GROUP ACTIONS AND STABILITY NOTIONS ON ALGEBRAIC VARIETIES

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2024

Valentin Boboc
Department of Mathematics
School of Natural Sciences

Contents

Abstract **5**

Declaration **6**

Copyright Statement **7**

Acknowledgements **9**

1 Motivation **11**

1.1 Overview 11

1.2 Contents of the thesis 15

2 Preliminaries **18**

2.1 Slope stability of coherent sheaves 18

2.2 T-varieties 21

2.3 Coherent sheaves over toric varieties 33

2.4 Galois covers 39

3 The case of $\mathbb{C}^*$-surfaces with fibrewise action **42**

3.1 Intersection numbers in complexity one 43

3.2 A combinatorial criterion for stability of $\mathsf{T_X}$ 50

4 The case of Hirzebruch surfaces in general **57**

4.1 Minimal surfaces 57

4.2 A remark vis-à-vis the ample cone 60

4.3 A general criterion for semistability of $\mathsf{T_X}$ 61

4.4 Applying the criterion to concrete examples 63

4.5 A special surface 70

5 The case of Weierstrass fibrations **87**

6 Conclusions and future work **94**

Bibliography **98**

Word count 15,479

List of Figures

2.1 A decomposition of a polyhedron into the Minkowski sum of a polytope and its recession cone 23

2.2 Downgrade of a Hirzebruch surface using a natural subtorus action . . 26

2.3 Slices of the resulting divisorial fan $\mathcal{S}$ 26

2.4 The reflexive polytope associated to dP_3 30

2.5 The divisorial polytope of the downgrade of dP_3 31

3.1 Intersecting a vertical divisor with a prime divisor on a $\mathbb{C}^*$-surface . . . 45

3.2 Expressing X^1 as two different blow ups 53

4.1 A schematic of an admissible blowup 64

4.2 A schematic of a generic blowup 64

4.3 A special downgrade of $\mathbb{F}_2$ 71

4.4 Slices of the divisorial fan of X 72

4.5 A diagram of the example surface $\mathsf{X}(\mathcal{M})$ 72

4.6 Slices of the divisorial fan of Y 74

The University of Manchester

Valentin Boboc
Doctor of Philosophy
Group Actions and Stability Notions on Algebraic Varieties
September 16, 2024

In this thesis, we investigate the stability of the tangent bundles of various types of algebraic surfaces. We begin by introducing the notion of slope stability in the sense of Mumford and Takemoto and review some preliminaries concerning the tools and techniques we employ to prove later results. We generalise a result by Hering-Nils-Süß concerning toric varieties to the case of complexity one $\mathbb{C}^*$-surfaces with fibrewise group actions. Next, we derive a criterion for the semistability of the tangent bundle of blow ups of Hirzebruch surfaces in the general setting and explore some examples and counterexamples. Finally, we give a full description or the tangent bundles of smooth Weierstrass fibrations using the topological Euler characteristic and the genus of the base curve.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Library's regulations (see http://www.manchester.ac.uk/library/about/regulations) and in The University's Policy on Presentation of Theses.

Acknowledgements

First and foremost, I would like to thank my supervisor, Hendrik Süß, for his continued support all throughout these four years. It is hard to imagine completing this work without his patient guidance and mentoring.

I appreciate my family's efforts to stay enthusiastic about my research in spite of the fact that the topic remains a mystery to them – as it does to me, but to a different extent. In particular, I am grateful beyond words for my grandmother's love, for the wisdom she imparted, and for being the first person to encourage me to study mathematics. Her memory is a blessing.

I am grateful for my secondary school teachers, Aura and Ion, who instilled in me a love for geometry and algebra, respectively. Your efforts and your dedication is not forgotten.

My gratitude also extends to all the colleagues and friends who in spite of it all made these four years special. From London (and previously Bristol), Ania has been both a source of support and an enabler of my best (and worst) Netflix habits. You're an amazing friend, and I truly appreciate how close we still are after all these years.

In Birmingham, I'd like to thank Hannah and Jamie for not letting me win at Catan that one time. It would have gone to my head.

All the way back to Romania, special thanks goes to my gym buddy Nico who has known me for twenty years now and who has been a great friend for quite literally as long as my memory serves.

As far as Manchester is concerned, there are many people to name, so you will all forgive the following, non-exhaustive list. I'd like to thank Tom and Gabriel for starting the Lockdown Chess Club with me. We learned that we should always play f3, no matter what. I would also like to mention all the second years who share too much of my terrible sense of humour: Shez, Miko, Georgia, and of course Alice, my loyal partner unto binge watching Bridgerton and the related spin-offs. Thank you all for the support and for the good times we spent together.

My deepest appreciation also extends to my bridge therapists, Hollie and Frank. You couldn't model how enjoyable it is to vent to you at 9 a.m. in the morning.

The list would feel worse than incomplete without mentioning my first ever office mates. Ulla, Jacob, and Kai all helped me figure things out in the early days when I needed it the most. And I could never forget Mahah. Ever since you left, the office became markedly quieter and less musical.

A final shout-out in no particular order goes to: Joe, whom I wish the best of luck with those pesky finite groups; the other Joe, who doesn't need too much luck with his chemistry; Sunny for her excellent musical tastes; David for the storytelling; Calum, who will hopefully figure out those weak equivalencies soon; Dale, for encouraging me to apply for that job with DBT; Barbara Fantechi, for replying to some more or less desperate online correspondence; and finally Amir for being a great new housemate and friend.

I could say more, but to quote a better author, my paper reminds me to conclude.

Chapter 1

Motivation

1.1 Overview

This thesis explores new results concerning the slope stability of the tangent bundles of algebraic surfaces. Our aim is to further the understanding of the structure of the tangent bundles on surfaces which are birational to ruled surfaces or strictly elliptic surfaces. The results shrink the gap in the knowledge present in the literature concerning these types of surfaces, featuring both intuitively predictable conclusions and new counter-examples. The methods we use are an eclectic mix of combinatorial and algebraic techniques that demonstrate the applicability of a wide selection of concepts, ranging from Galois covers of algebraic varieties to the combinatorial language of T-varieties.

We lay out a brief primer concerning the relevance of the concept of stability in algebraic geometry. When one studies a type of mathematical object, a question that arises naturally is the question of classifying such objects. In the most basic sense, a classification is a process which collects objects into "classes," that is intensionally defined sets of objects whose members all fulfil certain criteria or possess certain properties. In other words, a classification requires first and foremost a criterion for distinguishing between different classes of objects.

The main motivation behind studying different stability notions on coherent sheaves is given by the philosophy of classification. While the literature on this topic itself is rather scant (see e.g. [20]), the philosophy of classification addresses foundational questions in multiple scientific disciplines, although biology is perhaps the most prominent and the most well known such discipline.

There are many classifications that most students of mathematics are aware of. Take the example of classifying finite sets up to bijection. The only invariant required is the cardinality of the set. Up to bijection, there is a unique set of cardinality n for all $n \in \mathbb{N}$. In fact, we can collect all the classes of finite sets and put them together into a parameter space (or moduli space). This is in our case the set $\mathbb{N}$ whose elements are in bijection with equivalence classes of finite sets.

Note that if we change the criterion and ask to classify finite sets up to equality rather than bijection, then the classification is much less aesthetically pleasing. This is because there are plenty of sets with n elements which are bijective to each other but not equal. Consider for example the four element bijective sets:

$$\{1, 2, 3, 4\} \neq \{a, b, c, d\} \neq \{\text{parsely, sage, rosemary, thyme}\} \neq \dots$$

Other elementary examples of moduli spaces are the projective line $\mathbb{P}^1$, which parametrises all the lines in the affine plane that pass through the origin, and the projective space $\mathbb{P}^5$, which can (alternatively) be regarded as the moduli space of homogeneous quadrics in three variables as every homogeneous quadric can be defined by six ordered coefficients up to scalar multiples. In mathematical symbols: Let x, y, z be the three variables. Then a homogeneous quadric

$$Q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz$$

is in bijective correspondence with the tuple

$$p_Q = [A : B : C : D : E : F] \in \mathbb{P}^5$$

which represents the coefficients of $Q(x, y, z)$ in the stipulated order. The equivalence relation is given by scaling by a non-zero scalar: if $\lambda \neq 0$, then $\lambda \cdot p_Q \sim p_Q$.

Definition 1. Informally, a parametrisation problem (or a moduli problem) consists of the following data:

1. a notion of object together with an equivalence relation (a classification criterion),
2. a notion of family of objects over a base variety B together with a notion of equivalence of families, and
3. a notion of pullback of families which is compatible with the equivalence relation on families.

Definition 2. A fine moduli space for a parametrisation problem as above is a variety M which admits a universal family X, that is X is a family over M (a flat morphism $f : X \to M$) such that every other family over a variety B is obtained up to equivalence by pulling back X via a unique morphism $p : B \to M$.

This gives the desired bijection

$$\{\text{equivalence classes of families over } M\} \longleftrightarrow \{\text{morphisms } B \to M\}.$$

Remark 3. The universal family is tautological in the following sense. Let $x \in M$ be a point, let $p : \{x\} \hookrightarrow M$ be the inclusion morphism, and let O_x be the object that corresponds to the point x. Using the universal family property, we have that $O_x \simeq p^*X$. Here, we think of O_x as a family over a point. However, the fibre is given by $p^*X \simeq f^{-1}(x)$, and we recall that $f : X \to M$ is the universal family. This gives us $f^{-1}(x) \simeq O_x$.

Example 4. In the example of $\mathbb{P}^5$ as the moduli space of homogeneous quadrics in three variables, consider the universal family $f : X \to \mathbb{P}^5$. Then the quadric Q can be identified with the fibre $f^{-1}(p_Q)$.

The contents of this thesis are partially motivated by the following parametrisation problem.

Question 5. Let X be a smooth algebraic surface over the complex numbers. Let $\mathrm{VB}(\mathsf{X})$ be the set of isomorphism classes of algebraic vector bundles over X. Can we endow $\mathrm{VB}(\mathsf{X})$ with a natural structure of a scheme?

The answer is negative (see e.g. [30]). But it turns out that there exists a sub-class of coherent torsion-free sheaves which give a positive answer to this question. In other words, if we remove a certain number of sheaves from $\mathrm{VB}(\mathsf{X})$ then we can construct a fine moduli space and use all the machinery that comes with having a moduli space endowed with the structure of a scheme. One immediate example is that a curve in $\mathrm{VB}(\mathsf{X})$ traces out a one-parameter family of torsion-free coherent sheaves over X. The correct coherent heaves we need to consider are precisely the ones that have the property of being stable.

It is perhaps worth considering the point of view of the philosophy of classification. The suggested classification doesn't fall under the point of view of either essentialism (the view that classes are discrete and membership can be decided by inspecting empirical properties or "essences") or cluster analysis (the view that objects can be put into the same class if they share enough properties that have the tendency to occur together). It is rather something closer to "pragmatic realism" [18]. In loc. cit., Dupreé argues that for most classification problems, there will likely be multiple criteria which yield reasonable classifications of objects and that none of these criteria are privileged "in a natural sense." This approach is realist in that the criterion we choose (stability) is to some extent an objective and verifiable property of the mathematical objects we study. The approach is pragmatic because we choose to privilege stability due to the fact that it enables us to carry out a further classification which has functionally and aesthetically desirable properties, i.e. we obtain a fine moduli space and benefit from the machinery provided by algebraic geometry.

Indeed, many authors have worked on alternative classifications where either the notion of a fine moduli space is weakened or the requirement of being a scheme is weakened to the requirement of being a stack. These are all different methods which give results which feel relevant in different settings.

There has also been plenty of work in the field of mathematical physics, in particular particle physics and string theory, where the notion of coherent sheaf is used to model various physical objects and phenomena. One example is thinking of vector bundles as gauge fields on D-branes. For a comprehensive introduction, the reader is directed to consult [17]. In this setting, the slope of a vector bundle is nothing other than the "charge density" of the corresponding D-brane configuration. A D-brane is stable if it is a BPS state, which is the higher dimensional generalisation of the classical concept of a charged black hole being an extremal black hole, i.e. it carries maximum charge.

Such connections also provide a pseudo-essentialist reason for why the reader should care about different notions of stability on coherent sheaves, that is there exist applications to other applied disciplines.

Finally, the tangent sheaf is in some sense the most canonical coherent sheaf that one can construct on an algebraic surface. From the point of view of a geometer studying algebraic surfaces, understanding more about the structure of this fundamental object is a task which is interesting in and of itself.

In the sequel, we summarise the contents of the remaining chapters.

1.2 Contents of the thesis

Chapter 2 - Preliminaries

We fix some notation and collect some useful facts and results that are relevant in subsequent chapters. We begin by defining the notion of slope stability of coherent sheaves together with an overview of known results and connections to special metrics on vector bundles. We continue with an overview of the language of divisorial fans and polyhedral divisors necessary for describing T-varieties. Next, we discuss the Klyachko description of vector bundles over toric varieties and briefly describe parliaments of polytopes – combinatorial objects that allow us to compute global sections of toric vector bundles. We give a brief overview of log pairs and log Fano varieties together and finish the chapter with a primer on Galois covers of algebraic varieties and interactions with slope stability.

Chapter 3 - The case of $\mathbb{C}^*$-surfaces with fibrewise action

We investigate the stability of the tangent bundles of $\mathbb{C}^*$-surfaces with a very particular type of group action. We begin by proving some combinatorial results concerning intersection numbers of T-invariant divisors in complexity one. We proceed to describe a combinatorial criterion for semistability of the tangent bundles of such surfaces. The result demonstrates a parallel between $\mathbb{C}^*$-surfaces with fibrewise action and toric varieties.

Chapter 4 - The case of Hirzebruch surfaces in general

This chapter attempts to make a more programmatic analysis of the stability of tangent bundles of Hirzebruch surfaces and their blow ups. In the first part, we investigate minimal ruled surfaces and determine that the semistability of tangent bundles is determined by the size of certain invariants. Next, we remark that the ample cone of a surface obtained by blowing up a Hirzebruch surface can be partitioned into pairwise-disjoint convex sets determined by regions where the tangent bundle is semistable or unstable. We deduce a criterion for deciding whether semistable regions within the ample cone exist or not. We then apply our criterion to some specific examples. In particular, we show that certain “nicely arranged” blow ups of Hirzebruch surfaces always have unstable tangent bundles. This provides an alternative (not combinatorial) proof of the main theorem from Chapter 3. We conclude this chapter by analysing in detail a blow up of the second Hirzebruch surface. We use a combination of tools to show that we have constructed an example of a surface with stable tangent bundle which cannot be obtained as an iterated blow up of $\mathbb{P}^2$, but rather only as an iterated blow up of a minimal singular surface with stable tangent bundle.

Chapter 5 - The case of Weierstrass fibrations

Smooth Weierstrass fibrations are a large and special class of strictly elliptic surfaces. In particular, they are surfaces of Kodaira dimension one, about which significantly less is known concerning their tangent bundles and various notions of stability. Using the fact that such fibrations can be expressed as branched double covers of minimal ruled surfaces, we provide a full classification for the stability of the tangent bundles of such surfaces according to their topological Euler characteristic. We close the chapter with some observations concerning blow ups of Weierstrass fibrations, but we fail to offer a complete classification.

Chapter 7 - Conclusions and future work

The final chapter consists of some final reflections together with a list of problems that remain open and which the author would like to undertake in the future.

Chapter 2

Preliminaries

2.1 Slope stability of coherent sheaves

Let X be a smooth variety, $\mathcal{F}$ a coherent sheaf over X, and $\mathsf{H} \in \mathrm{Pic}(\mathsf{X})$ an ample line bundle. The concept of slope stability was introduced by Mumford and Takemoto for the purpose of constructing moduli spaces of sheaves. The slope of $\mathcal{F}$ with respect to H is the rational number

$$\mu_{\mathsf{H}}(\mathcal{F}) := \frac{\mathsf{H}^{\dim(\mathsf{X})-1} \cdot c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}.$$

A coherent sheaf $\mathcal{F}$ is said to be semistable if for all coherent subsheaves $\mathcal{E} \subset \mathcal{F}$ we have $\mu_{\mathsf{H}}(\mathcal{E}) \leqslant \mu_{\mathsf{H}}(\mathcal{F})$. The sheaf $\mathcal{F}$ is stable if the former inequality is strict. A sheaf is polystable if it splits into a direct sum of stable sheaves with identical slopes. By convention, we say that a sheaf is unstable if it is not semistable.

Besides the geometric aim of classifying stable vector bundles, another important motivation for studying the stability of coherent sheaves on varieties is given by a sequence of famous results on moduli spaces of connections in the presence of complex and symplectic structure. The most famous of these assertions is the following.

Donaldson-Uhlenbeck-Yau Theorem ([16][51]). On a compact Kähler manifold, any semistable holomorphic vector bundle with trivial determinant line bundle admits a Hermite-Einstein connection.

This ties in to the problem of finding Kähler-Einstein metrics on complex manifolds. The proof of the following conjecture makes another connection between analytical methods in complex geometry and algebraic geometry.

Yau-Tian-Donaldson Conjecture ([10][11][12][13]). A Fano manifold X admits a Kähler-Einstein metric if and only id the polarised algebraic variety $(X, -K_X)$ is K-polystable.

We deduce the following chain of implications.

$$X \text{ is Kähler-Einstein} \Leftrightarrow X \text{ is K-polystable} \Rightarrow T_X \text{ is polystable} \Rightarrow T_X \text{ is semistable.}$$

The first arrow is the Yau(丘)-Tian(田)-Donaldson Conjecture. The second arrow follows from the Donaldson-Uhlenbeck-Yau Theorem, and the last arrow follows from a simple exercise, presented below.

Proposition 6. *Let X be a smooth algebraic variety and let $\mathcal{F}$ be a coherent sheaf on X. If $\mathcal{F}$ is polystable with respect to H then $\mathcal{F}$ is semistable with respect to H.*

Proof. Let $\mathcal{F}$ be a polystable sheaf with respect to some ample H. By definition, we can write $\mathcal{F}$ as the direct sum

$$\mathcal{F} = \oplus_{i=1}^{k} F_i,$$

where F_i are all stable coherent subsheaves and the following chain of equalities holds

$$\mu_H(F_1) = \mu_H(F_2) = \ldots = \mu_H(F_k) = \mu_H(\mathcal{F}).$$

Let $\mathcal{E} \subset \mathcal{F}$ be a coherent subsheaf. In particular, there exists a non-trivial morphism $\mathcal{E} \to \mathcal{F}$. In our case, this is the same as a non-trivial morphism $\mathcal{E} \to F_i$ for some $1 \leqslant i \leqslant k$. Since F_i is stable, we have that $\mu_H(E) \leqslant \mu_H(F_i)$, where equality can only be achieved when $\mathcal{E} \simeq F_i$. □

Corollary 7. *Let X be a smooth Fano variety. If T_X is unstable with respect to the anticanonical polarisation, then X does not admit a Kähler-Einstein metric.*

Note that stability of the tangent bundle does not imply the existence of a Kähler-Einstein metric. This prompts the following problem.

Question 8. Classify which complex varieties have (semi)stable tangent bundles.

The following is a summary of known results in the case of smooth surfaces.

Proposition 9. *Let X be a smooth complex surface.*

1. *If X is del Pezzo, then T_X is polystable with respect to $-K_X$.*
2. *If X is a $K3$ surface, then T_X is polystable with respect to any $H \in \mathrm{Pic}(X)$.*

For rational toric surfaces, we have a complete description due to Hering, Nill, and Süß [29].

Proposition 10. *Let X be a smooth toric surface. Then there exists an ample $H \in \mathrm{Pic}(X)$ such that T_X is semistable with respect to H if and only if X is an iterated blow up of $\mathbb{P}^2$.*

Recall the following facts about filtrations.

Proposition 11 (see [26])**.** *Let $\mathcal{F}$ be a coherent, torsion-free sheaf on a smooth projective variety X, and let H be an ample divisor. Then there exists a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

of $\mathcal{F}$ depending on H, called the Harder-Narasimhan filtration, such that:

1. *the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are torsion-free;*
2. *the slopes of the quotients satisfy $\mu_H(\mathcal{F}_1) > \mu_H(\mathcal{F}_2/\mathcal{F}_1) > \ldots > \mu_H(\mathcal{F}/\mathcal{F}_{k-1})$.*

Definition 12. Let $\mathcal{F}$ be a coherent, torsion-free sheaf on a smooth projective variety X, and let H be an ample divisor. The unique sheaf $\mathcal{F}_1$ featuring in the Harder-Narasimhan filtration of $\mathcal{F}$ is called the maximal destabilising subsheaf of $\mathcal{F}$.

We end this section by fixing some notation. In particular, for an algebraic surface X, we define the following subsets of the ample cone of X, which we refer to as the *stable locus* and *semistable locus*, respectively.

$$\mathrm{Stab}(\mathcal{F}) := \{\mathsf{H} \in \mathrm{Amp}(\mathsf{X}) : \ \mathcal{F} \text{ is stable with respect to } \mathsf{H}\},$$
$$\mathrm{sStab}(\mathcal{F}) := \{\mathsf{H} \in \mathrm{Amp}(\mathsf{X}) : \ \mathcal{F} \text{ is semistable with respect to } \mathsf{H}\}.$$

2.2 T-varieties

A T-variety is a complex algebraic variety equipped with an effective action of an algbebraic torus T. The difference between the dimension of the variety and the dimension of the torus acting on it is called complexity. Zero complexity T-varieties are toric varieties. By now classical objects, there is vast literature dedicated to studying their properties from multiple points of view. The reader is directed to [14] and [22] for a comprehensive introduction.

We focus exclusively on T-varieties of complexity one. Akin to toric varieties, T-varieties can also be described using combinatorial data. In the case of complexity one, the most basic combinatorial data consists of a collection of polyhedra with identical recession cones parametrised by the points of a curve, which happens to be the Chow quotient by the torus action. A very precise introduction can be found by consulting [2] [4] [49] and the references therein. We give a brief primer.

The algebraic torus

By an algebraic torus, we understand $T = (\mathbb{C}^*)^n$. To such a torus we associate the lattice of characters M and the lattice of one-parameter subgroups N, defined as:

$$M := \mathrm{Hom}(T, \mathbb{C}^*),$$
$$N := \mathrm{Hom}(\mathbb{C}^*, T).$$

The associated $\mathbb{R}$-vector spaces are denoted $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. The lattices come with a perfect pairing $M \times N \to \mathbb{Z}$ which can be naturally extended to a bilinear pairing $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$, which we denote $(m, v) \mapsto \langle m, v \rangle$. The torus can be identified with the algebraic variety whose coordinate algebra is the $\mathbb{C}$-algebra generated by its character lattice. In symbols: $T \simeq \mathrm{Spec}(\mathbb{C}[M])$.

Polyhedral divisors

We assume familiarity with basic notions of convex geometry, chiefly cones, fans, and polyhedra living in the extended lattices $M_\mathbb{R}$ and $N_\mathbb{R}$.

It is a well-known fact that any polyhedron Π can be decomposed as

$$\Pi = P + \sigma,$$

where P is a polytope (a bounded convex region), and σ is a potentially empty cone (an unbounded convex region), which is uniquely determined. See Figure 2.1 for an example. The cone σ is called a recession cone, however in [4], the authors refer to it as a tail cone.

We write $\mathrm{tail}(\Pi) = \sigma$. We define the set of σ-tailed lattice polyhedra in N to be

$$\mathrm{Pol}_\mathbb{R}(N, \sigma) = \{\Pi \subset N_\mathbb{R} : \ \mathrm{tail}(\Pi) = \sigma\}.$$

Under Minkowski sum, i.e.

$$\Pi + \Pi' = \{u + u' : \ u \in \Pi \text{ and } u' \in \Pi'\}$$

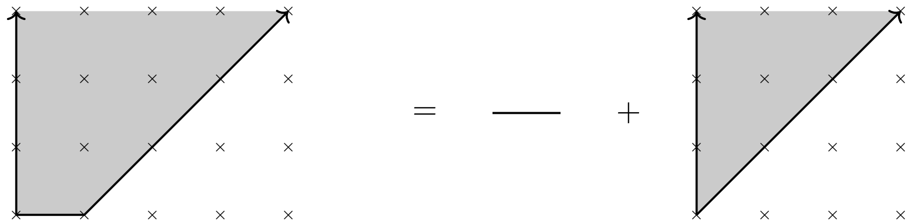

Figure 2.1: A decomposition of a polyhedron into the Minkowski sum of a polytope and its recession cone

the set $\mathrm{Pol}_{\mathbb{R}}(N, \sigma)$ becomes an abelian semigroup. The cone σ acts as an identity under Minkowski sum, and we furthermore append the empty polyhedron $\emptyset \subset N_{\mathbb{R}}$ as an absorbant element, i.e. $\emptyset + \Pi = \emptyset$ for all $\Pi \subset N_{\mathbb{R}}$.

Definition 13. A polyhedral divisor $\mathfrak{D}$ supported on a smooth curve Y with tail cone σ is a finite formal sum

$$\mathfrak{D} = \sum_{P} \mathfrak{D}_P \otimes P$$

where P runs through all the prime divisors of Y and $\mathfrak{D}_P \in \mathrm{Pol}_{\mathbb{R}}(N, \sigma)$. In other words, a polyhedral divisor is an element

$$\mathfrak{D} \in \mathrm{Pol}_{\mathbb{R}}(N, \sigma) \otimes \mathrm{CDiv}_{\mathbb{R}}(Y).$$

The locus of a polyhedral divisor $\mathfrak{D}$ supported on Y is

$$\overline{Y} = Y \setminus \bigcup_{\mathfrak{D}_P = \emptyset} P.$$

For every element $u \in \sigma^\vee \cap M$, the evaluation of a polyhedral divisor $\mathfrak{D}$ at u is

$$\mathfrak{D}(u) = \sum_{P} \min_{v \in \mathfrak{D}_P} \langle v, u \rangle \; P.$$

This procedure yields an ordinary Weil divisor on the locus $\overline{Y}$.

