problem with given normal and shear stresses on all or part of the sphere given by \( \sigma(\theta) \) and \( \tau(\theta) \), we may express them as a series of Legendre
polynomials using the following expansions as explained in (2.6.19),

$$\sigma(\theta) = \sum_{n=0}^{\infty} \sigma_n P_n(\mu), \quad \tau(\theta) = \sum_{n=1}^{\infty} \tau_n \frac{dP_n(\mu)}{d\theta} = -\sum_{n=1}^{\infty} \tau_n \frac{dP_n(\mu)}{d\mu} \sin \theta,$$

(8.2.13)

where the coefficients $\sigma_n$ and $\tau_n$ are given by:

$$\sigma_n = \frac{2n+1}{2} \int_0^{\pi} \sigma(\theta) P_n(\cos \theta) \sin \theta d\theta,$$

$$\tau_n = \frac{2n+1}{2n(n+1)} \int_0^{\pi} \tau(\theta) \frac{dP_n(\cos \theta)}{d\theta} \sin \theta d\theta,$$

(8.2.14)

as was discussed in Section 2.6.5, and the fact that the derivatives of the Legendre polynomials are also an orthogonal set has been used. Inserting these expansions into (8.2.12), using the orthogonality properties for Legendre polynomials given in (2.6.15) along with the identity $\mu \equiv P_1(\mu)$, we find (8.2.12) reduces to:

$$Z^* = \frac{4\pi R_0^2}{3} (\sigma_1 + 2\tau_1),$$

(8.2.15)

and hence for a self-equilibrated set of boundary conditions the condition

$$\sigma_1 + 2\tau_1 = 0$$

(8.2.16)

must be imposed on the expansions of $\sigma(\theta)$ and $\tau(\theta)$. We will use this condition in the following calculations, but first we need to derive appropriate forces to be applied to the surface of the sphere.

### 8.2.3 Flow Around a Rigid Sphere

For flow at low Reynolds number, such as that in blood flow, the Stokes’ equations may be used as an approximation to the full Navier-Stokes equations governing the motion
of a viscous fluid. These Stokes’ equations are given by Ockendon and Ockendon (1995),

\[ \bar{\mu} \nabla^2 \mathbf{v} = \nabla p_0 \]
\[ \nabla \cdot \mathbf{v} = 0, \tag{8.2.17} \]

where \( \mathbf{v} \) is the fluid velocity, rather than the more common \( u \), as the displacements are already denoted \( u \), \( \bar{\mu} \) is the dynamic viscosity and \( p_0 \) is the pressure of the fluid.

For a rigid sphere of radius \( R_0 \) subject to a uniform flow of velocity \( U \) parallel to the \( Z \)-axis at low Reynolds number such that the inertia terms can be neglected, the velocity at all points can be found by using a streamfunction of the form:

\[ v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}, \quad v_\phi = 0, \tag{8.2.18} \]

where \( \psi(r, \theta) \) is the streamfunction, the symmetry of the sphere has been used to remove the \( \phi \)-dependence (Ockendon and Ockendon 1995). This streamfunction automatically satisfies the conservation of mass equation, and reduces the Stokes’ equations (8.2.17) to

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 \psi(r, \theta) = 0. \tag{8.2.19} \]

Using separation of variables and applying the no-slip boundary conditions as well as the condition \( \psi \sim \frac{1}{2} Ur^2 \sin^2 \theta \) as \( r \to \infty \) to ensure the flow is uniform far away from the sphere, the solution is given by

\[ \psi(r, \theta) = \frac{U}{2} \left( r^2 - \frac{3R_0 r}{2} + \frac{R_0^3}{2r} \right) \sin^2 \theta, \tag{8.2.20} \]

from which the velocities and the stresses can be determined. The resulting normal and shear stresses on the surface of the sphere are given by,

\[ \sigma(\theta) = -p_0 + \frac{3U \bar{\mu} \cos \theta}{2R_0}, \quad \tau(\theta) = -\frac{3U \bar{\mu} \sin \theta}{2R_0}. \tag{8.2.21} \]

The net force on the sphere is given by \( F = 6\pi \bar{\mu} UR_0 \) in the direction of the positive \( Z \)
axis, and this is hence the force required in the opposite direction to hold the sphere in place. This force is linear in all three parameters involved in the flow, and this result is known as Stokes’ drag coefficient (Ockendon and Ockendon 1995).

