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F -SPLIT AND F -REGULAR VARIETIES WITH A DIAGONALIZABLE GROUP ACTION

PIOTR ACHINGER, NATHAN ILTEN, AND HENDRIK SÜSS

ABSTRACT. Let H be a diagonalizable group over an algebraically closed field k of positive characteristic, and X a normal k -variety with an H -action. Under a mild hypothesis, e.g. H a torus or X quasiprojective, we construct a certain quotient log pair (Y, Δ) and show that X is F -split (F -regular) if and only if the pair (Y, Δ) is F -split (F -regular). We relate splittings of X compatible with H -invariant subvarieties to compatible splittings of (Y, Δ) , as well as discussing diagonal splittings of X . We apply this machinery to analyze the F -splitting and F -regularity of complexity-one T -varieties and toric vector bundles, among other examples.

1. INTRODUCTION

Let k be an algebraically closed field of positive characteristic p . An F -splitting of a k -scheme X is an \mathcal{O}_X -linear map splitting the map $F^* : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ induced by the absolute Frobenius morphism; X is F -split if such a splitting exists. Originally introduced by Mehta and Ramanathan in their study of Schubert varieties [MR85], a scheme being F -split has remarkable consequences, including the vanishing of all higher cohomology groups of any ample line bundle. The slightly stronger notion of (global) F -regularity¹ [HH88, Smi00] (see Definition 3.1) is closely connected to the property of being log-Fano [SS10]. Both notions have been extended to pairs (X, Δ) of a normal variety X and an effective \mathbb{Q} -divisor Δ [SS10].

In this article, we study the F -splitting and F -regularity properties of normal varieties equipped with an effective action by a diagonalizable group. On one end of the spectrum, normal toric varieties are always F -regular [Smi00]. On the other hand, characterizations of F -split and F -regular normal singularities with good \mathbb{G}_m action have been given by Watanabe [Wat91] in terms of their Demazure representations. Moving to the case H finite, any elliptic curve E can be realized as a double cover of \mathbb{P}^1 (as long as $\text{char } k \neq 2$) inducing a μ_2 -action, and there is a classical characterization in terms of this cover when E is F -split (see Example 7.2). Our main result, which we state below, allows us to uniformly treat these three above cases, along with those of many other varieties, including toric vector bundles.

Let X be a normal variety with an effective action by a diagonalizable group H . Let X° be the open subvariety of X consisting of those points with finite stabilizers, and assume that X° admits a geometric quotient $\pi : X^\circ \rightarrow Y$, $Y = X^\circ/H$. This is the case if e.g. H is a torus, or X is quasiprojective. We define an effective \mathbb{Q} -divisor

¹What we call F -regularity is called global F -regularity in [Smi00].

Δ on Y by

$$\sum_{P \subset Y} \frac{\mu(P) - 1}{\mu(P)} P,$$

where $\mu(P)$ is the order of the stabilizer of the generic point of any irreducible component of $\pi^{-1}(P) \subset X^\circ$.

Main Theorem (Theorem 4.1). *Let X be an H -variety as above. Then X is F -split (F -regular) if and only if the pair (Y, Δ) is F -split (F -regular).*

The machinery we develop actually gives a bijection between H -invariant F -splittings of X and F -splittings of (Y, Δ) , as well as giving a partial description of the set of all F -splittings of X in terms of the quotient pair (Y, Δ) (see Remark 4.11). Furthermore, we relate F -splittings of X compatible with H -invariant subvarieties to certain splittings of (Y, Δ) (Propositions 5.1 and 5.2). The main obstruction to applying our main theorem in practice is that the quotient Y is potentially non-separated. To deal with this, we show that (Y, Δ) can be replaced by a pair $(Y^{\text{sep}}, \Delta^{\text{sep}})$ such that Y^{sep} is a variety, and (Y, Δ) is F -split (F -regular) if and only if $(Y^{\text{sep}}, \Delta^{\text{sep}})$ is, see Proposition 6.2 and Theorem 6.11.

The main theorem has a number of applications. We recover that normal toric varieties are F -regular, along with Watanabe's characterization of normal singularities with \mathbb{G}_m action which are F -split or F -regular (Theorem 7.10). Given a torus T , a *complexity-one* T -variety is a T -variety X for which $\dim T = \dim X - 1$; we give an explicit characterization of F -split and F -regular complexity-one T -varieties, see Theorem 7.12. We also are able to give combinatorial criteria for the F -splitting or F -regularity of a large class of toric vector bundles. In particular, we characterize F -split and F -regular rank two vector bundles (Corollary 8.10), recover Xin's result [Xin14] that the cotangent bundle of a smooth toric variety is F -split (Corollary 8.5), and answer a question of Lauritzen by providing an example of an F -split toric vector bundle \mathcal{E} such that \mathcal{E}^* is not F -split (Example 8.11). Further applications include a better understanding of the F -splitting and F -regularity of cyclic covers (§7.1), H -varieties with toroidal affine quotients (§7.2), surjectively graded algebras (§7.5), and Cox rings (§7.6).

We also study *diagonal splittings* of a T -variety X , that is, splittings of $X \times X$ which are compatible with the diagonal. Payne showed that normal toric varieties are not always diagonally split, and gave a combinatorial characterization of those which are [Pay09]. We give a necessary and sufficient criterion for a T -invariant splitting of $X \times X$ to be compatible with the diagonal, generalizing Payne's result to higher complexity T -varieties, see Theorem 9.4. While certainly less explicit than Payne's characterization of diagonally split toric varieties, our criterion can be effectively applied in many instances, particularly for complexity-one T -varieties. We also deduce two easier-to-check necessary criteria for the existence of a diagonal splitting.

The rest of the paper is organized as follows. In §2, we discuss the action of a diagonalizable group on a normal variety, as well as constructing the log pair (Y, Δ) . Preliminaries on the Frobenius morphism are contained in §3. We prove our main result in §4, and discuss invariant compatible splittings in §5. We show how to replace our potentially non-separated quotient Y by a variety in §6. In §7, we consider a number of special cases: cyclic covers, T -varieties with toroidal affine quotients, \mathbb{G}_m actions, complexity-one actions, surjectively graded algebras, and

Cox rings. We dedicate all of §8 to the special case of toric vector bundles. Finally, §9 contains our results on diagonal splittings of T -varieties.

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2. DIAGONALIZABLE GROUP SCHEME ACTIONS

2.1. Preliminaries. We will work over an algebraically closed field k . Let H be a *diagonalizable group scheme* over k , that is, a subgroup scheme of a torus \mathbb{G}_m^r for some $r \geq 0$. For general facts about diagonalizable group schemes, see e.g. [GP11, Exp. I §4.4], [Wat79, §2.2], [Jan03]. Thus H is isomorphic to a product of copies of the multiplicative group \mathbb{G}_m and group schemes of n -th roots of unity $\mu_n = \mathbb{G}_m[n]$. We denote by M the character group $\mathfrak{X}(H) = \text{Hom}_{k\text{-gp.sch.}}(H, \mathbb{G}_m)$ of H . By an H -variety we mean a normal variety² X together with an *effective* action $H \times X \rightarrow X$. We say that H acts *almost freely* if for all $x \in X(k)$, the stabilizer H_x is finite (as a group scheme over k). Note that the set of all points $x \in X$ such that H_x is finite forms an open subvariety X° of X which we call the *almost-free locus*. We will always suppose that the following holds:

- (1) X° admits an open cover by H -invariant affine open subsets.

Condition (1) is not automatically fulfilled (see e.g. [Har77, B.3.4.1]), but it is always satisfied if H is connected [Sum74], or if X is quasi-projective:

Lemma 2.1. *Condition (1) is fulfilled if X , or more generally, X° is quasi-projective.*

Proof. The group scheme H splits as a product $H \cong H_{\text{red}}^0 \times G$, where H_{red}^0 is the reduced connected component of the identity, and G is finite. If X° is quasi-projective, it is well known [BBCM02, Theorem 4.3.1] that there is a good geometric quotient $X^\circ \rightarrow X^\circ/G$, where $Y = X^\circ/G$ is quasi-projective. Furthermore, Y is normal [Sha13, pp. 126]. Since the action of H_{red}^0 on X° commutes with that of G , it descends to an action on Y , and Y has an H_{red}^0 -invariant affine cover by [Sum74]. Pulling this back to X° gives the necessary H -invariant affine cover. \square

Suppose now that the H -action on X is almost free, that is $X = X^\circ$. In this situation, there is a normal (potentially non-separated) scheme $Y = X/H$ which is a geometric quotient of X . We denote by $\pi : X \rightarrow Y$ the quotient map. Let $\mathcal{A} = \pi_* \mathcal{O}_X$, with the associated M -grading $\mathcal{A} = \bigoplus_{u \in M} \mathcal{A}_u$, so that $X = \text{Spec}_Y \mathcal{A}$. Our first goal is to describe the H -variety X , or equivalently the graded algebra \mathcal{A} , in terms of divisors on Y . We treat the case of tori first. Let $\text{CaDiv}_{\mathbb{Q}} Y$ denote the group of \mathbb{Q} -divisors on Y with Cartier multiple.

Proposition 2.2. *Let T be a torus, and let $X = \text{Spec}_Y \mathcal{A}$, be a T -variety with an almost free action with quotient $\pi : X \rightarrow Y$. Then there exists a homomorphism $\mathcal{D} : \mathfrak{X}(T) \rightarrow \text{CaDiv}_{\mathbb{Q}} Y$ and a T -equivariant isomorphism*

$$X \cong \text{Spec}_Y \bigoplus_{v \in \mathfrak{X}(T)} \mathcal{O}_Y([\mathcal{D}(v)]) \cdot \chi^v.$$

²An integral separated scheme of finite type over k .

The above proposition follows almost immediately from Theorem 3.4 of [AH06]. However, the authors of loc. cit. only state and prove this theorem for the case that the ground field has characteristic zero. We believe that their proof applies essentially unchanged in the case of positive characteristic. Instead of verifying all the details here, we present a slightly different argument here for the special case in which we are interested.

Lemma 2.3. *Suppose that a torus T acts freely on a T -variety V . Then V is a Zariski locally trivial T -torsor over V/T .*

Proof. By Luna's étale slice theorem [Lun73], which holds for tori in arbitrary characteristic (cf. [Alp10, Remark 1.1]), there is an étale cover $Y \rightarrow V/T$ such that the pullback of V to Y is a T -torsor. But by étale descent for tori (cf. e.g. [Mil80, III.4]), $V \rightarrow V/T$ must already be a T -torsor in the Zariski topology. \square

Proof of Proposition 2.2. We adapt the proof of Theorem 3.4 of [AH06]. Since $\mathfrak{X}(T)$ is free and the action of T on X is effective (so that each \mathcal{A}_v is non-zero), there exists a (non-unique) homomorphism $\mathfrak{X}(T) \rightarrow k(X)^*$, $v \mapsto f_v$ satisfying $f_v \in k(X)_v$ for all $v \in \mathfrak{X}(T)$.

There exists a finite subgroup scheme $S \subset T$ containing all stabilizer groups T_x . Indeed, by [Sum74], it suffices to show that a linear action of a torus on \mathbb{A}^n only admits finitely many different stabilizer groups, and this is a straightforward calculation. Let $T' = T/S$, leading to an inclusion $\mathfrak{X}(T') \subset \mathfrak{X}(T)$. Set $X' = X/S = \text{Spec}_Y \bigoplus_{u \in \mathfrak{X}(T')} \mathcal{A}_u$. Then T' acts freely on X' , so by Lemma 2.3, X' is a T' -torsor over $X'/T' = X/T = Y$ in the Zariski topology. Equivalently, \mathcal{A}_v is an invertible sheaf for $v \in \mathfrak{X}(T')$ and the multiplication maps $\mathcal{A}_v \otimes \mathcal{A}_{v'} \rightarrow \mathcal{A}_{v+v'}$ are isomorphisms for $v, v' \in \mathfrak{X}(T')$. Thus there exists a unique homomorphism $\mathcal{D} : \mathfrak{X}(T') \rightarrow \text{CaDiv}(Y)$ such that for all $v \in \mathfrak{X}(T')$ the map

$$\mathcal{A}_v \rightarrow k(Y), \quad f \mapsto f f_v^{-1} \in k(Y)$$

identifies \mathcal{A}_v with $\mathcal{O}_Y(\mathcal{D}(v))$. Since $\mathfrak{X}(T)/\mathfrak{X}(T')$ is torsion and \mathbb{Q} is uniquely divisible, there exists a unique extension $\mathcal{D} : \mathfrak{X}(T) \rightarrow \text{CaDiv}_{\mathbb{Q}} Y$. If f is a local section of $\mathcal{O}_Y([\mathcal{D}(v)])$ and $n > 0$ is such that $nv \in \mathfrak{X}(T')$, we have $f^n f_v^n = f^n f_{nv} \in \mathcal{A}_{nv}$, and hence $f f_v \in \mathcal{A}_v$ since \mathcal{A} is normal. Thus, multiplication by f_v defines homomorphisms

$$\beta_v : \mathcal{O}_Y([\mathcal{D}(v)]) \rightarrow \mathcal{A}_v$$

as desired, and it's clear that they are multiplicative.

We check that the induced homomorphism $\beta = \bigoplus \beta_v$ is an isomorphism. This is clearly local on Y , so we can assume that $Y = \text{Spec } A_0$, $X = \text{Spec } A$, $A = \bigoplus_{v \in \mathfrak{X}(T)} \mathcal{A}_v$. Let $B_v = f_v^{-1} \mathcal{A}_v \subseteq k(Y)$ be the fractional ideal corresponding to $\mathcal{O}_Y([\mathcal{D}(v)])$. Then β corresponds to the map

$$\beta : B = \bigoplus_{v \in \mathfrak{X}(T)} B_v \cdot \chi^v \rightarrow \bigoplus_{v \in \mathfrak{X}(T)} \mathcal{A}_v = A.$$

which multiplies elements of B_v by f_v . It suffices to check that B is a normal domain and that β induces an isomorphism on fraction fields. This follows by the last paragraph of the proof of Theorem 3.4 in [AH06]. \square

In general, if H is a subgroup scheme of a torus T , passing from X to $X' = (T \times X)/H$ allows us to reduce questions about H -varieties to questions about T -varieties:

Lemma 2.4. *Let $\phi : H \rightarrow G$ be a homomorphism of diagonalizable group schemes whose cokernel is a torus. In the situation above, consider the $\mathfrak{X}(G)$ -graded \mathcal{O}_Y -algebra*

$$\mathcal{A}' = \bigoplus_{v \in \mathfrak{X}(G)} \mathcal{A}_{\phi^*(v)} \cdot \chi^v = (\mathcal{O}_Y[\mathfrak{X}(G)] \otimes \mathcal{A})^H$$

where $\mathcal{O}_Y[\mathfrak{X}(G)]$ is given the $\mathfrak{X}(H)$ -grading in which χ^v has weight $-\phi^*(v)$. The kernel of the map $f : \mathcal{A}' \rightarrow \mathcal{A}$ identifying $\mathcal{A}'_v = \mathcal{A}_{\phi^*(v)} \cdot \chi^v$ with $\mathcal{A}_{\phi^*(v)}$ is generated by $\chi^v - 1$ for $v \in \ker(\phi^*)$. Let $X' = \text{Spec}_Y \mathcal{A}'$, with the induced H -equivariant map $f : X \rightarrow X'$. Then X' is a G -variety with an almost free action satisfying condition (1), and is identified by construction with the quotient $(G \times X)/H$ where H acts on G via the inverse of ϕ . Moreover, for any two points $x \in X(k)$, $x' \in X'(k)$ with the same image in Y , the G -stabilizer of x' is the image of the H -stabilizer of x under ϕ .

Proof. Self-evident. □

Theorem 2.5. *Let H be a diagonalizable group scheme, and choose an injective homomorphism $\phi : H \rightarrow T$ into a torus T . As before, let $X = \text{Spec}_Y \mathcal{A}$, $\mathcal{A} = \bigoplus_{u \in M} \mathcal{A}_u$ be an H -variety with an almost free action satisfying condition (1), with quotient $\pi : X \rightarrow Y$. Let $X' = (T \times X)/H = \text{Spec}_Y \mathcal{A}'$, $\mathcal{A}' = \bigoplus_{v \in \mathfrak{X}(T)} \mathcal{A}_{\phi^*(v)} \cdot \chi^v$ be as in Lemma 2.4.*

(a) *There exists a homomorphism $\mathcal{D} : \mathfrak{X}(T) \rightarrow \text{CaDiv}_{\mathbb{Q}} Y$ and an H -equivariant isomorphism*

$$X \cong \text{Spec}_Y \left(\bigoplus_{v \in \mathfrak{X}(T)} \mathcal{O}_Y([\mathcal{D}(v)]) \cdot \chi^v \right) / (\chi^v - 1 : v \in \ker(\phi^*)).$$

(b) *Let $s : M \rightarrow \mathfrak{X}(T)$ be a set-theoretic section of ϕ^* , and let $z(u, u') = s(u) + s(u') - s(u + u')$ be the associated 1-cocycle. There exists a 1-cocycle $g : M \times M \rightarrow k(Y)^*$ with $g_{u, u'}$ a section of $\mathcal{O}_Y([\mathcal{D}(z(u, u'))])$ for all $u, u' \in M$, and an H -equivariant isomorphism*

$$X \cong \text{Spec}_Y \bigoplus_{u \in M} \mathcal{O}_Y([\mathcal{D}(s(u))]) \cdot \chi^u,$$

where the multiplication on the right hand side is defined by the formula

$$(f \cdot \chi^u)(f' \cdot \chi^{u'}) = (g_{u, u'}^{-1} f f') \cdot \chi^{u+u'}.$$

(c) *There exists a unique homomorphism $\bar{\mathcal{D}} : M \rightarrow \text{CaDiv}_{\mathbb{Q}/\mathbb{Z}} Y$ making the diagram*

$$\begin{array}{ccc} \mathfrak{X}(T) & \xrightarrow{\mathcal{D}} & \text{CaDiv}_{\mathbb{Q}} Y \\ \phi^* \downarrow & & \downarrow \\ M & \xrightarrow{\bar{\mathcal{D}}} & \text{CaDiv}_{\mathbb{Q}/\mathbb{Z}} Y \end{array}$$

commute. In other words, $\mathcal{D}(\ker(\phi^*)) \subseteq \text{CaDiv}_{\mathbb{Z}} Y$, so in particular $\mathcal{D}(z(u, u'))$ is integral for $u, u' \in M$.

Proof. Apply Proposition 2.2 to X' , taking as S used in the proof a subgroup scheme of H , and use Lemma 2.4 to obtain the first isomorphism. For the second assertion, take the multiplicative system $v \mapsto f_v : \mathfrak{X}(T) \rightarrow k(X')^*$ used in the proof

of Proposition 2.2 and consider it as a homomorphism $\mathfrak{X}(T) \rightarrow k(X)^*$ with f_v semi-invariant of weight $\phi^*(v)$. This makes sense because each f_v is in $k(Y) \cdot \mathcal{A}'_v = k(Y) \cdot \mathcal{A}_{\phi^*(v)}$. Define $g_{u,u'} = f_{z(u,u')}^{-1} = f_{s(u)}^{-1} f_{s(u')}^{-1} f_{s(u+u')}$. Then $f \mapsto f f_{s(u)}^{-1}$ defines a homomorphism as desired, which is an isomorphism on the graded pieces. Finally, $\mathcal{D}(v)$ is integral for $v \in \ker(\phi^*)$ by construction (this is where we use the fact that $S \subseteq H$), which shows the last assertion. \square

Remark 2.6. In [AP12, §3] Altman and Petersen construct finite covers of \mathbb{P}^1 with abelian Galois group using so-called A -divisors. Such an A -divisor is a special instance of the map $\bar{\mathcal{D}}$ from Theorem 2.5 in the case $Y = \mathbb{P}^1$ and H a finite group scheme.

