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THE MODULI OF SMOOTH HYPERSURFACES WITH LEVEL STRUCTURE

A. JAVANPEYKAR AND D. LOUGHRAN

ABSTRACT. We construct the moduli space of smooth hypersurfaces with level N structure over $\mathbb{Z}[1/N]$. As an application we show that, for N large enough, the stack of smooth hypersurfaces over $\mathbb{Z}[1/N]$ is uniformisable by a smooth affine scheme. To prove our results, we use the Lefschetz trace formula to show that automorphisms of smooth hypersurfaces act faithfully on their cohomology. We also prove a global Torelli theorem for smooth cubic threefolds over fields of odd characteristic.

1. INTRODUCTION

The moduli of smooth proper curves of genus g with $g \geq 2$, or principally polarized abelian schemes of fixed dimension, or polarized K3 surfaces of fixed degree are smooth finite type separated Deligne-Mumford stacks over \mathbb{Z} . All these stacks admit *level structures* [23, 30, 31]. Such structures are usually introduced to help rigidify the moduli problem and lead to interesting theory and applications [29].

The aim of this note is to construct a moduli stack of smooth hypersurfaces with level structure. We will define a level N structure on a smooth hypersurface to be a trivialization of its cohomology with $\mathbb{Z}/N\mathbb{Z}$ -coefficients (see §3 for details). Key to our construction is the following result on the action of an automorphism of a smooth hypersurface on its cohomology.

Theorem 1.1. *Let $d \geq 3$ and $n \geq 1$ be integers with $(d, n) \neq (3, 1)$. Let k be a field and let ℓ be a prime number which is invertible in k . Let X be a smooth hypersurface of degree d in \mathbb{P}_k^{n+1} , and let $\sigma \in \text{Aut}(X)$ be non-trivial. If $\text{char}(k) = 0$ or the order of σ is coprime to $\text{char}(k)$, then σ acts non-trivially on $H^n(X_{\bar{k}, \text{ét}}, \mathbb{Q}_\ell)$.*

The question of whether the automorphism group of a variety acts faithfully on its cohomology has been investigated for other families of varieties, such as Enriques surfaces [24], hyperkähler varieties [4, Prop. 9], [9, §3], [30, §2.4], and some surfaces of general type [11, 28].

Let $d \geq 3$ and let $n \geq 1$ be integers with $(d, n) \neq (3, 1)$. Let $\mathcal{C}_{d,n}$ be the stack of smooth hypersurfaces of degree d in \mathbb{P}^{n+1} (see §3). Mumford has shown that $\mathcal{C}_{d,n}$ is a smooth finite type separated Deligne-Mumford stack over \mathbb{Z} whose coarse moduli space is an affine scheme. For an integer $N \geq 1$, we will define an algebraic stack $\mathcal{C}_{d,n}^{[N]}$ over $\mathbb{Z}[1/N]$ parametrizing smooth hypersurfaces with a level N structure, and our

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main result (Theorem 3.2) is that it is representable by a smooth affine scheme when N is large enough.

To state our result, let $a_{d,n}$ be the product of all primes p for which there exist an algebraically closed field k and a smooth hypersurface X of degree d in \mathbb{P}_k^{n+1} with a linear automorphism of order p (Definition 3.1). We will show in Section 3 that $a_{d,n}$ is a well-defined positive integer. It divides the product of all orders of linear automorphism groups of smooth hypersurfaces of degree d in \mathbb{P}_k^{n+1} , as k runs over all algebraically closed fields. In particular, Theorem 1.1 shows that any non-trivial automorphism σ acts faithfully on the cohomology of X as long as the characteristic of k is coprime to $a_{d,n}$.

Theorem 1.2. *Let $d \geq 3$ and let $n \geq 1$ be integers. For all $N \geq 3$ coprime to $a_{d,n}$, the stack $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}$ is uniformisable by a smooth affine scheme U over $\mathbb{Z}[1/N]$.*

Here, following Noohi [26, Def. 6.1], we say that an algebraic stack X is uniformisable if there exist an algebraic space U and a finite étale morphism $U \rightarrow X$.

