Weak pairwise justifiability as a common root of Arrow'sand the Gibbard–Satterthwaite theorems

DOI:
10.1007/s00355-024-01575-6

Social Choice and Welfare
https://doi.org/10.1007/s00355-024-01575-6

ORIGINAL PAPER

Weak pairwise justifiability as a common root of Arrow's and the Gibbard–Satterthwaite theorems

Salvador Barberà [1,2] · Dolors Berga [3] · Bernardo Moreno [4] · Antonio Nicolò [5,6]

Received: 20 July 2022 / Accepted: 11 December 2024

Abstract

We introduce a novel principle that we call weak pairwise justifiability, which applies to a large class of collective choice rules, including the social choice functions and the social welfare functions about which the Gibbard–Satterthwaite theorem and Arrow's impossibility theorem are predicated, respectively. We prove that, under appropriate qualifications, our principle is a common root for these two classical results, when applied to rules defined over the full domain of weak preference orders (also for strict).

1 Introduction

The relationship between Arrow's (Arrow 1963) and the Gibbard–Satterthwaite's theorems has been a matter of interest for a long time. Well before the work of Gibbard (1973) and Satterthwaite (1975), Vickrey (1960) had already conjectured that there was a strong connection between strategy-proofness and Arrow's condition of Independence of Irrelevant Alternatives. He stated that "social welfare functions that satisfy the nonperversity and the independence postulates, and are limited to rankings as arguments are (...) immune to strategy. It can be plausibly conjectured that the converse

We appreciate the comments and suggestions by Walter Bossert, Luis Corchón, Anke Gerber, Hervé Moulin, Matthew Jackson ,and Arunava Sen. Early versions of our results were encompassed in a work entitled "Pairwise justificable changes in collective choices", presented at the World Congress of the Econometric Society 2020,and the following seminars: Online Social Choice and Welfare Seminar Series 2020, Leicester Weekly Seminar Series 2021, and UAB Microeconomics seminar 2020. Salvador Barberà acknowledges financial support from the Spanish Agencia Estatal de Investigación (AEI), through the Severo Ochoa Programme for Centres of Excellence in R&D (Barcelona School of Economics CEX2019-000915-S), and from grant PID2020-116771GB-I00 funded by MCIN/AEI/10.13039/501100011033 and from the Generalitat de Catalunya, through grant 2021SGR00194. Dolors Berga thanks the support from project PID2019-106642GB-I00 financed by MCIN/AEI/10.13039/501100011033, project PID2022-138003NB-I00/ MCIN/ AEI / 10.13039/501100011033/FEDER,UE, and from the Generalitat de Catalunya, through grant 2021SGR00570. Bernardo Moreno acknowledges the support from project PID2020-114309GB-I00 financed by MCIN/AEI/10.13039/501100011033. The usual disclaimer applies.

Extended author information available on the last page of the article

Published online: 07 January 2025

is also true". Our purpose in this paper is to provide a theorem that encompasses the impossibility result obtained in each one of these two fundamental theorems when stated for the full domains of weak individual preferences and strict individual preferences (see Theorem 3). Undoubtedly, due to abundant discussions and to the existence of previous partial results, that we shall survey below, the profession is aware that there exists a strong connection between them, and we don't expect the reader to be surprised by this purpose. What is novel is that we establish the connection by analyzing the consequences of a condition, that we call weak pairwise justifiability, on the performance of collective choice correspondences.

Clearly, each one of the classical results we want to relate start from different formulations, since Arrow's social welfare functions focus on aggregate social preferences and the Gibbard–Satterthwaite theorem is about collective choices from a fixed set of alternatives. This requires establishing a bridge between the two. To this purpose, we frame our results in terms of collective choice rules, defined as follows.

Given a fixed set of alternatives, and a fixed set of agents endowed with preference orders over them, preference profiles assign one preference to each agent. Subsets of alternatives are called agendas. Situations are combinations of one single profile and one agenda. They are interpreted as instances where a society composed by agents with the given preferences must choose a subset of the elements in the given agenda. Collective choice rules are defined as correspondences that, for each situation in their domain, select a subset of the agenda.[1]

As announced, our analysis focuses on a property that collective choice correspondences may or may not satisfy, that we call weak pairwise justifiability. Informally, it states the following. Let x and y be two alternatives that are in the agenda for two different situations, and let x be selected in the first but not in the second, in which y is chosen. Then, there must exist some alternative z and some agent i for whom z has improved its relative position with respect to x, when going from the first to the second situation. What the condition requires is that x's fall in the social appreciation requires that some other alternative has improved upon it in someone's preferences.

Our main result (Theorem 3) proves that no collective choice correspondence defined, either on the domain of situations where all preference profiles of weak orders are admissible, or on the domain of situations where all preference profiles of strict orders are admissible can satisfy simultaneously the conditions of non-dictatorship, weak pairwise justifiability, weak decisiveness, and full range. From there we derive the same non-existence result for social welfare functions and for social choice functions satisfying the same conditions used in Theorem 3, since both of them are subclasses of collective choice correspondences to which our non-existence theorem applies (see Corollaries 1 and 2). Since the result also applies to the case of individual weak preferences, it extends the coverage of preceding results that we will discuss below.

Let us elaborate a bit more about the relevance of weak pairwise justifiability. In our view, it is an attractive requirement, and one that can be used for other purposes, beyond the one we contemplate here.[2] Actually, the condition is reminiscent, but different than

[1] Fishburn (1973) already proposed the same notion of collective choice correspondences under the name of social choice functions.

[2] In a companion paper, Barberà et al. (2024) consider collective choice functions and view a stronger version of this property as a new defense for Condorcet consistency: when defined on properly restricted

other classical normative requirements, like Maskin monotonicity or strong positive association when stated for rules operating on the universal set of weak preference profiles. Yet, there is no wonder that many of these conditions may collapse into one under more restricted domains, like that of strict preferences.

Let us now mention some important works that have approached Arrow's and the Gibbard–Satterthwaite theorem from different angles. Gibbard (1973) used Arrow's theorem as an intermediate step in the proof of his own theorem. Satterthwaite (1973) explicitly analyzed the mutual implications between the conditions involved in these two theorems. The latter presented further evidence of the parallelism between strategy-proofness and Arrow's conditions, as well as in their respective proofs, and so did Pattanaik (1978), Muller and Satterthwaite (1977) and later Reny (2001), among other authors. A direct and conclusive way to connect the two results was proposed by Eliaz (2004), and consists in proving that they both can be derived from a single general theorem predicated on rules that contain Arrowian social welfare functions and strategy-proof social choice functions as particular cases.[3] He defined a class of rules when individual preferences are strict, that he called social aggregators, and proved that they must be dictatorial if they satisfy a property termed preference reversal, which is implied by Arrow's conditions and by strategy-proofness. This is also our approach, in a different and larger framework because we consider collective choice correspondences instead of social aggregators, and we admit indifferences. Moreover, weak pairwise justifiability is weaker than preference reversal (see Sect. 4 for a detailed comparison among pairwise justifiability and other conditions that have been proposed in the literature, including Maskin monotonicity and preference reversal).

A similar line of reasoning was also taken by Man and Takayama (2013), who proved that any social choice correspondence defined on the universal preference domain, ranging over more than three alternatives and satisfying the axioms of strong unanimity, independence of unfeasible alternatives and independence of losing alternative, is serially dictatorial.[4] Akbarpour and Nariman (2016) considered a novel property imposing that whenever all voters change their opinions, the outcome of a social choice mechanism necessarily changes. They showed that the only social choice mechanism that satisfies the above-mentioned property is dictatorship, and they suggest that the Arrow and Gibbard–Satterthwaite theorems might be deduced from this theorem.

From here on, the paper proceeds as follows. In Sect. 2 we provide notation and definitions and the basics for comparison with Arrow and Gibbard–Satterthwaite's frameworks. In Sect. 3 we state our impossibility result for collective choice correspondences and state the relationship with Gibbard–Satterthwaite and Arrow's theorems. Section 4 discusses the connections between weak pairwise justifiability and other

Footnote 2 continued
domains of preferences, pairwise justifiability is very closely related to the possibility of respecting the desirable objective of Condorcet consistency.

3 An earlier but still unpublished paper by Barberà (2001) proposed an alternative approach in the same vein.

4 Note that they assume that preferences are strict in some cases, which blurs the comparison between the two theorems to be unified. For example, they do not embed Gibbard and Satterthwaite's result when individual indifferences are allowed.

conditions proposed in the social choice literature. Although some proofs are outlined in the text, they are collected in their formal and complete form in the Appendix.

2 Notation and definitions

2.1 The basic framework

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$. Let A be a finite set of *alternatives* with $\#A \geq 3$. We denote subsets of alternatives as B, B',... and we call them *agendas*. We denote by $\mathcal{A}$ the set of all nonempty subsets of A and by $\mathcal{B} \subseteq \mathcal{A}$ a *collection of subsets of alternatives*, or equivalently, *a collection of agendas.*

Let $\mathcal{R}$ be the set of all *preferences* on A (that is, all complete, reflexive, and transitive binary relations on A). Elements of $\mathcal{R}$ are denoted by R_i, R_j,... The top of a preference $R_i \in \mathcal{R}$ in $B \in \mathcal{B}$, denoted by $t(R_i, B)$, is the set of alternatives $x \in B$ such that $x R_i y$ for all $y \in B$. As usual, P_i and I_i denote the strict and indifference preference relation induced by R_i, respectively.

Let $\mathcal{R}^n$ be the set of all possible *preference profiles,* also called *the universal domain,* and $\mathcal{D} \subseteq \mathcal{R}^n$ be a subset of preference profiles. Elements of $\mathcal{R}^n$ are denoted by $R = (R_1, R_2, ..., R_n)$. When we have a partition of N into different sets $S_1, S_2, ..., S_k$, we write a preference profile as $R = (R_{S_1}, R_{S_2}, ..., R_{S_k})$.

Let $\mathcal{P}^n \subseteq \mathcal{R}^n$ denote the subset of all preference profiles where agents' preferences on A are *strict* (that is, also antisymmetric), also called *the strict universal domain*.

A *situation* is a pair $(R, B) \in \mathcal{D} \times \mathcal{B}$.[5]

A *collective choice correspondence* on $\mathcal{D} \times \mathcal{B}$ is a mapping $C : \mathcal{D} \times \mathcal{B} \to \mathcal{A}$ that for each situation $(R, B) \in \mathcal{D} \times \mathcal{B}$ assigns a non-empty subset of alternatives $C(R, B) \in 2^B \backslash \{\varnothing\}$. When the correspondence is single valued, hence a function, we can speak of *collective choice functions*.