Definition 14. A polyhedral divisor $\mathfrak{D}$ is called a p-divisor if the evaluation $\mathfrak{D}(u)$ is semiample for all $u \in \sigma^\vee \cap M$ and big for all $u \in \mathrm{int}(\sigma^\vee)$. Moreover, if the locus $\overline{Y}$ is an affine curve, these conditions are automatically satisfied.

To such a p-divisor we can associate a sheaf of algebras

$$\mathcal{O}(\mathfrak{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\overline{Y}}(\mathfrak{D}(u)) \cdot \chi^u,$$

which carries an N-grading. The conditions of Definition 14 ensure that $\mathrm{H}^0(\overline{\mathsf{Y}}, \mathfrak{D}(\mathsf{u}))$ is a finitely generated algebra for all $\mathsf{u} \in \sigma^\vee \cap \mathsf{M}$.

This leads us to the following two constructions:

$$\mathsf{X}(\mathfrak{D}) := \mathrm{Spec}(\mathrm{H}^0(\overline{\mathsf{Y}}, \mathcal{O}(\mathfrak{D}))),$$
$$\tilde{\mathsf{X}}(\mathfrak{D}) := \mathrm{Spec}_{\overline{\mathsf{Y}}}\, \mathcal{O}(\mathfrak{D}).$$

The morphism $\tilde{\mathsf{X}}(\mathfrak{D}) \to \mathsf{Y}$ is a good quotient in the sense of geometric invariant theory. The map $\tilde{\mathsf{X}}(\mathfrak{D}) \to \mathsf{X}(\mathfrak{D})$ is a rational map which potentially contracts some T-orbits. If Y is an affine curve, we have $\tilde{\mathsf{X}}(\mathfrak{D}) \simeq \mathsf{X}(\mathfrak{D})$. One can define morphisms between p-divisors and an equivalence relation such that up to certain modifications, two p-divisors $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent if and only if $\mathsf{X}(\mathfrak{D}) \simeq \mathsf{X}(\mathfrak{D}')$.

Proposition 15 ([2] Thms 3.1, 3.4)**.** *There is an equivalence of categories between normall affine T-varieties and p-divisors up to modifications.*

The cone to fan analogy in toric geometry can be extended to T-varieties in the following manner.

Definition 16. The intersection of two polyhedral divisors is taken to be the pointwise intersection of corresponding polyhedral coefficients. We say a polyhedral divisor is a face of another polyhedral divisor if the map between their corresponding T-varieties is an open embedding. A divisorial fan is a finite set $\mathcal{S}$ of polyhedral divisors closed under intersections such that the intersection of any two polyhedral divisors is a common face of both. We can write in symbols

$$\mathsf{X}(\mathcal{S}) = \bigcup_{\mathcal{D} \in \mathcal{S}} \mathsf{X}(\mathfrak{D})$$

where $\mathsf{X}(\mathfrak{D} \cap \mathfrak{D}') = \mathsf{X}(\mathfrak{D}) \cap \mathsf{X}(\mathfrak{D}')$. More details about the gluing procedure can be found in [3].

Given a toric variety X corresponding to a fan $\Sigma \subset \mathsf{N}$, choose a subtorus $\mathsf{T}_{\mathsf{N}'} \subset \mathsf{T}_{\mathsf{N}}$. Then we can produce the divisorial fan $\mathcal{S}$ of the corresponding $\mathsf{T}_{\mathsf{N}'}$-variety $\mathsf{X}(\mathcal{S})$ using a procedure called downgrading. The embedding of tori is given by the following short exact sequence of lattices

$$0 \to N' \xrightarrow{F} N \xrightarrow{P} N'' \to 0.$$

Lattices are free abelian groups, so we may choose a splitting $N \simeq N' \oplus N''$ with projections $s : N \to N'$ and $P : N \to N''$. We let Σ' be the fan which is the common refinement of all the images $P(\delta)$ of all faces $\delta \in \Sigma$. We call Y the toric variety associated to the fan Σ' which will act as the base curve for the polyhedral divisors. Now, let n_ρ be the primitive generators of the rays $\rho \in \Sigma'$. For every maximal cone σ, set

$$\mathfrak{D}^\sigma = \sum_{\rho \in \Sigma'(1)} \Delta_\rho(\sigma) \otimes D_\rho,$$

where $\Delta_\rho(\sigma) = s(P^{-1}(n_\rho) \cap \sigma)$ and D_ρ is the torus-invariant prime divisor of Y corresponding to the ray ρ. Finally, observe that $\{\mathfrak{D}^\sigma\}_{\sigma \in \Sigma(n)}$ is a divisorial fan.

The next example studies a downgrade of the Hirzebruch surface.

Example 17. Consider the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$. Its toric fan Σ is given by the rays

$$\rho_1 = (1, 0)^T, \quad \rho_2 = (0, 1)^T, \quad \rho_3 = (-1, n)^T, \quad \rho_4 = (0, -1)^T.$$

In the downgrade construction, set

$$F = (0, 1)^T, \quad P = (1, 0), \quad s = (0, 1).$$

The divisorial fan consists of the following polyhedral divisors glued together:

$$\mathfrak{D}^{\sigma_0} = \emptyset \otimes \{-1\} + [0, \infty) \otimes \{1\},$$

$$\mathfrak{D}^{\sigma_1} = [n, \infty) \otimes \{-1\} + \emptyset \otimes \{1\},$$

$$\mathfrak{D}^{\sigma_2} = (-\infty, n] \otimes \{-1\} + \emptyset \otimes \{1\},$$

$$\mathfrak{D}^{\sigma_3} = \emptyset \otimes \{-1\} + (-\infty, 0] \otimes \{1\}.$$

Figure 2.2 shows the details. And Figure 2.3 shows the resulting divisorial fan over $\mathsf{Y} \simeq \mathbb{P}^1$. Vertical bars are used to indicate subdivisions of the slices, and F^+, F^- are the two horizontal divisors which originate from the two vertical rays ρ_2, ρ_4 of Σ.

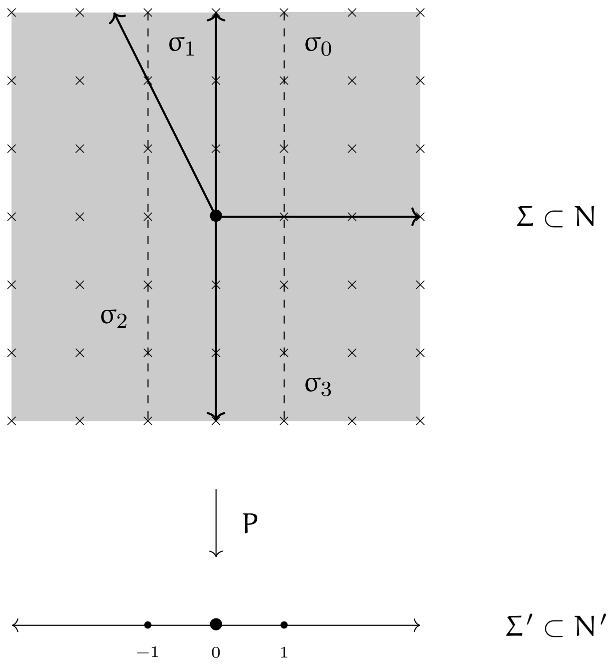

Figure 2.2: Downgrade of a Hirzebruch surface using a natural subtorus action

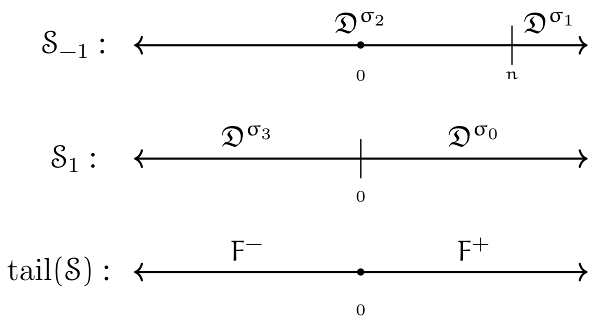

Figure 2.3: Slices of the resulting divisorial fan $\mathcal{S}$

T-invariant divisors

We now describe Weil and Cartier divisors on complexity one T-varieties via support functions. This is also an extension of the usual support function interpretation in the toric setting.

Definition 18. Let X be a T-variety and let $k = \dim(T)$. A vertical T-invariant divisor is the closure of a union of k-dimensional orbits. A horizontal T-invariant divisor is the closure of the union of $k-1$-dimensional orbits.

Proposition 19. *Let $\mathfrak{D}$ be a p-divisor over a smooth curve Y with tailcone σ. Vertical divisors are in bijection with pairs (P, ν), where $P \in Y$ and ν is a vertex of $\mathfrak{D}_P$. The corresponding divisor is denoted $D_{P,\nu}$. Horizontal divisors are in bijection with extremal rays, i.e. rays ρ of the tailcone σ such that $\rho \cap \deg(\mathfrak{D}) = \emptyset$. The corresponding divisor is denoted D_ρ.*

Proof. See [2]. □

Example 20. For the divisorial fan $\mathcal{S}$ of the downgraded Hirzebruch surface in Example 17, we have two prime vertical divisors: $D_{-1,n}$, $D_{1,0}$, and two prime horizontal divisors: D_{-1}, D_1, which we denoted F^+ and F^- in Figure 2.2.

Definition 21. Let $\Sigma \subset N_{\mathbb{R}}$ be a polyhedral subdivision of $N_{\mathbb{R}}$ consisting of tailed polyhedra. A support function is a continuous function $h : |\Sigma| \to \mathbb{R}$, affine on every cone of Σ, which has integer slope and integer translation, i.e. for $\nu \in |\Sigma|$ and $n \in \mathbb{Z}_{\geqslant 0}$ such that $k\nu$ is a lattice point, we have $k \cdot h(\nu) \in \mathbb{Z}$. The group of such support functions is denoted $\mathrm{SF}(\Sigma)$.

Definition 22. Let $h \in \mathrm{SF}(\Sigma)$ and consider a polyhedron $\delta \in \Sigma$ with $\mathrm{tail}(\delta) = \sigma$. The linear part of h is

$$\mathrm{lin}\left(h|_\delta(\nu)\right) = h(p+\nu) - h(p)$$

for some $p \in \delta$.

Definition 23. Let $\mathcal{S}$ be a divisorial fan on a curve Y. Then the slices $\mathcal{S}_P$ are polyhedral subdivisions. So we can define the support functions $\mathrm{SF}(\mathcal{S})$ as

$$h = \sum_{P \in Y} h_P \otimes P$$

where $h_P \in \mathrm{SF}(\mathcal{S}_P)$ and the following two conditions are satisfied

1. All h_P have the same linear part h_t.

2. h_P differs from h_t only for finitely many $P \in Y$.

We call SF($\mathcal{S}$) the group of divisiorial support functions.

Definition 24. A divisorial support function $h \in \mathrm{SF}(\mathcal{S})$ is principal if

$$h(v) = \langle u, v \rangle + D$$

with $u \in M$ and D is a principal divisor on Y where we view D as an element of SF($\mathcal{S}$) by assigning the value $\mathrm{coeff}_P(D)$ to every slice $\mathcal{S}_P$.

A divisorial support function $h \in \mathrm{SF}(\mathcal{S})$ is Cartier if for every $\mathcal{D} \in \mathcal{S}$ the restriction $h|_{\mathcal{D}}$ is principal. We denote the corresponding group by CaSF($\mathcal{S}$).

The following proposition tells us Cartier divisorial support functions give a complete description of Cartier divisors on $X(\mathcal{S})$.

Proposition 25. *([46, Prop 3.10])* $\mathrm{CDiv}(X(\mathcal{S}))$ *and* $\mathrm{CaSF}(\mathcal{S})$ *are isomorphic as abelian groups.*

We now describe the space of global sections of an invariant Cartier divisor. Let D be a T-invariant Cartier divisor on X. The action of T extends to the space of global sections $\mathrm{H}^0(X, \mathcal{O}(D))$ and from elementary representation theory we know we can decompose this into eigenspaces as such:

$$\mathrm{H}^0(X, \mathcal{O}(D)) = \bigoplus_{u \in M} \mathrm{H}^0(X, \mathcal{O}(D))_u.$$

Definition 26. Let h be a Cartier support function with linear part h_t. We define the associated polyhedron to be

$$\square_h = \{u \in M_{\mathbb{R}} : \ \langle u, v \rangle \geqslant h_t(v) \text{ for all } v \in |\mathrm{tail}(\mathcal{S})|\}.$$

We also define the dual function $h^* : \square_h \to \mathrm{Div}_{\mathbb{Q}}(Y)$ to be

$$h^*(u) = \sum_{P \in Y} h_P^*(u) \otimes P = \sum_{P \in Y} \min_{\mathrm{vert}}(u - h_p) \otimes P.$$

Proposition 27. *Let* h *represent a Cartier divisor* D *on* $X(\mathcal{S})$ *with linear part* h_t. *Then for* $u \in \Box_h$ *we have*

$$H^0(X, \mathcal{O}(D))_u = H^0(Y, \mathcal{O}_Y(h^*(u))).$$

Divisorial polytopes

Definition 28. Let Y be a smooth projective curve. A divisorial polytope Ψ supported on Y consists of a lattice polyope $\Box \subset M_{\mathbb{R}}$ (the base polytope) and a piecewise affine concave function

$$\Psi = \sum_P \Psi_P \otimes P : \Box \to \mathrm{Div}_{\mathbb{Q}}(Y),$$

such that

1. For u in the interior of $\Box$, deg $\Psi(u) > 0$,
2. For any vertex u of $\Box$ with deg $\Psi(u) = 0$, $\Psi(u)$ is trivial,
3. For all $P \in Y$, the graph of Ψ_P has its vertices in $M \times \mathbb{Z}$.

Proposition 29 (see [31])**.** *There is a correspondence between polyhedral divisors* Ψ *and polarised complexity one T-varieties* (X, D).

We may construct a polarised T-variety straight from the divisorial polytope Ψ in the following way:

$$\mathbb{P}(\Psi) = \mathrm{Proj}\left(\mathrm{Sym}^\bullet \bigoplus_{u \in \Box \cap M} H^0(Y, \mathcal{O}(\Psi(u)))\right).$$

We can perform the downgrade procedure similarly to Example 17. Consider a polytope $P \subset M_{\mathbb{R}}$ representing a toric variety X_P with an effective action of $T = \mathrm{Spec}(\mathbb{C}[M])$ and a codimension one subtorus $T' \subset T$ with character lattice M'. Starting from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{j} M \xrightarrow{f} M' \to 0,$$

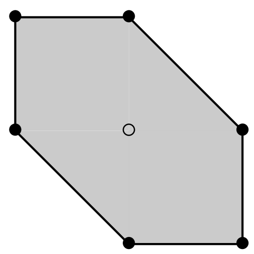

Figure 2.4: The reflexive polytope associated to dP_3

choose a section $s : M' \to M$. We obtain a divisorial polytope $\Psi : \Box \to \mathrm{Div}_{\mathbb{Q}}(\mathbb{P}^1)$:

$$\Psi := \Psi_0 \otimes \{0\} + \Psi_\infty \otimes \{\infty\},$$

where

$$\begin{aligned} \Box &:= f(P), \\ \Psi_0(u) &:= \max_{u \in M} \left\{ i^{-1} \left(f^{-1}(u) \cap P - s(u) \right) \right\}, \\ \Psi_\infty(u) &:= -\min_{u \in M} \left\{ i^{-1} \left(f^{-1}(u) \cap P - s(u) \right) \right\} - 2. \end{aligned}$$

The corresponding complexity one T-variety $\mathbb{P}(\Psi)$ is isomorphic to the starting toric variety X_P.

Example 30. Consider the toric variety $X = \mathrm{dP}_3$, the del Pezzo surface obtained by blowing up the projective plane $\mathbb{P}^2$ in three distinct torus-fixed points. The corresponding lattice polytope $P \simeq M_{\mathbb{Q}}$ is given in Figure 2.4. In this setting $M \simeq \mathbb{Z}^2$ and $M' \simeq \mathbb{Z}$. Set

$$j = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Then we obtain the following divisorial polytope (see Figure 2.5).

$$\begin{aligned} \Box &= f(P) = [-1, 1] \subset M'_{\mathbb{Q}} \simeq \mathbb{Q}, \\ \Psi_0 &= \min_{u \in \Box} \{1,\ 1 - u\}, \\ \Psi_\infty &= \min_{u \in \Box} \{-1,\ u - 2\}. \end{aligned}$$

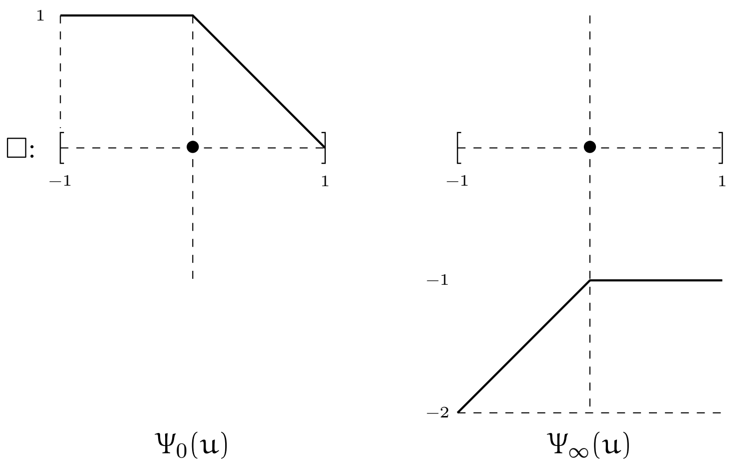

Figure 2.5: The divisorial polytope of the downgrade of dP_3

Volumes and intersections

Divisorial polytopes substitute classical polytopes when studying T-varieties. There is a natural extension of the notion of volume of a function which extends known results concerning toric intersection theory.

Definition 31. Henceforth, μ is the Lebesgue measure of the extended lattice $M_{\mathbb{R}}$, $\square \subset M_{\mathbb{R}}$ is a lattice polytope, and $f : \square \to \mathrm{Div}_{\mathbb{Q}}(Y)$ is a function. The volume of such a function is defined to be

$$\mathrm{vol}(f) := \sum_{P \in Y} \int_{\square} f_P \; d\mu.$$

Proposition 32 ([46, Prop 3.31])**.** *Let Ψ be a divisorial polytope on a complexity one T-variety X of dimension n. Denote by D the semiample divisor associated to Ψ. The top self-intersection number of D is given by*

$$D^n = n! \;\; \mathrm{vol}(\Psi).$$

Example 33. For the divisorial polytope in Figure 2.5, we calculate:

$$\begin{aligned}\mathrm{vol}(\psi) &= \sum_{P \in Y} \int_{\square} \Psi_P \, d\mu \\ &= \int_{[-1,1]} \Psi_0 \, d\mu + \int_{[-1,1]} \Psi_\infty \, d\mu \\ &= \frac{3}{2} + \frac{3}{2} = 3.\end{aligned}$$

Hence $(-K_X)^2 = 2! \ \mathrm{vol}(\Psi) = 2 \times 3 = 6$, which is a standard calculation for $X = \mathrm{dP}_3$.

Cox rings for complexity one T-varieties

For a normal complete variety X, the Cox ring is defined to be

$$\mathrm{Cox}(X) := \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathrm{H}^0(X, \mathcal{O}_X(D)).$$

It can be equipped with a multiplication map which turns $\mathrm{Cox}(X)$ into a complex algebra graded by the class group $\mathrm{Cl}(X)$. Describing the Cox ring of a Mori Dream Space in terms of generators and relations is a question of fundamental interest.

We briefly describe the procedure in the case of complexity one T-varieties. Start with a complexity one T-variety $X = X(\mathcal{S})$, where $\mathcal{S} = \sum_{p \in \mathbb{P}^1} \mathcal{S}_p \otimes \{p\}$.

Let $\mathrm{Vert}(p)$ be the set of vertical divisors $D_{p,\nu}$ lying over $p \in \mathbb{P}^1$. Let $\mathcal{R}$ be the set of rays ρ of $\mathrm{tail}(\mathcal{S})$ such that the corresponding horizontal divisor D_ρ is not contracted by the rational map $\tilde{X}(\mathcal{S}) \to X(\mathcal{S})$. Let $\mathcal{P}$ be the set of points $p \in \mathbb{P}^1$ such that the slice $\mathcal{S}_p$ is not trivial, i.e. $\mathcal{S}_p \neq \mathrm{tail}(\mathcal{S})$. Using the cross-section, we can order and label the elements of $\mathcal{P}$ as such $\mathcal{P} = \{0, \infty, a_1, a_2, \ldots, a_r\}$ with $a_i \in \mathbb{C}^*$.

We have the following result.

Theorem 34 ([28, Thm 1.2])**.** *Let $X = X(\mathcal{S})$ be a complexity one T-variety as above. Introduce a variable S_ρ for each ray $\rho \in \mathcal{R}$ and a variable $T_{p,\nu}$ for each vertical divisor $D_{p,\nu}$. Then the Cox ring of X is given by*

$$\mathrm{Cox}(X) := \frac{\mathbb{C}\left[T_{p,\nu}, S_\rho : \ p \in \mathcal{P}, \ \rho \in \mathcal{R}\right]}{\langle T^{\mu(0)} + a_i T^{\mu(\infty)} + T^{\mu(a_i)} : \ i = 1, 2, \ldots, r \rangle},$$

where

$$T^{\mu(p)} := \prod_{v \in \mathrm{Vert}(p)} T_{p,v}^{\mu(v)}$$

and $\mu(v)$ *is the smallest positive integer such that* $\mu(v) \cdot v \in \mathbb{Z}$*.*

Remark 35. In order to calculate the Cox ring of a singular T-variety X, consider a resolution $r : \tilde{X} \to X$. Calculate $\mathrm{Cox}(\tilde{X})$ using Theorem 34, then $\mathrm{Cox}(X)$ will be given by $\mathrm{Cox}(\tilde{X})$ after removing the variables $T_{p,v}$ which correspond to divisors contracted by the resolution r.

2.3 Coherent sheaves over toric varieties

We describe Klyachko's celebrated combinatorial description of torus equivariant coherent sheaves on simplicial toric varieties. The classification of toric vector bundles in terms of vector spaces with associated filtrations was introduced in the late 1980's in a series of papers by Klyachko [32] [33] [34]. This classification was notably used to investigate the stability of vector bundles over the projective plane $\mathbb{P}^2$ [35].

We recount the basics of the classification in this section. A more comprehensive introduction into the mathematics and the historical background of Klyachko's vector spaces with filtrations can be found in Payne's survey [45].

Let X be a simplicial complete toric variety of dimension n, determined by a strongly convex polyhedral fan $\Sigma \subset N_{\mathbb{R}}$. Order the rays $\rho_i \in \Sigma(1)$ and write v_i for the unique minimal generator of ρ_i.

With this notation in mind, a toric vector bundle is a locally free $\mathcal{O}_X$-module $\mathcal{V}$ of finite rank r, equipped with an action of the dense torus T inherited from X. In more detail, there exists an action of the torus T on the variety $\mathrm{Spec}(\mathrm{Sym}\mathcal{V})$ such that the projection map $\mathrm{Spec}(\mathrm{Sym}\mathcal{V}) \to X$ is T-equivariant and T acts linearly on the fibres.

The following statement concerns equivariant vector bundles over affine toric varieties.

Proposition 36. *Every toric vector bundle on an affine toric variety splits equivariantly as a direct sum of toric line bundles.*

Proof. Let $\mathcal{V}$ be a toric vector bundle on the affine toric variety given by a single cone $\sigma \subset N_{\mathbb{R}}$, i.e.

$$X = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

Let x_σ be the torus-fixed point in the minimal T-orbit $O_\sigma \subset X$. Next, choose characters $s_1, \ldots, s_n$ such that the set $\{s_i(x_\sigma)\}$ is a basis of the fibre $\mathcal{V}_{x_\sigma}$. The set of points $x \in X$ such that $\{s_i(x)\}$ is not a basis for the fibre $\mathcal{V}_x$ is closed, invariant under the action of T, and does not contain O_σ, hence it must be empty. Therefore, $\mathcal{V}$ splits equivariantly as the direct sum of the line subbundles spanned by the characters $s_i \in M$. □

This proposition is a simpler statement of the more famous result by Gubeladze [24], which showed that an arbitrary vector bundle on an affine toric variety must be trivial. This ties in with a theorem due to Quillen and Suslin which states that projective modules over polynomial rings are free.

We now proceed with the more general description of vector bundles over complete toric varieties.

Theorem 37 ([33, Thm 2.2.1])**.** *The category of toric vector bundles over $X = X(\Sigma)$ is equivalent to the category of finite-dimensional complex vector spaces V equipped with separated exhaustive decreasing filtrations $\{V^{\rho_i}(j)\}_{j\in\mathbb{Z}}$, for all $1 \leqslant i \leqslant n$, that satisfy the following compatibility condition:*

For every maximal cone $\sigma \in \Sigma(n)$, there exists a decomposition

$$V = \bigoplus_{u \in u(\sigma)} L_u \quad \textit{such that} \quad V^{\rho_i}(j) = \sum_{\langle u, v_i\rangle \geqslant j} L_u.$$

Proof. One direction of the equivalence is given by associating to the vector bundle $\mathcal{V}$ the vector space of the fibre $\mathcal{V}_{x_0}$ over the identity of the torus together with the filtrations defined in the statement. Then a morphism of toric vector bundles $f : \mathcal{V} \to \mathcal{W}$ maps fibres linearly to fibres, and hence induces a linear map $\overline{f} : V \to W$. Moreover, f respects the eigenspace decompositions of the modules of sections and, in particular, takes $H^0(U_{\rho_i}, \mathcal{V})_u$ to $H^0(U_{\rho_i}, \mathcal{W})_u$ for each ray $\rho_i \in \Sigma(1)$ and $u \in M$. Hence $\overline{f}$ takes $V^{\rho_i}(j)$ to $W^{\rho_i}(j)$ for every $j \in \mathbb{Z}$, as expected.