Versions of Theorem 1.1 and Theorem 1.2 were obtained by the authors in [19] for other complete intersections in projective space *over* \mathbb{C} . These constructions used the infinitesimal Torelli theorem for smooth complete intersections and spreading out arguments, hence only showed that $\mathcal{C}_{d,n}$ becomes uniformisable after base-changing to some non-explicit arithmetic curve $B = \text{Spec } \mathcal{O}_K[S^{-1}]$ (cf. the proof of [19, Prop. 2.12]). The significance of Theorem 1.2 is that we obtain an uniformisation over $\mathbb{Z}[1/N]$ for sufficiently large N .

To illustrate how one can use the faithfulness of the action of the automorphism group on étale cohomology, we finish with an application to the global Torelli problem for smooth cubic threefolds (here it is already known that any automorphism acts faithfully on cohomology, by work of Pan [27, Thm. 1.2]). A famous theorem of Clemens and Griffiths [14] states that any cubic threefold over \mathbb{C} is uniquely determined by its *intermediate Jacobian*. The intermediate Jacobian is usually defined via transcendental techniques, however it has been known for some time that this theory can be made to work for cubic threefolds over other fields [10, 17], with a definitive construction over schemes of characteristic not equal to 2 being given recently by Achter in [2]. In [19] we obtained an extension of Clemens and Griffiths' Torelli theorem to arbitrary fields of characteristic 0. We will use the Torelli theorem of Beauville [3] over algebraically closed fields of characteristic not equal to 2 to obtain the following.

Theorem 1.3. *Let k be a field of characteristic not equal to 2 and let X_1, X_2 be cubic threefolds over k . If the intermediate Jacobians $J(X_1)$ and $J(X_2)$ of X_1 and X_2 are isomorphic as principally polarised abelian varieties over k , then $X_1 \cong X_2$.*

Whilst finishing this paper, we learned of the recent results of Chen-Pan-Zhang [12]. Here the aforementioned methods relying on infinitesimal Torelli are used in a way similar to [19] to prove a version of Theorem 1.1 for complete intersections, including some cases of positive characteristic for which the infinitesimal Torelli theorem holds.

Our paper deals with tame automorphisms in arbitrary characteristic and gives a new and completely different elementary proof using the Lefschetz trace formula.

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Conventions. For X a scheme over a field k , we let $\text{Aut}(X)$ be the group of automorphisms of X over k . If B is a scheme and $N \neq 0$ is an integer, then we write $B_{\mathbb{Z}[1/N]}$ for $B \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}[1/N]$.

2. TAME LINEAR AUTOMORPHISMS ACT FAITHFULLY ON COHOMOLOGY

Let k be a field and let ℓ be a prime number which is invertible in k . Let X be a smooth hypersurface of degree d in \mathbb{P}_k^{n+1} . Recall that, by the Lefschetz hyperplane section theorem, the cohomology ring $H^*(X_{\bar{k}, \text{ét}}, \mathbb{Z}_\ell)$ is torsion free, and that for $i \neq n$, the group $H^i(X_{\bar{k}, \text{ét}}, \mathbb{Z}_\ell)$ is trivial if i is odd and isomorphic to \mathbb{Z}_ℓ if i is even. In particular, the only “interesting” cohomology group is $H^n(X_{\bar{k}, \text{ét}}, \mathbb{Z}_\ell)$.

Denote by $\text{Lin}(X)$ the group of *linear automorphisms* of the k -scheme X , i.e. those automorphism which are induced by an automorphism of the ambient projective space. When $d \geq 3$, it is known that $\text{Lin}(X)$ is finite [7, Thm. 3.1]. Moreover, if $(d, n) \neq (3, 1), (4, 2)$, then $\text{Lin}(X) = \text{Aut}(X)$ [7, Thm. 3.1].

