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TROPICAL MATRIX GROUPS

ZUR IZHAKIAN¹, MARIANNE JOHNSON² and MARK KAMBITES³

ABSTRACT. We study the structure of groups of finitary tropical matrices under multiplication. We show that the maximal groups of $n \times n$ tropical matrices are precisely the groups of the form $G \times \mathbb{R}$ where G is a group admitting a 2-closed permutation representation on n points. Each such maximal group is also naturally isomorphic to the full linear automorphism group of a related tropical polytope. Our results have numerous corollaries, including the fact that every automorphism of a projective (as a module) tropical polytope of full rank extends to an automorphism of the containing space.

1. INTRODUCTION

Tropical algebra is the algebra of the real numbers (sometimes augmented with an extra element denoted by $-\infty$) under the operations of addition and maximum. It has applications in areas such as combinatorial optimisation and scheduling, control theory, and algebraic geometry to name but a few (see [2] for a survey of applications). Many problems arising from these application areas are naturally expressed using (tropical) linear equations, so much of tropical algebra concerns matrices.

In this paper, we study the abstract structure of the semigroup \mathcal{M}_n of real $n \times n$ square matrices under the operation of *tropical matrix multiplication*. An important step in understanding tropical algebra is to understand the maximal subgroups of this semigroup (that is, the maximal groups of tropical matrices), in terms of both their abstract group structure and the geometry of their natural actions on tropical space. It is a basic fact of semigroup theory that every subgroup of a semigroup S lies in a unique maximal subgroup; moreover, the maximal subgroups of S are precisely the \mathcal{H} -classes of S which contain an idempotent element (see, for example, [11]). Recent research of the authors into the structure of (tropically) idempotent matrices [12] provides a useful basis for studying maximal subgroups of \mathcal{M}_n using the tools of semigroup theory. The main result of the present paper is that the maximal subgroups of \mathcal{M}_n are, up to isomorphism, exactly the groups $G \times \mathbb{R}$ where G is a finite group with a 2-closed permutation representation on n points. There are a number of significant open questions about 2-closed permutation groups. Most notably, the *polycirculant conjecture* of Marušić, Jordan and Klin, which is important in graph theory, asserts that every transitive, non-trivial 2-closed permutation group has a fixed-point-free element of prime order (see for example [4, Section 6.6] or [5]). In light of our results, it seems possible that tropical methods may have a role to play in studying such groups.

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Our results are related to (although obtained independently of) work of Shitov [19], which established a conjecture of the second and third authors [13] by proving that every subgroup of tropical matrices has a torsion free abelian subgroup of index $n!$ or less. More precisely, Shitov showed that every group of tropical matrices (with a zero element $-\infty$ admitted) is isomorphic to a group of tropical monomial matrices, and hence embeds into the permutation wreath product $\mathbb{R} \wr S_n$. Here, by contrast, we obtain an *exact* description of the groups which arise as maximal subgroups of tropical matrices, but in the slightly more restricted setting of the finitary tropical semiring without $-\infty$. We believe the methods developed here should ultimately suffice for a comparable understanding of the case with $-\infty$, but this is contingent upon the extension to this case of some preliminary results upon which we rely, which have so far been established only in the finitary case.

In addition to this introduction, this article comprises five sections. In Section 2 we introduce some preliminary definitions. Section 3 gives a brief account of Green's relations for the semigroup \mathcal{M}_n and proves that every maximal subgroup H is isomorphic to the automorphism group of a particular tropical polytope (namely, the column space of any element of H). In Section 4 we summarise a number of results on (tropically) idempotent matrices which, combined with the results of the previous section, reduce the problem of understanding maximal subgroups to the full rank case. Finally, Section 5 establishes our strongest results. We shall exhibit a natural and canonical embedding of each full rank maximal subgroup (and hence of each subgroup) of \mathcal{M}_n into the group of units of $M_n(\mathbb{T})$. An analysis of this embedding allows us to give an exact description of the maximal subgroups. Other corollaries of our results include the fact that every automorphism of a projective (as a module) tropical polytope of maximal dimension extends to an automorphism of the containing space, and that every group of full rank tropical matrices has a common eigenvector.

2. PRELIMINARIES

We write \mathbb{FT} for the set \mathbb{R} equipped with the operations of maximum (denoted by \oplus) and addition (denoted by \otimes , by $+$ or simply by juxtaposition). Thus, we write $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. It is readily verified that \mathbb{FT} is an abelian group (with neutral element 0) under \otimes and a commutative semigroup of idempotents (without a neutral element) under \oplus , and that \otimes distributes over \oplus . These properties mean \mathbb{FT} has the structure of an *idempotent semifield*.

It will sometimes be convenient to work with the extended tropical semifield $\mathbb{T} = \mathbb{FT} \cup \{-\infty\}$, where we extend the definitions of \oplus and \otimes in the obvious way (namely, $a \oplus -\infty = -\infty \oplus a = a$ and $a \otimes -\infty = -\infty \otimes a = -\infty$, for all $a \in \mathbb{T}$).

Let $M_n(\mathbb{FT})$ denote the set of all $n \times n$ matrices with entries in \mathbb{FT} . The operations \oplus and \otimes induce corresponding operations on $M_n(\mathbb{FT})$ in the usual way. We write $A^{\otimes n}$ for the n th power of a square matrix A under tropical matrix multiplication (denoted \otimes). Tropical matrix multiplication gives $M_n(\mathbb{FT})$ the structure of a semigroup. We shall see in the following sections that this semigroup has a rich and interesting structure.

We shall be interested in the space \mathbb{FT}^n consisting of n -tuples x with entries in \mathbb{FT} ; we write x_i for the i th component of x . We call \mathbb{FT}^n (*affine*) *tropical n -space*. The space \mathbb{FT}^n admits an addition and a scaling action of \mathbb{FT} given by $(x \oplus y)_i = x_i \oplus y_i$ and $(\lambda \otimes x)_i = \lambda \otimes (x_i)$ respectively. These operations give \mathbb{FT}^n the structure of an *\mathbb{FT} -module*⁴. There is a natural action of the semigroup $M_n(\mathbb{FT})$ on the left [respectively, right] of \mathbb{FT}^n , obtained by viewing elements of

⁴Some authors use the term *semimodule*, to emphasise the non-invertibility of addition, but since no other kind of module exists over \mathbb{FT} we have preferred the more concise term.

the latter as column vectors [respectively, row vectors] and applying tropical matrix multiplication.

The space \mathbb{FT}^n also has the structure of a lattice, under the partial order given by $x \leq y$ if $x_i \leq y_i$ for all i .

From affine tropical n -space we obtain *projective tropical* $(n-1)$ -space, denoted \mathbb{PFT}^{n-1} , by identifying two vectors if one is a tropical multiple of the other by an element of \mathbb{FT} . We identify \mathbb{PFT}^{n-1} with \mathbb{R}^{n-1} via the map

$$(x_1, \dots, x_n) \mapsto (x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n).$$

Submodules of \mathbb{FT}^n (that is, subsets closed under tropical addition and scaling) are termed *tropical convex sets*. Finitely generated tropical convex sets are called *tropical polytopes*. Since tropical convex sets are closed under scaling, each tropical convex set $X \subseteq \mathbb{FT}^n$ induces a subset of \mathbb{PFT}^{n-1} , termed the *projectivisation* of X and denoted $\mathcal{P}X$.

For $A \in M_n(\mathbb{FT})$ we let $R(A)$ denote the tropical polytope in \mathbb{FT}^n generated by the rows of A and let $C(A)$ denote the tropical polytope in \mathbb{FT}^n generated by the columns of A . We call these tropical polytopes the *row space* and *column space* of A respectively.