For the other direction, consider a vector space V with filtrations $\{V^{\rho_i}(j)\}$ satisfying the compatibility condition in the statement. Then let

$$V^\sigma_u = \bigcap_{\rho \preceq \sigma} V^{\rho_i}(\langle u, v_i \rangle),$$

for cones $\sigma \in \Sigma$ and $u \in M$ and set

$$V^\sigma = \bigoplus_{u \in M} V^\sigma_u.$$

We have that V^σ has a natural $\mathbb{C}[U_\sigma]$-module structure, where multiplication by a character $\chi^{u'}$, for $u' \in (\sigma^\vee \cap M)$, is the sum of the inclusions $V^\sigma_u \subset V^\sigma_{u-u'}$. Let T act on V^σ such that V^σ_u is the χ^u-isotypical part. Then it is easy to check using the compatibility condition that the induced quasicoherent sheaf $\tilde{V}^\sigma$ is equivariantly isomorphic to

$$\bigoplus_{[u] \in M_\sigma} \mathcal{L}_{[u]} \otimes_{\mathbb{C}} V_{[u]}.$$

In particular, $\tilde{V}^\sigma$ is locally free. Furthermore, the direct sum decomposition implies that $V^{\rho_i}(j) = V$ for $j << 0$, from which it follows that the natural inclusions

$$V^\sigma_u \subset V^\tau_{u-u'},$$

for $\tau \preceq \sigma$ and $u \perp \tau$, induce toric isomorphisms

$$\tilde{V}^\sigma|_{U_\tau} \simeq \tilde{V}^\tau.$$

These isomorphisms represent the gluing data of a vector bundle over X. We skip the details. Finally, observe that morphisms of vector spaces with compatible filtrations induce morphisms of toric vector bundles, and that the functor so defined is inverse to the functor $\mathcal{V} \mapsto (V, \{V^{\rho_i}(j)\})$, up to natural isomorphism. This gives the desired equivalence of categories. □

Remark 38. The compatibility condition tells us that on the open subset U_σ, the vector bundle equivariantly splits into a direct sum of line bundles. The filtrations prescribe a gluing of such splittings into a vector bundle. The adjectives "separated" and "exhaustive" tell us that $V^{\rho_i}(j) = 0$ for $j << 0$ and $V^{\rho_i}(j) \simeq V \simeq \mathbb{C}^r$ for $j >> 0$. In other words, every filtration may only contain finitely many distinct linear subspaces.

Proposition 39. *We describe Klyachko filtrations for the tangent bundle and for line bundles.*

1. *([33, Ex 2.3.1]) Let $X = X(\Sigma)$ be a smooth toric variety and let T_X be its tangent sheaf. By construction, T_X is equivariant with respect to the action of the dense torus. The filtration corresponding to T_X is $(V, \{V^\rho(j)\})$ where*

$$V^\rho(j) = \begin{cases} N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if} \quad j \leqslant 0, \\ \text{span}(\rho) & \text{if} \quad j = 1, \\ 0 & \text{if} \quad j > 1. \end{cases}$$

2. *([33, Ex 2.3.5]) Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a Weil divisor on the smooth toric variety $X = X(\Sigma)$. Then the filtration associated to the line bundle $\mathcal{O}_X(D)$ is $(W, \{W^\rho(j)\})$ where*

$$W^\rho(j) = \begin{cases} \mathbb{C} & \text{if} \quad j \leqslant a_\rho, \\ 0 & \text{if} \quad j > a_\rho. \end{cases}$$

Proposition 40. *Let $\mathcal{V}, \mathcal{W}$ be toric vector bundles over $X = X(\Sigma)$ determined by the filtrations $(V, \{V^\rho(j)\})$ and $(W, \{W^\rho(j)\})$, respectively. Then the following hold:*

1. *The tensor product $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{W}$ corresponds to the filtration $(V \otimes_{\mathbb{C}} W, \{T^\rho(j)\})$, where*

$$T^\rho(j) = \sum_{s+t=j} V^\rho(s) \otimes W^\rho(t).$$

2. *The second exterior power* $\mathcal{V} \wedge \mathcal{V}$ *corresponds to the filtration* $(V \wedge V, \{E^\rho(j)\})$, *where*

$$E^\rho(j) = \sum_{s+t=j} V^\rho(s) \wedge V^\rho(t).$$

The language of filtrations also provides a simple method for calculating the vector spaces of global sections of a toric vector bundle.

Proposition 41 ([33, Cor 4.1.3])**.** *Let* $X = X(\Sigma)$ *be a simplicial toric variety and let* $\mathcal{V}$ *be a toric vector bundle over* X*. Fix a character* $u \in M$*. Then the* u*-graded piece of the vector space of global sections of* $\mathcal{V}$ *is given by*

$$H^0(X, \mathcal{V})_u = \bigcap_{\rho \in \Sigma(1)} E^\rho\left(\langle u, \rho \rangle\right).$$

Remark 42. We clarify some of the details of this correspondence for line bundles. Let $\mathcal{O}_X(D)$ be a line bundle on the toric variety X. Then D has the form $D = a_1D_1 + a_2D_2 + \ldots a_nD_n$. By [14, Theorem 6.1.7], the Cartier divisor D is determined by a choice of character $u_\sigma \in M$ for each maximal cone $\sigma \in \Sigma(n)$. The Cartier divisor D has an associated piecewise linear function $\phi_D : N_\mathbb{R} \to \mathbb{R}$ which satisfies $\phi_D(v_i) = -a_i$ and $\phi_D(v) = \langle u_\sigma, v\rangle$ for all vectors $v \in \sigma$. The global sections are determined by the polytope

$$P_D = \{u \in M \otimes_\mathbb{Z} \mathbb{R} : \ \langle u, v_i \rangle \leqslant a_i\}.$$

Then looking at the filtration $(W \simeq \mathbb{C}, \{W^\rho(j)\})$, we see that $W_{u_\sigma} \simeq W \simeq \mathbb{C}$ for all maximal cones $\sigma \in \Sigma(n)$. Then the u-graded part of the vector space of global sections is $H^0(X, \mathcal{O}_X(D))_u = \mathbb{C}$ for $\langle u, v_i \rangle \leqslant a_i$ for all $1 \leqslant i \leqslant n$ and $H^0(X, \mathcal{O}_X(D))_u = 0$ otherwise. We can write this conclusion more compactly as

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P \cap M} \mathrm{span}(\chi^{-u}).$$

We use the opposite sign convention to [14] and [22].

Another way of describing global sections of toric vector bundles is via parliaments of polytopes, as described by di Rocco-Jabbusch-Smith.

Proposition 43 ([15, Prop 3.1]). *For a toric vector bundle* $\mathcal{V}$ *with associated Klyachko filtration* $(V, \{V^{\rho}(j)\})$*, there exists a unique matroid* $M(\mathcal{V})$*, representable over* $\mathbb{C}$*, such that*

1. *The poset* $L(\mathcal{V})$ *consisting of all linear subspaces of the form* $\bigcap_{\rho} V^{\rho}(j_{\rho}) \subset V$ *with* $j_{\rho} \in \mathbb{Z}^{n}$*, ordered by inclusion, is isomorphic to a meet-subsemilattice in the lattice of flats,*
2. *among all the matroids which satisfy the previous property, the number of elements in the ground set is minimal, and*
3. *among all the matroids which satisfy the previous two properties, the number of circuits is minimal.*

Definition 44. For a toric vector bundle $\mathcal{V}$ with Klyachko filtration $(V, \{V^{\rho}(j)\})$, the associated parliament of polytopes $\mathcal{P}(\mathcal{V})$ is defined as

$$\mathcal{P}(\mathcal{V}) := \{P_{\mathbf{e}} \subset M \otimes_{\mathbb{Z}} \mathbb{R} : \ \mathbf{e} \in M(\mathcal{V})\},$$

where individual polytopes are determined by

$$P_{\mathbf{e}} := \{\mathbf{u} \in N \otimes_{\mathbb{Z}} \mathbb{R} : \ \langle \mathbf{u}, \rho \rangle \leqslant \max \{j \in \mathbb{Z} : \ \mathbf{e} \in E^{\rho}(j)\} \text{ for all } \rho\}.$$

Proposition 45 ([15] Prop 1.1). *The lattice points in the polytopes of the parliament for* $\mathcal{V}$ *correspond to torus-equivariant generators for the space of global sections of* $\mathcal{V}$*:*

$$H^{0}(X, \mathcal{V}) \simeq \sum_{\mathbf{e} \in M(\mathcal{V})} \operatorname{span}\left(\mathbf{e} \otimes \chi^{-\mathbf{u}} : \ \mathbf{u} \in P_{\mathbf{e}} \cap M\right) \subset V \otimes_{\mathbb{C}} T.$$

We conclude this section by giving an example of how to calculate with Klyachko filtrations.

Example 46. The tangent bundle $T_{\mathbb{P}^2}$ of the projective plane has the associated filtration $(\mathbb{C}^2, T^{\rho}(j))$ by Proposition 39:

$$T^{\rho}(j) = \begin{cases} \mathbb{C}^2 & \text{if} \quad j \leqslant 0, \\ \operatorname{span}(\rho) & \text{if} \quad j = 1, \\ 0 & \text{if} \quad j > 1. \end{cases}$$

The toric fan of $\mathbb{P}^2$ contains the rays $\rho_1 = (1,0)^T$, $\rho_2 = (0,1)^T$, and $\rho_3 = (-1,-1)^T$. We calculate directly the top exterior power $\bigwedge^2 T_{\mathbb{P}^2}$ using Proposition 40.

For any given ray $\rho \in \Sigma(1)$, if $j \geqslant 2$, then the pairs of integers s, t such that $s + t = 2$ can be assumed without loss of generality to be such that $s \geqslant 2$ and $t \leqslant 0$. Then every term has the form $T^\rho(s) \wedge T^\rho(t) = 0 \wedge \mathbb{C}^2 = 0$.

We then obtain that $\bigwedge^2 T_{\mathbb{P}^2}$ corresponds to the filtration $(\mathbb{C}, \{E^\rho(j)\})$, where

$$E^\rho(j) = \begin{cases} \mathbb{C} & \text{if} \quad j \leqslant 1, \\ 0 & \text{if} \quad j > 1. \end{cases}$$

By Proposition 39, $(\mathbb{C}, \{E^\rho(j)\})$ corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^2}(D)$ which has the coefficient 1 at every torus-fixed prime divisor, i.e.

$$D = D_{\rho_1} + D_{\rho_2} + D_{\rho_3}.$$

Since the class group of the projective plane is known to be $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z} \simeq Z[\mathcal{O}_{\mathbb{P}^2}(1)]$, then we can make the identifications

$$\bigwedge^2 T_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2}(D) \simeq \mathcal{O}_{\mathbb{P}^2}(3).$$

This confirms the known standard computation:

$$\bigwedge^2 T_{\mathbb{P}^2} \simeq \det(T_{\mathbb{P}^2}) \simeq -K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3).$$

2.4 Galois covers

The abelian cover is a fundamental concept in topology. The analysis of general covers of fixed degree of an algebraic variety is a question that poses many difficulties. The general philosophy is to establish a correspondence between abelian covers of an algebraic variety and subgroups of the group of divisors of that variety. A comprehensive picture of the more classical questions surrounding abelian covers can be obtained by consulting Rita Pardini's extensive work on this topic [42] [43] [44]. Other authors worked on finding applications of this construction to number theory, K3 surfaces, and moduli spaces of curves [1] [8] [25] [52].

In this section we recall some basic definitions concerning Galois covers and in particular cyclic covers and how they relate to the question of stability for coherent sheaves.

Definition 47. Let X, Y be a normal varieties. A cover is a finite morphism $f : Y \to X$. An abelian cover is a Galois morphism $f : Y \to X$ with abelian Galois group. Note that if X is smoooth, then f is also a flat morphism.

Definition 48. Let X be a smooth projective variety and let $B \in \mathrm{Pic}(X)$ be an effective divisor. We say that B is divisible by $n \in \mathbb{N}$ in $\mathrm{Pic}(X)$ if and only if there exists a line bundle $L \in \mathrm{Pic}(X)$ such that $L^n = \mathcal{O}_X(B)$.

The divisor B gives a morphism $L^{-n} \to \mathcal{O}_X$, which endows

$$\mathcal{A} = \mathcal{O}_X \oplus L^{-1} \oplus L^{-2} \oplus \ldots \oplus L^{-n+1}$$

with the structure of a commutative unital ring.

Definition 49. A cyclic cover of order n branched along B is the natural projection

$$f : Y = \mathrm{Spec}(\mathcal{A}) \to X.$$

We refer to B as the branching locus and we denote by B' the reduced divisor of the preimage $f^{-1}(B)$.

Proposition 50 (see e.g. [43])**.** *With notation as above, the following hold.*

1. $f^*L = \mathcal{O}_Y(B_1)$*;*

2. $K_Y = f^*(K_X + L^{n-1})$*;*

3. $f_*\mathcal{O}_Y = \mathcal{O}_X \oplus L^{-1} \oplus L^{-2} \oplus \ldots \oplus L^{1-n}$.

Denote by G the Galois group of the cover f. Given a coherent sheaf $\mathcal{F}$ on X, there is an isomorphism $\sigma^* f^* \mathcal{F} \simeq f^* \mathcal{F}$ for all $\sigma \in G$. Consider a subsheaf $\mathcal{E} \subset f^* \mathcal{F}$. The image of this subsheaf under the inclusion $\sigma^* \mathcal{E} \simeq \sigma^* f^* \mathcal{F}$ is denoted by $\mathcal{E}^\sigma$.

The following is a known result in Galois descent theory.

Proposition 51 ([21, Thm 28])**.** *Let $f : Y \to X$ be a finite flat Galois morphism with Galois group G and both X and Y are integral schemes. Let $\mathcal{F}$ be a locally free sheaf on X and $\mathcal{G}'$ be a coherent subsheaf of $f^* \mathcal{F}$ satisfying the following properties:*

1. *For all $\sigma \in G$, $(\mathcal{G}')^\sigma \simeq \mathcal{G}'$ as a subsheaf of $f^* \mathcal{F}$;*
2. *$(f^* \mathcal{F})/\mathcal{G}'$ is torsion-free.*

Then there exists a coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G}' = f^* \mathcal{G}$ as a subsheaf of $f^* \mathcal{F}$.

The following propositions concern the stability of vector bundles on cyclic coverings given that their stability is known on the base of the cover. These theorems provide some inspiration and some techniques that are used in Chapter 5.

Proposition 52 ([53, Thm 3.2])**.** *Let X be a smooth projective variety and let $f : Y \to X$ be a cyclic cover of order n branched along a smooth effective divisor $B \in H^0(X, L^n)$. If $\mathcal{F}$ is a locally free sheaf on X and is (semi)stable with respect to some ample polarisation $H \in \mathrm{Pic}(X)$, then $f^* \mathcal{F}$ is (semi)stable with respect to $f^* H$.*

Proposition 53 ([53, Thm 3.4])**.** *Let X be a smooth projective variety and let $f : Y \to X$ be a cyclic cover of order n branched along a smooth effective divisor B. Let $H \in \mathrm{Pic}(X)$ be an ample line bundle which satisfies $H \sim \ell B$ for some rational $\ell > 0$. Assume that Ω^1_X is (semi)stable and the inequality*

$$n \; H \cdot c_1(\Omega^1_X) + (n+1) \; H \cdot B \geqslant 0$$

holds. Then Ω^1_Y is (semi)stable with respect to $f^* H$.

Chapter 3

The case of $\mathbb{C}^*$-surfaces with fibrewise action

Consider a Hirzebruch surface $\mathbb{F}_\mathsf{d}$ with fan as in Figure 2.2 and the downgrade induced by the fan morphism $\binom{0}{-1} : \mathbb{Z}^2 \to \mathbb{Z}$ which yields the quotient morphism $\mathbb{F}_\mathsf{d} \to \mathbb{P}^1$. As a $\mathbb{C}^*$-surface, $\mathbb{F}_\mathsf{d}$ possesses two horizontal divisors which we denote by F^+ and F^-, and we can associate a divisorial fan $\mathcal{D}$ as in Figure 2.3.

From the classical classification of surfaces, we know that surfaces constructed by blowing up $\mathbb{F}_\mathsf{d}$ in $\mathbb{C}^*$-fixed points form an important category of algebraic surfaces. It turns out that surfaces constructed in this way allow for a more intrinsic description.

Definition 54. Let $\mathsf{X} = \mathsf{X}(\mathcal{D})$ be a $\mathbb{C}^*$-surface. We say X has *fibrewise group action* if the quotient map to the base curve $\mathsf{X} \to \mathsf{Y}$ is a morphism of algebraic varieties, i.e. here we have $\mathsf{X}(\mathcal{D}) \simeq \tilde{\mathsf{X}}(\mathcal{D})$, and $\mathbb{C}^*$ acts transitively on the fibres of $\mathsf{X} \to \mathsf{Y}$.

This section is dedicated to answering the following question.

Question 55. Determine the stability of the tangent bundle T_X for $\mathbb{C}^*$-surfaces X with fibrewise group action.

3.1 Intersection numbers in complexity one

In preparation for results concerning surfaces with fibrewise action, recall the following result from toric geometry.

Proposition 56 ([22, Section 5.3]). *Let $X = X(\Sigma)$ be the toric variety associated to the fan $\Sigma \subset N$, let D be the torus-fixed Cartier divisor associated to the polytope $P \in M$, and let V be a torus-fixed prime divisor associated to some cone $\sigma \in \Sigma$. Let P_σ be the intersection of P with the corresponding translation of the linear subspace orthogonal to σ, i.e.*

$$P_\sigma = P \cap \left(\sigma^\perp + u(\sigma)\right).$$

The lattice $M(\sigma)$ is the translation of M by $u(\sigma)$ and defines a volume form on the linear space $\sigma^\perp + u(\sigma)$ which has dimension equal to the dimension of V. We then have

$$\mathrm{vol}_{M(\sigma)}(P_\sigma) = \int_{[X]} \frac{1}{k!}\, D^k \frown [V].$$

This gives a combinatorial description for the intersection number of an arbitrary torus-fixed Cartier divisor and a prime divisor in terms of the volume of a certain polytope. The remainder of this subsection extends this result to complexity one T-varieties and the language of divisorial fans and divisorial polytopes.

The following lemma is a technical observation. It will help us connect the combinatorial data for global sections of a divisor to the combinatorial data for the ideal presentation of a divisor.

Lemma 57. *Let X be a smooth projective variety, D an ample divisor on X, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf corresponding to a subvariety V of X. Then, for sufficiently high r, we have*

$$H^0(X, \mathcal{O}(rD)/\mathcal{I}) = H^0(V, \mathcal{O}_V \otimes \mathcal{O}(rD)) \simeq H^0(X, \mathcal{O}(rD))/\, H^0(X, \mathcal{I} \otimes \mathcal{O}(rD)).$$

Proof. Given the ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, we have the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_V \to 0.$$

Tensoring the sequence by $\mathcal{O}(D)^{\otimes r} = \mathcal{O}(rD)$ and then using the long exact sequence for sheaf cohomology, we obtain

$$0 \to H^0(X, \mathcal{I} \otimes \mathcal{O}(rD)) \to H^0(X, \mathcal{O}(rD)) \to H^0(V, \mathcal{O}_V \otimes \mathcal{O}(rD)) \to H^1(X, \mathcal{I} \otimes \mathcal{O}(rD)) \to \dots.$$

Since $\mathcal{O}_X$ and $\mathcal{I}$ are coherent sheaves, then from the ampleness of D it follows that for sufficiently high r we have

$$H^i(X, \mathcal{I} \otimes \mathcal{O}(rD)) = 0 \text{ for all } i > 0, \text{ and}$$
$$H^i(X, \mathcal{O}_X \otimes \mathcal{O}(rD)) = 0 \text{ for all } i > 0.$$

Inspecting the long exact sequence, we see that $H^i(V, \mathcal{O}_V \otimes \mathcal{O}(rD)) = 0$ for all $i > 0$ and thus the long exact sequence is reduced to a short exact sequence

$$0 \to H^0(X, \mathcal{I} \otimes \mathcal{O}(rD)) \to H^0(X, \mathcal{O}(rD)) \to H^0(V, \mathcal{O}_V \otimes \mathcal{O}(rD)) \to 0.$$

As this is a short exact sequence of finitely generated vector spaces, the sequence splits and so we conclude that

$$H^0(V, \mathcal{O}_V \otimes \mathcal{O}(rD)) \simeq H^0(X, \mathcal{O}(rD)) / H^0(X, \mathcal{I} \otimes \mathcal{O}(rD)).$$

□

Our aim is to prove the following result. This is the complexity one analogue of Proposition 56 when we take the Cartier divisor D to be a vertical divisor.

Proposition 58. *Let $X = (X, D)$ be an n-dimensional, smooth, polarised complexity one T-variety with D given by the divisorial polytope $\Psi = (\Psi, \square)$. Fix a vertical prime divisor $D_{P,v}$. Then the intersection number $D^n \cdot D_{P,v}$ is given by*

$$D^{n-1} \cdot D_{P,v} = \mathrm{vol}(F),$$

where F is the facet of the graph of Ψ_P which corresponds to $D_{P,v}$.

Before we proceed with the proof, let's give an example of what applying this proposition looks like in practice.

Example 59. Consider the del Pezzo surface $X = \mathrm{dP}_3$. We consider X to be a $\mathbb{C}^*$-surface as in Example 30 and take the divisorial polytope $(\Psi, \Box)$ described therein to be the Cartier divisor D. We wish to intersect D with the prime divisor $D_{0,0}$. We take this to be the prime divisor $D_{0,0} = \mathrm{orb}(\sigma)$, where $\sigma = (0,1)^T$ is the ray from the corresponding toric fan of $X = \mathrm{dP}_3$.

Figure 3.1 shows the facet F (bolded) of the graph of Ψ_0 corresponding to the prime divisor $D_{0,0}$. Applying Proposition 58 helps us calculate that

$$D \cdot D_{0,0} = D|_{D_{0,0}} = \mathrm{vol}(F) = 1.$$

This can be easily verified using classical intersection theory or convex geometry in the toric setting using Proposition 56.

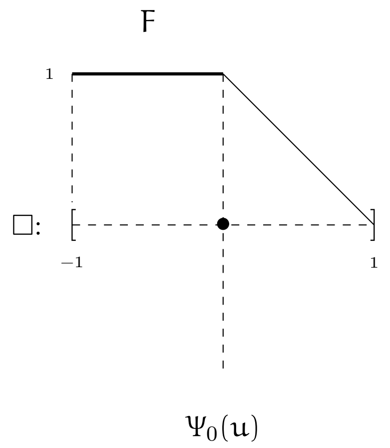

Figure 3.1: Intersecting a vertical divisor with a prime divisor on a $\mathbb{C}^*$-surface

We now prepare the proof of Proposition 58 with some intermediary results.

Denote $E = D|_{D_{P,\nu}}$. We are interested in the behaviour of the limit

$$E^n = \lim_{\ell \to \infty} \frac{1}{\ell^n} \mathrm{h}^0 \left(D_{P,\nu}, \mathcal{O}(\ell \cdot E) \right).$$

To this end, we use the ideal description of $D_{P,\nu}$ from [46]. Motivated by the contents of Lemma 57, it suffices to understand the quotient

$$\mathcal{A} := \bigoplus_{u \in \Box \cap M} H^0(Y, \mathcal{O}(\Psi(u))) \Big/ \bigoplus_{u \in \Box \cap M} H^0(Y, \mathcal{O}(\Psi(u))) \cap \{f \in \mathbb{C}(Y) : \operatorname{ord}_P(f) > -\langle u, v\rangle\}.$$

The next lemma gives an upper bound to $h^0(D_{P,v}, \mathcal{O}(E))$.

Lemma 60. *In the setting of Proposition 58, we have that*

$$h^0(D_{P,v}, \mathcal{O}(E)) \leqslant \#(F),$$

where $\#(F)$ is the number of lattice points on the facet F.

Proof. Fix a weight $u \in \Box \cap M$ and consider the graded component $\mathcal{A}_u$.

$$\begin{aligned} \text{We have } s \in H^0(Y, \mathcal{O}(\Psi(u))) &\Leftrightarrow \operatorname{div}(s) + \Psi(u) \geqslant 0 \\ &\Leftrightarrow \sum_{Q \in Y} \operatorname{ord}_Q(s) \cdot Q + \sum_{Q \in Y} \langle m_Q, u\rangle \cdot Q \geqslant 0 \\ &\Leftrightarrow \sum_{Q \in Y} (\operatorname{ord}_Q(s) + \langle m_Q, u\rangle) \cdot Q \geqslant 0 \\ &\Leftrightarrow \text{for all } Q: \ \operatorname{ord}_Q(s) \geqslant -\langle m_Q, u\rangle. \end{aligned}$$

Now a section $s \in \mathcal{A}_u$ is nontrivial if and only if $\operatorname{ord}_P(s) = -\langle u, v\rangle$. For fixed v, $\langle -, v\rangle : \Box \to \operatorname{CDiv}_{\mathbb{Q}}(Y)$ is a linear function and thus a maximal facet of the graph of Ψ_P. Call this facet F. Moreover, $\operatorname{ord}_P(s) = -\langle u, v\rangle$ must be an integer. Thus given a nonzero section $s \in \mathcal{A}_u$, we can obtain a lattice point $(u, -\langle u, v\rangle) \in F \cap (M \times \mathbb{Z})$. □

Next, we provide a sufficient condition for a lattice point $u \in \Box \cap M$ to yield a global section of E.

