Feedback stability analysis via frequency dependent constraints

DOI:
10.1016/j.automatica.2024.112000

Document Version
Accepted author manuscript

Citation for published version (APA):
Khong, S. Z., & Lanzon, A. (2025). Feedback stability analysis via frequency dependent constraints. *IEEE Transactions on Automatic Control*, *172*, Article 112000. https://doi.org/10.1016/j.automatica.2024.112000

Published in:
IEEE Transactions on Automatic Control

Authors' Camera Ready Manuscript. To appear in IEEE Transactions on Automatic Control.
Please cite using bibliographic data of the associated published version.

Feedback stability analysis via frequency dependent constraints

Sei Zhen Khong and Alexander Lanzon

Abstract—**We establish robust stability of feedback interconnections of linear time-invariant systems with potentially marginally stable open-loop poles via frequency-varying quadratic constraints. These constraints may be used to characterise numerous system properties, such as small-gain, passivity, negative imaginariness, and their weighted variants, that are manifested on different band-limited frequencies. Such systems arise in practical mechanical settings where force actuators and displacement/velocity sensors are not colocated. We demonstrate that frequency-limited system properties may be characterised using the powerful Iwasaki–Hara lemma in terms of convex feasibility problems involving linear matrix inequalities, thereby complementing the utility of the Iwasaki–Hara lemma for robust feedback stability analysis involving mixed system properties on the frequency axis.**

Index Terms—**Feedback stability, frequency dependent inequalities, integral quadratic constraints, Iwasaki–Hara lemma**

I. Introduction

Linear time-invariant (LTI) systems that display varying properties across different frequency bands are ubiquitous in the field of systems and control [1], [2]. Of particular interest are systems that are passive [3], [4] or negative imaginary [5], [6] in certain frequency ranges, in which imaginary-axis poles may be present. Negative imaginary dynamics naturally arise in mechanical systems with colocated force actuators and position sensors [5], otherwise they are only manifested in limited frequency ranges [1], [2]. This paper examines robust feedback stability of LTI systems with potential poles on the imaginary axis via frequency-varying quadratic constraints. Various system properties across distinct frequency ranges, such as weighted passivity, small-gain, and negative imaginariness, may be characterised by such quadratic constraints.

The theory of integral quadratic constraints (IQCs) [7] was originally developed for open-loop stable systems, and is hence inapplicable to the setting in this paper. To accommodate potential imaginary-axis poles, either hard IQCs [8], [9], [10], [11] or soft IQCs equipped with homotopy in the gap metric [12], [13], [14], [15], [11] may be used. A third method [16], [17] employs a modification of the integration contour of the $\mathbf{L}_2$-space in the frequency domain. All these approaches involve the use of Hilbert signal spaces, which, from the perspective of LTI systems, is indirect and opaque. This paper establishes robust feedback stability of LTI systems with possible imaginary-axis poles *directly* in the frequency domain via frequency-dependent constraints and the multivariable Nyquist criterion [18], [19], [20], without recourse to hard IQCs or the graph topology. The technical difficulty involved in applying the latter methods will also be laid out.

This work was supported in part by the National Science and Technology Council of Taiwan (grant number: 113-2222-E-110-002-MY3) and the Engineering and Physical Sciences Research Council (EPSRC) [grant number EP/R008876/1]. All research data supporting this publication are directly available within this publication. (Corresponding author: S. Z. Khong.)

S. Z. Khong is with the Department of Electrical Engineering, National Sun Yat-sen University, Kaohsiung 80424, Taiwan. szkhong@mail.nsysu.edu.tw

A. Lanzon is with the Department of Electrical and Electronic Engineering, School of Engineering, University of Manchester, Manchester M13 9PL, UK. Alexander.Lanzon@manchester.ac.uk

The frequency-varying constraints considered in this paper may be used to characterise various system properties on the different frequency bands. They incorporate passivity [21], [22], [23], small-gain [24], negative imaginariness [25], [26], [27], [28], and many others [29], [30]. Related mixed system properties have been investigated in [31], [32], [33] from the input-output perspective and [34], [35] from the viewpoint of dissipativity. In particular, [32] falls under the general framework of hard IQCs in [11] while [31], [33] establish $\mathbf{L}_\infty$ stability of the closed-loop transfer function. By contrast, this paper provides frequency-dependent conditions under which the stronger notion of $\mathbf{H}_\infty$ feedback stability can be guaranteed by making use of certain structured robustness results in [36] and the Nyquist stability criterion.

The content of this paper is strongly related with the influential works by Iwasaki & Hara. In particular, the equivalence between feedback stability and quadratic graph separation is established in [36] for matrices, upon which the results in this paper are built. Furthermore, these robust stability results rely on mixing frequency-dependent properties on the imaginary axis. Various properties on distinct frequency bands, such as weighted small-gain and passivity properties, may be characterised in a computationally tractable manner via linear matrix inequalities (LMIs) by the seminal Iwasaki–Hara lemma [37]. The Iwasaki–Hara lemma importantly generalises the classical Kalman–Yakubovich–Popov (KYP) lemma [38] in numerous aspects by allowing for quadratic constraints defined on curves on the complex plane. It has its roots in [39], and also [2], where lightly damped mechanical systems with noncolocated force actuators and position sensors (e.g. a magnetic storage device with a swing-arm) are considered. Such systems exhibit negative imaginariness only in a certain frequency range, and feedback stabilisation of such systems may be achieved with the frequency-dependent results in this paper; see also [40], [41] for studies of control performance of these systems. It is noteworthy that time-domain interpretation of such dynamical behaviours has been considered in [42]. A specialisation to the negative imaginary setting appeared in [43].

The paper is organised as follows. The next section introduces the notation used in the paper and contains mathematical preliminaries. The main results on robust feedback stability based on frequency-dependent quadratic constraints for LTI systems with potentially marginally stable poles are derived in Section III. Application of the Iwasaki–Hara lemma for characterising system properties on specified frequency bands is detailed in Section IV. Systems having mixed positive real, negative imaginary, and bounded real properties are considered in Section V. Section VI contains an example of a flexible mechanical structure that motivates the utility of the main results. Finally, the paper is concluded in Section VII.

II. Notation and preliminaries

Denote by $\mathbb{R}, \mathbb{R}_+, \bar{\mathbb{R}}_+, j\mathbb{R}, \mathbb{C}$, $\mathbb{C}_-$, $\mathbb{C}_+$, and $\bar{\mathbb{C}}_+$ the reals, the positive reals, the nonnegative reals, the imaginary axis, the complex plane, the open left-half complex plane, the open right-half complex plane, and the closed right-half complex plane, respectively. The real part and imaginary part of $s \in \mathbb{C}$ are denoted by $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$, respectively.

A. Matrix theory

For $M \in \mathbb{C}^{n\times m}$, denote by $\bar{\sigma}(M)$ (resp. $\underline{\sigma}(M)$) the largest (resp. smallest) singular value of M. Denote by $\bar{M}$ the (element-wise) complex conjugate and M^* the complex conjugate transpose of M. Given $x \in \mathbb{C}^n$, let $|x| = \sqrt{x^*x}$. An $M \in \mathbb{C}^{n\times n}$ is said to be Hermitian if $M = M^*$. The set of $n \times n$ Hermitian matrices is denoted by $\mathbb{H}_n$. An $M \in \mathbb{H}_n$ is said to be positive semidefinite (resp. definite), denoted $M \geq 0$ (resp. $M > 0$), if $v^*Mv \geq 0$ (resp. > 0) for all nonzero $v \in \mathbb{C}^n$. Given $M \in \mathbb{C}^{n\times n}$, define its field of values (a.k.a. numerical range) [44, Def. 1.1.1] as

$$F(M) = \{x^*Mx : x \in \mathbb{C}^n, |x| = 1\},$$

and the angular field of values [44, Def. 1.1.2] as

$$F'(M) = \{x^*Mx : x \in \mathbb{C}^n, x \neq 0\}.$$

The angular field of values is invariant to Hermitian congruence transformation, i.e. $F'(TMT^*) = F'(M)$ for all invertible $T \in \mathbb{C}^{n\times n}$ [44, Property 1.2.12]. Denote by $\lambda(M)$ the spectrum of M, i.e. the set of its eigenvalues. Denote by $\lambda_i(M)$, $i \in \{1, \ldots, n\}$ the elements in $\lambda(M)$. Note that $\lambda(M) \subset F(M)$ [44, Property 1.2.6]. The following results are useful.