Remark 2.7. The use of $X' = (T \times X)/H$ in order to understand the action of a diagonalizable group H with torsion is reminiscent of the construction of the *Cox sheaf* of a variety Y when $\text{Cl}(Y)$ has torsion; see [IAL15, §1.4] for details.

2.2. Basic setup. In the rest of the article, unless stated otherwise, we fix the following setup. The base field k is algebraically closed of characteristic $p > 0$, H is a diagonalizable group scheme over k with character group M , $\phi : H \rightarrow T$ is an injective homomorphism into a torus T , $s : M \rightarrow \mathfrak{X}(T)$ is a set-theoretic section of ϕ^* , and $z(u, u') = s(u) + s(u') - s(u + u')$. We consider an H -variety X such that the almost-free locus X° satisfies condition (1), and $\pi : X^\circ \rightarrow Y$ is the quotient map. We let $X' = (T \times X^\circ)/H$ as in Lemma 2.4, considered as a T -variety, with quotient map π' . Let $\mathcal{A} = \pi_* \mathcal{O}_{X^\circ} = \bigoplus_{u \in M} \mathcal{A}_u$, $\mathcal{A}' = \pi'_* \mathcal{O}_{X'} = \bigoplus_{v \in \mathfrak{X}(T)} \mathcal{A}_{\phi^*(v)} \cdot \chi^v$. We fix a homomorphism $v \mapsto f_v : \mathfrak{X}(T) \rightarrow k(X)$ with f_v semi-invariant of weight v , and define $g_{u,u'} = f_{z(u,u')}^{-1}$. If H itself is a torus, we can always assume that $H = T$, so that $z = 0$ and $g_{u,u'} = 1$. Theorem 2.5 gives us $\mathcal{D} : \mathfrak{X}(T) \rightarrow \text{CaDiv}_{\mathbb{Q}} Y$, $\bar{\mathcal{D}} : M \rightarrow \text{CaDiv}_{\mathbb{Q}/\mathbb{Z}} Y$, and the isomorphism

$$(2) \quad X^\circ \cong \text{Spec}_Y \bigoplus_{u \in M} \mathcal{O}_Y(\lfloor \mathcal{D}(s(u)) \rfloor) \cdot \chi^u, \quad \chi^u \cdot \chi^{u'} = g_{u,u'}^{-1} \chi^{u+u'}.$$

This representation of X induces an isomorphism of k -algebras

$$(3) \quad k(X)^{\text{s-inv}} \cong \bigoplus_{u \in M} k(Y) \chi^u, \quad \chi^u \cdot \chi^{u'} = g_{u,u'}^{-1} \cdot \chi^{u+u'}$$

Here $k(X)^{\text{s-inv}}$ is the subalgebra of $k(X)$ which is generated by the semi-invariant functions. We write \mathcal{D} and $\bar{\mathcal{D}}$ in the form

$$(4) \quad \mathcal{D}(v) = \sum_P \alpha_P(v) \cdot P, \quad \bar{\mathcal{D}}(u) = \sum_P \bar{\alpha}_P(u) \cdot P$$

where the sums range over all prime divisors P in Y , and homomorphisms $\alpha_P : \mathfrak{X}(T) \rightarrow \mathbb{Q}$, $\bar{\alpha}_P : M \rightarrow \mathbb{Q}/\mathbb{Z}$ with $\bar{\alpha}_P(\phi^*(v)) \equiv \alpha_P(v)$ modulo 1.

For any $\bar{\alpha} : M \rightarrow \mathbb{Q}/\mathbb{Z}$, let $\mu(\bar{\alpha})$ denote the order of $\bar{\alpha}$, i.e., the smallest natural number $n > 0$ such that $n \cdot \bar{\alpha}(u) = 0$ for all $u \in M$. For a prime divisor $P \subseteq Y$, we denote by $\mu(P)$ the order of the stabilizer of a generic point of $\pi^{-1}(P)$. We denote by Δ the \mathbb{Q} -divisor

$$\Delta = \sum_P \frac{\mu(P) - 1}{\mu(P)} \cdot P$$

on Y . We let $B = X \setminus X^\circ$, and for a prime divisor $D \subseteq X$ contained in B , we denote by $\rho_D : M \rightarrow \mathbb{Z}$ the unique homomorphism satisfying $\nu_D(f) = -\rho_D(u)$ if

$f \in k(X)^*$ has weight u (cf. Lemma 2.9 below). We also define a polytope

$$P_X = \{u \in M_{\mathbb{Q}} \mid \rho_D(u) \leq 1\}$$

where D ranges over all prime divisors D contained in B . For $u \in M$ and $n \in \mathbb{Z}$, we will write $u \in n \cdot P_X$ meaning that the image of u in $M_{\mathbb{Q}}$ is in $n \cdot P_X$.

Proposition 2.8 (cf. [AH06, Corollary 7.11]). *In the above situation, let P be a prime divisor on Y . Then the stabilizer of every generic point of the preimage of P has character group $\ker(\bar{\alpha}_P : M \rightarrow \mathbb{Q}/\mathbb{Z})$, and hence is isomorphic to μ_n where $n = \mu(\bar{\alpha}_P)$. In particular, $\mu(P) = \mu(\bar{\alpha}_P)$.*

Proof. We can assume that $H = T$, by replacing X by $X' = (T \times X)/H$, which has the same \mathcal{D} and stabilizers as X . We may shrink Y until it contains no $P' \neq P$ in the support of \mathcal{D} (i.e., $\alpha_{P'} = 0$ for $P' \neq P$). If P itself is not in the support of \mathcal{D} , we see that X is a T -torsor over Y . In any case, we may shrink Y further so that Y and X are affine with coordinate rings A_0 and A , respectively, and P is principal. Choosing a basis of M such that all but one basis element is contained in $\ker(\alpha_P)$, we may reduce to the case $M = \mathbb{Z}$. But then the stabilizer must be of the form μ_n , and by the proof of Proposition 2.2, n is exactly the smallest integer such that $n\alpha_P P$ is a \mathbb{Z} -divisor, that is, $n = \mu(\alpha_P)$. \square

Lemma 2.9. *Let X be an H -variety as above, D any prime divisor not intersecting the almost-free locus X° , and H_{red}^0 the reduced connected component of the identity in H . Then the stabilizer of H_{red}^0 (and hence of H) at the generic point of D is one-dimensional. If*

$$\rho_D \in \mathfrak{X}(H_{\text{red}}^0)^* = \text{Hom}(\mathfrak{X}(H_{\text{red}}^0), \mathbb{Z}) = \text{Hom}(M, \mathbb{Z})$$

is the associated primitive co-character such that the generic point is attractive under the corresponding one-parameter subgroup, then any non-zero rational function f of weight $u \in M$ vanishes to order $-\rho_D(u)$ on D .

Proof. Both claims follow from the proof of [HS10, Proposition 3.2]. \square

Example 2.10 (Blow up of a flag variety $F(1, 1, 1)$). We consider the variety $W = F(1, 1, 1) = SL_3/B$ of complete flags in k^3 . It is well known that W is isomorphic to the hypersurface

$$W = V(x_0 z_0 - x_1 z_1 + x_2 z_2) \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

We denote \mathbb{G}_m^2 by T and obtain a T -action on W given by the weight matrix

$$\begin{array}{cccccc} x_0 & x_1 & x_2 & z_0 & z_1 & z_2 \\ \left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right). \end{array}$$

It is easy to see that the locus W° of finite stabilizers is covered by the two open subsets $U_1 = [x_0 x_1 x_2 \neq 0]$ and $U_2 = [z_0 z_1 z_2 \neq 0]$. In particular, there are no divisors contained in $W \setminus W^\circ$. We have

$$U_1 \cong T \times \underbrace{V(z_0 - z_1 + z_2)}_{\cong \mathbb{P}^1}; \quad U_2 \cong \underbrace{V(x_0 - x_1 + x_2)}_{\cong \mathbb{P}^1} \times T$$

with the canonical T -action on the right-hand-sides. In particular the torus acts with trivial stabilizers on W° . The quotient morphisms are both induced by

$$\pi : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1; \quad (x_0 : x_1 : x_2, z_0 : z_1 : z_2) \mapsto (x_0 z_0 : x_1 z_1).$$

The image of the intersection $U_1 \cap U_2$ under this quotient is $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Hence, $Y = W^\circ/T$ is the projective line with doubled points $0, 1$, and ∞ . Let us choose y_0, y_1 as coordinates for \mathbb{P}^1 . Via the embedding of function fields induced by the dominant morphism π we have $y_0 = x_0 z_0$ and $y_1 = x_1 z_1$. For the structure sheaves of U_1 and U_2 we obtain

$$\mathcal{O}_{U_1} = \mathcal{O}_{\mathbb{P}^1}[(x_1/x_0)^{\pm 1}, (x_2/x_0)^{\pm 1}]$$

and

$$\mathcal{O}_{U_2} = \mathcal{O}_{\mathbb{P}^1}[(z_1/z_0)^{\pm 1}, (z_2/z_0)^{\pm 1}]$$

with generators living in degrees $\pm(0, 1)$ and $\pm(1, 0)$. We have $y_1/y_0 = \frac{x_1 z_1}{x_0 z_0}$ and using the equation $x_0 z_0 - x_1 z_1 + x_2 z_2 = 0$ we obtain $\frac{x_2 z_2}{x_0 z_0} = y_1/y_0 - 1$. This gives

$$\frac{z_1}{z_0} = \frac{y_1}{y_0} \cdot \frac{x_0}{x_1}, \quad \frac{z_2}{z_0} = \left(\frac{y_1}{y_0} - 1 \right) \cdot \frac{x_0}{x_2}.$$

Note, that $\operatorname{div}(y_1/y_0) = [0] - [\infty]$ and $\operatorname{div}(y_1/y_0 - 1) = [1] - [\infty]$. Hence, setting $\mathcal{D}_1(a, b) = 0$ and $\mathcal{D}_2(a, b) = (a + b) \cdot [\infty] - a \cdot [0] - b \cdot [1]$ gives

$$\mathcal{O}_{U_1} = \bigoplus_{u \in \mathbb{Z}^2} \mathcal{O}(\mathcal{D}_1(u)), \quad \mathcal{O}_{U_2} = \bigoplus_{u \in \mathbb{Z}^2} \mathcal{O}(\mathcal{D}_2(u)).$$

Since $\mathcal{D}_1(u)$ and $\mathcal{D}_2(u)$ coincide on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, they induce a divisor $\mathcal{D}(u)$ on the non-separated prevariety Y (which was covered by two instances of \mathbb{P}^1). We obtain

$$W^\circ = U_1 \cup U_2 = \operatorname{Spec}_Y \bigoplus_{u \in \mathbb{Z}^2} \mathcal{O}(\mathcal{D}(u)).$$

Now, consider the one-parameter subgroup $\lambda : \mathbb{G}_m \hookrightarrow T$ acting with weights

$$\begin{pmatrix} x_0 & x_1 & x_2 & z_0 & z_1 & z_2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

The fixed point set of these action consists of two connected components: the lines $(0 : 0 : 1, * : * : 0)$ and $(* : * : 0, 0 : 0 : 1)$, which are in fact both T -invariant. The first one contains sources and the second one contains sinks of the \mathbb{G}_m -action, which is free in a neighborhood of these sets. A local calculation shows that the exceptional divisors of the blowup $\widetilde{W} \rightarrow W$ along these lines consist of λ -fixed points, as well. In particular we have $\widetilde{W}^\circ = W^\circ$ and we then obtain two prime divisors D_+ and D_- in $\widetilde{W} \setminus \widetilde{W}^\circ$. Lemma 2.9 implies that $\rho_{D_+} = (0, 1)$ and $\rho_{D_-} = -(0, 1)$ holds. We obtain $P_W = M_{\mathbb{Q}}$ and $P_{\widetilde{W}} = \{(a, b) \in M_{\mathbb{Q}} \mid -1 \leq b \leq 1\}$. The boundary divisor Δ is trivial in both cases (since $\mathcal{D}(u)$ was integral).

We continue this example and discuss the F -splitting and F -regularity of W and \widetilde{W} in Example 6.13.

Example 2.11 (Cyclic covers). Set $H = \mu_n \subseteq \mathbb{G}_m = T$, and X be an H -variety satisfying (1). In this case, $M = \mathbb{Z}/n\mathbb{Z}$, and we choose the “elementary school arithmetic” section $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}$ with image in $[0, n - 1)$. Then Theorem 2.5 states that

$$X \cong \operatorname{Spec}_Y \bigoplus_{i=0}^{n-1} \mathcal{O}_Y \left(\left[\begin{array}{c} i \\ n \end{array} D \right] \right)$$

for some divisor D on Y , with multiplication of the i -th and j -th graded piece defined by the usual product if $i + j < n$, and using division (“carrying”) by a section g of $\mathcal{O}_Y(D)$ if $i + j \geq n$ (in which case $z(i, j) = n$). This can be seen

in an elementary way if $X = \text{Spec } A$, $A = \bigoplus_{i=0}^{n-1} A_i$ is affine: let $f_1 \in A_1$ be a nonzero element, and let $g = f_1^n \in A_0$. This defines a homomorphism $A_0[t]/(t^n - g) \rightarrow A$ sending t to f_1 , inducing an isomorphism of fraction fields, and hence identifying A with the integral closure of A_0 in $\text{Frac}(A_0)(g^{1/n})$. This also gives us maps $A_i \rightarrow \text{Frac}(A_0)$ sending f to f/f_1^i , and it is easily seen that the image is $\{h \in \text{Frac}(A_0) : n \text{div}(h) + i \text{div}(g) \geq 0\}$. If we define $\mathcal{D}(i) = \frac{i}{n} \cdot \text{div}(g)$, we now get the desired isomorphisms $\tilde{A}_i \cong \mathcal{O}_Y([\mathcal{D}(i)])$. Moreover, we have $g_{i,j} = g$ if $i+j \geq n$, $g_{i,j} = 1$ otherwise.

Suppose that the divisor $D = \text{div}(g)$ is reduced, so that $X = \text{Spec } A[t]/(t^n - g)$. Then $X' = \text{Spec } A[t, q, q^{-1}]/(t^n - gq)$ where t has weight 1 and q has weight n , and the map $A[t, q, q^{-1}]/(t^n - gq) \rightarrow A[t]/(t^n - g)$ sends q to 1. The stabilizer at a point of X' mapping to D is $H = \mu_n$. In particular, if n is divisible by the characteristic of k , this gives an example of a T -variety with a point whose stabilizer is non-reduced.

3. PRELIMINARIES ON FROBENIUS

We fix now a prime p , and assume that our algebraically closed field k has characteristic p . Let X be a k -scheme. By $F_X : X \rightarrow X$ (or simply F) we denote the *absolute Frobenius* of X , that is, the identity map on the underlying topological space and the p -th power map $F^* : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X = \mathcal{O}_X$ on the structure sheaf. This means that $F_*\mathcal{O}_X$ is just \mathcal{O}_X as a sheaf of rings, but has an \mathcal{O}_X -module structure defined by $x * f = x^p f$.

Definition 3.1. Let X be a k -scheme.

- (1) (Mehta–Ramanathan [MR85], see also [BK05, §1.1]) A *Frobenius splitting* (or *F-splitting*) of X is an \mathcal{O}_X -linear map $\sigma : F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ satisfying $\sigma \circ F^* = \text{id}$.
- (2) We say that an *F-splitting* σ is *compatible* with a closed subscheme $Z \subseteq X$ defined by a sheaf of ideals \mathcal{I}_Z if $\sigma(F_*\mathcal{I}_Z) \subseteq \mathcal{I}_Z$.
- (3) (Ramanan–Ramanathan, cf. [BK05, 1.4.1]) Assume X is normal and let D be an effective divisor on X , giving rise to a reflexive sheaf $\mathcal{O}_X(D)$ and a section $s : \mathcal{O}_X \rightarrow \mathcal{O}_X(D)$. We say that a *F-splitting* $\sigma : F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ is a *D-splitting* if it extends along s to a map $F_*(\mathcal{O}_X(D)) \rightarrow \mathcal{O}_X$.
- (4) Assume that X is normal, and let Δ be an effective \mathbb{Q} -divisor on X . By an *F-splitting* of the pair (X, Δ) we mean a *D-splitting* of X , where $D = \lceil (p-1)\Delta \rceil$.
- (5) ([SS10, Definition 3.1]) Assume that X is normal, Δ an effective \mathbb{Q} -divisor on X . We say (X, Δ) is *F-regular*³ if for every effective divisor $D > 0$, there exists some $e > 0$ such that the map $\mathcal{O}_X \rightarrow F_*^e\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D)$ splits as a map of \mathcal{O}_X modules. We say that X is *F-regular* if $(X, 0)$ is *F-regular*.

Note that if a pair (X, Δ) is *F-regular*, then it is automatically *F-split*. The following theorem provides a useful criterion for checking *F-regularity*:

Theorem 3.2 ([SS10, Theorem 3.9]). *The pair (X, Δ) is F-regular if and only if there exists an effective divisor $C > 0$ on X satisfying the following two properties:*

- (1) *There exists an $e > 0$ such that the map*

$$\mathcal{O}_X \rightarrow F_*^e\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + C)$$

splits.

³In [SS10], what we call *F-regularity* is called *global F-regularity*.

(2) The pair $(X \setminus C, \Delta_{X \setminus C})$ is F -regular.

Remark 3.3. Suppose that a normal scheme is D -split for some effective divisor $D = \sum a_P P$. Then $a_P < p$ for all P . In particular, if (X, Δ) is F -split for a \mathbb{Q} -divisor $\Delta = \sum b_P P$, then $b_P \in [0, 1]$ for all P . Indeed, if $D' \leq D$ is an effective divisor and X is D -split, then it is D' -split as well, so the claim is that X cannot be D -split for $D = pP$ with a single prime divisor P . Shrinking X , we can moreover assume that P is Cartier. In this situation, $F_* \mathcal{O}_X(D) = F_* \mathcal{O}_X(pP) = F_*(F^* \mathcal{O}_X(P)) = (F_* \mathcal{O}_X) \otimes \mathcal{O}_X(P)$ by the projection formula. Using this identification, $\mathcal{O}_X \rightarrow F_* \mathcal{O}_X(D)$ is the composition of the canonical section $s_P : \mathcal{O}_X \rightarrow \mathcal{O}_X(P)$ and $F^* \otimes id : \mathcal{O}_X(P) = \mathcal{O}_X \otimes \mathcal{O}_X(P) \rightarrow (F_* \mathcal{O}_X) \otimes \mathcal{O}_X(P) = F_* \mathcal{O}_X(D)$. Thus if $\mathcal{O}_X \rightarrow F_* \mathcal{O}_X(D)$ splits, so does $s_P : \mathcal{O}_X \rightarrow \mathcal{O}_X(P)$, which is impossible.