### 8.3 Deformation of a Solid Elastic Sphere

We now look at the case of an elastic sphere held by optical tweezers, as discussed in the introduction to this chapter. We model the bead which is held in the optical tweezers as a force which is applied normal to a small area of the surface of the sphere, without including the shape of the bead. Therefore we apply the following surface tractions to the deformation of the elastic sphere,

$$\sigma(\theta) = -p_0 + \frac{3U\bar{\mu}\cos^2 \theta}{2R_0} + \zeta(\theta), \quad \tau(\theta) = -\frac{3U\bar{\mu}\sin^2 \theta}{2R_0},$$  

(8.3.1)

for some function \(\zeta(\theta)\) representing the holding force of the optical tweezers which depends on the flow parameters \(U\) and \(\bar{\mu}\). We model the optical tweezers effect as a stress of strength \(\Gamma\) acting normal to the surface over an arc \(2\delta\) radians wide. Hence as \(\delta\) decreases the stress will become more concentrated, corresponding to a smaller silica bead size relative to the radius of the cell in the physical experiments. To this end we therefore choose \(\zeta(\theta)\) to be

$$\zeta(\theta) = \begin{cases} 
\Gamma & \theta \in [\pi - \delta, \pi + \delta] \\
0 & \text{else}
\end{cases}$$  

(8.3.2)

where \(\Gamma\) is the constant normal pressure applied to the surface. In order for the sphere to be held immobile, we require that \(\Gamma\) balances the Stokes’ drag, that is

$$\Gamma = \frac{6\pi\bar{\mu}UR_0}{R_0^2 \sin^2(\delta)} = \frac{6\pi\bar{\mu}U}{R_0 \sin^2 \delta},$$  

(8.3.3)

where \(R_0^2 \sin^2 \delta\) is the area over which the pressure is applied. This choice of \(\Gamma\) satisfies the zero resultant condition, (8.2.16), whereas the Dirac delta function for \(\zeta(\theta)\) does not.
Having made this choice of $\zeta(\theta)$, we may calculate the coefficients $\sigma_n$ and $\tau_n$ in (8.2.14) as,

$$\sigma_0 = \frac{1}{2} \int_0^{\pi} \sigma(\theta) \sin \theta d\theta = p_0 + \frac{\Gamma}{2} \int_{-\delta}^{\pi} \sin \theta d\theta = -p_0 + \frac{3U\bar{\mu}(1 - \cos \delta)}{R_0 \sin \delta}, \quad (8.3.4a)$$

$$\sigma_1 = -\frac{3U\bar{\mu}}{R_0}, \quad \tau_1 = \frac{3U\bar{\mu}}{2R_0}, \quad (8.3.4b)$$

$$\sigma_n = \frac{(2n + 1)\Gamma}{2} \int_{-\delta}^{\pi} P_n(\cos \theta) \sin \theta d\theta \quad (8.3.4c)$$

$$= \frac{(2n + 1)\Gamma(-1)^n}{2} \int_{\cos \delta}^{1} P_n(x) dx = \frac{(2n + 1)\Gamma(-1)^n}{2} P_n^{-1}(\cos \delta) \sin \delta, \quad \tau_n = 0,$$

where $P_n^{-1}(\mu)$ is an Associated Legendre function, and the last two equations hold for all values of $n \geq 2$. In evaluating (8.3.4c) the relations $P_n(-x) = (-1)^nP_n(x)$ and $\int_{\cos \alpha}^{1} P_n(x) dx = P_n^{-1}(\cos \alpha) \sin \alpha$ have been used, the second of these relations coming from Abramowitz and Stegun (1964), along with the standard orthogonality relations given in Section 2.6.4.