Lemma II.1 ([44, Property 1.2.5, Thm. 1.7.8, and Problem 12 on page 15])**.** *Let $M, N \in \mathbb{C}^{n\times n}$. Then the following hold:*

(i) *$F'(M) \subset \bar{\mathbb{C}}_+$ if and only if $M + M^* \geq 0$;*
(ii) *$F'(M) \subset \mathbb{C}_+$ if and only if $M + M^* > 0$;*
(iii) *If $0 \notin F(N)$, then $\lambda(MN) \subset F'(M)F'(N)$;*
(iv) *$F'(M) \subset \mathbb{R}$ if and only if M is Hermitian.*

Lemma II.2 ([36, Cor. 1])**.** *Let $M \in \mathbb{C}^{p\times q}$ and $N \in \mathbb{C}^{q\times p}$. Then,*

$$\det(I - \tau MN) \neq 0 \quad \text{for all } \tau \in [0, 1]$$

if there exists $\Pi \in \mathbb{H}_{q+p}$ such that

$$\begin{bmatrix} I \\ \tau M \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \tau M \end{bmatrix} \geq 0 \quad \text{for all } \tau \in [0, 1] \tag{1}$$

and

$$\begin{bmatrix} N \\ I \end{bmatrix}^* \Pi \begin{bmatrix} N \\ I \end{bmatrix} < 0.$$

The following corollary is a direct consequence from Lemma II.2 when the muliplier Π is specialised to capture bounded real, positive real and negative imaginary conditions.

Corollary II.3. *Let $M \in \mathbb{C}^{p\times q}$ and $N \in \mathbb{C}^{q\times p}$. Then,*

$$\det(I - MN) \neq 0$$

if

$$\begin{bmatrix} I \\ M \end{bmatrix}^* \Pi \begin{bmatrix} I \\ M \end{bmatrix} \geq 0$$

$$\text{and} \quad \begin{bmatrix} N \\ I \end{bmatrix}^* \Pi \begin{bmatrix} N \\ I \end{bmatrix} < 0$$

with $\Pi = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$ for some $\gamma > 0$, or $\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, or $\Pi = \begin{bmatrix} 0 & jI \\ -jI & 0 \end{bmatrix}$, or any conic combination of them.

Proof: The result is trivially obtained directly from Lemma II.2 by taking Π to be the specified forms provided in this corollary. ∎

B. Transfer functions

Let $\mathbf{L}_\infty$ consist of transfer functions G satisfying $\operatorname{ess\,sup}_{\omega\in\mathbb{R}} \bar{\sigma}(G(j\omega)) < \infty$. Let $\mathbf{R}^{n\times m}$ denote the set of real-rational proper transfer function matrices of dimensions $n \times m$ and $\mathbf{RH}_\infty^{n\times m}$ be its stable subset containing elements with no poles in $\bar{\mathbb{C}}_+$. Every $G \in \mathbf{R}^{n\times m}$ can be realised as $G(s) = C(sI - A)^{-1} + D$, and we denote such a realisation by $G = (A, B, C, D)$. Denote by $\mathbf{M}^{n\times m} \subset \mathbf{R}^{n\times m}$ the set of stable and marginally stable transfer functions such that $G \in \mathbf{M}^{n\times m}$ if and only if

(i) G has no poles in $\mathbb{C}_+$;
(ii) For any $\omega_0 \geq 0$, if $j\omega_0$ is a pole of G, then it is a simple pole.

Let $\mathbf{C}_m \subset \mathbf{L}_\infty$ be the class of functions $\Pi : \mathbb{R} \to \mathbb{C}^{m\times m}$ that are piecewise continuous and for which $\lim_{\omega\to\infty} \Pi(\omega)$ exists, $\Pi(\omega) = \Pi(\omega)^*$ for all $\omega \in [0, \infty]$, and $\Pi(-\omega) = \overline{\Pi(\omega)}$ for all $\omega \in (0, \infty)$.

The positive feedback interconnection of two transfer functions $P \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{m\times n}$, denoted by $[P, C]$, is described by:

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} I & -C \\ -P & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This is illustrated in Figure 1.

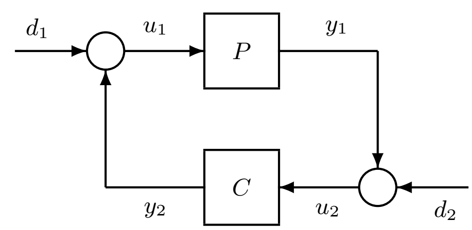

Fig. 1. A standard feedback configuration $[P, C]$.

Definition II.4. $[P, C]$ *is said to be* stable *if*

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I + C(I - PC)^{-1}P & C(I - PC)^{-1} \\ (I - PC)^{-1}P & (I - PC)^{-1} \end{bmatrix} \in \mathbf{RH}_\infty.$$

The following three new definitions on frequency-band-limited positive real, negative imaginary, and bounded real properties will be critical to describing mixed positive real, negative imaginary, and bounded real systems in this paper.

First, we define a class of systems that has a positive real-type property on a subset of positive frequencies. When considering potential poles on the imaginary axis, we have to restrict attention only to those poles inside the frequency set of interest.

Definition II.5. *Given $\Omega \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$, a system $G \in \mathbf{R}^{m\times m}$ is said to be positive real (a.k.a. passive) on Ω if*

(i) *G has no poles in the open right half plane;*
(ii) *for all $\omega \in \Omega$ such that $j\omega$ is not a pole of G,*

$$G(j\omega) + G(j\omega)^* \geq 0;$$

(iii) *for any $\omega_0 \in \Omega \cap \bar{\mathbb{R}}_+$, if $j\omega_0$ is a pole of G, then it is a simple pole and*

$$\lim_{s\to j\omega_0} (s - j\omega_0)G(s) \geq 0.$$

If G is positive real on the entire $\bar{\mathbb{R}}_+ \cup \{\infty\}$, it is said to be a positive real (a.k.a. passive) system.

Second, we define a class of systems that has a negative imaginary-type property on a subset of positive frequencies. When considering potential poles on the imaginary axis, we have to restrict attention only to those poles inside the frequency set of interest.

Definition II.6. *Given $\Omega \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$, a system $G \in \mathbf{R}^{m\times m}$ is said to be negative imaginary on Ω if*

(i) *G has no poles in the open right half plane;*

(ii) *for all $\omega \in \Omega$ such that $j\omega$ is not a pole of G,*

$$j\left(G(j\omega) - G(j\omega)^*\right) \geq 0;$$

(iii) *for any $\omega_0 \in \Omega \cap \mathbb{R}_+$, if $j\omega_0$ is a pole of G, then it is a simple pole and*

$$\lim_{s \to j\omega_0} (s - j\omega_0) jG(s) \geq 0.$$

(iv) *if $0 \in \Omega$ and is a pole of G, then it is at most a double pole and*

$$\lim_{s \to 0} s^2 G(s) \geq 0.$$

If G is negative imaginary on the entire $\bar{\mathbb{R}}_+ \cup \{\infty\}$, it is said to be a negative imaginary system.

Third and last, we define a class of systems that has a bounded real-type property (with a specified gain γ) on a subset of positive frequencies. Note that poles on the imaginary axis are permitted in this class of systems, but these imaginary axis poles must be outside the frequency band where the bounded real property holds.