Lemma 3.4. Let X be an integral normal k -scheme, K its function field, $D = \sum a_P P$ a divisor on X , and $\sigma_K : F_*^e K \rightarrow K$ a K -linear map. Denote by ν_P the valuation of K of X corresponding to a prime divisor P . Then σ_K restricts to a map $F_* \mathcal{O}_X(D) \rightarrow \mathcal{O}_X$ if and only if for all prime divisors P on X

$$\nu_P(f) \geq -a_P + p^e \quad \Rightarrow \quad \nu_P(\sigma_K(f)) \geq 1 \quad \text{for all } f \in K.$$

Proof. As $\mathcal{O}_X(D)$ can be identified with the sheaf of rational functions with poles of order $\leq a_P$ along each prime divisor P , we see that σ_K restricts as desired if and only if $\nu_P(f) \geq -a_P \Rightarrow \nu_P(\sigma_K(f)) \geq 0$. Since $\sigma_K(g^{p^e} f) = g \sigma_K(f)$, substituting fg^{p^e} for f where $\nu_P(g) = -1$ yields the desired result. \square

Remark 3.5. When we calculate examples, it will often be convenient to relate F -splittings to sections of the $(p-1)$ -st power of the anticanonical sheaf. Let X be normal, Δ an effective \mathbb{Q} -divisor, and D any divisor on X . If $U \subseteq X$ is the smooth locus, the relative dualizing sheaf of $F^e : U \rightarrow U$ is $\omega_U \otimes (F^e)^* \omega_U^{-1} = \omega_U^{1-p^e}$. By Grothendieck duality, we have for any $e \geq 0$ an \mathcal{O}_U -linear isomorphism

$$\mathrm{Hom}_{\mathcal{O}_U}(F_*^e \mathcal{O}_U([(p^e - 1)\Delta + D]), \mathcal{O}_U) \cong \mathcal{O}_U([(1 - p^e)(K_U + \Delta) - D]),$$

Using the S_2 -property, we can push this isomorphism forward to X , see e.g. [SS10, Remark 2.5]. Taking global sections, we obtain an identification

$$\mathrm{Hom}_{\mathcal{O}_X}(F_*^e \mathcal{O}_X([(p^e - 1)\Delta + D]), \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X([(1 - p^e)(K_X + \Delta) - D])).$$

Example 3.6. If X is a toric variety defined by a fan Σ , then $-K_X$ can be chosen to be the complement of the open orbit, in which case a basis for its sections is given by monomials χ^{-u} , where u is a lattice point in the polytope

$$P_X = \{u \in M_{\mathbb{Q}} \mid \rho(u) \leq 1 \ \forall \rho \in \Sigma(1)\}.$$

Here, M is the character lattice of the torus acting on X , $\Sigma^{(1)}$ is the set of rays of Σ , and $\rho(u)$ denotes the value of the primitive generator of ρ on u . By the above remark, Laurent polynomials $\sum_{u \in M \cap (p-1)P_X} a_u \chi^u$ correspond to maps $F_* \mathcal{O}_X \rightarrow \mathcal{O}_X$. For such a map to be a splitting, the coefficient of χ^0 must be equal to one; this condition is also sufficient if X is complete [Pay09]. See also Lemma 4.9.

4. TORUS ACTIONS AND FROBENIUS

Consider the setup and notation of §2.2, and assume that H has no p -torsion (see Remark 4.13 below for what we can say without this assumption). Our main result on Frobenius splittings and F -regularity is the following:

Theorem 4.1. *The H -variety X is F -split (F -regular) if and only if (Y, Δ) is F -split (F -regular).*

We start by endowing the sheaves $F_*\mathcal{O}_X$ and $\mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X)$ with an H -equivariant structure. This is rather straight-forward, but can cause some confusion, as we work with the absolute Frobenius morphisms, which are *not* morphisms of k -schemes. To remedy this, one usually introduces the relative (k -linear) Frobenius morphisms $F_{X/k} : X \rightarrow X'$ where $X' = X \otimes_{k, F_k} k$ is the “Frobenius twist” of X . On the other hand, in commutative algebra and in the literature on F -splittings and F -singularities, it is customary to work with the absolute Frobenius morphisms, and indeed it would be annoying to have to keep track of the various twists of everything in sight, especially since we will be interested in iterates of the Frobenius.

Fortunately, in our situation the group $H = \text{Spec } k[M]$ is naturally defined over \mathbb{F}_p (that is, we are given an \mathbb{F}_p -group scheme $H_0 = \text{Spec } \mathbb{F}_p[M]$ and an isomorphism $H \cong H_0 \otimes_{\mathbb{F}_p} k$). We can now view the action of H on X over k as an action of H_0 on X considered as an \mathbb{F}_p -scheme. The Frobenius $F_{H_0} : H_0 \rightarrow H_0$ is simply the multiplication by p map on the group scheme, and induces the multiplication by p map on M . From the point of view of H_0 , an iterate of the absolute Frobenius $F_X^e : X \rightarrow X$ is $F_{H_0}^e$ -equivariant. In particular, the push-forward $F_*^e\mathcal{O}_X$ has a canonical H_0 -equivariant structure, *when we view X as an H_0 -scheme with H_0 acting via $F_{H_0}^e$* . In particular, as $\ker(F_{H_0}^e)$ acts trivially on X in this action, the push-forward decomposes as $F_*^e\mathcal{O}_X = \bigoplus_{u \in M/p^e M} (F_*^e\mathcal{O}_X)_u$.

If $X = \text{Spec } A$ is affine, with $A = \bigoplus_{u \in M} A_u$, then the twisted action corresponds to the grading $A = \bigoplus_{u \in M} A_{u/p^e}$, with the convention that $A_{u/p^e} = 0$ if u is not a multiple of p^e (note the absence of p -torsion in M). The push-forward $F_*^e\mathcal{O}_X$ corresponds to A with the usual grading, and for $u \in M$ the u -graded piece of the graded module $\text{Hom}(F_*^e\mathcal{O}_X, \mathcal{O}_X)$ consists of $\sigma : A \rightarrow A$ satisfying

$$\sigma(f^{p^e}g) = f\sigma(g) \quad \text{and} \quad \sigma(A_{u'}) \subseteq A_{(u'-u)/p^e}.$$

The isomorphism (3) of §2.2 induces for all $u \in M$ homomorphisms of $k(Y)$ -vector spaces:

$$(5) \quad \sigma \mapsto \bar{\sigma} : \text{Hom}_{k(X)}(F_*^e k(X)^{s\text{-inv}}, k(X)^{s\text{-inv}})_u \rightarrow \text{Hom}_{k(Y)}(F_*^e k(Y), k(Y)),$$

defined by $\bar{\sigma}(f) = \sigma(f \cdot \chi^u)_0 \in k(Y)$, the degree 0 part of $\sigma(f \cdot \chi^u)$ with respect to the M grading.

Lemma 4.2. *The homomorphisms (5) are isomorphisms.*

Proof. The inverse map $\bar{\sigma} \mapsto \sigma$ is defined by

$$(6) \quad \sigma(f \cdot \chi^{u'}) = \sigma(f) \cdot \chi^{(u'-u)/p^e},$$

if $u' - u \in p^e M$, and zero otherwise. □

Since we will be dealing with \mathbb{Q} -divisors throughout, we need several lemmas pertaining to rounding their coefficients.

Lemma 4.3. *For every integer $\mu \geq 1$ and every $n \in \mathbb{Z}$, we have*

$$\left\lceil \frac{n+1}{\mu} \right\rceil = \left\lfloor \frac{n}{\mu} \right\rfloor + 1.$$

Proof. Omitted. □

Lemma 4.4. *Let q, b, ν and μ be integers, with $\mu, q \geq 1$. Then*

$$\min \left\{ \left\lfloor \frac{n}{\mu} \right\rfloor \mid n \in \mathbb{Z} \text{ s.t. } \nu \leq \left\lfloor \frac{nq + b}{\mu} \right\rfloor \right\} = \left\lfloor \frac{1}{\mu} \left\lceil \frac{\mu\nu - b}{q} \right\rceil \right\rfloor.$$

Proof. Omitted. □

Lemma 4.5. *Let S be a non-empty set, $\nu : S \rightarrow \mathbb{Z} \cup \{\infty\}$ a function, $T : S \rightarrow S$ a bijection satisfying $\nu(Tx) = \nu(x) + 1$ for all $x \in S$. Let $q \geq 1$ be an integer and let $\tau : S \rightarrow S$ be a function satisfying $\tau(T^q x) = T\tau(x)$. Finally, let $\mu \geq 1$ be an integer and let $b \in \mathbb{Z}$. The following are equivalent:*

- (a) *for every $x \in S$ such that $\nu(x) \geq -\lceil \frac{b}{\mu} + (q-1)\frac{\mu-1}{\mu} \rceil$, we have $\nu(\tau(x)) \geq 0$,*
- (b) *for every $x \in S$ and $n \in \mathbb{Z}$ such that $\nu(x) \geq -\lfloor \frac{nq+b}{\mu} \rfloor$, we have $\nu(\tau(x)) \geq -\lfloor \frac{n}{\mu} \rfloor$.*

Proof. The second condition is equivalent to the requirement that

$$\nu(\tau(x)) \geq \max \left\{ -\left\lfloor \frac{n}{\mu} \right\rfloor \mid n \in \mathbb{Z} \text{ s.t. } \nu(x) \geq -\left\lfloor \frac{nq + b}{\mu} \right\rfloor \right\}$$

for every $x \in S$. By Lemma 4.4 (with $\nu = -\nu(x)$), the above inequality can be rewritten as

$$(7) \quad \nu(\tau(x)) \geq -\left\lfloor \frac{1}{\mu} \left\lceil \frac{\mu(-\nu(x)) - b}{q} \right\rceil \right\rfloor = \left\lfloor \frac{1}{\mu} \left\lfloor \frac{\mu\nu(x) + b}{q} \right\rfloor \right\rfloor.$$

Note that since $\nu(Tx) = \nu(x) + 1$ and $\tau(T^q x) = T\tau(x)$, (7) holds for $x \in S$ if and only if it holds for $T^q x$. By replacing x with $T^{qk}x$ for an appropriate $k \in \mathbb{Z}$ we can thus assume that $\nu(x) \in [\frac{q-b}{\mu} - q, \frac{q-b}{\mu})$, i.e., $q(1-\mu) \leq \mu\nu(x) + b < q$. In this situation, the right hand side of (7) is 0. We conclude that the system of inequalities (7) for all $x \in S$ can be reduced to

$$(8) \quad \nu(x) < \frac{q-b}{\mu} \quad \& \quad \nu(x) \geq \frac{q-b}{\mu} - q \quad \Rightarrow \quad \nu(\tau(x)) \geq 0.$$

We claim that (8) is equivalent to the analogous statement without the first inequality, that is, to

$$(9) \quad \nu(x) \geq \frac{q-b}{\mu} - q \quad \Rightarrow \quad \nu(\tau(x)) \geq 0.$$

Indeed, (9) clearly implies (8). On the other hand, if $x \in S$ satisfies $\nu(x) \geq \frac{q-b}{\mu} - q$, we can write $x = T^{qk}y$ for some $k \geq 0$ and $y \in S$ satisfying $\nu(y) < \frac{q-b}{\mu}$ and $\nu(y) \geq \frac{q-b}{\mu} - q$. Thus by (8), we have $\nu(\tau(y)) \geq 0$, and hence $\nu(\tau(x)) = \nu(T^k\tau(y)) = k + \nu(\tau(y)) \geq 0$.

Finally, by Lemma 4.3 applied to $n = q - b - q\mu - 1$, we have

$$\begin{aligned}
 \nu(x) \geq \frac{q-b}{\mu} - q &\Leftrightarrow \tilde{\nu}(x) \geq \left\lceil \frac{q-b}{\mu} - q \right\rceil \\
 &= \left\lceil \frac{(q-b-q\mu-1)+1}{\mu} \right\rceil \\
 &= \left\lceil \frac{q-b-q\mu-1}{\mu} \right\rceil + 1 \\
 &= \left\lceil -\frac{b}{\mu} - (q-1)\frac{\mu-1}{\mu} \right\rceil \\
 &= -\left\lfloor \frac{b}{\mu} + (q-1)\frac{\mu-1}{\mu} \right\rfloor,
 \end{aligned}$$

which is the inequality in the first condition. \square

For the purpose of the next lemma, let us say that a \mathbb{Q} -divisor $E = \sum c_P \cdot P$ on Y has Δ -denominators if $c_P \cdot \mu(P) \in \mathbb{Z}$ for all prime divisors P on Y . Equivalently, E has Δ -denominators if its pullback \tilde{E} to X° is integral.

Lemma 4.6. *Let E be a \mathbb{Q} -divisor on Y with Δ -denominators. Then the isomorphism (5) induces isomorphisms*

$$\mathrm{Hom}_{\mathcal{O}_{X^\circ}}(F_*^e \mathcal{O}_{X^\circ}(\tilde{E}), \mathcal{O}_{X^\circ})_u \cong \mathrm{Hom}_{\mathcal{O}_Y}(F_*^e \mathcal{O}_Y([(p-1)\Delta + \mathcal{D}(s(u)) + E]), \mathcal{O}_Y).$$

Proof. Since X° and Y are normal, it is enough to check the statement over the localizations of Y at its codimension 1 points. We can thus assume that $Y = \mathrm{Spec} R$ where R is a discrete valuation ring with uniformizer t . We denote the prime divisor $\mathrm{div}(t)$ by P and the valuation on R by ν .

Assume first that $H = T$, that is, H is a torus (in particular, $s = \mathrm{id}$). We have $\mathcal{D}(u) = \alpha(u) \cdot P$ for some $\alpha : M \rightarrow \mathbb{Q}$. Then $\mu := \mu(P)$ is the order of the composition $\bar{\alpha} : M \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$, and

$$X = \mathrm{Spec}_Y \bigoplus_{w \in M} \chi^w t^{-\lfloor \alpha(w) \rfloor} R.$$

We also have $E = \beta \cdot P$ for some $\beta \in \frac{1}{\mu}\mathbb{Z}$. Since E has Δ -denominators, \tilde{E} is the principal divisor associated to $t^{\beta'} \chi^v$ for some $\beta' \in \mathbb{Z}$, $v \in M$. The principal divisor on X° associated to a function χ^v is exactly the pullback of $\mathcal{D}(v)$, so we obtain that

$$\beta = \beta' + \alpha(v).$$

Consider a pair consisting of a $\sigma : F_*^e k(X)^{s^{-\mathrm{inv}}} \rightarrow k(X)^{s^{-\mathrm{inv}}}$ of degree u and the corresponding $\bar{\sigma} : F_*^e k(Y) \rightarrow k(Y)$ under the isomorphism 4.2. Thus,

$$\sigma(f \cdot \chi^w) = \sigma(f) \cdot \chi^{(w-u)/p^e}$$

if $w - u \in p^e M$, 0 otherwise.

Then $\bar{\sigma}$ restricts to an element of $\mathrm{Hom}_{\mathcal{O}_Y}(F_*^e \mathcal{O}_Y([(p^e-1)\Delta + \mathcal{D}(s(u)) + E]), \mathcal{O}_Y)$ if and only if

$$(10) \quad \nu(f) \geq \left\lceil (p^e - 1)\frac{\mu-1}{\mu} + \alpha(u) + \beta \right\rceil \Rightarrow \nu(\bar{\sigma}(f)) \geq 0.$$

On the other hand, σ restricts to an element of $\mathrm{Hom}_{\mathcal{O}_{X^\circ}}(F_*\mathcal{O}_{X^\circ}(\tilde{E}), \mathcal{O}_{X^\circ})_u$ if and only if

$$\sigma(\chi^w t^{-\beta' - \lfloor \alpha(w+v) \rfloor} R) \subset \chi^{(w-u)/p^e} t^{-\lfloor \alpha((w-u)/p^e) \rfloor} R,$$

where we put $\chi^w = 0$ if $w \notin M$. But by (6) this is equivalent to requiring that for every $f \in R$, $w \in M$,

$$(11) \quad \nu(f) \geq -\lfloor \alpha(p^e w + u) + \beta \rfloor \quad \Rightarrow \quad \nu(\bar{\sigma}(f)) \geq -\lfloor \alpha(w) \rfloor.$$

Consider (11) for all $w \in M$ and for a fixed f . Then the quantity $\alpha(w)$ (appearing here twice) traces all possible values $\frac{n}{\mu}$, $n \in \mathbb{Z}$. Thus (11) (for all f and w) equivalent to

$$(12) \quad \nu(\tau(f)) \geq -\left\lfloor p^e \frac{n}{\mu} + \alpha(u) + \beta \right\rfloor \quad \Rightarrow \quad \nu(\bar{\sigma}(f)) \geq -\left\lfloor \frac{n}{\mu} \right\rfloor$$

(for all f and n).

We can now apply Lemma 4.5 to $S = R$, $Tx = tx$, $\tau = \bar{\sigma}$, $q = p^e$ and $b = \mu(\alpha(u) + \beta)$. Then the condition (a) of that lemma is equivalent to (10), and condition (b) is equivalent to (12). Hence the two conditions are equivalent, which concludes the proof in the case $H = T$.

To treat the general case, we first apply the above argument to $X' = (T \times X^\circ)/H$. Note that the isomorphisms (5) for X and X' induce identifications for $v \in \mathfrak{X}(T)$

$$\mathrm{Hom}_{k(X)}(F_*^e k(X)^{s-\mathrm{inv}}, k(X)^{s-\mathrm{inv}})_{\phi^*(v)} \cong \mathrm{Hom}_{k(X')} (F_*^e k(X')^{s-\mathrm{inv}}, k(X')^{s-\mathrm{inv}})_v$$

$$\sigma \mapsto \sigma'$$

which can be explicitly described as $\sigma'(f \cdot \chi^{v'}) = \sigma(f \cdot \chi^{\phi^*(v')})_{\phi((v'-v)/p^e)}$ if $v - v' \in p^e \mathfrak{X}(T)$, 0 otherwise. It is clear from this description that

$$\sigma \in \mathrm{Hom}_{\mathcal{O}_{X^\circ}}(F_*\mathcal{O}_{X^\circ}(\tilde{E}), \mathcal{O}_{X^\circ})$$

if and only if $\sigma' \in \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_{X'}(\tilde{E}'), \mathcal{O}_{X'})$, where \tilde{E}' is the pull-back of E to X' . \square

Remark 4.7. Our proof of Lemma 4.6, while direct, is perhaps not too illuminating. Let us explain why we expected Lemma 4.6 and Theorem 4.1 to be true in the first place. In the case when H is a torus, there is a relation between K_X and K_Y , along with a formula relating sections of their integral multiples which implicitly involves the divisor Δ [AIP⁺12, §8.1 and 8.3]. The relation between sections of $(1-p)K$ and F -splittings (Remark 3.5) then suggests our main theorem. To turn this expectation into a proof, one would need to check that the identifications of [AIP⁺12] are compatible with the Frobenius trace maps. This is the approach taken in [ST10] for the situation of finite covers.

The next goal is to relate F -splittings on X and X° .

Lemma 4.8. *We have*

$$\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) = \bigoplus_{u \in M \cap (p-1)P_X} \mathrm{Hom}_{\mathcal{O}_{X^\circ}}(F_*\mathcal{O}_{X^\circ}, \mathcal{O}_{X^\circ})_u$$

Proof. Consider any non-zero eigensection $\sigma_u \in \mathrm{Hom}_{\mathcal{O}_{X^\circ}}(F_*\mathcal{O}_{X^\circ}, \mathcal{O}_{X^\circ})_u$. The claim is that this section extends to X if and only if $u \in (p-1)P_X$. Now, a semi-invariant $f \in F_*\mathcal{O}_{X^\circ}$ of weight w is regular on a general point of a prime divisor D from above exactly if $-\rho_D(w) \geq 0$. On the one hand, $\sigma_u(f)$ has weight $u + w$,

so is regular if and only if $\sigma_u(f)$ is zero, or $-\frac{1}{p}\rho_D(w+u) \geq 0$. But $\sigma_u(f) = 0$ if $u+w \notin pM$. Hence σ_u extends to X if $u \in (p-1)P_X$.