Proposition 2.1. *Let $d \geq 3$ and $n \geq 1$ be integers such that $(d, n) \neq (3, 1)$. Let k be a field and let ℓ be a prime number which is invertible in k . Let X be a smooth hypersurface of degree d in \mathbb{P}_k^{n+1} and let $\sigma \in \text{Lin}(X)$ be non-trivial. If $\text{char}(k) = 0$ or the order of σ is coprime to $\text{char}(k)$, then σ acts non-trivially on $H^n(X_{\bar{k}, \text{ét}}, \mathbb{Q}_\ell)$.*

Proof. To prove the result, we may assume that k is algebraically closed. We may also assume that $\text{char}(k) > 0$, as when $\text{char}(k) = 0$ the result is a special case of [19, Prop. 2.16].

Let $\sigma \in \text{Lin}(X)$ be non-trivial. As the order of σ is coprime to $\text{char}(k)$, we may decompose $H^0(X, \mathcal{O}_X(1))$ into a direct sum of t eigenspaces $H^0(X, \mathcal{O}_X(1)) = \bigoplus_{i=1}^t V_i$ with dimensions $m_i \geq 1$. Note that

$$m_1 + \cdots + m_t = n + 2. \quad (2.1)$$

The fixed locus X^σ of σ acting on X is the disjoint union $\bigsqcup_{i=1}^t X_i$, where $X_i = X \cap \mathbb{P}(V_i)$ and $\mathbb{P}(V_i) \subset \mathbb{P}_k^{n+1}$ denotes the corresponding projective subspace. As the order of σ is coprime to $\text{char}(k)$, the fixed locus X^σ is smooth [16, Prop. A.8.10], hence each X_i is smooth. An elementary argument, given by choosing a basis for each V_i , reveals three possible cases for the X_i . Namely, on reordering the X_i if necessary, there exists $0 \leq r \leq s \leq t$ such that the following hold.

- If $1 \leq i \leq r$, then X_i is a smooth hypersurface of degree d in $\mathbb{P}_k^{m_i-1}$.

- If $r < i \leq s$, then $X_i \cong \mathbb{P}_k^{m_i-1}$.
- If $s < i \leq t$, then $X_i = \emptyset$.

Assume now that σ acts trivially on $H^n(X_{\bar{k},\text{ét}}, \mathbb{Q}_\ell)$. Then $\text{tr}(\sigma^*, H^*(X_{\bar{k},\text{ét}}, \mathbb{Q}_\ell))$ equals the ℓ -adic Euler characteristic $\chi(X)$ of X . The Lefschetz trace formula (see e.g. (III.4.11.4), p.111 of [18]) then implies that

$$\chi(X) = \text{tr}(\sigma^*, H^*(X_{\bar{k},\text{ét}}, \mathbb{Q}_\ell)) = \chi(X^\sigma). \quad (2.2)$$

We will derive a contradiction using the well-known formula

$$\chi(X) = n + 2 + \frac{(1-d)^{n+2} - 1}{d}, \quad (2.3)$$

which can be found in the discussion following [13, Cor. 2.5], for example. Applying the Lefschetz trace formula (2.2) and using (2.3) we obtain

$$n + 2 + \frac{(1-d)^{n+2} - 1}{d} = \sum_{i=1}^r \left(m_i + \frac{(1-d)^{m_i} - 1}{d} \right) + \sum_{i=r+1}^s m_i.$$

Using (2.1) and rearranging, we find that

$$(1-d)^{n+2} = 1 - r - d(t-s) + \sum_{i=1}^r (1-d)^{m_i}. \quad (2.4)$$

We shall use this formula to obtain a contradiction. We first consider some special cases. If $n = 1$ then the Lefschetz trace formula yields

$$\#X^\sigma = 2 - (d-1)(d-2),$$

which is a contradiction, as we assume that $d > 3$ when $n = 1$. If $t = s$ and $r = 0$ then (2.4) cannot be satisfied, since $d \geq 3$.

