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Lattice Hamiltonian approach to the Schwinger model

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Outline

1. The Schwinger model on lattice
2. Strong coupling expansion (SCE)
3. Ground state energy
4. Mass gaps
5. Chiral condensate
6. Oscillations of chiral condensate
7. Summary & outlook

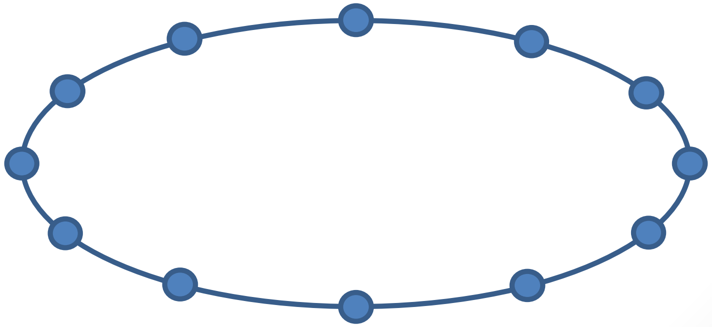
The Schwinger model

Hamiltonian of the Schwinger model in the Kogut-Susskind staggered discretization [1,2]:

$$\mathcal{H} = -\frac{i}{2a}\sum_{n=1}^{M}\Big(\phi^{\dagger}(n)e^{i\theta(n)}\phi(n+1) - \phi^{\dagger}(n+1)e^{-i\theta(n)}\phi(n)\Big)$$

$$+\; m\sum_{n=1}^{M}(-1)^{n}\phi^{\dagger}(n)\phi(n) + \frac{ag^{2}}{2}\sum_{n=1}^{M}L^{2}(n)$$

- $\phi(n)$ – single-component fermion field on a circle with M sites
- $\theta(n) = agA_1(n)$ – gauge field variable related to the Abelian vector potential
- $L(n) = E(n)/g$ – variable related directly to the electric field
- m – fermion mass
- a – lattice spacing
- g – gauge coupling constant



[1] Kogut and Susskind, Phys. Rev. D 11 (1975) 395

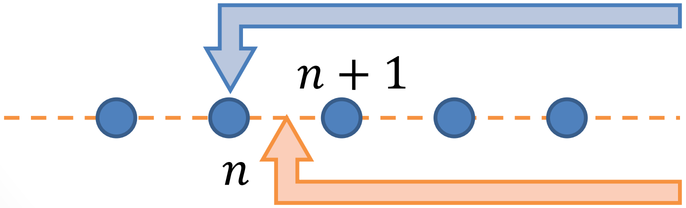
[2] Banks, Susskind and Kogut, Phys. Rev. D 13 (1976) 1043

The Schwinger model

Hamiltonian of the Schwinger model in lattice representation after the Jordan-Wigner transformation [3]:

$$\mathcal{H}_{JW} = -\frac{1}{2a}\sum_{n=1}^{M}\left(\sigma^{+}(n)e^{i\theta(n)}\sigma^{-}(n+1) + \text{h.c.}\right) + \frac{m}{2}\sum_{n=1}^{M}\left(1 + (-1)^{n}\sigma^{3}(n)\right) + \frac{ag^{2}}{2}\sum_{n=1}^{M} L^{2}(n)$$

- $\sigma^{i}(n)$ – Pauli matrices residing on the sites
- $L(n)$ – gauge field excitations defined between sites n and $n+1$
- $e^{\pm i\theta(n)}$ – ladder operators for gauge field excitations



Spin matrices $\sigma^{3}(n), \sigma^{\pm}(n)$
$\sigma^{3}(n) = \pm 1$

Gauge field excitations $L(n), e^{i\theta(n)}$
$L(n) = 0, \pm 1, \pm 2, \ldots$

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[3] Jordan, Wigner, Z. Phys. 47 (1928) 631.

Strong coupling expansion on the Schwinger model

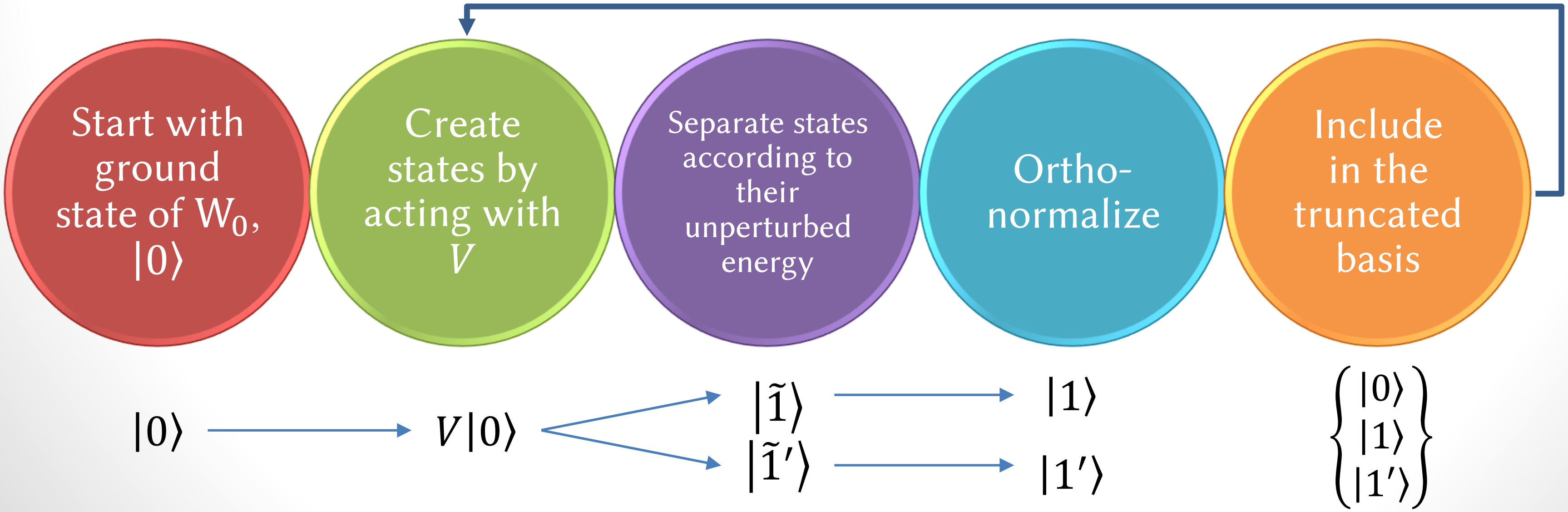
Rewrite the Hamiltonian in a dimensionless form:

$$\sum_{n=1}^{M}\left(\sigma_n^+ e^{i\theta(n)}\sigma_{n+1}^- + \text{h.c.}\right)$$

$$W = \frac{2}{ag^2}\mathcal{H}_{JW} = \boxed{W_0} + x\boxed{V}$$

$$\frac{m}{ag^2}\sum_{n=1}^{M}\left(1+(-1)^n\sigma_n^3\right) + \sum_{n=1}^{M} L^2(n)$$

- If $x \equiv \beta = \frac{1}{a^2g^2}$ is small, we can treat W_0 as an unperturbed Hamiltonian and V as a perturbation.
- SCE creates the truncated basis of W