On the other hand, since $\sigma_u \neq 0$, there locally exists a semi-invariant function f of some weight w such that $\sigma_u(f) \neq 0$. This implies that for any weight $w' \in w+pM$ there locally is a semi-invariant function f' of weight w' with $\sigma_u(f') \neq 0$. We can choose w' such that $0 \leq -\rho_D(w') < p$, in which case we must have $-\frac{1}{p}\rho_D(w'+u) \geq 0$, that is, $\rho_D(u) \leq -\rho_D(w') < p$. \square

Lemma 4.9. *Consider a section*

$$\sigma \in \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

with decomposition $\sigma = \bigoplus_{u \in M} \sigma_u$ into eigensections. If σ is an F -splitting, then so is σ_0 . Conversely, if σ_0 is an F -splitting and $\sigma_u = 0$ for all $u \in pM$, $u \neq 0$, then so is σ . Finally, $\bar{\sigma}_0$ is a splitting if and only if σ_0 is.

Proof. Such a section σ is an F -splitting if and only if $\sigma(1) = 1$. Since 1 is an eigenfunction of weight 0 in both \mathcal{O}_X and $F_*\mathcal{O}_X$, $\sigma(1) = 1$ implies that $\sigma_0(1) = 1$ as well, hence σ_0 is an F -splitting. On the other hand, since $\sigma_u(1)$ has weight u in \mathcal{O}_X , and the weight of any semi-invariant function in \mathcal{O}_X is a multiple of p , we get that $\sigma(1) = \sum_{u \in pM} \sigma_u(1)$ and the second claim follows.

For the final claim, note that $\sigma_0(1) = \bar{\sigma}_0(1) \in k(Y) \subset k(X)$. \square

Lemma 4.10. *If X° is F -regular, then so is X .*

Proof. Suppose that X° is F -regular. Since X is normal, the property of being F -regular is independent of sets of codimension at least two, and we may assume that X is non-singular and $B := X \setminus X^\circ$ is a Cartier divisor. By Theorem 3.2, it suffices to show that the map $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X(B)$ splits.

Since X° is F -regular, it is F -split. Let σ be any splitting, which we may assume to be H -invariant (Lemma 4.9). Hence, σ extends to a splitting $F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ (Lemma 4.8). Working locally on an affine invariant chart, consider any $f \in F_*\mathcal{O}_X(B)$ homogeneous of weight u . We must show that $\sigma(f) \in \mathcal{O}_X$. Now, $f \in F_*\mathcal{O}_X(B)$ implies that $\rho_D(u) \leq 1$ for any component D of B . But then

$$\left\lfloor \frac{1}{p}u(\rho_D) \right\rfloor \leq 0$$

so $\sigma(f)$ must be regular on X , since $\sigma(f)$ has weight u/p , and equals 0 if $u/p \notin M$. Hence, σ gives a splitting of $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X(B)$. \square

Proof of Theorem 4.1. We first deal with the statement concerning F -splitting. By Lemma 4.9, if X has an F -splitting, it has an invariant F -splitting. By Lemma 4.8, X has an invariant F -splitting if and only if X° has an invariant F -splitting. Finally, X° has an invariant F -splitting if and only if (Y, Δ) has an F -splitting by Lemma 4.6 applied in the case $u = 0$, $E = 0$.

We now deal with F -regularity. By Lemma 4.10, we may assume that $X = X^\circ$. Firstly, assume that X is F -regular, and let D be an effective divisor on Y . Then there is a splitting of $\mathcal{O}_X \rightarrow F_*^e\mathcal{O}(\tilde{D})$ which we may assume to be H -invariant (cf. Lemma 4.9) which leads to a splitting of $\mathcal{O}_Y \rightarrow F_*^e\mathcal{O}_Y(\lceil (p^e-1)\Delta \rceil + D)$ by Lemma 4.6. Hence, (Y, Δ) is F -regular.

Conversely, assume that (Y, Δ) is F -regular. Since X and Y are normal, we may remove a set of codimension at least two to arrive at the situation that $Y = U \cup C$

for some effective divisor C and some non-singular affine U over which X is a torsor. Now, since (Y, Δ) is F -regular, the map $\mathcal{O}_Y \rightarrow F_*^e \mathcal{O}_Y(\lceil(p^e - 1)\Delta\rceil + C)$ splits for some e , so by Lemma 4.6 the map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}(\tilde{C})$ splits as well, where \tilde{C} is the preimage of C in X . Furthermore, $X \setminus \tilde{C}$ is affine, and non-singular since it is an H -torsor over U and H is smooth [Mil80, §4]. It follows that $X \setminus \tilde{C}$ is F -regular [SS10, Remark 3.3]. Hence, by Theorem 3.2, X is F -regular. \square

Remark 4.11. Our proof of Theorem 4.1 actually shows that H -invariant F -splittings of X are in bijection with F -splittings of (Y, Δ) . Furthermore, combining Lemmas 4.8 and 4.6 gives a graded isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus_{u \in M \cap (p-1)P_X} \mathrm{Hom}_{\mathcal{O}_Y}(F_* \mathcal{O}_Y(\lceil(p-1)\Delta + \mathcal{D}(u)\rceil), \mathcal{O}_Y).$$

Lemma 4.9 providing a sufficient criterion for a section σ of the right hand side to correspond to a splitting.

Remark 4.12. Consider an effective \mathbb{Q} -Cartier divisor B on Y . Then our machinery actually shows that the pair (X, \tilde{B}) is F -split or F -regular if and only if $(Y, \Delta + B)$ shares the same property. Here, \tilde{B} will denote both the pullback of B to X° , as well as its closure in X .

Indeed, for any divisor B on Y , let $\lceil B \rceil^\mu$ denote the smallest \mathbb{Q} -divisor $B' \geq B$ on Y such that the B' has Δ -denominators. If $B = \sum c_P P$, then

$$\lceil B \rceil^\mu = \sum \frac{\lceil \mu(P) c_P \rceil}{\mu(P)} P.$$

Now, as in the proof of Theorem 4.1, (X, \tilde{B}) is F -split (F -regular) if and only if (X°, \tilde{B}) is, with an H -invariant splitting. But by Lemma 4.6 applied with $E = \lceil(p^e - 1)B\rceil^\mu$, H -invariant maps $F_* \mathcal{O}_{X^\circ}(\lceil(p^e - 1)\tilde{B}\rceil) \rightarrow \mathcal{O}_{X^\circ}$ are in bijection to H -invariant maps $F_* \mathcal{O}_Y(\lceil(p^e - 1)\Delta + \lceil(p^e - 1)B\rceil^\mu) \rightarrow \mathcal{O}_Y$. Since Δ has Δ -denominators,

$$\lceil(p^e - 1)\Delta + \lceil(p^e - 1)B\rceil^\mu = \lceil(p^e - 1)(\Delta + B)\rceil.$$

Applying the remaining arguments in the proof of Theorem 4.1 will show the claim.

Remark 4.13. Many of the statements above continue to hold if we allow H to have p -torsion. Note that in this generality, if $X = \mathrm{Spec} A$, $A = \bigoplus_{u \in M} A_u$, the twisted action (using the e -th Frobenius on H) on X corresponds to the grading

$$A = \bigoplus_{u \in M} A'_u \quad \text{where} \quad A'_u = \bigoplus_{w: p^e w = u} A_u,$$

and $\mathrm{Hom}(F_*^e \mathcal{O}_X, \mathcal{O}_X)_u$ consists of those $\sigma : A \rightarrow A$ which satisfy $\sigma(f^{p^e} g) = f \sigma(g)$ and $\sigma(A_{u'}) \subseteq A'_{u'-u}$, i.e., if $f \in A_u$ then $\sigma(f)_w = 0$ unless $p^e w = u' - u$. In this case, the map (5) is defined as $\bar{\sigma}(f) = \sigma(f \cdot \chi^u)_0$, the degree 0 part of $\sigma(f \cdot \chi^u)$ with respect to the original grading on A (note that A'_0 itself is graded by $M[p^e]$). Lemma 4.2 is still true, with the inverse map $\bar{\sigma} \mapsto \sigma$ given by the more complicated formula

$$(13) \quad \sigma(f \cdot \chi^u) = \sum_{w: p^e w = u' - u} \bar{\sigma}(\lambda_{u,w} f) \cdot \chi^w,$$

where

$$\lambda_{u,w} = \frac{\chi^{u+p^e w}}{\chi^u (\chi^w)^{p^e}} = g_{u,p^e w} g_{(p^e-1)w,w} g_{(p^e-2)w,w} \cdots g_{2w,w} g_{w,w} \in k(Y).$$

Moreover, Lemmas 4.6 and 4.8 continue to hold, as does the first statement of Lemma 4.9. Furthermore, any invariant F -splitting σ of X induces an F -splitting $\bar{\sigma}$ of (Y, Δ) . Hence, X F -split (or F -regular) implies the same for (Y, Δ) . The problem with the other direction in Theorem 4.1 is that $\sigma : F_* \mathcal{O}_X \rightarrow \mathcal{O}_X$ (of weight $u = 0$) does not have to be a splitting if $\bar{\sigma}$ is, as Example 4.14 below shows. In fact, σ is a splitting if and only if $\bar{\sigma}$ is a splitting satisfying $\bar{\sigma}((\chi^u)^{p^e}) = 0$ for every $u \in M[p^e]$. In more intrinsic terms, this condition is equivalent to $\bar{\sigma}(f) = 0$ for every $f \in k(Y)$ which is not a p -th power but which becomes a p -th power in $k(X)$. We do not know if Theorem 4.1 still holds if H has p -torsion.

Example 4.14. Let $H = \mu_p$, $X = \mathbb{A}_k^1$ with coordinate x and the standard μ_p -action. Then $Y = \mathbb{A}_k^1$ with coordinate y , and $\pi^* y = x^p$. In this case, for $u = 0$ and $f = 1$, the formula (13) simplifies to

$$\sigma(1) = \sum_{i=0}^{p-1} \bar{\sigma}(y^i) x^i.$$

In particular, σ is a splitting if and only if $\bar{\sigma}(y^i) = 0$ for $0 < i < p$, $\bar{\sigma}(1) = 1$, that is, if $\bar{\sigma}$ is \mathbb{G}_m -invariant for the standard action of \mathbb{G}_m on Y .

5. COMPATIBLE SPLITTINGS

We again consider an H -variety X , and use notation established in §2.2, assuming again that H has no p -torsion. We now establish two results concerning compatible splittings. Recall that H_{red}^0 is the reduced connected component of the identity in H .

Proposition 5.1. *Consider an H_{red}^0 -invariant splitting $\sigma \in \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$. Then σ is compatible with $B := X \setminus X^\circ$, that is, $\sigma(F_* \mathcal{I}_B) = \mathcal{I}_B$. In particular, any H -invariant splitting is compatible with B .*

Proof. To begin, assume that $H = H_{\text{red}}^0$. Without loss of generality, X is affine with coordinate ring $A = \bigoplus_{u \in M} A_u$. Let ω be the cone in $M_{\mathbb{Q}}$ generated by those $u \in M$ with $A_u \neq 0$, and ω' the face of elements invertible in the monoid ω . Then the ideal I_B of B is given by $\bigoplus_{u \in M \cap (\omega \setminus \omega')} A_u$. Since σ is H -invariant, it maps homogeneous elements of degree u to degree u/p , so $\sigma(F_*(I_B)) = I_B$.

Now to conclude the proof note that for general H , any H -invariant splitting is also H_{red}^0 -invariant. \square

Proposition 5.2. *Now let S be any closed subscheme of Y and*

$$\sigma \in \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$$

an F -splitting of X .

- (1) *If the splitting σ is compatible with $\overline{\pi^{-1}(S)} \subset X$, then its H -invariant part σ_0 is also compatible with $\overline{\pi^{-1}(S)} \subset X$.*
- (2) *Suppose the H -invariant part σ_0 is compatible with $\overline{\pi^{-1}(S)} \subset X$. Then $\bar{\sigma}_0 \in \text{Hom}_{\mathcal{O}_Y}(F_* \mathcal{O}_Y, \mathcal{O}_Y)$ is compatible with S .*

- (3) *Conversely, suppose that no component of S is contained in the support of Δ and $\bar{\sigma}_0$ is compatible with $S \subset Y$. Then σ_0 is compatible with $\overline{\pi^{-1}(S)}$.*

Proof. By [BK05, Lemma 1.1.7] we may assume that $X = X^\circ$ and that X and Y are affine with respective coordinate rings A_0 and $A = \bigoplus_{u \in M} A_u$. Let $I_S \subset A_0$ be the ideal of S ; then the ideal of $\pi^{-1}(S) = \overline{\pi^{-1}(S)}$ is $A \cdot I_S = \bigoplus_{u \in M} A_u \cdot I_S$. Let $\sigma = \sum_{u \in M} \sigma_u$ be the isotypical decomposition of σ .

First, assume that σ is compatible with $\overline{\pi^{-1}(S)}$. Consider any $f \in F_*(A \cdot I_S)$, without loss of generality homogeneous of degree w . Then $\sigma(f) \in A \cdot I_S$, and so we have that $\sigma_u(f) \in A_{(w-u)/p} \cdot I_S$. In particular, $\sigma_0(f) \in A_{w/p} \cdot I_S \subset A \cdot I_S$, so σ_0 is compatible with $\overline{\pi^{-1}(S)}$.

Now if σ_0 is compatible with $\overline{\pi^{-1}(S)}$, then for any degree zero element $f \in F_*(A_0 \cdot I_S) = F_*(I_S)$, we have $\sigma_0(f) \in I_S$, so $\bar{\sigma}_0 \in \text{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y)$ is compatible with S .

On the other hand, suppose that $\bar{\sigma}_0(I_S) = I_S$, and that S is contained in the support of Δ . Then again by [BK05, Lemma 1.1.7], we may shrink Y and only consider the case that \mathcal{D} is trivial, that is, $A = \bigoplus_{u \in M} A_0 \cdot \chi^u$. But then for $f \in F_*(I_S \cdot A_0 \cdot \chi^u)$, $\sigma_0(f) \in I_S \cdot A_0 \cdot \chi^{u/p}$ as desired, where $\chi^{u/p} = 0$ if $u/p \notin M$. \square

6. SEPARATIONS

We have been able to characterize F -regularity and the existence of an F -splitting for an H -variety X in terms of the quotient pair (Y, Δ) in Theorem 4.1. However, the quotient Y need not in general be separated. We now describe how to replace the pair (Y, Δ) with a pair $(Y^{\text{sep}}, \Delta^{\text{sep}})$ such that Y^{sep} is separated. Recall that an open subscheme $U \subset Y$ is *big* if $\text{codim}_Y(Y \setminus U) > 1$.

Definition 6.1. A *separation* of a k -scheme Y is rational map $\mathfrak{s} : Y \dashrightarrow Y^{\text{sep}}$, such that

- (1) Y^{sep} is separated.
- (2) The map \mathfrak{s} is defined on a big open subset $U \subset Y$ which maps locally isomorphically to a big open subset of Y^{sep} .

Recall that a *prevariety* is an integral scheme of finite type over k . We will use the following proposition to replace our quotient $Y = X^\circ/T$ with a variety.

Proposition 6.2. *Every normal prevariety admits a separation.*

Remark 6.3. In [HS10], separated quotients of T -varieties are produced by considering the inverse limit of GIT-quotients. In this setting, the image of the quotient map into the GIT-limit gives a separation of X°/T and the distinguished component of the limit which contains the image coincides with the Chow-quotient introduced in [AH06].

To prove the proposition, we need several facts about centers of valuations.

Definition 6.4. Consider a valuation ν of $k(Y)$. A center of ν is an irreducible closed subset $C \subset Y$ such that $\mathcal{O}_{C,Y} \subseteq \mathcal{O}_\nu$ and $\mathcal{O}_{C,Y} \hookrightarrow \mathcal{O}_\nu$ is a local ring homomorphism.

Lemma 6.5. *A prevariety Y is separated if and only if every valuation of $k(Y)$ has at most one center.*

Proof. See [Har77, Theorem 4.3] □

Lemma 6.6. *Consider dominant morphism $\phi : Y \rightarrow Y'$ from a prevariety Y to a prevariety Y' and a valuation ν of $k(Y)$ with center $C \subset Y$. Then $C' := \overline{\phi(C)}$ is a center of $\nu|_{k(Y')}$.*

Proof. Note that we have a local ring homomorphism $\mathcal{O}_{Y',C'} \hookrightarrow \mathcal{O}_{Y,C}$ induced by ϕ and a local ring homomorphism $\mathcal{O}_{Y,C} \hookrightarrow \mathcal{O}_\nu$ by the definition of a center. Hence, the composition is a local ring homomorphism as well. □

Definition 6.7. A *multiple center* of a prevariety Y is a closed subset $C \subset Y$ which is the center of some valuation ν , such that ν has more than one center. We define the *non-separated locus* of a prevariety Y to be the union of all multiple centers.

Lemma 6.8. *The locus of non-separateness of a prevariety Y is a Zariski closed subset.*

Proof. We consider some open affine covering $\{U_i\}_{i \in I}$ of Y and denote $U_i \cap U_j$ by U_{ij} . We set $A_i = \mathcal{O}_Y(U_i)$ and denote the sub-algebra of the function field generated by A_i and A_j by $A_i A_j$ and its spectrum by \tilde{U}_{ij} . We have a birational map $\phi_{ij} : \tilde{U}_{ij} \dashrightarrow U_{ij}$ and a commutative diagram as follows

$$\begin{array}{ccc}
 & \tilde{U}_{ij} & \\
 f_{ji} \swarrow & \vdots & \searrow f_{ij} \\
 U_i & & U_j \\
 \nwarrow & \phi_{ij} & \nearrow \\
 & U_{ij} &
 \end{array}$$

Now, we denote the indeterminacy locus of ϕ_{ij} by V_{ij} . We claim that

$$\bigcup_{i,j \in I} \overline{f_{ij}(V_{ij})} \subset Y$$

equals the locus of non-separateness.

Assume we have a point y in this finite union. This means there is a pair (i, j) and component of the C of V_{ij} such that $y \in \overline{f_{ij}(C)}$. Now, we may choose a valuation ν , which has center C . This implies that $f_{ij}(C)$ and $f_{ji}(C)$ are centers of ν , as well. Since C lies in the locus of indeterminacy, $f_{ij}(C)$ and $f_{ji}(C)$ do not intersect U_{ij} . Hence, $\overline{f_{ij}(C)} \neq \overline{f_{ji}(C)}$. Hence, y lies in the locus of non-separateness.