Definition II.7. *Given $\Omega \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$, a system $G \in \mathbf{R}^{m \times m}$ is said to be bounded real on Ω with gain $\gamma \in \bar{\mathbb{R}}_+$ if*

(i) *G has no poles in the open right half plane;*

(ii) *for all $\omega \in \Omega$,*

$$\gamma^2 I - G(j\omega)^* G(j\omega) \geq 0.$$

If G is bounded real on the entire $\bar{\mathbb{R}}_+ \cup \{\infty\}$, it is said to be a bounded real system.

III. Feedback stability

The main result of this paper is derived in this section. It is concerned with the robust stability of a feedback interconnection of LTI systems with marginally stable poles and mixed system properties on the frequency axis, as characterised by quadratic constraints that vary with the frequency parameter. The result covers the well-known passivity theorem as a special instance.

A. Open-loop stable systems

First, the following known result for open-loop stable systems is reviewed and its limitations discussed.

Lemma III.1. *Given $P \in \mathbf{RH}_\infty^{p \times q}$ and $C \in \mathbf{RH}_\infty^{q \times p}$, $[P, C]$ is stable if there exists $\Pi \in \mathbf{C}_{p+q}$ such that for all $\omega \in [0, \infty], \tau \in [0, 1]$,*

$$\begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix} \geq 0$$

and

$$\begin{bmatrix} C(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} C(j\omega) \\ I \end{bmatrix} < 0.$$

Proof: The claim may be established using the IQC theorem [7, Thm. 1] together with the Plancherel-Parseval theorem as in [45, Thm. 3.1]. A more direct frequency-domain proof may be found in [46, Prop. II.1]. ∎

Lemma III.1 guarantees feedback stability of $[P, C]$ via two frequency-dependent inequalities/constraints. It provides a powerful framework within which various system properties across different frequency bands may be captured via a frequency-varying multiplier Π. When Π is expressed as

$$\Pi(\omega) = \begin{bmatrix} Q(\omega) & S(\omega) \\ S(\omega)^* & R(\omega) \end{bmatrix},$$

it is often known as a dynamical $(Q(\omega), S(\omega), R(\omega))$-multiplier in the theory of dissipativity, as is considered in [31], [35]. A great number of commonly encountered uncertainty may be captured by such a multiplier; see [7, Section VI].

Lemma III.1 is only applicable to open-loop stable P and C. In the case where P has poles on the imaginary axis, the Plancherel-Parseval theorem cannot be applied, in the proof of Lemma III.1, to obtain the conditions in the IQC theorem [7, Thm. 1]. One way to handle imaginary-axis poles involves using hard IQCs [11, Thm. III.1] as in the passivity theorem [3, Chapter VI], where the integrals are taken over all finite intervals for extended $\mathbf{L}_2$ signals. This works well when the multiplier Π is static, as in the case of passivity, but when Π is dynamical, it is unclear how one may obtain a hard IQC from a frequency-domain inequality when the transfer function involved has imaginary-axis poles.

An alternative method to accommodating imaginary-axis poles makes use of homotopies that are continuous in the graph topology, as in [13], [15]. This method, however, relies on the existence of a nominal feedback system that is stable and the satisfaction of an IQC along the entire homotopy from the nominal system to the perturbed system under study. The search for such a homotopy is often a challenge and varies from one case to another.

In the next subsection, feedback stability of systems with possible marginally stable poles is established directly in the frequency domain using the Nyquist stability criterion. Doing so bypasses the difficulty involved in employing hard IQCs or the graph topology.

B. Open-loop systems with possible marginally stable poles

Robust stability conditions for open-loop systems that admit marginally stable poles are derived next in terms of frequency-dependent quadratic constraints.

Theorem III.2. *Given $P \in \mathbf{M}^{n \times n}$ and $C \in \mathbf{RH}_\infty^{n \times n}$, suppose there exist $\epsilon > 0$ and $\Pi = \left[\begin{smallmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{smallmatrix}\right] \in \mathbf{C}_{2n}$ such that*

(i) *for all $\tau \in [0, 1]$ and $\omega \in \bar{\mathbb{R}}_+$ such that $j\omega$ is not a pole of P,*

$$\begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix} \geq 0; \tag{2}$$

(ii) *for every $j\omega_0$ that is a pole of P, Π is continuous at ω_0,*

$$\Pi_{11}(\omega) = 0 \text{ and } \Pi_{22}(\omega) = 0 \quad \text{for all } |\omega - \omega_0| \leq \epsilon, \tag{3}$$

and

$$\Pi_{12}(\omega_0) K + K^* \Pi_{12}(\omega_0)^* \geq 0, \tag{4}$$

where

$$K = \lim_{s \to j\omega_0} (s - j\omega_0) P(s); \tag{5}$$

(iii) *for all $\omega \in [0, \infty]$,*

$$\begin{bmatrix} C(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} C(j\omega) \\ I \end{bmatrix} < 0. \tag{6}$$

Then $[P, C]$ is stable.

Proof: If P has no poles on the imaginary axis, then the claim follows straightforwardly from Lemma III.1. Thus, denote by $j\Omega$ the set of imaginary-axis poles of P. Consider a Nyquist contour $\mathcal{N}$ on the imaginary axis indented into $\mathbb{C}_+$ around every element of $j\Omega$ via sufficiently small semicircles. We show below that the eigenloci of PC along the aforementioned Nyquist contour, i.e. $\{\lambda_i(P(s)C(s)) : s \in \mathcal{N}\}$, $i \in \{1, \ldots, n\}$, does not intersect with the real interval $[1, \infty)$ on the complex plane. This implies that $(I - PC)^{-1}$ has no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ by the multivariable Nyquist stability criterion [20, Thm. L3], since P and C have no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ and the indexed family of circuits formed by the juxtaposition of the eigenloci of PC do not encircle the critical point $1 + j0$.

First, for any $\omega \in [0,\infty]$ such that $j\omega$ is not a pole of P, (2) holding for all $\tau \in [0,1]$ and (6) imply via Lemma II.2 that $\det(I - \tau P(j\omega)C(j\omega)) \neq 0$ for all $\tau \in [0,1]$, from which it follows that $\lambda(P(j\omega)C(j\omega)) \cap [1,\infty) = \emptyset$. For $\omega \in (-\infty, 0)$ such that $j\omega$ is not a pole of P, it also holds that $\lambda(P(j\omega)C(j\omega)) \cap [1,\infty) = \emptyset$ since $P, C \in \mathbf{R}^{n\times n}$, whereby $P(-j\omega) = \overline{P(j\omega)}$ and $C(-j\omega) = \overline{C(j\omega)}$.

Next, for every $j\omega_0$ that is a pole of P and the corresponding K defined in (5), note that $P(s) \approx \frac{1}{s-j\omega_0}K$ when s is sufficiently close to $j\omega_0$. By (3), (2), and Lemma II.1(i), it holds that $F'(\Pi_{12}(\omega)P(j\omega)) \subset \bar{\mathbb{C}}_+$ for $|\omega - \omega_0| \leq \epsilon$ and $\omega \neq \omega_0$, where continuity of Π at ω_0 has been used. Putting these together implies that $F'(\Pi_{12}(\omega_0)K) \subset \mathbb{R}$. Together with (4), it follows that $F'(\Pi_{12}(\omega_0)K) \subset \bar{\mathbb{R}}_+$, which is equivalent to $\Pi_{12}(\omega_0)K$ being positive semidefinite Hermitian by Lemma II.1(i) and (iv). Observe that (3) and (6) imply that $F'(\Pi_{12}(\omega)^*C(j\omega)) \subset \mathbb{C}_-$ for $|\omega - \omega_0| \leq \epsilon$ by Lemma II.1(ii). Applying Lemma II.1(iii) then yields that for sufficiently small $\rho > 0$,

$$\begin{aligned}
\lambda(P(s)C(s)) &= \lambda(\Pi_{12}(\omega_0)P(s)C(s)\Pi_{12}(\omega_0)^{-1}) \\
&\subset F'(\Pi_{12}(\omega_0)P(s))F'(C(s)\Pi_{12}(\omega_0)^{-1}) \\
&= F'(\Pi_{12}(\omega_0)P(s))F'(\Pi_{12}(\omega_0)^*C(s)) \\
&\approx F'\left(\frac{1}{s-j\omega_0}\Pi_{12}(\omega_0)K\right)F'(\Pi_{12}(\omega_0)^*C(j\omega_0))
\end{aligned}$$

for all $s = j\omega_0 + \rho e^{j\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, whereby

$$\lambda(P(s)C(s)) \cap \mathbb{R}_+ = \emptyset.$$

Finally, we show that $(I - PC)^{-1}$ has no poles in $j\Omega$ under (2) and (6), so that $(I - PC)^{-1} \in \mathbf{RH}_\infty$ follows from the Nyquist-based analysis above. To this end, let $j\omega_0 \in j\Omega$ and note from (6) that for all $|\omega - \omega_0| \leq \epsilon$, $C(j\omega)$ is nonsingular and

$$-C(j\omega)^{-*}\Pi_{12}(\omega)^* - \Pi_{12}(\omega)C(j\omega)^{-1} > 0. \tag{7}$$