A point x in a tropical convex set X is called *extremal in X* if the set

$$X \setminus \{\lambda \otimes x : \lambda \in \mathbb{FT}\}$$

is a submodule of X . Clearly some scaling of every extremal point must lie in every generating set for X . In fact, every tropical polytope is generated by its extremal points considered up to scaling [3, 20].

3. GREEN'S RELATIONS, IDEMPOTENTS AND REGULARITY

Green's relations are five equivalence relations (\mathcal{L} , \mathcal{R} , \mathcal{H} , \mathcal{D} and \mathcal{J}) and three partial orders ($\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$), which can be defined on any semigroup, and which describe the structure of its maximal subgroups and principal left, right and two-sided ideals; for further details we refer the reader to an introductory text such as [11].

The study of Green's relations for the full tropical matrix semigroups was initiated (in the case of $M_2(\mathbb{T})$) by the second and third authors [13]. In [10], Hollings and the third author gave a complete description of the \mathcal{D} -relation for $M_n(\mathbb{FT})$, using the *duality* between the row and column space of a tropical matrix. In [14] the second and third authors described the equivalence relation \mathcal{J} and pre-order $\leq_{\mathcal{J}}$ in $M_n(\mathbb{FT})$ and $M_n(\mathbb{T})$. The main results of these papers, for the case $M_n(\mathbb{FT})$, are summarised in the following theorem (see [10, Proposition 3.1 and Theorem 5.1] and [14, Theorem 5.3 and Theorem 6.1] for full details and proofs).

Theorem 3.1. *Let $A, B \in M_n(\mathbb{FT})$.*

- (i) $A \leq_{\mathcal{L}} B$ if and only if $R(A) \subseteq R(B)$; $A \mathcal{L} B$ if and only if $R(A) = R(B)$;
- (ii) $A \leq_{\mathcal{R}} B$ if and only if $C(A) \subseteq C(B)$; $A \mathcal{R} B$ if and only if $C(A) = C(B)$;
- (iii) $A \mathcal{H} B$ if and only if $R(A) = R(B)$ and $C(A) = C(B)$;
- (iv) $A \mathcal{D} B$ if and only if $C(A)$ and $C(B)$ are isomorphic as \mathbb{FT} -modules (or, equivalently, $R(A)$ and $R(B)$ are isomorphic as \mathbb{FT} -modules);
- (v) $A \leq_{\mathcal{J}} B$ if and only if there exists a tropical polytope $X \subseteq \mathbb{FT}^n$ such that $R(A)$ (or, equivalently, $C(A)$) embeds linearly into X and $R(B)$ (or, equivalently, $C(B)$) surjects linearly onto X ;
- (vi) $A \mathcal{J} B$ if and only if $A \mathcal{D} B$.

Parts (i)-(iii) of the above theorem are straightforward and apply where \mathbb{FT} is replaced by a semiring in much greater generality; parts (iv)-(vi) require considerably more work and are more particular to the tropical case. We note that the

fundamental difference between the structure of the full tropical matrix semigroup and that of the full linear monoid $M_n(K)$ over a field K in some sense stems from the lack of a cohesive definition of the *rank* of a tropical matrix (or equivalently, the *dimension* of a tropical module), which in turn is due to the more flamboyant nature of tropical polytopes, as compared with finite-dimensional vector spaces. Indeed, the \mathcal{J} -classes of the full linear monoid $M_n(K)$ are precisely the sets of matrices of a specified rank (giving $n + 1$ classes in total), with $\leq_{\mathcal{J}}$ corresponding to the natural ordering of ranks (see [17] for further details). The structure of $M_n(\mathbb{FT})$ is rather more involved, owing to the fact that there are no longer finitely many \mathcal{J} -classes (or in other words, isomorphism types of tropical polytopes) - in fact each \mathcal{J} -class is contained in infinite ascending and descending chains within the highly non-trivial \mathcal{J} -order. We note that the study of isomorphism types of tropical polytopes (or, equivalently, the study of \mathcal{J} -classes for the tropical matrix semigroup) is an area of research in its own right, known as *tropical convexity* (see, for example, [6]).

We note the following interesting correspondence between inclusions of row spaces and certain surjections of column spaces; see [10] for full details.

Theorem 3.2. [10, Theorem 4.2, Corollary 4.3].

- (i) $R(A) \subseteq R(B)$ if and only if there is a surjective \mathbb{FT} -linear morphism from $C(B)$ to $C(A)$ taking the i th column of B to the i th column of A for all i .
- (ii) $R(A) = R(B)$ if and only if there is an \mathbb{FT} -linear isomorphism from $C(A)$ to $C(B)$ taking the i th column of B to the i th column of A for all i .

A complete description of the idempotent elements of $M_2(\mathbb{FT})$ was given in [13] and it follows from the results given there that the semigroup $M_2(\mathbb{FT})$ is regular (that is, every \mathcal{R} class and every \mathcal{L} class contains an idempotent) and each maximal subgroup is isomorphic to either \mathbb{R} (under addition) or $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. For $n \geq 3$ it is known that the semigroups $M_n(\mathbb{FT})$ are no longer regular. In [12] the present authors gave a geometric characterisation of the regular elements of $M_n(\mathbb{FT})$. In the present paper we turn our attention to the maximal subgroups of $M_n(\mathbb{FT})$, in other words, the \mathcal{H} -classes containing an idempotent tropical matrix. We shall show that each maximal subgroup is isomorphic to a direct product of \mathbb{R} with some finite group. Given an idempotent tropical matrix E , we first note that the \mathcal{H} -class containing E is isomorphic to the full group of \mathbb{FT} -module automorphisms of $C(E)$. This result mirrors the behaviour seen in the full linear monoid $M_n(K)$ over a field K , where each maximal subgroup in the \mathcal{J} -class of rank r may easily be realised as an automorphism group of a vector space of dimension r .

Theorem 3.3. *Let E be an idempotent in $M_n(\mathbb{FT})$, with corresponding \mathcal{H} -class denoted by H_E , and let $\text{Aut}(C(E))$ denote the group of \mathbb{FT} -module automorphisms of $C(E)$. Then*

- the map $\psi : \text{Aut}(C(E)) \rightarrow H_E$ sending each automorphism f to the matrix obtained by applying f to the columns of E ; and
- the map $\varphi : H_E \rightarrow \text{Aut}(C(E))$ sending each element $A \in C(E)$ to the map from $C(E)$ to itself given by left multiplication by A

are mutually inverse group isomorphisms.

Proof. It suffices to show that ψ is an isomorphism and that $\varphi \circ \psi$ is the identity map on $\text{Aut}(C(E))$.

Let $f : C(E) \rightarrow C(E)$ be an \mathbb{FT} -module automorphism and let $A = \psi(f)$. Since f is surjective, it is clear that $C(A) = C(E)$ and hence $A \mathcal{R} E$, by Theorem 3.1 (ii). Now, since f is an \mathbb{FT} -linear isomorphism of column spaces, taking the i th column of E to the i th column of A , it follows from Theorem 3.2 (ii) that $R(A) = R(E)$, giving $A \mathcal{L} E$ and hence $A \mathcal{H} E$. Thus ψ is well-defined.

We first notice that if f is an automorphism of $C(E)$ with $\psi(f) = A$, then $f(x) = A \otimes x$ for all $x \in C(E)$. Indeed, since $x \in C(E)$, we have that $x = E \otimes x = \bigoplus_{i=1}^n x_i \otimes E_i$, where E_i denotes the i th column of E . Thus

$$f(x) = \bigoplus_{i=1}^n x_i \otimes f(E_i) = \bigoplus_{i=1}^n x_i \otimes A_i = A \otimes x$$

for all $x \in C(E)$.