Start with ground state of W$_0$, $|0\rangle$

Create states by acting with V

Separate states according to their unperturbed energy

Ortho-normalize

Include in the truncated basis

$$|0\rangle \rightarrow V|0\rangle \rightarrow |\tilde{1}\rangle,\ |\tilde{1}'\rangle \rightarrow |1\rangle,\ |1'\rangle \qquad \begin{Bmatrix}|0\rangle\\|1\rangle\\|1'\rangle\end{Bmatrix}$$

Observables

- Ground state energy:

$$E_0 = \frac{\omega_0}{2Mx} \xrightarrow[M\to\infty]{a\to 0} -\frac{1}{\pi}$$

- Scalar mass gap ($m = 0$):

$$\frac{M_S}{g} = \frac{\omega_1 - \omega_0}{2\sqrt{x}} \xrightarrow[M\to\infty]{a\to 0} \frac{2}{\sqrt{\pi}}$$

- Vector mass gap ($m = 0$):

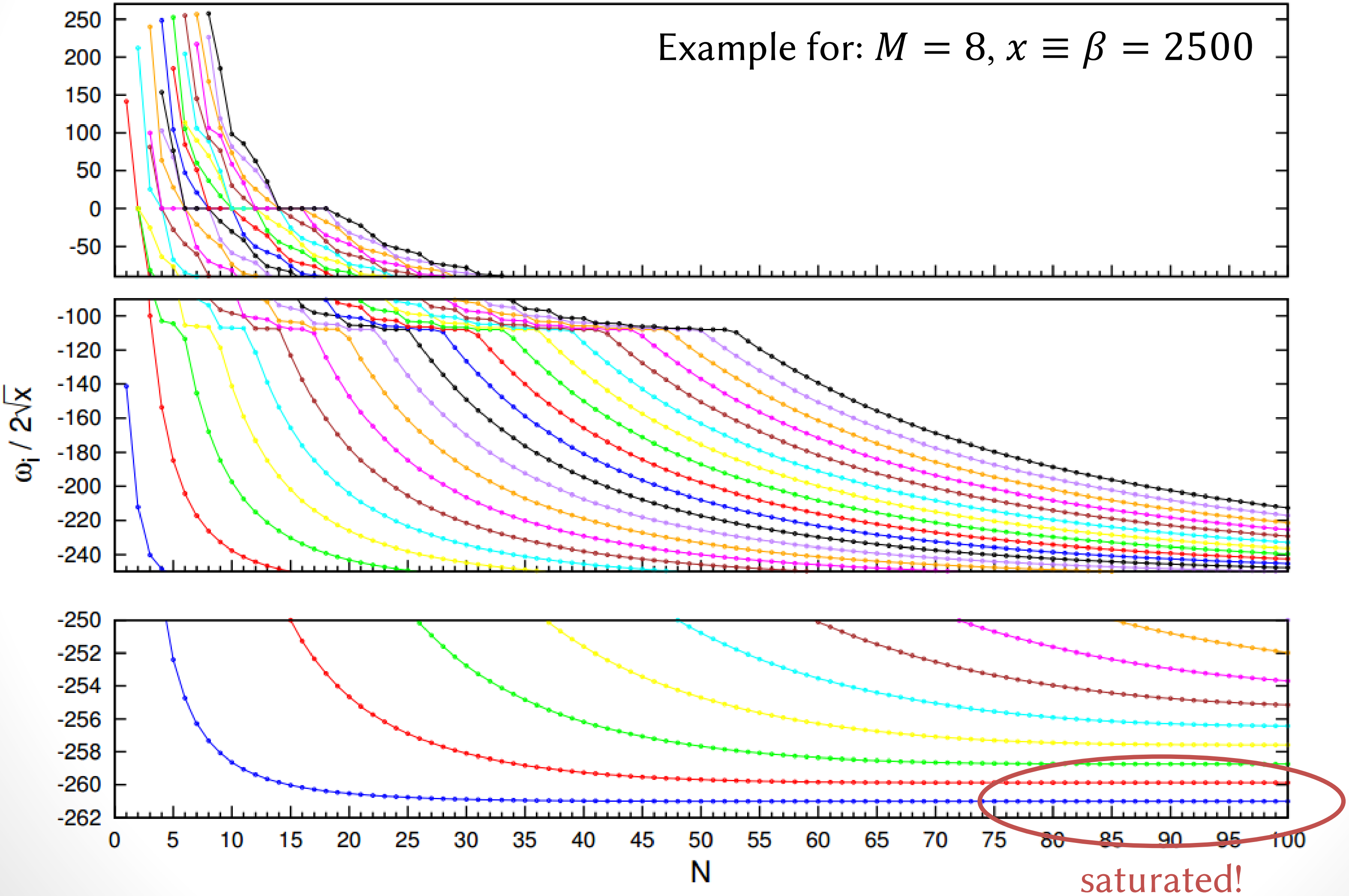
$$\frac{M_V}{g} = \frac{\omega_0^V - \omega_0}{2\sqrt{x}} \xrightarrow[M\to\infty]{a\to 0} \frac{1}{\sqrt{\pi}}$$

- Chiral condensate (chiral order parameter):

$$\frac{\langle\bar{\psi}\psi\rangle_0}{g} = \frac{\sqrt{x}}{2M}\langle 0|\sum_{n=1}^{M}(-1)^n\sigma^3(n)\,|0\rangle$$

- ○ ω_i – eigenvalues of W_0
- ○ ω_i^V – eigenvalues of vector Hamiltonian created using SCE with first state $V^-|0\rangle$.