A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ has *full range* if for each $B \in \mathcal{B}$ and $x \in B$ there exists $R \in \mathcal{D}$ such that $x \in C(R, B)$.

We now formalize the principle whose analysis is at the center of our work, that we call *weak pairwise justifiability.* A comparison between this condition and other important ones in the literature, including Maskin monotonicity, strong positive association, and strong monotonicitiy is provided in Sect. 4.

Definition 1 A *collective choice correspondence* C on $\mathcal{D} \times \mathcal{B}$ satisfies **weak pairwise justifiability** on $\mathcal{D}' \times \mathcal{B}$, $\mathcal{D}' \subseteq \mathcal{D}$ if, for any two situations $(R, B), (R', B') \in \mathcal{D}' \times \mathcal{B}$ such that $x \in C(R, B)$, $x \notin C(R', B')$, $y \in C(R', B')$ and $x, y \in B \cap B'$ then either (1) there is some agent $i \in N$ and some alternative $z \in A \backslash \{x\}$ such that $x P_i z$ and $z R'_i x$, or (2) there is some agent $i \in N$ such that $x I_i y$ and $R_i \neq R'_i$.

Although the first part of the property is plausible and with an intuitive meaning, the second part is technical. Concerning the first part, the condition requires that a decrease in the social appreciation of x necessitates that some other alternative has improved upon it in someone's preferences. Because this other alternative z can be

[5] Le Breton and Weymark (2011) refer to a situation as a pair formed by a preference profile and an agenda.

outside B or B', it implies that *irrelevant* alternatives can have an influence on a social outcome.[6]

Let us also introduce a special case of collective choice correspondences that result from imposing a restriction on their range when their domain is restricted. We refer to the case where, when defined on the subdomains with strict individual preferences, the correspondence is required to select singletons, and only allowed to become multivalued when some agents' preferences admit indifferences. We say that these correspondences satisfy *weak decisiveness*. Formally:

Definition 2 A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$, $\mathcal{D} \subseteq \mathcal{R}^n$ satisfies **weak decisiveness** if for any situation (R, B) such that $R \in \mathcal{D} \cap \mathcal{P}^n$, $\#C(R, B) = 1$.

Although weak decisiveness may be seen as a demanding requirement, let's mention that any social welfare function satisfying Arrow's conditions satisfies it (as shown in Proposition 4) and, by definition, any social choice function too.

Note that a collective choice function trivially satisfies weak decisiveness but not any collective choice correspondence satisfying weak decisiveness is a collective choice function. The latter is because when individual indifferences exist, more than one alternative can be chosen.

We conclude this subsection with three examples. The reader will find additional examples of collective choice functions satisfying weak pairwise justifiability in specific subdomains of preferences in Sect. 4.

Example 1 illustrates a serial dictatorship that satisfies weak pairwise justifiability.

Example 1 Let $n = 3$, $\mathcal{B} = \{A\} = \{x, y, z\}$, and $\mathcal{D} = \mathcal{R}^n$. Consider the serial dictator collective choice function C with the following orders of dictators $1 > 2 > 3$ and a tie-breaking rule $b : x > y > z$. Namely, rule C works as follow: for any R, $C(R, A) \in t(R_1, A)$, that is, the chosen alternative belongs to agent 1's set of best alternatives. If $t(R_1, A)$ is a singleton, this is the outcome. Otherwise, agent 2 picks her favorite alternatives in this set: $C(R, A) \in t(R_2, \{t(R_1, A)\})$. If it is a singleton, this is the outcome. Otherwise, agent 3 picks her best alternatives and if it is not a singleton, the chosen alternative is selected using the tie-breaking rule b. This rule satisfies weak pairwise justifiability. We prove it for two particular cases that may be helpful to understand when each part of the condition appears to be decisive.

Case 1: Let R be such that $t(R_1, A) = \ell$, thus, $C(R, A) = \ell$ and R' such that $C(R', A) = y \neq \ell$. Then, $\ell P_1 y$ and $y R'_1 \ell$ meaning that part (1) of pairwise justifiability holds.

Case 2: Let R and R' be such that $t(R_1, A) = \{x, y\}$, $C(R, A) = x$, and $C(R', A) = y$. Observe that if $R'_1 \neq R_1$, since $x I_1 y$, part (2) of weak pairwise justifiability holds. If $R'_1 = R_1$, $t(R'_1, A) = t(R_1, A)$. By definition of C, either $x P_2 y$ and $y R'_2 x$, or $x I_2 y$. In the former case, part (1) of weak pairwise justifiability holds. In the latter case, if $R'_2 \neq R_2$, part (2) of weak pairwise justifiability holds. If $R'_2 = R_2$, then $R'_3 \neq R_3$, and two subcases may arise: either $x P_3 y$ and $y P'_3 x$ (by definition of C) or $x I_3 y$ where part (1) and (2) hold, respectively. All the remaining cases follow similar arguments and the proof is left to the reader.

[6] See Sects. 2.2 and 4, where we deal with Arrow's theorem.

Example 2 presents a collective choice correspondence that violates weak pairwise justifiability.

Example 2 Let $n = 2$, $A = \{x, y, z\}$ and consider a collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$, with $\mathcal{B} = \{A, B\}$, $B = \{x, y\}$, and $\mathcal{D} = \mathcal{R}^n$ such that agent 1 is a dictator when the agenda is A and agent 2 is a dictator when the agenda is B. Consider the profile R such that $t(R_1, A) = x$ and $t(R_2, A) = y$; it follows that $C(R, A) = x$ and $C(R, B) = y$ and weak pairwise justifiability is clearly violated.

Finally, Example 3 defines a weakly pairwise justifiable collective choice correspondence that violates weak decisiveness.

Example 3 Let $n \geq 3$, $\mathcal{B} = \{A\}$ where $A = \{x, y, z\}$, and $\mathcal{D} = \mathcal{P}^n$. Let C be such that for any R, $C(R, A) = \cup_{i \in N} t(R_i, A)$.

2.2 Some further definitions and a tour to Arrow and Gibbard–Satthertwaite's results

Our main task in the paper is to prove that no collective choice correspondence can satisfy simultaneously the conditions of non-dictatorship, weak pairwise justifiability, weak decisiveness, and full range, when defined either on the universal domain or on the strict universal domain. From there we can connect it with the result of non-existence of social welfare functions as in Arrow's and of social choice functions as in Gibbard–Satterthwaite's.

To this end, we start by defining *dictatorships*.

Definition 3 A *collective choice correspondence* C on $\mathcal{D} \times \mathcal{B}$ is **dictatorial** (on $\mathcal{D} \times \mathcal{B}$) if there exists an agent $i \in N$, the dictator on $\mathcal{D} \times \mathcal{B}$, such that for any $B \in \mathcal{B}$ and any $R \in \mathcal{D}$, $C(R, B) \subseteq t(R_i, B)$.

When a collective choice correspondence is dictatorial we also refer to it as a dictatorship.

The objects about which the Gibbard–Satterthwaite theorem are predicated are usually called social choice functions, and in fact can be viewed as specific collective choice rules that always deliver a singleton and only operate for the agenda that includes all alternatives.

More formally, a **social choice function** $f : \mathcal{D} \to A$ is a collective choice function C on $\mathcal{D} \times \mathcal{B}$ for $\mathcal{B} = \{A\}$. Note that properties on f can be trivially translated as properties on C, and viceversa. When convenient, we shall use f and C indistinctly.

Strategy-proofness, then is defined for social choice functions, as follows:

Definition 4 Let $\mathcal{D} \subseteq \mathcal{R}^n$. A social choice function $f : \mathcal{D} \to A$ is **strategy-proof** on $\mathcal{D}$ if for any agent $i \in N$, any preference profile $R \in \mathcal{D}$, and any agent i's preference R'_i such that $(R'_i, R_{N\setminus\{i\}}) \in \mathcal{D}$, $f(R) R_i f(R'_i, R_{N\setminus\{i\}})$.

And the Gibbard–Satterthwaite result reads like:

Theorem 1 (Gibbard–Satterthwaite impossibility Theorem) *Let A be the set of alternatives such that $\#A \geq 3$ and let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. There is no full-range social choice function $f : \mathcal{D} \to A$ that is strategy-proof and non-dictatorial on $\mathcal{D}$.*

The connection between our collective choice rules and Arrowian social welfare functions is a bit more subtle because our statements are initially about social choices and Arrow's are about social preferences. Here, the use of pairwise justifiability is crucial.

In Arrow's terms, a **social welfare function** F on $\mathcal{D}$ is a mapping from $\mathcal{D}$ to $\mathcal{R}$. For any $R \in \mathcal{D}$, $F(R) \in \mathcal{R}$ denotes the binary relation that F assigns to R.

We use the following notation for social preferences associated to F: given $R \in \mathcal{R}^n$, $\succsim_R$ denotes the social preference $F(R)$. Specifically, for any $x, y \in A$, $x \succsim_R y$ means that x is weakly socially preferred to y and $x \succ_R y$ means that x is strictly socially preferred to y.

Definition 5 A *social welfare function* F on $\mathcal{D}$ is **dictatorial** if for any $R \in \mathcal{D}$ and any $x, y \in A$, there exists $i \in N$, the dictator, such that if $x P_i y$ then $x \succ_R y$.

Definition 6 A *social welfare function* F on $\mathcal{D}$ satisfies the **weak Pareto** condition if for any $R \in \mathcal{D}$ and any $x, y \in A$, if $x P_i y$ for every $i \in N$, then $x \succ_R y$.

Definition 7 A *social welfare function* F on $\mathcal{D}$ satisfies **independence of irrelevant alternatives** if for any $R, R' \in \mathcal{D}$ and any $x, y \in A$, if [for any $i \in N$, $x R_i y \iff x R'_i y$] then [$x \succsim_R y \iff x \succsim_{R'} y$].

Theorem 2 (Arrow impossibility Theorem) *Let A be the set of alternatives such that $\#A \geq 3$ and let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. There is no social welfare function F on $\mathcal{D}$ satisfiying weak Pareto, independence of irrelevant alternatives and non-dictatorship.*

To connect social welfare functions with collective choice correspondences we need the following definition.

Definition 8 A *collective choice correspondence C on $\mathcal{D} \times \mathcal{A}$ is **transitively rationalizable** if for any $R \in \mathcal{D}$, there exists a transitive binary relation on A*, say $\mathbf{R}_R \in \mathcal{R}$, *that* rationalizes C (that is, *for any agenda $B \in \mathcal{A}$, $C(R, B) = t(\mathbf{R}_R, B)$*).