Consider now the remaining case where $r + (t-s) \geq 1$ and $n \geq 2$. Note that $t \geq 2$ as σ is non-trivial. We find that

$$\begin{aligned} & |1 - r - d(t-s) + \sum_{i=1}^r (1-d)^{m_i}| \\ & \leq (t-s) + r - 1 + (d-1)(t-s) + \sum_{i=1}^r (d-1)^{m_i} \\ & \leq t - 1 + \sum_{i=1}^t (d-1)^{m_i}. \end{aligned} \quad (2.5)$$

To proceed, we require the following elementary inequalities.

$$n + 2 \leq (1/4) \cdot x^{n+2} \quad \text{for } n \geq 2 \text{ and } x \geq 2. \quad (2.6)$$

$$\sum_{i=1}^t x^{m_i} \leq (3/4) \cdot x^{n+2} \quad \text{for } n \geq 1, m_i \geq 1, t \geq 2 \text{ and } x \geq 2. \quad (2.7)$$

The inequality (2.6) is trivial. For (2.7), on using (2.1) we have

$$\begin{aligned} \sum_{i=1}^t x^{m_i} &= x^{n+2} \left(\frac{1}{x^{m_2+\dots+m_t}} + \dots + \frac{1}{x^{m_1+\dots+m_{t-1}}} \right) \\ &\leq x^{n+2} \left(\frac{1}{2^{m_2+\dots+m_t}} + \dots + \frac{1}{2^{m_1+\dots+m_{t-1}}} \right). \end{aligned}$$

For $t \geq 3$ we obtain

$$\frac{1}{2^{m_2+\dots+m_t}} + \dots + \frac{1}{2^{m_1+\dots+m_{t-1}}} \leq \frac{1}{2^{t-1}} + \dots + \frac{1}{2^{t-1}} = \frac{t}{2^{t-1}} \leq \frac{3}{4},$$

as required. For $t = 2$, using (2.1) we find that there is some i for which $m_i \geq 2$ (as $n \geq 1$), which also yields (2.7), as required.

Applying (2.6) and (2.7) to (2.5) we find that

$$\begin{aligned} |1 - r - d(t - s) + \sum_{i=1}^r (1 - d)^{m_i}| &\leq (1/4) \cdot (d - 1)^{n+2} - 1 + (3/4) \cdot (d - 1)^{n+2} \\ &< |1 - d|^{n+2}, \end{aligned}$$

which contradicts (2.4). This completes the proof. \square

Proof of Theorem 1.1. If $(d, n) \neq (4, 2)$ then this follows immediately from Proposition 2.1, as we have $\text{Lin}(X) = \text{Aut}(X)$ in such cases. So let $(d, n) = (4, 2)$ and let $\sigma \in \text{Aut } X$ act trivially on $H^2(X_{\bar{k}, \acute{e}t}, \mathbb{Q}_\ell)$. Then it also acts trivially on $\text{Pic } X_{\bar{k}}$, as the cycle class map is injective here. Hence $\sigma \in \text{Lin}(X)$, and so the result again follows from Proposition 2.1. \square

We obtain the following corollary, which will be required for attaching level structure.

Corollary 2.2. *Let $d \geq 3$ and $n \geq 1$ be integers such that $(d, n) \neq (3, 1)$. Let k be a field and let $N \geq 3$ be an integer which is invertible in k . Suppose that $\text{char}(k) = 0$ or $\text{char}(k) > 0$ is coprime to $a_{d,n}$. If X is a smooth hypersurface of degree d in \mathbb{P}_k^{n+1} , then the homomorphism*

$$\text{Aut}(X) \rightarrow \text{Aut}(H^n(X_{\bar{k}, \acute{e}t}, \mathbb{Z}/N\mathbb{Z}))$$

is injective.

