Spiders' webs in the Eremenko-Lyubich class

SPIDERS' WEBS IN THE EREMENKO–LYUBICH CLASS

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ABSTRACT. Consider the entire function $f(z) = \cosh(z)$. We show that the escaping set $I(f)$ – that is, the set of points whose orbits tend to infinity under iteration of f – has a structure known as a "spider's web". This disproves a conjecture of Sixsmith from 2020. In fact, we show that the *fast escaping set* $A(f)$, i.e. the subset of $I(f)$ consisting of points whose orbits tend to infinity at an iterated exponential rate, is a spider's web. This answers a question of Rippon and Stallard from 2012. We also discuss a wider class of functions to which our results apply, and state some open questions.

1. INTRODUCTION

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. The set

$$I(f) := \{z \in \mathbb{C}\colon \lim_{n\to\infty} f^n(z) = \infty\}$$

is called the *escaping set* of f, where

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$$

denotes the n-th iterate of f.

The *fast escaping set* $A(f) \subset I(f)$ consists of those points for which $|f^n(z)|$ tends to infinity at the fastest possible rate. More precisely, let

$$M(R) := M(R, f) := \max_{|z|\leq R} |f(z)|$$

denote the maximum modulus function of f, and write $M^n(R) = M^n(R, f)$ for the n-th iterate of this function. If R is sufficiently large, then $M(r) > r$ for all $r \geq R$; it follows that $M^n(R) \to \infty$. Let us fix such an R, and define

$$A_R(f) := \{z \in \mathbb{C}\colon |f^n(z)| \geq M^n(R) \text{ for all } n \geq 0\}$$

and

$$A(f) := \bigcup_{n\geq 0} f^{-n}(A_R(f)).$$

It can be shown that the set $A(f)$ is independent of the choice of R; see [RS12, Theorem 2.2].

The escaping set and the fast escaping set have been central objects of study in transcendental dynamics in the past decades; see [BR24] for a survey on the escaping set, its history and its properties. One of the reasons that the escaping set is useful in the study of transcendental dynamics is that it often contains structures that can be used to facilitate the study of the overall dynamics. In particular, Rippon and

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Stallard discovered that, for many entire functions (particularly those of *small growth*), the escaping set, and even $A_R(f)$, has the structure of a "spider's web". (See [RS12, Theorem 1.9] for a precise statement, and [BR24, §7.4] for further discussion.)

1.1. Definition ([RS12, Definition 1.2])**.** A set $E \subset \mathbb{C}$ is called a *spider's web* if E is connected, and if there exists an increasing sequence $(G_n)_{n=0}^{\infty}$ of simply-connected domains with $\bigcup_{n=0}^{\infty} G_n = \mathbb{C}$ and $\partial G_n \subset E$ for all $n \geq 0$.

We note that the condition that $A_R(f)$ is a spider's web is independent of the choice of R; see [RS19, Theorem 1.3]. Moreover, if $A_R(f)$ is a spider's web, then so is $A(f)$, and if $A(f)$ is a spider's web, then so is $I(f)$ (see Theorem 1.4 of [RS12] and the remark after its proof).

The *Eremenko–Lyubich class* consists of those entire functions f for which the set of critical and asymptotic values is bounded; see [EL92] and [Six18]. If $f \in \mathcal{B}$, then f is bounded on a curve tending to infinity, and hence $A_R(f)$ is never a spider's web (with R as above); see [RS12, Theorem 1.8]. In particular, $A_R(f)$ does not separate the plane (see [RS12, Theorem 1.4]). It can be shown that $A_R(f)$ has uncountably many connected components for sufficiently large R (this follows e.g. from [Rem19, Proposition 8.1] and [Rem09, Theorem 1.1]; we omit the details). For many $f \in \mathcal{B}$, even $I(f)$ has uncountably many connected components [ARS22, Corollary 1.6]. On the other hand, there are also many values of a for which the escaping set of the exponential map $E_a(z) := e^z + a$ is connected [Rem11]. However, $I(E_a)$ is never a spider's web. Indeed, a is a logarithmic asymptotic value of E_a, and Sixsmith [Six20, Theorem 1.4] proved that $I(f)$ is never a spider's web if $f \in \mathcal{B}$ has a finite logarithmic asymptotic value.

In view of these results, Sixsmith [Six20, Conjecture after Theorem 1.4] conjectured that $I(f)$ is not a spider's web for any $f \in \mathcal{B}$. We disprove this conjecture, even for the fast escaping set $A(f)$.