Using these expressions for the boundary conditions we can find the values of the arbitrary constants $A_n$ and $B_n$ in the equations for the displacements in (8.2.8) and (8.2.9). So, using the expressions for the stresses in (8.2.10) and (8.2.11), we find for $n \geq 2$,

$$\sigma_n = A_n(n + 1)(n^2 - n - 2 - 2\nu)R_0^n + B_n(n - 1)R_0^{n-2}, \quad (8.3.5)$$

$$0 = A_n(n^2 + 2n - 1 + 2\nu)R_0^n + B_n(n - 1)R_0^{n-2}, \quad (8.3.6)$$

where $\sigma_n$ is given by (8.3.4c). Thus we find the following expressions for the coefficients $A_n$ and $B_n$,

$$A_n = \frac{(2n + 1)\Gamma(-1)^nP_n^{-1}(\cos \delta) \sin \delta}{8GR_0^n(n^2 + n + 2n\nu + 1 + \nu)}, \quad (8.3.7)$$

$$B_n = \frac{(2n + 1)(n^2 + 2n - 1 + 2\nu)\Gamma(-1)^nP_n^{-1}(\cos \delta) \sin \delta}{8GR_0^{n-2}(n - 1)(n^2 + n + 2n\nu + 1 + \nu)}, \quad (8.3.8)$$
Considering the cases when \( n = 0, 1 \) separately, we find the following two constants,

\[
A_0 = -\frac{\sigma_0}{4G(1+\nu)} = -\frac{1}{4G(1+\nu)} \left[ -p_0 + \frac{\Gamma(1 - \cos \delta)}{2} \right] \\
A_1 = \frac{\tau_1}{4GR_0(1+\nu)} = \frac{3U\bar{\mu}}{8GR_0^2(1+\nu)}
\]  

(8.3.9)  

(8.3.10)

The coefficient \( B_1 \) represents a rigid body motion corresponding to the displacements \( u_R = B_1 \cos \theta, u_\theta = B_1 \sin \theta \), and hence does not produce any stresses and may be superimposed on any stress-produced deformation, and therefore we set \( B_1 = 0 \) without any loss of generality. We note that in the expressions above, the constants \( U, \bar{\mu} \) and \( G \) only occur in the formulae for the displacements and stresses in the dimensionless combination \( \eta = \frac{U\bar{\mu}}{\mbox{GR}_0^2} \), and hence it is this ratio which is relevant rather than the individual values. An equivalent ratio is denoted the \textit{capillary number} by Lac et al. (2007) and expresses the relative importance of the fluid viscous stresses to the elastic property of the material, and it is therefore the size of this number which determines how much the sphere deforms. Similarly, the initial radius of the sphere \( R_0 \) only scales the values of the resulting sphere and hence can be taken as unity without loss of generality.

### 8.3.1 Results

With all the arbitrary coefficients determined we may find the displacements, \( u_R, u_\theta \), as a function of the coordinates \( (r, \theta) \), and then plot the resulting deformed sphere after the deformation.

To plot the deformed shape \textit{Mathematica} was used to calculate the displacement \( u \) at the surface of the sphere, then the values of \( x = X + u \) were plotted to give the deformed sphere. For the infinite sums in (8.2.8) and (8.2.9) the first 42 terms were taken, as this was sufficient to give smooth functions for the displacements without causing numerical precision errors as was found when taking more terms. In the following, the diagrams shown are planes through the ‘sphere’ at a fixed value of \( \phi \), the complete deformed sphere being made by revolving this plane around the \( z \)-axis as it has been assumed that
the resulting shape is axisymmetric. Figure 8.3.1 shows the undeformed sphere with a

![Figure 8.3.1: Sphere of unit radius with bold arc corresponding to the area over which the restraining force is applied when \( \delta = 0.1 \)](image)

bold arc corresponding to the \( \delta \) radians where the force from the optical tweezers applies.

The exterior pressure, \( p_0 \), does not enter the formulae for the stresses or the displacements in the incompressible case when \( \nu = 1/2 \). This stems from the fact that the pressure is distributed evenly around the exterior of the sphere, and incompressibility implies that no deformation is possible from such a pressure. When \( \nu = 1/2 \) the deformation is mostly concentrated around the optical tweezers force, with little change to the shape of the sphere elsewhere. This can be seen clearly in figure 8.3.2, where the increase of the external flow speed primarily increases the elongation of the ‘tendril’ on the negative \( z \)-axis with little significant change elsewhere. Varies of \( \eta \) which are less than 0.001 show very little deformation, and we may therefore see \textit{a posteriori} that the assumptions of linear elasticity are appropriate in this case. Note that \( \nu = 1/2 \) corresponds to incompressibility, though the cell shows a change of volume for \( \eta = 0.01, 0.03 \) and higher. This is a consequence of the fact that the deformations are becoming large, and the specification \( \nu = 1/2 \) is only an approximation to the nonlinear requirement of conservation
of mass, \( \det \mathbf{F} = 1 \). When \( \nu < 1/2 \) the material is compressible and hence the volume