First, note that transitively rationalizable collective choice correspondences defined on $\mathcal{D} \times \mathcal{A}$ induce social welfare functions. Then, observe that each social welfare function uniquely defines a transitively rationalizable collective choice correspondence. The following proposition proves that collective choice correspondences satisfying weak pairwise justifiability are transitive rationalizable. Therefore, each collective choice correspondence satisfying weak parwise justifiability identifies a social welfare function.

Proposition 1 *Any collective choice correspondence C on $\mathcal{D} \times \mathcal{A}$ satisfying weak pairwise justifiability on $\mathcal{D} \times \mathcal{A}$ is transitively rationalizable on $\mathcal{D} \times \mathcal{A}$.*

The proof is in the Appendix. In what follows, abusing of the language, we will use the following Remark:

Remark 1 When we say that a social welfare function F satisfies either weak pairwise justifiability, weak decisiveness, or full range we mean that the associated collective choice correspondence satisfies either of them.

3 Impossibility result

In this section we explore the consequences of imposing our condition of weak pairwise justifiability on collective choice correspondences for two cases, one where the set of preference profiles is the universal domain and the other where it is the strict universal domain. We offer a result in this vein, showing that, as it is known to happen in other contexts and under different conditions, our is also too demanding and precipitates dictatorship.[7] Although full proofs are relegated to the Appendix, we would like to note that the strategy we use introduces a new concept, that of a determinant individual, which is related but different than that of a pivot, and may be useful for other uses beyond the one we exploit here.

Theorem 3 *Let $A \in \mathcal{B}$ and let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. There is no full range collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ satisfying weak decisiveness, weak pairwise justifiability and non-dictatorship.*

Theorem 3 applies to the frameworks of Gibbard–Satterthwaite and Arrow (as we have clarified in Sect. 2.2) and when we restrict the attention either to social choice functions or social welfare functions, the following corollaries hold. Corollary 1 is obtained from Theorem 3, because it refers to social choice functions, and this directly implies that weak decisiveness must be satisfied and $\mathcal{B} = \{A\}$. Corollary 2 is obtained from Theorem 3, because it refers to social welfare functions, and this implies that $\mathcal{B} = \mathcal{A}$.

Corollary 1 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. There is no full range social choice function f on $\mathcal{D}$ satisfying weak pairwise justifiability and non-dictatorship.*

Corollary 2 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. There is no full range social welfare function F on $\mathcal{D}$ satisfying weak decisiveness, weak pairwise justifiability and non-dictatorship.*

The statement in Theorem 3 is robust in the sense that when ruling out only one of the properties imposed, we can define rules satisfying the other properties.

Collective choice functions that in any situation $(R, B) \in \mathcal{D} \times \mathcal{B}$ select an arbitrary alternative $\ell_B \in B$ for any $B \in \mathcal{B}$ violate full range. Collective choice functions that, in any situation $(R, B) \in \mathcal{D} \times \mathcal{B}$, select the alternative in the agenda B that comes first relative to a given linear ordering of the alternatives, except when there is unanimity on the unique best alternative (in which case they select it), violate weak pairwise justifiability,[8] Finally, the collective choice correspondence described in Example 3 that selects all alternatives in the top of every agent violates weak decisiveness.

[7] Examples are abundant. To mention one, consider for instance the notion of self-selectivity introduced by Koray (2000) which requires that a social choice function should choose itself from among other rival such functions when it is employed by the society to make this choice as well. Koray (2000) shows that a unanimous and neutral social choice function is universally self-selective if and only if it is dictatorial.

[8] To show it, consider, for simplicity, 2 agents and 3 alternatives $\{x, y, z\}$. Let the linear ordering of alternatives be $z > y > x$. Let (R, B) and $(R'B)$ be such that $B = \{x, z\}$, $R_1 = R'_1$: xP_1yP_1z, R_2: yP_2xP_2z, R'_2: $xP'_2yP'_2z$. By definition of C, $C(R, B) = z$, $C(R', B) = x$. But note that from R to R', no alternative has improved its relative position with respect to z, for any agent.

The detailed proof of Theorem 3 is in the Appendix and we provide here an outline of how we proceed.

We start by proving that when individual indifferences are not part of the domain of preference profiles, there is no weak decisive collective choice correspondence (that is, a collective choice function) satisfying full range, weak pairwise justifiability and non-dictatorship. We prove this in two steps.

In the first one we fix an agenda with at least three alternatives and show that our property implies the existence of a dictator on such fixed agenda. The proof of this step contains the novel definition of an agent who is determinant at a given profile, which differs from the more common and weaker notion of being pivotal.[9]

In the second step we compare the outcomes of the rule for varying agendas. The argument involves two cases, depending on whether or not both agendas have two alternatives each or at least one of them contains three or more alternatives. The common starting point for both cases is that, since $A \in \mathcal{B}$, for any pair of agendas there exists a third one with at least three alternatives, containing both agendas. Applying the previous step to such inclusive agenda we prove that the dictator is the same agent at all admissible profiles and for all relevant agendas. This ends the proof for the case of strict preferences.

Then, we prove the result in case we allow for indifferences in the preferences of individuals. The following lemmas (1 and 2) are used in the transition between one result to the other. The proofs are in the Appendix.

Lemma 1 *If C on $\mathcal{R}^n \times \{B\}$, $\#B \geq 3$, is a full range collective choice correspondence satisfying weak pairwise justifiability and weak decisiveness on $\mathcal{R}^n \times \{B\}$ then C is dictatorial on $\mathcal{P}^n \times \{B\}$.*

Lemma 2 *Let C be a collective choice correspondence satisfying weak pairwise justifiability on $\mathcal{R}^n \times \{B\}$. If there is a dictator i for the restriction of C to $\mathcal{P}^n \times \{B\}$, then C is dictatorial and i is the dictator on $\mathcal{R}^n \times \{B\}$.*

As mentioned above, Corollaries 1 and 2 are straightforward consequences of Theorem 3. From them we obtain the same result of non-existence of social welfare functions as in Arrow's theorem and of social choice functions as in Gibbard–Satterthwaite theorem, since both fall into the class of collective choice correspondences to which our theorem applies. Proposition 2 joint with Corollary 1 clarify why Gibbard–Satterthwaite's result can be obtained as a direct corollary of Theorem 3. Propositions 3, 4, and 5 below joint with Corollary 2 elucidate why Arrow's result can be obtained as a corollary of ours (see their proofs in the Appendix).

Proposition 2 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. A social choice function $f : \mathcal{D} \to A$ is strategy-proof on $\mathcal{D}$ if and only if it satisfies weak pairwise justifiability on $\mathcal{D}$.*

Proposition 3 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. If a social welfare function F on $\mathcal{D}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then F satisfies weak pairwise justifiability.*

[9] Note that we could have resorted to Muller and Satterthwaite's proof of dictatorship, for example, and avoided the proof of this step. However, we believe that this is an opportunity to use a different technique of proof, more in spirit of those that directly exploit the notion of pivotal voters and identifies the dictator as an agent who is pivotal in all environments.

Proposition 4 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. If a social welfare function F on $\mathcal{D}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then F satisfies weak decisiveness.*

Proposition 5 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. If a social welfare function F on $\mathcal{D}$ satisfies the weak Pareto condition, then F satisfies full range.*

As noted by Mossel and Tamuz (2012), the connection between Arrowian social welfare functions and dictatorial rules is no longer one of equivalence when individuals are allowed to be indifferent among alternatives. Some dictatorial rules may not satisfy the conditions of Arrow's theorem. Likewise, it is also possible to construct a dictatorial social welfare function that violates weak pairwise justifiability when the dictator is indifferent between two alternatives.

4 Connections between weak pairwise justifiability and other conditions

Weak pairwise justifiability resembles several properties that have been proposed in the social choice literature. Nevertheless, most of them apply to situation with a fixed agenda and to preference domains that do not admit indifferences. Among the properties that are defined on preference domains admitting individual indifferences (with a fixed agenda), we deem important to start comparing weak pairwise justifiability with Maskin monotonicity.

Maskin monotonicity is a property that plays a fundamental role in the implementation literature (see Maskin 1999). Let $L(a, R_i) = \{x \in A | a R_i x\}$ be the lower contour set of an alternative $a \in A$ for an agent $i \in N$ with a given preference R_i. Let $\mathcal{D} \subseteq \mathcal{R}^n$ be a preference domain.

Definition 9 A social choice function $f : \mathcal{D} \to A$ satisfies **Maskin monotonicity** on $\mathcal{D}$ if given any pair of preference profiles $R, R' \in \mathcal{D}$ and any $a \in A$ we have that $f(R) = a$ implies $f(R') = a$ whenever $L(a, R_i) \subseteq L(R', a)$ for each agent $i \in N$

When indifferences are allowed, Maskin monotonicity implies weak pairwise justifiability but the converse does not hold. Proposition 6, proved in the Appendix, and Example 4 show that our property is weaker than Maskin Monotonicity.

Proposition 6 *Any Maskin monotonic social choice function $f : \mathcal{D} \to A$ satisfies weak pairwise justifiability on $\mathcal{D}$.*

Example 4 Let $N = \{1, 2\}$ and $A = \{x, y, z\}$. There are only two admissible preference profiles and indifferences are allowed: $\mathcal{D} = \{(R_1, R_2), (R'_1, R'_2)\}$ where R_1: $x P_1 y I_1 z$, R'_1: $x I'_1 y P'_1 z$, and $R_2 = R'_2 \in \mathcal{R}$. Let f be such that $f(R) = x$ and $f(R') = y$. It is easy to check that f satisfies weak pairwise justifiability (from R to R', $x P_1 y$ and $y R'_1 x$ while from R' to R, $y I'_1 x$ and $x P_1 y$). However, f violates Maskin monotonicity: for both R and R', all alternatives are worse or indifferent to x but $f(R) = x \neq y = f(R')$.

Sanver (2006) proposed a weakening of Maskin monotonicity named "almost monotonicity".[10] Let $L^*(a, R_i) = \{x \in A | a P_i x\}$ be the strict lower contour set of an alternative a for an agent i with preference R_i.

Definition 10 A social choice function $f : \mathcal{D} \to A$ satisfies **almost monotonicity** on $\mathcal{D}$ if given any pair of preference profiles $R, R' \in \mathcal{D}$ and any $a \in A$ we have that $f(R) = a$ implies $f(R') = a$ whenever $L(a, R_i) \subseteq L(R', a)$ as well as $L^*(a, R_i) \subseteq L^*(R', a)$ for each agent $i \in N$.