Summing (2) and (7) then yields the existence of $\nu > 0$ and $\eta > 0$ such that for all $|\omega - \omega_0| < \nu$ and $\omega \neq \omega_0$,

$$(P(j\omega) - C(j\omega)^{-1})^*\Pi_{12}(\omega)^* + \Pi_{12}(\omega)(P(j\omega) - C(j\omega)^{-1}) > \eta I,$$

by which

$$\begin{aligned}
C(j\omega)^{-*}&(P(j\omega)C(j\omega) - I)^*\Pi_{12}(\omega)^* \\
&+ \Pi_{12}(\omega)(P(j\omega)C(j\omega) - I)C(j\omega)^{-1} > \eta I.
\end{aligned}$$

This implies the existence of $\gamma > 0$ such that

$$\underline{\sigma}(I - P(j\omega)C(j\omega)) > \gamma$$

for all $|\omega - \omega_0| < \nu$ and $\omega \neq \omega_0$. Since $(I - PC)^{-1} \in \mathbf{R}$, it follows that

$$\sup_{|\omega-\omega_0|<\nu} \bar{\sigma}((I - P(j\omega)C(j\omega))^{-1}) < \infty,$$

i.e. $j\omega_0$ is not a pole of $(I - PC)^{-1}$.

Since there are no pole-zero cancellations between P and C in $\bar{\mathbb{C}}$, the stability of $[P, C]$ is equivalent to $(I - PC)^{-1} \in \mathbf{RH}_\infty$ [47, Thm. 5.7]. This completes the proof. ■

The condition in (3) signifies that the diagonal blocks of Π are zero when $j\omega$ is sufficiently close to an imaginary-axis pole of P, whereby Π reduces to a multiplier that corresponds to weighted passivity on its residue in (4) and no gain information about P would be exploited in establishing feedback stability.

Note that Theorem III.2 is a robust stability result in that P and C may be arbitrary transfer functions that satisfy the conditions in the theorem for the stability of $[P, C]$ to hold. In other words, closed-loop stability is robust against perturbations that satisfy the frequency-domain inequalities in the theorem.

Remark III.3. *Note that if* $\Pi = \left[\begin{smallmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{smallmatrix}\right] \in \mathbf{C}$ *is such that* $\Pi_{11}(\omega) \geq 0$ *and* $\Pi_{22}(\omega) \leq 0$ *for all* $\omega \in \bar{\mathbb{R}}_+$*, then* (2) *holds for all* $\tau \in [0,1]$ *if and only if it holds for* $\tau = 1$.

IV. The Iwasaki–Hara lemma

In this section, we elucidate how Theorem III.2 nicely complements the Iwasaki–Hara lemma in [37, Theorem 2]. In particular, by directly applying the Iwasaki–Hara lemma, we demonstrate that system properties that hold in a prespecified range of frequencies can be characterised via linear matrix inequalities (LMIs).

Consider the main Theorem III.2. The inequality in (4) is straightforward to verify because it has no frequency dependence. Next, we demonstrate how the Iwasaki–Hara lemma may be used to characterise (2) and (3). For simplicity, we assume that $\Pi_{11}(\omega) \geq 0$ and $\Pi_{22}(\omega) \leq 0$ for all $\omega \in \bar{\mathbb{R}}_+$, so that Remark III.3 is applicable and (2) only needs to be verified for $\tau = 1$. This is not a restrictive assumption because most multipliers, such as those corresponding to weighted small-gain, passivity, and negative imaginary properties, satisfy this assumption. The difficulty in verifying (2) lies in the fact that $\Pi \in \mathbf{C}$ is only piecewise continuous on $\mathbb{R}$, and thus may not admit a realisation $(A_\Pi, B_\Pi, C_\Pi, D_\Pi)$ on the entirety of $j\mathbb{R}$. Nonetheless, for any segment $[\omega_1, \omega_2] \subset \mathbb{R} \cup \{\infty\}$ on which Π is continuous, Π may be approximated arbitrarily closely by $\tilde{\Pi} \in \mathbf{RL}_\infty$ on this frequency segment [48, Lem. A.6.11]. Such a $\tilde{\Pi}$ should satisfy $\tilde{\Pi}(j\omega) = \tilde{\Pi}(j\omega)^*$ for all $\omega \geq 0$, in which case it may be factorised into $\tilde{\Pi}(j\omega) = \Psi(j\omega)^*M\Psi(j\omega)$ for all $\omega \geq 0$, where $M = M^T$ is a real matrix and $\Psi = \begin{bmatrix}\Psi_1 & \Psi_2\end{bmatrix} \in \mathbf{RH}_\infty$ [49, Section 7.3]. Such factorisations are not unique.

Suppose $\Pi(\omega) = \Psi(j\omega)^*M\Psi(j\omega)$ for $\omega \in [\omega_1, \omega_2] \subset [0,\infty]$ and one would like to verify (2) for $\omega \in [\omega_1, \omega_2]$ and $\tau = 1$.

Theorem IV.1. *Let a minimal realisation of* $\Psi\left[\begin{smallmatrix}I\\P\end{smallmatrix}\right]$ *be* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$*. Then,* (2) *holds for all* $\omega \in [\omega_1, \omega_2]$ *and* $\tau = 1$ *if and only if there exist complex matrices* $Y = Y^*$ *and* $Z = Z^* \geq 0$ *such that*

(i) *(for* $\omega_2 < \infty$*)*

$$\begin{bmatrix}\hat{A} & \hat{B}\\ 0 & I\end{bmatrix}^T \begin{bmatrix}-Z & Y + j\omega_c Z\\ Y - j\omega_c Z & -\omega_1\omega_2 Z\end{bmatrix} \begin{bmatrix}\hat{A} & \hat{B}\\ 0 & I\end{bmatrix} \leq \begin{bmatrix}\hat{C}^T\\ \hat{D}^T\end{bmatrix} M \begin{bmatrix}\hat{C} & \hat{D}\end{bmatrix},$$

where $\omega_c = (\omega_1 + \omega_2)/2$*;*

(ii) *(for* $\omega_2 = \infty$*)*

$$\begin{bmatrix}\hat{A} & \hat{B}\\ 0 & I\end{bmatrix}^T \begin{bmatrix}0 & Y + jZ\\ Y - jZ & -2\omega_1 Z\end{bmatrix} \begin{bmatrix}\hat{A} & \hat{B}\\ 0 & I\end{bmatrix} \leq \begin{bmatrix}\hat{C}^T\\ \hat{D}^T\end{bmatrix} M \begin{bmatrix}\hat{C} & \hat{D}\end{bmatrix}.$$

Proof: This follows directly from the state-space nonstrict inequality version of the Iwasaki–Hara lemma [37, Theorem 4]. ■

Observe that the LMIs above may also be used to verify (2) under (3). Finally, it is worth noting that (6) in Theorem III.2 may be verified in a similar manner by alluding to [37, Thm. 3] in lieu of [37, Thm. 4]. In this case, minimality of realisations is no longer a concern and pole-zero cancellations in $\Psi\left[\begin{smallmatrix}C\\I\end{smallmatrix}\right]$ are allowed. The details are omitted for brevity.