Proof. We may assume that N is a power of some prime number ℓ . Let $\sigma \in \text{Lin}(X)$ be a linear automorphism which acts trivially on $H^n(X_{\bar{k}, \acute{e}t}, \mathbb{Z}/N\mathbb{Z})$. Consider the action of σ on $H^n(X_{\bar{k}, \acute{e}t}, \mathbb{Z}_\ell)$, which we view as given by some \mathbb{Z}_ℓ -matrix A such that $A \bmod N$ is the identity matrix. As $\text{Lin}(X)$ is finite and $H^n(X_{\bar{k}, \acute{e}t}, \mathbb{Z}_\ell)$ is torsion free, we see that A is semi-simple and that the eigenvalues of $A \otimes \bar{\mathbb{Z}}_\ell$ are roots of unity which are equal to 1 mod N , as $A \bmod N$ is the identity matrix. Therefore, as $N \geq 3$, a lemma of Minkowski and Serre (see the appendix of [31] or the more general [33, Thm. 6.7]) implies that each eigenvalue of $A \otimes \bar{\mathbb{Z}}_\ell$ is in fact equal to 1, and hence A , being semi-simple, is the identity matrix. Thus σ acts trivially on $H^n(X_{\bar{k}, \acute{e}t}, \mathbb{Z}_\ell)$.

However, by our assumptions, the order of σ is coprime to the characteristic of k and hence σ is trivial by Theorem 1.1, as required. \square

Remark 2.3. Throughout this section we excluded the necessary case $(d, n) = (3, 1)$. Indeed, let ℓ be a prime and let k be an algebraically closed field with $\ell \in k^*$. If X is a smooth cubic in \mathbb{P}_k^2 , then

$$\ker(\mathrm{Lin}(X) \rightarrow \mathrm{Aut}(H^1(X_{\bar{k}, \text{ét}}, \mathbb{Q}_\ell)) \cong J[3](\bar{k})),$$

where $J[3]$ denotes the 3-torsion group of the Jacobian J of X . This kernel is non-trivial if and only if either $\mathrm{char}(k) \neq 3$ or $\mathrm{char}(k) = 3$ and J is ordinary. In particular, the hypothesis $(d, n) \neq (3, 1)$ can not be omitted in Theorem 1.1. Quadric hypersurfaces are also easily seen not to satisfy the conclusion of Proposition 2.1.

Remark 2.4. The proof of Proposition 2.1 breaks down for wild automorphisms, i.e. those automorphisms σ whose order is divisible by $\mathrm{char}(k)$. Here X^σ need no longer be smooth, hence the Lefschetz trace formula (2.2) does not take such a simple form (see [18, §III.4.11]). The tame case is however sufficient for the application to level structures for hypersurfaces.

3. LEVEL STRUCTURE

In this section we give the promised application to the moduli space of smooth hypersurfaces with level structure.

3.1. The stack of smooth hypersurfaces. Let $d \geq 3$ and $n \geq 1$. Let $\mathrm{Hilb}_{d,n}$ be the Hilbert scheme of degree d smooth hypersurfaces in \mathbb{P}^{n+1} over \mathbb{Z} . There is a natural left PGL_{n+2} -action on $\mathrm{Hilb}_{d,n}$. The following quotient stack

$$\mathcal{C}_{d,n} := [\mathrm{PGL}_{n+2} \backslash \mathrm{Hilb}_{d,n}]$$

is the (moduli) stack of smooth hypersurfaces of degree d in \mathbb{P}^{n+1} . (We use [22, §2.4.2] as our main reference for quotient stacks. Note that, however, we consider left actions, whereas *loc. cit.* considers right actions.) A precise description of the functor of points of $\mathcal{C}_{d,n}$ is given in [6, §2.3.2]. For a smooth hypersurface X of degree d in \mathbb{P}^{n+1} over an algebraically closed field k , the group $\mathrm{Lin}(X)$ is the group of k -points of the inertia group scheme of the corresponding object in $\mathcal{C}_{d,n}(k)$.