Lemma 61. *Given $u \in \Box \cap M$, then we can construct a non-zero section*

$$s \in H^0(D_{P,v}, \mathcal{O}(E))_u$$

provided that $\deg\lfloor \Psi(u) \rfloor \geqslant 0$.

Proof. Fix a weight $u \in \Box \cap M$. We need to construct a non-zero section $s \in \mathcal{A}_u$. Denote its principal divisor $\operatorname{div}(s)$ by H. In order for s to belong the graded piece $\mathcal{A}_u$, it must satisfy two conditions: $H + \Psi(u) \geqslant 0$, and the coefficient of H at P must be equal to $-\langle u, v\rangle$.

Since $s \in k(\mathbb{P}^1)$, an eligible section can be identified by constructing the corresponding degree zero principal divisor H.

To this end, set $H = -\lfloor \Psi(u) \rfloor$. This has $\operatorname{coeff}_P(H) = -\langle u, v\rangle$. Assuming that $\deg\lfloor \Psi(u) \rfloor \geqslant 0$ then either $\deg H = 0$, in which case we are done, or $\deg H < 0$, in which case we set $H' = H + (-\deg(H)) \cdot Q$, where $P \neq Q \in Y$ is in the support of $\Psi(u)$. Then $\deg(H') = 0$, as required.

□

Lemma 62. *Let $f : \Box \to \mathbb{R}$ be a piece-wise linear, continuous, concave function, and fix a positive integer q. Then for a sufficiently large dilation $\ell \cdot \Box$, there exists a convergent sequence $c(\ell) \to 1$ such that*

$$\ell \cdot f\left(\frac{x}{\ell}\right) > q \text{ for all } x \in \ell \cdot c(\ell) \cdot \Box.$$

Proof. First we prove the statement in the one-dimensional setting $\Box = [a, 0] \subset \mathbb{R}$. Here, we distinguish two cases. First, we assume that $f(0) > 0$ and $f(a) > 0$. Then for $\ell > \frac{q}{\min\{f(0), f(a/\ell)\}}$ and $c(\ell) = 1$, we have that $\ell \cdot f\left(\frac{x}{\ell}\right) > q$ for $x \in \{0, a\}$. By concavity, the result follows for all $x \in \ell \cdot \Box$.

In the second case, we assume that $f(0) > 0$ and $f(a) = 0$. Set $c(\ell) = \left(\frac{q}{\ell m} + a\right)/a$, where m is the linear part of f at a. Inspect what happens at $\ell \cdot c(\ell) \cdot a$. Note that for $\ell \to \infty$ we have that $c(\ell) \to 1$ and $(\ell \cdot a) - (\ell \cdot c(\ell) \cdot a) \to 0$. We can then assume that the linear parts of f at $\ell \cdot a$ and $\ell \cdot c(l) \cdot a$ are the same.

$$
\begin{aligned}
\text{Then } \ell \cdot f\left(\frac{\ell \cdot c(\ell) \cdot a}{\ell}\right) &= \ell \cdot f\left(\frac{q}{\ell m} + a\right) = \ell\left(m\left(\frac{q}{\ell m} + a\right) + b\right) \\
&= q + \ell(ma + b) \\
&= q + \ell \cdot f(a) \\
&= q.
\end{aligned}
$$

For ℓ sufficiently large, we have $\ell \cdot f(0) > q$. By concavity, we obtain again that $\ell \cdot f\left(\frac{x}{\ell}\right) > q$ on $\ell \cdot c(\ell) \cdot \square$.

Now we deal with the case $\square \subset \mathbb{R}^n$. We denote by V the set of vertices of $\square$. For every $x \in V$, consider the restriction of f to the line segment between 0 and x. Using the one dimensional case, there exists a contraction factor $c(\ell, x) \to 1$ such that $c(\ell, x) \cdot \ell \cdot f\left(\frac{x}{\ell}\right) > q$. This holds for every $x \in V$, and since the restriction of f to any line segment is again concave, it follows that $c \cdot \ell \cdot f\left(\frac{x}{\ell}\right) > q$ for any $c \leqslant \min_{x \in V}\{c(\ell, x)\}$. □

Lemma 63. *In the setting of Proposition 58, we have*

$$\#\left(\ell \cdot c(\ell) \cdot F\right) \leqslant h^0\left(D_{P,\nu}, \mathcal{O}(\ell E)\right).$$

Proof. We abuse notation by denoting the both the facet and its projection to the base polytope by F. In that case, for any point $x \in \ell \cdot c(\ell) \cdot F$ we have that $x \in \ell \cdot c(\ell) \cdot \square$. By fixing q to be the number of non-integral coefficients of Ψ and applying Lemma 62, it follows that $\ell \cdot \Psi\left(\frac{x}{\ell}\right) > q$ for all such x. Finally, Lemma 61 says that x corresponds to a non-vanishing global section. □

We now have all the instruments required to prove the main statement of this subsection.

Proof of Proposition 58. The lemmas from before yield the inequality

$$\#\left(\ell \cdot c(\ell) \cdot F\right) \leqslant h^0\left(D_{P,\nu}, \mathcal{O}(\ell E)\right) \leqslant \#\left(\ell \cdot F\right),$$

and after dividing by ℓ^n and taking limits we obtain

$$\lim_{\ell\to\infty} \frac{\#(\ell \cdot c(\ell) \cdot F)}{\ell^n} \leqslant E^n \leqslant \lim_{\ell\to\infty} \frac{\#(\ell \cdot F)}{\ell^n}.$$

It remains to show that the flanking terms are equal. Set q to be potentially larger than the number of non-integral coefficients of Ψ in a manner such that $\ell \cdot c(\ell)$ becomes an integer. In this situation, we can see that

$$\left(\frac{\#(\ell \cdot c(\ell) \cdot F)}{(\ell \cdot c(\ell))^n}\right)_{\ell\in\mathbb{N}} \text{ is a subsequence of } \left(\frac{\#(\ell \cdot F)}{\ell^n}\right)_{\ell\in\mathbb{N}}.$$

Since the latter sequence converges, then the former converges to the same limit.

Observing that $c(\ell) \to 1$ as $\ell \to \infty$ allows us to rewrite

$$\frac{\#(\ell \cdot c(\ell) \cdot F)}{(\ell \cdot c(\ell))^n} = \frac{\#(\ell \cdot c(\ell) \cdot F)}{\ell^n} \cdot \frac{1}{(c(\ell))^n}.$$

Finally, the algebra of limits gives

$$\lim_{\ell\to\infty} \frac{\#(\ell \cdot c(\ell) \cdot F)}{\ell^n} = \lim_{\ell\to\infty} \frac{\#(\ell \cdot c(\ell) \cdot F)}{(\ell \cdot c(\ell))^n} = \lim_{\ell\to\infty} \frac{\#(\ell \cdot F)}{\ell^n}.$$

This concludes the proof. □

The following result is the complexity one analogue of Proposition 56 where D is taken to be a horizontal divisor. The case of horizontal divisors turns out to be less involved.

Proposition 64. *Let $X = (X, D)$ be an n-dimensional, smooth, polarised complexity one T-variety with D given by the divisorial polytope $\Psi = (\Psi, \Box)$, and fix a horizontal prime divisor D_ρ. Then the intersection number $D^{n-1} \cdot D_\rho$ is given by*

$$D^{n-1} \cdot D_\rho = (n-1)! \ \operatorname{vol}(\Phi),$$

where $\Phi = (\Phi, F)$ is given by the functions $\Phi_P(u) = \Psi_P(u)$ for all $P \in Y$ and the base polytope is given by the facet $F \leqslant \Box$ which satisfies $\rho \perp \mathbb{Q}_{\geqslant 0}(F \times \{1\})$.

Proof. Following the main idea of the proof of Proposition 58, we need to analyse the quotient

$$\mathcal{B} := \bigoplus_{(u,1)\in(\sigma^\vee\cap M)\times\{1\}} H^0\left(Y, \mathcal{O}(\Psi((u,1)))\right) \Big/ \bigoplus_{(u,1)\in\left((\sigma^\vee\setminus\rho^\perp)\cap M\right)\times\{1\}} H^0\left(Y, \mathcal{O}(\Psi((u,1)))\right).$$

Fix a weight u and consider the graded piece $\mathcal{B}_u$. A section $s \in \mathcal{B}_u$ is nonzero if and only if

$$(u,1) \in \left(\sigma^\vee \setminus (\sigma^\vee \setminus \rho^\perp) \cap M\right) \times \{1\} = \left(\rho^\perp \cap M\right) \times \{1\}.$$

The ray ρ then corresponds to a facet $F \leqslant \Box$ such that $\rho \perp \mathbb{Q}_{\geqslant 0}(F \times \{1\})$. So a section $s \in \mathcal{B}_u$ is nonzero if and only if $u \in F \subset \Box$. In this setting, Proposition 32 gives the conclusion. □

We demonstrate this proposition with an example.

Example 65. Let $X = \mathrm{dP}_3$ like in Example 30 and Figure 3.1. We now want to intersect D with F^-, the horizontal divisor that corresponds to the vertex $-1 \in \Box$ of the base polytope. Applying Proposition 64, we see that

$$D \cdot F^- = 1!\ \mathrm{vol}(\Phi) = \Psi_0(-1) + \Psi_\infty(1) = 1 + (-2) = -1.$$

3.2 A combinatorial criterion for stability of T_X

Let X be a $\mathbb{C}^*$-surface with fibrewise torus action as in Definition 54 and consider a divisorial polytope $(\Psi, \Box)$ which represents a polarisation D of X. Let I be a finite index set. The function $\deg\Psi$ can be expressed as the minimum of finitely many linear functions $\{a_i + m_i x\}_{i\in I}$, or equivalently as the min-plus tropical polynomial

$$\deg\Psi = \bigoplus_{i\in I} a_i \odot x^{m_i}.$$

The Fundamental Theorem of Tropical Algebra says there exists a tropical polynomial

$$f = \bigoplus_{i\in I} A_i \odot x^{m_i}$$

which has the same graph as $\deg \Psi$ and can be factored into linear tropical polynomials. We require every linear function to contribute non-trivially to the graph of $\deg \Psi$. This amounts to asking that the tropical roots $r_i = A_i - A_{i+1}$ of f all be distinct. This is achieved when $2A_i < A_{i-1} + A_{i+1}$ for all $i \in I$ and $\{m_i\}_{i \in I}$ is a decreasing sequence. Denote by m_s the smallest element in that sequence.

Since facets of the graph Ψ_P are in one-to-one correspondence with vertices $v \in \mathcal{D}_P$ (see Proposition 19), we note that in the case of $X = \mathbb{F}_d$, we have

$$\deg \Psi = 0 \odot x^d.$$

Let's first prove that the algebraic surfaces with fibrewise $\mathbb{C}^*$-action from Definition 54 are precisely iterated blow ups of Hirzebruch surfaces $\mathbb{F}_d$ equipped with the subordinate action given by the $(0, -1)^T : \mathbb{Z}^2 \to \mathbb{Z}$ lattice morphism described earlier.

Lemma 66. *A smooth algebraic surface* $X = X(\mathcal{D})$ *has fibrewise* $\mathbb{C}^*$*-action if and only if* X *is a* $\mathbb{C}^*$*-equivariant iterated blow up of a Hirzebruch surface* $\mathbb{F}_d$ *with* $d \geqslant 0$*, whose* $\mathbb{C}^*$*-action is the subordinate torus action given by the character* $(0, -1)^T \in M \simeq \mathbb{Z}^2$.

Proof. The "if" direction is immediate. Endowing $\mathbb{F}_d$ with the specified torus action gives $\mathbb{F}_d$ the structure of a $\mathbb{C}^*$-surface. By the downgrade procedure, we see that there exists a morphism $\mathbb{F}_d \to \mathbb{P}^1$ and $\mathbb{C}^*$ acts along the fibres.

For the "only if" direction, let X be a $\mathbb{C}^*$-surface with fibrewise action. Then there exists a divisorial fan $\mathcal{D}$ such that $X = X(\mathcal{D})$. Because X is assumed to be projective and smooth, the tail fan of $\mathcal{D}$ consists of the origin $\{0\}$ and two rays $\{\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\leqslant 0}\}$. By Proposition 19, the two rays correspond to the two horizontal divisors, F^+ and F^-, which are rational curves isomorphic to the Chow quotient $\mathbb{P}^1$ consisting of $\mathbb{C}^*$-fixed points. Now proceed as follows. By the Castelnuovo Contraction Criterion, we can blow down every $\mathbb{C}^*$-invariant -1-curve until we obtain a relatively minimal surface Y, i.e. no fibre of the morphism to the Chow quotient contains -1-curves.

We are now left with two options. Either $Y \simeq \mathbb{P}^2$ or $Y \simeq \mathbb{F}_d$. Since all the blow downs performed in the anterior step were $\mathbb{C}^*$-equivariant, it follows that Y is equipped with a morphism $Y \to \mathbb{P}^1$ to the same Chow quotient. However, it is elementary to observe that $\mathbb{P}^2 \to \mathbb{P}^1$ must be a constant map and the tailfan of the corresponding divisorial fan cannot contain two rays. This means that we must have $Y \simeq \mathbb{F}_d$.

Finally, $Y \simeq \mathbb{F}_d$ inherits a $\mathbb{C}^*$-action from X such that $\mathbb{C}^*$ acts identically on each fibre of the morphism $Y \to \mathbb{P}^1$. A $\mathbb{C}^*$-action on Y is nothing other than an injective group homomorphism

$$\mathbb{C}^* \hookrightarrow \mathrm{Aut}(Y) = \mathrm{Aut}(\mathbb{F}_d) = \mathrm{PGL}_2 \times \mathbb{C}^*.$$

Since $\mathrm{Aut}(\mathbb{F}_d)$ has a maximal torus of dimension 2, it follows that every $\mathbb{C}^*$-action on $\mathbb{F}_d$ is a subordinate action of $\mathbb{C}^* \times \mathbb{C}^*$ which we can denote by $s_1 \times s_2$, where s_1 is the action on the fibres and s_2 is the action on the base of the fibre bundle. By assumption, a fibrewise action is an injection $\mathbb{C}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^*$ of some one dimensional torus into the two-dimensional torus, where the action s_2 on the base is the trivial action.

Since the $\mathbb{C}^* \times \mathbb{C}^*$-action is toric, we can see from the toric fan of $\mathbb{F}_d$ that the only options for a subordinate $\mathbb{C}^*$-action with these properties are the actions corresponding to the characters $(0, -1)$ and $(0, 1) \in M \simeq \mathbb{Z}^2$. But these yield equivalent divisorial fans. □

For the remainder of the section, equip $\mathbb{F}_d$ with a fibrewise $\mathbb{C}^*$-action.

Lemma 67. *Let X be an iterated $\mathbb{C}^*$-equivariant blow up of $\mathbb{F}_d$ such that $0 \leqslant k \leqslant d$ centres of the blow up lie on the horizontal divisor F^+. Then X can be expressed as a blow up of $\mathbb{F}_{d-k}$.*

Proof. We show this by induction on k. Note that for $k = 0$, the statement is trivial. For the induction hypothesis, assume that for some n, the statement is true for all $k \leqslant n$. To investigate the case $k = n + 1$, consider a sequence of blow ups of $\mathbb{F}_d$, denoted X. The induction hypothesis says that X can be expressed as an iterated blow

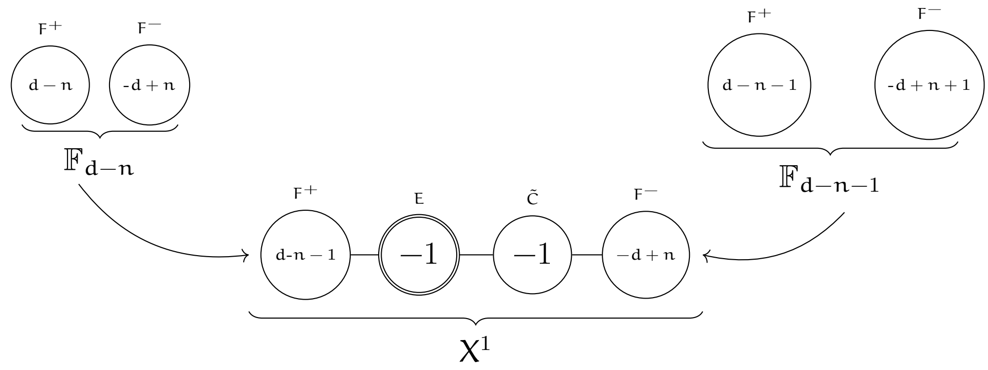

Figure 3.2: Expressing X^1 as two different blow ups

up of $\mathbb{F}_{d-n}$, i.e. we can explicitly write out the sequence of morphisms as

$$X \to \ldots \to X^1 \to \mathbb{F}_{d-n},$$

where we assume that blow ups with centre on F^+ occur first. In particular $X^1 \to \mathbb{F}_{d-n}$ is such a blow up. The centre of this blow up is the intersection of F^+ with an invariant curve C with self-intersection number $C^2 = 0$.

The surface X^1 features two new invariant curves: an exceptional divisor E with $E^2 = -1$, and the strict transform $\tilde{C}$ of C with $\tilde{C}^2 = -1$. The strict transforms of the horizontal divisors have $(F^+)^2 = d - n - 1$ and $(F^-)^2 = -(d - n)$.

Next, we can blow down the strict transform curve $\tilde{C}$ to obtain a surface Y with two horizontal divisors characterised by $(F^+)^2 = d-n-1$ and $(F^-)^2 = -(d-n-1)$. Since the Chow quotient of Y is $\mathbb{P}^1$, [41, Prop 4.1] tells us that Y is a $\mathbb{P}^1$-bundle of degree $d-n-1$ over $\mathbb{P}^1$, i.e. Y is isomorphic to $\mathbb{F}_{d-n-1}$. We can recover X^1 by performing a blow up on $\mathbb{F}_{d-n-1}$ with centre lying on F^- (see Figure 3.2). Replacing the final map in the chain defining X with the new blow up $X^1 \to Y = \mathbb{F}_{d-n-1}$ completes the proof.

□

Lemma 68. *Let $X = X(\mathcal{D})$ be a $\mathbb{C}^*$-equivariant iterated blow up of $\mathbb{F}_d$ such that k centres of the blow up lie on F^+. Fix a polarisation $D = (\Psi, \square)$. Then the minimal slope of* $\deg \Psi$ *is $m_s = d - k$.*

Proof. We check the case $k = 0$. For any polarisation $(\Psi, [0, a])$ of $X = \mathbb{F}_d$, we know that the facets of the graph Ψ_P are in one-to-one correspondence to the vertices of $\mathcal{D}_P$ by Proposition 19. For $P = 0$, the graph Ψ_0 has a unique facet F such that $\Psi(u) = \langle (u, 1), (d, h) \rangle$ for some h, while for $P \neq 0$, the contributions are trivial. Hence $\deg \Psi$ is itself a linear function with slope equal to d. This gives $m_s = d$.

Blow ups with centre on F^- will insert either vertices $v < -d$ on $\mathcal{D}_0$ or $v < 0$ on $\mathcal{D}_P$ for $P \neq 0$. The degree function will then consist of more facets with slopes $m_i \geqslant d$, so $m_s = d$ remains unchanged in this case.

For $k > 0$, we can use Lemma 67 to remark that X can be expressed as a $\mathbb{C}^*$-equivariant iterated blow up of $\mathbb{F}_{d-k}$ where none of the blow ups have centres on F^+ or its strict transforms. This takes us back to the case $k = 0$ where we swap $d - k$ for d. Then the conclusion follows. □

Next we describe a combinatorial criterion for the instability of the tangent bundle of an algebraic surface with fibrewise $\mathbb{C}^*$-action. The result is reminiscent of Proposition 10. The contents of Proposition 69 and Theorem 70 extend the conclusions of Herring-Nill-Süß [29] about toric surfaces to surfaces with fibrewise $\mathbb{C}^*$-action.

Proposition 69. *Let $X = X(\mathcal{D})$ be an algebraic surface with fibrewise $\mathbb{C}^*$-action. Let $D = (\Psi, \square = [0, a])$ be an ample polarisation. Then the tangent bundle T_X is slope unstable with respect to D provided that*

$$\deg \Psi(0) + \deg \Psi(a) > 2a.$$

Proof. Let $f : X \to \mathbb{P}^1$ be the morphism to the Chow quotient. First, we claim that the relative tangent sheaf $T_{X/\mathbb{P}^1}$ has the form

$$T_{X/\mathbb{P}^1} \simeq \mathcal{O}(F^+ + F^-),$$

where F^+, F^- are the strict transforms of the horizontal divisors of $\mathbb{F}_d$. Taking the first Chern class in the formula from [48, Thm 1.1] we obtain

$$c_1(T_{X/\mathbb{P}^1}) = -K_X + f^* K_{\mathbb{P}^1} + R(f),$$

where we introduce the ramification divisor $R(f) = \sum_i (m_i - 1)C_i$. The sum runs over all the vertical divisors C_i and m_i denotes the multiplicity of C_i in the fibre containing C_i. In this setting, for $C_i = D_{P,\nu}$ we have $m_i = \mu(\nu)$. The claim follows by construction since the canonical divisor is given by (see e.g. [4, Section 8.1])

$$K_X = -F^+ - F^- + f^*K_{\mathbb{P}^1} + R(f).$$

Next, we aim to show that the relative tangent sheaf destabilises the tangent bundle. To this end, we find a numerical bound for the slope of T_X as follows.

$$\begin{aligned}
2\mu_D(\mathcal{T}_X) &= -K_X \cdot D \\
&= (F^+ \cdot D) + (F^- \cdot D) - \sum_{(P,\nu)} \left(\mu(\nu)\mathrm{coeff}_P(K_{\mathbb{P}^1}) + m(\nu) - 1\right) D_{P,\nu} \cdot D \\
&= (F^+ \cdot D) + (F^- \cdot D) + \sum_{P \in \mathrm{Supp}(K_{\mathbb{P}^1})} \left(\sum_{\nu \in \mathbb{Z}} c_{P,\nu} D_{P,\nu} \cdot D - \sum_{\nu \in \mathbb{Q}\setminus\mathbb{Z}} d_{P,\nu} D_{P,\nu} \cdot D \right) \\
&\quad - \sum_{P \notin \mathrm{Supp}(K_{\mathbb{P}^1}), \nu \in \mathbb{Q}\setminus\mathbb{Z}} e_{P,\nu} D_{P,\nu} \cdot D \\
&\leqslant (F^+ \cdot D) + (F^- \cdot D) + \sum_{P \in \mathrm{Supp}(K_{\mathbb{P}^1}), \nu \in \mathbb{Z}} c_{P,\nu} D_{P,\nu} \cdot D \\
&\leqslant (F^+ \cdot D) + (F^- \cdot D) + 2\,\mathrm{vol}(\square).
\end{aligned}$$

Above, we introduced the notation $c_{P,\nu}, d_{P,\nu}, e_{P,\nu}$ for the respective coefficients of the form $-\left(\mu(\nu)\mathrm{coeff}_P(K_{\mathbb{P}^1}) + \mu(\nu) - 1\right)$. We obtained the first inequality by noting that $d_{P,\nu}, e_{P,\nu} \geqslant 0$ so their respective terms contribute negatively to $\mu_D(T_X)$. Regarding the second inequality, we know that $\mathrm{Supp}(K_{\mathbb{P}^1}) = \{0, \infty\}$, so $c_{P,\nu} = 1$ for all vertices ν. Then, using the vertical divisor-facet correspondence together with Proposition 58, we see that

$$c_{P,\nu} \sum_{(P,\nu)} D_{P,\nu} \cdot D = \sum_{P,\nu} \mathrm{vol}(F_{P,\nu}) \leqslant 2\,\mathrm{vol}(\square).$$

We can assume without loss of generality that $\square = [0, a]$. Then Proposition 64 tells us how to intersect D with the horizontal divisors, giving

$$F^+ \cdot D = \deg \Psi(a) \qquad \text{and} \qquad F^- \cdot D = \deg \Psi(0).$$

Then the condition for instability becomes

$$\begin{aligned}\mu_D(T_{X/\mathbb{P}^1}) = \mu_D(F^+ + F^-) &= F^+ \cdot D + F^- \cdot D \\ &= \deg \Psi(0) + \deg \Psi(a) \\ &> \frac{1}{2}\left(F^+ \cdot D + F^- \cdot D + 2\,\mathrm{vol}(\Box)\right) \\ &\geqslant \mu_D(T_X).\end{aligned}$$

Rearranging the middle inequality yields the desired relation

$$\deg \Psi(0) + \deg \Psi(a) > 2a.$$

□

Theorem 70. *Let* X *be an algebraic surface with fibrewise* $\mathbb{C}^*$*-action which can be expressed as a* $\mathbb{C}^*$*-equivariant iterated blow up of* $\mathbb{F}_d$ *where* $d \geqslant 2$*. Then the tangent bundle* T_X *is unstable with respect to any polarisation.*

Proof. Let $(\Psi, \Box)$ be any polarisation of X where we assume without loss of generality that the base polytope has the form $\Box = [0, a]$. Following Proposition 69, we need to check that

$$\deg \Psi(0) + \deg \Psi(a) > 2a.$$

Since Ψ is a divisorial polytope, we know that $\deg \Psi(0) \geqslant 0$ and by concavity of the function deg, we have that $\deg \Psi(a) - m_s a \geqslant 0$. Combining these facts we see immediately that $\deg \Psi(0) + \deg \Psi(a) \geqslant \deg \Psi(a) \geqslant m_s a$. Let k be the number of $\mathbb{C}^*$-fixed points blown up on the minimal surface $\mathbb{F}_d$. We may assume that all k centres lie on the positive section F^+ and its strict transforms. By Lemma 68, we have that $m_s = d - k$, and the instability condition is satisfied provided that $k \leqslant d - 2$. □

Chapter 4

The case of Hirzebruch surfaces in general

4.1 Minimal surfaces

Let C be a curve of genus g. We consider ruled surfaces $\pi : X \to C$, where we can write $X \simeq \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ a rank two vector bundle over C which satisfies the property that $H^0(C, \mathcal{E}) \neq 0$, but for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$. In these conditions, we have that $e = -\deg(\bigwedge^2 \mathcal{E})$ is an invariant of X.