We claim that ψ is a homomorphism of groups. Indeed, let f, g be \mathbb{FT} -module automorphisms of $C(E)$ and let $\psi(f) = A$ and $\psi(g) = B$. Since $B \mathcal{H} E$, we have $B \otimes E = B$ giving

$$\begin{aligned} \psi(f \circ g) &= (f \circ g(E_1), \dots, f \circ g(E_n)) \\ &= (A \otimes B \otimes E_1, \dots, A \otimes B \otimes E_n) \\ &= A \otimes B \otimes E = A \otimes B = \psi(f) \otimes \psi(g). \end{aligned}$$

Next we show that ψ is injective. Indeed, suppose $\psi(f) = \psi(g)$. Then, by definition, $f(E_i) = g(E_i)$ for all i . But the columns E_i generate $C(E)$, so by linearity it must be that $f = g$.

Now we show that ψ is surjective. Let $A \in H_E$ and define $f : C(E) \rightarrow C(E)$ by $f(x) = A \otimes x$. Since H_E is a group, there exists $A' \in H_E$ such that $A \otimes A' = E = A' \otimes A$ and it is easy to see that f is a bijection, with inverse f' defined by $f'(x) = A' \otimes x$. Thus $f \in \text{Aut}(C(E))$ and it is clear that $\psi(f) = A$, giving that ψ is surjective.

It remains to show that $\varphi \circ \psi$ is the identity map on $\text{Aut}(C(E))$. Suppose then that $f \in \text{Aut}(C(E))$, and let $A = \psi(f)$. Write E_i for the i th column of E and A_i for the i th column of A . Then by the definition of ψ , $A_i = f(E_i)$ for all i . Now let $x \in C(E)$ for each i let x_i denote the i th component of x . Then

$$\begin{aligned} [\varphi(\psi(f))](x) &= [\varphi(A)](x) = A \otimes x = \bigoplus_{i=1}^n x_i \otimes A_i = \bigoplus_{i=1}^n x_i \otimes f(E_i) \\ &= f\left(\bigoplus_{i=1}^n x_i \otimes E_i\right) = f(E \otimes x) = f(x), \end{aligned}$$

as required to show that $f = \varphi(\psi(f))$. \square

We note that Theorem 3.3 has a dual statement in terms of the right action of H_E on the row space of E .

Example. Consider the following elements of $M_3(\mathbb{FT})$:

$$E = \begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ -3 & -3 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & -3 & 0 \\ 0 & -3 & 0 \\ -3 & -3 & -3 \end{pmatrix}.$$

It is easily verified by direct computation that E is an idempotent, that $A \otimes E = E \otimes A = A$, that $B \otimes E = E \otimes B = B$ and that $A \otimes B = B \otimes A = E$. It follows that A and B are mutually inverse elements of the maximal subgroup H_E around E . Let E_1, E_2, E_3 and A_1, A_2, A_3 denote the columns of E and A respectively. The column space $C(E)$ (which by Theorem 3.1 is equal to $C(A)$) is illustrated in projective 2-space in Figure 1 below. Note that the point $(x_1, x_2, 0)$ is the projectivisation of all points of the form $(x_1 + x_3, x_2 + x_3, x_3)$ in affine tropical 3-space. The shaded triangular region then indicates the set of all tropical linear combinations of the columns of E . For example, the marked point at $(1, 2, 0)$ is easily seen to be equal to $(1 \otimes E_1) \oplus (2 \otimes E_2) \oplus (0 \otimes E_3)$. We note that in general the column (or row) space of an idempotent turns out to have a particularly nice structure; it was shown in [12, Proposition 5.5] that each such column space is min-plus (as well as max-plus)

convex, from which it follows it is a convex polytope in the usual (Euclidean) sense. (In fact, any max-plus convex polytope which is also Euclidean convex must be the column (or row) space of an idempotent [15, Corollary C].)

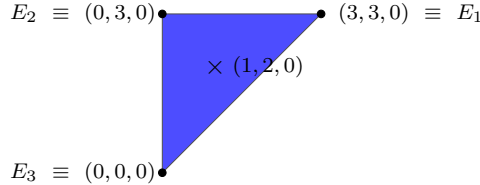


FIGURE 1. The column space of the idempotent matrix E , shown in projective tropical 2-space $\mathbb{P}\mathbb{FT}^2$.

Since E is idempotent it fixes its column space pointwise; the corresponding automorphism is thus the identity map, which of course is as one would expect since E is the identity of H_E . Now notice that the automorphism corresponding to A maps the extremal point E_1 to $A_1 = E_3$, the extremal point E_2 to $A_2 = E_1$ and the extremal point E_3 to $A_3 = 3 \otimes E_2$. Thus, it is the map which cyclically permutes the extremal points in projective space, but with a non-trivial scaling which gives it infinite order (for example, we note that the marked point is a fixed point of the action in projective space, since $(1 \otimes E_3) \oplus (2 \otimes E_1) \oplus (0 \otimes (3 \otimes E_2)) = 1 \otimes (1, 2, 0) \equiv (1, 2, 0)$). The inverse automorphism naturally corresponds to B , of course. We shall return to this example later.

4. DIMENSION, PROJECTIVITY, IDEMPOTENTS AND REGULARITY

There are several important notions of dimension for a tropical convex set $X \subseteq \mathbb{FT}^n$. The *tropical dimension* is the topological dimension of X , viewed as a subset of \mathbb{R}^n with the usual topology. Note that, in contrast to the classical (Euclidean) case, tropical convex sets may have regions of different topological dimension. We say that X has *pure dimension* k if every open (within X with the induced topology) subset of X has topological dimension k . The *generator dimension* of X is the minimal cardinality of a generating subset, under the linear operations of scaling and addition. If X is a tropical polytope, this is equal to the number of extremal points of X considered up to scaling [3, 20]. The *dual dimension* [12] of X is the minimal cardinality of a generating set under scaling and the induced operation of greatest lower bound within X . (Notice that, in general, the greatest lower bound of two elements within a tropical convex set X need not be the same as their component-wise minimum, which may not be contained in X .) In fact, it follows from [12, Proposition 3.1] that if $X = C(A)$ for some matrix $A \in M_n(\mathbb{FT})$, then the dual dimension of X is equal to the generator dimension of $R(A)$. In general, these three notions of dimension do not coincide.

In [12], the present authors gave a characterisation of *projectivity* (in the sense of ring theory) for tropical polytopes in terms of the geometric and order-theoretic structure on these sets. We briefly recall that a module P is called *projective* if every morphism from P to another module M factors through every surjective module morphism onto M . One of the main results of [12] can be summarised as follows.

Theorem 4.1. [12, Theorems 1.1 and 4.5]. *Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then the following are equivalent:*

- (i) X is projective as an \mathbb{FT} -module;
- (ii) X is the column space of an idempotent matrix in $M_n(\mathbb{FT})$;
- (iii) X has pure dimension equal to its generator dimension and dual dimension.

Since all three notions of dimension coincide for projective tropical polytopes, we define the *dimension* of a projective tropical polytope to be this common value. For example, the projective tropical polytope shown in Figure 1 has dimension 3. We shall refer to projective tropical polytopes of dimension k as *projective k -polytopes*. Numerous definitions of rank have been introduced and studied for tropical matrices (see for example [1, 6, 8, 9] for more details), mostly corresponding to different notions of “dimension” of the row or column space. In light of Theorem 4.1, we shall focus on the following three definitions of rank. The *tropical rank* of a matrix is the tropical dimension of its row space (or equivalently, by [7, Theorem 23] for example, its column space). It also has a characterisation in terms of the computation of the matrix permanent [6]. The *row rank* is the generator dimension of the row space, which by [12, Proposition 3.1] is also the dual dimension of the column space. Dually, the *column rank* is the generator dimension of the column space and also the dual dimension of the row space. In particular, it is clear from the definitions that all three notions of rank are \mathcal{H} -class invariants; in fact they are \mathcal{J} -class invariants (see [14, Corollary 8.5 and Corollary 8.11]). Whilst these three notions of rank can in general differ, it follows from Theorem 4.1 that they must coincide for any *idempotent* matrix. Thus we shall refer without ambiguity to the *rank* of an idempotent matrix, and by a slight abuse of terminology, the *rank* of a maximal subgroup.