1.2. Theorem (A spider's web escaping set in the Eremenko–Lyubich class)**.** *The set* $A(\cosh)$ *is a spider's web.*

The function cosh has critical values 1 and -1, and no asymptotic values; hence it belongs to the class $\mathcal{B}$. Theorem 1.2 also provides the first example of a function for which $A(f)$ is a spider's web, but $A_R(f)$ is not. This answers a question of Rippon and Stallard [RS12, p.807, Question 2]. As we briefly discuss in Section 4, the result holds, with the same proof, for a larger class of transcendental entire functions. In particular, the same result is true with cosh replaced by $z \mapsto (\cosh z)^2$, which was studied in [RS12, Section 5] and [PS23, Example 5.10].

The idea of the proof of Theorem 1.2 is as follows. Sixsmith observed (see Theorem 3.3) that $A(f)$ is a spider's web if and only if $\hat{\mathbb{C}} \setminus A(f)$ is disconnected, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. Pardo-Simón [PS23] has given a topological model for the dynamics of the map $h(z) := \cosh(z)/2$ in terms of the simpler "disjoint-type" function $g(z) := \cosh(z)$, whose dynamics is well-understood. In particular, it follows from her work that there is a continuous and surjective map $\hat{\mathbb{C}} \setminus A(h) \to (J(g) \setminus A(g)) \cup \{\infty\}$; see Proposition 3.2. It was previously shown by Evdoridou and Sixsmith [ES21] that the latter set is disconnected; in fact, the union of $A(g)$ with the Fatou set $F(g)$ is a spider's web. (For the map g, the set $F(g)$ is an unbounded connected set, consisting of the immediate basin of an attracting fixed point. In particular, the set $A(g) \cup F(g)$ is not a subset of the escaping set, and $A(g)$ and $I(g)$ are not spiders' webs.)

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2. Iterated exponential growth

For a large class of transcendental entire functions, including exponential and trigonometric functions, points in the fast escaping set are characterised as those whose orbits exhibit *iterated exponential growth*. Here we briefly review this fact and the properties of iterated exponential growth for the reader's convenience.

Let us define $F\colon [0,\infty) \to [0,\infty); t \mapsto \exp(t) - 1$. Then $F(t) > t$ for $t > 0$, and hence the sequence $F^n(t)$ tends to infinity. We are interested in the rate at which these orbits grow.

2.1. Definition (Iterated exponential growth)**.** A sequence $(a_n)_{n=0}^{\infty}$ of non-negative real numbers has *iterated exponential growth* if

$$0 < \liminf_{n\to\infty} F^{-n}(a_n) \leq \limsup_{n\to\infty} F^{-n}(a_n) < \infty.$$

The specific function F is not relevant here; any exponentially growing function gives rise to the same notion of iterated exponential growth:

2.2. Proposition (Elementary properties of iterated exponential growth)**.**

(a) Let $\delta > 0$ and define $\Omega_\delta(t) := \exp(\delta t)$ for $t \in \mathbb{R}$. Let t_0 be such that $\Omega_\delta(t) > t$ for $t \geq t_0$. Then the sequence $(\Omega_\delta^n(t_0))_{n=0}^{\infty}$ has iterated exponential growth.

(b) Let $C > 1$. A sequence $(a_n)_{n=0}^{\infty}$ has iterated exponential growth if and only if the sequence $\left(a_n^C\right)_{n=0}^{\infty}$ has iterated exponential growth.

Proof. Observe that $\Omega_1(t) > F(t)$ for all $t \geq 0$. Let t_0 and δ be as in (a). There is $T \geq t_0$ such that $\Omega_\delta^n(T) > \Omega_1^n(1) > F^n(1)$ for $n \geq 0$ (see e.g. [RRS10, Lemma 3.4]). Choose k so large that $\Omega_\delta^k(t_0) \geq T$; then

$$F^{-n}(\Omega_\delta^n(t_0)) \geq F^{-n}(\Omega_\delta^{n-k}(T)) \geq F^{-k}(1) > 0$$

for $n \geq k$. Similarly, $\Omega_{1/2}(t) < F(t)$ for $t \geq 1$. Taking $T \geq 1$ such that $\Omega_{1/2}^n(T) > \Omega_\delta^n(t_0)$ for $n \geq 0$, we see that

$$\limsup F^{-n}(\Omega_\delta^n(t_0)) < T < \infty.$$

To prove part (b), observe that $(a_n)_{n=0}^{\infty}$ has iterated exponential growth if and only if $(F(a_n))_{n=0}^{\infty}$ does, and the latter sequence is larger than a_n^C when $|a_n|$ is sufficiently large. □

It follows easily that, for a trigonometric function such as $h(z) \cosh z$ or $g(z) = \cosh(z)/2$ (the two functions that will play an important role in the proof of Theorem 1.2), a point z is in the fast escaping set if and only if its orbit has iterated exponential growth.