V. Mixed systems

In this section, we specialise the results from the earlier sections to a class of systems having mixed positive real, negative imaginary, and bounded real properties on the frequency axis.

First, we define what we mean by mixed systems.

Definition V.1. *$P \in \mathbf{R}^{m\times m}$ is said to be "mixed positive real, negative imaginary, and bounded real with gain γ" if*

(i) *it is positive real on $\Omega^P \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$;*
(ii) *it is negative imaginary on $\Omega^N \subset \mathbb{R}_+$;*
(iii) *it is bounded real on $\Omega^B \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with gain $\gamma \in \bar{\mathbb{R}}_+$;*
(iv) $\Omega^P \cup \Omega^N \cup \Omega^B = \bar{\mathbb{R}}_+ \cup \{\infty\}$.

Observe that — for a mixed positive real, negative imaginary, and bounded real (with gain γ) system — the frequency sets Ω^P, Ω^N, and Ω^B are allowed to intersect in any combination. Indeed, an overlap of frequencies where systems properties occur is very common due to the inherent continuity of the frequency responses. Such an overlap of system properties is beneficial as it means that we do not need to know the transition frequencies between system properties with any precision.

The reason why we choose to exclude the values 0 and ∞ from the negative imaginary property in Definition V.1 and insist that these two frequency values must be either positive real or bounded real is so that we can derive an unconditional stability result (see Theorem V.2). Indeed, purely negative imaginary results (i.e. not mixed systems), such as those in [25], [26], result in conditional stability statements with extra conditions imposed at 0 and ∞.

Next, we provide a feedback stability result when two mixed positive real, negative imaginary, and bounded real (with gain γ) systems are interconnected in a positive feedback loop. One of the two systems (C in Theorem V.2) must have strict properties. The result simply states that if the positive real, negative imaginary, and bounded real properties of the two mixed systems occur at the same frequency regions Ω^P, Ω^N, and Ω^B respectively, then the feedback interconnection of these two mixed systems is stable.

Theorem V.2. *Let $[a_i^P, b_i^P] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with $a_i^P < b_i^P$ for all $i \in \{1,\dots,r_1\}$ and define $\Omega^P = \bigcup_{i=1}^{r_1}[a_i^P, b_i^P]$. Let $[a_i^N, b_i^N] \subset \mathbb{R}_+$ with $a_i^N < b_i^N$ for all $i \in \{1,\dots,r_2\}$ and define $\Omega^N = \bigcup_{i=1}^{r_2}[a_i^N, b_i^N]$. Let $[a_i^B, b_i^B] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with $a_i^B < b_i^B$ for all $i \in \{1,\dots,r_3\}$ and define $\Omega^B = \bigcup_{i=1}^{r_3}[a_i^B, b_i^B]$. Suppose that $\Omega^P \cup \Omega^N \cup \Omega^B = \bar{\mathbb{R}}_+ \cup \{\infty\}$.*

Let $P \in \mathbf{R}^{m\times m}$ be positive real on Ω^P, negative imaginary on Ω^N, and bounded real on $[a_i^B, b_i^B] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with gain $\gamma_i \in \bar{\mathbb{R}}_+$ for all $i \in \{1,\dots,r_3\}$. Suppose also that the poles of P on $j\mathbb{R}_+$ do not coincide with the boundary points of $j\Omega^P$ and $j\Omega^N$.

Let $C \in \mathbf{RH}_\infty^{m\times m}$ satisfy

(i) $-C(j\omega) - C(j\omega)^* > 0 \quad \forall \omega \in \Omega^P$*;*
(ii) $j[C(j\omega) - C(j\omega)^*] > 0 \quad \forall \omega \in \Omega^N$*;*
(iii) $I - \gamma_i^2 C(j\omega)^* C(j\omega) > 0 \quad \forall \omega \in [a_i^B, b_i^B], i \in \{1,\dots,r_3\}$.

Then, $[P, C]$ is stable.

Proof: Let $\mathit{\Pi}(\omega) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ for all $\omega \in \Omega^P$, $\mathit{\Pi}(\omega) = \begin{bmatrix} 0 & jI \\ -jI & 0 \end{bmatrix}$ for all $\omega \in \Omega^N \backslash \Omega^P$ and $\mathit{\Pi}(\omega) = \begin{bmatrix} \gamma_i^2 I & 0 \\ 0 & -I \end{bmatrix}$ for all $\omega \in [a_i^B, b_i^B] \backslash (\Omega^P \cup \Omega^N)$ and $i \in \{1,\dots,r_3\}$. Then, $\mathit{\Pi} \in \mathbf{C}$ by definition. Furthermore, the suppositions on P guarantee that $P \in \mathbf{M}^{m\times m}$ because P has only simple poles on $j\mathbb{R}$ and no poles in $\mathbb{C}_+$, and also guarantee that parts (i) and (ii) of Theorem III.2 hold. Finally, the suppositions on C guarantee that part (iii) of Theorem III.2 holds. The conclusion follows via application of Theorem III.2. ∎

The number of intervals as stated in Theorem V.2 is restricted to be finite for computational purposes. Even though it is indeed possible to handle countably infinite closed intervals, doing so would amount to solving an infinite number of LMIs, which is not computationally tractable.

Since Theorem V.2 considers a positive feedback interconnection, the strict positive real property of C on Ω^P in condition (i) has to be specified on $-C$ instead to compensate for the negative sign of the feedback loop in the passivity theorem.

Observe also that a different gain γ_i is allowed for each different continuous interval $[a_i^B, b_i^B]$ within Ω^B where the bounded real property holds as long as the controller C is bounded by gain $\frac{1}{\gamma_i}$ for the same connected frequency interval where P is bounded by gain γ_i.

Note again that the frequency regions Ω^P, Ω^N, and Ω^B — where the positive real, negative imaginary, and bounded real properties hold — can overlap in any combination. This is beneficial as it means that we do not need to know precisely the frequencies where system properties transition.

Next, we provide a convenient equivalent LMI characterisation via the Iwasaki–Hara lemma using a realisation of plant P to check whether P has mixed positive real, negative imaginary, and bounded real properties on the specified frequency regions Ω^P, Ω^N and Ω^B. Note that the plant P can potentially have poles on the imaginary axis.

Theorem V.3. *Let $[a_i^P, b_i^P] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with $a_i^P < b_i^P$ for all $i \in \{1,\dots,r_1\}$ and define $\Omega^P = \bigcup_{i=1}^{r_1}[a_i^P, b_i^P]$. Let $[a_i^N, b_i^N] \subset \mathbb{R}_+$ with $a_i^N < b_i^N$ for all $i \in \{1,\dots,r_2\}$ and define $\Omega^N = \bigcup_{i=1}^{r_2}[a_i^N, b_i^N]$. Let $[a_i^B, b_i^B] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with $a_i^B < b_i^B$ for all $i \in \{1,\dots,r_3\}$ and define $\Omega^B = \bigcup_{i=1}^{r_3}[a_i^B, b_i^B]$.*

Furthermore, let $P = (A_p, B_p, C_p, D_p) \in \mathbf{R}^{m\times m}$ with (A_p, B_p) controllable and (C_p, A_p) detectable. Define $\Omega_0 = \{\omega_0 \in \bar{\mathbb{R}}_+ : \det(j\omega_0 I - A_p) = 0\}$.