The following result is a consequence of [12, Theorem 4.2].

Theorem 4.2. [12, Theorem 4.2]. *Let $X \subseteq \mathbb{FT}^n$ be a projective k -polytope. Then X is isomorphic to the column space of a $k \times k$ idempotent matrix over \mathbb{FT} .*

The following theorem implies that, up to isomorphism, the maximal subgroups of $M_n(\mathbb{FT})$ are *precisely* the full rank maximal subgroups of $M_k(\mathbb{FT})$, for $k \leq n$.

Theorem 4.3. *Let $E \in M_n(\mathbb{FT})$ be idempotent of rank k . Then for every $r \geq k$ there is an idempotent $F \in M_r(\mathbb{FT})$ such that F has rank k and $H_E \cong H_F$.*

Proof. By Theorem 4.1, $C(E)$ is a projective k -polytope. Hence by Theorem 4.2, $C(E)$ is isomorphic to the column space of an idempotent in $M_k(\mathbb{FT})$. In particular, $C(E)$ embeds linearly in \mathbb{FT}^k and it follows easily (for example, by duplicating rows) that $C(E)$ embeds linearly in \mathbb{FT}^r . Let $X \subseteq \mathbb{FT}^r$ be the image of such an embedding. Since X is isomorphic to $C(E)$, it is projective as an \mathbb{FT} -module, so applying Theorem 4.1 again tells us that X is the column space of an idempotent $F \in M_r(\mathbb{FT})$. Notice that the dimension of X must be k and hence F has rank k . Now applying Theorem 3.3 twice, H_E and H_F are isomorphic to the automorphism groups of $C(E)$ and X respectively; but since $C(E)$ and X are isomorphic, their automorphism groups are isomorphic. \square

Thus in order to compute the isomorphism types of maximal subgroups of $M_n(\mathbb{FT})$ it is enough to consider the maximal subgroups of full rank in $M_k(\mathbb{FT})$ for all $k \leq n$. The rest of the paper will therefore concentrate on the case where E is an idempotent of full rank n in $M_n(\mathbb{FT})$, so that $C(E)$ is a projective n -polytope in \mathbb{FT}^n .

5. THE \mathcal{H} -CLASS OF AN IDEMPOTENT OF FULL RANK

Let E be an idempotent in $M_n(\mathbb{FT})$ and let H_E denote the \mathcal{H} -class of E . We can consider the matrices in H_E as maps from \mathbb{FT}^n to \mathbb{FT}^n , acting by left multiplication, and it follows from Theorem 3.3 that these maps restrict to \mathbb{FT} -module automorphisms of $C(E)$. We shall show that when E has rank n , these automorphisms are affine linear maps *in the classical sense*. It then follows that every automorphism of $C(E)$ extends to an automorphism of \mathbb{FT}^n .

From now on we assume that E is an idempotent of full rank n . In particular, by Theorem 4.1, the column space $C(E)$ has pure tropical dimension n . It is well known that the (topological) boundary of $C(E)$ is precisely the set of all points $y \in C(E)$ for which the equation $E \otimes x = y$ has multiple solutions (see for example [2, Chapter 6.2] or [7]). As a consequence, we obtain the following useful fact:

Lemma 5.1. *Let E be an idempotent of rank n in $M_n(\mathbb{FT})$, and consider the column space $C(E)$ as a subset of \mathbb{R}^n equipped with the usual topology. Then left multiplication by E maps all points exterior to $C(E)$ onto the boundary of $C(E)$.*

Proof. Suppose false for a contradiction, and let $x \notin C(E)$ be such that $E \otimes x$ does not lie on the boundary of $C(E)$. Certainly $E \otimes x \in C(E)$, so it must be that $E \otimes x$ lies in the interior of $C(E)$. Since E is idempotent, it has eigenspace $C(E)$ with eigenvalue 0, and $E = E^{\otimes k}$ for all k . Thus, by [2, Theorem 6.2.14], each point in the interior of $C(E)$ has a unique preimage under the action of E . But since E is idempotent we have $E \otimes x = E \otimes (E \otimes x)$, giving $x = E \otimes x$, and hence contradicting the assumption that $x \notin C(E)$. \square

Lemma 5.2. *Let E be an idempotent of rank n in $M_n(\mathbb{FT})$, and let A be an element in the \mathcal{H} -class of E . Define $\varphi_A : \mathbb{FT}^n \rightarrow \mathbb{FT}^n$ to be the map given by left multiplication by A . Then*

- (i) φ_A maps interior points of $C(E)$ to interior points;
- (ii) φ_A maps boundary points of $C(E)$ to boundary points and
- (iii) φ_A maps all points exterior to $C(E)$ onto the boundary of $C(E)$.

Proof. It is clear that the image of φ_A is $C(E)$ and it follows from Theorem 3.3 that φ_A restricts to an automorphism of $C(E)$. Since H_E is a group, there exists $A' \in H_E$ such that $A \otimes A' = A' \otimes A = E$ and hence φ_A and $\varphi_{A'}$ are mutually inverse on $C(E)$.

(i) Let $x \in C(E)$ be an interior point of the column space. Suppose for contradiction that φ_A maps x to some point y on the boundary of $C(E)$. Then $\varphi_{A'}$ must map y back to x , so that $E \otimes x = (A' \otimes A) \otimes x = A' \otimes y = x$. Consider the equation $A \otimes z = y$. We note that $z = x$ is the unique solution in \mathbb{FT}^n (since φ_A is an automorphism of $C(E)$, no other element of $C(E)$ can be a solution; if $z \notin C(E)$ were a solution, then we would have $E \otimes z = A' \otimes A \otimes z = A' \otimes y = x$, contradicting Lemma 5.1). However, we have seen that given any point y on the boundary there exists some $z \notin C(A)$ such that $A \otimes z = y$.

(ii) It follows immediately that A must map boundary points to boundary points too; if A maps a boundary point y to an interior point x , then A' must map x back to y , contradicting part (i).

(iii) If x is exterior to $C(E)$ then by Lemma 5.1, $E \otimes x$ lies on the boundary. But $A \otimes E = A$ so now $A \otimes x = A \otimes E \otimes x$ lies on the boundary by part (ii) above. \square

Recall that $\mathbb{T} = \mathbb{FT} \cup \{-\infty\}$. We briefly consider the semigroup $M_n(\mathbb{T})$. It is clear that this is a monoid, whose identity element, denoted I_n , is the $n \times n$ matrix with 0 entries on the diagonal and $-\infty$ entries off the diagonal. It is well known that the units of $M_n(\mathbb{T})$ are precisely the *tropical monomial matrices*, that is, those matrices with exactly one entry not equal to $-\infty$ in each row and in each column. Thus it is clear that every unit has the form $D(\lambda_1, \dots, \lambda_n)P_\sigma$, where $D(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ and P_σ is a tropical permutation matrix whose i th row has a 0 in the $\sigma(i)$ th position and $-\infty$ entries elsewhere. We shall now show that given an idempotent E of rank n in $M_n(\mathbb{FT})$, the corresponding \mathcal{H} -class H_E is isomorphic to a certain subgroup of the group of units in $M_n(\mathbb{T})$.