We fix a section $s : C \to X$, and denote $s = s(C)$, with the property $\mathcal{O}_X(s) \simeq \mathcal{O}_X(1)$. We call this a *minimal section*, and note that for $e > 0$, this minimal section is unique (see e.g. [37, Cor 1.17]). Whilst for $e < 0$ this is not the case, it is true however that all minimal sections are numerically equivalent.

For any divisor D on C, we write by abuse of notation $\pi^* D = Df$. Any element of $\mathrm{Pic}(X)$ can be written as $as + Df$ with $a \in \mathbb{Z}$ and $D \in \mathrm{Pic}(C)$, and any element of $\mathrm{Num}(X)$ as $as + df$ with $a, d \in \mathbb{Z}$. Furthermore, denoting the class of a fibre by f, the intersection form is given by $f^2 = 0$, $s^2 = -e$, and $s \cdot f = 1$.

The invariants g and e impose one important restriction onto each other, namely that if $e < 0$ then $g > 0$ ([27, V.2.13]). If $g = 0$, the base curve is $\mathbb{P}^1$ and all vector bundles are split. Moreover, we must have $e \geqslant 0$ and all such surfaces have the form

$$\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)).$$

These are known as *rational ruled surfaces* or *Hirzebruch surfaces*.

Remark 71. We are interested in the stability of the tangent bundle T_X of ruled surfaces of the form $\pi : X \to C$. The main observation that enables us to attack this question is that the *relative tangent bundle* $T_{X/C} = \ker(T_X \to \pi^* T_C)$ is in all cases of interest a destabilising subsheaf. This is partly inspired by a similar strategy being adapted to the case of certain Fano threefolds in [47] and the conclusions of [50].

Definition 72. Let $\pi : X \to C$ be a ruled surface and fix an ample divisor H. We define the *difference of slopes* to be

$$\Delta_H(T_X) = \left(\mu_H(T_{X/C}) - \mu_H(T_X)\right).$$

This is positive whenever the relative tangent bundle is a destabilising subsheaf. We omit to write H if it is evident from context.

For the remainder of this section, we focus on minimal ruled surfaces $X \neq \mathbb{F}_0, \mathbb{F}_1$, i.e. X either has $g = 0$ and $e \geqslant 2$ or $g > 0$. We proceed to show that such ruled surfaces have unstable tangent bundle. Before that, we collect some relevant facts.

Proposition 73. *Let $\pi : X \to C$ be a ruled surface. Then in* $\mathrm{Num}(X)$ *we have*

1. $-K_X \equiv 2s - (2g - 2 - e)f$*;*
2. $c_1(\mathcal{T}_{X/C}) \equiv 2s + ef$*.*

Proof. For part (1), see [27, V.2.10]. For part (2), we use the modified Euler exact sequence for projectivised vector bundles:

$$0 \to \Omega^1_{X/C} \to \mathcal{O}_X(-1) \otimes \pi^* \mathcal{E}^\vee \to \mathcal{O}_X \to 0.$$

Using standard facts about Chern classes, we see that

$$\begin{aligned} c_1(\Omega^1_{X/C}) &= c_1(\mathcal{O}_X(-1)\otimes \pi^*\mathcal{E}^\vee) \\ &= \mathrm{rk}(\mathcal{E})c_1(\mathcal{O}_X(-1)) + c_1(\pi^*\mathcal{E}^\vee) \\ &= -2c_1(\mathcal{O}_X(1)) + \pi^*(c_1(\mathcal{E}^\vee)) \\ &= -2c_1(\mathcal{O}_X(1)) + c_1(\mathcal{E}^\vee)f. \end{aligned}$$

By construction, $\mathcal{E}^\vee \simeq \pi_*\mathcal{O}_X(1)$, hence

$$c_1(\Omega^1_{X/C}) = -2c_1(\mathcal{O}_X(1)) + \pi_*c_1(\mathcal{O}_X(1))f \equiv -2s - ef.$$

□

Proposition 74. *Let* $H = as + bf \in \mathrm{Num}(X)$ *be a divisor. For* $e \geqslant 0$*, we have that* H *is ample if and only if* $a > 0$ *and* $b > ae$*. For* $e < 0$*, we have that* H *is ample if and only if* $a > 0$ *and* $b > \frac{1}{2}ae$*.*

Proof. See [27, V.2.18]. □

Proposition 75. *Let* $\pi : X \to C$ *be some ruled surface other than* $\mathbb{F}_0$ *and* $\mathbb{F}_1$*. Then the tangent bundle* T_X *is unstable with respect to any polarisation* H*, i.e.* $\mathrm{sStab}(T_X) = \emptyset$*. Moreover, the relative tangent bundle* $T_{X/C} \subset T_X$ *is a destabilising subsheaf.*

Proof. Set $H = as + bf$, an ample divisor. We have to show that the difference of slopes $\Delta_H(T_X)$ is positive. First, we compute the slopes directly.

$$\begin{aligned} 2\mu(T_X) &= -K_X \cdot H = (2s + (e+2)f - 2gf)\cdot(as+bf) \\ &= 2a + 2b - ae - 2ga. \\ \mu(T_{X/C}) &= c_1(T_{X/C})\cdot H = (2s+ef)\cdot(as+bf) = 2b - ae. \end{aligned}$$

Second, we analyse each situation of interest.

Case 1: $g = 0$ and $e \geqslant 2$. We have $\Delta(T_X) = 2b - 2a - ae > 2ae - 2a - ae = ae - 2a = a(e-2) \geqslant 0$, where we use the ampleness of H as in Proposition 74.

Case 2: $g > 0$ and $e \geqslant 0$. We have $\Delta(T_X) = 2b - 2a - ae + 2ga > 2ae - ae - 2a + 2ga = ae - 2a + 2ga = ae + 2a(g-1) \geqslant 0$, where we use the fact that $a > 0$ and $b > ea$.

Case 3: $g > 0$ and $e < 0$. We have $\Delta(T_X) = 2b - 2a - ae + 2ga > ae - ae - 2a + 2ga = -2a + 2ga = 2a(g-1) \geqslant 0$, since $a > 0$ and $b > \frac{1}{2}ae$ due to the ampleness of H. This concludes the analysis.

□

4.2 A remark vis-à-vis the ample cone

We can study how the stability of a rank two coherent sheaf on an algebraic surface varies as we vary the choice of polarisation. We make the observation that in this very specific case we can partition the ample cone of the surface into pairwise disjoint convex sets. This observation helps us demonstrate a criterion for semistability based off testing semistability with respect to the relative tangent bundle.

Definition 76. Let $\mathcal{E}$ be a coherent, torsion-free sheaf of rank two on some smooth projective surface X. If $\mathcal{E}$ is H-unstable, then there exists a unique maximal destabilising subsheaf $\mathcal{F} \subset \mathcal{E}$. We define the *unstable region* $R(H)$ to be

$$R(H) = \left\{ M \in \mathrm{Amp}(X) : \left(c_1(\mathcal{F}) - \frac{1}{2} c_1(E) \right) \cdot M > 0 \right\} \subset \mathrm{Amp}(X).$$

The *semistable locus* $\mathrm{sStab}(\mathcal{E})$ can then be identified with the complement of the union of all unstable regions. In symbols:

$$\mathrm{sStab}(\mathcal{E}) := \mathrm{Amp}(X) \Bigg\backslash \left(\bigcup_{H \in \mathrm{Amp}(X)} R(H) \right).$$

Lemma 77 (see also [40]). *Let $\mathcal{E}$ be a rank two coherent torsion-free sheaf on a smooth projective surface X. The unstable regions $R(H)$ together with the semistable locus partition the ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$. More precisely, the following hold.*

1. *Unstable regions and the semistable locus are convex cones in* $\mathrm{Amp}_\mathbb{R}(\mathsf{X})$*;*

2. *Unstable regions are open subsets in* $\mathrm{Amp}_\mathbb{R}(\mathsf{X})$*;*

3. *Unstable regions and the semistable locus partition* $\mathrm{Amp}_\mathbb{R}(\mathsf{X})$*, i.e. the union of all regions is equal to* $\mathrm{Amp}_\mathbb{R}(\mathsf{X})$ *and regions are pairwise disjoint;*

4. *The semistable locus is a closed set in* $\mathrm{Amp}_\mathbb{R}(\mathsf{X})$*.*

Proof. Part (1) follows from the linearity of the intersection product. Part (2) follows from the continuity of the intersection product. To see part (3), note that each polarisation must appear in at least one region. The uniqueness of the Harder-Narasimhan filtration then implies that each polarisation must appear in exactly one region. Part (4) then follows from parts (2) and (3). □

4.3 A general criterion for semistability of T_X

Theorem 78. *Let* X *be an iterated blow up of some Hirzebruch surface* $\mathbb{F}_\mathsf{e}$ *with degree* $\mathsf{e} \geqslant 2$*. Then* $\mathrm{sStab}(\mathsf{T}_\mathsf{X}) = \emptyset$ *if and only if* $\mathsf{T}_{\mathsf{X}/\mathbb{P}^1}$ *is a destabilising subsheaf for all* $\mathsf{H} \in \mathrm{Amp}_\mathbb{R}(\mathsf{X})$*.*

Proof. The "if" direction is clear. In order to prove the "only if" direction, we need to understand how the ample cone $\mathrm{Amp}_\mathsf{R}(\mathsf{X})$ is partitioned into convex sets. First, we show that that the relative tangent bundle $\mathsf{T}_{\mathsf{X}/\mathbb{P}^1}$ is always a destabilising subsheaf at least for some ample polarisations H. In other words, there exists an unstable region $\mathsf{R}(\mathsf{H}_1) \subset \mathrm{Amp}_\mathbb{R}(\mathsf{X})$ where the Harder-Narasimhan filtration is given by

$$0 \subset \mathsf{T}_{\mathsf{X}/\mathbb{P}^1} \subset \mathsf{T}_\mathsf{X}.$$

Denote by $\mathsf{f}: \mathsf{X} \to \mathbb{F}_\mathsf{e}$ the blowup morphism. We already know by Proposition 75 that $\mathrm{sStab}(\mathsf{T}_{\mathbb{F}_\mathsf{e}}) = \emptyset$ for $\mathsf{e} \geqslant 2$. Therefore, $\mathrm{Amp}_\mathbb{R}(\mathsf{T}_{\mathbb{F}_\mathsf{e}})$ is covered by a single unstable region $\mathsf{R}(\mathsf{H}')$ where $\mathsf{T}_{\mathbb{F}_\mathsf{e}/\mathbb{P}^1}$ is the maximal destabilising subsheaf.

Next, consider a polarisation of the form

$$H = f^*H' + \sum_{i=1}^{k} \varepsilon_i E_i \in \mathrm{Amp}_{\mathbb{R}}(X),$$

where $H' \in \mathrm{Amp}_{\mathbb{R}}(\mathbb{F}_e)$, $\varepsilon_i > 0$, and E_i are the pullbacks of the exceptional divisors associated to the blowup $f : X \to \mathbb{F}_e$. We can show that $R(H) \subset \mathrm{Amp}_{\mathbb{R}}(X)$ is an unstable region and $T_{X/\mathbb{P}^1}$ is the maximal destabilising subsheaf.

We calculate the difference of slopes

$$\begin{aligned}
\left(c_1(T_{X/\mathbb{P}^1}) - \frac{1}{2}c_1(T_X)\right) \cdot H &= \left(f^* c_1(T_{\mathbb{F}_e/\mathbb{P}^1}) - \sum_{i=1}^{k} E_i - \frac{1}{2}\left(f^* c_1(T_{\mathbb{F}_e}) - \sum_{i=1}^{k} E_i\right)\right) \cdot H \\
&= \left(f^* c_1(T_{\mathbb{F}_e/\mathbb{P}^1}) - \frac{1}{2} f^* c_1(T_{\mathbb{F}_e})\right) \cdot f^*H' - \frac{1}{2}\sum_{i=1}^{k} \varepsilon_i E_i^2 \\
&= \left(c_1(T_{\mathbb{F}_e/\mathbb{P}^1}) - \frac{1}{2}c_1(T_{\mathbb{F}_e})\right) \cdot H' + \frac{1}{2}\sum_{i=1}^{k} \varepsilon_i \\
&= \Delta_{H'}(T_{\mathbb{F}_e}) + \frac{1}{2}\sum_{i=1}^{k} \varepsilon_i \\
&> 0,
\end{aligned}$$

where positivity follows from the assumption that $T_{\mathbb{F}_e}$ is H'-unstable. As $T_{\mathbb{F}_e/\mathbb{P}^1}$ is a saturated, destabilising subsheaf, it follows from the uniqueness of the Harder-Narasimhan filtration (see Proposition 11) that the maximal destabilising subsheaf in the region $R(H)$ is indeed $T_{X/\mathbb{P}^1}$.

By Lemma 77, there exists a decomposition of the ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$ into convex sets: $\mathrm{sStab}(T_X)$, $R(H_1), R(H_2), \ldots, R(H_\ell)$, where $R(H_i)$ is the unstable region associated to the ample polarisation H_i. We fix $R(H_1) = R(H)$ to be the unstable region where $T_{X/\mathbb{P}^1}$ is the maximal destabilising subsheaf.

Now, assume toward a contradiction that $\mathrm{sStab}(T_X) = \emptyset$ and the regions $R(H_i) \neq \emptyset$ are all pairwise disjoint. This means there exist maximal destabilising subsheaves of T_X which are different from $T_{X/\mathbb{P}^1}$. This also implies that the ample cone is covered by these unstable regions as such:

$$\mathrm{Amp}_{\mathbb{R}}(X) = \bigcup_{i=1}^{\ell} R(H_i).$$

Again by Lemma 77, we know that $R(H_i)$ are all pairwise disjoint open subsets of $\mathrm{Amp}_{\mathbb{R}}(X)$. Since $\mathrm{Amp}_{\mathbb{R}}(X)$ is a connected topological space, it cannot be covered by pairwise disjoint open sets. This yields a contradiction. So it must be the case that either $\mathrm{sStab}(T_X) \neq \emptyset$ or $R(H_2) = R(H_3) = \ldots = R(H_\ell) = \emptyset$. In the latter case, the ample cone consists of a single unstable region, namely $\mathrm{Amp}_{\mathbb{R}}(X) = R(H)$, the only unstable region that is confirmed to be non-empty.

□

4.4 Applying the criterion to concrete examples

With Theorem 78 now proved, we have a criterion that verifies the semistability of an iterated blowup of a Hirzebruch surface. The situation is more complex than the case of surfaces with fibrewise $\mathbb{C}^*$-action as in Theorem 70.

In particular, we can replicate some results that show that blowing up a Hirzebruch surface in specific centres yields a surface which has everywhere unstable tangent bundle. On the other hand, we construct an algebraic surface ruled over an elliptic curve whose tangent bundle is semistable. We also construct a blowup of $\mathbb{F}_2$ which has stable tangent bundle. The latter is treated as a $\mathbb{C}^*$-surface whose action is *not* a fibrewise action as explained in Definition 54.

Let's define two types of blowups of Hirzebruch surfaces in the general setting (the non-equivariant case) that we can investigate later.

Definition 79. An infinitely near point (of order n) p_n on a surface X^0 is given by a sequence of points $p_0, p_1, \ldots, p_n$ on surfaces $X^0, X^1, \ldots, X^n$ such that X^i is given by blowing up X^{i-1} at the point p_{i-1} and p_i is a point of the surface X^i with image p_{i-1}.

Definition 80. Let $X = X^k \to X^{k-1} \to \ldots \to X^1 \to Y$ be a sequence of blowups, where Y is a minimal ruled surface.

We say that the sequence $X \to Y$ is *admissible* if every surface X^i is obtained by

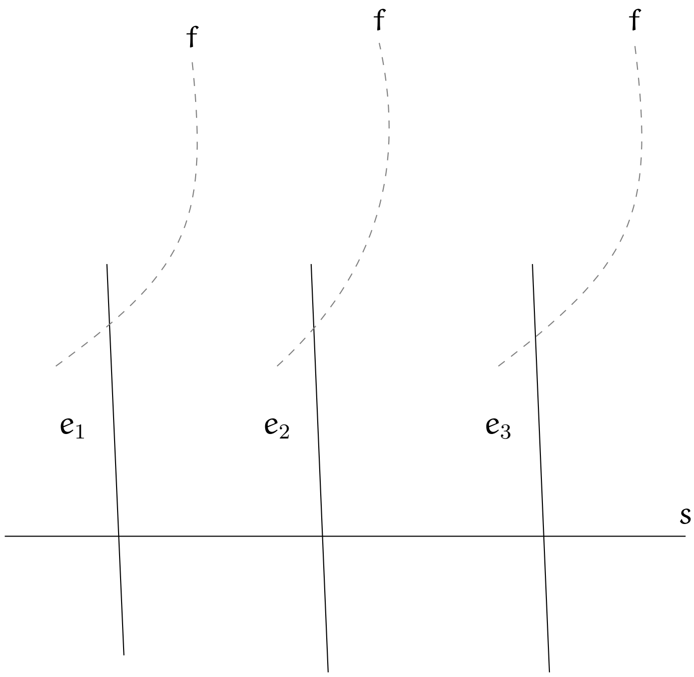

Figure 4.1: A schematic of an admissible blowup

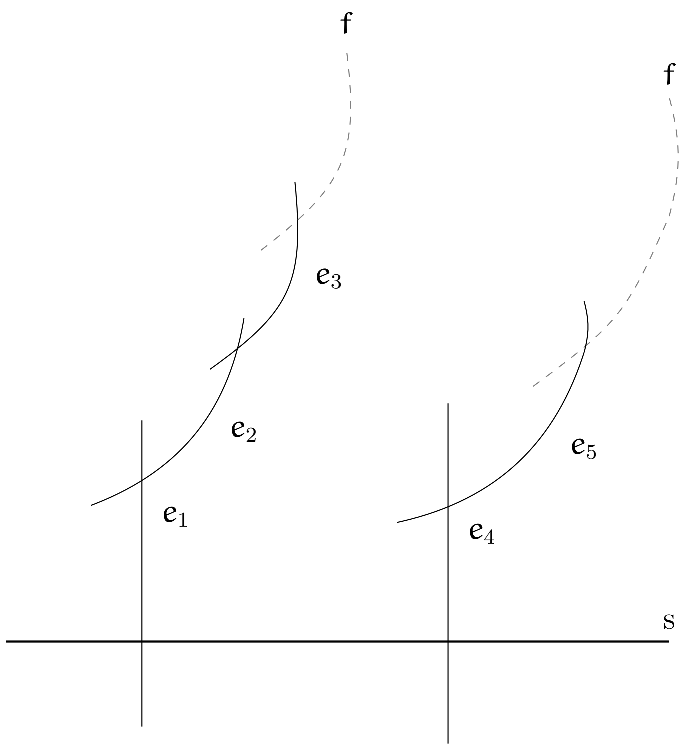

Figure 4.2: A schematic of a generic blowup

blowing up X^{i-1} in a point p_i which lies on the minimal section of X^{i-1}.

We say that the sequence $X \to Y$ is *generic* if the centre p_i of the blow up $X^i \to X^{i-1}$ either lies on the minimal section of X^{i-1} or it is infinitely near to a point p_j with $j < i$ which lies on the minimal section of X^j.

Finally, define the following sets of exceptional divisors:

$$\mathcal{E} \coloneqq \{E_1, E_2, \dots, E_k\},$$
$$\mathcal{S} \coloneqq \{E_i \in \mathcal{E} : \; p_i \in \tilde{s} \in \mathrm{Pic}(Y)\},$$

where $\tilde{s}$ denotes the strict transform of the minimal section s.

We prepare the result concerning such blowups of Hirzebruch surfaces with the following two observations. The first concerns a simple application in characteristic zero of a fact concerning arithmetic surfaces that is true in the broadest sense.

Proposition 81. *Let $\pi : X \to C$ be a smooth surface. Then*

1. $c_1(T_{X/C}) = -K_X + \pi^* K_C$ *if every fibre of π is reduced;*
2. $c_1(T_{X/C}) = -K_X + \pi^* K_C + R(\pi)$ *if π has nonreduced fibres and the ramification divisor is $R(\pi) = \sum_i (m_i - 1) D_i$, where m_i is the multiplicity of D_i in the fibre containing D_i and the sum runs over all nonreduced prime divisors $D_i \subset X$.*

Proof. Relation (2) is obtained from [48, Thm 1.1] by rearranging the isomorphism $K_{X/C} \simeq (\Omega^1_{X/C})^{\vee\vee} \otimes \mathcal{O}_X(R(\pi))$. From the same reference, we know that $R(\pi) = \sum_i r_i D_i$, where, taking x_i to be the generic point of D_i, we have by construction $r_i = r(\mathcal{O}_{x_i}/\mathcal{O}_{\pi(x_i)}) \geqslant m_i - 1$, and equality holds if and only if m_i is not a multiple of the characteristic of the residue field of $\pi(D_i)$. Since we work in characteristic zero, this condition is always satisfied. Finally, relation (1) can be obtained from relation (2) by observing that if all D_i are reduced, then $m_i = 1$ for all i and thus $R(\pi) = 0$. □

The second observation concerns computing a formula for the first Chern class of the relative tangent bundle of an admissible blowup.

Proposition 82. *Let* $\pi : X \to C$ *be an admissible blow up of a ruled surface. Then we have that*

$$c_1(T_{X/C}) = 2s + ef - \sum_{i \in \mathcal{S}} E_i.$$

Proof. We proceed by induction. In the base case $X = \mathbb{P}(\mathcal{E})$, there are no exceptional divisors. This situation is analysed in Proposition 73.

Next, assume the claim holds for some surface X which is an admissible blow up in $n > 0$ points of some ruled surface. Then let $f : Y \to X$ be the blow up of X in one additional point $p \in X$, and denote the exceptional divisor by E.

Using Proposition 81, we know that $c_1(T_{X/C}) = -K_{X/C} + R$. Denote the respective ramification divisors by R_X and R_Y. We write $R_X = \sum_{i,j} R_{ij} \tilde{E}_{ij}$ and we make the notation $\text{coeff}_{\tilde{E}_{ij}}(R_X) = R_{ij}$. Here E_{ij} are the exceptional divisors produced by blowing up in a point which is infinitely near to the point associated to the exceptional divisor E_i.

We take the first Chern classes of the relative tangent bundles and compute the difference.

$$\begin{aligned} f^* c_1(T_{X/C}) - c_1(T_{Y/C}) &= f^* \left(-K_X + R_X\right) - \left(-K_Y + R_Y\right) \\ &= (K_Y - f^* K_X) + f^* R_X - R_Y \\ &= E + f^* R_X - R_Y. \end{aligned}$$

Suppose that p lies on the intersection of two distinct exceptional curves, $\tilde{E}_{ij}$ and $\tilde{E}_{ik}$. Then by construction we have that

$$\begin{aligned} \text{coeff}_E(f^* R_X) &= R_{ij} + R_{ik}, \text{ and} \\ \text{coeff}_E(R_Y) &= (R_{ij} + 1) + (R_{ik} + 1) - 1 = R_{ij} + R_{ik} + 1. \end{aligned}$$

Hence we get that $E + f^* R_X - R_Y = E - E = 0$, so $c_1(T_{Y/C}) = f^* \left(c_1(T_{X/C})\right)$. By the induction hypothesis, we are done.

Now suppose that p lies on the strict transform of the minimal section. Then

$$\mathrm{coeff}_E(f^*R_X) = \mathrm{coeff}_E(R_Y) = (R_{ij}+1) - 1 = R_{ij}.$$

So $c_1(T_{Y/C}) = f^*\left(c_1(T_{X/C})\right) - E$ and $E \in \mathcal{S}$, as required. ☐

We now show that surfaces obtained as generic blowups of ruled surfaces have unstable tangent bundles.