Eigenvalue flow with the order of strong coupling expansion, N

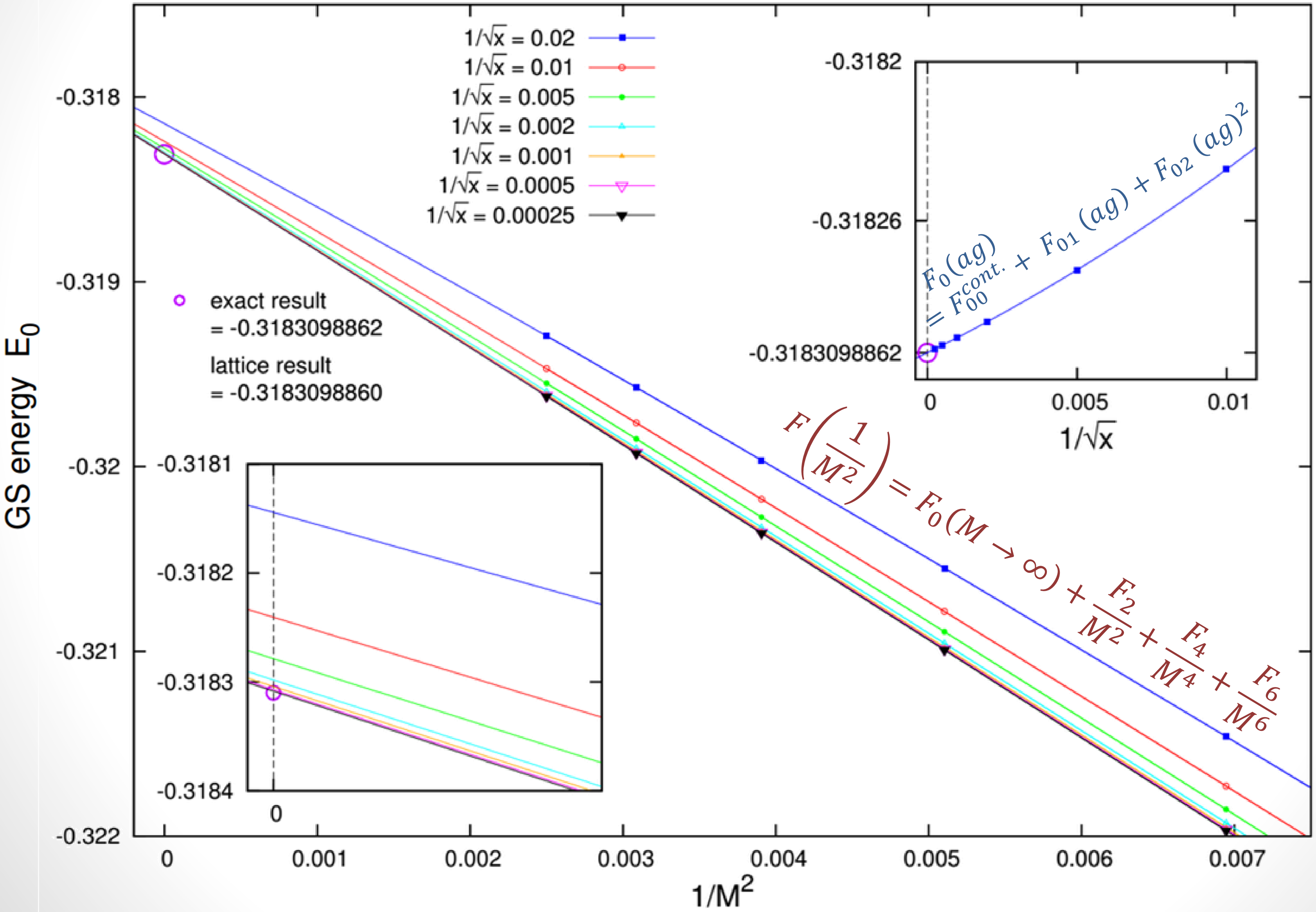


saturated!

Ground state energy

$$E_0 = \frac{\omega_0}{2Mx}$$

$m = 0$



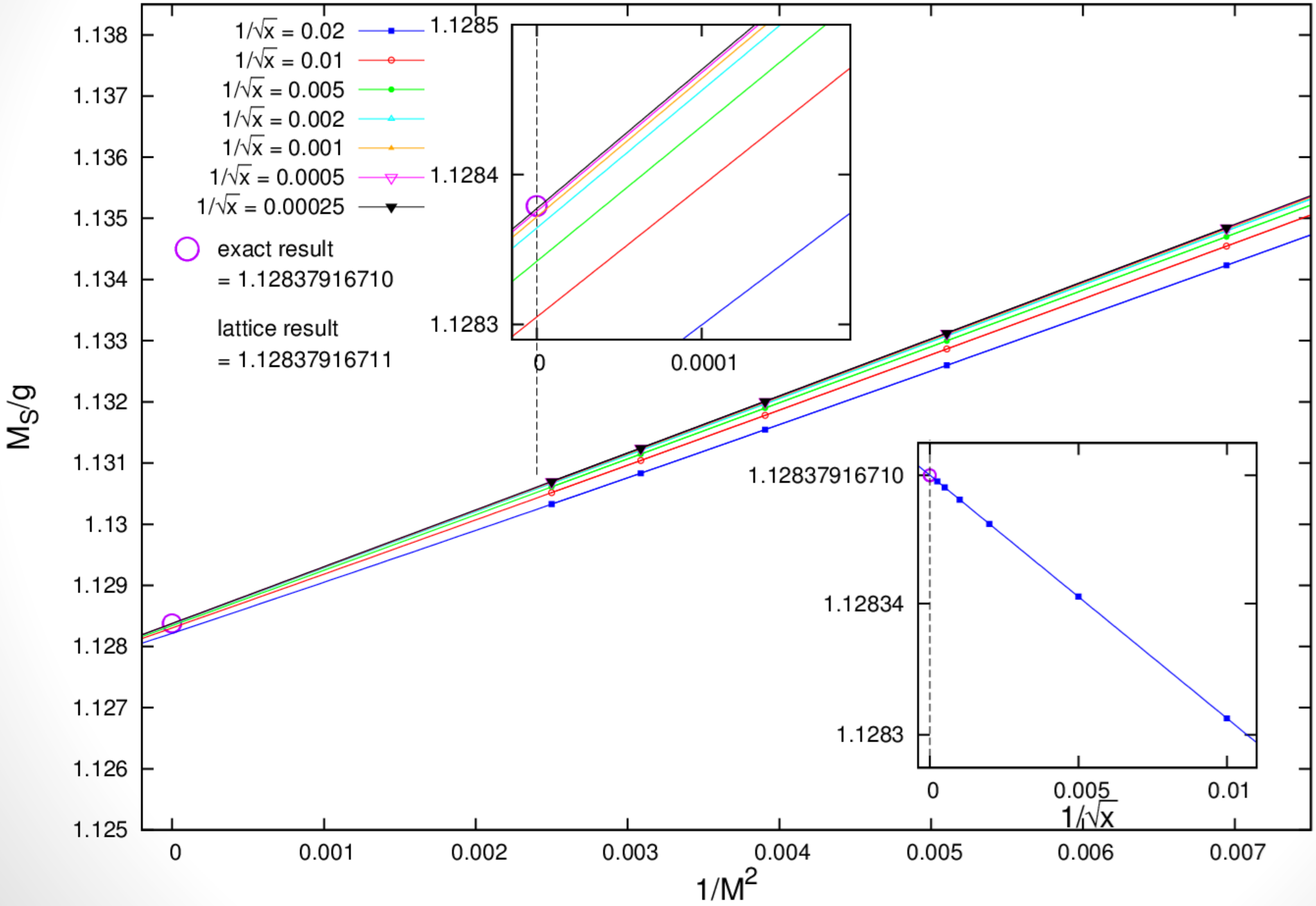
$$F_0(ag) = F_{00}^{cont.} + F_{01}\,(ag) + F_{02}\,(ag)^2$$

$$F\left(\frac{1}{M^2}\right) = F_0(M \to \infty) + \frac{F_2}{M^2} + \frac{F_4}{M^4} + \frac{F_6}{M^6}$$