Theorem 5.3. *Let E be an idempotent of rank n in $M_n(\mathbb{FT})$ and let H_E denote the \mathcal{H} -class of E . Let G_E be the set of all units $P \in M_n(\mathbb{T})$ which commute with E . Then there is a group isomorphism:*

$$\gamma : G_E \rightarrow H_E, \quad \gamma(P) = P \otimes E \quad (= E \otimes P).$$

Proof. Let P be a unit commuting with E . We first note that $C(P \otimes E) = C(E \otimes P) = C(E)$ and $R(P \otimes E) = R(E)$, so $P \otimes E \in H_E$. Notice that γ is a homomorphism of groups; indeed, for $P, Q \in G_E$,

$$\gamma(P \otimes Q) = E \otimes (P \otimes Q) = E \otimes E \otimes (P \otimes Q) = (E \otimes P) \otimes (E \otimes Q) = \gamma(P) \otimes \gamma(Q).$$

For injectivity, suppose $E \otimes P = E \otimes Q$ for some units $P, Q \in G_E$. Then $E \otimes P \otimes Q^{-1} = E$. Note that since $P \otimes Q^{-1}$ is a monomial matrix, the columns of $E \otimes P \otimes Q^{-1}$ are a scaling and permutation of the columns of E . But since E has rank n , no column of E is a scaling of another column. It follows that $P \otimes Q^{-1}$ must be the tropical identity matrix, giving that $P = Q$.

It remains to show that γ is surjective. Let $A \in H_E$. Since E has rank n it follows that the columns of A provide a minimal generating set for the column space of E . Thus we may choose a unit $P \in M_n(\mathbb{T})$ such that $A = E \otimes P$. Let x be an interior point of $C(E)$. By Lemma 5.2, A maps x to an interior point. Moreover, P must also map x to an interior point; if not then, by Lemma 5.1, $E \otimes P$ maps x to the boundary of $C(E)$, contradicting $A = E \otimes P$. In fact, it is easy to see that $P \otimes x = A \otimes x$ for all interior points x . Since $C(E)$ has pure dimension, every boundary point is a limit of interior points. Since P and A are continuous maps, it follows that $P \otimes x = A \otimes x$ for all $x \in C(E)$. In particular, by Theorem 3.3, the action of P restricted to $C(E)$ is an \mathbb{FT} -module automorphism. Now for all $x \in C(E)$ we have $E \otimes P \otimes x = P \otimes x = P \otimes E \otimes x$, giving $E \otimes P \otimes E = P \otimes E \otimes E$ and hence

$$E \otimes P = A = A \otimes E = E \otimes P \otimes E = P \otimes E \otimes E = P \otimes E.$$

□

Example. Returning to our example matrices E , A and B from Section 3, notice that $A = \bar{A} \otimes E = E \otimes \bar{A}$ and $B = \bar{B} \otimes E = E \otimes \bar{B}$ where

$$\bar{A} = \begin{pmatrix} -\infty & 0 & -\infty \\ -\infty & -\infty & 3 \\ 0 & -\infty & -\infty \end{pmatrix} \quad \text{and} \quad \bar{B} = \begin{pmatrix} -\infty & -\infty & 0 \\ 0 & -\infty & -\infty \\ -\infty & -3 & -\infty \end{pmatrix}$$

are unit matrices which commute with E .

Theorem 3.3 and Theorem 5.3 combine to prove some results which may be of independent interest.

Theorem 5.4. *Every automorphism of a projective n -polytope in \mathbb{FT}^n*

- (i) *extends to an automorphism of \mathbb{FT}^n ; and*
- (ii) *is a (classical) affine linear map.*

Proof. Let $X \subseteq \mathbb{FT}^n$ be a projective n -polytope and let $f : X \rightarrow X$ be an \mathbb{FT} -module automorphism. By Theorem 4.1, $X = C(E)$ for some full rank idempotent $E \in M_n(\mathbb{FT})$. By Theorem 3.3, there is a matrix $A \in H_E$ such that $f(x) = A \otimes x$ for all $x \in C(E) = X$. By Theorem 5.3 there is a unit $P \in M_n(\mathbb{T})$ such that $A = E \otimes P = P \otimes E$. Now for any $x \in X = C(E)$ we have

$$P \otimes x = P \otimes E \otimes x = A \otimes x = f(x),$$

so the map

$$\mathbb{FT}^n \rightarrow \mathbb{FT}^n, \quad x \mapsto P \otimes x$$

is an automorphism of \mathbb{FT}^n extending f as required to establish (i).

Now since P is a monomial matrix, we have

$$P_{i,j} = \begin{cases} \lambda_i & \text{if } j = \sigma(i) \\ -\infty & \text{otherwise.} \end{cases}$$

Thus, for all $x \in C(E)$,

$$(P \otimes x)_i = \lambda_i \otimes x_{\sigma(i)} = x_{\sigma(i)} + \lambda_i,$$

giving $f(x) = P_\sigma \cdot x + \lambda$, where P_σ is the (classical) permutation matrix corresponding to σ and $\lambda = (\lambda_1, \dots, \lambda_n)^T$. \square

Since an \mathbb{FT} -module automorphism f of $C(E)$ respects scaling, it induces a well-defined map \hat{f} on the projectivisation of $C(E)$. It turns out that, in the case E is an idempotent matrix of full rank, this map too is affine linear.

Corollary 5.5. *Let E be an idempotent of rank n in $M_n(\mathbb{FT})$, let A be an element in the \mathcal{H} -class of E and let $\widehat{\varphi}_A : \mathcal{PC}(E) \rightarrow \mathcal{PC}(E)$ denote the corresponding map induced by left multiplication by A . Then $\widehat{\varphi}_A$ is a (classical) affine linear map on $\mathcal{PC}(E)$, regarded as a subset of \mathbb{R}^{n-1} .*

Proof. From the proof of Theorem 5.4, left multiplication by A yields an affine linear map $\varphi_A : \mathbb{FT}^n \rightarrow \mathbb{FT}^n$ of the form $\varphi_A(x) = x_{\sigma(i)} + \lambda_i = P_\sigma \cdot x + \lambda$, where P_σ is a (classical) permutation matrix, corresponding to a permutation $\sigma \in S_n$, and $\lambda = (\lambda_1, \dots, \lambda_n)^T$ is a constant vector. Recall that we may identify $\mathbb{P}\mathbb{FT}^{(n-1)}$ with $\mathbb{R}^{(n-1)}$ via the map

$$(x_1, \dots, x_n) \mapsto (x_1 - x_n, \dots, x_{n-1} - x_n)$$

and hence

$$\widehat{\varphi}_A \begin{pmatrix} x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} - x_{\sigma(n)} \\ \vdots \\ x_{\sigma(n-1)} - x_{\sigma(n)} \end{pmatrix} + \begin{pmatrix} \lambda_1 - \lambda_n \\ \vdots \\ \lambda_{n-1} - \lambda_n \end{pmatrix}.$$

If $\sigma(n) = n$ then it is immediate that $\widehat{\varphi}_A$ is an affine linear map on $\mathcal{PC}(E)$ regarded as a subset of \mathbb{R}^{n-1} . Suppose that $\sigma(n) \neq n$. Then $\sigma(k) = n$ for some $k \in \{1, \dots, n-1\}$. Let B be the $(n-1) \times (n-1)$ matrix given by

$$B_{i,j} = \begin{cases} 1 & \text{if } i \neq k \text{ and } j = \sigma(i), \\ -1 & \text{if } j = \sigma(n), \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that

$$\widehat{\varphi}_A \begin{pmatrix} x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \end{pmatrix} = B \cdot \begin{pmatrix} x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \end{pmatrix} + \begin{pmatrix} \lambda_1 - \lambda_n \\ \vdots \\ \lambda_{n-1} - \lambda_n \end{pmatrix}.$$

\square

We shall use Theorem 5.3 to prove our first main result; that is, that every maximal subgroup H_E is isomorphic to a direct product $\mathbb{R} \times \Sigma$, where Σ is a subgroup of S_n . We first recall a few basic facts about tropical eigenvalues (and refer the reader to [2, Chapter 4] for proofs of these results). To every matrix $M \in M_n(\mathbb{T})$ we associate a weighted directed graph Γ_M , with edges labelled by the finite entries of M . It is well known that the *maximal cycle mean* of M (that is, the maximum average weight of a path from a node to itself in Γ_M) is always a tropical eigenvalue of M . Moreover, if the directed graph Γ_M is strongly connected, then this is the *only* eigenvalue. In particular, every element $A \in M_n(\mathbb{FT})$ has unique

(tropical) eigenvalue. On the other hand, if P is a unit in $M_n(\mathbb{T})$, the mean weight of any cycle in Γ_P is an eigenvalue for P . Thus P may have up to k distinct eigenvalues, where k is the number of disjoint cycles in the underlying permutation of P .