Proposition 83. *Let $\pi : Y \to C$ be the generic blow up of some ruled surface $X \to C$ with invariant $e \geqslant 2-2g$. Then the tangent bundle T_Y is unstable with respect to any polarisation,*

Proof. Assume without loss of generality that Y is a sequence of n admissible blow ups and all k generic blow ups have centres on the strict interior of $\tilde{e}_n$. Then the anti-canonical divisor has numerical class

$$-K_Y = 2s - (2g-2-e)f - \sum_{i\in[n]} e_i - \sum_{j\in[k]} e_j.$$

By Lemma 82, the first Chern class of the relative tangent bundle is

$$c_1(T_{Y/C}) = 2s + ef - \sum_{i\in\mathcal{S}} e_i - \sum_{j\in[k]} e_j.$$

As before, we fix a polarisation $H = As + Bf - \sum_{i\in[n]} C_i e_i - \sum_{j\in[k]} C_j e_j$, then we calculate the difference of slopes:

$$\begin{aligned}
\Delta_H(T_Y) &= \left(2\,c_1(T_{Y/C}) + K_Y\right)\cdot H \\
&= \left(2s + (2g-2+e)f + \sum_{[n]\setminus\mathcal{S}} e_i - \sum_{\mathcal{S}} e_i - \sum_{[k]} e_j\right)\cdot H \\
&= 2B - 2eA + (2g-2+e)A + \sum_{[n]\setminus\mathcal{S}} C_i - \sum_{\mathcal{S}} C_i - \sum_{[k]} C_j \\
&= 2\left(B - eA - \sum_{\mathcal{S}} C_i\right) + (2g-2+e)\,A + \sum_{[n-1]} C_i + \left(C_n - \sum_{[k]} C_j\right).
\end{aligned}$$

By the Nakai-Moishezon criterion for ampleness, we see that

$$B - eA - \sum_{\mathcal{S}} C_i = \tilde{s} \cdot H > 0,$$

$$A = f \cdot H > 0,$$

$$C_i = e_i \cdot H > 0 \text{ for all } i \in [n-1], \text{ and}$$

$$C_n - \sum_{[k]} C_j = \tilde{e}_n \cdot H > 0.$$

Hence $\Delta(Y) > 0$, as required. □

Remark 84. Set $Y = \mathbb{F}_e$ with $e \geqslant 2$. Assuming that the sequence of blowups is admissible in Proposition 83 yields an alternative proof to the statements in Proposition 69 and Theorem 70. It turns out that first principles are sufficient to derive this result in itself. Working with $\mathbb{C}^*$-surfaces and their combinatorial language yields a proof that amounts to bounding the volume of a divisorial polytope emulating the proof concerning toric surfaces from [29]. We also use the language of divisorial fans and polytopes in the next section where we construct a blowup of a Hirzebruch surface which has stable tangent bundle.

Remark 85. We can rephrase the stability condition in terms of the positivity of a certain curve class. More precisely, given a polarisation by a line bundle H we can express $\Delta(X)$ as the intersection number of H with the curve class

$$\alpha = 2c_1(T_{X/C}) + K_X = 2s + (2g - 2 + e)f - 2\sum_{i \in \mathcal{S}} e_i + \sum_{i \in \mathcal{E}} e_i.$$

Now, $\Delta(X) > 0$ holds for every ample polarisation if and only if α is a pseudo-effective curve class.

We give an example of a blowup of a ruled surface with semistable tangent bundle.

Proposition 86. *Let* $f\colon X \to Z$ *is a blowup of a ruled surface* Z *of genus* $g = 1$ *and degree* $e = -1$ *in at least* 2 *distinct points lying on a minimal section* s*. Then there exists an ample polarisation such that the relative tangent bundle does not destabilise* T_X*.*

Proof. It is enough to prove the claim for a blowup in exactly two points. The claim will then follow from the fact that $\alpha = 2s - f - e_1 - e_2$ from the above consideration is not pseudo-effective. If it would be pseudo-effective then for every $\delta > 0$ there exists a curve C from some class $\alpha' = \ell\big(2s + (\varepsilon - 1)f + (1 - \varepsilon_1)e_1 - (1 - \varepsilon_2)e_2\big)$ with $\varepsilon, \varepsilon_1, \varepsilon_2 < \delta$. Now f_*C is an effective curve of class $\ell\big(2s + (\varepsilon - 1)f\big)$ on Z which passes through both centres of the blowup with multiplicity at least $(1 - \delta)\ell$. If now ms is a component of f_*C then $\alpha' - ms$ must be an effective class and therefore $2\ell - m \geqslant 2\ell(1 - \varepsilon)$ must hold. The latter implies $m \leqslant 2\ell\varepsilon < 2\ell\delta$.

Now, we calculate the intersection number $s.(\alpha' - ms) = \ell(1 + \varepsilon) - m$. Hence, the multiplicities of the two centres in f_*C add to at most

$$\ell(1 + \varepsilon) - m + 2m = \ell(1 + \varepsilon) + m < (1 + 3\delta)\ell < 2(1 - \delta)\ell$$

for δ sufficiently small. Hence, f_*C cannot pass through each of the centres with multiplicity at least $(1-\delta)\ell$ and therefore C cannot be effective. Hence, our assumption on the pseudo-effectiveness of α must be wrong. □

Next we give an example of a sequence of blowups that is not generic. We show that $\mathrm{sStab}(T_X) \neq \emptyset$.

Proposition 87. *Let $\pi : X \to \mathbb{P}^1$ be a Hirzebruch surface of degree $e \geqslant 2$, and let $p : Y \to X$ be the blow up in three points such that the first point lies on the minimal section, the second point lies on the strict interior of the first exceptional divisor, and the third point lies on the strict interior of the second exceptional divisor. Then there exists an ample polarisation such that $T_{Y/\mathbb{P}^1}$ does not destabilise T_Y.*

Proof. Using Lemma 82 we can compute $c_1(T_{Y/\mathbb{P}^1}) = 2s + ef - e_1 - e_2 - e_3$. The anti-canonical divisor has numerical class $-K_Y = 2s + (e + 2)f - e_1 - e_2 - e_3$. Then fixing a polarisation $H = As + Bf - C_1e_1 - C_2e_2 - C_3e_3$, we can calculate the difference of slopes $\Delta(T_Y) = 2B - (e + 2)A - C_1 - C_2 - C_3$. Rearranging this in the usual way, we see

$$\Delta(T_Y) = 2\,(B - eA - C_1) + (e - 2)A + (C_1 - C_2) - C_3.$$

Using the Nakai-Moishezon criterion, we can tell that the first three terms are positive because they are the intersection of H with the effective curves $\tilde{s} = s - e_1$, f, and $\tilde{e}_1 = e_1 - e_2$, respectively. Since C_3 is an arbitrary positive coefficient, we can choose a polarisation H with sufficiently large C_3 such that $\Delta(T_Y) < 0$. □

Remark 88. Summing up, Proposition 86 demonstrates the problems that occur in the special case where the base curve of the ruled surface is an elliptic curve, whilst Proposition 87 examines what goes wrong when considering non-generic blow ups more generally.

These examples prompt the following question.

Question 89. Does there exist a sequence of blowups $X \to \mathbb{F}_e$ of a Hirzebruch surface $\mathbb{F}_e$ with $e \geqslant 2$ such that T_X is indeed stable?

The next section focuses on providing an affirmative answer.

4.5 A special surface

In the previous section, we saw several examples of blow ups of Hirzebruch surfaces which have unstable tangent bundle. In 3, we proved a more general result concerning $\mathbb{C}^*$-surfaces with effective fibrewise action. This section will focus on constructing an example of a surface with unstable tangent bundle which cannot be produced as an iterated blow up of $\mathbb{P}^2$ but only as the blow up of a minimal singular surface wit stable tangent bundle.

The counterexample surface

Start with the toric Hirzebruch surface $\mathbb{F}_2$. This corresponds combinatorially to a fan $\Sigma \subset M \simeq \mathbb{Z}^2$ as in Figure 2.2. We blow up two toric fixed points. The first is the orbit corresponding to the two-dimensional cone σ_0, whose exceptional divisor corresponds to the ray $e_1 = (1,1)^T$. The second corresponds to the ray $e_2 = (1,2)^T$. For the downgrade, we choose the subtorus action given by

$$F = (2,-1)^T, \quad P = (-1,2), \quad s = (1,-1)^T.$$

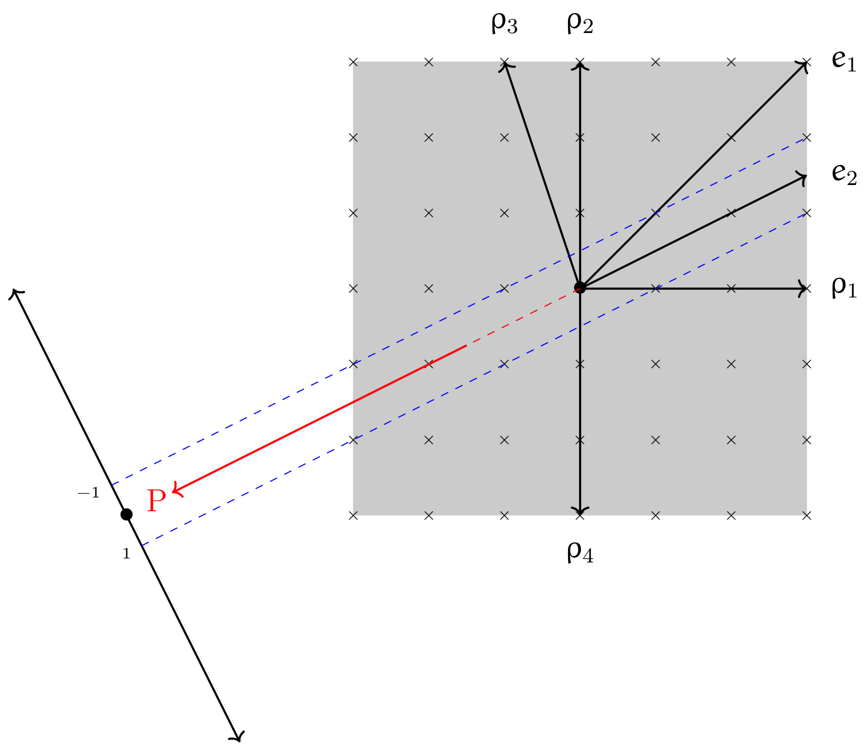

Figure 4.3: A special downgrade of $\mathbb{F}_2$

A visual representation is given in detail in Figure 4.3. The red arrow indicates the projection P. The blue dashed lines are given by

$$P^{-1}(-1) = \{(x,y) \in \mathbb{Q}^2 : \; x - 2y + 1 = 0\},$$
$$P^{-1}(1) = \{(x,y) \in \mathbb{Q}^2 : \; x - 2y - 1 = 0\}.$$

Calculating the intersections of $P^{-1}(1)$ and $P^{-1}(-1)$ with the different rays of Σ, we obtain the divisorial fan $\mathcal{M}$ supported on $Y = \mathbb{P}^1$ consisting of the p-divisors:

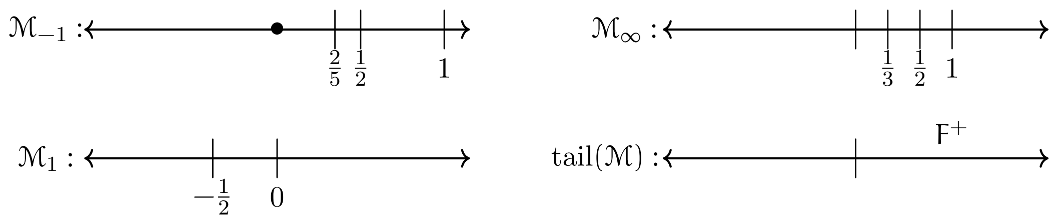

Figure 4.4: Slices of the divisorial fan of X

$$\mathfrak{D}^0 = [0,\infty)\otimes\{1\} \qquad \mathfrak{D}^1 = [1,\infty)\otimes\{-1\}$$

$$\mathfrak{D}^2 = \left[\frac{1}{2},1\right]\otimes\{-1\} \qquad \mathfrak{D}^3 = \left[\frac{2}{5},\frac{1}{2}\right]\otimes\{-1\}$$

$$\mathfrak{D}^4 = \left(-\infty,\frac{2}{5}\right]\otimes\{-1\} \qquad \mathfrak{D}^5 = \left(-\infty,-\frac{1}{2}\right]\otimes\{1\}$$

$$\mathfrak{D}^6 = \left[-\frac{1}{2},0\right]\otimes\{1\}.$$

The desired counterexample surface X is obtained by further blowing up three $\mathbb{C}^*$-invariant points on the slice $\mathcal{M}_\infty$ as in Figure 4.4. Note that $\mathsf{X}(\mathcal{M})$ is a strict $\mathbb{C}^*$-surface, i.e. it is not toric.

Figure 4.5 shows a schematic drawing of the configuration of exceptional divisors on the $\mathsf{X}(\mathcal{M})$. here we note that the surface $\mathsf{X}(\mathcal{M})$ is not generic.

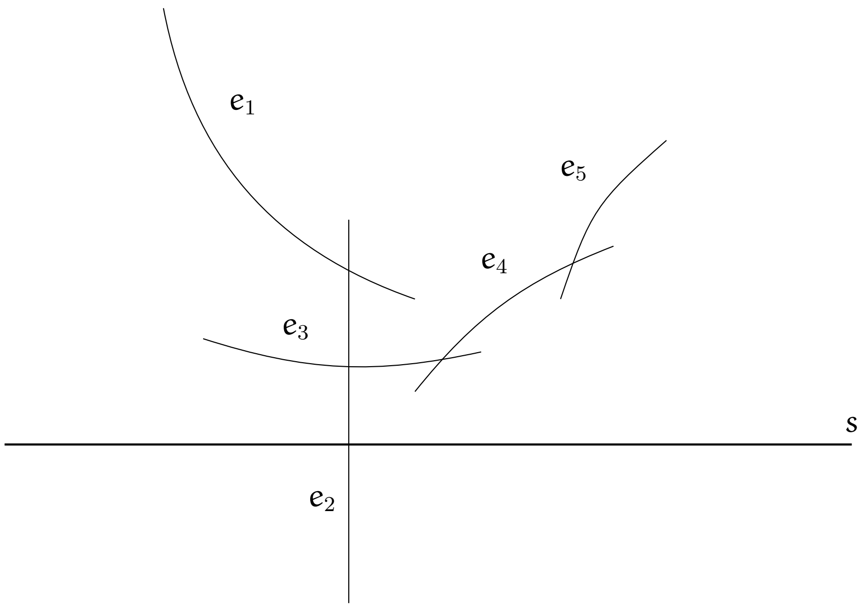

Figure 4.5: A diagram of the example surface $\mathsf{X}(\mathcal{M})$

Proposition 90. *The surface* $\pi : X \to \mathbb{P}^1$ *has the structure of a ruled surface over* $\mathbb{P}^1$ *and has the following invariants:*

$$-K_X = 2s + 4f - \sum_{i=1}^{5} e_i,$$

$$c_1(T_{X/\mathbb{P}^1}) = 2s + 2f - e_1 - e_3 - e_4 - e_5.$$

Proof. The anticanonical divisor is calculated using the standard formula for blow ups. The second formula for the Chern class of the relative tangent bundle follows from the formula in [48, Thm 1.1]. □

Proposition 91. *There exists an ample polarisation* H *such that the surface* X *has semistable tangent bundle.*

Proof. We show that the relative tangent bundle is not a destabilising subsheaf. To this end, fix an ample class $H = as + bf - \sum_{i=1}^{5} c_i e_i$. Next, compute the difference of slopes:

$$\begin{aligned}
\Delta_H(T_X) &= \big(2\,c_1(T_{X/\mathbb{P}^1}) - c_1(T_X)\big) \cdot H \\
&= (4s + 4f - 2e_1 - 2e_3 - 2s - 4f + e_1 + e_2 + e_3 + e_4 + e_5) \cdot H \\
&= (2s + e_2 - e_1 - e_3 - e_4 - e_5) \cdot H \\
&= 2b - 4a + c_2 - c_1 - c_3 - c_4 - c_5.
\end{aligned}$$

From the Nakai-Moishezon criterion for ampleness, we know that $2b - 4a > 0$ and $c_i > 0$ for all $i \in \{1, 2, \ldots, 5\}$. Set $a = 1$, $b = 3$, and $c_i = 1$ for all i. Then $\Delta_H(T_X) = -1$.

This shows that there exists an ample class $H \in \mathrm{Amp}(X)$ such that $T_{X/\mathbb{P}^1}$ is not a destabilising subsheaf of T_X. By Theorem 78 it follows T_X is semi-stable with respect to H. In symbols: $H \in \mathrm{sStab}(T_X)$, and in particular $\mathrm{sStab}(T_X) \neq \emptyset$. □

The arguments concerning partitioning the ample cone only reveal information about the presence or absence of the property of being semistable. This prompts the following question.

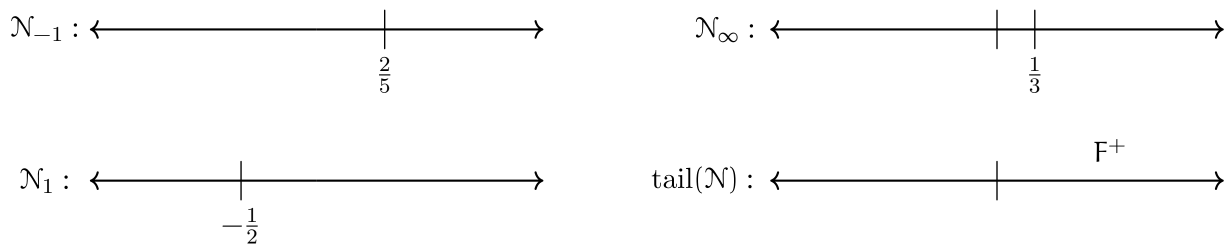

Figure 4.6: Slices of the divisorial fan of Y

Question 92. For the surface $X = X(\mathcal{M})$, is it the case that $\mathrm{Stab}(T_X) = \emptyset$?

We approach this question by studying a different $\mathbb{C}^*$-surface $Y = X(\mathcal{N})$ associatied to the divisorial fan $\mathcal{N}$ supported on the curve $Y' = \mathbb{P}^1$ as in Figure 4.6. There is a birational morphism of $\mathbb{C}^*$-surfaces $f : X = X(\mathcal{M}) \to X(\mathcal{N}) = Y$, which acts by contracting the curves $D_{-1,1}$, $D_{-1,\frac{1}{2}}$, $D_{1,0}$, and $D_{\infty,1}$, $D_{\infty,\frac{1}{2}}$ on $X = X(\mathcal{M})$.

In this setting, stability of T_Y implies stability of T_X by the openness property.

Proposition 93. *The surface* Y *is the weighted projective hypersurface*

$$\left\{x^3 w + z^2 + y^5 = 0\right\} \subset \mathbb{P}(1, 2, 5, 7).$$

Proof. As complexity one T-varieties, X and Y are Mori Dream Spaces. Their Cox rings are finitely generated complex algebras and in particular Y is given as an embedding into weighted projective space by its Cox ring.

We have that $X \to Y$ is a resolution of singularities (see e.g. [36, Cor 2.11]). Using Theorem 34 and the subsequent remark, we first compute the Cox ring of X. For the $\mathcal{M}_{-1}$ slice, we have the vertical divisors $D_{-1,2/5}$, $D_{-1,1/2}$, and $D_{-1,1}$. We introduce the variables $T_{-1,2/5}$, $T_{-1,1/2}$, and $T_{-1,1}$. We have $\mu\left(\frac{2}{5}\right) = 5$, $\mu\left(\frac{1}{2}\right) = 2$, and $\mu(1) = 1$. We obtain the monomial

$$T^{\mu(p=-1)} = T^5_{-1,2/5} T^2_{-1,1/2} T_{-1,1}.$$

Repeating the procedure for the slices $\mathcal{M}_1$ and $\mathcal{M}_\infty$, we obtain the monomials $T^{\mu(p=1)} = T^2_{1,-1/2} T_{1,0}$ and $T_{\mu(p=\infty)} = T^{\infty,0} T^3_{\infty,1/3} T^2_{\infty,1/2} T_{\infty,1}$. Relabelling the set of points with nontrivial slices as $1 \mapsto 0$, $-1 \mapsto \infty$, $\infty \mapsto 1$, Theorem 34 tells us that the

Cox ring is given by

$$\mathrm{Cox}(X) = \frac{\mathbb{C}[T_{p,\nu}, S_+ :\ p \in \{0,1,\infty\}]}{\left\langle T_{0,-1/2}^2 T_{0,0} + T_{\infty,2/5}^5 T_{\infty,1/2}^2 T_{\infty,1} + T_{1,1/3}^3 T_{1,1/2}^2 T_{1,0} T_{1,1} \right\rangle}.$$

The divisors being contracted by the resolution $X \to Y$ are $T_{\infty,1/2}$, $T_{\infty,1}$, $T_{0,0}$, $T_{1,1/2}$, and $T_{1,0}$. This gives

$$\mathrm{Cox}(Y) = \frac{\mathbb{C}\left[T_{p,\nu}, S_+ :\ p \in \{0,1,\infty\}\right]}{\left\langle T_{\infty,2/5}^5 + T_{0,-1/2}^2 + T_{1,1/3}^3 T_{1,1} \right\rangle}.$$

After calculating the degrees of each $T_{p,\nu}$ and relabelling, the claim follows. □

The next section will prove that there exists an ample H such that T_Y is H-stable.

Proposition 94. *The class group is given by* $\mathrm{Cl}(Y) \simeq \mathbb{Z}[D]$ *with* $[D] = [x = 0]$*, and the canonical divisor is* $K_Y = -5\ [D]$.

Proof. Standard calculations. See e.g. [4]. □

Stability of vector bundles on orbifolds

Let X be a $\mathbb{Q}$-factorial normal variety and let H be an ample class. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, which is torsion-free in codimension one, the determinant

$$\det(\mathcal{F}) = \left(\bigwedge^{\mathrm{rank}(\mathcal{F})} \mathcal{F} \right)^{\vee\vee}$$

is a $\mathbb{Q}$-Cartier divisor. We define the slope of $\mathcal{F}$ with respect to H as

$$\mu_H(\mathcal{F}) := \frac{H \cdot [\det(\mathcal{F})]}{\mathrm{rank}(\mathcal{F})},$$

where $[\det(\mathcal{F})] \in \mathrm{N}^1(X)_{\mathbb{Q}}$ is the equivalence class of the corresponding $\mathbb{Q}$-Cartier divisor and the product is defined by the non-degenerate bilinear pairing

$$\mathrm{N}^1(X)_{\mathbb{Q}} \times \mathrm{N}_1(X)_{\mathbb{Q}} \to \mathbb{Q}.$$

In these conditions, set the maximal slope to

$$\mu_H^{\max}(\mathcal{F}) := \sup\{\mu_H(\mathcal{G}) : \ \mathcal{G} \subset \mathcal{F} \text{ is a coherent subsheaf}\}.$$

Note that the maximal slope is well-defined as subsheaves of $\mathcal{F}$ are also torsion-free in codimension one. This allows us to make the following definition.

Definition 95. We say that $\mathcal{F}$ is semistable with respect to H if $\mu_H^{\max}(\mathcal{F}) = \mu_H(\mathcal{F})$. Furthermore, stability and instability are defined analogously to the smooth case.

Remark 96. Consider the natural quotient map

$$t : \mathcal{F} \to \mathcal{F}^t \simeq \mathcal{F}/\mathrm{tors}.$$

Let $\mathcal{G} \subset \mathcal{F}$ be a coherent subsheaf. The sheaves $\mathcal{G}$ and $\mathcal{G}^t := t(\mathcal{G})$ differ only on a codimension two subset. From the properties of reflexive hulls, it follows that $\mu_H(\mathcal{G}) = \mu_H(\mathcal{G}^t)$. An identical argument can be made in the opposite direction. Consider a coherent subsheaf of the torsion-free part $\mathcal{G}^t \subset \mathcal{F}^t$. Set $\mathcal{G} := t^{-1}(\mathcal{G}^t)$. As $\mathcal{G}$ and $\mathcal{G}^t$ again only differ along a codimension two subset, the equality of slopes follows $\mu_H(\mathcal{G}) = \mu_H(\mathcal{G}^t)$.

Proposition 97. *There exists a subsheaf $\mathcal{G} \subset \mathcal{F}$ such that $\mu_H(\mathcal{G}) = \mu_H^{\max}(\mathcal{F})$ if and only if there exists a subsheaf $\mathcal{G}^t \subset \mathcal{F}^t$ such that $\mu_H(\mathcal{G}^t) = \mu_H^{\max}(\mathcal{F}^t)$. Moreover $\mu_H^{\max}(\mathcal{F}) = \mu_H^{\max}(\mathcal{F}^t)$ is a finite rational number.*

Let $\pi : \tilde{X} \to X$ be a resolution of singularities. The pushforward map respects $\mathbb{Q}$-linear equivalence and hence extends to a surjective linear map

$$\pi_* : \ N^1(\tilde{X})_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}}.$$

The dual map π^* of π_* obtained via the intersection pairing is then an injective linear map

$$\pi^* : \ N_1(X)_{\mathbb{Q}} \to N_1(\tilde{X})_{\mathbb{Q}}.$$

For any $L \in N_1(X)_{\mathbb{Q}}$ and $M \in N^1(\tilde{X})_{\mathbb{Q}}$, the pair of linear maps π_* and π^* satisfies the projection formula

$$\pi^*(L) \cdot M = L \cdot \pi_*(M).$$

Proposition 98. *Let* $\pi : \tilde{X} \to X$ *be a resolution of singularities and let* $\tilde{\mathcal{F}}$ *be a coherent sheaf on* $\tilde{X}$*, which is torsion-free in codimension one. Then the following equality of slopes holds*

$$\mu_{\pi^* H}(\tilde{\mathcal{F}}) = \mu_H(\pi_* \tilde{\mathcal{F}}).$$

Proof. Begin by remarking that the pushforward of a torsion-free sheaf is again torsion-free. The image of a sheaf which is torsion-free in codimension one will again be torsion-free in codimension one. This follows from the standard observation that the image of a codimension two set must be of codimension two.