of the material can be decreased by the deformation. The changing shape of the deformation is shown for the value of \( \eta = 0.01 \) in figure 8.3.3 for differing values of \( \nu \). The effect of the external pressure, \( p_0 \), on a compressible sphere is shown in figure 8.3.4 for a value of \( \eta = 0.01 \). Here the compressible material can be seen to drastically reduce in volume with a corresponding change in density, but the shape remains unaltered with just the scale changing. When the sphere is held fixed in the optical trap, the point \( \theta = \pi \) does not move while the rest of the sphere deforms around it, and therefore the shapes in Figure 8.3.4 should be shifted appropriately. Figure 8.3.5 shows the effect of varying \( \delta \) while keeping \( \eta \) fixed. Allowing the point force to be applied over a large range has the
expected effect of causing the ‘tendril’ to be spread over a larger area such that it merely appears to be a bulge in the sphere.

8.4 Deformation of a Hollow Elastic Sphere

We now consider a hollow fluid filled elastic sphere, where the elastic material now occupies the space $R_1 \leq R \leq R_0$, where $R_1 > 0$ is the internal radius. In the previous sections, the requirement that the formulae were valid at the origin excluded the second set of spherical harmonics of the form $R^{-(n+1)} P_{-(n+1)}(\mu) = R^{-(n+1)} P_n(\mu)$, which we now need to include in order to have enough arbitrary constants to satisfy the additional boundary conditions at the inside of the sphere. To do this, we make the substitution $n \rightarrow -(n + 1)$ in (8.2.8) and (8.2.9), and add the corresponding expressions, with a different arbitrary
constant, to (8.2.8) and (8.2.9), giving,

\[
\begin{align*}
    u_R &= \sum_{n=0}^{\infty} \left[ A_n (n+1)(n-2+4\nu)R^{n+1} + B_n nR^{n-1} + \frac{C_n}{R^n} n(n+3-4\nu) - \frac{D_n(n+1)}{R^{n+2}} \right] P_n(\mu), \\
    u_{\theta} &= \sum_{n=1}^{\infty} \left[ A_n (n+5-4\nu)R^{n+1} + B_n R^{n-1} + \frac{C_n}{R^n} (-n+4-4\nu) + \frac{D_n}{R^{n+2}} \right] \frac{dP_n(\mu)}{d\theta},
\end{align*}
\]

(8.4.1) (8.4.2)

where \(C_n\) and \(D_n\) are two additional sets of arbitrary constants. Therefore (8.2.10) and (8.2.11) become,

\[
\begin{align*}
    \sigma_R &= \sum_{n=0}^{\infty} \left[ A_n (n+1)(n^2-n-2-2\nu)R^n + B_n n(n-1)R^{n-2} \\
        &\quad - \frac{C_n}{R^{n+1}} (n^2+3n-2\nu) + \frac{D_n(n+1)(n+2)}{R^{n+3}} \right] P_n(\mu) \\
    \tau_{R\theta} &= \sum_{n=1}^{\infty} \left[ A_n (n^2+2n-1+2\nu)R^n + B_n (n-1)R^{n-2} \\
        &\quad + \frac{C_n}{R^{n+1}} (n^2-2+2\nu) - \frac{D_n(n+2)}{R^{n+3}} \right] \frac{dP_n(\mu)}{d\theta},
\end{align*}
\]

(8.4.3) (8.4.4)

which we can then evaluate at \(R = R_0\) where \(\sigma(\theta), \tau(\theta)\) are as given in (8.3.1), and at \(R = R_1\) where \(\sigma(\theta) = -p_1, \tau(\theta) = 0\), with \(p_1\) the internal pressure exerted by the fluid on the inner surface of the sphere. We therefore find the following expressions for the first few coefficients, along with \(B_1 = 0, C_1 = 0,\)

\[
\begin{align*}
    A_0 &= \frac{-1}{4G(1+\nu)(R_0^3-R_1^3)} \left( \frac{-p_0 + 3U\bar{\mu}(1-\cos\delta)}{R_0 \sin^2\delta} \right) R_0^3 + p_1 R_1^3 \\
    A_1 &= \frac{-R_0^3 R_1^3}{8G(1+\nu)(R_0^3-R_1^3)} \\
    D_0 &= \frac{-R_0^3 R_1^3}{4G(R_0^3-R_1^3)} \left( \frac{-p_0 + p_1 + 3U\bar{\mu}(1-\cos\delta)}{R_0 \sin^2\delta} \right) \\
    D_1 &= \frac{R_0^3 R_1^3}{12G(R_0^3-R_1^3)} \frac{3U\bar{\mu} R_1}{2R_0}
\end{align*}
\]