Assume instead that we have a point y in the non-separated locus. This means it belongs to some multiple center V of some valuation ν . Hence, we have another center V' of the same valuation. They cannot both intersect the same affine chart, since affine varieties are separated. Hence, we have two charts U_i and U_j , such that $U_i \cap V = U_j \cap V = \emptyset$ but $U_i \cap V \neq \emptyset$ and $U_j \cap V' \neq \emptyset$. In particular ν has no center on the intersection U_{ij} . The fact that ν has a center on U_i and U_j is equivalent to the inclusions of the coordinate ring $A_i, A_j \subset \mathcal{O}_\nu$. But then we have $A_i A_j \subset \mathcal{O}_\nu$ as well. Hence, ν has a center C on \tilde{U}_{ij} and we have $\overline{f_{ij}(C)} = V$ and $\overline{f_{ij}(C)} = V'$, by Lemma 6.6. Since ν has no center on U_{ij} , it follows that C is contained in the indeterminacy locus V_{ij} of ϕ_{ij} . Hence, V and V' are contained in $\overline{f_{ij}(V_{ij})}$ and $\overline{f_{ji}(V_{ij})}$ respectively. □

Proof of Proposition 6.2. Consider the non-separated locus inside the prevariety Y . From the components of the non-separated locus of codimension 1, several components have the same local ring. For every one of the local rings occurring, choose one of these components and remove the rest. The remaining prevariety Y' is “separated in codimension one”, i.e. the non-separated locus V of Y' has codimension > 1 . If we remove this locus from Y' we obtain a variety $Y^{\text{sep}} = Y' \setminus V$. Now, the rational map \mathfrak{s} is just the inverse of the inclusion $Y^{\text{sep}} \hookrightarrow Y$. Let D be a prime divisor in Y . Then by construction, there is a prime divisor $D' \subset Y^{\text{sep}}$ with the same local ring as D . In other words, for every prime divisor in Y there is an open subset intersecting D which is mapped isomorphically to an open subset of Y^{sep} . \square

Remark 6.9. If Y is a smooth prevariety of dimension one, then it admits a unique separation $\mathfrak{s} : Y \dashrightarrow Y^{\text{sep}}$, and \mathfrak{s} is a morphism.

Consider a separation $\mathfrak{s} : Y \dashrightarrow Y^{\text{sep}}$. For any \mathbb{Q} -divisor $D = \sum_{P \subset Y} a_P \cdot P$ on Y , we define

$$\mathfrak{s}_{\max} D := \sum_{P' \subset Y^{\text{sep}}} \max\{a_P \mid P \subset \mathfrak{s}^{-1}(P')\} \cdot P'.$$

With this we have $\mathfrak{s}_* \mathcal{O}(-D) = \mathcal{O}(-\mathfrak{s}_{\max} D)$.

A separation of a pair (Y, Δ) consists of a separation $\mathfrak{s} : Y \dashrightarrow Y^{\text{sep}}$ along with the \mathbb{Q} -divisor $\Delta^{\text{sep}} := \mathfrak{s}_{\max} \Delta$ on Y^{sep} . By Proposition 6.2, such a separation always exists, although it is not necessarily unique.

Remark 6.10. Note that Δ^{sep} is the unique minimal divisor on Y^{sep} such that $\mathfrak{s}^*(\Delta^{\text{sep}}) \geq \Delta$.

We may use the following result, coupled with Theorem 4.1 and Proposition 5.2 to characterize (compatible) F -splittings and F -regularity of a T -variety in terms of properties of a separated quotient.

Theorem 6.11. *Consider a normal pair (Y, Δ) and a separation $\mathfrak{s} : (Y, \Delta) \dashrightarrow (Y^{\text{sep}}, \Delta^{\text{sep}})$.*

- (1) *The pair (Y, Δ) is F -split (F -regular) if and only if $(Y^{\text{sep}}, \Delta^{\text{sep}})$ is F -split (F -regular).*
- (2) *Let $S \subset Y$ be a closed subscheme such that $\overline{U \cap S} = S$, where U is the open subset of Y on which \mathfrak{s} is regular. Then (Y, Δ) is F -split is compatible with S if and only if $(Y^{\text{sep}}, \Delta^{\text{sep}})$ is F -split compatible with $\overline{\mathfrak{s}(S)}$.*

Proof. Due to our normality assumption, we may without loss of generality assume that \mathfrak{s} is regular on all of Y with image Y^{sep} . Now, since \mathfrak{s} is a local isomorphism between the pairs $(Y^{\text{sep}}, \Delta^{\text{sep}})$ and $(Y, \mathfrak{s}^* \Delta^{\text{sep}})$, we have an isomorphism between

$$\text{Hom}_{\mathcal{O}_{Y^{\text{sep}}}}(F_*^e \mathcal{O}_{Y^{\text{sep}}}([(p^e - 1)\Delta^{\text{sep}}] + D), \mathcal{O}_{Y^{\text{sep}}})$$

and

$$(14) \quad \text{Hom}_{\mathcal{O}_Y}(F_*^e \mathcal{O}_Y([(p^e - 1)\mathfrak{s}^*(\Delta^{\text{sep}})] + \mathfrak{s}^*(D)), \mathcal{O}_Y)$$

for any effective divisor D on Y which preserves the property of being a splitting. Furthermore, (14) is equal to

$$\text{Hom}_{\mathcal{O}_Y}(F_*^e \mathcal{O}_Y([(p^e - 1)\Delta] + \mathfrak{s}^*(D)), \mathcal{O}_Y)$$

by definition of Δ^{sep} . This proves claim 1 with regards to F -splitting. For F -regularity, note that $\mathfrak{s}^*(\mathfrak{s}_*(C)) \geq C$ for any divisor C on Y , and the claim follows. Claim 2 follows immediately from the above isomorphism and [BK05, Lemma 1.1.7]. \square

We are now going to reformulate the description of splittings of X° (Lemma 4.6) in terms of Y^{sep} . Remember from §2.2 that X° has a description as

$$X^\circ \cong \text{Spec}_Y \bigoplus_{u \in M} \mathcal{O}([\mathcal{D}(s(u))]) \cdot \chi^u$$

where $\mathcal{D} : \mathfrak{X}(T) \rightarrow \text{CaDiv}_{\mathbb{Q}} Y$. We define $h : M \rightarrow \text{CaDiv}_{\mathbb{Q}} Y^{\text{sep}}$ by $h(u) = \mathfrak{s}_{\max}((p-1)\Delta + \mathcal{D}(s(u)))$. If H is a torus, i.e. M is torsion-free and s is the identity, we may view h as a convex and piecewise linear function on $M_{\mathbb{Q}}$.

Lemma 6.12. *For $u \in (p-1)P_X$ we have*

$$\text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_u \cong \text{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_{Y^{\text{sep}}}(\lceil h(u) \rceil), \mathcal{O}_{Y^{\text{sep}}}).$$

Proof. As in the proof of Theorem 6.11 we have

$$\text{Hom}_{\mathcal{O}_{Y^{\text{sep}}}}(F_*\mathcal{O}_{Y^{\text{sep}}}(\mathfrak{s}_{\max}(\lceil (p-1)\Delta \rceil + \mathcal{D}(s(u)))), \mathcal{O}_{Y^{\text{sep}}})$$

being equal to

$$\text{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y(\lceil (p-1)\Delta \rceil + \mathcal{D}(s(u))), \mathcal{O}_Y). \quad \square$$

Example 6.13 (Blowup of a flag variety (continued)). For $W^\circ = \widetilde{W}^\circ$ from Example 2.10 we had as a non-separated quotient the projective line with doubled points $\{0, 1, \infty\}$. The separation is just the ordinary \mathbb{P}^1 and the morphism $W^\circ \rightarrow Y \rightarrow Y^{\text{sep}} = \mathbb{P}^1$ is again given by

$$\pi : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1; \quad (x_0 : x_1 : x_2, z_0 : z_1 : z_2) \mapsto (x_0 z_0 : x_1 z_1).$$

The piecewise linear function $h : P_W \rightarrow \text{Div}_{\mathbb{Q}} \mathbb{P}^1$ defined by $h(u) := \mathfrak{s}_{\max} \mathcal{D}(u)$ is given by

$$h(a, b) = \max\{-a, 0\}[1] + \max\{-b, 0\}[1] + \max\{a + b, 0\}[\infty].$$

For \widetilde{W} we obtain just the restriction $h|_{P_{\widetilde{W}}}$.

Since $(Y^{\text{sep}}, \Delta^{\text{sep}}) = (\mathbb{P}^1, 0)$ for W and \widetilde{W} we deduce by Theorem 4.1 and Theorem 6.11 that both varieties are F -regular (and hence F -split) for every prime p .

We continue our discussion in Example 9.11, showing that both varieties are *diagonally split*.

7. SPECIAL CASES

In this section, we consider some special cases and examples of H -varieties where criteria for F -splitting and F -regularity simplify.

7.1. Cyclic Covers. Let X be a normal n -fold cyclic cover of a normal variety Y with reduced branch divisor D , and assume that n is relatively prime to p . Let Δ be the boundary divisor as in §2.2. Then by Theorem 4.1, we have that X is F -split (F -regular) if and only if (Y, Δ) is F -split (F -regular). Note that the support of Δ is exactly D , and Δ is of the form

$$\Delta = \sum_i \frac{n_i - 1}{n_i} D_i,$$

where the D_i are the irreducible components of D and each n_i divides n . If the ramification index of every point $x \in X$ in the ramification locus is equal to n , then we simply have

$$\Delta = \frac{n-1}{n} D.$$

Note that our result for cyclic covers is simply a special case of [ST10], which gives criteria for F -splitting and F -regularity to be preserved under arbitrary finite morphisms with tame ramification.

Proposition 7.1. *Let X, Y, Δ and D be as above with $\Delta = \frac{n-1}{n} D$ and X, Y projective. Suppose that $\mathcal{O}_Y((n-1)D) \cong \omega_Y^{-n}$. Then X is F -split if and only if:*

- (1) *We have $p \equiv 1 \pmod{n}$, that is, $p-1 = \alpha n$ for some $\alpha \in \mathbb{N}$; and*
- (2) *The isomorphism $\mathcal{O}_Y((n-1)D) \cong \omega_Y^{-n}$ induces a non-zero map*

$$\phi : \mathcal{O}_Y(\alpha(n-1)D) \rightarrow \omega_Y^{1-p},$$

and a multiple of $\phi(1)$ corresponds to an F -splitting of Y under Grothendieck duality.

Furthermore, X is never F -regular.

Proof. By Theorem 4.1, we are checking whether or not the pair $(Y, \frac{n-1}{n} D)$ is F -split (or F -regular). By Remark 3.5, maps $F_* \mathcal{O}_Y(\lceil (p-1) \frac{n-1}{n} D \rceil) \rightarrow \mathcal{O}_Y$ are given by sections of

$$\mathcal{L} = \omega_Y^{1-p} \left(- \left\lceil (p-1) \frac{n-1}{n} D \right\rceil \right).$$

Now, \mathcal{L}^n is a sub-bundle of

$$(15) \quad \omega^{n(1-p)}(-(p-1)(n-1)D) \cong \mathcal{O}_Y$$

with equality if and only if $(p-1) \frac{n-1}{n} D$ is an integral divisor, that is $p \equiv 1 \pmod{n}$. Since Y is projective, $H^0(Y, \mathcal{L})$ is at most one-dimensional, and must vanish unless $p \equiv 1 \pmod{n}$. Hence, condition (1) must hold for X to be F -split.

If \mathcal{L} has no sections, then X is not F -split; assume instead that the space of global sections is generated by some non-zero $f \in H^0(Y, \mathcal{L})$. Then $f^n \in H^0(Y, \mathcal{L}^n)$ corresponds to the isomorphism $\mathcal{O}_Y(\lceil (p-1)(n-1)D \rceil) \cong \omega_Y^{(1-p)n}$ via (15), and f induces a map ϕ as in condition (2). Hence, assuming condition (1), condition (2) is necessary and sufficient for F -splitting. Furthermore, X is never F -regular, since again by duality,

$$\mathrm{Hom}_{\mathcal{O}_Y} \left(F_*^e \mathcal{O}_Y \left(\left\lceil (p^e - 1) \frac{n-1}{n} D \right\rceil + E \right), \mathcal{O}_Y \right) = 0$$

for any non-trivial effective divisor E . □

Example 7.2 (Elliptic curves as double covers). Let X be a smooth elliptic curve, and $p > 2$. Then an affine model of X can be given by

$$y^2 = x(x-1)(x-\lambda),$$

for $\lambda \neq 0, 1$ which realizes X as a double cover of \mathbb{P}^1 with branch divisor $D = \{0\} + \{1\} + \{\lambda\} + \{\infty\}$. The curve X is F -split if and only if it is ordinary [BK05, 1.3.9], and it is classically known that this is the case if and only if the coefficient of $x^{(p-1)/2}$ in $(x-\lambda)^{(p-1)/2}(x-1)^{(p-1)/2}$ is non-zero [Har77, Corollary 4.2].

We can easily recover this result using Proposition 7.1. Indeed, taking $1, x$ as a basis of $\mathcal{O}(1)$ with $(1)_0 = \{\infty\}$ and $(x)_0 = \{0\}$, we have an isomorphism $\mathcal{O}(D) \rightarrow \mathcal{O}(4)$ sending 1 to $x(x-1)(x-\lambda)$. The section $\phi(1) = (x(x-1)(x-\lambda))^{(p-1)/2} \in \mathcal{O}(2(p-1))$ corresponds to a splitting of \mathbb{P}^1 if and only if the coefficient of x^{p-1} is non-zero, cf. Example 3.6. But this is the same as requiring that the coefficient of $x^{(p-1)/2}$ in $(x-\lambda)^{(p-1)/2}(x-1)^{(p-1)/2}$ is non-zero.

Definition 7.3. Based on the above example, we say that the pair $(\mathbb{P}^1, \frac{1}{2}(c_1 + c_2 + c_3 + c_4))$ is *ordinary* if and only if the coefficient of $x^{(p-1)/2}$ in $(x-\lambda)^{(p-1)/2}(x-1)^{(p-1)/2}$ is non-zero, where λ is the cross-ratio $(c_1, c_2; c_3, c_4)$. By the above, the pair $(\mathbb{P}^1, \frac{1}{2}(c_1 + c_2 + c_3 + c_4))$ is F -split if and only if it is ordinary.

Example 7.4 (Elliptic curves as triple covers). In the situation of Proposition 7.1, we can also take $Y = \mathbb{P}^1$, $n = 3$, $D = \{0\} + \{1\} + \{\infty\}$. The curve X is a triple cover of \mathbb{P}^1 , and is a smooth elliptic curve as long as $p > 3$. By Proposition 7.1, X is F -split if and only if $p \equiv 1 \pmod{3}$. Indeed, we again have an isomorphism $\mathcal{O}(D) \rightarrow \mathcal{O}(3)$ sending 1 to $(x(x-1))$ where we take a basis of $\mathcal{O}(1)$ as in Example 7.2. Then $\phi(1) \in \mathcal{O}(2(p-1))$ is given by $x^{2(p-1)/3}(x-1)^{2(p-1)/3}$, and the coefficient of $x^{(p-1)}$ is clearly non-zero.

Example 7.5 (K3 double covers). Let $Y = \mathbb{P}^2$, D be a smooth sextic curve in Y , and X a double cover of Y ramified over D . Then X is a smooth K3 surface. If $f \in k[x, y, z]$ is a sextic polynomial such that $D = V(f)$ and $p > 2$, then Proposition 7.1 implies that X is F -split if and only if the coefficient of $(xyz)^{p-1}$ in f^{p-1} is non-zero.

More generally, similar statements can be made for K3 surfaces arising as double covers of smooth toric surfaces Y . Indeed, let P_Y be as in Example 3.6. A smooth section f of ω_Y^{-2} can be written as

$$f = \sum_{u \in 2P \cap M} a_u \chi^u$$

and for $p > 2$ the corresponding K3 double cover is F -split if and only if the constant term of f^{p-1} is non-zero.

Example 7.6 (A Fano threefold). Consider a homogeneous quartic polynomial $f \in k[x, y, z, w]$, $\text{char } k > 2$, and let X be a double cover of \mathbb{P}^3 with branch locus $V(f)$. Then X is a smooth Fano threefold of degree 16, which is F -split if and only if $(\mathbb{P}^3, \frac{1}{2}V(f))$ is F -split by Theorem 4.1. By Remark 3.5 and Example 3.6, $(\mathbb{P}^3, V(f))$ is F -split if and only if the coefficient of $x^\alpha y^\beta z^\gamma w^\delta$ in $f^{(p-1)/2}$ is non-zero for some $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ with $\alpha, \beta, \gamma, \delta \leq p-1$. For example, taking $f = x^4 + y^4 + z^4 + w^4$, we see that X is F -split if and only if $\text{char } k \geq 5$.

A similar analysis can be carried out for any cyclic cover of a toric variety.

7.2. Affine Quotients.

Definition 7.7. We say that a pair (Y, Δ) is a *toroidal* if the formal completion of (Y, Δ) at every closed point y is isomorphic to the formal completion of a pair (V_y, B_y) , where V_y is toric and B_y is the toric boundary divisor.

Theorem 7.8. *Let X be an H -variety where H has no p -torsion, (Y, Δ) as in §2.2, and $(Y^{\text{sep}}, \Delta^{\text{sep}})$ any separation. Assume that Y^{sep} is affine and $(Y^{\text{sep}}, [\Delta^{\text{sep}}])$ is toroidal. Then X is F -regular.*

Proof. Combine Theorem 4.1, Theorem 6.11, and Lemma 7.9 below. \square

Lemma 7.9. *Let Y be a normal affine variety, Δ an effective \mathbb{Q} -divisor, and assume that the coefficients Δ are all less than 1. If $(Y, [\Delta])$ is toroidal, then the pair (Y, Δ) is F -regular.*

Proof. Since Y is affine, we can argue as in the proof of [BK05, Proposition 1.1.6] to show that (Y, Δ) is F -regular if the pair is F -regular in a formal neighborhood of each closed point. Hence, we are reduced to showing the following: let Y be toric, B the toric boundary divisor, and Δ an effective \mathbb{Q} -divisor with $\Delta < B$ and whose coefficients are all smaller than 1. Then (Y, Δ) is F -regular.

Due to the assumption on the coefficients of Δ , there exists some $e \in \mathbb{N}$ such that

$$[(p^e - 1)\Delta] + B \leq (p^e - 1)B.$$

Now, the canonical toric splitting $\chi^u \rightarrow \chi^{u/p^e}$ (with $\chi^{u/p^e} = 0$ if $u/p^e \notin M$) splits $\mathcal{O}_Y \rightarrow F_*^e \mathcal{O}_Y((p^e - 1)B)$, hence also $\mathcal{O}_Y \rightarrow F_*^e \mathcal{O}_Y([(p^e - 1)\Delta] + B)$, and by Theorem 3.2 we conclude that (Y, Δ) is F -regular. \square

7.3. \mathbb{G}_m -Actions. The F -splitting and F -regularity of normal affine \mathbb{G}_m -varieties X with $\mathcal{O}_X^{\mathbb{G}_m} = k$ was studied in [Wat91]. By a classical result of Demazure [Dem88], such X may be described as

$$X = \text{Spec}_Y \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}([nD])$$

where

$$D = \sum_i \frac{p_i}{q_i} P_i$$

is a \mathbb{Q} -Cartier divisor on a projective variety Y . Assuming that p_i, q_i are relatively prime, the *fractional part* of D is

$$D' = \sum_i \frac{q_i - 1}{q_i} P_i.$$

Theorem 7.10 ([Wat91, Theorem 3.3]). *Let X be as above. Then X is F -split (F -regular) if and only if (Y, D') is F -split (F -regular).*

Theorem 7.10 is a special case of our Theorem 4.1. Indeed, for $H = \mathbb{G}_m$ and X as above, $X^\circ/H = Y$ and our Δ is exactly the D' from above.