Scalar mass gap

$$\frac{M_S}{g} = \frac{\omega_1 - \omega_0}{2\sqrt{x}}$$

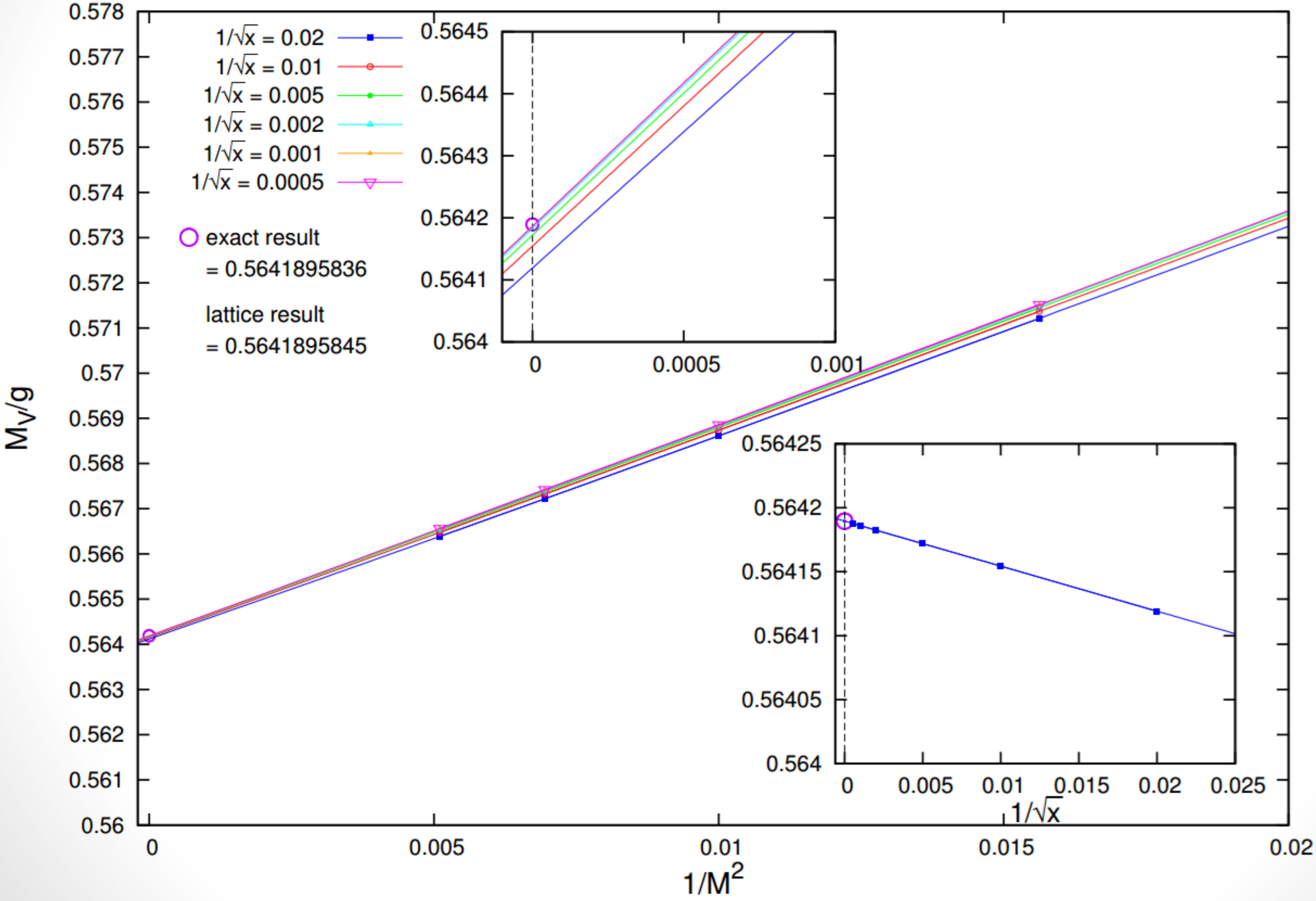
$m = 0$



Vector mass gap

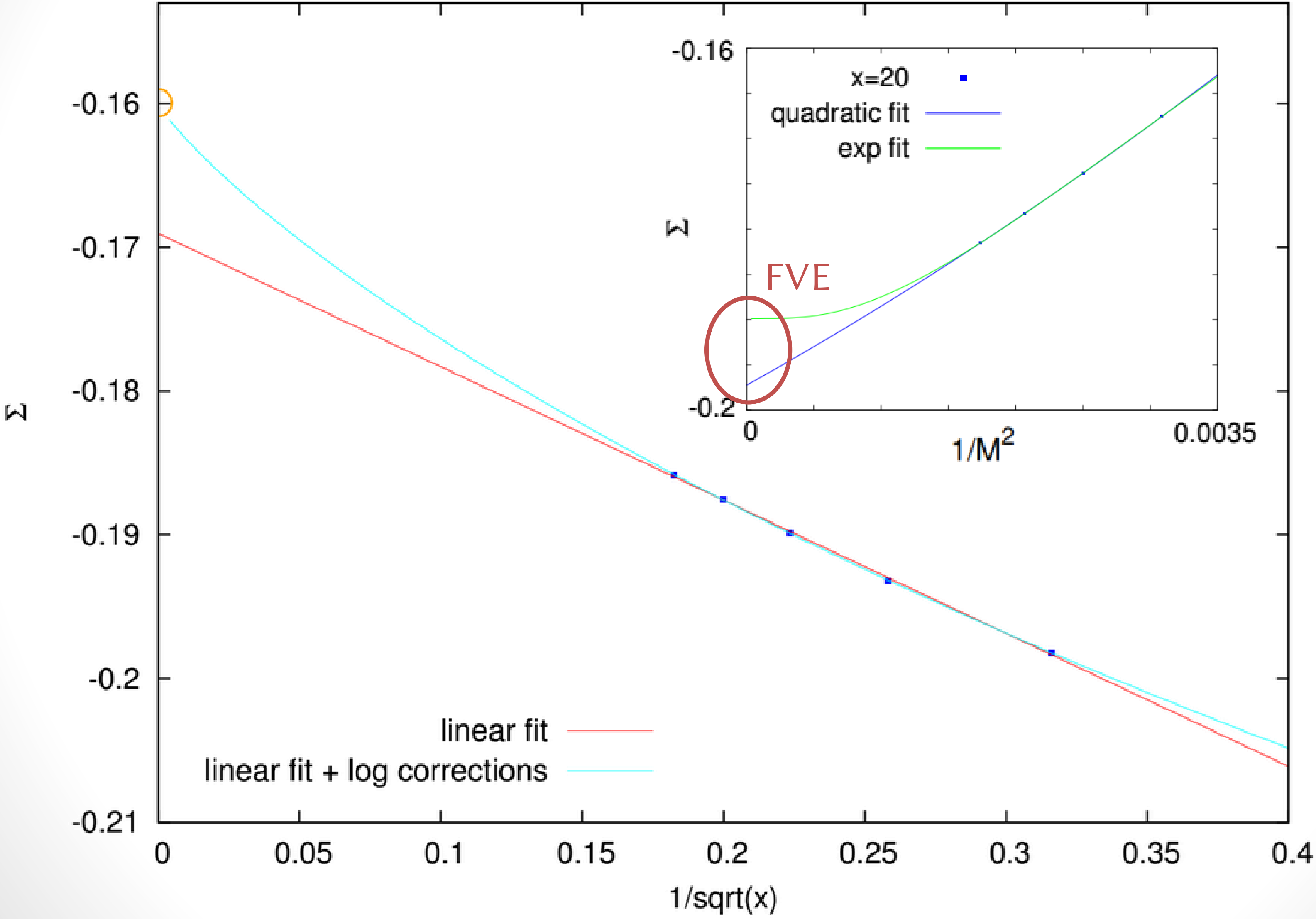
$$\frac{M_V}{g} = \frac{\omega_0^V - \omega_0}{2\sqrt{x}}$$