This is true far more generally. A transcendental entire function has *finite order* if there is a constant $C > 0$ such that

$$M(r, f) \leq \exp(r^C) \tag{2.1}$$

for all sufficiently large r. It has *positive lower order* if there is a constant $c > 0$ such that

$$M(r, f) \geq \exp(r^c) \tag{2.2}$$

for all sufficiently large r.

2.3. Proposition (Fast escaping points of finite-order functions)**.** *If f has finite order and positive lower order, then $z \in A(f)$ if and only if $|f^n(z)|$ has iterated exponential growth. In particular, this holds whenever $f \in \mathcal{B}$ has finite order.*

Proof. If (2.1) holds and $z \in \mathbb{C}$, then

$$|f^n(z)| \leq M^n(r, f) \leq (\Omega_C^n(r^C))^{1/C}$$

for sufficiently large r. Similarly, if (2.2) holds, then

$$M^n(r, f) \geq (\Omega_c^n(r^c))^{1/c}.$$

So the claim follows from Proposition 2.2 and the definition of $A(f)$.

It is well-known that functions in the Eremenko–Lyubich class have positive lower order; see e.g. [Rem23, Corollary 1.2]. This implies the final claim. ☐

3. Proof of the theorem

Define the entire functions h and g by $h(z) := \cosh(z)$ and $g(z) := \cosh(z)/2$. We begin with some observations.

(a) Both functions belong to the Eremenko–Lyubich class. Indeed, h has critical values 1 and -1 and no asymptotic values, while g has critical values $1/2$ and $-1/2$ and no asymptotic values.
(b) Both functions have finite order in the sense of (2.1).
(c) The map g has two real fixed points $p_a < p_r$, with $p_a \approx 0.589$ attracting and $p_r \approx 2.127$ repelling. The interval $(-p_r, p_r)$ belongs to the immediate basin of attraction of p_a; in particular both critical values belong to this basin. Hence g is of "disjoint type" in the sense of [Rem09, Remark after Definition 2.2]; see [MB12, Proposition 2.8].
(d) Both critical values of h belong to the real axis. Points on the real axis escape to $+\infty$, and $h(x)$ has exponential growth as $x \to \infty$. So $\mathbb{R} \subset A(h)$ by the previous section. It follows that the Julia set of h is the entire complex plane (see Theorems 7, 12 and 15 in [Ber93]).

The dynamics of the function g is well-understood. We refer to [BR24, §2] for an elementary discussion of the dynamics of $z \mapsto \sin(z)/2$, which was studied already by Fatou [Fat26], and which is known to be topologically conjugate to g. We require the following fact regarding the complement of $A(g)$. Denote the closure of $J(g)$ in $\hat{\mathbb{C}}$ by

$$\hat{J}(g) := J(g) \cup \{\infty\}.$$

3.1. Theorem (The complement of $A(g)$)**.** *The set $\hat{J}(g) \setminus A(g)$ is disconnected.*

Proof. This follows from a theorem of Evdoridou and Sixsmith [ES21, Theorem 1.1(b)] (which generalises a result of Evdoridou and the author for exponential maps [ERG18]). They prove that, for any disjoint-type entire function f of finite order, the set $X := \hat{J}(f) \setminus A(f)$ is *totally separated*, which means that for any two points $z, w \in X$, there exists an open and closed subset U of X with $z \in U$ and $w \notin U$. In particular, (since $J(g) \setminus A(g)$ is always non-empty), X is disconnected. □

In [PS22] and [PS23], Pardo-Simón shows that the dynamics of h on $J(h) = \mathbb{C}$ can be understood using the dynamics of g on its Julia set. More precisely, she modifies $J(g)$, and the dynamics of g thereon, in an appropriate manner to give a complete model for the topological dynamics of cosh, and indeed, for a much more general class of functions. We do not describe the construction directly, but will explain how Pardo-Simón's results and their proofs imply the following key fact.

3.2. Proposition (The complement of $A(h)$)**.** *There exists a continuous and surjective function* $\psi\colon \hat{\mathbb{C}} \setminus A(h) \to \hat{J}(g) \setminus A(g)$ *with* $\psi(z) = \infty$ *if and only if* $z = \infty$.