(8.4.5)
where the remaining coefficients $A_n, B_n, C_n, D_n$, for $n \geq 2$ are given by the solutions to the following equation,

$$
\begin{pmatrix}
A_n \\
B_n \\
C_n \\
D_n
\end{pmatrix} = 
\begin{pmatrix}
\sigma_n \\
0 \\
0 \\
0
\end{pmatrix} \cdot (8.4.6)
$$

where $\sigma_n$ is given by (8.3.4c), and $M$ is given by the following matrix, which is non-singular provided $n > 2$ and $R_0 > R_1$,

$$
M = \begin{pmatrix}
(n + 1)(n^2 - n - 2 - 2\nu)R_0^n & n(n - 1)R_0^{n-2} - \frac{n}{R_0^{n+1}}(n^2 + 3n - 2\nu) & \frac{(n+1)(n+2)}{R_0^{n+3}} \\
(n^2 + 2n - 1 + 2\nu)R_0^n & (n - 1)R_0^{n-2} - \frac{n}{R_0^{n+1}}(n^2 - 2 + 2\nu) & -\frac{n+2}{R_0^{n+3}} \\
(n + 1)(n^2 - n - 2 - 2\nu)R_1^n & n(n - 1)R_1^{n-2} - \frac{n}{R_1^{n+1}}(n^2 + 3n - 2\nu) & \frac{(n+1)(n+2)}{R_1^{n+3}} \\
(n^2 + 2n - 1 + 2\nu)R_1^n & (n - 1)R_1^{n-2} - \frac{n}{R_1^{n+1}}(n^2 - 2 + 2\nu) & -\frac{n+2}{R_1^{n+3}}
\end{pmatrix}
$$

With all the coefficients determined this way we may evaluate the displacements, (8.4.1) and (8.4.2), and therefore the entire deformation is known, and we can plot the deformed state in the same way as was done before for the solid sphere. We now have two more parameters than in the solid sphere, $R_1$ and $p_1$, and define a new dimensionless parameter $\epsilon = (R_0 - R_1)/R_0$. The external and internal pressures, $p_0$ and $p_1$, only occur in the expressions for $A_0$ and $D_0$, and in particular in the incompressible case where $\nu = 1/2$, then the $A_0$ term drops out in (8.4.3), and therefore the deformation is controlled only by the pressure difference, $\Delta p = p_0 - p_1$. For compressible materials the individual pressures are important, rather than just the difference.

Therefore, in the incompressible case we have the following 5 dimensionless parameters, $\eta, \Delta p/G, \delta, \nu, \epsilon = (R_0 - R_1)/R_0$.

### 8.4.1 Results

We now show the effect of varying the five parameters upon which the deformation depends. However, there is a greater amount of coupling between the effects of the pa-
rameters in this case than in the solid sphere studied previously. It should be noted that, although large deformations may be seen with certain values of the parameters are chosen, the linear elasticity requirement of small strains will not be satisfied in these cases. They do however give information as to what values require nonlinear elasticity to be valid. First, we vary $\eta$, increasing the amount of deformation which is applied to the sphere. In Figure 8.4.1 $\eta$ is varied for a moderately thin sphere with $\epsilon = 0.1$. Comparison with 8.3.2 shows that the value of $\eta$ required for a deformation to occur is reduced by approximately an order of magnitude. Larger values than $\eta = 0.003$ cause very large deformations for this choice of parameters, which violate the linear theory. Figure 8.4.2 shows a thinner membrane, $\epsilon = 0.05$, which shows that the value of $\eta$ required for significant deformations is again reduced. As may be seen in 8.4.3, increasing the value of $\delta$ makes the deformation less concentrated around $\theta = \pi$, with the three-dimensional shape changing from a more pinched shape into a more rounded egg-like shape.