$m = 0$



Chiral condensate

$m = 0$



Comparison with MPS results

- Ground state energy and mass gaps – massless model:

Observable	SCE+ED	MPS [4]
E_0	-0.3183098860(2)	-0.318338(24)
M_S/g	1.12837916711(1)	1.1279(12)
M_V/g	0.5641895845(9)	0.56421(9)

- Chiral condensate - massless case:

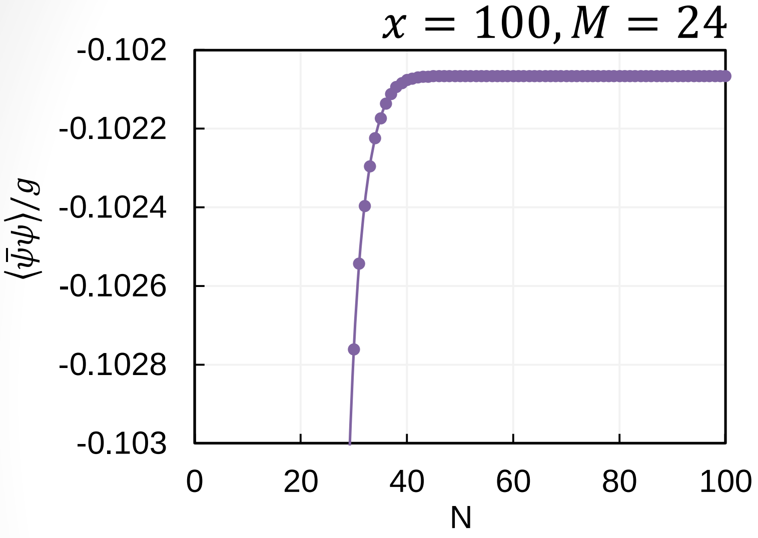
x	SCE+ED	MPS [5]	Difference
20	-0.189878819389204	-0.190252558470094	0.00037
25	-0.187519020840406	-0.187968793405922	0.00045
30	-0.185829589660617	-0.186208219358035	0.00038
cont.	-0.16(1)	-0.159930(8)	

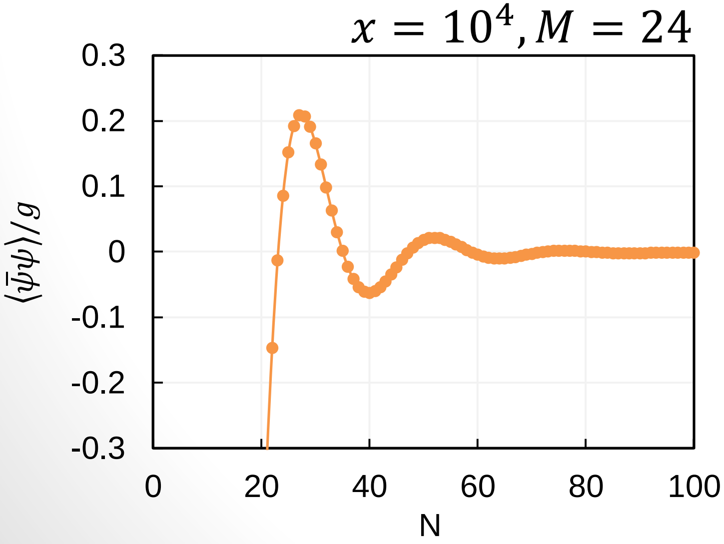
- C.c. - massive $m = 0.125$ (this is after subtracting log divergence [6]):

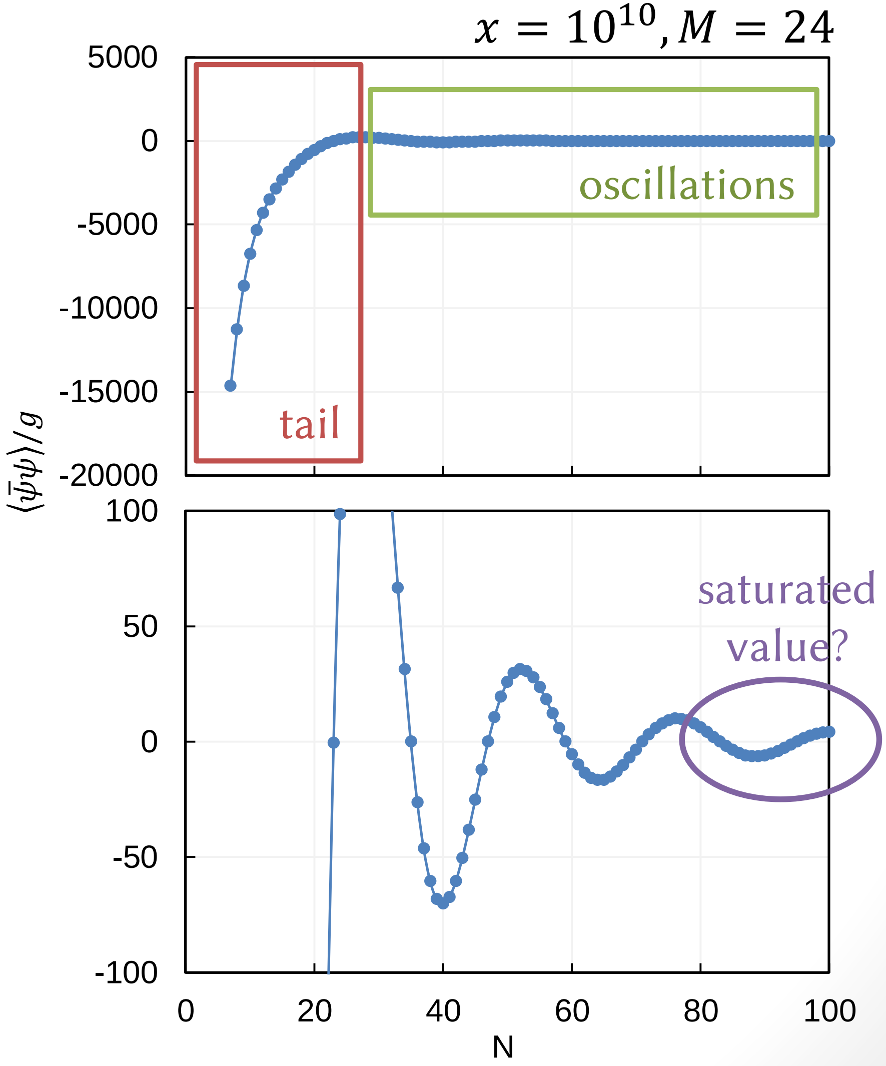
x	SCE+ED	MPS [5]
cont.	-0.091(5)	-0.092023(4)

[4] Bañuls, et al., JHEP 11 (2013) 158..
[5] Bañuls, et al., PoS(Lattice2013)332.
[6] de Forcrand, Nucl.Phys.Proc.Suppl. 63 (1998) 679-681.