Regarding the main statement, the sheaves $\det(\pi_* \tilde{\mathcal{F}})$ and $\pi_* \det(\tilde{\mathcal{F}})$ are both rank one torsion-free sheaves by construction. Since they also agree away from the singular locus, we have the linear equivalences

$$\pi_*[\det(\tilde{\mathcal{F}})] = [\pi_* \det(\tilde{\mathcal{F}})] = [\det(\pi_* \tilde{\mathcal{F}})] \in N^1(X)_{\mathbb{Q}}.$$

The desired equality of slopes is now immediate by the projection formula. □

Proposition 99. *Let* $\pi : \tilde{X} \to X$ *be a resolution of singularities and let* $\mathcal{F}$ *be a coherent sheaf on* X *which is torison-free in codimension one. Denote by* $\pi^{[*]}(\mathcal{F}) = (\pi^* \mathcal{F})^{\vee\vee}$ *the reflexive pullback. Then the following equality of slopes holds*

$$\mu_{\pi^* H}\left(\pi^{[*]}(\mathcal{F})\right) = \mu_H(\mathcal{F}).$$

Proof. Proposition 98 tells us that $\mu_{\pi^* H}\left(\pi^{[*]}\mathcal{F}\right) = \mu_H\left(\pi_*\left(\pi^{[*]}\mathcal{F}\right)\right)$. Observe that $\mathcal{F}$ and $\pi_*\left(\pi^{[*]}\mathcal{F}\right)$ agree on X except potentially away from a codimension one subset. This implies that their reflexive determinants are isomorphic and so the claim follows. □

Next comes a generalisation of the Kodaira-Nakano-Akizuki (小平–中野–秋月) vanishing theorem to surfaces with quotient singularities.

Proposition 100 ([5, Thm 4])**.** *Let* X *be an orbifold surface and let* $\mathcal{L}$ *be an ample line bundle. Then for all* $i \geqslant 2$ *we have the vanishing*

$$H^i\left(X, \tilde{\Omega}_X \otimes \mathcal{L}\right) = 0$$

Proposition 101. *Let $\mathbb{P} = \mathbb{P}(a_0, a_1, \dots, a_n)$ be an n-dimensional weighted projective space and let $Y \subset \mathbb{P}$ be a hypersurface of degree d. Then the vanishing*

$$H^{n-1}(Y, \tilde{\Omega}^r_Y(\ell)) = 0$$

holds provided that the following conditions are all satisfied:

1. $\ell + d > 0$,
2. $r < \frac{n(\ell+\tau)}{\tau} - 1$, *and*
3. *the tangent bundle $T_{\mathbb{P}}$ is semistable.*

Here $\tau = [-K_{\mathbb{P}}] \in \mathrm{Cl}(\mathbb{P}) \simeq \mathbb{Z}$.

Proof. Fix a generator $\mathcal{O}(1)$ for the class group $\mathrm{Cl}(\mathbb{P}) \simeq \mathbb{Z}$. Then a hypersurface Y of degree d is an element of the linear system $|\mathcal{O}(d)|$. We have the following short exact sequence on Y

$$0 \to \mathcal{O}_Y(-d) \to \tilde{\Omega}^1_{\mathbb{P}}|_Y \to \tilde{\Omega}^1_Y \to 0,$$

where $\mathcal{O}_Y(-d) := (\mathcal{O}(d)^\vee)|_Y$ is the normal bundle to Y by the Poincaré adjunction formula. Tensoring the sequence by $\mathcal{O}_Y(\ell + d)$ yields

$$0 \to \tilde{\Omega}^r_Y(\ell) \to \tilde{\Omega}^{r+1}_{\mathbb{P}}(\ell + d)|_Y \to \tilde{\Omega}^{r+1}_Y(\ell + d) \to 0.$$

There is an induced long exact sequence of cohomologies. From the segment

$$\to H^{n-2}(Y, \tilde{\Omega}^{r+1}_Y(\ell + d)) \to H^{n-1}(Y, \tilde{\Omega}^r_Y(\ell)) \to H^{n-1}(Y, \tilde{\Omega}^{r+1}_{\mathbb{P}}(\ell + d)|_Y) \to$$

we see that in order to prove the proposition it suffices to show that under our three conditions, the following hold:

1. $H^{n-2}(Y, \tilde{\Omega}^{r+1}_Y(\ell + d)) = 0$, and
2. $H^{n-1}(Y, \tilde{\Omega}^{r+1}_{\mathbb{P}}(\ell + d)|_Y) = 0$.

The generalised Kodaira-Nakano-Akizuki vanishing from Proposition 100 says that $H^{n-2}(Y, \tilde{\Omega}_Y^{r+1}(\ell+d)) = 0$ provided that $\mathcal{O}_Y(\ell+d)$ is ample, i.e. $\ell+d > 0$, and $n-2+r+1 > \dim(Y) = n-1$.

Given that Y is a hypersurface, we also have the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_P|_Y = \mathcal{O}_Y \to 0.$$

Tensoring this with $\tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell+d)$ and going to the long exact sequence of cohomology, we analyse the segment

$$\to H^{n-1}(\mathbb{P}, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell+d)) \to H^{n-1}(Y, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell+d)|_Y) \to H^n(\mathbb{P}, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell)) \to .$$

By Proposition 100, the leftmost cohomology vanishes $H^{n-1}(\mathbb{P}, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell+d)) = 0$ precisely when $\mathcal{O}_{\mathbb{P}}(\ell+d)$ is ample, i.e. $\ell+d > 0$, and $n-1+r+1 > \dim(\mathbb{P}) = n$, which always holds.

Next we analyse the rightmost cohomology $H^n(\mathbb{P}, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell))$. By Serre duality, this is isomorphic to

$$H^0(\mathbb{P}, \wedge^{r+1} T_{\mathbb{P}}(-\ell-\tau))^{\vee}.$$

We now use the assumption that $T_{\mathbb{P}}$ is semistable. Then the second reflexive exterior power $\wedge^{r+1} T_{\mathbb{P}}$ is also semistable. Moreover, we have that the reflexive tensor product $\wedge^{r+1} T_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-\ell-\tau) = \wedge^{r+1} T_{\mathbb{P}}(-\ell-\tau)$ is semistable.

Remark that a semistable vector bundle of negative degree does not admit nonzero sections. Indeed, suppose that there exists a nonzero section of a semistable vector bundle V over some manifold X. This is equivalent to an injective homomorphism $\mathcal{O}_X \to V$. If the degree of V is negative, then the image of this homomorphism has degree zero, but this contradicts semistability.

We can compute the slope of the reflexive tensor product above to be

$$\mu_{\mathcal{O}(1)}\left(\wedge^{r+1} T_{\mathbb{P}}(-\ell-\tau)\right) = \frac{(r+1)\tau}{n} - \ell - \tau,$$

where we use the standard formulae: $\mu_{\mathcal{O}(1)}(\wedge^k V) = k \cdot \mu_{\mathcal{O}(1)}(V)$ and $\mu_{\mathcal{O}(1)}(V \otimes W) = \mu_{\mathcal{O}(1)}(V) + \mu_{\mathcal{O}(1)}(W)$. Observe that the slope is negative, i.e.

$$\frac{(r+1)\tau}{n} - \ell - \tau < 0$$

if and only if

$$r < \frac{n(\ell+\tau)}{\tau} - 1.$$

But this is one of the conditions in our proposition. We thus conclude that

$$H^n(\mathbb{P}, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell)) = 0.$$

In view of the long exact sequences we investigated and the new vanishings above, we also conclude that

$$H^{n-1}(Y, \tilde{\Omega}_{\mathbb{P}}^{r+1}(\ell + d)|_Y) = 0$$

if the three conditions from the proposition hold. Moreover, if the conditions hold, we finally have that

$$H^{n-1}(Y, \tilde{\Omega}_Y^r(\ell)) = 0.$$

This concludes the proof. □

Proposition 102. *Let $\mathbb{P}$ be a weighted projective space of dimension $m+1$ such that $T_{\mathbb{P}}$ is semistable. Let $Y \subset X$ be an irreducible orbifold hypersurface of $\mathbb{P}$ of degree d such that $-K_Y$ is ample and $\mathrm{Cl}(Y) \simeq \mathrm{Cl}(\mathbb{P}) \simeq \mathbb{Z}$. Then the tangent bundle T_Y is stable provided that*

$$d > \frac{\deg(T_{\mathbb{P}})(\dim(Y)-1)}{2\dim(Y)-1}.$$

Proof. Consider a hypersurface Y which satisfies the conditions in the statement. Suppose that T_Y is unstable. Then there exists a coherent subsheaf

$$F \subset T_Y$$

which contradicts the stability condition. Denoting $\delta := \deg(F)$, $r := \mathrm{rank}(F)$, and $m := \dim(Y)$, the instability condition $\mu_{\mathcal{O}(1)}(F) \geqslant \mu_{\mathcal{O}(1)}(T_Y)$ takes the form

$$\frac{\delta}{r} \geqslant \frac{\tau - d}{m}.$$

Since we have the isomorphism $\mathrm{Cl}(Y) \simeq \mathrm{Cl}(\mathbb{P}) \simeq \mathbb{Z}$ induced by the inclusion morphism $Y \hookrightarrow \mathbb{P}$ and the degree of F is equal to δ, it follows that

$$\bigwedge^r F \simeq \mathcal{O}_Y(\delta).$$

In view of this and the instability of T_Y, the coherent subsheaf $F \subset T_Y$ induces a nonzero section

$$s \in H^0(Y, (\wedge^r F)^\vee \otimes \wedge^r T_Y) = H^0(Y, (\wedge^r T_Y)(-\delta)).$$

The aim is to show that such a nonzero section does not exist.

The canonical divisor of the hypersurface Y is $K_Y = \mathcal{O}_Y(d - \tau)$. Serre duality then gives

$$H^0(Y, (\wedge^r T_Y)(-\delta)) = H^n(Y, \tilde{\Omega}^r_Z(\delta - \tau + d))^\vee.$$

By Proposition 101, it suffices to show that:

$$\delta - \tau + d + d > 0, \tag{4.1}$$

$$r < \frac{n(\tau + \delta - \tau + d)}{\tau} - 1 = \frac{n(\delta + d)}{\tau} - 1. \tag{4.2}$$

Using the instability condition, we can see that in order to show 4.1, it suffices to prove that

$$\frac{r(\tau - d)}{m} > \tau - 2d. \tag{4.3}$$

As $r \geqslant 1$ and $\tau > d$ (as $-K_Y$ is assumed to be ample), we already know that

$$\frac{r(\tau - d)}{m} \geqslant \frac{\tau - d}{m}.$$

Hence 4.1 holds when

$$\frac{\tau - d}{m} > \tau - 2d.$$

Rearranging, this is equivalent to

$$d > \frac{\tau(m-1)}{2m-1}.$$

But this is precisely the inequality from the statement. Hence 4.1 is proved.

Now we focus on inequality 4.2. Rearranging, it is sufficient to prove the equivalent inequality

$$\frac{\tau(r+1)}{m+1} - d < \delta.$$

Using the instability condition, it is sufficient to prove that

$$\frac{\tau(r+1)}{m+1} - d < \frac{r(\tau - d)}{m-1}.$$

Rearranging, we obtain the equivalent inequality

$$(r+1)(dm + d - \tau) < (m+1)(dm + d - \tau).$$

We know that $r < m$. Then inequality 4.2 holds if

$$dm + d - \tau > 0.$$

A theorem of Kobayashi gives the inequality $\tau \leqslant m + 2$. Observing that we only consider the case when $m \geqslant 2$ concludes the proof. □

We now turn our attention again to the surface $Y = X(\mathcal{N})$.

Theorem 103. *Let* $X = X(\mathcal{M})$ *be the prescribed blowup of* $\mathbb{F}_2$ *defined in the beginning of the section. Then* $\mathrm{Stab}(T_X) \neq \emptyset$.

Proof. Proposition 91 already shows that $\mathrm{sStab}(T_X) \neq \emptyset$. In order to prove the stronger statement, we regard $X = X(\mathcal{M})$ as a resolution of $Y = X(\mathcal{N})$. By the openness property of stability, given the birational morphism $X \to Y$, it suffices to show that $\mathrm{Stab}(T_Y) \neq \emptyset$.

To this end, we apply Proposition 102 to the inclusion

$$Y = \{x^3 w + z^2 + y^5 = 0\} \hookrightarrow X = \mathbb{P}(1, 2, 5, 7).$$

Both Y and $\mathbb{P}(1, 2, 5, 7)$ are Fano, and we verified directly that $\mathrm{Cl}(Y) \simeq \mathrm{Cl}(\mathbb{P}(1, 2, 5, 7)) \simeq \mathbb{Z}$. The hypersurface Y has degree $\deg(Y) = 10$. Its dimension is $\dim(Y) = 2$, and we can verify the inequality in Proposition 102 directly:

$$d = 10 > \frac{15(2-1)}{2 \times 2 - 1} = \frac{15}{3} = 5.$$

The one impediment to applying the proposition is that $\mathbb{P}(1,2,5,7)$ does not have semistable tangent sheaf. Looking at the relation between Proposition 102 and Proposition 101, we need to show in lieu of semistability of $T_{\mathbb{P}(1,2,5,7)}$ that a certain twist of the second exterior power of the tangent budle

$$\wedge^2 T_{\mathbb{P}(1,2,5,7)}(-\ell - \tau)$$

does not have global sections when $r < \frac{n(\ell+\tau)}{\tau} - 1$. In the proof of Proposition 101, we can see that in our example we have $r = 1$ and $\tau = \deg(T_{\mathbb{P}(1,2,5,7)}) = 15$. Hence the condition we are looking for is $\ell > -5$.

The weighted projective space $X = \mathbb{P}(1,2,5,7)$ is determined by the fan Σ consisting of the rays

$$\rho_1 = (1,0,0)^T, \rho_2 = (0,1,0)^T, \rho_3 = (0,0,1)^T, \rho_4 = (-2,-5,-7)^T.$$

Moreover, we assume that the class group $\mathrm{Cl}(\mathbb{P}(1,2,5,7)) \simeq \mathbb{Z}$ is generated by the prime divisor corresponding to the first ray D_{ρ_1}.

Now we calculate the Klyachko description of $\mathcal{V} = \wedge^2 T_{\mathbb{P}(1,2,5,7)}(-\ell - \tau)$. Start with the simple tangent bundle $T_{\mathbb{P}(1,2,5,7)}$. By Proposition 39, this is given by the filtered vector space $(\mathbb{C}^3, \{T^\rho(j)\})$ with

$$T^\rho(j) = \begin{cases} \mathbb{C}^3 & \text{if} \quad j \leqslant 0 \\ \mathrm{span}(\rho) & \text{if} \quad j = 1 \\ 0 & \text{if} \quad j > 1. \end{cases}$$

Applying Proposition 40, the second exterior power of the tangent bundle $\wedge^2 T_{\mathbb{P}(1,2,5,7)}$

is given by the filtered vector space $(\mathbb{C}^3, \{E^\rho(j)\})$ with

$$E^{\rho_1}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant 0 \\ \langle \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3 \rangle & \text{if } j = 1 \\ 0 & \text{if } j > 1, \end{cases} \qquad E^{\rho_2}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant 0 \\ \langle \mathbf{e}_2 \wedge \mathbf{e}_1, \mathbf{e}_2 \wedge \mathbf{e}_3 \rangle & \text{if } j = 1 \\ 0 & \text{if } j > 1, \end{cases}$$

$$E^{\rho_3}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant 0 \\ \langle \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_3 \wedge \mathbf{e}_2 \rangle & \text{if } j = 1 \\ 0 & \text{if } j > 1, \end{cases} \qquad E^{\rho_4} = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant 0 \\ \langle 5\mathbf{e}_1 \wedge \mathbf{e}_2 + 7\mathbf{e}_1 \wedge \mathbf{e}_3, & \\ 2\mathbf{e}_1 \wedge \mathbf{e}_2 - 7\mathbf{e}_2 \wedge \mathbf{e}_3 \rangle & \text{if } j = 1 \\ 0 & \text{if } j > 1. \end{cases}$$

By Proposition 39, the line bundle $\mathcal{O}_{\mathbb{P}(1,2,5,7)}(1)$ has the associated filtration $(\mathbb{C}, \{L^\rho(j)\})$ with

$$L^{\rho_1}(j) = \begin{cases} \mathbb{C} & \text{if } j \leqslant 0 \\ 0 & \text{if } j > 0, \end{cases} \qquad L^{\rho_{2,3,4}}(j) = \begin{cases} \mathbb{C} & \text{if } j \leqslant 1 \\ 0 & \text{if } j > 1. \end{cases}$$

Applying the rule for tensor products from Proposition 40, we get that $\mathcal{O}_{\mathbb{P}(1,2,5,7)}(m)$ corresponds to the filtered vector space $(\mathbb{C}, \{M^\rho(j)\})$ with

$$M^{\rho_1}(j) = \begin{cases} \mathbb{C} & \text{if } j \leqslant 0 \\ 0 & \text{if } j > 0, \end{cases} \qquad M^{\rho_{2,3,4}}(j) = \begin{cases} \mathbb{C} & \text{if } j \leqslant m \\ 0 & \text{if } j > m. \end{cases}$$

Finally, $\mathcal{V} = \wedge^2 T_{\mathbb{P}(1,2,5,7)}(-\ell - \tau)$ corresponds to the filtration associated to the tensor product $\wedge^2 T_{\mathbb{P}(1,2,5,7)} \otimes \mathcal{O}_{\mathbb{P}(1,2,5,7)}(-\ell - \tau)$. Set $m = \ell + \tau$. By Proposition 40, this corresponds to tensor product of filtered vector spaces $(\mathbb{C}^3, \{E^\rho(j)\}) \otimes (\mathbb{C}, \{M^\rho(j)\})$. We denote the result of the calculation by $(\mathbb{C}^3, \{V^\rho(j)\})$, where $V^\rho(j)$ is given by

$$V^{\rho_1}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant 0 \\ \langle \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3 \rangle & \text{if } j = 1 \\ 0 & \text{if } j > 1, \end{cases} \qquad V^{\rho_2}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant -m \\ \langle \mathbf{e}_2 \wedge \mathbf{e}_1, \mathbf{e}_2 \wedge \mathbf{e}_3 \rangle & \text{if } j = -m+1 \\ 0 & \text{if } j > -m+1, \end{cases}$$

$$V^{\rho_3}(j) = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant -m \\ \langle \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_3 \wedge \mathbf{e}_2 \rangle & \text{if } j = -m+1 \\ 0 & \text{if } j > -m+1, \end{cases} \qquad V^{\rho_4} = \begin{cases} \mathbb{C}^3 & \text{if } j \leqslant -m \\ \langle 5\mathbf{e}_1 \wedge \mathbf{e}_2 + 7\mathbf{e}_1 \wedge \mathbf{e}_3, & \\ 2\mathbf{e}_1 \wedge \mathbf{e}_2 - 7\mathbf{e}_2 \wedge \mathbf{e}_3 \rangle & \text{if } j = -m-1 \\ 0 & \text{if } j > -m+1. \end{cases}$$

Finally, to compute global sections, we construct the parliament of polytopes associated to $\mathcal{V} = \wedge^2 T_{\mathbb{P}(1,2,5,7)}(-\ell - \tau)$. Applying Algorithm 3.2 from [15] to the filtered vector space $(\mathbb{C}^3, \{V^\rho(j)\})$, we can associate a matroid $M(\mathcal{V})$ to the coherent sheaf $\mathcal{V}$ whose ground set is given by

$$M(\mathcal{V}) = \{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, 5\mathbf{e}_1 \wedge \mathbf{e}_2 + 7\mathbf{e}_1 \wedge \mathbf{e}_3, 2\mathbf{e}_1 \wedge \mathbf{e}_2 - 7\mathbf{e}_2 \wedge \mathbf{e}_3\}.$$

The associated parliament of polytopes $\mathcal{P}(\mathcal{V})$ consists of a polytope $P_\mathbf{e}$ for every $\mathbf{e} \in M(\mathcal{V})$, where

$$P_\mathbf{e} := \{\mathbf{u} \in M \otimes_\mathbb{Z} \mathbb{R} : \ \langle \mathbf{u}, \rho \rangle \leqslant \max\{j \in \mathbb{Z} : \ \mathbf{e} \in V^\rho(j)\} \text{ for all } \rho\}.$$

We claim that $P_\mathbf{e} = \emptyset$ for all $\mathbf{e} \in M(\mathcal{V})$ and $\ell > -5$, hence $\mathcal{V}$ has no global sections when $\ell > -5$. Set $\mathbf{u} = (u_1, u_2, u_3)$. Take $\mathbf{e} = \mathbf{e}_1 \wedge \mathbf{e}_2$. The corresponding polytope $P_{\mathbf{e}_1 \wedge \mathbf{e}_2} \in \mathcal{P}(\mathcal{V})$ is cut out by the inequalities

$$\begin{aligned} P_{\mathbf{e}_1 \wedge \mathbf{e}_2} = \{&u_1 \leqslant 1, \ u_2 \leqslant -\ell - \tau + 1, \ u_3 \leqslant -\ell - \tau, \\ &-2u_1 - 5u_2 - 7u_3 \leqslant -\ell - \tau + 1\}. \end{aligned}$$

Assume $P_{\mathbf{e}_1 \wedge \mathbf{e}_2} \neq \emptyset$. Multiply the second relation by 5 and the third relation by 7 and add both to the fourth relation. This gives:

$$-2u_1 \leqslant -13(\ell + \tau) + 4 \Leftrightarrow u_1 \geqslant \frac{13}{2}(\ell + \tau) - 4.$$

Since $\tau = 15$ and $\ell \geqslant -4$ by assumption, we get that

$$u_1 \geqslant \frac{13}{2}(\ell + \tau) - 4 = \frac{1}{2}(13(\ell + 15) - 8) \geqslant \frac{1}{2}(13 \times 11 - 8) = \frac{151}{2}.$$

But the first relation says that $u_1 \leqslant 1$. Contradiction. Hence $P_{e_1 \wedge e_2} = \emptyset$.

Similarly, the remaining polytopes are defined by

$$\begin{aligned}
P_{e_1 \wedge e_3} &= \{u_1 \leqslant 1,\ u_2 \leqslant -\ell - \tau,\ u_3 \leqslant -\ell - \tau + 1, -2u_1 - 5u_2 - 7u_3 \leqslant -\ell - \tau\},\\
P_{e_2 \wedge e_3} &= \{u_1 \leqslant 0,\ u_2 \leqslant -\ell - \tau + 1,\ u_3 \leqslant -\ell - \tau + 1, -2u_1 - 5u_2 - 7u_3 \leqslant -\ell - \tau\},\\
P_{5e_1 \wedge e_2 + 7e_1 \wedge e_3} &= \{u_1 \leqslant 1,\ u_2 \leqslant -\ell - \tau,\ u_3 \leqslant -\ell - \tau, -2u_1 - 5u_2 - 7u_3 \leqslant -\ell - \tau + 1\},\\
P_{2e_1 \wedge e_2 - 7e_2 \wedge e_3} &= \{u_1 \leqslant 0,\ u_2 \leqslant -\ell - \tau + 1,\ u_3 \leqslant -\ell - \tau, -2u_1 - 5u_2 - 7u_3 \leqslant -\ell - \tau + 1\}.
\end{aligned}$$

Following the same strategy as in the first example, we can deduce from the latter three relations that $u_1 \geqslant 68$, $u_1 \geqslant \frac{131}{2}$, $u_1 \geqslant 71$, or $u_1 \geqslant \frac{137}{2}$, respectively.

Hence $\mathcal{P}(\mathcal{V}) = \{\emptyset\}$ for $j > -5$, as required.

□

Chapter 5

The case of Weierstrass fibrations

Let C be a smooth projective curve of genus g.

Definition 104. We say $\pi : X \to C$ is a *Weierstrass fibration* if π is a proper, flat morphism such that every fibre is a Gorenstein curve of genus one, i.e. each fibre is either an elliptic curve, a rational curve with an ordinary double point, or a rational curve with a cusp. Furthermore, the generic fibre is smooth and only finitely many singular fibres are allowed.

We insist that the fibration have a section $s_0 : C \hookrightarrow X$ whose image $s_0 = s_0(C)$ does not intersect any of the singular points of the singular fibres. Under these assumptions, the Néron-Severi lattice NS(X) is generated by the class of a fibre f together with a finite set of sections $s_0, s_1, \ldots, s_m$ [38, VII.2.1].

The sections are in one-to-one correspondence with the $\mathbb{C}(C)$-rational points of the generic fibre, hence they form a group. Using this group law, we can express any divisor $M \in$ NS(X) as

$$M = s + rf,$$

where $s = s_i$ for some $0 \leqslant i \leqslant m$ and $r \in \mathbb{Z}$, and the intersection form is given by $f^2 = 0$, $s_i \cdot f = 1$ for all i, and $s_i^2 = -\chi$ for all i, where χ is the degree of the fundamental bundle $L = (R^1\pi_*\mathcal{O}_X)^\vee$, which is a line bundle over the base curve [38, II.3.6].

We can represent X as a double cover $p : X \to R$ of the ruled surface

$$R = \mathbb{P}\left(\mathcal{O}_C \oplus L^{-2}\right)$$

with structure morphism $q : R \to C$ such that $q \circ p = f$ [38, III.2]. The double cover p is branched along a divisor of class

$$B = q^* L^6 \otimes \mathcal{O}_R(4) = 4s + 6\chi f,$$

which we express in the basis of Pic(R) as described in Chapter 2. Since R is the projectivisation of a decomposable rank two vector bundle, R has invariant

$$e = -\deg(L^{-2}) = 2\chi \geqslant 0.$$

Remark 105. Since X is assumed to be smooth, then the branch divisor B is smooth and p is a Galois cover. There is a Galois group $G = \{1, \sigma\}$ acting on X and $G \simeq \mathbb{Z}/2$. We wish to check the stability of the cotangent bundle Ω^1_X. By uniqueness of the Harder-Narasimhan filtration, it suffices to check subsheaves $\mathcal{F} \subset \Omega^1_X$ which are stable under the action of G, i.e. $\sigma \bullet \mathcal{F} \simeq \mathcal{F}$.