Remark 7.11. Suppose now that in the above setting, Y is a complete intersection in \mathbb{P}^n , cut out by hypersurfaces Y_i . Assume furthermore that the fractional part of D is of the form

$$D' = \frac{1-a}{a}(V \cap Y)$$

triple	F -split	F -regular
$(1, *, *)$	$p \geq 2$	$p \geq 2$
$(2, 2, *)$	$p \geq 3$	$p \geq 3$
$(2, 3, 3)$	$p \geq 5$	$p \geq 5$
$(2, 3, 4)$	$p \geq 5$	$p \geq 5$
$(2, 3, 5)$	$p \geq 7$	$p \geq 7$
$(2, 3, 6)$	$p \equiv 1 \pmod{3}$	No
$(2, 4, 4)$	$p \equiv 1 \pmod{4}$	No
$(3, 3, 3)$	$p \equiv 1 \pmod{3}$	No

TABLE 1. Stabilizer orders for F -split and F -regular complexity-one T -varieties

for some reduced hypersurface $V \subset \mathbb{P}^n$. Hara [Har95, Theorem 4.2] shows that

$$\mathrm{Spec}_Y \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}([nD])$$

is F -split if and only if

$$\mathrm{Spec}_{\mathbb{P}^n} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}([nE])$$

is F -split for some (or equivalently, for all) ample divisor(s) E on \mathbb{P}^n with fractional part

$$E' = \frac{1-a}{a}V + \frac{p-1}{p} \sum Y_i.$$

Reinterpreted using our notation here, this shows that (Y, D') is F -split if and only if (\mathbb{P}^n, E') is F -split.

7.4. Complexity-One Actions. Let X be a T -variety of complexity one, that is, X is a normal variety with an effective action by an algebraic torus T satisfying $\dim X = \dim T + 1$. Using notation as in §2.2, we have that Y is a potentially non-separated smooth curve. Then there is a unique smooth quasiprojective curve C which is a separation of Y . Let $\psi : X^\circ \rightarrow C$ be the composition of the quotient map π with the separation $Y \rightarrow C$. For any point $c \in C$, let $\mu(c)$ be the maximal order of the stabilizer of a general point of $\psi^{-1}(c)$.

We can completely characterize F -split and F -regular complexity one T -varieties in terms of the curve C and the stabilizers of the fibers of ψ :

Theorem 7.12. *The complexity-one T -variety X is F -split in exactly the following cases:*

- (1) C is affine.
- (2) C is an ordinary elliptic curve, and T acts freely on X° .
- (3) $C = \mathbb{P}^1$, $\mu(c) = 1$ for all but at most three points c_1, c_2, c_3 , and $\mu(c_1), \mu(c_2), \mu(c_3)$ is one of the triples in Table 1.
- (4) $C = \mathbb{P}^1$, $\mu(c) = 1$ for all but four points c_1, c_2, c_3, c_4 which have $\mu(c_i) = 2$, $p \geq 3$, and the pair $(\mathbb{P}^1, \sum \frac{1}{2}c_i)$ is ordinary (see Definition 7.3).

Furthermore, X is F -regular exactly in case 1, or case 3 as described in Table 1.

Remark 7.13. In the case that $\dim X = 2$, X is affine, and Y is projective, we recover [Wat91, Theorem 4.2].

Proof of Theorem 7.12. Consider the separation (C, Δ^{sep}) of (Y, Δ) . Then by Theorem 6.11, X is F -split or F -regular if and only if (C, Δ^{sep}) has the same property. Suppose first that C is affine. The separation (C, Δ^{sep}) of (Y, Δ) is toroidal, so X is F -split and F -regular by Theorem 7.8.

For the other cases, we may appeal to [Wat91, Theorem 4.2] coupled with our Theorem 4.1. However, since the proof of loc. cit. is rather terse, we include a proof here for completeness. We now consider the case of C projective. Using the duality of Remark 3.5, we see that a necessary condition for (C, Δ^{sep}) to be F -split (or F -regular) is that

$$(16) \quad \deg[(p-1)\Delta^{\text{sep}}] \leq (p-1)(2g-2) \quad \text{if } F\text{-split}$$

$$(17) \quad \deg[(p^e-1)\Delta^{\text{sep}}] < (p^e-1)(2g-2) \quad \text{for some } e \text{ if } F\text{-regular.}$$

Here, g is the genus of the curve C .

Since Δ^{sep} is effective, we immediately conclude that $g \leq 1$ if X is F -split, with $\Delta^{\text{sep}} = 0$ in the case $g = 1$. In the case $g = 1$, we thus conclude that X is F -split if and only if $\Delta^{\text{sep}} = 0$ and C is F -split. The condition on Δ^{sep} is equivalent to saying that T acts freely on X , and an elliptic curve is F -split if and only if it is ordinary, see e.g. [BK05, 1.3.9]. By the above degree requirement, we also see that if X is F -regular, then we must have $C = \mathbb{P}^1$.

We now analyze the case $C = \mathbb{P}^1$. Let S be the finite subset of \mathbb{P}^1 containing those points y with $\mu(y) \neq 1$. Note that we have

$$\Delta^{\text{sep}} = \sum_{y \in S} \frac{\mu(y) - 1}{\mu(y)} y.$$

Assuming that X is F -split, the above degree bound leads to

$$\sum_{y \in S} \frac{1}{\mu(y)} \geq \#S - 2,$$

with strict inequality if X is F -regular. A straightforward calculation shows that the only possible multiplicities $\mu(y)$ which can occur for X F -split are the triples listed in Table 1 or $(2, 2, 2, 2)$. Equation (16) shows that the stated conditions on p are also necessary. Likewise, by (17) the only multiplicities $\mu(y)$ which can occur for X F -regular are the triples listed in Table 1. Note that the case $(2, 2, 2, 2)$ is covered in Example 7.2, and the case $(3, 3, 3)$ is covered in Example 7.4.

It remains to show that for each triple, the condition on p is also sufficient for F -splitting (or F -regularity). Fix the anticanonical divisor $-K_{\mathbb{P}^1} = \{0\} + \{\infty\}$ as in Example 3.6. A section

$$\sum_{i=1-p}^{p-1} a_i \chi^i \in H^0(\mathbb{P}^1, \mathcal{O}((1-p)K_{\mathbb{P}^1}))$$

has a multiple which splits $\mathcal{O}_{\mathbb{P}^1} \rightarrow F_* \mathcal{O}_{\mathbb{P}^1}$ if and only if $a_0 \neq 0$. Now, we may assume that the points c_1, c_2, c_3 are respectively $0, \infty$, and 1 . Let $(\mu(c_1), \mu(c_2), \mu(c_3))$ be one of the triples from Table 1 with p satisfying the requisite bound. Set

$$\alpha_i = \frac{\mu(c_i) - 1}{\mu(c_i)}.$$

Then $\Delta^{\text{sep}} = \sum \alpha_i c_i$, and $\sum [(p-1)\alpha_i] \leq 2(p-1)$. Hence, there exists $\beta \in \mathbb{N}$ such that

$$(\chi^1 - 1)^{\lceil (p-1)\alpha_3 \rceil} \cdot \chi^{-\beta}$$

is a section of $H^0(\mathbb{P}^1, F_* \mathcal{O}(\lceil (1-p)(K_{\mathbb{P}^1} + \Delta^{\text{sep}}) \rceil))$ and the coefficient a_0 of χ^0 is non-zero. We conclude that a multiple of this section corresponds to a splitting σ of $(\mathbb{P}^1, \Delta^{\text{sep}})$, so X is F -split.

If we are in the situation where we are claiming that X is F -regular, then there exists $e \in \mathbb{N}$ such that $\sum [(p^e - 1)\alpha_i] < 2(p^e - 1)$. Composing the splitting σ from above with itself, we get that σ^e splits $\mathcal{O}_{\mathbb{P}^1} \rightarrow F_*^e \mathcal{O}_{\mathbb{P}^1}(\lceil (p^e - 1)\Delta^{\text{sep}} \rceil)$; Let τ be the corresponding section of $H^0(\mathbb{P}^1, F_*^e \mathcal{O}_{\mathbb{P}^1}(\lceil (1-p^e)(K_{\mathbb{P}^1} + \Delta^{\text{sep}}) \rceil))$. By choice of e , we have

$$\lceil (1-p^e)(K_{\mathbb{P}^1} + \Delta^{\text{sep}}) \rceil > 0,$$

so there exists an effective divisor $D > 0$ with τ a section of $F_*^e \mathcal{O}_{\mathbb{P}^1}(\lceil (1-p^e)(K_{\mathbb{P}^1} + \Delta^{\text{sep}}) \rceil - D)$. Hence, σ^e splits $\mathcal{O}_{\mathbb{P}^1} \rightarrow F_*^e \mathcal{O}_{\mathbb{P}^1}(\lceil (p^e - 1)\Delta^{\text{sep}} \rceil - D)$, so by Theorem 3.2, $(\mathbb{P}^1, \Delta^{\text{sep}})$ and X are F -regular. \square

Remark 7.14. We can define the genus of a pair (C, Δ^{sep}) by

$$g(C, \Delta^{\text{sep}}) = \frac{\deg \Delta^{\text{sep}} + 2g(C)}{2}.$$

By the above theorem, F -split implies that $g(C, \Delta^{\text{sep}}) \leq 1$, and F -regular that $g(C, \Delta^{\text{sep}}) < 1$.

Example 7.15. By [Smi00, Proposition 6.3], any smooth Fano variety in characteristic zero is F -regular after reducing to characteristic p for p sufficiently large. We illustrate this with the list of complexity-one smooth Fano threefolds from [Süß14]. For the threefolds 2.24, 3.8, and 3.10, the stabilizer orders are given by the triple $(2, 2, 2)$.⁴ Hence, these threefolds are F -split and F -regular exactly in characteristics p with $p \geq 3$. All other threefolds on the list have stabilizer orders given by the triple $(1, *, *)$ and are F -split and F -regular in arbitrary characteristic.

7.5. Surjectively Graded Algebras. Let A be an F -finite noetherian \mathbb{Z}^n -graded normal integral domain of characteristic p . Then A is *surjectively graded* [Has03] if for all $u, u' \in \mathbb{Z}^n$ with $A_u, A_{u'} \neq 0$, the multiplication map $A_u \otimes_{A_0} A_{u'} \rightarrow A_{u+u'}$ is surjective. Then Hashimoto shows the following:

Theorem 7.16 (cf. [Has03, Theorem 5.1]). *Assume that $\bigoplus_{n \in \mathbb{Z}} A_{nu}$ is F -regular for some u in the interior of the weight cone of A with $A_u \neq 0$. Then A is F -regular as well.*

Surjectively graded algebras fit nicely into our framework as well. Let A be a surjectively graded finitely generated normal k -algebra, $X = \text{Spec } A$. Constructing X° , Y , and Δ as in §2.2, we have that $\Delta = 0$. Indeed, if A is surjectively graded, then the sheaf

$$\mathcal{A} \cong \bigoplus_{v \in M} \mathcal{O}_Y(\lfloor \mathcal{D}(v) \rfloor)$$

of \mathcal{O}_Y -algebras is also (locally) surjectively graded. But it is straightforward to check that this implies that $\mathcal{D}(v)$ is integral for all $v \in M$, and hence $\Delta = 0$. We may thus conclude by our Theorem 4.1 that $X = \text{Spec } A$ is F -regular if and only if X°/T is F -regular.

⁴Note that only special elements in these three deformation families admit a two-torus action.

7.6. Cox Rings and Related Constructions. Let Y be a normal variety with finitely generated class group $\text{Cl}(Y)$. The *Cox sheaf* of Y is the $\text{Cl}(Y)$ -graded sheaf

$$\mathcal{R}(Y) = \bigoplus_{[D] \in \text{Cl}(Y)} \mathcal{O}_Y(D).$$

This definition appears to depend on choice of representatives D of classes $[D] \in \text{Cl}(Y)$, but any two choices lead to isomorphic sheaves. Furthermore, choosing representatives for a generating set of $\text{Cl}(Y)$ leads to an \mathcal{O}_Y -algebra structure on \mathcal{O}_Y , and any two choices lead to isomorphic \mathcal{O}_Y -algebras [IAL15, §1.4].

The *Cox ring* of Y is the ring $R(Y) = H^0(Y, \mathcal{R}(Y))$. It is a natural generalization of the homogeneous coordinate ring of projective space. Note that in general it need not be finitely generated. However, it is always integral and normal [IAL15, §1.5].

Proposition 7.17 ([GOST12, cf. Proposition 4.6]). *Suppose that $R(Y)$ is finitely generated, and assume that $\text{Cl}(Y)$ has no p -torsion. Then*

$$X = \text{Spec } R(Y)$$

is F -split (F -regular) if and only if Y is F -split (F -regular).

Proof. Let $H = \text{Spec } k[\text{Cl}(Y)]$. In this situation, $U = \text{Spec}_{\mathcal{O}_Y} \mathcal{R}(Y)$ is an H -invariant open subset of X of codimension at least one [IAL15, §1.6], hence X is F -split (F -regular) exactly when U is. But by construction, H acts on U with finite stabilizers, $U/H = Y$, and the boundary divisor $\Delta \subset Y$ is trivial. The claim now follows from our Theorem 4.1. \square

A related situation occurs when considering a normal variety Z embedded in some other variety Y as above with $R(Y)$ finitely generated. For simplicity, we shall assume that Y is toric, in which case $R(Y)$ is a polynomial ring; generalizations are left to the reader. Consider $Z \subset Y$ a normal variety. Let $U \subset \text{Spec } R(Y)$ be as in the proof of Proposition 7.17, and let $I \subset R(Y)$ be the $\text{Cl}(Y)$ -homogeneous ideal of $\overline{\pi^{-1}(Z)} \subset \text{sec } R(Y)$, where $\pi : U \rightarrow Y$ is the quotient map.

Proposition 7.18. *Let Z , Y , and I be as above.*

- (1) *If $V(I)$ is F -split (F -regular), then so is Z .*
- (2) *Suppose that Y is smooth, $V(I)$ normal, and any component of $V(I) \setminus U$ of dimension $\dim V(I) - 1$ has infinite H -stabilizer. Then Z being F -split (F -regular) implies that $V(I)$ is as well.*

Proof. The first claim is a straightforward application of our Theorem 4.1. For the second claim, note that Y smooth implies that H acts freely on U . Then Z is F -split (F -regular) if and only if $V(I) \cap U$ is by loc. cit. Under the further assumptions, $V(I)$ is F -split (F -regular) if and only if $V(I) \cap U$ is. \square

Example 7.19 (Elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$). Any form $f \in k[x_0, x_1, y_0, y_1]$ of bidegree $(2, 2)$ defines a (possibly singular) elliptic curve E embedded in $\mathbb{P}^1 \times \mathbb{P}^1$. Assume that $V(f) \subset \mathbb{A}^4$ is normal. Then the corresponding curve E is F -split if and only if the coefficient of $(x_0 x_1 y_0 y_1)^{p-1}$ in f^{p-1} is non-zero. Indeed, by the above proposition, E is F -split if and only if $k[x_0, x_1, y_0, y_1]/(f)$ is F -split, and Fedder's criterion [Fed83] implies that the latter is F -split if and only if $f^{p-1} \notin \langle x_0^p, x_1^p, y_0^p, y_1^p \rangle$.

8. TORIC VECTOR BUNDLES

Toric vector bundles and their projectivizations provide a natural class of normal varieties with action by a lower-dimensional torus. We apply our general results here to discuss the F -splitting and F -regularity of certain toric vector bundles. Note that in relation to positivity properties of toric vector bundles, it was asked in [HMP10] exactly which toric vector bundles are F -split. Our Theorems 4.1 and 8.4 give a complete answer in the special case of *two-step* bundles defined below.

Given a vector bundle \mathcal{E} , we denote by $\mathbb{P}(\mathcal{E})$ the corresponding projective bundle, whose fibers are the spaces of lines in the fibers of \mathcal{E} .⁵ To begin with, we have the following well known result.

Proposition 8.1 (cf. [BK05, §1.1]). *A vector bundle \mathcal{E} is F -split (F -regular) if and only if its projectivization $\mathbb{P}(\mathcal{E})$ is F -split (F -regular).*

Proof. There is a natural \mathbb{G}_m -action on \mathcal{E} given by the diagonal action on each fiber. The fixed point set is given by the zero-section and \mathbb{G}_m acts with trivial stabilizers elsewhere. Now, $(\mathbb{P}(\mathcal{E}), 0)$ is the corresponding quotient pair and we obtain the result by applying Theorem 4.1. \square

Now, let X be a toric variety corresponding to a fan Σ with embedded torus T , see [Ful93] for details. Throughout this section, M will be the character lattice of T , and $\Sigma^{(1)}$ the set of rays of Σ . A *toric vector bundle* on X is a vector bundle \mathcal{E} on X equipped with a T -equivariant structure. This equivariant structure turns both \mathcal{E} and $\mathbb{P}(\mathcal{E})$ into T -varieties.

To a toric vector bundle \mathcal{E} of rank r , Klyachko [Kly89] associated a k -vector space E of dimension r and a full decreasing filtration $E^\rho(\lambda)$ of E for every ray $\rho \in \Sigma^{(1)}$

$$\cdots \supset E^\rho(\lambda - 1) \supset E^\rho(\lambda) \supset E^\rho(\lambda + 1) \supset \cdots .$$

fulfilling the following compatibility condition: For each maximal cone $\sigma \in \Sigma$, there are lattice points $u_1, \dots, u_r \in M$ and a decomposition into one-dimensional subspaces $E = L_1 \oplus \cdots \oplus L_r$ such that

$$E^\rho(\lambda) = \bigoplus_{\rho(u_i) \geq \lambda} L_i,$$

for each $\rho \preceq \sigma$ and all $\lambda \in \mathbb{Z}$. Here $\rho(u_i)$ denotes the value of a primitive generator of ρ on u_i . From this data one can reconstruct \mathcal{E} as follows. The sections of \mathcal{E} on the chart U_σ of X corresponding to σ are given as a submodule of $k[\sigma^\vee \cap M] \otimes E$ via

$$H^0(U_\sigma, \mathcal{E})_u = \bigcap_{\rho \preceq \sigma} E^\rho(\rho(u)).$$

Note, that the description of toric vector bundles by filtration behaves well with standard constructions as tensor product and dualization. Indeed, the dual bundle corresponds to the filtrations $E^{*\rho}(\lambda) = E^\rho(-\lambda)^\perp$ of the dual vector space E^* .

Definition 8.2. We say that a toric vector bundle \mathcal{E} is a *two-step bundle* if every filtration $E^\rho(\lambda)$ has most two steps where the dimension jumps (i.e. at most one proper subset of E occurs).

⁵By some authors, this bundle is denoted by $\mathbb{P}(\mathcal{E}^*)$.

Clearly, any rank two toric vector bundle is a two-step bundle, since E is two-dimensional in this case.

Example 8.3. By [Kly89] the tangent and cotangent bundles are examples of two-step bundles, since their filtrations have the following form:

$$\mathcal{T}^\rho(\lambda) = \begin{cases} N \otimes k & \text{for } \lambda < 0 \\ \langle \rho \rangle & \text{for } \lambda = 0, \\ 0 & \text{for } \lambda > 0, \end{cases}$$

$$\Omega^\rho(\lambda) = \begin{cases} M \otimes k & \text{for } \lambda < 0 \\ \rho^\perp & \text{for } \lambda = 0, \\ 0 & \text{for } \lambda > 0. \end{cases}$$

For a given two-step bundle \mathcal{E} , let E_1, \dots, E_ℓ be the proper subspaces of E occurring in the filtrations $E^\rho(\lambda)$. For every E_i and every ray $\rho \in \Sigma^{(1)}$ we define

$$\mu_i(\rho) = \max\{\lambda \mid E_i \subset E^\rho(\lambda)\} - \min\{\lambda \mid E^\rho(\lambda) \neq E\}.$$

Now we set

$$\mu_i = \max\{\mu_i(\rho) \mid \rho \in \Sigma^{(1)}\}.$$

Consider $Y = \text{Bl}_{E_1, \dots, E_\ell} \mathbb{P}(E)$, the successive blowup of $\mathbb{P}(E)$ along the strict transforms of the subspaces E_i . Let the corresponding (strict transforms) of the exceptional divisors be denoted by D_i . If E_i is a hyperplane, the corresponding blowup is just the identity morphism; we define the exceptional divisor in this case to be the hyperplane E_i itself. Note that Y and this configuration of divisors is independent of the ordering of the E_i on a big open subset.