Oscillations of chiral condensate while changing the SCE order N


$x = 100, M = 24$


$x = 10^4, M = 24$


$x = 10^{10}, M = 24$

Oscillations: fitting ansatz

- We have chosen the following fitting function:

$$\Sigma(N) = \Sigma(N \to \infty) + a\left(\frac{b}{N^3} + e^{-\alpha N}\right)\sin\left(\frac{2\pi}{T}N + \varphi\right)$$

for small x

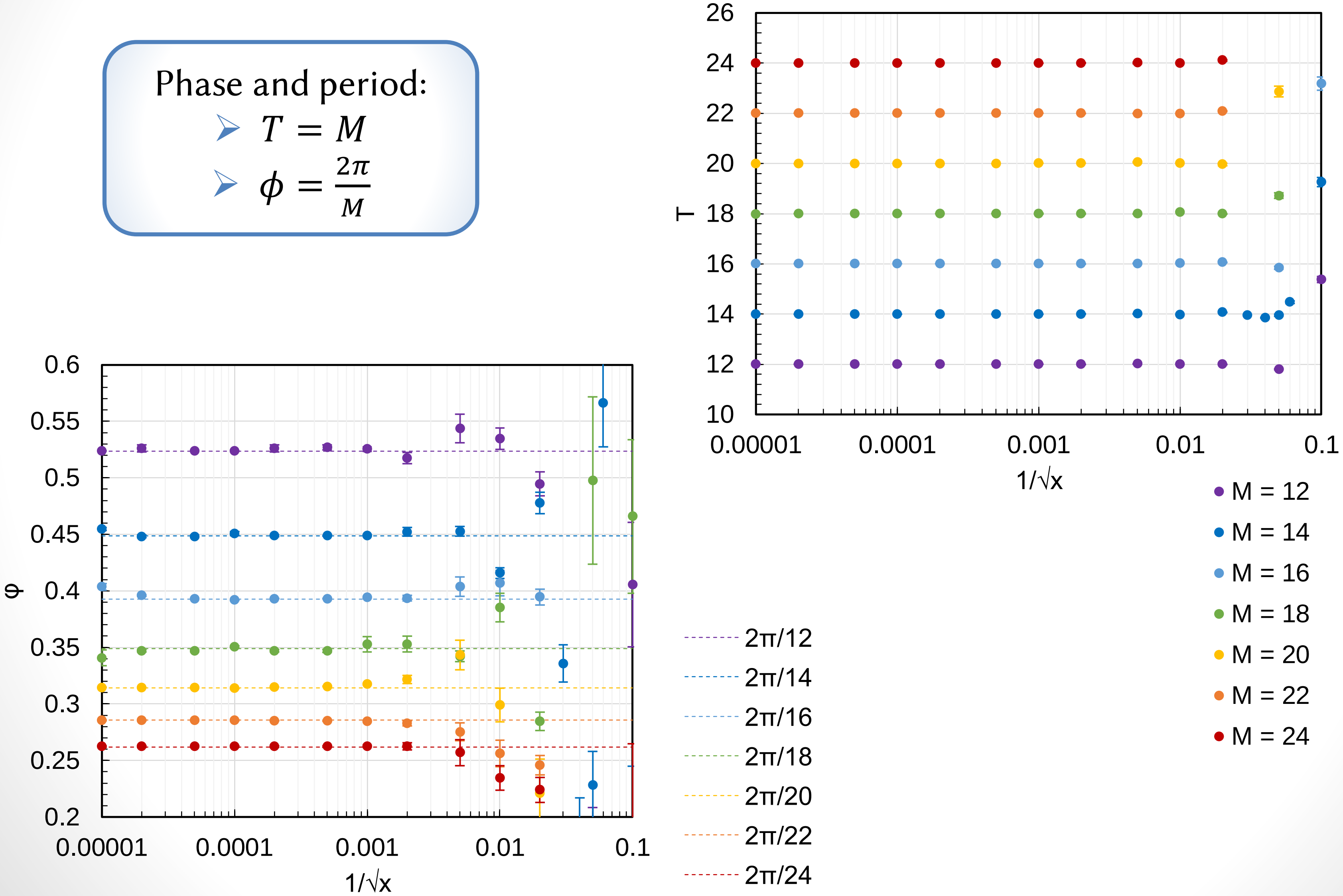
for huge x

- If we can guess the fitting ansatz correctly, we can use small number of points to approximate saturated values, $\Sigma(N \to \infty)$.