We shall require the following two lemmas, concerning units that commute with elements of $M_n(\mathbb{FT})$, the first of which can be seen as a generalisation of the fact that a diagonal matrix commutes with any element of $M_n(\mathbb{FT})$ if and only if it is a scalar matrix, and hence has a single tropical eigenvalue.

Lemma 5.6. *Let $A \in M_n(\mathbb{FT})$ and let $P \in M_n(\mathbb{T})$ be a unit which commutes with A . Then P has only one (tropical) eigenvalue.*

Proof. Suppose false, that is, that P has multiple distinct eigenvalues. Since P is monomial, it is clear that some power $P^{\otimes k}$ of P is a diagonal matrix. Since P has distinct eigenvalues, so does $P^{\otimes k}$, and so $P^{\otimes k}$ cannot be a scalar matrix. It follows that $P^{\otimes k}$ cannot commute with any matrix in $M_n(\mathbb{FT})$, which contradicts the assumption that P commutes with A . \square

Example. Recall the unit matrices \bar{A} and \bar{B} from our running examples. These matrices have maximum cycle means 1 and -1 respectively, and hence by Lemma 5.6 these are the unique eigenvalues of the respective matrices.

Lemma 5.7. *Let $A \in M_n(\mathbb{FT})$ and let P and Q be units which commute with A . Suppose also that P and Q have maximal cycle mean 0. Then $P \otimes Q$ has maximal cycle mean 0.*

Proof. It follows from Lemma 5.6 that every cycle in the graphs corresponding to P and Q has mean weight 0. In particular, the (classical) sum of all the finite entries in P is equal to 0, and the sum of the finite entries in Q is equal to 0. By a simple calculation, the sum of the finite entries in the product $P \otimes Q$ must also be 0. Now, since the product $P \otimes Q$ also commutes with A , applying Lemma 5.6 again yields that every cycle in $P \otimes Q$ has the same average weight. Since the sum of the entries (which is 0) is a weighted sum of the cycle means, and the cycle means are all the same, we deduce that the cycle means are all 0. In particular, the maximum cycle mean is 0. \square

Theorem 5.8. *Let E be an idempotent of rank n in $M_n(\mathbb{FT})$. The group G_E of units commuting with E is the internal direct product of its subgroups $R = \{\lambda \otimes I_n : \lambda \in \mathbb{FT}\}$ and $\Sigma = \{P \in G_E : P \text{ has eigenvalue } 0\}$, where $R \cong \mathbb{R}$, and Σ is a finite group embeddable in the symmetric group S_n .*

Proof. It is easy to see that $R \trianglelefteq G_E$. Moreover, the only diagonal matrices which commute with E are those contained in R . Since I_n has eigenvalue 0 we have $I_n \in \Sigma$. Let $A, B \in \Sigma$. By Lemma 5.7 we see that $A \otimes B \in \Sigma$ and it is also clear that $A^{-1} \in \Sigma$, giving that $\Sigma \leq G_E$. Thus $R, \Sigma \leq G_E$ with $R \cap \Sigma = \{I_n\}$. Let $P \in G_E$. By Lemma 5.6, P has a unique eigenvalue, λ say. Hence we may write $P = \lambda \otimes I_n \otimes P_\lambda$, where $P_\lambda \in \Sigma$. In other words, $G_E = R\Sigma$. Moreover, it is clear that every element of R commutes with every element of Σ , giving that G_E is the internal direct product of R and Σ .

Now $R \cong \mathbb{R}$, and from the definition of matrix multiplication it is easy to show that the map $\phi : G_E \rightarrow S_n$, sending a monomial matrix to its underlying permutation is a homomorphism of groups, with kernel R . Hence

$$\Sigma \cong G_E/R \cong \text{Im } \phi \leq S_n.$$

\square

Example. Returning once more to our running example, recall that in Section 3 we introduced an idempotent E , together with mutually inverse matrices A and B in H_E . Earlier in this section we found units $\bar{A}, \bar{B} \in G_E$ such that $\bar{A} \otimes E = E \otimes \bar{A} = A$ and $\bar{B} \otimes E = E \otimes \bar{B} = B$, and later saw, as a consequence of Lemma 5.6 that \bar{A} and \bar{B} each have a unique eigenvalue (1 and -1 respectively). The unit matrix $C = -1 \otimes \bar{A}$ therefore has eigenvalue 0, and hence lies in the finite subgroup Σ from the statement of Theorem 5.8. In fact, one can check that $C^{\otimes 3} = E$ and $C, C^{\otimes 2}$ and E itself are the only units with eigenvalue 0 commuting with E , so that $\Sigma \cong \mathbb{Z}/3\mathbb{Z}$. Hence, the maximal subgroup around E is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{R}$.

Proposition 5.9. *Let E be an idempotent of rank n in $M_n(\mathbb{FT})$ and let H_E denote the \mathcal{H} -class of E . The classical mean of the columns of E is an eigenvector for all $A \in H_E$.*

Proof. Let E_1, \dots, E_n denote the columns of E , and let $m = \frac{1}{n} \sum_{i=1}^n E_i$ be their classical mean. We claim that m is an eigenvector for every element of H_E . Let $A \in H_E$. By Theorem 5.4, left multiplication by A is a classical affine linear map, from which it follows that

$$A \otimes m = A \otimes \left(\frac{1}{n} \sum_{i=1}^n E_i \right) = \frac{1}{n} \sum_{i=1}^n A \otimes E_i.$$

By Theorem 5.3, we know that $A = P \otimes E = E \otimes P$ for some monomial matrix P , and hence left multiplication by P permutes and scales the columns of E according to the finite entries of P :

$$A \otimes m = \frac{1}{n} \sum_{i=1}^n (P \otimes E) \otimes E_i = \frac{1}{n} \sum_{i=1}^n P \otimes E_i = \frac{1}{n} \sum_{i=1}^n \lambda_i \otimes E_{\sigma(i)},$$

where the λ_i are the finite entries of P . Now, since $\lambda_i \otimes E_{\sigma(i)} = \underline{\lambda}_i + E_{\sigma(i)}$, where $\underline{\lambda}_i$ denotes the vector with all entries equal to λ_i , we see that

$$A \otimes m = \frac{1}{n} \sum_{i=1}^n (\underline{\lambda}_i + E_{\sigma(i)}) = \frac{1}{n} \sum_{i=1}^n \underline{\lambda}_i + \frac{1}{n} \sum_{i=1}^n E_{\sigma(i)} = \frac{1}{n} \sum_{i=1}^n \underline{\lambda}_i + \frac{1}{n} \sum_{i=1}^n E_i = \lambda \otimes m,$$

where λ denotes the average of the λ_i (the finite entries of P). \square

Notice that since the classical mean of the columns of E is an eigenvector for all elements of H_E , it is in particular an element of $C(E)$. We note that the existence of a common eigenvector may also be deduced from results of Merlet [16].

Example. Returning for a final time to our running example, the subgroup H_E has common eigenspace $\{\lambda \otimes (1, 2, 0)^T\}$. This gives a single fixed point of the corresponding action on the projective column space; this is the point marked with a cross in Figure 1.