Almost monotonicity resembles the weakening of our property compared to Maskin monotonicity, because it admits a change in the alternative selected by the social choice function when an alternative ranked below the chosen alternative becomes indifferent to it. However, our property admits that an alternative a that is selected at a given profile R is not anymore selected at profile R' even if all alternatives in the range of the social choice function did not change their ranking relative to a at R', because an "unfeasible" alternative, not in the range of the social choice function, has changed its ranking relative to a at R' (for instance in case there is a reference alternative that is not feasible).

When only strict preferences are admissible, additional properties have been defined. In the strict universal domain of preferences, weak pairwise justifiability is not only equivalent to strategy-proofness and Maskin monotonicity but also to other well-known properties that have been defined in the literature, like strong positive association (see Muller and Satterthwaite 1977) or strong monotonicity (see Moulin 1988).[11] Among the properties proposed for the strict universal domain, the preference reversal property proposed by Eliaz (2004) is of upmost importance for our setting because he presents a framework that encompasses, like our own, social choice functions and social welfare functions. Preference reversal can be rephrased as follows to facilitate comparison with ours: "If a rule chooses x to be socially better than y in situation 1, and y better than x in situation 2, it must be that at least one member of society prefers x to y in 1 and y to x in 2." Formally, for social choice functions:

Definition 11 Let $\mathcal{D} \subseteq \mathcal{P}^n$. A social choice function $f : \mathcal{D} \to A$ satisfies **preference reversal** on $\mathcal{D}$ if for any pair of preference profiles $R, R' \in \mathcal{D}$ such that $f(R) = x$ and $f(R') = y$, then there must exist one agent $i \in N$ such that $x P_i y$ and $y P'_i x$.

And for social welfare functions:

Definition 12 A social welfare function $F : \mathcal{D} \to \mathcal{R}$ satisfies **preference reversal** if for any pair of preference profiles R and $R' \in \mathcal{D}$ and for any pair of alternatives $x, y \in A$, such that $x F(R) y$ and $y F(R') x$, there is some agent $i \in N$ such that $x P_i y$ and $y P'_i x$.

Eliaz proved his condition imply impossibility if there are at least three alternatives, as we also do. However, our notion of weak pairwise justifiability is strictly weaker.

[10] Sanver (2006) proposed this property for the implementation by awards of collective choice correspondences. We rephrase his property for social choice functions.

[11] See a summary in Sect. 5 in Barberà et al. (2012) and also their Proposition 4 that shows that these equivalences break down when considering smaller domains of strict preferences.

We also enlarge the scope of our analysis to include the case in which individual indifferences are allowed.

It is straightforward to notice that, by definition, preference reversal implies weak pairwise justifiability. However, the converse does not always hold as shown for social choice functions and for social welfare functions in the following example.

Example 5 Let $N = \{1, 2\}$, $\mathcal{B} = \{A\}$ where $A = \{x, y, z, w\}$, and the set of admissible preference profiles is $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, $\mathcal{D}_1 = \mathcal{D}_2 = \{R^1, R^2, R^3\}$ where $xP^1wP^1yP^1z$, $zP^2yP^2wP^2x$, and $yP^3zP^3xP^3w$.[12]
Consider the Borda Count with the tie-breaking $w > z > y > x$ which defines a social welfare function F.[13]
We show that the social choice function f rationalized by F on A violates preference reversal, but it satisfies weak pairwise justifiability. To show the latter, consider $R = (R_1^1, R_2^3)$, and $R' = (R_1^1, R_2^2)$. Observe that $f(R) = y$ and $f(R') = w$. Note that weak pairwise justifiability from R to R' is satisfied because yP_2^3z and zP_2^2y, and from R' to R is satisfied because wP_2^2x and xP_2^3w. A similar argument can be repeated for each pair of preference profiles, which would prove that weak pairwise justifiability holds. To check that f violates preference reversal, note that no agent has changed her preferences between w and y from R to R'.

Now, we show that the social welfare function F violates preference reversal and satisfies weak pairwise justifiability. The score at R of w is 2 while that of y is 4, and therefore, $yF(R)w$. The score at R' of w and y is 3, thus $wF(R')y$. Since no agent has changed her preferences from R to R' between w and y, this is a violation of preference reversal. Note that weak pairwise justifiability from R to R' is satisfied because yP_2^3z and zP_2^2y, and from R' to R is satisfied because xP_2^3w and wP_2^2x. A similar argument can be repeated for each pair of preferences profiles and alternatives, which would prove that weak pairwise justifiability holds. Therefore, the Borda Count applied to each feasible agenda defines a collective choice function that satisfies weak pairwise justifiability.

Appendix

Proposition 1 *Any collective choice correspondence C on $\mathcal{D} \times \mathcal{A}$ satisfying weak pairwise justifiability on $\mathcal{D} \times \mathcal{A}$ is transitively rationalizable on $\mathcal{D} \times \mathcal{A}$.*

Proof of Proposition 1 Let C be a collective choice correspondence on $\mathcal{D} \times \mathcal{A}$ satisfying weak pairwise justifiability on $\mathcal{D} \times \mathcal{A}$. Let $R \in \mathcal{D}$ be any preference profile. Since $\mathcal{A}$

[12] Notice that these preference profiles satisfy one of the three forms of value restriction defined by Sen and Pattanaik (1969), called intermediate. Namely, for any triple of alternatives, there is one that never appears in the second place.

[13] This rule F is defined as follows: for each agent it allocates 3 points to the alternative at the top of the agent's preference order, 2 points to the alternative in the second place, and 1 point to the alternative in the third place. Then, the social welfare function is constructed by ranking alternative a over b when a's total score (summing points for a over all agents) is greater than b's total score, and use the tie-breaking rule when the two scores coincide.

contains every pair of alternatives as a possible agenda, define a binary relation $\mathbf{R}_R$ on A as follows: for any $x, y \in A$, $x\mathbf{P}_R y$ if and only if $C(R, \{x, y\}) = x$ and $x\mathbf{I}_R y$ if and only if $C(R, \{x, y\}) = \{x, y\}$ $(*)$. Note that $\mathbf{R}_R$ is complete. We now show that $\mathbf{R}_R$ transitively rationalizes C, that is, for any agenda $\overline{B} \in \mathcal{A}$, $C(R, \overline{B}) = t(\mathbf{R}_R, \overline{B})$ and $\mathbf{R}_R$ is transitive.

We first show that for any agenda $\overline{B} \in \mathcal{A}$, $C(R, \overline{B}) = t(\mathbf{R}_R, \overline{B})$ (**). Consider any agenda $\overline{B}$ containing at least two alternatives (otherwise, the choice is unique). First, we show $C(R, \overline{B}) \subseteq t(\mathbf{R}_R, \overline{B})$: take any $x \in C(R, \overline{B})$ and show that $x \in t(\mathbf{R}_R, \overline{B})$. Take any $y \in \overline{B}\backslash\{x\}$ such that $\{x, y\} \subseteq \overline{B}$ and observe that the following holds: if $x \in C(R, \overline{B})$ then $x \in C(R, \{x, y\})$ by weak pairwise justifiability since agents' preferences do not change. Thus, $x \in t(\mathbf{R}_R, \{x, y\})$ by $(*)$. Repeating the same argument for all $y \in \overline{B}\backslash\{x\}$, we obtain that $x \in t(\mathbf{R}_R, \overline{B})$.

Second, we prove $t(\mathbf{R}_R, \overline{B}) \subseteq C(R, \overline{B})$: take any $x \in t(\mathbf{R}_R, \overline{B})$ and suppose, by contradiction, that $x \notin C(R, \overline{B})$. Consider $\{x, y\}$ such that $y \in C(R, \overline{B})$ (which always exists). By weak pairwise justifiability, since $y \in C(R, \overline{B})$ then $y \in C(R, \{x, y\})$. Moreover, since $x \notin C(R, \overline{B})$ then $x \notin C(R, \{x, y\})$. By definition of $\mathbf{R}_R$ on pairs of alternatives $(*)$, $y\mathbf{P}_R x$ which is a contradiction to the fact that $x \in t(\mathbf{R}_R, \overline{B})$.

Now, we prove that $\mathbf{R}_R$ transitively rationalizes C, that is $\mathbf{R}_R$ is transitive: take any triple of alternatives $x, y, z \in A$ we have to show that if $x\mathbf{R}_R y$ and $y\mathbf{R}_R z$ then $x\mathbf{R}_R z$. Observe that by definition of $\mathbf{R}_R$ on pairs stated in $(*)$, each one of the three relationships can be written using $C(R, \cdot)$, where $\cdot$ refers to the corresponding pair of compared alternatives being $B = \{x, y\}$, $B'' = \{y, z\}$, or $B' = \{x, z\}$. Distinguish the two cases concerning the choice in B:
(1) $C(R, B) = x$ $(x\mathbf{P}_R y)$ or (2) $C(R, B) = \{x, y\}$ $(x\mathbf{I}_R y)$.
For each one of the two cases we distinguish subcases depending on the choices in B'', that is, $C(R, B'') \in \{\{y\}, \{y, z\}\}$ $(y\mathbf{R}_R z)$. Define $\widetilde{B} = \{x, y, z\}$ and start with case (1):
(1.1) $C(R, B) = x$, $C(R, B'') = y$, then we show that $C(R, B') = x$. Since $y \notin C(R, B)$, $x, y \in B \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B), by weak pairwise justifiability we obtain $y \notin C(R, \widetilde{B})$. Similarly, since $z \notin C(R, B'')$, $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B''), by weak pairwise justifiability we obtain $z \notin C(R, \widetilde{B})$. Thus, $x = C(R, \widetilde{B})$. By (**), $x = t(\mathbf{R}_R, \widetilde{B})$ and thus $x = t(\mathbf{R}_R, B')$ and by $(*)$, $x = C(R, B')$.
(1.2) $C(R, B) = x$, $C(R, B'') = \{y, z\}$, then we show that $C(R, B') = x$. We prove it by contradiction. First, observe that since $y \notin C(R, B)$, $x, y \in B \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B), by weak pairwise justifiability we obtain $y \notin C(R, \widetilde{B})$ and by (**) $y \notin t(\mathbf{R}_R, \widetilde{B})$. Now suppose, by contradiction, that $z \in C(R, B')$. By (**), $z \in t(\mathbf{R}_R, B')$ and since $y \notin t(\mathbf{R}_R, \widetilde{B})$, we obtain that $z \in t(\mathbf{R}_R, \widetilde{B})$. Again by (**), $z \in C(R, \widetilde{B})$. Since $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from (R, B'') to $(R, \widetilde{B})$, by weak pairwise justifiability we obtain $y \in C(R, \widetilde{B})$ which is a contradiction to what we have previously obtained. Therefore, we have proved that $C(R, B') = x$.
We now consider case (2):
(2.1) $C(R, B) = \{x, y\}$, $C(R, B'') = y$, then we show that $C(R, B') = x$. We prove it by contradiction. First, observe that since $z \notin C(R, B'')$, $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B''), by weak pairwise justifiability we