Theorem 8.4. *Let (Y, Δ) be the quotient pair for $\mathbb{P}(\mathcal{E})$, where \mathcal{E} is a two-step toric vector bundle. A separation of (Y, Δ) is given by*

$$Y^{\text{sep}} = \text{Bl}_{E_1, \dots, E_\ell} \mathbb{P}(E), \quad \Delta^{\text{sep}} = \sum_i \frac{\mu_i - 1}{\mu_i} D_i.$$

Proof. The claim follows directly from the arguments of [GHPS12, Proposition 3.5 and §6.2] and [HS10, Theorem 5.9]. \square

For two-step toric vector bundles \mathcal{E} , we can thus apply Theorem 6.11 together with Theorem 4.1 to determine when $\mathbb{P}(\mathcal{E})$ and \mathcal{E} are F -split or F -regular. In the following, we consider several special cases.

Note that the following Corollary was obtained by [Xin14] for the case of F -splitting using arguments different than ours:

Corollary 8.5. *The cotangent bundle of a smooth toric variety X is always F -regular. In particular, it is F -split.*

Proof. In this case, we have $(Y^{\text{sep}}, \Delta^{\text{sep}}) = (\mathbb{P}^{n-1}, 0)$, where $n = \dim X$. \square

On the other hand, the Frobenius pullback of the cotangent bundle is not even F -split:

Example 8.6. We consider the vector bundle $\mathcal{E} = F^* \Omega_X$ on a smooth complete toric variety $X = X_\Sigma$ of dimension n . This bundle is given by the filtrations

$$E^\rho(\lambda) = \begin{cases} M \otimes k & \lambda < 0 \\ \rho^\perp & 0 \leq \lambda \leq p \\ 0 & \lambda > p \end{cases}$$

In particular it is a two-step bundle and we see that a separation of the corresponding quotient pair is given by $(Y^{\text{sep}}, \Delta^{\text{sep}})$, where $Y^{\text{sep}} = \mathbb{P}(M \otimes k)$ and

$$\Delta^{\text{sep}} = \sum_{\rho \in \Sigma^{(1)}} \frac{p-1}{p} \mathbb{P}(\rho^\perp).$$

We obtain

$$\deg[(1-p)(K_{Y^{\text{sep}}} + \Delta^{\text{sep}})] = (p-1)(n - (\#\Sigma^{(1)})).$$

Since $\#\Sigma^{(1)} > n$ for X complete, the right hand side is negative, and we conclude using Remark 3.5 that \mathcal{E} cannot be F -split.

It is known that the cotangent bundle for flag varieties is also F -split, see [KLT99]. We ask:

Question 8.7. Let X be any smooth F -split (or F -regular) variety. Is Ω_X always F -split (or F -regular)?

The tangent bundle on a smooth toric variety is not always F -split (see Example 8.12 below), but it is in the case of projective space.

Corollary 8.8. *The tangent bundle of \mathbb{P}^n is always F -regular. In particular, it is F -split.*

Proof. In this case, we have Y^{sep} is the blowup of \mathbb{P}^{n-1} at $n+1$ general points, and $\Delta^{\text{sep}} = 0$. The claim now follows from the above discussion and the following lemma. \square

Lemma 8.9. *The blowup of \mathbb{P}^n at $n+2$ general points is F -regular.*

Proof. After applying a projective transformation, we can take the $n+2$ points to be $n+1$ toric fixed points of \mathbb{P}^n , along with the point 1. The blowup X of \mathbb{P}^n at the $n+1$ fixed points is toric, and choosing $-K_X$ to be the standard toric anticanonical divisor, a basis for $H^0(X, \mathcal{O}(-K_X))$ is given by monomials χ^u for $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$ satisfying

$$\begin{aligned} -1 \leq u_i \leq n-1 & & i = 1, \dots, n; \\ 1 - n \leq \sum u_i \leq 1. & & \end{aligned}$$

Consider the global section

$$\tau = (1 - \chi^{e_n}) \prod_{i=1}^{n-1} (1 - \chi^{-e_i})$$

Here e_i is the standard basis of \mathbb{Z}^n . The coefficient of χ^0 in τ^{p-1} is 1. Hence, under the Grothendieck duality used in Remark 3.5, τ^{p-1} corresponds to an F -splitting of X . Furthermore, this lifts to an F -splitting of the blowup \tilde{X} of X at the point 1, since τ vanishes to order n at the point 1, see [BK05, Exercise 1.3.13].

But in fact, τ is a global section of $\mathcal{O}(-K_X - E)$ for E any one of the exceptional divisors of $X \rightarrow \mathbb{P}^n$, excluding one. Hence, by Grothendieck duality, we have a splitting of $\mathcal{O}_{\tilde{X}} \rightarrow F_* \mathcal{O}_{\tilde{X}}(E)$. But $\tilde{X} \setminus E$ is an open subvariety of a toric variety, so \tilde{X} is F -regular by Theorem 3.2. \square

We can give the most precise answer as to when \mathcal{E} is F -split or F -regular in the case of rank two toric vector bundles.

Corollary 8.10. *Let \mathcal{E} be a rank two toric vector bundle on an arbitrary toric variety, with associated vector space E and proper lines E_i . Then \mathcal{E} is F -split if and only if either there are at most three lines E_i with values $\mu_i > 1$, and the μ_i form a triple as in Table 1; or $p \geq 3$, there are exactly four lines E_1, E_2, E_3, E_4 with $\mu_i > 1$, for these four lines we have $\mu_i = 2$, and the coefficient of $y^{(p-1)/2}$ in $(y - \lambda)^{(p-1)/2}(y - 1)^{(p-1)/2}$ is non-zero, where λ is the cross-ratio of four colinear points $v_i \in E_i$, $i = 1, 2, 3, 4$.*

Likewise, \mathcal{E} is F -regular if and only if either there are at most three lines E_i with values $\mu_i > 1$, and the μ_i form a F -regular triple as in Table 1.

Proof. This is a direct application of Theorem 7.12. \square

N. Lauritzen raised the question if there is an F -split vector bundle \mathcal{E} such that the dual bundle \mathcal{E}^* is not F -split [ope10]. For rank two bundles this cannot happen, since in that case $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}^*)$ are isomorphic. However, in the following examples, we will see a number of two-step toric bundles of higher rank satisfying this property.

Example 8.11. Let $X = X_\Sigma$ be a toric variety, $\Sigma^{(1)} = \{\rho_1, \dots, \rho_\ell\}$. We consider the toric vector bundle \mathcal{E} on X given by

$$E^{\rho_i}(\lambda) = \begin{cases} E & \lambda < 0 \\ E_i & \lambda = 0 \\ 0 & \lambda > 0 \end{cases}$$

where the E_i are hyperplanes in E in sufficiently general position, and $\dim E = n + 1$. If for example X is regular, this collection of filtrations fulfills the necessary compatibility condition.

Now, a separation for the quotient pair of $\mathbb{P}(\mathcal{E})$ is given by $(\mathbb{P}^n, 0)$, and for $\mathbb{P}(\mathcal{E}^*)$ by $(\text{Bl}_\ell \mathbb{P}^n, 0)$, where $\text{Bl}_\ell \mathbb{P}^n$ is the blowup of \mathbb{P}^n at ℓ general points. Since \mathbb{P}^n is F -regular, $\mathbb{P}(\mathcal{E})$ is as well; in particular, it is F -split. On the other hand, on $Y^{\text{sep}} = \text{Bl}_\ell \mathbb{P}^n$ the sheaf $\mathcal{O}((1-p)K_{Y^{\text{sep}}})$ has no global section if $\ell \geq h^0(\mathbb{P}^n, \mathcal{O}(n+1))$, so in this case $\mathbb{P}(\mathcal{E}^*)$ cannot be F -split.

We can modify this example to give a counterexample where $\mathbb{P}(\mathcal{E})$ is F -split but not F -regular, and $\mathbb{P}(\mathcal{E}^*)$ is not F -split. Indeed, consider the bundle \mathcal{E} as above, except that for $2(n+1)$ of the rays the filtrations $E^{\rho_i}(\lambda)$ have value E_i for two steps in the filtration instead of just one. In this case, the boundary divisor Δ is $\sum_i \frac{1}{2} E_i$, the sum being over the indices for those $2(n+1)$ rays. Since $\deg \Delta = n + 1$, (Y, Δ) cannot be F -regular, but it will be F -split if the E_i are sufficiently general. On the other hand, $\mathbb{P}(\mathcal{E}^*)$ will still not be F -split.

Example 8.12. In [GHPS12, Example 4.2], a smooth toric variety is constructed such that the quotient for the tangent bundle is given by $Y^{\text{sep}} = \text{Bl}_{14} \mathbb{P}^{n-1}$. In characteristic $p \neq 2, 3, 9$ of the 14 points form the complete base locus of a pencil of cubics [Tot08]. Hence, $\omega_{Y^{\text{sep}}}^{1-p}$ does not admit any global sections and Y^{sep} and hence \mathcal{T}_X is not F -split. On the other hand, Ω_X is always F -split by Corollary 8.5.

9. DIAGONAL SPLITTINGS

Definition 9.1. A *diagonal splitting* of a scheme X is a splitting of $X \times X$ compatible with the diagonal [Ram87]. By a *diagonal splitting of a pair* (X, Δ) we mean

a splitting of

$$(X \times X, \Delta \times X + X \times \Delta)$$

which is compatible with the diagonal. More generally, by a *diagonal splitting of a triple* $(X; \Delta_+, \Delta_-)$ we mean a splitting of

$$(X \times X, \Delta_+ \times X + X \times \Delta_-)$$

which is compatible with the diagonal.

Note that X being diagonally split has strong consequences for the syzygies of X , see e.g. [BK05, 1.5]

Example 9.2. If C is a complete curve and (C, Δ) is diagonally split, then $g(C, \Delta) \leq 1/2$ has to hold (see Remark 7.14 for a definition of $g(C, \Delta)$). Likewise, if $(C; \Delta_+, \Delta_-)$ is diagonally split, then we must have $g_+ := g(C, \Delta_+) \leq 1/2$ and $g_- := g(C, \Delta_-) \leq 1/2$.

Proof. The diagonal has bidegree $(1, 1)$ in $C \times C$. Hence,

$$D := -K_{C \times C} - (\Delta_+ \times X + X \times \Delta_-)$$

has bidegree $(2 - 2g_+, 2 - 2g_-)$. Hence, if g_+ or g_- is larger than $1/2$, then D and all its positive multiples have empty linear systems. By Remark 3.5 this implies that there is no such splitting. For (C, Δ) we get the claim by considering $(C; \Delta, \Delta)$. \square

Now, let X be an H -variety as in §2.2 and assume that H has no p -torsion. The product $X \times X$ admits a natural $H \times H$ -action. However, the diagonal is invariant only with respect to the diagonal subgroup $H \subset H \times H$. This embedding of groups corresponds to the surjection of character lattices

$$M \times M \rightarrow M; \quad (u_1, u_2) \mapsto u_1 + u_2.$$

Hence, semi-invariant functions of degree $(u, -u)$ with respect to the $H \times H$ -action are exactly the invariant functions with respect to the diagonal action. Now, by using Proposition 5.2, we see that we may assume that a diagonal splitting of X is of the form

$$(18) \quad \sigma = \sum_{w \in M} \sigma_{(w, -w)}$$

where $\sigma_{(w, -w)} \in \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_{X \times X}, \mathcal{O}_{X \times X})_{(w, -w)}$.

Remark 9.3. Note that by §4, given an element $\sigma_w \in \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)_w$ we may interpret it as an element of $\text{Hom}_{\mathcal{O}_Y}(F_* k(Y), k(Y))$ which we as before we will denote by $\bar{\sigma}_w$ in the following. Remember that Lemma 4.6 ensures that

$$\bar{\sigma}_w \in \text{Hom}_{\mathcal{O}_Y}(F_* \mathcal{O}_Y(\lceil (p-1)\Delta + \mathcal{D}(w) \rceil), \mathcal{O}_Y).$$

This extends to $X \times X$ with the full $H \times H$ -action as follows. For

$$\sigma_{(w, -w)} = \sum_i \sigma_i \otimes \sigma^i \in \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)_w \otimes \text{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)_{-w}$$

we have $\bar{\sigma}_{(w, -w)} = \sum \bar{\sigma}_i \otimes \bar{\sigma}^i$. This defines an element of $\text{Hom}_{\mathcal{O}_V}(F_* \mathcal{O}_V, \mathcal{O}_V)$ of some open subset of $V \subset Y \times Y$ intersecting the diagonal.

We now give a characterization of those invariant splittings σ of $X \times X$ which are compatible with the diagonal. For simplicity, we will assume that H is equal to a torus T . For every class $[w] \in M/pM$ we define

$$\bar{\sigma}_{[w]} = \sum_{u \in [w]} \bar{\sigma}_{(u, -u)}.$$

In the following we denote the ideal sheaves of the diagonals in $X \times X$ and $Y \times Y$ by I_X and I_Y , respectively.

Theorem 9.4. *A Frobenius splitting σ of $X \times X$ is compatible with the diagonal if and only if for every $[w] \in M/pM$ we have $\bar{\sigma}_{[w]} \equiv \bar{\sigma}_{[0]} \pmod{I_Y}$ and $\bar{\sigma}_{[w]}$ is compatible with the diagonal, that is, $\bar{\sigma}_{[w]}(F_*I_Y) \subset I_Y$.*

Proof. We consider generators of F_*I_X as an $\mathcal{O}_{X \times X}$ -module. There are two types of generators we have to take into account. One coming from the diagonal of $T \times T$ the other one from the diagonal of $Y \times Y$:

$$(19) \quad f \cdot \chi^0 \otimes \chi^0 - f \cdot \chi^u \otimes \chi^{-u}, \quad u \in M; f \in k(Y) \times k(Y)$$

$$(20) \quad f \cdot (\chi^w \otimes \chi^u), \quad u, w \in M \text{ and } f \in I_Y.$$

In fact, these elements generate F_*I_X as a k -vector space.

Assume first that we have an element g of the form (20). Then $\sigma(g)$ will vanish if $u \neq -w$. Assume that $u = -w$. We obtain

$$\begin{aligned} \sigma(f \cdot (\chi^w \otimes \chi^{-w})) &= \sum_{u \in [w]} \sigma_{(u, -u)}(f \cdot (\chi^w \otimes \chi^{-w})) \\ &= \sum_{u \in [w]} \bar{\sigma}_{(u, -u)}(f) \cdot \chi^{w-u} \otimes \chi^{u-w} \\ &= \left(\sum_{u \in [w]} \bar{\sigma}_{(u, -u)}(f) \right) \chi^0 \otimes \chi^0 + \\ &\quad + \sum_{u \in [w]} \bar{\sigma}_{(u, -u)}(f) \cdot (\chi^{w-u} \otimes \chi^{u-w} - \chi^0 \otimes \chi^0). \end{aligned}$$

Note that the first summand of the right-hand-side is an element of I_X if and only if $\sum_{u \in [w]} \bar{\sigma}_{(u, -u)}(f) = \bar{\sigma}_{[w]}(f)$ is an element of I_Y . The second summand is always an element of I_X , since $(\chi^{w-u} \otimes \chi^{u-w} - \chi^0 \otimes \chi^0)$ lies in I_X .

Assume instead we have an element g of the form (19). Then we obtain

$$\begin{aligned} \sigma(g) &= \sigma(f \cdot \chi^0 \otimes \chi^0 - f \cdot \chi^u \otimes \chi^{-u}) \\ &= \sigma(f \cdot \chi^0 \otimes \chi^0) - \sigma(f \cdot \chi^u \otimes \chi^{-u}) \\ &= \sum_{u \in [0]} \sigma_{(u, -u)}(f \cdot \chi^0 \otimes \chi^0) - \sum_{u \in [w]} \sigma_{(u, -u)}(f \cdot \chi^u \otimes \chi^{-u}) \\ &= \sum_{u \in [0]} \bar{\sigma}_{(u, -u)}(f) \cdot \chi^{-u} \otimes \chi^u - \sum_{u \in [w]} \bar{\sigma}_{(u, -u)}(f) \cdot \chi^{w-u} \otimes \chi^{u-w} \\ &\equiv (\bar{\sigma}_{[0]}(f) - \bar{\sigma}_{[w]}(f)) \cdot \chi^0 \otimes \chi^0. \end{aligned}$$

Here, the congruence is modulo elements of the form $(\chi^0 \otimes \chi^0 - \chi^u \otimes \chi^{-u}) \in I_X$ as above. Now, the right-hand-side lies in I_X if and only if $(\bar{\sigma}_{[0]}(f) - \bar{\sigma}_{[w]}(f))$ is an element of I_Y . \square

We obtain the following corollary, which is a simple generalization of the corresponding result on toric varieties in [Pay09].

Corollary 9.5. *Consider a T -variety X , and suppose σ is a splitting of $X \times X$ compatible with the diagonal. Then for every class $[w] \in M/pM$ there must be a representative $u \in [w]$ such that the homogeneous component of weight $(u, -u)$ in σ is non-trivial. In particular $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_u \neq 0$ and $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_{-u} \neq 0$.*

Proof. For σ to be a splitting, $\bar{\sigma}_{[0]}$ must be non-trivial, and the result follows by Theorem 9.4. \square

Remark 9.6 (The toric case). In the toric case, the criterion that for all $[w] \in M/pM$, there must be a representative $u \in [w]$ such that $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_{\pm u} \neq 0$ is exactly the criterion that the polytope $\mathbb{F}_X := P_X \cap -P_X$ contains a representative of every class $[w] \in M/pM$, cf. Lemma 4.8. Payne shows that this criterion is both necessary and sufficient [Pay09]. The sufficiency of this criterion is easily seen: for any lattice point $u \in \mathbb{F}_X$, $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)_{\pm u} \cong k$ by Remark 4.11. Since X is complete, $[0] \cap (P_X \cap -P_X) = 0$, so by Lemma 4.9, $\sum_u 1 \cdot \chi^u \otimes \chi^{-u}$ corresponds to an invariant splitting σ of $X \times X$, where the sum is taken over a choice of representative u for each class of M/pM . Now, by Theorem 9.4, this splitting is compatible with the diagonal.

It was Payne's result which was one of our original motivations for studying F -splittings of higher complexity T -varieties. As Payne points out, the diagonal of $X \times X$ is not $T \times T$ -invariant, but it is invariant with respect to the action of the diagonal torus. We were struck by the fact that Payne's polytope $\mathbb{F}_X = P_X \cap -P_X$ is exactly the polytope corresponding to the anticanonical divisor on the Chow quotient Z of $X \times X$ by the diagonal torus T . In fact, our machinery (§4 and Proposition 5.2) can be used to show that a toric variety X is F -split if and only if the above quotient Z is split compatibly with some point in the interior of Z (note that Z is a toric variety with respect to the quotient torus $(T \times T)/T$). This is easily seen to be equivalent to Payne's criterion discussed above. We leave the details to the reader.