It follows from Theorem 5.8 that every maximal subgroup of $M_n(\mathbb{FT})$ is isomorphic to a group of the form $G \times \mathbb{R}$ for some finite group $G \leq S_n$. We now proceed to consider exactly which groups G can arise.

Recall (from for example [4, Section 6.6]) that a permutation group $G \leq S_n$ is called *2-closed* if G contains every element of S_n which preserves the orbits of ordered pairs. Equivalently, G is 2-closed if it is the full permutation group of colour-preserving automorphisms of an n -vertex edge-coloured directed graph without loops [5].

Theorem 5.10. *The maximal subgroups of $M_n(\mathbb{FT})$ are, up to isomorphism, exactly groups $G \times \mathbb{R}$ where G is a finite group with a 2-closed permutation representation on n points.*

Proof. Suppose first that $G \times \mathbb{R}$ is isomorphic to a maximal subgroup of $M_n(\mathbb{FT})$. Let F be the identity matrix in this subgroup, so that the subgroup H_F is isomorphic to $G \times \mathbb{R}$. If F has rank k , then by Theorem 4.3, H_F is isomorphic to H_E for some idempotent matrix $E \in M_k(\mathbb{FT})$ of full rank k .

Now by Proposition 5.9, the elements of H_E have a common (tropical) eigenvector m ; the classical mean of the columns of E . Let D be the $k \times k$ diagonal matrix with entries given by the entries of the vector m . Then it is easily verified that $D^{-1}ED$ is an idempotent, conjugation by D gives an isomorphism between H_E and $H_{D^{-1}ED}$, and every element of the latter has eigenvector $(0, \dots, 0)$, so we may replace E by $D^{-1}ED$. (This trick is an elementary application of the technique of *visualisation*; see for example [18].) Thus we may assume without loss of generality that $(0, \dots, 0)$ is a common eigenvector for H_E .

Consider now the set Σ of unit matrices commuting with E and having eigenvalue 0. By Theorem 5.8 and Theorem 5.3, Σ is a group and $\Sigma \times \mathbb{R}$ is isomorphic to H_E , which by assumption is isomorphic to $G \times \mathbb{R}$. It follows easily (for example, by considering the set of torsion elements) that Σ is isomorphic to G . Moreover, matrices in Σ are monomial matrices which fix the eigenvector $(0, \dots, 0)^T$, from which it follows easily that they are tropical permutation matrices, and hence induce a permutation representation of G on n points.

Next, consider the complete directed graph with vertex set $\{1, \dots, k\}$, and the edge from i to j coloured by E_{ij} . It is easily seen that a permutation matrix commutes with E if and only if the corresponding permutation is a colour-preserving automorphism of the coloured graph. Thus, Σ , and hence also G , is isomorphic to the full automorphism group of a coloured directed graph on k points, and hence has a faithful 2-closed representation on k points. This can be extended to a 2-closed representation on n points, by the addition of $n - k$ additional vertices which are fixed pointwise by the group.

Conversely, let $G \leq S_n$ be a 2-closed permutation group. We may assume G is the colour-preserving automorphism group of an edge-coloured directed graph Γ on vertices $1, \dots, n$. For each edge-colour used in Γ , pick a distinct real number in the interval $[-1.1, -0.9]$ in such a way that the numbers chosen form a basis for a free abelian group under addition (for example, by choosing the numbers sequentially so that each is outside \mathbb{Q} extended by the preceding numbers). Define $E \in M_n(\mathbb{FT})$ by setting E_{ij} to be the number corresponding to the colour of the edge from i to j , and the diagonal entries to be 0. Then E is tropically idempotent: indeed, for any i and j we have

$$(E^{\otimes 2})_{ij} = \max_k \{E_{ik} + E_{kj}\},$$

where $E_{ii} + E_{ij} = 0 + E_{ij} = E_{ij}$, $E_{ij} + E_{jj} = 0 + E_{jj} = E_{jj}$ and $E_{ik} + E_{ki} \leq -0.9 - 0.9 \leq -1.8 < -1.1 \leq E_{ij}$ for $k \neq i, j$.

We claim that the maximal subgroup H_E is isomorphic to $G \times \mathbb{R}$. By Theorem 5.3, it will suffice to show that the group G_E of unit matrices commuting with E is isomorphic to $G \times \mathbb{R}$. We will show this by showing that G_E consists of exactly the scalings of permutation matrices corresponding to permutations in G .

First, note that if $\sigma \in G$, then the fact that σ is an automorphism of the graph means that the corresponding permutation matrix, and hence any scaling thereof, commutes with E . Now suppose M is another unit matrix commuting with E , say $M_{\sigma(i),j} = \lambda_j$ for some $\sigma \in S_n$ and $\lambda_i \in \mathbb{FT}$. Since E and M commute, for all i and j we have

$$(E \otimes M)_{i,j} = E_{i,\sigma(j)} + \lambda_j = \lambda_i + E_{\sigma^{-1}(i),j} = (M \otimes E)_{i,j}. \quad (5.1)$$

Consider first the case $\sigma \in G$. Then the edges $(i, \sigma(j))$ and $(\sigma^{-1}(i), j)$ have the same colour in Γ , which by the construction of E forces $E_{i,\sigma(j)} = E_{\sigma^{-1}(i),j}$ for all i

and j . The equations given by (5.1) therefore yield $\lambda_i = \lambda_j$ for all i and j , so M is a scaling of the permutation matrix corresponding to σ .

Now consider the case $\sigma \notin G$. Then by the definition of G , there are distinct vertices r and s in Γ such that the edges (r, s) and $(\sigma(r), \sigma(s))$ have different colours. By the definition of E , this gives

$$E_{r,s} \neq E_{\sigma(r),\sigma(s)}. \quad (5.2)$$

Using the fact that $\lambda_i - \lambda_j = (\lambda_i - \lambda_k) + (\lambda_k - \lambda_j)$, it follows from equation (5.1) that

$$E_{i,\sigma(j)} - E_{\sigma^{-1}(i),j} = (E_{i,\sigma(k)} - E_{\sigma^{-1}(i),k}) + (E_{k,\sigma(j)} - E_{\sigma^{-1}(k),j}) \quad (5.3)$$

for all $i, j, k \in [n]$. Setting $k = \sigma(r), j = s$ and $i = \sigma(k) = \sigma^2(r)$ in equation (5.3) and rearranging yields

$$E_{\sigma^2(r),\sigma(s)} + E_{r,s} = E_{\sigma(r),\sigma(s)} + E_{\sigma(r),s}.$$

Now, since the distinct entries of E generate a free abelian group, there are only two ways that this can happen: either

- (1) $E_{\sigma^2(r),\sigma(s)} = E_{\sigma(r),\sigma(s)}$ and $E_{r,s} = E_{\sigma(r),s}$; or
- (2) $E_{\sigma^2(r),\sigma(s)} = E_{\sigma(r),s}$ and $E_{r,s} = E_{\sigma(r),\sigma(s)}$.

Case (2) is prohibited by equation (5.2), so the equations of Case (1) must hold.

On the other hand, setting $i = \sigma(r), k = s$ and $j = \sigma^{-1}(k) = \sigma^{-1}(s)$ in equation (5.3) and rearranging yields

$$E_{\sigma(r),s} + E_{r,s} = E_{\sigma(r),\sigma(s)} + E_{r,\sigma^{-1}(s)}.$$

Using again the fact that distinct entries of E generate a free abelian group, there are only two ways that this can happen: either

- (1)' $E_{\sigma(r),s} = E_{\sigma(r),\sigma(s)}$ and $E_{r,s} = E_{r,\sigma^{-1}(s)}$; or
- (2)' $E_{\sigma(r),s} = E_{r,\sigma^{-1}(s)}$ and $E_{r,s} = E_{\sigma(r),\sigma(s)}$.