Proof. In [PS22, Definition 5.5], Pardo-Simón introduces a space $J(g)_\pm = J(g) \times \{+, -\}$, with a certain topology that is locally compact and Hausdorff, but not second countable and hence not metrizable. We do not require a full description of the topology, but rather will be using the following facts (compare Proposition 5.7 and Lemma 5.8 of [PS22]).

(1) The projection $\pi\colon J(g)_\pm \to J(g)$ is continuous.
(2) We have $\pi(x_n) \to \infty$ in $\hat{J}(g)$ if and only if $x_n \to \infty$ in $\hat{J}(g)_\pm$, where $\hat{J}(g)_\pm$ is the one-point compactification of $\hat{J}(g)_\pm$. In particular, π extends continuously to a map $\pi\colon \hat{J}(g)_\pm \to \hat{J}(g)$.

We define $\tilde{g}\colon J(g)_\pm \to J(g)_\pm$ to be the function that acts on the first component as g and as the identity on the second component (i.e., it preserves sign). Compare [PS22, p.13414].

The function h is *strongly postcritically separated* in the sense of [PS22, Definition 4.1]. By [PS22, Theorem 6.5], there is a continuous and surjective function $\varphi\colon J(g)_\pm \to \mathbb{C}$ with $h \circ \varphi = \varphi \circ \tilde{g}$. Moreover (see [PS22, Formula (6.16)]), $\varphi(x_n) \to \infty$ if and only if $x_n \to \infty$ in $J(g)_\pm$. In particular, φ extends continuously to a surjection $\varphi\colon \hat{J}(g)_\pm \to \hat{\mathbb{C}}$ with $\varphi(\infty) = \infty$.

Claim 1. Let $x \in J(g)_\pm$. Then $\varphi(x) \in A(h)$ if and only if $\pi(x) \in A(g)$.

Proof. By the discussion following formula (6.15) on p.13420 of [PS22], the distance between the point $\varphi(x)$ and the point $\pi(x)$ is uniformly bounded, independently of x. Here the distance is measured with respect to a certain hyperbolic orbifold metric; see [PS22, §4].

The underlying surface of the orbifold in question is $\mathbb{C}$, and hence the orbifold has a puncture at ∞. (See [PS22, Theorem 4.6].) Therefore, for $n \geq 0$, the moduli $|h^n(\varphi(x))| = |\varphi(\tilde{g}^n(x))|$ and $|g^n(\pi(x))| = |\pi(\tilde{g}^n(x))|$ differ from each other at most by a fixed power. Using Proposition 2.2, we see that these sequences either both have iterated exponential growth, or neither does. The claim follows by Proposition 2.3. △

In [PS23, §5], the author gives a complete description of when two points of $J(g)_\pm$ have the same image under φ. We require the following.

Claim 2. Suppose that $x, y \in J(g)_\pm$ with $\varphi(x) = \varphi(y)$ and $\pi(x) \notin A(g)$. Then $\pi(x) = \pi(y)$.

Proof. By Propositions 5.2 and 5.8 of [PS23], the claim holds more generally when $z = \pi(x) \in J(g)$ has the property that $g^n(z) \notin (p_r, \infty)$ for all $n \geq 0$. Here p_r is the real repelling fixed point of g.

To see that this implies the claim as stated, recall that $\cosh(t) > \exp(t)/2$ for all t. Hence the g-orbit of any point in (p_r, ∞) has iterated exponential growth, and (p_r, ∞) and its iterated preimages belong to $A(g)$. △

Now we can complete the proof. It follows from Claims 1 and 2 that setting

$$\psi(\varphi(x)) := \pi(x)$$

yields a well-defined surjective function

$$\psi\colon \hat{\mathbb{C}} \setminus A(h) \to \hat{J}(g) \setminus A(g).$$

We claim that ψ is continuous. Continuity at ∞ is immediate from the continuity properties of φ and π at infinity, noted above. Indeed, if $\varphi(x_n) \to \infty$, then $x_n \to \infty$ and thus $\pi(x_n) \to \infty$.

Now suppose that $z_n = \varphi(x_n) \to z$ in $J(h) \setminus A(h)$, and write $w_n = \psi(z_n) = \pi(x_n)$ and $w = \psi(z) = \pi(x)$. Since $\hat{J}(g)$ is compact, we may suppose that $w_n \to w' \in \hat{J}(g)$, and must prove that $w = w'$.

By compactness, the sequence (x_n) has an accumulation point $y \in \hat{J}(g)_\pm$. By continuity of φ, we see that $\varphi(y) = z = \varphi(x)$. We have $\pi(x) = \psi(z) \in J(g) \setminus A(g)$. By Claim 2 and continuity of π, we conclude that $w' = \pi(y) = \pi(x) = w$, as claimed. □

To complete the proof, we use the following observation of Sixsmith [Six20, Theorem 1.5].