Then, P is positive real on Ω^P, negative imaginary on Ω^N, and bounded real on $[a_i^B, b_i^B]$ with gain $\gamma_i \in \bar{\mathbb{R}}_+$ for all $i \in \{1,\dots,r_3\}$ if and only if all of the following conditions hold:

1) *A_p has at most simple eigenvalues on $j\bar{\mathbb{R}}_+$ and there exists a real matrix $X = X^T \geq 0$ such that $A_p X + X A_p^T \leq 0$;*
2) $\forall \omega_0 \in \Omega_0 \cap \Omega^P$, $\lim_{s\to j\omega_0}(s - j\omega_0)P(s) \geq 0$*;*
3) $\forall \omega_0 \in \Omega_0 \cap \Omega^N$, $\lim_{s\to j\omega_0}(s - j\omega_0)jP(s) \geq 0$*;*
4) *$\forall i \in \{1,\dots,r_1\}$, there exist complex matrices $Y_i^P, Z_i^P \in \mathbb{H}_n$ with $Z_i^P \geq 0$ such that*
$$\begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^P Z_i^P & Y_i^P + jZ_i^P \\ Y_i^P - jZ_i^P & -2\lambda_i^P Z_i^P \end{bmatrix} \begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix},$$
where $\mu_i^P = \frac{1}{a_i^P + b_i^P}$ and $\lambda_i^P = \frac{a_i^P b_i^P}{a_i^P + b_i^P}$;
5) *$\forall i \in \{1,\dots,r_2\}$, there exist complex matrices $Y_i^N, Z_i^N \in \mathbb{H}_n$ with $Z_i^N \geq 0$ such that*
$$\begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^N Z_i^N & Y_i^N + jZ_i^N \\ Y_i^N - jZ_i^N & -2\lambda_i^N Z_i^N \end{bmatrix} \begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} 0 & jI_m \\ -jI_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix},$$
where $\mu_i^N = \frac{1}{a_i^N + b_i^N}$ and $\lambda_i^N = \frac{a_i^N b_i^N}{a_i^N + b_i^N}$;
6) *$\forall i \in \{1,\dots,r_3\}$, there exist complex matrices $Y_i^B, Z_i^B \in \mathbb{H}_n$ with $Z_i^B \geq 0$ such that*
$$\begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^B Z_i^B & Y_i^B + jZ_i^B \\ Y_i^B - jZ_i^B & -2\lambda_i^B Z_i^B \end{bmatrix} \begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} \gamma_i^2 I_m & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix},$$

where $\mu_i^B = \frac{1}{a_i^B + b_i^B}$ *and* $\lambda_i^B = \frac{a_i^B b_i^B}{a_i^B + b_i^B}$.

Proof: Since the realisation of P is stabilisable and detectable, hidden modes can only occur in $\mathbb{C}_-$. This ensures that there is no difference between the eigenvalues of A_p and the poles of P in $\bar{\mathbb{C}}_+$.

Now, note that condition (iv) in Definition II.6 is automatically fulfilled since $0 \notin \Omega^N$. Next, note that conditions 1)-3) are equivalent to condition (i) in Definition II.7 and conditions (i) and (iii) in Definitions II.5 and II.6.

Next, observe that condition 4) is equivalent to condition (ii) in Definition II.5 via [37, Theorem 4]. Similarly, condition 5) is equivalent to condition (ii) in Definition II.6 via [37, Theorem 4]. Finally, condition 6) is equivalent to condition (ii) in Definition II.7 via [37, Theorem 4]. This concludes the proof. ■

Remark V.4. *If* $a_i^P = 0$ *for some* $i \in \{1, \ldots, r_1\}$, *then* $\mu_i^P = \frac{1}{b_i^P} > 0$ *and* $\lambda_i^P = 0$. *On the other hand, if* $b_i^P = \infty$ *for some* $i \in \{1, \ldots, r_1\}$, *then* $\mu_i^P = 0$ *and* $\lambda_i^P = a_i^P < \infty$. *Indeed, it is easy to see that* $\mu_i^P, \lambda_i^P \in \bar{\mathbb{R}}_+$ *for all* $i \in \{1, \ldots, r_1\}$. *Similarly for* μ_i^B *and* λ_i^B.

On the other hand, $\mu_i^N, \lambda_i^N \in \mathbb{R}_+$ *for all* $i \in \{1, \ldots, r_2\}$ *since* $a_i^N, b_i^N \in \mathbb{R}_+$ *for all* $i \in \{1, \ldots, r_2\}$ *(i.e.* a_i^N *and* b_i^N *cannot take values of* 0 *and* ∞*).*

The LMIs in conditions 4)-6) involve complex variables due to the fact that the frequency regions Ω^P, Ω^N, and Ω^B are not symmetric about 0 (i.e. they admit one-sided frequencies) [37]. These complex LMIs can be equivalently transformed into real LMIs via the following equivalence: Given $R \in \mathbb{H}_n$, $R \le 0$ (resp. < 0) $\Leftrightarrow$ $\begin{bmatrix} \mathrm{Re}(R) & \mathrm{Im}(R) \\ -\mathrm{Im}(R) & \mathrm{Re}(R) \end{bmatrix} \le 0$ (resp. < 0) .

Next, we provide a convenient equivalent LMI characterisation via the Iwasaki–Hara lemma using a realisation of controller C to check whether the chosen C fulfills the required mixed strict positive real, strict negative imaginary, and strict bounded real properties on the specified frequency regions Ω^P, Ω^N, and Ω^B. Stability of the controller C together with stabilisability and detectibility of its realisation are automatically guaranteed by the equivalent LMIs.

Theorem V.5. *Let* $[a_i^P, b_i^P] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ *with* $a_i^P < b_i^P$ *for all* $i \in \{1, \ldots, r_1\}$ *and define* $\Omega^P = \bigcup_{i=1}^{r_1} [a_i^P, b_i^P]$. *Let* $[a_i^N, b_i^N] \subset \mathbb{R}_+$ *with* $a_i^N < b_i^N$ *for all* $i \in \{1, \ldots, r_2\}$ *and define* $\Omega^N = \bigcup_{i=1}^{r_2} [a_i^N, b_i^N]$. *Let* $\gamma_i \in \bar{\mathbb{R}}_+$ *and* $[a_i^B, b_i^B] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ *with* $a_i^B < b_i^B$ *for all* $i \in \{1, \ldots, r_3\}$ *and define* $\Omega^B = \bigcup_{i=1}^{r_3} [a_i^B, b_i^B]$.

Then, $C \in \mathbf{RH}_\infty^{m \times m}$ *with stabilisable and detectable realisation* (A_c, B_c, C_c, D_c) *satisfies*

(i) $-C(j\omega) - C(j\omega)^* > 0 \quad \forall \omega \in \Omega^P$,
(ii) $j[C(j\omega) - C(j\omega)^*] > 0 \quad \forall \omega \in \Omega^N$,
(iii) $I - \gamma_i^2 C(j\omega)^* C(j\omega) > 0 \quad \forall \omega \in [a_i^B, b_i^B], i \in \{1, \ldots, r_3\}$,

if and only if all of the following conditions hold:

1) $\exists$ *a real matrix* $X = X^T > 0$ *such that* $A_c X + X A_c^T < 0$*;*
2) $\forall i \in \{1, \ldots, r_1\}$, *there exist complex matrices* $Y_i^P, Z_i^P \in \mathbb{H}_n$ *with* $Z_i^P > 0$ *such that*
$$\begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^P Z_i^P & Y_i^P + jZ_i^P \\ Y_i^P - jZ_i^P & -2\lambda_i^P Z_i^P \end{bmatrix} \begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix} < \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} 0 & -I_m \\ -I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix},$$
where $\mu_i^P = \frac{1}{a_i^P + b_i^P}$ *and* $\lambda_i^P = \frac{a_i^P b_i^P}{a_i^P + b_i^P}$*;*
3) $\forall i \in \{1, \ldots, r_2\}$, *there exist complex matrices* $Y_i^N, Z_i^N \in \mathbb{H}_n$ *with* $Z_i^N > 0$ *such that*
$$\begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^N Z_i^N & Y_i^N + jZ_i^N \\ Y_i^N - jZ_i^N & -2\lambda_i^N Z_i^N \end{bmatrix} \begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix} < \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} 0 & jI_m \\ -jI_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix},$$
where $\mu_i^N = \frac{1}{a_i^N + b_i^N}$ *and* $\lambda_i^N = \frac{a_i^N b_i^N}{a_i^N + b_i^N}$*;*
4) $\forall i \in \{1, \ldots, r_3\}$, *there exist complex matrices* $Y_i^B, Z_i^B \in \mathbb{H}_n$ *with* $Z_i^B > 0$ *such that*
$$\begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^B Z_i^B & Y_i^B + jZ_i^B \\ Y_i^B - jZ_i^B & -2\lambda_i^B Z_i^B \end{bmatrix} \begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix} < \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} I_m & 0 \\ 0 & -\gamma_i^2 I_m \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix},$$
where $\mu_i^B = \frac{1}{a_i^B + b_i^B}$ *and* $\lambda_i^B = \frac{a_i^B b_i^B}{a_i^B + b_i^B}$.