obtain $z \notin C(R, \widetilde{B})$. By (**) $z \notin t(\mathbf{R}_R, \widetilde{B})$. Moreover, since $C(R, B) = \{x, y\}$, we get that $t(\mathbf{R}_R, \widetilde{B}) = \{x, y\}$ and by (**) $C(R, \widetilde{B}) = \{x, y\}$. Now suppose, by contradiction, that $z \in C(R, B')$. Since $x, z \in B' \cap \widetilde{B}$, and agents' preferences do not change from (R, B') to $(R, \widetilde{B})$, by weak pairwise justifiability we obtain $z \in C(R, \widetilde{B})$ which is a contradiction to what we have previously obtained. Therefore, we have proved that $C(R, B') = x$.
(2.2) $C(R, B) = \{x, y\}$, $C(R, B'') = \{y, z\}$, then we show that $C(R, B') = \{x, z\}$. By contradiction, if $C(R, B') = x$ then $z \notin C(R, \widetilde{B})$. If $z \in C(R, \widetilde{B})$, since $x, z \in B' \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B'), by weak pairwise justifiability we obtain $z \in C(R, B')$ which is a contradiction to our hypothesis. Suppose that $y \in C(R, \widetilde{B})$, then since $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from (R, B'') to $(R, \widetilde{B})$, by weak pairwise justifiability we obtain $z \in C(R, \widetilde{B})$ which is a contradiction, thus $y \notin C(R, \widetilde{B})$. Then, $x \in C(R, \widetilde{B})$. Since $x, y \in B \cap \widetilde{B}$, and agents' preferences do not change from (R, B) to $(R, \widetilde{B})$, by weak pairwise justifiability we obtain $y \in C(R, \widetilde{B})$ which is a contradiction. Thus, $C(R, \widetilde{B})$ is not well-defined. A similar argument holds and non-definiteness of $C(R, \widetilde{B})$ would be obtained if we suppose that $C(R, B') = z$. This ends the proof. □

Proposition 2 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. A social choice function $f : \mathcal{D} \to A$ is strategy-proof on $\mathcal{D}$ if and only if it satisfies weak pairwise justifiability on $\mathcal{D}$.*

Proof of Proposition 2 As mentioned in Sect. 2.2, properties on f can be trivially translated as properties on C, and viceversa and we use f and C indistinctly. First, as indicated in Sect. 4, the equivalence for $\mathcal{D} = \mathcal{P}^n$ comes from Muller and Satterthwaite (1977).[14]

We show the "if part" for $\mathcal{D} = \mathcal{R}^n$. By contradiction, suppose that f violates strategy-proofness on $\mathcal{R}^n$, that is, there exist $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $y = f(R'_i, R_{N\backslash\{i\}}) P_i f(R) = x$. Let $\widetilde{R}_i \in \mathcal{R}^n$ be such that (i) the set of alternatives indifferent to x at $\widetilde{R}_i$ is the lower contour set at x of R_i (denoted as $L(R_i, x)$), (ii) $[z \widetilde{P}_i w \iff z P'_i w]$ for all $z \in A \backslash L(R_i, x)$ and $w \in L(R_i, x)$, and (iii) $[z_1 \widetilde{R}_i z_2 \iff z_1 R'_i z_2]$ for all $z_1, z_2 \in A \backslash L(R_i, x)$. Thus, by weak pairwise justifiability from R to $\widetilde{R} = (\widetilde{R}_i, R_{N\backslash\{i\}})$, then $f(\widetilde{R}) = s$ where $s \in L(R_i, x)$. Since condition (1) in weak pairwise justifiability can not hold from $\widetilde{R}$ to $R' = (R'_i, R_{N\backslash\{i\}})$, condition (2) must hold which imposes that $f(R'_i, R_{N\backslash\{i\}}) \in L(R_i, x)$ which is the desired contradiction since $y \notin L(R_i, x)$.

We show the "only if part" for $\mathcal{D} = \mathcal{R}^n$ by contradiction: suppose that f violates weak pairwise justifiability on $\mathcal{R}^n$, that is, there exist two preference profiles $R, R' \in \mathcal{R}^n$ such that $f(R) = x$, $f(R') = y$, $x, y \in A$, and for no agent $i \in N$ and no alternative $z \in A \backslash \{x\}$, $x P_i z$ and $z R'_i x$, and for no agent $i \in M$ where

[14] Let $\mathcal{D} = \mathcal{P}^n$ and by contradiction, suppose that f violates strategy-proofness on $\mathcal{P}^n$, that is, there exist $R \in \mathcal{P}^n$, $i \in N$, and $R'_i \in \mathcal{P}$ such that $y = f(R'_i, R_{N\backslash\{i\}}) P_i f(R) = x$. Define $\widehat{R}_i$ and $\widetilde{R}_i$ such that y is the best alternative and the rest of alternatives are ordered as in R_i and R'_i, respectively. By weak pairwise justifiability from R to $\widehat{R} = (\widehat{R}_i, R_{N\backslash\{i\}})$, $f(\widehat{R}) = x$. Similarly, by weak pairwise justifiability from R to $\widetilde{R} = (\widetilde{R}_i, R_{N\backslash\{i\}})$, $f(\widetilde{R}) = y$. Then, by weak pairwise justifiability from $\widetilde{R}$ to $\widehat{R}$, $f(\widehat{R}) = y$ which is the desired contradiction.

$M = \{i \in N : R_i \neq R'_i\}$, $x I_i y$. Therefore, at R either $x P_i y$ or $y P_i x$ for all agents $i \in M$. Define $\widetilde{R}$ as follows: for each agent $j \in M$ such that $x P_j y$, x is the top in A of $\widetilde{R}$ and y in the second place, and for each agent $k \in M$ such that $y P_k x$, y is the top in A of $\widetilde{R}$ and x in the second place. Start from R and change, one by one, the preference of each agent $j \in M$ such that $x P_j y$ from R_j to $\widetilde{R}_j$. In each step, strategy-proofness implies that the outcome is x (otherwise, agent $j \in M$ would gain by saying R_j instead of $\widetilde{R}_j$). Now, change the preference of each agent $k \in M$ such that $y P_k x$ from R_k to $\widetilde{R}_k$. In each step, strategy-proofness implies that the outcome is x (otherwise, suppose first that at some step the outcome v is neither x nor y. By definition of $\widetilde{R}_k$, $y \widetilde{P}_k x \widetilde{R}_k v$, agent $k \in M$ would gain by saying R_k instead of $\widetilde{R}_k$. If the outcome v was y, agent $k \in M$ would gain by saying $\widetilde{R}_k$ instead of R_k). Thus, $f(\widetilde{R}) = x$. Note that since weak pairwise justifiability is violated, for each agent $j \in M$ such that $x P_j y$, $x P'_j y$, while for each agent $k \in M$ such that $y P_k x$, either $x P'_k y$, $y P'_k x$, or $x I'_k y$. Define $\widehat{R}$ as follows: for each agent $j \in M$ such that $x P_j y$ and $x P'_j y$, x is the top in A of $\widehat{R}$ and y in the second place, for each agent $k \in M$ such that $y P_k x$ and $x P'_k y$, x is the top in A of $\widehat{R}$ and y in the second place, for each agent $k \in M$ such that $y P_k x$ and $y R'_k x$, y is the top in A of $\widehat{R}$ and x in the second place. Start from R' and change, one by one, the preference of each agent $j \in M$ such that $x P_j y$ and $x P'_j y$ from R_j to $\widehat{R}_j$. In each step, strategy-proofness implies that the outcome is y (otherwise, agent $j \in M$ would gain by saying R_j instead of $\widehat{R}_j$ or the converse). Now, change the preference of each agent $k \in M$ such that $y P_k x$ and $x P'_k y$ from R'_k to $\widehat{R}_k$. In each step, strategy-proofness implies that the outcome is y (otherwise, agent $k \in M$ would gain by saying R_k instead of $\widehat{R}_k$, or the converse). Finally, change the preferences of agents $k \in M$ such that $y P_k x$ and $y R'_k x$ from R'_k to $\widehat{R}_k$. In each step, strategy-proofness implies that the outcome is y (otherwise, agent $k \in M$ would gain by saying R'_k instead of $\widehat{R}_k$). Thus, $f(\widehat{R}) = y$. Finally, change the preference of each agent $i \in N$ from $\widetilde{R}_i$ to $\widehat{R}_i$ starting first with type j agents in M. Remember that all such agents have x as top and y as second in both preferences**.** Therefore, strategy-proofness implies that the outcome is x (otherwise, agent $j \in M$ would gain by saying $\widetilde{R}_j$ instead of $\widehat{R}_j$). We now change type k agents in M. Remember that all such agents have y as top and x is second in $\widetilde{R}$ and x and y in the first and second position in $\widehat{R}$. By strategy-proofness, the outcome must be either x or y. Note that if the outcome is y, then this agent k would gain by saying $\widehat{R}_k$ instead of $\widetilde{R}_k$). Therefore, $f(\widehat{R}) = x$ which is the desired contradiction. □

Proposition 3 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. If a social welfare function F on $\mathcal{D}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then F satisfies weak pairwise justifiability.*