Our next goal is to give a simpler necessary condition for a complexity-one T -variety to be diagonally split. To begin with, suppose that Y is any complete variety, and let $\mathcal{D} : M \rightarrow \mathrm{Div}_{\mathbb{Q}}(Y)$ be as in (4). We set

$$U = \mathrm{Spec}_Y \bigoplus_{u \in M} \mathcal{O}(\mathcal{D}(u)).$$

Then the quotient pair of U is (Y, Δ) .

Lemma 9.7. *Assume we are given a diagonal splitting σ of U of the form (18). Let Δ_+, Δ_- , be effective \mathbb{Q} -divisors on Y . Suppose that for every $w \in pM$ with $\sigma_{(w, -w)}$ non-trivial there are functions $f_w^p \in k(Y) \subset F_*k(Y)$ satisfying $f_0^p = 1$ and with*

$$\mathrm{div}(f_w^{\pm p}) + [(p-1)\Delta + \mathcal{D}(\pm w)] \geq [(p-1)\Delta_{\pm}].$$

Then there is a diagonal splitting of $(Y; \Delta_+, \Delta_-)$.

Proof. Remember that using Remark 9.3 we obtain $\bar{\sigma}_{(w, -w)}$ as an element of $\mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y([\!(p-1)\Delta + \mathcal{D}(w)\!]), \mathcal{O}_Y) \otimes \mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y([\!(p-1)\Delta + \mathcal{D}(-w)\!]), \mathcal{O}_Y)$.

By our hypothesis on the f_w^p , multiplying with $f_w^p \otimes f_w^{-p}$ gives an element of

$$\mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y([(p-1)\Delta_+], \mathcal{O}_Y) \otimes \mathrm{Hom}_{\mathcal{O}_Y}(F_*\mathcal{O}_Y([(p-1)\Delta_-], \mathcal{O}_Y).$$

We set

$$\sigma' = \sum_w (f_w^p \chi^w \otimes f_w^{-p} \chi^{-w}) \cdot \sigma_{(w, -w)}.$$

By definition, this is a homogeneous element in

$$\mathrm{Hom}_{\mathcal{O}_{X \times X}}(F_*K(X \times X), K(X \times X))_{(0,0)}.$$

We obtain

$$\begin{aligned} \bar{\sigma}' &= \sum_w (f_w^p \otimes f_w^{-p})(\chi^w \otimes \chi^{-w}) \sigma_{(w, -w)} \\ &= \sum_w (f_w^p \otimes f_w^{-p}) \bar{\sigma}_{(w, -w)}. \end{aligned}$$

Now, we claim that $\bar{\sigma}'$ gives the desired splitting on $Y \times Y$. To see that it is indeed a splitting, note, that $\sigma(1) = \sigma_{(0,0)}(1) = 1$. In particular, all other homogeneous components of $\sigma(1)$ vanish. Hence, multiplying one of these components with some element of the form $f_w^p \chi^w \otimes f_w^{-p} \chi^{-w}$ does not contribute to the degree-(0, 0) part of $\sigma'(1)$. We thus obtain $\sigma'(1) = \sigma(1) = 1$. The same holds for $\bar{\sigma}'(1)$ which is just the restriction of $\sigma'(1)$ to the invariant functions.

It remains to show that $\bar{\sigma}'$ is compatible with the diagonal. We have

$$(1 - f_w^p \chi^w \otimes f_w^{-p} \chi^{-w}) \in I_X,$$

so $\sigma(g)$ and $\sigma'(g)$ differ only by an element of I_X . On the other hand, for some element $g \in F_*I_X$ we obtain $\sigma(g) \in I_X$, since σ is compatible with the diagonal. Hence, $\sigma'(F_*I_X) \subset I_X$ holds. Since σ' is of degree (0, 0) we also have $\sigma'(F_*I_{(0,0)}) \subset I_{(0,0)}$. Then we are done, since $I_{(0,0)}$ gives the ideal sheaf for the diagonal of $Y \times Y$ and $\bar{\sigma}'$ is just the restriction of σ' to the degree (0, 0) part. \square

Let us denote by $\mathrm{supp}_1 \sigma$ the set of degrees $w \in M$ such that the homogeneous component of degree $(w, -w)$ of σ is non-trivial.

Lemma 9.8. *Consider the subset $\bar{M} \subset M$ of those $u \in M$ such that $\mathcal{D}(u)$ is principal. Suppose there is a diagonal splitting σ of U satisfying*

$$\mathrm{supp}_1 \sigma \cap pM \subset p\bar{M}.$$

Then there exists a diagonal splitting of (Y, Δ) .

Proof. This is just Lemma 9.7 applied to the case $\mathrm{div}(f_w^p) = \mathcal{D}(w)$. \square

Let's now consider the case that the torus action on X is of complexity one. This means that $C = Y^{\mathrm{sep}}$ is a curve. If X is diagonally split, it is F -split as well and by Theorem 7.12 we know that $g(C, \Delta) \leq 1$, i.e. the curve is either elliptic or \mathbb{P}^1 .

Proposition 9.9. *If $g(C, \Delta) > 1/2$ then an invariant diagonal splitting has to have a non-trivial component in a non-zero degree $(w, -w) \in pM \times pM$.*

Proof. If an invariant splitting doesn't have a non-trivial component in a non-zero degree $(w, -w) \in pM \times pM$, then (by restricting to an open subset subset of X) Lemma 9.8 would provide us with a diagonal splitting of (C, Δ) , which is impossible by Example 9.2. \square

Theorem 9.10. *Let X be a complete diagonally split T -variety of complexity one. Then $C = \mathbb{P}^1$ and we are in the cases $(1, *, *)$ or $(2, 2, 2)$ from Theorem 7.12.*

Proof. By restricting to an open subset U of X we assume that $Y = Y^{\text{sep}} = C$. Given a diagonal splitting σ of form (18) let $M' \subset M$ be the sublattice generated by $\text{supp}_1 \sigma$. Note that by Proposition 9.5 the quotient $M'/(M' \cap pM)$ surjects to $M/pM \cong M \otimes_{\mathbb{Z}} \mathbb{F}_p$ but this implies that $M' \cap pM = pM'$. Indeed, given a \mathbb{Z} -basis e'_1, \dots, e'_ℓ of M' we may consider its image in M/pM . By our condition on $\text{supp}_1 \sigma$, the images of the basis vectors span M/pM . But this implies that they are linearly independent over \mathbb{F}_p . Now, given an integral linear combination $u' = \sum_i \lambda_i e'_i$ of the basis elements which lies in pM gives rise to a linear combinations $0 = \sum_i \bar{\lambda}_i \bar{e}'_i$ in M/pM . By linear independence the coefficients $\bar{\lambda}_i$ have to vanish. Hence, their representatives λ_i are elements of $p\mathbb{Z}$ and u' is an element of pM' .

Let us now consider the case of pairs (C, Δ) of genus 1. Remember that

$$\text{Hom}_{\mathcal{O}_C}(F_* \mathcal{O}_C((p-1)\Delta), \mathcal{O}_C)^* \cong H^0(C, \mathcal{O}_C([(1-p)(K_C + \Delta)])).$$

Hence, by Lemma 4.6 we have $\deg \mathcal{D}(w) \leq 0$ and $\deg \mathcal{D}(-w) \leq 0$ for every w in the support of σ . By linearity this implies $\deg \mathcal{D}(w) = 0$ for $w \in M'$. Again by Lemma 4.6, $\mathcal{D}(w)$ has to be a principal divisor for w in the support of σ and hence for every $w \in M'$, as well. Hence, we can take $\bar{M} = M'$ and apply Lemma 9.8, using that $M' \cap pM = pM'$. We obtain a diagonal splitting of (C, Δ) . But this is impossible by Example 9.2.

By Table 1, the remaining cases we must rule out are those of pairs (C, Δ) of genus larger than $3/2$, that is, the cases of the triples $(2, 2, r)$ ($r > 2$), $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$. We set \bar{M} to be the sublattice of M' consisting of those u such that $\deg \mathcal{D}(u) = 0$. Our first claim is that $\text{supp}_1 \sigma \cap pM \subset \bar{M}$. Indeed, if $\deg \mathcal{D}(w) > 0$ for some $w \in pM$, one can check case by case that $\deg \mathcal{D}(w)$ would be at least $p \cdot (2 - \deg \Delta)$. Now, we would have

$$(1-p) \deg(\Delta + K_C) - \deg \mathcal{D}(w) < 0$$

and there cannot be a non-trivial homomorphism in degree w . On the other hand, if $\deg \mathcal{D}(w) < 0$ than we have $\deg \mathcal{D}(-w) > 0$. Hence, we must have $\deg \mathcal{D}(w) = 0$ for all degrees in $\text{supp}_1 \sigma \cap pM$.

We can apply the same methods as in the genus 1 case if $\mathcal{D}(w)$ is integral for every $w \in \bar{M}$. If we are in the case $(2, 3, 5)$ this has to hold true, since there is no way to obtain $a/2 + b/3 + c/5$ being an integer without all the summands being integers.

For the remaining cases, we will use the diagonal splitting of U to construct a diagonal splitting of $(C; \Delta_+, \Delta_-)$. Here, writing $\Delta = a_1[c_1] + a_2[c_2] + a_3[c_3]$, we take $\Delta_+ = a_1[c_1] + a_3[c_3]$ and $\Delta_- = a_2[c_2] + a_3[c_3]$. Note that by properly ordering a_1, a_2, a_3 , we have $\deg \Delta_+ = a_1 + a_3 > 1$ and $\deg \Delta_- = a_2 + a_3 > 1$ so as before, by Example 9.2 we will obtain a contradiction.

We will discuss the case $\Delta = 2/3[c_1] + 2/3[c_2] + 1/2[c_3]$ in detail; the other cases follow similarly. We wish, for any $w \in \text{supp}_1 \sigma \cap pM$, to produce a function f_w as in Lemma 9.7. Write such w as $w = (\ell p)w'$ with $\ell \in \mathbb{N}$ and $w' \in M'$ a primitive lattice element. Now, we have seen above that $\mathcal{D}(w')$ is of degree 0. If $\mathcal{D}(\ell w')$ is integral, then we set $D_w = \mathcal{D}(\ell w')$ and have $\mathcal{D}(w) = pD_w$. Since D_w has degree zero, it is principal, that is, $D_w = \text{div } f_w$ for some rational function f_w . Furthermore, this f_w

satisfies the requirements of Lemma 9.7, since

$$\lceil (p-1)\Delta + \mathcal{D}(\pm w) \mp pD_w \rceil = \lceil (p-1)\Delta \rceil \geq \lceil (p-1)\Delta_+ \rceil, \lceil (p-1)\Delta_- \rceil.$$

Assume now instead that $\mathcal{D}(\ell w')$ is not integral. Since it has degree 0, up to changing the roles of c_1 and c_2 we have $\mathcal{D}(w') = 1/3[c_1] - 1/3[c_2] + D_0$, with D_0 some integral divisor of degree 0. This means that

$$\lceil (p-1)\Delta + \mathcal{D}(w) \rceil = \left\lceil \frac{(2+\ell)p-2}{3} \right\rceil [c_1] + \left\lceil \frac{(2-\ell)p-2}{3} \right\rceil [c_2] + \left\lceil \frac{p}{2} \right\rceil [c_3] + \ell p D_0.$$

Now, if $\ell \equiv 0 \pmod{3}$, then $\mathcal{D}(\ell w')$ is integral so the case above applies. Suppose instead that $\ell \equiv 2 \pmod{3}$. Then we obtain

$$\begin{aligned} \lceil (p-1)\Delta + \mathcal{D}(w) \rceil &= \left\lceil \frac{(2+2)p-2}{3} \right\rceil [c_1] + \left\lceil \frac{(2-2)p-2}{3} \right\rceil [c_2] + \left\lceil \frac{p}{2} \right\rceil [c_3] + pD'_w \\ &= \left\lceil \frac{4p-2}{3} \right\rceil [c_1] + \left\lceil \frac{p}{2} \right\rceil [c_3] + pD'_w; \\ \lceil (p-1)\Delta + \mathcal{D}(-w) \rceil &= \left\lceil \frac{(2-2)p-2}{3} \right\rceil [c_1] + \left\lceil \frac{(2+2)p-2}{3} \right\rceil [c_2] + \left\lceil \frac{p}{2} \right\rceil [c_3] - pD'_w \\ &= \left\lceil \frac{4p-2}{3} \right\rceil [c_2] + \left\lceil \frac{p}{2} \right\rceil [c_3] + pD'_w \end{aligned}$$

with D'_w being an integral divisor of degree 0, hence of the form $\text{div } f_w$ for some rational function f_w . Since

$$\begin{aligned} \lceil (p-1)\Delta + \mathcal{D}(w) - pD'_w \rceil &\geq \lceil (p-1)\Delta_+ \rceil \\ \lceil (p-1)\Delta + \mathcal{D}(-w) + pD'_w \rceil &\geq \lceil (p-1)\Delta_- \rceil \end{aligned}$$

the function f_w fulfills the requirements for Lemma 9.7.

If instead $\ell \equiv -2 \pmod{3}$, a similar analysis also produces a function f_w satisfying the requirements of Lemma 9.7. Now, applying Lemma 9.7 we obtain a diagonal splitting of $(C; \Delta_+, \Delta_-)$. But as we have seen, this is impossible. \square

Example 9.11 (Blowup of a flag variety (continued)). Once more consider the variety \widetilde{W} from Example 2.10. Remember, that the piecewise linear function from Lemma 6.12 describing the homogeneous components of $\text{Hom}(F_*\mathcal{O}_{\widetilde{W}}, \mathcal{O}_{\widetilde{W}})$ was given in Example 6.13 by

$$h(a, b) = \max\{-a, 0\}[1] + \max\{-b, 0\}[1] + \max\{a+b, 0\}[\infty].$$

Now, for every pair of integers $w = (a, -b)$ with $0 \leq a, b \leq p-1$ we set $w' = (a-p, -b)$ and we have $w, w' \in (p-1)P_{\widetilde{W}}$. Moreover, we obtain

$$\begin{aligned} h(w) &= b[1] + \max\{a-b, 0\}[\infty], & h(-w) &= a[0] + \max\{b-a, 0\}[\infty], \\ h(w') &= (p-a)[0] + b[1], & h(-w') &= (b+p-a)[\infty], \end{aligned}$$

Recall (Remark 3.5) that there is an isomorphism

$$\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(F_*\mathcal{O}_{\mathbb{P}^1}(D), \mathcal{O}_{\mathbb{P}^1}) \cong H^0(\mathbb{P}^1, \mathcal{O}((1-p)K_{\mathbb{P}^1} - D)).$$

We will denote this correspondence by the symbol \triangleq .

For w and w' as above and $K = K_{\mathbb{P}^1} = -[0] - [\infty]$ we consider

$$\begin{aligned}\sigma_{(w,-w)} &\triangleq \sum_{i=0}^{p-a-1} y^{1-p+i}(y-1)^{p-1} \otimes y^{-i} \\ &\in H^0(\mathcal{O}(-K - [h(w)])) \otimes H^0(\mathcal{O}(-K - [h(-w)])), \\ \sigma_{(w',-w')} &\triangleq \sum_{i=p-a}^{p-1} y^{1-p+i}(y-1)^{p-1} \otimes y^{-i} \\ &\in H^0(\mathcal{O}(-K - [h(w')])) \otimes H^0(\mathcal{O}(-K - [h(-w)']))\end{aligned}$$

as elements of

$$\mathrm{Hom}(F_*\mathcal{O}_{\widetilde{W}}, \mathcal{O}_{\widetilde{W}})_w \otimes \mathrm{Hom}(F_*\mathcal{O}_{\widetilde{W}}, \mathcal{O}_{\widetilde{W}})_{-w} = \mathrm{Hom}(F_*\mathcal{O}_{\widetilde{W} \times \widetilde{W}}, \mathcal{O}_{\widetilde{W} \times \widetilde{W}})_{(w,-w)}$$

and $\mathrm{Hom}(F_*\mathcal{O}_{\widetilde{W} \times \widetilde{W}}, \mathcal{O}_{\widetilde{W} \times \widetilde{W}})_{(w',-w')}$, respectively.

We set σ to be the sum of all these $\sigma_{(w,-w)}$ and $\sigma_{(w',-w')}$. Then we obtain

$$\begin{aligned}(21) \quad \bar{\sigma}_{[w]} &\triangleq \sum_{i=0}^{p-1} y^{1-p+i}(y-1)^{p-1} \otimes y^{-i} \\ &= \frac{(y \otimes 1 - 1 \otimes y)^{p-1}}{y^{p-1} \otimes 1} \cdot (y-1)^{p-1} \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-K_{\mathbb{P}^1 \times \mathbb{P}^1} - \mathrm{diag}))\end{aligned}$$

Hence, we have $\bar{\sigma}_{[w]} = \bar{\sigma}_{[0]}$ for every $w \in M$. Moreover, $\bar{\sigma}_{[w]}$ is compatible with the diagonal. It remains to show that σ is actually a splitting. To see this, note, that σ_0 is the only non-trivial homogeneous component $\sigma_{(w,-w)}$ with $w \in pM$. Moreover, $\bar{\sigma}_0 = \bar{\sigma}_{[0]}$ defines a splitting for $\mathbb{P}^1 \times \mathbb{P}^1$, since the monomial 1 occurs with coefficient 1 in (21). Hence, we have $\sigma(1) = \sigma_0(1) = 1$.

We just proved that the blowup \widetilde{W} of the flag variety W is diagonally split. This implies also that the blow up in only one of the curves and W itself are diagonally split. The latter was previously known, since all flag varieties are diagonally split by [Ram87].

Example 9.12. Consider the blowup X of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of degree $(0, 1, 1)$; this is number 4.8 in the classification of Fano threefolds by Mori and Mukai [MM82]. There is a \mathbb{G}_m^2 -action here defined by the weight matrix

$$\begin{matrix} & u_0 & u_1 & v_0 & v_1 & w_0 & w_1 \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} \end{matrix}$$

where the u_i, v_i, w_i are homogeneous coordinates on the three factors of \mathbb{P}^1 . We may assume that the center of the blow up is the curve $C = \{1\} \times V(v_0w_0 - v_1w_1)$. The quotient is again a non-separated \mathbb{P}^1 with the points $0, 1, \infty$ doubled. The separation is just \mathbb{P}^1 and the corresponding quotient map is given by

$$(u_0 : u_1, v_0 : v_1, w_0 : w_1) \mapsto (v_0w_0 : v_1w_1)$$

and we have two prime divisors in $X \setminus X^\circ$ with corresponding one-parameter subgroups $\pm\rho \in N \cong \mathbb{Z}^2$, with $\rho = (0, 1)$. This and the piecewise linear function $h : P_X \rightarrow \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ can be obtained similarly to Example 2.10 or read off from the data given in [Süß14]. For h we obtain

$$h(a, b) = \max\{-a, 0\}[1] + \max\{-b, 0\}[1] + \max\{a, 0\}[\infty].$$

One checks that $\sigma_{(w,-w)}$ as defined in Example 9.11 is again an element of $H^0(\mathcal{O}(-K - [h(w)])) \otimes H^0(\mathcal{O}(-K - [h(-w)]))$ and similarly for $\sigma_{(w',-w')}$. Hence, we can again take the sum of all $\sigma_{(w,-w)}$ and $\sigma_{(w',-w')}$ to obtain a diagonal splitting for X .

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