The effects of the Poisson’s ratio $\nu$, which gives the compressibility of the material, and the external pressure $p_0$ are coupled together, as may be seen in Figures 8.4.4 and 8.4.5. In Figure 8.4.4 $\nu$ is changed subject to the same external pressure, and the deformed shape

Figure 8.4.1: Effect of varying $\eta$ for a hollow sphere with $\nu = 1/2, \delta = 0.1, \epsilon = 0.1$: 
(a) $\eta = 0.0001$, (b) $\eta = 0.0005$, (c) $\eta = 0.001$, (d) $\eta = 0.003$
Figure 8.4.2: Effect of varying $\eta$ for a hollow sphere with $\nu = 1/2, \delta = 0.1, \epsilon = 0.05$: (a) $\eta = 0.0001$, (b) $\eta = 0.0005$, (c) $\eta = 0.001$, (d) $\eta = 0.003$

is squeezed further when the material is more compressible, as expected. Figure 8.4.5 shows how increasing the external pressure has a very similar effect as decreasing $\nu$. The scaled thickness of the membrane, $\epsilon$, plays a very significant role in the deformation. Figure 8.4.6 shows how changing this thickness for a given value of $\eta$ very markedly
Figure 8.4.4: Effect of varying $\nu$ for a hollow sphere with $\eta = 0.001, \epsilon = 0.1, \delta = 0.1, p_0/G = 0.1, p_1 = 0$: (a) $\nu = 0.1$, (b) $\nu = 0.2$, (c) $\nu = 0.33$, (d) $\nu = 0.5$

Figure 8.4.5: Effect of varying $p_0/G$ for a hollow sphere with $\eta = 0.001, \epsilon = 0.1, \delta = 0.1, \nu = 0.33, p_1 = 0$: (a) $p_0/G = 0.1$, (b) $p_0/G = 0.2$, (c) $p_0/G = 0.3$, (d) $p_0/G = 0.5$

increases the amount of deformation caused. Therefore knowledge of the thickness of the membrane is important as to providing the resulting shape.
Figure 8.4.6: Effect of varying $\epsilon$ for a hollow sphere with $\eta = 0.001, \delta = 0.1, \nu = 0.5$: 
(a) $\epsilon = 0.5$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.1$, (d) $\epsilon = 0.05$
8.5 Conclusion

We now consider the case of stem cells deformed in an optical trap, applying experimental values of the various parameters. We take the fluid viscosity as that of blood plasma, $\mu_f = 1.6 \times 10^{-3}{Nsm}^{-2}$ (Haidekker et al. 2002), and assume that the cell is incompressible due to its high water content, $\nu = 0.5$. The radius of an individual stem cell varies between 8 $-$ 16 micrometres, and we therefore take $R_0 = 12 \mu m$. The contact radius between the silica bead and a cell is given by approximately $1.17 \mu m$ (Li and Liu 2008), and therefore we have $\delta = 1.17/12 = 0.0975$. We assume that the internal and external pressures are equal, as when the cells are in vivo they should not be swelling or contracting, and we therefore set $p_1 = p_0$, and as we are considering an incompressible material the pressures are therefore irrelevant as discussed earlier. Experimentally, the optical trap can be used to hold cells in speeds of up to $100 \mu ms^{-1}$, and we therefore have a maximum of $U = 100 \mu ms^{-1}$. The Young’s modulus of the cell has been given by Titushkin and Cho (2007) as $E = 3.2 \pm 1.4 kPa$, with the shear modulus deduced from $G = E/3$. We therefore have just one parameter in the hollow sphere left to specify, the inner radius $R_1$ or equivalently the membrane thickness. In the case where we model the cell as a solid sphere, we find that no significant deformation is expected within this parameter regime, and therefore linear elasticity is an appropriate theory to use here. In fact, we find that, for these values of the parameters, that no significant deformation will occur unless $\eta$ is increased by two orders of magnitude, which would require flow speeds which are much greater than those the optical tweezers may currently substan. If the hollow model is used, then the membrane thickness becomes a key parameter as may be seen between Figures 8.4.1 and 8.4.2, and we also require a different value for the shear modulus of the membrane as the value given above is for the whole cell.

In the experiments which were performed, the cell does not undergo any significant deformation, which is consistent with the predictions of the solid sphere model. Thus the above analysis enables us to predict the stresses which the cell is subjected to under this deformation.
References


using limiting chain extensibility constitutive models. *Biomechanics and Modeling in Mechanobiology* 1(4), 251–266.