Proof of Proposition 3 Let F on $\mathcal{D}$ be a social welfare function satisfying the weak Pareto condition (WP) and independence of irrelevant alternatives (IIA). Suppose, to get a contradiction, that F violates weak pairwise justifiability. That is, there exist two situations $(R, B), (R', B') \in \mathcal{D} \times \mathcal{B}$ and a pair of alternatives $x, z \in B, B'$ such that $x \in C(R, B)$, $x \notin C(R', B')$, $z \in C(R', B')$ and $x, z \in B \cap B'$, that is, $x \succsim_R z$ and $z \succ_{R'} x$ holds. Therefore, we have that $x \succsim_R z$ and $z \succ_{R'} x$. Note that the violation of weak pairwise justifiability requires that there is no agent i and no alternative $y \in A\backslash\{x\}$ such that xP_iy and $yR_i'x$ and also for no agent, xI_iz and $R_i \neq R_i'$ holds. The latter means that for any agent i such that $R_i \neq R_i'$ either xP_iz or zP_ix. By the former, we also have that for those agents i such that xP_iz then $xP_i'z$. Moreover, note that by WP, there must be at least one agent $j \in N$ such that xP_jz (otherwise, $x \notin C(R, B)$). By IIA, there must be an agent changing the relative position between x and z when going from R to R' (otherwise, $x \in C(R', B')$). Since, for all for any $i \in N$ such that xI_iz we have that $R_i = R_i'$ and for any agent i such that xP_iz we have that $xP_i'z$, there must be an agent $k \in N$ such that zP_kx and $xR_k'z$. We now define $\widehat{R}$ as follows: for any $i \in N$ such that xI_iz we have that $x\widehat{I}_iz$ and $z\widehat{P}_iy$, for any agent $j \in N$ such that xP_jz we have $x\widehat{P}_jz$ and $z\widehat{P}_jy$, and for agents $k \in N$ such that zP_kx we have $z\widehat{P}_ky$ and $y\widehat{P}_kx$. By IIA, $x \succsim_{\widehat{R}} z$ and by WP, $z \succ_{\widehat{R}} y$. We now define $\widetilde{R}$ as follows: for any agent $j \in N$ such that $xP_j'z$ we have $x\widetilde{P}_jy$ and $y\widetilde{P}_jz$, for any agent $k \in N$ such that $zR_k'x$, then $y\widetilde{P}_kz$ and $y\widetilde{P}_kx$, and the order between x and z in $\widetilde{R}_k$ is the same as that in R_k'. By IIA, $z \succ_{\widetilde{R}} x$ and by WP, $y \succ_{\widetilde{R}} z$. Since $x \succ_{\widehat{R}} y$ and $y \succ_{\widetilde{R}} x$, but when going from $\widehat{R}$ to $\widetilde{R}$ no agent has changed the relative position between x and y we get a contradiction to IIA. □

Proposition 4 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. If a social welfare function F on $\mathcal{D}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then F satisfies weak decisiveness.*

Proof of Proposition 4 Let F on $\mathcal{D}$ be a social welfare function satisfying the weak Pareto condition (WP) and independence of irrelevant alternatives (IIA). Suppose, to get a contradiction, that F violates weak decisiveness. Therefore, for some situation (R, B) and some $x, z \in B$, $x \sim_R z$ and there is no agent i such that xI_iz. This means that for any agent i either xP_iz or zP_ix. We now define $\widehat{R}$ as follows: for any agent $j \in N$ such that xP_jz we have $x\widehat{P}_jy$ and $y\widehat{P}_jz$, and for agent $k \in N$ such that zP_kx we have $y\widehat{P}_kz$ and $z\widehat{P}_kx$. By IIA, $x \sim_{\widehat{R}} z$ and by WP, $y \succ_{\widehat{R}} z$. We now define $\widetilde{R}$ as follows: for any agent $j \in N$ such that xP_jz we have $z\widetilde{P}_jy$ and $x\widetilde{P}_jz$, and for agent $k \in N$ such that zP_kx, we have $z\widetilde{P}_ky$ and $y\widetilde{P}_kx$. By IIA, $z \sim_{\widetilde{R}} x$ and by WP, $z \succ_{\widetilde{R}} y$. Since $y \succ_{\widehat{R}} x$ and $x \succ_{\widetilde{R}} y$, but when going from $\widehat{R}$ to $\widetilde{R}$ no agent has changed the relative position between x and y we get a contradiction to IIA. □

Proposition 5 *Let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. If a social welfare function F on $\mathcal{D}$ satisfies the weak Pareto condition, then F satisfies full range.*

Proof of Proposition 5 Take $B \in \mathcal{A}$, $x \in B$ and define $R \in \mathcal{P}^n \subset \mathcal{R}^n$ such that for all $i \in N$, $t(R_i, A) = x$. By the weak Pareto condition, $x \succ_R y$ for all $y \in B\backslash\{x\}$. Thus the corresponding collective choice correspondence is such that $C(R, B) = x$ which shows full range. □

Theorem 3 *Let $A \in \mathcal{B}$ and let $\mathcal{D} = \mathcal{P}^n$ or $\mathcal{D} = \mathcal{R}^n$. There is no full range collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ satisfying weak decisiveness, weak pairwise justifiability and non-dictatorship.*

Proof of Theorem 3 We distinguish two parts.

Part I. $\mathcal{D} = \mathcal{P}^n$. In this case, a collective choice correspondence satisfying weak decisiveness is a collective choice function. By contradiction, suppose that there exists a full range collective choice function satisfying weak pairwise justifiability and non-dictatorship on $\mathcal{P}^n \times \mathcal{B}$. We proceed by proving that any full range collective choice function satisfying weak pairwise justifiability is dictatorial which is the desired contradiction.

We divide the proof in two steps. In the first step we fix an agenda with at least three alternatives (which exists since $A \in \mathcal{B}$) and show that our property implies the existence of a dictator on such fixed agenda. In the second step we show that this dictator is the same for all possible agendas.

Step 1: Let $B \in \mathcal{B}$. For any full range collective choice function C on $\mathcal{P}^n \times \{B\}$ satisfying weak pairwise justifiability on $\mathcal{P}^n \times \{B\}$ where $\#B \geq 3$, there is an agent that is a dictator on $\mathcal{P}^n \times \{B\}$.

Since this statement refers to the case where individual preferences are strict, we only need to use the first part of Definition 1 of weak pairwise justifiability.

In this step we concentrate on the consequences of weak pairwise stability when individual preferences change but the agenda remains the same.

Claim 1. Let R be a preference profile where alternative x is the top of R_i in B for each agent i. Then, $C(R, B) = x$.

Proof of Claim 1: Since all alternatives in B are in the range of C, where C is a full range collective choice function, then there exists a profile $\widetilde{R}$ for which $C(\widetilde{R}, B) = x$. Consider now the profile $\widehat{R}$ where all agents place x at their top in A, thus also in B, while keeping the same ordering among the rest of alternatives as in $\widetilde{R}$. No alternative in A has improved, for no agent, its position relative to x when society's profile changes from $\widetilde{R}$ to $\widehat{R}$, hence $C(\widehat{R}, B) = x$ by weak pairwise justifiability. Since in all profiles where all agents have x as their top alternative in A, no alternative in A has improved, for no agent, its position relative to x, again by weak pairwise justifiability, the proof is complete.

Claim 2. Let R be a preference profile where all agents have either z or w as their top alternative in B. Then, the choice at this profile in B must be either z or w.

Proof of Claim 2: Let J be the set of agents whose top in B is z and K be the set of those whose top in B is w at profile R, and assume that $C(R, B) = x \notin \{z, w\}$. Consider now the profile $\widehat{R}$ where all agents in J place z at their top in A and all agents in K place w at their top in A, while keeping the same ordering among the rest of alternatives as in R. Since no alternative in A has improved, for no agent, its

position relative to $C(R, B)$, by weak pairwise justifiability, $C(\widehat{R}, B) = x$. Now, we shall arrive at a contradiction through several assertions.
(2.1) It cannot be that $z\widehat{R}_j w\widehat{R}_j x$ for all $j \in J$, nor that $w\widehat{R}_k z\widehat{R}_k x$ for all $k \in K$.
Suppose that $z\widehat{R}_j w\widehat{R}_j x$ for all $j \in J$. Consider the profile $\widetilde{R}$ where all agents in J have w as their top in B, keeping the rest of their ranking unchanged as in $\widehat{R}$, and agents in K have not changed preferences. Then, the choice at profile $\widetilde{R}$ in B should be w by Claim 1, because w is the top in B for all agents in $\widetilde{R}$. Yet, no alternative has improved, for no agent, its position relative to x when society's profile changes from $\widehat{R}$ to $\widetilde{R}$, hence $C(\widetilde{R}, B) = x$ by weak pairwise justifiability. A contradiction. By a similar argument, it cannot be the case that $w\widehat{R}_k z\widehat{R}_k x$ for all $k \in K$.
(2.2) Now, consider the partition of profile $\widehat{R}$ into four sets of preferences, corresponding to agents whose preferences rank x, z, w as follows taking into account that, without loss of generality, by weak pairwise justifiability, we can assume that x is ranked as the alternative in the second place in A in those profiles for J where it is above w, and in those for K where it is above z.
J_1 are agents who rank z as the top, followed by x as the second alternative in A (thus, in B).
K_1 are agents who rank w as the top, followed by x as the second alternative in A (thus, in B).
$J\backslash J_1$ is the set of those agents i for whom z is the top and $w\widehat{R}_i x$.
$K\backslash K_1$ is the set of those agents i for whom w is the top and $z\widehat{R}_i x$.
If J_1 or K_1 are empty, we would be in Claim (2.1). Our starting assumption is that $C(\widehat{R}, B) = C(\widehat{R}_{J_1}, \widehat{R}_{K_1}, \widehat{R}_{J\backslash J_1}, \widehat{R}_{K\backslash K_1}, B) = x$.
We shall now consider the possible choices in B under several profiles. In all of them, the preferences of $J\backslash J_1$ and $K\backslash K_1$ remain unchanged.
Let R' be such that, all the rest being unchanged with respect to $\widehat{R}$, agents in J_1 have w as the alternative in the second place, between z and x.
Let R'' be such that, all the rest being unchanged with respect to $\widehat{R}$, agents in K_1 have z as the alternative in the second place, between w and x.
Let R''' be such that both agents in J_1 and K_1 have changed in the way described when defining R' and R''. That is, those in J_1, have w as the alternative in the second place, between z and x, and those in K_1, have z as the alternative in the second place, between w and x.
Remark that, $C(R', B) \neq x$, by the argument we used in (2.1), and that w is the only alternative whose ranking has improved over some alternative and for some agent from $\widehat{R}$ to R' (equivalently, w is the unique alternative that gets worse from R' to $\widehat{R}$ for some agent). Hence, by weak pairwise justifiability $C(R', B) = w$. For the same reasons, it must be that $C(R'', B) = z$. But then, $C(R''', B)$ must be w, because passing from R' to R''', the relationship between w and all other alternatives has not changed for any agent. And, for the same reasons, passing from R'' to R''', $C(R''', B)$ must be z. Since C is a collective choice function, that is a contradiction.
Remark Before we develop our third claim, let us introduce some notation and definitions that will be useful. For $B' \subseteq B$, we will say that an *agent* i is B'*-determinant at profile* R if and only if $C\left((R'_i, R_{N\backslash\{i\}}), B'\right) = t(R'_i, B')$ for all R'_i. Also remark that if i is B'-determinant at R, it is also B'-determinant at all profiles $\widetilde{R}$ such that

$\widetilde{R}_j = R_j$ for all $j \neq i$.
Given Claims 1 and 2, there will exist a profile where agents' preferences have alternative z and w as the only top in B and one of the agents is (z, w)-determinant.