Fix an ample $H \in$ Pic(R). Hereafter, unless otherwise specified, we compute the slope of a coherent sheaf on R with respect to H, and the slope of a coherent sheaf on X with respect to p^*H, which is also ample.

Lemma 106. *Let $p : X \to R$ be an abelian cover, and let H be an ample divisor on R. Then p^*H is an ample divisor on X.*

Proof. Suppose otherwise. Then p^*H is nef and there exists an effective curve α in X such that $p^*H \cdot \alpha = 0$. By the projection formula, this is the same as saying that $H \cdot p_*\alpha = 0$. But, as H is assumed to be ample, this can only be true if p contracts α. The morphism p is finite, and finite morphisms cannot contract curves. Contradiction. □

Lemma 107. *The slope of the cotangent sheaf* Ω^1_X *is given by*

$$\mu_{p^*H}(\Omega^1_X) = \left(K_R + \frac{1}{2}B\right) \cdot H.$$

Proof. The slope of the cotangent sheaf is $\mu(\Omega^1_X) = \frac{1}{2}K_X \cdot p^*H$. Substitute the formula for the canonical divisor $K_X = p^*\left(K_R + \frac{1}{2}B\right)$ (see e.g. [43]) and use the fact that for all divisors $D \in \mathrm{Pic}(R)$ we have $p^*D \cdot p^*H = 2\, D \cdot H$, since p is a finite morphism of degree two. □

Theorem 108. *Let* $\pi : X \to C$ *be a smooth Weierstrass fibration with invariant* χ *and base curve of genus* g. *Set an ample divisor* $H = 4s + 9\chi f$. *Then the tangent bundle* T_X *is*

1. *unstable for any ample polarisation if* $\chi < 2g - 2$;
2. H*-semistable if* $\chi = 2g - 2$;
3. H*-stable if* $\chi > 2g - 2$.

Proof. Let $\chi \geqslant 2g - 2$. Consider X as a double cover $p : X \to R$ branched along the smooth divisor $B = 4s + 6\chi f$, and fix an ample divisor

$$H = B + 3\chi f = 4s + 9\chi f \in \mathrm{Pic}(R).$$

Assume towards a contradiction that Ω^1_X is unstable with respect to p^*H. Then there exists a unique maximal destabilising rank one subsheaf $\mathcal{F} \subset \Omega^1_X$, which has torsion-free quotient $\Omega^1_X/\mathcal{F}$ and is stable under the action of $\mathbb{Z}/2$.

Assume that $\mathcal{F} \not\subset p^*\Omega^1_R$. We adapt some of the ideas of [53, Theorem 3.4] to derive a contradiction. To this end, consider the short exact sequence

$$0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{N} = p^*\Omega_R \cap \mathcal{F}$. This is itself a rank one sheaf with torsion-free quotient, stable under the action of $\mathbb{Z}/2$. Therefore, by [53, Sec 3.2], there exists a line bundle $N \subset \Omega_R$

such that $\mathcal{N} = p^*N$. Furthermore, under our assumptions, $\mathcal{Q}$ is a rank one $\mathcal{O}_{B_1}$-module, where B_1 is the reduced divisor associated to $p^{-1}(B)$.

We can compute the slope of $\mathcal{F}$ as follows:

$$\begin{aligned}
\mu_{p^*H}(\mathcal{F}) &= c_1(\mathcal{F}) \cdot p^*H && \text{(definition of slope)} \\
&= c_1(\mathcal{N}) \cdot p^*H + c_1(\mathcal{Q}) \cdot p^*H && \text{(additivity of Chern classes)} \\
&= c_1(p^*N) \cdot p^*H + B_1 \cdot p^*H && \text{(assumptions on } \mathcal{N} \text{ and } \mathcal{Q}) \\
&= 2\, c_1(N) \cdot H + 2\, \frac{1}{2}\, B \cdot H && (p \text{ is finite of degree 2)} \\
&= 2\, c_1(N) \cdot (B + 3\chi f) + B \cdot H && (H = B + 3\chi f) \\
&= 2\, c_1(N) \cdot B + 3\chi\ \ c_1(N) \cdot f + B \cdot H \\
&\leqslant -2B^2 + 3\chi \cdot 0 + B \cdot H \\
&= -8\chi.
\end{aligned}$$

In the last step, we use the conclusions of [53, Sec 3.3] to see that $c_1(N) \cdot B \leqslant -B^2$. To see that $c_1(N) \cdot f = \deg(N|_f) \leqslant 0$, consider the Euler sequence restricted to a fibre

$$0 \to q^*\Omega_C|_f \to \Omega_R|_f \to \omega_{R/C}|_f \to 0.$$

Combining this with the fact that $f \simeq \mathbb{P}^1$, we see that $\Omega_R|_f \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Then we have $\mathrm{Hom}(N|_f, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathrm{Hom}(N|_f, \mathcal{O}_{\mathbb{P}^1}) \oplus \mathrm{Hom}(N|_f, \mathcal{O}_{\mathbb{P}^1}(-2))$. For a non-trivial homomorphism to exist, we require that $\deg(N|_f) \leqslant 0$.

Finally for this case, using Lemma 107, we compute the difference of slopes.

$$\begin{aligned}
\mu_{p*H}(\Omega_X) - \mu_{p^*H}(\mathcal{F}) &\geqslant \left(K_R + \frac{1}{2}B\right) \cdot H - (-8\chi) \\
&= (2g - 2 + \chi)f \cdot (4s + 9\chi f) + 8\chi \\
&= 4(2g - 2) + 12\chi > 0.
\end{aligned}$$

This shows that $\mathcal{F}$ is not a destabilising subsheaf, yielding a contradiction.

Then it must be the case that $\mathcal{F} \subset p^*\Omega^1_R$. By [53, Sec 3.2], there exists a line bundle $F \subset \Omega_R$ such that $\mathcal{F} = p^*F$. Taking the dual picture in Proposition 10, it suffices to check the case when $\mathcal{F} = p^*(q^*\Omega^1_C)$, i.e. the pullback of the maximal destabilising subsheaf of Ω^1_R.

Check the difference of slopes:

$$\begin{aligned} \mu_{p^*H}(\Omega^1_X) - \mu_{p^*H}(\mathcal{F}) &= \left(K_R + \frac{1}{2}B\right) \cdot H - 2q^*\Omega^1_C \cdot H \\ &= ((2g-2+\chi)f - 2(2g-2)f) \cdot (Cs + Df) \\ &= (\chi - (2g-2))\, C. \end{aligned}$$

Note that if $\chi > 2g-2$, then $\mathcal{F}$ cannot be a destabilising subsheaf, and thus we obtain that Ω^1_X is stable with respect to p^*H. If $\chi = 2g-2$, then $\mathcal{F} = p^*(q^*\Omega^1_C)$ is a subsheaf with $\mu_{p^*H}(\mathcal{F}) = \mu_{p^*H}(\Omega^1_X)$. Then Ω^1_X is semistable with respect to p^*H.

Now let $\chi < 2g-2$. We return to the philosophy of Remark 71 and show that $T_{X/C} \subset T_X$ is a destabilising subsheaf. Recall that $K_X = (2g-2+\chi)f$ and $K_{X/C} = \chi f$ (see e.g. [6]). Fix a divisor $M = s + rf \in \mathrm{NS}(X)$. By [27, II.7.10], there exists $r_0 > 0$ such that for $r \geqslant r_0$, M is ample. Then we can calculate the difference of slopes:

$$\Delta(T_X) = 2\,c_1(T_{X/C}) \cdot M - c_1(T_X) \cdot M = (-2\chi f + (2g-2+\chi)f) \cdot M = 2g-2-\chi,$$

and remark that given $\chi < 2g-2$, we have $\Delta(X) > 0$. Hence T_X is unstable. □

We obtain the following corollaries.

Corollary 109. *Let $\beta : Y \to X$ be an iterated blow up of a smooth Weierstrass fibration $\pi : X \to C$ with $\chi > 2g-2$. Then there exists an ample divisor $\mathcal{M} \in \mathrm{Pic}(Y)$ such that the tangent bundle T_Y is stable.*

Proof. We know from Theorem 108 that we can find an ample divisor $M \in \mathrm{Pic}(X)$ such that T_X is stable. Then T_Y is stable with respect to β^*M, which is a big and nef divisor. Suppose otherwise. Then there exists a maximal destabilising rank one subsheaf $\mathcal{F} \subset T_Y$. By the projection formula, this gives a destabilising subsheaf $\beta_*\mathcal{F} \subset T_X$, which gives a contradiction.

Finally, since stability is an open property (see e.g. [23]), there exists a small perturbation $\mathcal{M}$ of β^*M which is ample and T_Y is stable with respect to $\mathcal{M}$. □

Remark 110. In the setting above, if $\chi = 2g - 2$, then we can only ascertain the semi-stability of T_Y with respect to the big and nef divisor β^*M.

Definition 111. Let $\beta : Y \to C$ be an iterated blow up of a smooth Weierstrass fibration $\pi : X \to C$ with $n > 0$ centres lying on smooth fibres, and let χ be the degree of the fundamental bundle of X. We define the ε-number $\varepsilon(Y)$ of Y as $\varepsilon(Y) = \chi + \rho$ where ρ is the smallest positive integer that makes the curve

$$\rho f - \sum_{i=1}^{n} e_i \in \mathrm{NS}(Y)$$

an effective curve.

Corollary 112. *Let* $\beta : Y \to C$ *be an iterated blow up of a smooth Weierstrass fibration* $\pi : X \to C$ *with centres lying on smooth fibres, and* Y *has epsilon number* $\varepsilon(Y) < 2g-2$. *Then the tangent bundle* T_Y *is unstable.*

Proof. We show that $T_{Y/C} \subset T_Y$ is a destabilising subsheaf. The canonical divisor is given by $K_Y = (2g - 2 + \chi)f + \sum_{i=1}^{n} e_i$ and the dualising sheaf is $K_\pi = \chi f$. Then Proposition 81 gives

$$c_1(T_{Y/C}) = -\chi f - \sum_{i=1}^{n} e_i + \sum_{i=1}^{n} (\mu_i - 1)\tilde{e}_i,$$

where $\tilde{e}_i$ is the strict transform of the exceptional divisor e_i.

Fix an ample divisor $M = s + rf - \sum_{i=1}^{n} C_i e_i$ with $r \gg 0$. The difference of slopes is given by

$$\begin{aligned}\Delta_M(T_Y) &= 2\left(-\chi f - \sum_{i=1}^{n}(\mu_i - 1)\tilde{e}_i\right) \cdot M + \left((2g - 2 + \chi)f + \sum_{i=1}^{n} e_i\right) \cdot M \\ &= (2g - 2 - \chi) - \sum_{i=1}^{n} e_i \cdot M + 2\left(\sum_{i=1}^{n}(\mu_i - 1)\tilde{e}_i\right) \cdot M \\ &= (2g - 2 - \chi - \rho) + \left(\rho f - \sum_{i=1}^{n} e_i\right) \cdot M + 2\left(\sum_{i=1}^{n}(\mu_i - 1)\tilde{e}_i\right) \cdot M.\end{aligned}$$

Since $\sum_{i=1}^{n}(\mu_i - 1)\tilde{e}_i$ is an effective curve and M is ample, it follows from the Nakai-Moishezon criterion that the third term of $\Delta_M(T_Y)$ must be positive. By Definition 111, the curve $\rho f - \sum_{i=1}^{n} e_i$ is also effective, so the second term must be positive. Finally, we have by assumption that $\varepsilon(Y) < 2g - 2$, so $2g - 2 - \chi - \rho = 2g - 2 - \varepsilon(Y) > 0$. Hence $\Delta_M(T_Y) > 0$ and T_Y is unstable. $\square$

Chapter 6

Conclusions and future work

To briefly summarise, the aim of this thesis was to further the understanding of the structure of tangent bundles on algebraic varieties to fill in some gaps in the literature. We succesfully achieved our goal. In Chapter 3, we produced a combinatorial criterion for determining whether the tangent bundle of a $\mathbb{C}^*$-surface with fibrewise action is semistable. This result extends and generalises a similar result previously obtained for toric surfaces. In Chapter 4, we investigated the stability of the tangent bundles of minimal ruled surfaces together with general blow ups of Hirzebruch surfaces. We described a class of blow ups of Hirzebruch surfaces which always have unstable tangent bundles. We also constructed an example of a blow up of $\mathbb{F}_2$ which has stable tangent bundle. Finally, in Chapter 5, we investigate this question for the case of smooth Weierstrass fibrations. Here we provided a full classification for minimal Weierstrass fibrations according to their topological Euler characteristic and drew some conclusions about blow ups of such minimal surfaces.

We conclude with a few words concerning some questions for the future.

Further work

More on elliptic surfaces

We investigated the stability of the tangent bundles of Weierstrass fibrations and provided a complete description for minimal fibrations in Theorem 108 and also explored some consequences concerning certain blow ups of such surfaces in Corollaries 109 and 112. There still exist some gaps concerning smooth Weierstrass fibrations.

Question 113. We know that the blow up of a Weierstrass fibration with $\chi = 2g - 2$ will have semistable tangent bundle with respect to some big and nef polarisation. Can we find an ample polarisation for which the tangent bundle is semistable? Is this strictly semistable? Can we find another invariant which can distinguish between polarisation which make the tangent bundle strictly semistable or stable?

Question 114. We saw that a certain class of blow ups of smooth Weierstrass fibrations with small enough epsilon number have unstable tangent bundles, agreeing with our general intuition. Can we find an example of such a blow up with $\varepsilon > 2g - 2$ such that that the tangent bundle is strictly semistable or even stable? Such an example would be an analogue of the example surface in Section 4.5 in the setting of elliptic surfaces.

Question 115. In Chapter 5 we only allowed smooth Weierstrass fibrations so we could exploit the branched cover description and its properties. What can be said about the stability of tangent bundles of general elliptic surfaces which have singular curves in the fibres?

Question 116. Elliptic surfaces serve an important role as instruments for investigating questions in number theory and algebraic geometry in positive characteristic. It would be interesting to explore similar questions in characteristic p.

More on blow ups of Hirzebruch surfaces

In Chapter 4, we explored several patterns of blow ups and their consequences for the stability of the tangent bundles of the resulting surfaces. The criterion in Theorem 78 proves to be quite powerful at evaluating many examples throughout the chapter. Some new questions arose from the observations at the end of the chapter.

Question 117. Can we find a more systematic way of describing the blow ups of Hirzebruch surfaces which end up having stable tangent bundle? We obtained the crucial example from Section 4.5. Is there a way to generalise this to a larger class of examples perhaps in terms of properties of the divisorial fan?

Question 118. The special example mentioned above also shows an instance of a surface with stable tangent bundle obtained from a smooth surface with everywhere unstable tangent bundle. The crux of the proof is showing that the surface can be obtained as a birational modification of a singular $\mathbb{C}^*$-surface with stable tangent bundle. Can this observation be turned into a more general rule for all algebraic surfaces? Namely, can we say that an algebraic surface has stable tangent bundle if and only if it can be obtained as a birational transformation of a (potentially singular) surface with stable tangent bundle?

Stability of syzygy bundles

Let X be a smooth algebraic surface, H a fixed ample polarisation and V a globally generated vector bundle over X. Then the syzygy bundle M_V is defined to be the kernel bundle of the evaluation map $\mathrm{eval} : \mathrm{H}^0(X, V) \otimes \mathcal{O}_X \to V$. This fits into a short exact sequence

$$0 \to M_V \to \mathrm{H}^0(X, V) \otimes \mathcal{O}_X \to V \to 0.$$

Such vector bundles arise naturally in a number of both algebraic and geometric questions. For instance, given two vector bundles V and W on X what are the optimal numerical conditions such that the natural product map of global sections

$$H^0(X, V) \times H^0(X, W) \to H^0(X, V \otimes_{\mathcal{O}_X} W)$$

becomes surjective.

Naturally, the question of stability of such vector bundles is also of interest for many numerical calculations. When X is a smooth algebraic curve of genus g, Butler [9] showed that if a vector bundle V is semistable and its slope is $\mu(V) \geqslant 2g$ then the syzygy bundle M_V is also semistable.

Syzygy bundles are well understood on algebraic curves of all types. Now let X be a smooth algebraic surfaces with a fixed ample polarisation H. If V is H-stable then there exists $m >> 0$ such that $M_{V(m)}$ is H-stable [7]. Here, $V(m) := V \otimes_{\mathcal{O}_X} \mathcal{O}_X(mH)$. Progress on more general vector bundles over the projective plane was made, for example, in [19], and for Enriques surfaces in [39].

Question 119. Let X be a smooth Weierstrass fibration with a fixed ample polarisation H. Let V be a globally generated vector bundle. What are the numerical conditions on a general V that make the syzygy bundle M_V (semi)stable? What if we begin with V itself being (semi)stable?

Question 120. Let X be a Hirzebruch surface with fixed ample polarisation H? Let V be a globally generated vector bundle over X. Consider a blow up morphism $\pi : Y \to X$. Determine conditions on X, Y and V that make π^*V is (semi)stable.

Bibliography

[1] Alexeev, V., and Pardini, R. Non-normal abelian covers. *Compositio Mathematica 148*, 4 (2012), 1051–1084.

[2] Altmann, K., and Hausen, J. Polyhedral divisors and algebraic torus actions. *Mathematische Annalen 334*, 3 (2006), 557–607.

[3] Altmann, K., Hausen, J., and Süss, H. Gluing affine torus actions via divisorial fans. *Transformation Groups 13*, 2 (2008), 215–242.

[4] Altmann, K., Ilten, N. O., Petersen, L., Süss, H., and Vollmert, R. The geometry of T-varieties. *Contributions to algebraic geometry* (2012), 17–69.

[5] Arapura, D. Vanishing theorems for V-manifolds. *Proceedings of the American Mathematical Society* (1988), 43–48.

[6] Bădescu, L. *Algebraic surfaces.* Springer Verlag NY, 2001. doi:10.1007/978-1-4757-3512-3.

[7] Basu, S., and Pal, S. Stability of syzygy bundles corresponding to stable vector bundles on algebraic surfaces. *arXiv preprint arXiv:2105.05433* (2021).

[8] Boggi, M., and Pikaart, M. Galois covers of moduli of curves. *Compositio Mathematica 120*, 2 (2000), 171–191.

[9] Butler, D. C. Normal generation of vector bundles over a curve. *Journal of differential geometry 39*, 1 (1994), 1–34.

[10] Chen, X., Donaldson, S., and Sun, S. Kähler-Einstein metrics on Fano manifolds. i: Approximation of metrics with cone singularities. *Journal of the American Mathematical Society 28*, 1 (2015), 183–197.

[11] Chen, X., Donaldson, S., and Sun, S. Kähler-Einstein metrics on Fano manifolds. ii: Limits with cone angle less than 2π. *Journal of the American Mathematical Society 28*, 1 (2015), 199–234.

[12] Chen, X., Donaldson, S., and Sun, S. Kähler-Einstein metrics on Fano manifolds. iii: Limits as cone angle approaches 2π and completion of the main proof. *Journal of the American Mathematical Society 28*, 1 (2015), 235–278.

[13] Chen, X.-X., Donaldson, S., and Sun, S. Kähler-Einstein metrics and stability. *arXiv preprint arXiv:1210.7494* (2012).

[14] Cox, D., Little, J., and Schenk, H. Toric varieties, volume 124 of Graduate Texts in Mathematics. *American Mathematical Society 1* (2011).

[15] Di Rocco, S., Jabbusch, K., and Smith, G. Toric vector bundles and parliaments of polytopes. *Transactions of the American Mathematical Society 370*, 11 (2018), 7715–7741.

[16] Donaldson, S. K. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of the London Mathematical Society 3*, 1 (1985), 1–26.

[17] Douglas, M. R., Fiol, B., and Römelsberger, C. Stability and bps branes. *Journal of High Energy Physics 2005*, 09 (2005), 006.

[18] Dupré, J. Natural kinds and biological taxa. *The Philosophical Review 90*, 1 (1981), 66–90.

[19] Ein, L., Lazarsfeld, R., and Mustopa, Y. Stability of syzygy bundles on an algebraic surface. *Mathematical Research Letters 20*, 1 (2013), 73–80. doi:10.4310/MRL.2013.v20.n1.a7.

[20] Ereshefsky, M. *The poverty of the Linnaean hierarchy: A philosophical study of biological taxonomy.* Cambridge University Press, 2000.

[21] Friedman, R. *Algebraic surfaces and holomorphic vector bundles.* Springer Science & Business Media, 2012.

[22] Fulton, W. *Introduction to toric varieties*, vol. 131. Princeton university press, 1993.

[23] Greb, D., and Kebekus, S., and Peternell, T. Movable curves and semistable sheaves. *International Mathematics Research Notices* (2016). doi:10.1093/imrn/rnv126.

[24] Gubeladze, I. D. Anderson's conjecture and the maximal monoid class over which projective modules are free. *Mathematics of the USSR-Sbornik 63*, 1 (1989), 165.

[25] Harbater, D. Galois covers of an arithmetic surface. *American Journal of Mathematics 110*, 5 (1988), 849–885.

[26] Harder, G., and Narasimhan, M. S. On the cohomology groups of moduli spaces of vector bundles on curves. *Mathematische Annalen 212*, 3 (1975), 215–248. doi:10.1007/BF01357141.

[27] Hartshorne, R. *Algebraic geometry.* Springer Science, 2013. doi:10.1007/978-1-4757-3849-0.

[28] Hausen, J., and Süss, H. The cox ring of an algebraic variety with torus action. *Advances in Mathematics 225*, 2 (2010), 977–1012.

[29] Hering, M., Nill, B., and Süss, H. Stability of tangent bundles on smooth toric picard-rank-2 varieties and surfaces, 2019. arXiv:1910.08848.

[30] Huybrechts, D., and Lehn, M. *The geometry of moduli spaces of sheaves.* Cambridge University Press, 2010.

[31] Ilten, N. O., and Süss, H. Polarized complexity-1 T-varieties. *Michigan mathematical journal 60*, 3 (2011), 561–578.

[32] Klyachko, A. Vector bundles, linear representations, and spectral problems. *arXiv preprint math/0304325* (2003).

[33] Klyachko, A. A. Toric bundles and problems of linear algebra. *Functional Analysis and Its Applications 23*, 2 (1989), 135–137.

[34] Klyachko, A. A. Equivariant bundles on toral varieties. *Mathematics of the USSR-Izvestiya 35*, 2 (1990), 337.

[35] Klyachko, A. A. Stable bundles, representation theory and hermitian operators. *Selecta Mathematica, New Series 4*, 3 (1998), 419–445.

[36] Liendo, A. Rational singularities of normal t-varieties. *arXiv preprint arXiv:0909.4134* (2009).

[37] Maruyama, M. *On classification of ruled surfaces*, vol. 3. Kinokuniya, 1970.

[38] Miranda, R. The basic theory of elliptic surfaces. *Dottorato di Ricerca in Matematica, ETS Editrice* (1989). math.colostate.edu/ miranda/BTES-Miranda.pdf.

[39] Mukherjee, J., and Raychaudhury, D. A note on stability of syzygy bundles on enriques and bielliptic surfaces. *Proceedings of the American Mathematical Society* (2022).

[40] Neumann, S. *A decomposition of the moving cone of a projective manifold according to the Harder-Narasimhan filtration of the tangent bundle*. PhD thesis, Thesis, Universität Freiburg. https://freidok. uni-freiburg. de/fedora . . . , 2009.

[41] Orlik, P., and Wagreich, P. Algebraic surfaces withk*-action. *Acta Mathematica 138*, 1 (1977), 43–81.

[42] Pardini, R. Triple covers in positive characteristic. *Arkiv för matematik 27*, 1 (1989), 319–341.

[43] Pardini, R. Abelian covers of algebraic varieties. *Walter de Gruyter, Berlin/New York Berlin, New York* (1991). eudml:153330.

[44] Pardini, R. On the period map for abelian covers of projective varieties. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 26*, 4 (1998), 719–735.

[45] Payne, S. Moduli of toric vector bundles. *Compositio Mathematica 144*, 5 (2008), 1199–1213.

[46] Petersen, L., and Süss, H. Torus invariant divisors. *Israel Journal of Mathematics 182*, 1 (2011), 481–504.

[47] Steffens, A. On the stability of the tangent bundle of Fano manifolds, 1994. arXiv:alg-geom/9407004.

[48] Sun, X. On relative canonical sheaves of arithmetic surfaces. *Mathematische Zeitschrift* (1996). doi:10.1007/BF02621626.

[49] Süss, H. Canonical divisors on T-varieties. *arXiv preprint arXiv:0811.0626* (2008).

[50] Tian, G. On Calabi's conjecture for complex surfaces with positive first chern class. *Inventiones mathematicae* (1990). doi:10.1007/BF01231499.

[51] Uhlenbeck, K., and Yau, S.-T. On the existence of hermitian-yang-mills connections in stable vector bundles. *Communications on Pure and Applied Mathematics 39*, S1 (1986), S257–S293.

[52] Xiao, G. Galois covers between k3 surfaces. In *Annales de l'institut Fourier* (1996), vol. 46, pp. 73–88.

[53] Zhang, Y. On the stability of tangent bundles on cyclic coverings, 2019. arXiv:1912.05851.