Again, Case (2)' is prohibited by equation (5.2), so the equations of Case (1)' must hold. But this gives

$$E_{r,s} = E_{\sigma(r),s} = E_{\sigma(r),\sigma(s)},$$

giving a contradiction.

Thus we have shown that M commutes with E if and only if M is a scaling of a tropical permutation matrix corresponding to a permutation from G . \square

Example. Consider the idempotents

$$E = \begin{pmatrix} 0 & -3 & -2 & -1 \\ -1 & 0 & -3 & -2 \\ -2 & -1 & 0 & -3 \\ -3 & -2 & -1 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & -2 \\ -2 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \end{pmatrix}.$$

in $M_4(\mathbb{FT})$, both of which have rank 4. We note that the classical mean of the columns of E (respectively, F) is a tropical scaling of $(0, 0, 0, 0)^T$ by -1.5 (respectively, -1). It follows that any unit commuting with E (or F) must be a permutation matrix. The maximum subgroups H_E and H_F (which by Theorem 3.3 are also the automorphism groups of the \mathbb{FT} -modules $C(E)$ and $C(F)$, illustrated in Figure 2) therefore have the form $G \times \mathbb{R}$ where G is the automorphism group of the appropriate coloured digraph from Figure 3. These are easily seen to be the cyclic group of order 4 and the dihedral group of order 8 respectively, so we have $H_E \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{R}$ and $H_F \cong D_4 \times \mathbb{R}$.

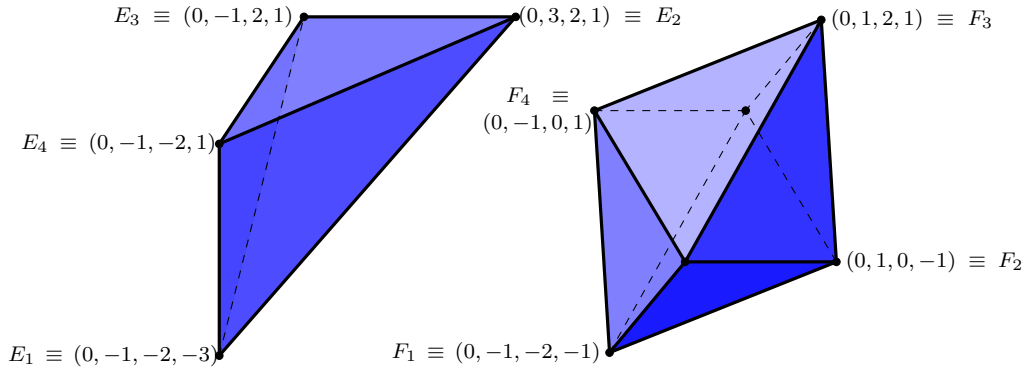


FIGURE 2. Two tropical polytopes in projective 3-space.

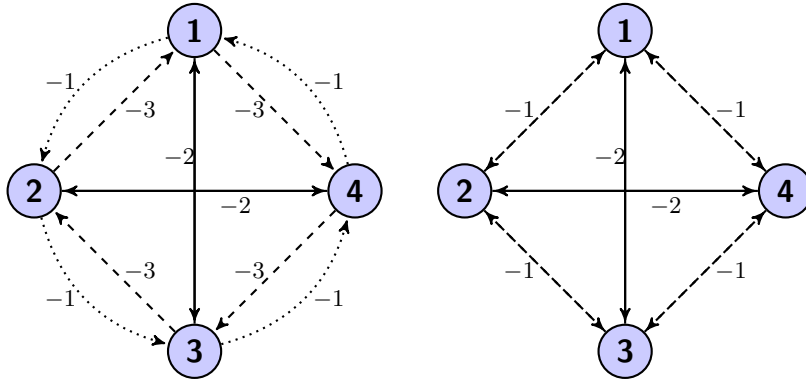


FIGURE 3. The coloured digraphs on 4 nodes corresponding to the idempotents E and F .

It is easily seen that the regular permutation representation of any finite group (that is, the group’s action on itself by translation) is a 2-closed permutation representation. Hence, we see that all groups of the form $G \times \mathbb{R}$, with G finite, are maximal groups of sufficiently large tropical matrices:

Corollary 5.11. *For any finite group G , $G \times \mathbb{R}$ arises as a maximal subgroup of $M_n(\mathbb{FT})$ for all $n \geq |G|$.*

In general the bound given by Corollary 5.11 is sharp; for example, computational calculations show that the alternating group A_4 does not have a 2-closed permutation representation on fewer than $12 = |A_4|$ points. Thus, although $A_4 \times \mathbb{R}$ is a subgroup of $M_4(\mathbb{FT})$, it does not arise as a *maximal* subgroup until $M_{12}(\mathbb{FT})$. (In fact, one can also verify with computational assistance that A_4 is the *smallest* group which does not admit a 2-closed permutation representation of degree its permutation degree.)

REFERENCES

- [1] M. Akian, R. Bapat, and S. Gaubert. Max-plus algebra. In *Handbook of Linear Algebra*. Chapman and Hall, 2006.
- [2] P. Butkovič. *Max-linear systems: theory and algorithms*. Springer, 2010.
- [3] P. Butkovič, H. Schneider, and S. Sergeev. Generators, extremals and bases of max cones. *Linear Algebra Appl.*, 421(2-3):394–406, 2007.
- [4] P. J. Cameron. *Permutation groups*, volume 45 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

- [5] P. J. Cameron, M. Giudici, G. A. Jones, W. M. Kantor, M. H. Klin, D. Marušič, and L. A. Nowitz. Transitive permutation groups without semiregular subgroups. *J. London Math. Soc. (2)*, 66(2):325–333, 2002.
- [6] M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 213–242. Cambridge Univ. Press, Cambridge, 2005.
- [7] M. Develin and B. Sturmfels. Tropical convexity. *Doc. Math.*, 9:1–27 (electronic), 2004.
- [8] P. Guillon, Z. Izhakian, Mairesse J., and Merlet G. The ultimate rank of tropical matrices,. *J. Algebra*, 437:222–248, 2015.
- [9] A. Guterman and Y. Shitov. Rank functions of tropical matrices,. *Linear Algebra Appl.*, 498:326–348, 2016.
- [10] C. Hollings and M. Kambites. Tropical matrix duality and Green’s \mathcal{D} relation. *J. London Math. Soc.*, 86:520–538, 2012.
- [11] J. M. Howie. *Fundamentals of Semigroup Theory*. Clarendon Press, 1995.
- [12] Z. Izhakian, M. Johnson, and M. Kambites. Pure dimension and projectivity of tropical polytopes. [arXiv:1106.4525 \[math.RA\]](https://arxiv.org/abs/1106.4525), 2011.
- [13] M. Johnson and M. Kambites. Multiplicative structure of 2×2 tropical matrices. *Linear Algebra Appl.*, 435:1612–1625, 2011.
- [14] M. Johnson and M. Kambites. Green’s \mathcal{J} -order and the rank of tropical matrices. *J. Pure Appl. Algebra*, 217:280–292, 2013.
- [15] M. Johnson and M. Kambites. Convexity of tropical polytopes,. *Linear Algebra Appl.*, 485:531–544, 2015.
- [16] G. Merlet. Semigroup of matrices acting on the max-plus projective space. *Linear Algebra Appl.*, 432(8):1923–1935, 2010.
- [17] J. Okniński. *Semigroups of matrices*, volume 6 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1998.
- [18] S. Sergeev, H. Schneider, and P. Butkovič. On visualization scaling, subeigenvectors and Kleene stars in max algebra. *Linear Algebra Appl.*, 431(12):2395–2406, 2009.
- [19] Y. Shitov. Tropical matrices and group representations. *J. Algebra*, 370:1–4, 2012.
- [20] E. Wagnaur. Moduloids and pseudomodules I. Dimension theory. *Discrete Math.*, 98(1):57–73, 1991.