Claim 3. If agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$ where all agents in J have z as the top in B, and all those in K have w as the top in B, then agent i is (z, w)-determinant at any profile $R' = (R_i, R_J, R'_K)$ where all agents in K have w as the top in B and $x \in A \backslash \{w\}$ as the alternative in the second place in B.

Proof of Claim 3: By weak pairwise justifiability, without loss of generality, we can assume that agent i is (z, w) -determinant at a profile $R = (R_i, R_J, R_K)$ where agents in J have z as the top in A, and those in K have w as the top in A. Since i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$, by weak pairwise justifiability, $C\left((R_i^w, R_J, R_K), B\right) = w$ where R_i^w is such that w is the top and z as the alternative in the second place in A. Again, by weak pairwise justifiability, $C\left((R_i^w, R_J, R'_K), B\right) = w$ where R'_K is such that w is the top and x as the alternative in the second place in A. By Claim 2, $C\left((R_i^z, R_J, R'_K), B\right) \in \{z, w\}$ where R_i^z is such that z is the top and w as the alternative in the second place in A. If $C\left((R_i^z, R_J, R'_K), B\right) = w$, by weak pairwise justifiability, $C\left((R_i^z, R_J, R_K), B\right) = w$ which is a contradiction to agent i being (z, w)-determinant at (R_i, R_J, R_K).

Claim 4. If agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R'_K)$ where agents in J have z as the top in B, and those in K have w as the top in B and $x \in A \backslash \{w\}$ as the alternative in the second place in B, then agent i is also (x, z, w)-determinant at profile $R = (R_i, R_J, R'_K)$.

Proof of Claim 4: We show that agent i is (x, z, w)-determinant at profile $R = (R_i, R_J, R'_K)$.
We first prove that $C\left((R'_i, R_J, R'_K), B\right) = x$ where R'_i ranks x as the first, z as the second in B. Suppose not. By Claim 3, agent i is (z, w)-determinant at (R_i, R_J, R'_K). Then, $C\left((R_i^z, R_J, R'_K), B\right) = z$, where R_i^z ranks z as the first, x as the second in B, and the relative order of the rest of alternatives in A as in R'_i. Note that if $C\left((R'_i, R_J, R'_K), B\right)$ was $y \neq z$, since y has the same relative order with respect to all alternatives in R_i^z and in R'_i, by weak pairwise justifiability, $C\left((R_i^z, R_J, R'_K), B\right) = y$ which is a contradiction to what we obtained above. Then, $C\left((R'_i, R_J, R'_K), B\right) = z$. For the same reason, when R''_i ranks x as the first, w as the second in B, and the rest of alternatives in A as in R'_i, it should be that $C\left((R''_i, R_J, R'_K), B\right) = w$.
Now consider the profile where agents in K switch the positions of x and w, and the relative order of the rest of alternatives in A as in R'_K, so that x is the top in B for all of them. At that new profile (R''_i, R_J, R''_K), the only two alternatives in top positions in B are z and x. Hence, by Claim 2, the choice must be either z or x. But z cannot be, because this would violate weak pairwise justifiability, because z has not improved for any alternative by any agent. Thus, $C\left((R''_i, R_J, R''_K), B\right) = x$.
Now, starting from this last profile, consider the one, say R'''_i, where i changes preferences so that z becomes the second to x in B, and the relative order of the rest of alternatives in A as in R''_i. Again, by weak pairwise justifiability, $C\left((R'''_i, R_J, R''_K), B\right) = x$ since no alternative and for no agent has improved rel-

ative to x. Finally, let agents in K change preferences to rank w as the first, x as the second, and the relative order of the rest of alternatives in A as in R''_K, say R'''_K. By weak pairwise justifiability, the choice cannot be z, and yet (R'''_i, R_J, R'''_K) is the same profile we start with, where z was to be chosen. This contradiction proves that $C\left((R'_i, R_J, R'_K), B\right) = x$.
We now show that for any R_i whose top is $x \in B$, then $C\left((R_i, R_J, R'_K), B\right) = x$. Suppose not. We have shown that $C\left((R'_i, R_J, R'_K), B\right) = x$ where R'_i ranks x the first, z the second in B. By weak pairwise justifiability, from (R'_i, R_J, R'_K) to (R''_i, R_J, R'_K) with x as the top in A and the relative order of the rest of alternatives in A as in R_i, then $C\left((R''_i, R_J, R'_K), B\right) = x$. Again, by weak pairwise justifiability, from (R_i, R_J, R'_K) to (R''_i, R_J, R'_K) we have that $C\left((R''_i, R_J, R'_K), B\right) \neq x$ which is a contradiction.

Claim 5. If an agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$ where all agents in J have z as the top in B, and all those in K have w as the top in B, then this agent i is B -determinant at that profile.
Proof of Claim 5: Suppose not. That is, there exists $\widehat{R}_i$ with x as the top alternative in B such that $C\left((\widehat{R}_i, R_J, R_K), B\right) \neq x$.
Let R'_i with x as the first and w as the second in B and the relative order of the rest of alternatives in A is as in $\widehat{R}_i$. By the same argument as the one at the end of Claim 4, since x is the top in B of $\widehat{R}_i$ and R'_i, $C\left((R'_i, R_J, R_K), B\right) \neq x$.
Since agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$, then $C\left((R^w_i, R_J, R_K), B\right) = w$, where R^w_i ranks w as the first, x as the second in B, and the relative order of the rest of alternatives in A as in $\widehat{R}_i$. If $C\left((R'_i, R_J, R_K), B\right)$ was $y \neq w$, since y has the same relative order with respect to all alternatives in R^w_i and in R'_i, by weak pairwise justifiability, $C\left((R^w_i, R_J, R_K), B\right) = y$ which is a contradiction to agent i being (z, w)-determinant at (R'_i, R_J, R_K). Thus, $C\left((R'_i, R_J, R_K), B\right) = w$. Let R'_K where all agents in K have w as the top in B and $x \in A\backslash\{w\}$ as the alternative in the second place in B, and the relative order of the rest of alternatives is as in R_K. Since w has the same relative order with respect to all alternatives and all agents, by weak pairwise justifiability from (R'_i, R_J, R_K) to (R'_i, R_J, R'_K), we have $C\left((R'_i, R_J, R'_K), B\right) = w$. But, this is a contradiction since by Claims 3 and 4, agent i is also (x, z, w)-determinant at profile $R = (R_i, R_J, R'_K)$ and thus $C\left((R'_i, R_J, R'_K), B\right) = x$.
Therefore, $C\left((R'_i, R_J, R_K), B\right) = C\left((\widehat{R}_i, R_J, R_K), B\right) = x$.

Claim 6. If an agent i is B-determinant at a profile $R = (R_i, R_J, R_K)$ where all agents in J have z as the top in B, and all those in K have w as the top in B, then i is B -determinant at all profiles.

Proof of Claim 6: We first show that agent i is B-determinant at all profiles. Consider R in the statement where agent i is B-determinant. Thus, i is also (z, w)-determinant at R. Change the preferences of all agents for $y \in B\backslash\{z, w\}$ so that y is the worst alternative, keeping the relative ordering of the rest of the alternatives. The choice is either z or w by Claim 2, depending on agent i's preferences. Then, by Claim 5, the modified profile still leaves i as being B-determinant. Now let y become the top alternative in B for i. The choice will be y, even if it is worse for all other agents since agent i is B-determinant at the modified profile. By weak pairwise justifiability, all profiles where y is the top in B for agent i gives y. This argument can be repeated for all alternatives y in B. Thus, agent i is B-determinant at all profiles.

Claim 7. For any $B \in \mathcal{B}$, there is an agent that is B-determinant at all profiles in $\mathcal{P}^n$. Thus, there is an agent that is a dictator on B at all profiles in $\mathcal{P}^n$.

Claim 7 follows from all previous claims. This ends the proof of Step 1.

Step 2. For any $B, B' \in \mathcal{B}$ such that $B' \neq B$ there exists an agent who is both a dictator on $\mathcal{P}^n \times \{B\}$ and on $\mathcal{P}^n \times \{B'\}$.

Take any two sets $B, B' \in \mathcal{B}$ such that $B' \neq B$. Let $B'' \in \mathcal{B}$ with at least three alternatives and such that $B \cup B' \subseteq B''$ (note that B'' exists since $A \in \mathcal{B}$). By Step 1 there is an agent, say i, who is a dictator on B''. We show that i is also a dictator on B and B'. Note that this is straightforward if an agenda B has only one alternative. Consider two cases.

Case 1: B and B' have both only 2 alternatives.

Take one of them, without loss of generality, $B = \{z, w\}$. Suppose, to get a contradiction, that agent i is not a dictator on B.

Subcase 1.1. There is a dictator on B, say agent $1 \neq i$. Let $R \in \mathcal{P}^n$ be such that z is the top of R_1 in A, w is the top of R_i in A, and any preference for other agents. Note that $C(R, B) = t(R_1, B) = z$ and $C(R, B'') = t(R_i, B'') = w$. By weak pairwise justifiability, from (R, B) to (R, B'') the choice must be the same for the two situations since agents' preferences do not change. Then, the dictator must be the same on B and on B'' which is the contradiction.

Subcase 1.2. There is no dictator on B. Let $R \in \mathcal{P}^n$ be such that for each agent $j \in N \backslash \{i\}$, $t(R_j, A) = z$ and $t(R_i, A) = w$. If $C(R, B) = w$ and since by Claim 1 in the proof of Step 1, when all agents have z as top, the choice is z, then agent i is (z, w)-determinant at R. By Claim 6 in the proof of Step 1, agent i is (z, w)-determinant at all profiles, meaning that i is a dictator on B, which is a contradiction. Therefore, $C(R, B) = z$. Since agent i is a dictator on B'', $C(R, B'') = w$. By weak pairwise justifiability, from (R, B) to (R, B'') the choice must be the same for the two situations, which is the contradiction.

Therefore, agent i must be the dictator on B.

Case 2: B and B' where at least one of them has three or more alternatives.

Suppose, without loss of generality, that B' has at least three alternatives. We first show that i is a dictator on B'.