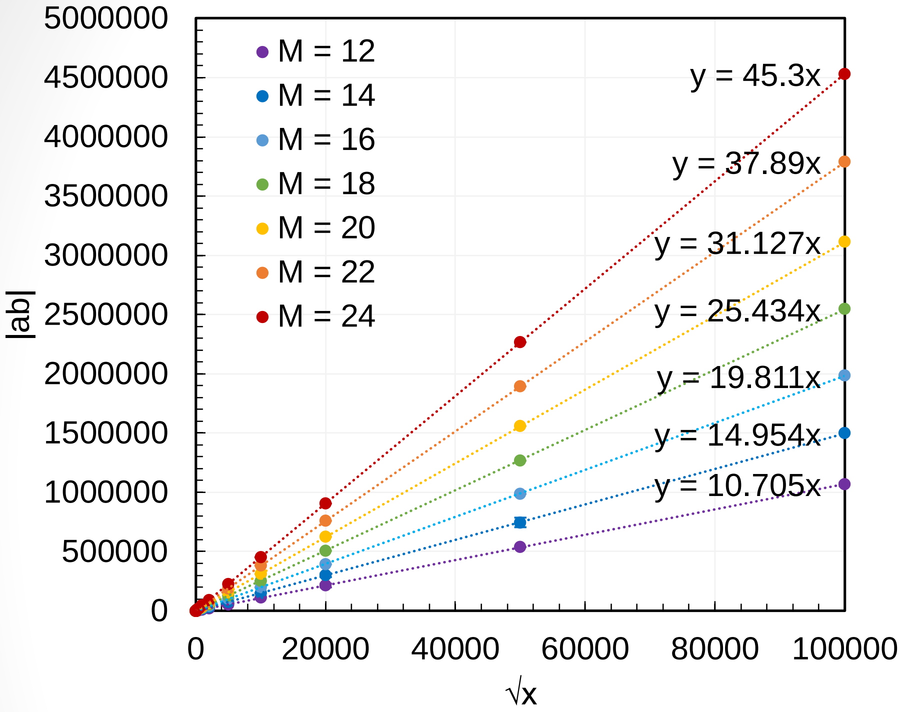
Oscillations: fitting parameters

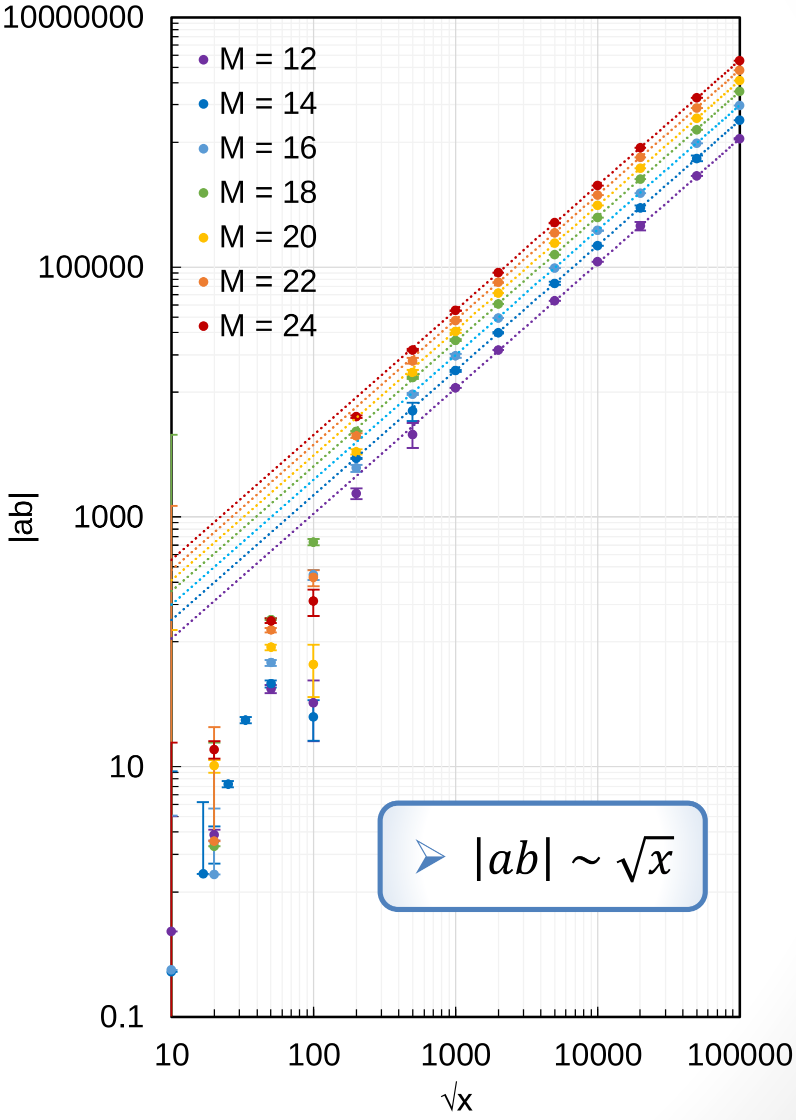
Phase and period:

- $T = M$
- $\phi = \dfrac{2\pi}{M}$

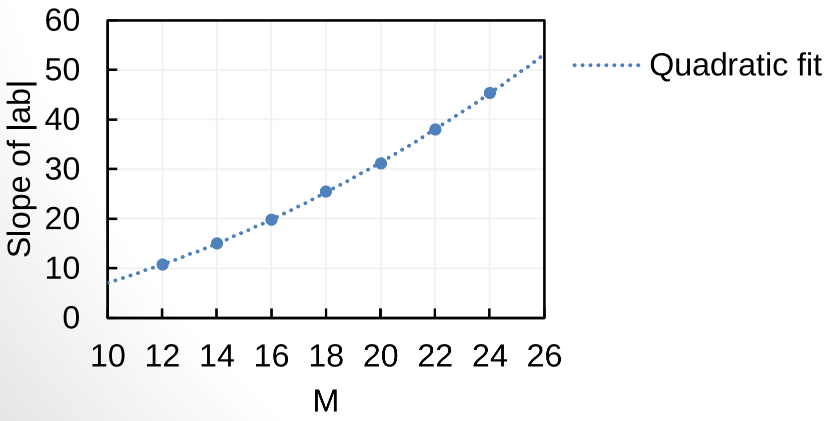


Fitting function: $\Sigma(N) = \Sigma(N \to \infty) + a\left(\frac{b}{N^3} + e^{-\alpha N}\right)\sin\left(\frac{2\pi}{T}N + \varphi\right)$