Proof: A stabilisable and detectable realisation has no hidden modes in $\bar{\mathbb{C}}_+$. Therefore, condition 1) is equivalent to A_c is Hurwitz, which in turn is equivalent to $C = (A_c, B_c, C_c, D_c) \in \mathbf{RH}_\infty^{m \times m}$ and (A_c, B_c, C_c, D_c) is stabilisable and detectable.

Next, observe that condition 2) is equivalent to condition (i) via [37, Theorem 3]. Similarly, condition 3) is equivalent to condition (ii) via [37, Theorem 3]. Finally, condition 4) is equivalent to condition (iii) via [37, Theorem 3]. This concludes the proof. ■

Theorem V.3 is based on the state-space nonstrict inequality version of the Iwasaki–Hara lemma, namely [37, Theorem 4] By contrast, Theorem V.5 makes use of the state space strict inequality version of the Iwasaki–Hara lemma, i.e. [37, Theorem 3]. The two results are distinct and built upon different assumptions on the state-space realisations.

VI. A MOTIVATING EXAMPLE

Consider a single-input-single-output flexible structure with force actuator and position sensor described in [1] by
$$P(s) = \frac{k_0}{s} + \sum_{i=1}^{m} \frac{k_i}{s^2 + 2\zeta_i \omega_i s + \omega_i^2},$$
where $k_0 > 0$, $0 \le \zeta_i < 1$, and $0 < \omega_i < \omega_{i+1}$ for all $i \in \{1, \ldots, m-1\}$. P is said to be in-phase if the coefficients of the flexible modes k_i have the same sign as k_0, i.e. $k_i > 0$ for all $i \in \{1, \ldots, m\}$. This happens when the force actuator is colocated with the position sensor, and P is a negative imaginary system that is amenable to the negative imaginary theory [26]. The reader is referred to [2, Section II], where a variety of justifications is provided from practical perspectives for the finite-frequency positive realness or negative imaginariness to be a crucial property for good control performance in the case where the control authority is limited, especially when the force actuator is not colocated with the position sensor. This renders some of the k_i's to be negative, and that P exhibits negative imaginariness only in a finite frequency band. Feedback control design for such a P may be performed with the aid of the main results in this paper, as demonstrated below. An alternative $\mathbf{H}_\infty$ loop shaping based design procedure may be found in [50].

Consider a simple, illustrative model for a flexible structure with noncolocated force actuator and position sensor:
$$P(s) = \frac{1}{s} + \frac{1}{s^2 + 1} + \frac{-0.2}{s^2 + 0.1s + 5^2}.$$
Note that P has marginally stable poles (i.e. not a bounded real system) and is neither a passive system nor a negative imaginary

system according to Definitions II.5 and II.6. Nonetheless, it may be verified with the aid of the Iwasaki–Hara lemma based Theorem V.3 from the preceding section that $P(j\omega)$ is positive real for $\omega \in [0, 0.1]$ as per Definition II.5, negative imaginary for $\omega \in [0.1, 4]$ as per Definition II.6, and $|P(j\omega)| \leq 0.5$ for $\omega \in [4, \infty]$ (c.f. Definition II.7). Note that in actual fact $P(j\omega)$ is negative imaginary also for $\omega \in [4, 4.94]$ and $|P(j\omega)| \leq 0.5$ for $\omega \in [2.3, \infty]$, i.e. there is an overlap of the frequencies where $P(j\omega)$ is negative imaginary and has a gain 0.5 or less. In accordance with Theorem III.2 and Remark III.3, by defining $\Pi(\omega) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ for $\omega \in [0, 0.1]$, $\Pi(\omega) = \left[\begin{smallmatrix} 0 & j \\ -j & 0 \end{smallmatrix}\right]$ for $\omega \in (0.1, 4)$, and $\Pi(\omega) = \begin{bmatrix} 0.5^2 & 0 \\ 0 & -1 \end{bmatrix}$ for $\omega \in [4, \infty]$, it follows that any $C \in \mathbf{RH}_\infty$ that satisfies $C(j\omega) + C(j\omega)^* < 0$ for $\omega \in [0, 0.1]$, $j(C(j\omega) - C(j\omega)^*) > 0$ for $\omega \in [0.1, 4]$, and $|C(j\omega)| < \frac{1}{0.5}$ for $\omega \in [4, \infty]$, would render $[P, C]$ stable. One such example is $C(s) = \frac{1}{s+1} - 1.1$.

VII. Conclusions and Future Research

We derived frequency-varying conditions for robust feedback stability of systems with marginally stable open-loop poles. Such conditions allow the "mixing" of various useful system properties on the imaginary axis in a seamless fashion, and are particularly useful for stabilising flexible structures with noncolocated force actuators and positions sensors, from which positive realness and negative imaginariness are manifested only on limited frequency bands. The results are intimately related to and complemented by the Iwasaki–Hara lemma, with which frequency-band-limited system properties may be characterised in a computationally effective manner using linear matrix inequalities.

Performing control synthesis that results in a closed-loop system manifesting frequency-limited negative imaginariness via the Iwasaki–Hara lemma following the approaches in [51] is an interesting future research direction. Specifically, characterisations of strictly negative imaginary and output strictly negative imaginary systems on an open frequency band is of significant importance from the perspective of control design [52]. Nevertheless, these systems are not amenable to the Iwasaki–Hara lemma in its present form and further investigations are urgently needed.

References

[1] K. Ono and T. Teramoto, "Design methodology to stabilize the natural modes of vibration of a swing-arm positioning mechanism," *ASME Adv. Info. Storage Syst.*, vol. 4, pp. 343–359, 1992.

[2] T. Iwasaki, S. Hara, and H. Yamauchi, "Dynamical system design from a control perspective: Finite frequency positive-realness approach," *IEEE Transactions on Automatic Control*, vol. 48, no. 8, pp. 1337–1354, 2003.

[3] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.* New York: Academic Press, 1975.

[4] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 3rd ed., ser. Communications and Control Engineering. Springer, 2016.

[5] I. R. Petersen and A. Lanzon, "Feedback control of negative-imaginary systems," *IEEE Contr. Syst. Magazine*, vol. 30, no. 5, pp. 54–72, 2010.

[6] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Trans. Automat. Contr.*, vol. 55, no. 10, pp. 2342–2347, 2010.

[7] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 42, no. 6, pp. 819–830, 1997.

[8] A. Megretski, U. Jönsson, C.-Y. Kao, and A. Rantzer, "The control handbook," in *Integral quadratic constraints*, 2nd ed., W. S. Levine, Ed. Springer-Verlag, 2011.

[9] P. Seiler, "Stability analysis with dissipation inequalities and integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 60, no. 6, pp. 1704–1709, 2015.

[10] J. Carrasco and P. Seiler, "Conditions for the equivalence between IQC and graph separation stability results," *International Journal of Control*, 2018.