By Step 1 there is an agent, say 1, who is a dictator on B'. Suppose to get a contradiction that $i \neq 1$. Let $z \in B' \cap B''$ be such that $z \neq w$. Let $R \in \mathcal{P}^n$ be such that z is the top of R_1 in A, w is the top of R_i in A, and any preference for other agents. Note that $C(R, B') = t(R_1, B') = z$, $C(R, B'') = t(R_i, B'') = w$. By weak pairwise justifiability, from (R, B') to (R, B'') the choice must be the same for the two situations. Then, the dictator must be the same on B' and on B'' which is the contradiction.

We now show that i is also a dictator on B.

If $\#B \geq 3$, repeat the same argument as for B'. Otherwise, if $\#B = 2$, repeat the same argument as in Case 1.

This ends the proof of Part I.

Part II. $\mathcal{D} = \mathcal{R}^n$. By contradiction, suppose that there exists a full range collective choice correspondence satisfying weak decisiveness, weak pairwise justifiability and non-dictatorship on $\mathcal{R}^n \times \mathcal{B}$. The next two lemmas will help to develop the proof for the case in which agents have indifferences.

Lemma 1 *If C on $\mathcal{R}^n \times \{B\}$, $\#B \geq 3$, is a full range collective choice correspondence satisfying weak decisiveness and weak pairwise justifiability on $\mathcal{R}^n \times \{B\}$ then C is dictatorial on $\mathcal{P}^n \times \{B\}$.*

Proof of Lemma 1 We first show that C has full range on $\mathcal{P}^n \times \{B\}$: for any $x \in B$ there exists $R \in \mathcal{P}^n$ such that $x \in C(R, B)$. Since all alternatives in B are in the range of C, there exists a profile $\widetilde{R} \in \mathcal{R}^n$ for which $x \in C(\widetilde{R}, B)$. Consider now a profile $\widehat{R} \in \mathcal{P}^n$ where all agents place x as the top in A, thus also in B. When agents' preferences change from $\widetilde{R}$ to $\widehat{R}$, no alternative in A has improved, for no agent, its position relative to x. Then, by weak pairwise justifiability, $x \in C(\widehat{R}, B)$. This shows that C has full range on $\mathcal{P}^n \times \{B\}$. Now, note first that weak pairwise justifiability is trivially translated to $\mathcal{P}^n \times \{B\}$. Moreover, weak decisiveness of C on $\mathcal{R}^n \times \{B\}$ implies that C is a collective choice funcion on $\mathcal{P}^n \times \{B\}$. Then, apply the result of Theorem 3 for strict preferences and obtain that C is dictatorial on $\mathcal{P}^n \times \{B\}$, which ends the proof of Lemma 1.

Lemma 2 *Let C be a collective choice correspondence satisfying weak pairwise justifiability on $\mathcal{R}^n \times \{B\}$. If there is a dictator i for the restriction of C to $\mathcal{P}^n \times \{B\}$, then C is dictatorial and i is the dictator on $\mathcal{R}^n \times \{B\}$.*

Proof of Lemma 2 Let i be a dictator for the restriction of C to $\mathcal{P}^n \times \{B\}$. Take $R \in \mathcal{P}^n$ such that $t(R_i, B) = x$ and x is the worst alternative for the rest of the agents. Since i is the dictator on $\mathcal{P}^n \times \{B\}$, then $C(R, B) = x$. Now, let $R' \in \mathcal{R}^n$ be any preference profile such that each agent's preferences keep the same relative order of x with respect to the rest of alternatives as in R, that is, x is the unique top alternative of i while x is the unique worst alternative for the rest of agents.[15] From R to R' no alternative in A has improved, for no agent, its position relative to x. Then, by weak pairwise justifiability, $x \in C(R', B)$. We now show that $C(R', B) = x$. By contradiction, let $y \in C(R', B)\backslash\{x\}$. Construct $R'' \in \mathcal{P}^n$ such that $t(R''_i, B) = x$, $y P''_i z$ for any $z \in A\backslash\{x, y\}$, and y is the unique top alternative while x is the worst alternative for the rest of the agents. When agents' preferences change from R' to R'', no alternative in A has improved, for no agent, its position relative to y. Hence, by weak pairwise justifiability, $y \in C(R'', B)$. Since i is a dictator on $\mathcal{P}^n \times \{B\}$, $C(R'', B) = x$ which is a contradiction. Thus, $C(R', B) = x$.

Now change the preferences of all agents different from i to any preference in $\mathcal{R}$, say $\widehat{R}$. Again, by weak pairwise justifiability from R' to $\widehat{R}$, the choice is still x since no alternative is weakly worse than x under R'_j for no agent different from i. We have shown that agent i is a dictator if her top is unique.

Finally, suppose now to get a contradiction that agent i is not a dictator when her

[15] Note that agents may be indifferent among alternatives different from x.

top is not unique. Then, there exists $\overline{R} \in \mathcal{R}^n$ such that $y \in C(\overline{R}, B)$ but $y \notin t(\overline{R}_i, B)$. Suppose, without loss of generality, that $x \in t(\overline{R}_i, B)$. Let R_i''' be such that x is the unique top alternative keeping the relative order of the rest of alternatives. When agents' preferences change from $\overline{R}$ to $(R_i''', \overline{R}_{N\setminus\{i\}})$, since no alternative in A has improved, for no agent, its position relative to y, by weak pairwise justifiability we have that $y \in C((R_i''', \overline{R}_{N\setminus\{i\}}), B)$. Given that agent i is the dictator when she has a unique top alternative, we get $C((R_i''', \overline{R}_{N\setminus\{i\}}), B) = x$ which is a contradiction. This completes the proof of Lemma 2.

We proceed by proving that any full range collective choice correspondence satisfying weak decisiveness and weak pairwise justifiability is dictatorial which is the desired contradiction. Given $\mathcal{B}$, we obtain that for any $B \in \mathcal{B}$, C has full range on $\mathcal{P}^n \times \{B\}$ by Lemma 1. Thus, C has full range on $\mathcal{P}^n \times \mathcal{B}$. Since weak pairwise justifiability and weak decisiveness are inherited in subsets of preference profiles, then C satisfies weak pairwise justifiability and weak decisiveness on $\mathcal{P}^n \times \mathcal{B}$. Then, we apply Part I and obtain that there is a dictator i for the restriction of C to $\mathcal{P}^n \times \mathcal{B}$. Finally observe that by Lemma 2 applied to any $B \in \mathcal{B}$, C is dictatorial and i is the dictator on $\mathcal{R}^n \times \mathcal{B}$. This ends the proof of Part II. □

Proposition 6 *Any social choice function $f : \mathcal{D} \to A$ satisfying Maskin monotonicity on $\mathcal{D}$ satisfies weak pairwise justifiability on $\mathcal{D}$.*

Proof of Proposition 6 By contradiction, if f violates weak pairwise justifiability on $\mathcal{D}$, there exist two preference profiles $R, R' \in \mathcal{D}$ such that $f(R) = x$, $f(R') = y$, $x, y \in A$, and for any $i \in N$ and any alternative $z \in A\setminus\{x\}$, either (1) $x P_i z$ and $x P_i' z$, or (2) $z P_i x$ and $z R_i' x$, or (3) $z P_i x$ and $x P_i' z$, or (4) $y I_i x$ and $R_i = R_i'$ holds. Start from R and change the preference of all agents R_i to R_i'. Note that by (1), (2), (3) and (4), for each agent $i \in N$, $\left[f(R) R_i z \Rightarrow f(R) R_i' z\right]$ thus, by Maskin monotonicity $f(R') = f(R)$ which is the desired contradiction. □

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest Authors have no conflict of interest to declare.

References

Akbarpour M, Nariman S (2016) A new interpretation of dictatorship with applications in social choice theory. Mimeo
Arrow K (1963) Social choice and individual values, 2nd edn. Wiley, New York (**1st edn, 1951**)
Barberà S (2001) A theorem on preference aggregation. Mimeo, Universitat Autònoma de Barcelona
Barberà S, Berga D, Moreno B (2012) Two necessary conditions for strategy-proofness: on what domains are they also sufficient? Games Econ Behav 75:490–509
Barberà S, Berga D, Moreno B, Nicolò A (2024) Condorcet consistency and pairwise justifiability under variable agendas. Int Econ Rev. https://doi.org/10.1111/iere.12728
Eliaz K (2004) Social aggregators. Soc Choice Welf 22(2):317–330
Fishburn PC (1973) The theory of social choice. Collections: Princeton Legacy Library
Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41:587–601
Koray S (2000) Self-selective social choice functions verify Arrow and Gibbard–Satterthwaite theorems. Econometrica 68:981–995
Le Breton M, Weymark J (2011) Arrovian social choice theory on economic domains. Handbook of social choice and welfare, vol II. Elsevier, Amsterdam
Man PTY, Takayama S (2013) A unifying impossibility theorem. Econ Theor 54:249–271
Maskin E (1999) Nash equilibrium and welfare optimality. Rev Econ Stud 66:23–38
Mossel E, Tamuz O (2012) Complete characterization of functions satisfying the conditions of Arrow's theorem. Soc Choice Welf 39:127–140
Moulin H (1988) Axioms of cooperative decision making, vol 15. Econometric society monographs. Cambridge University Press, Cambridge
Muller E, Satterthwaite MA (1977) The equivalence of strong positive association and strategy-proofness. J Econ Theory 14:412–418
Pattanaik P (1978) Strategy and group choice. North-Holland Publishing Co, Amsterdam, New York
Reny P (2001) Arrow's theorem and the Gibbard–Satterthwaite theorem: a unified approach. Econ Lett 70:99–105
Sanver R (2006) Nash implementing non-monotonic social choice rules by awards. Econ Theor 28:453–460
Satterthwaite M (1973) Manipulation of voting schemes: a general result. Econometrica 41:587–601
Sen AK, Pattanaik PK (1969) Necessary and sufficient conditions for rational choice under majority decision. J Econ Theory 1:178–202
Vickrey W (1960) Utility, strategy, and social decision rules. Q J Econ 74:507–535

Authors and Affiliations

Salvador Barberà [1,2] · Dolors Berga [3] · Bernardo Moreno [4] · Antonio Nicolò [5,6]

✉ Dolors Berga
dolors.berga@udg.edu

Salvador Barberà
salvador.barbera@uab.cat

Bernardo Moreno
bernardo@uma.es

Antonio Nicolò
antonio.nicolo@unipd.it

1 MOVE, Universitat Autònoma de Barcelona, Barcelona, Spain

[2] Barcelona SE, Barcelona, Spain
[3] Departament d'Economia, Universitat de Girona, Girona, Spain
[4] Departamento de Teoría e Historia Económica, Universidad de Málaga, Málaga, Spain
[5] University of Padova, Padova, Italy
[6] University of Manchester, Manchester, UK