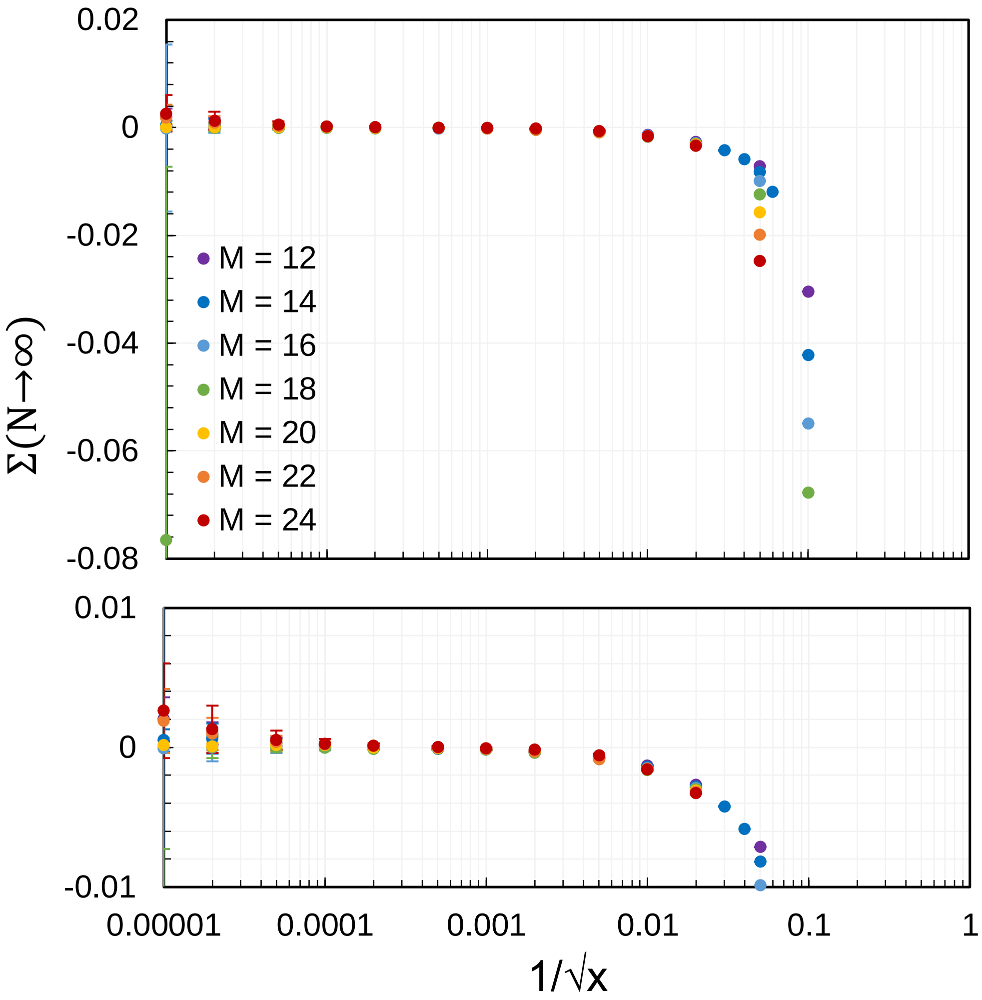


➢ $|ab| \sim \sqrt{x}$



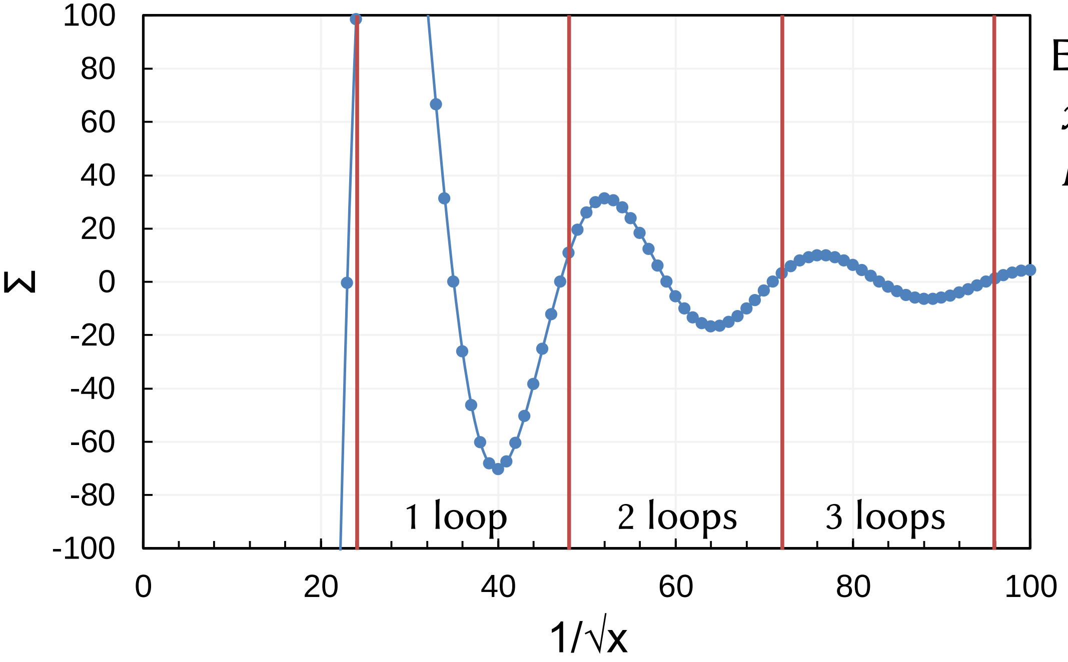
Fitting function: $\Sigma(N) = \Sigma(N \to \infty) + a\left(\frac{b}{N^3} + e^{-\alpha N}\right) \sin\left(\frac{2\pi}{T} N + \varphi\right)$

- High x: errors due to huge oscillations
- $\Sigma(N \to \infty)$ seems to go to zero – because of huge finite volume effects.



Oscillations of the chiral condensate and flux loops

- Every time $N = k\,M$, we reach the next flux loop in the system
- Period must reflect presence of the flux loops



Example:
$x = 10^{10}$
$M = 24$

- Final fitting ansatz:

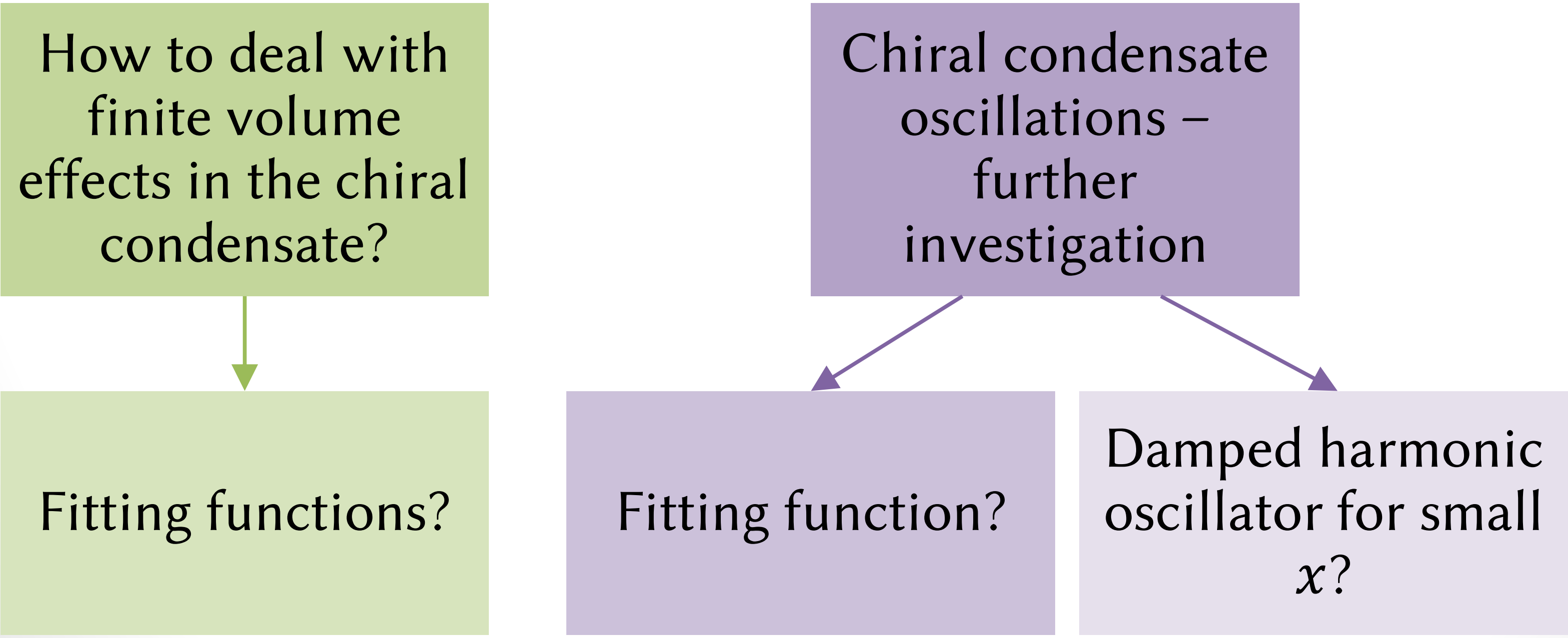
$$\Sigma(N,M,x) = \Sigma(N\to\infty,M,x) + \left(A(M)\frac{\sqrt{x}}{N^3} + B(M,x)e^{-\alpha(M,x)N}\right)\sin\frac{2\pi}{M}(N+1)$$

Summary and outlook

Lattice Hamiltonian results for massless Schwinger model:

- Almost machine precision for GS energy and mass gaps
- Chiral condensate, but we have a problem with FVE

Future:



How to deal with finite volume effects in the chiral condensate?

Fitting functions?

Chiral condensate oscillations – further investigation

Fitting function?

Damped harmonic oscillator for small x?

Thanks for listening!

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