3.3. Theorem (Characterisation of an escaping spider's web)**.** *Let f be a transcendental entire function. Then $A(f)$ (respectively $I(f)$) is a spider's web if and only if it separates some point of $J(f)$ from infinity.*

Proof of Theorem 1.2. By Theorem 3.1, the set $\hat{J}(g) \setminus A(g)$ is disconnected. By Proposition 3.2, this set is the continuous image of $\hat{\mathbb{C}} \setminus A(\cosh)$, hence the latter set is also disconnected.

In particular, $A(\cosh)$ separates some point of $\mathbb{C} \setminus A(\cosh) \subset J(\cosh)$ from infinity. The claim now follows from Theorem 3.3. □

4. Further remarks and questions

It is easy to see that $A(\cosh)$ has a dense path-connected subset, consisting of the real axis and its iterated preimages. In view of our result and the results of Pardo-Simón, it is plausible that, for $A(\cosh)$, the boundaries in Definition 1.1 can be chosen to be Jordan curves. This would imply that $A(\cosh)$ and $I(\cosh)$ are in fact path-connected; as far as we are aware, this would be the first example of a path-connected escaping set in the class $\mathcal{B}$.

4.1. Question. Are $I(\cosh)$ and $A(\cosh)$ path-connected?

Our proof of Theorem 1.2 generalises immediately to a large class of transcendental entire functions with escaping critical values. Indeed, suppose that $f \in \mathcal{B}$ is a function satisfying the following conditions.

(1) The function f has finite order of growth; i.e., (2.1) holds for some c.
(2) f has finitely many critical values and no finite asymptotic values, and the degree of the critical points of f is uniformly bounded.
(3) All critical points of f belong to $A(f)$, and the map f is strongly post-critically separated in the sense of [PS22].
(4) No two dynamic rays of f land at a common point.

Remark. We refer to [PS22] for the definition of strongly post-critically separated maps, as well as that of dynamic rays and their landing points.

4.2. Theorem (More fast escaping spiders' webs)**.** *Suppose that $f \in \mathcal{B}$ satisfies conditions (1)–(4). Then $A(f)$ is a spider's web.*

Proof. The proof proceeds exactly as the proof of Theorem 1.2. Indeed, Conditions (1), (2) and (3) imply that f satisfies the assumptions of [PS22, Theorem 6.5]. This yields a function $\varphi\colon \hat{J}(g)_{\pm} \to J(f)$ as discussed in the proof of Proposition 3.2, where $g(z) = \lambda f(z)$, with $|\lambda|$ sufficiently small. Claim 1 in the proof of Proposition 3.2 holds with the same proof. Properties (3) and (4) ensure that Claim 2 also holds. Hence we conclude as before that there is a continuous function $\varphi\colon \hat{\mathbb{C}} \setminus A(f) \to \hat{J}(g) \setminus A(g)$.

By [ES21, Theorem 1.1(b)], Condition (1) again implies that $\hat{J}(g) \setminus A(g)$ is disconnected. So $\hat{\mathbb{C}} \setminus A(f)$ is also disconnected, and $A(f)$ is a spider's web by Theorem 3.3. □

We may ask whether any of the conditions above can be omitted or weakened. If f has infinite order, the set $I(f) \setminus A(f)$ may contain curves to infinity; see Remark (3) after Theorem 5.2 of [RRS10]. Such an example may be chosen also to satisfy condition (2)–(4), using the techniques of Bishop [Bis15].

The function $f(z) = e^z$ satisfies all conditions except for (2), but it follows from [Dev93] that $\mathbb{C} \setminus I(f)$ contains unbounded connected sets. (In fact, as already mentioned in the introduction, if $f \in \mathcal{B}$ has a finite asymptotic value, then $I(f)$ is not a spider's web by [Six20, Theorem 1.4].)

Finally, the function $g(z) = \cosh(z)/2$ satisfies conditions (1), (2) and (4), but the Fatou set $F(f)$ is connected, and $A(f)$ and $I(f)$ have uncountably many connected components.

So conditions (1), (2) and (3) cannot be omitted entirely (although it is likely that some or all of them could be weakened). We do not know whether condition (4) can be omitted in Theorem 4.2.

4.3. Question. Suppose that $f \in \mathcal{B}$ satisfies (1), (2) and (3). Is $A(f)$ is a spider's web?

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