[11] S. Z. Khong, "On integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 67, no. 3, pp. 1603–1608, 2022.

[12] A. Rantzer and A. Megretski, "System analysis via integral quadratic constraints: Part II," Lund Institute of Technology, Tech. Rep. LUTFD2/TFRT–7559-SE, 1997.

[13] M. Cantoni, U. T. Jönsson, and C.-Y. Kao, "Robustness analysis for feedback interconnections of unstable distributed systems via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 57, pp. 302–317, 2012.

[14] M. Cantoni, U. T. Jönsson, and S. Z. Khong, "Robust stability analysis for feedback interconnections of time-varying linear systems," *SIAM J. Control Optim.*, vol. 51, no. 1, pp. 353–379, 2013.

[15] S. Z. Khong and M. Cantoni, "Reconciling ν-gap metric and IQC based robust stability analysis," *IEEE Trans. Autom. Contr.*, vol. 58, no. 8, pp. 2090–2095, 2013.

[16] S. Z. Khong, E. Lovisari, and A. Rantzer, "A unifying framework for robust synchronisation of heterogeneous networks via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 61, no. 5, pp. 1297–1309, 2016.

[17] S. Z. Khong, E. Lovisari, and C.-Y. Kao, "Robust synchronization in multi-agent networks with unstable dynamics," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 1, pp. 205–214, 2018.

[18] J. F. Barmanj and J. Katzenelson, "A generalized Nyquist-type stability criterion for multivariable feedback systems," *International Journal of Control*, vol. 20, no. 4, pp. 593–622, 1974.

[19] A. G. MacFarlane and I. Postlethwaite, "The generalized Nyquist stability criterion and multivariable root loci," *International Journal of Control*, vol. 25, no. 1, pp. 81–127, 1977.

[20] C. Desoer and Y.-T. Wang, "On the generalised Nyquist stability criterion," *IEEE Trans. Autom. Contr.*, vol. 25, no. 2, pp. 187–196, 1980.

[21] N. Kottenstette, M. J. McCourt, M. Xia, V. Gupta, and P. J. Antsaklis, "On relationships among passivity, positive realness, and dissipativity in linear systems," *Automatica*, vol. 50, no. 4, pp. 1003–1016, 2014.

[22] C.-Y. Kao, S. Z. Khong, and A. van der Schaft, "On the converse passivity theorems for LTI systems," In *Proc. 21st IFAC World Congress*, vol. 53, no. 2, pp. 6422–6427, 2020.

[23] S. Z. Khong and C.-Y. Kao, "Addendum to "Converse theorems for integral quadratic constraints"," *IEEE Trans. Automat. Contr.*, vol. 67, no. 1, pp. 539–540, 2022.

[24] S. Z. Khong and A. van der Schaft, "On the converse of the passivity and small-gain theorems for input–output maps," *Automatica*, vol. 97, pp. 58–63, 2018.

[25] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Trans. Autom. Contr.*, vol. 53, no. 4, pp. 1042–1046, 2008.

[26] A. Lanzon and H.-J. Chen, "Feedback stability of negative imaginary systems," *IEEE Trans. Automat. Contr.*, vol. 62, no. 11, pp. 5620–5633, 2017.

[27] S. Z. Khong, I. R. Petersen, and A. Rantzer, "Robust stability conditions for feedback interconnections of distributed-parameter negative imaginary systems," *Automatica*, vol. 90, pp. 310–316, 2018.

[28] D. Zhao, C. Chen, and S. Z. Khong, "A frequency-domain approach to nonlinear negative imaginary systems analysis," *Automatica*, vol. 146, p. 110604, 2022.

[29] S. Z. Khong and C.-Y. Kao, "Converse theorems for integral quadratic constraints," *IEEE Trans. Automat. Contr.*, vol. 66, no. 8, pp. 3695–3701, 2021.

[30] S. Z. Khong, "Feedback stability of generalised positive real and negative imaginary systems," *IEEE Trans. Automat. Contr.*, 2023.

[31] W. M. Griggs, B. D. O. Anderson, and A. Lanzon, "A "mixed" small gain and passivity theorem in the frequency domain," *Systems & Control Letters*, vol. 56, no. 9–10, pp. 596–602, 2007.

[32] W. M. Griggs, B. D. O. Anderson, A. Lanzon, and M. Rotkowitz, "Interconnections of nonlinear systems with "mixed" small gain and passivity properties and associated input-output stability results," *Systems & Control Letters*, vol. 58, no. 4, pp. 289–295, 2009.

[33] S. Patra and A. Lanzon, "Stability analysis of interconnected systems with "mixed" negative-imaginary and small-gain properties," *IEEE Trans. Automat. Contr.*, vol. 56, no. 6, pp. 1395–1400, 2011.

[34] A. Lanzon and P. Bhowmick, "Characterization of input-output negative imaginary systems in a dissipative framework," *IEEE Trans. Automat. Contr.*, vol. 68, no. 2, pp. 959–974, 2023.

[35] S. Z. Khong, C. Chen, and A. Lanzon, "Feedback stability analysis via dissipativity with dynamic supply rates," *Automatica*, 2024, under review.

[36] T. Iwasaki and S. Hara, "Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations," *IEEE Trans. Autom. Contr.*, vol. 43, no. 5, pp. 619–630, 1998.

[37] ——, "Generalized KYP lemma: Unified frequency domain inequalities with design applications," *IEEE Trans. Autom. Contr.*, vol. 50, no. 1, pp. 41 – 59, 2005.

[38] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Systems & Control Letters*, vol. 28, no. 1, pp. 7–10, 1996.

[39] T. Iwasaki, G. Meinsma, and M. Fu, "Generalized S-procedure and finite frequency KYP lemma," *Mathematical Problems in Engineering*, vol. 6, pp. 305–320, 2000.

[40] S. Hara, M. Kanno, and M. Onishi, "Finite frequency phase property versus achievable control performance in H_∞ loop shaping design," in *2006 SICE-ICASE International Joint Conference*, 2006, pp. 3196–3199.

[41] M. Kanno, S. Hara, and M. Onishi, "Characterization of easily controllable plants based on the finite frequency phase/gain property: A magic number $4+2\sqrt{2}$ in $\mathscr{H}_\infty$ loop shaping design," in *American Control Conference*, 2007, pp. 5816–5821.

[42] T. Iwasaki, S. Hara, and A. L. Fradkov, "Time domain interpretations of frequency domain inequalities on (semi) finite ranges," *Systems & control letters*, vol. 54, no. 7, pp. 681–691, 2005.

[43] J. Xiong, I. R. Petersen, and A. Lanzon, "Finite frequency negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 11, pp. 2917–2922, 2012.

[44] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge: Cambridge University Press, 1991.

[45] A. Megretski and S. Treil, "Power distribution inequalities in optimization and robustness of uncertain systems," *J. Math. Syst., Estimat. Control*, vol. 3, no. 3, pp. 301–319, 1993.

[46] A. Ringh, X. Mao, W. Chen, L. Qiu, and S. Z. Khong, "Gain and phase type multipliers for structured feedback robustness," *IEEE Transactions on Automatic Control*, 2024, conditionally accepted.

[47] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[48] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, ser. Texts in Applied Mathematics 21. Springer-Verlag, 1995.

[49] B. A. Francis, "Lecture notes in control and information sciences," 1987.

[50] K. Ohno, S. Hara, N. Yamahira, T. Kawabe, and T. Maruyama, "A practical loop shaping design procedure with classical control criteria and its application to hard disk drives," *IFAC Proceedings Volumes*, vol. 41, no. 2, pp. 2002–2007, 2008.

[51] T. Iwasaki and S. Hara, "Feedback control synthesis of multiple frequency domain specifications via generalized KYP lemma," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 5-6, pp. 415–434, 2007.

[52] S. Z. Khong, D. Zhao, and A. Lanzon, "Converse negative imaginary theorems," *Automatica*, 2024, in press.