It follows from the arguments given in [25, Cor. 2.5 and Prop. 4.2] and [32] that $\mathcal{C}_{d,n}$ is a smooth finite type separated algebraic stack with finite diagonal over \mathbb{Z} whose coarse moduli space $\mathcal{C}_{d,n}^{\mathrm{coarse}}$ is an affine scheme. Moreover, if $(d, n) \neq (3, 1)$, then $\mathcal{C}_{d,n}$ is a Deligne-Mumford stack over \mathbb{Z} ; see [7, Thm. 1.6].

3.2. Level structure. Let $N \geq 1$ be a positive integer. Let $\mathcal{C}_{d,n}^{[N]}$ be the category fibred in groupoids over $\mathbb{Z}[1/N]$ whose objects are triples $(S, f : X \rightarrow S, \phi)$, where S is a scheme over $\mathbb{Z}[1/N]$, $f : X \rightarrow S$ is an object of $\mathcal{C}_{d,n}(S)$ and $\phi : R_{\text{ét}}^n f_*(\mathbb{Z}/N\mathbb{Z}) \rightarrow (\mathbb{Z}/N\mathbb{Z})_S^{b_{d,n}}$ is an isomorphism of constructible sheaves on S . Here $b_{d,n}$ denotes the n th Betti number of some (hence any) smooth hypersurface of degree d in \mathbb{P}^{n+1} . A morphism from a triple $(S, f : X \rightarrow S, \phi)$ to a triple $(S', f' : X' \rightarrow S', \phi')$ is defined to be a pair (ρ, φ) , where:

- ρ is a morphism from $(S, f : X \rightarrow S)$ to $(S', f' : X' \rightarrow S')$ in $\mathcal{C}_{d,n}$, and
- φ is an isomorphism of sheaves from $R_{\text{ét}}^n f_*(\mathbb{Z}/N\mathbb{Z})$ to the pull-back of the sheaf $R_{\text{ét}}^n f'_*(\mathbb{Z}/N\mathbb{Z})$ to S which respects the isomorphisms

$$\phi : R_{\text{ét}}^n f_*(\mathbb{Z}/N\mathbb{Z}) \rightarrow (\mathbb{Z}/N\mathbb{Z})_S^{b_{d,n}} \quad \text{and} \quad \phi' : R_{\text{ét}}^n f'_*(\mathbb{Z}/N\mathbb{Z}) \rightarrow (\mathbb{Z}/N\mathbb{Z})_{S'}^{b_{d,n}}.$$

We call $\mathcal{C}_{d,n}^{[N]}$ the stack of smooth n -dimensional hypersurfaces of degree d with *level N structure*.

Definition 3.1. For $d \geq 3$ and $n \geq 1$, let $A_{d,n}$ be the set of prime numbers p such that there exist an algebraically closed field k and a smooth hypersurface X of degree d in \mathbb{P}_k^{n+1} with a linear automorphism of order p . As $\mathcal{C}_{d,n}$ is a finite type algebraic stack over \mathbb{Z} with finite diagonal, a standard stratification argument shows that the set $A_{d,n}$ is finite. Define

$$a_{d,n} := \prod_{p \in A_{d,n}} p.$$

Note that, if N is coprime to $a_{d,n}$, then $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}$ is a tame stack [1, Thm. 3.2].

We now show that, for $N \geq 3$ coprime to $a_{d,n}$ and $(d,n) \neq (3,1)$, the stack $\mathcal{C}_{d,n}^{[N]}$ is in fact an affine scheme over $\mathbb{Z}[1/N]$.

Theorem 3.2. *Let $N \geq 1$ be an integer. Suppose that $(d,n) \neq (3,1)$.*

- (1) *The stack $\mathcal{C}_{d,n}^{[N]}$ is a $\text{GL}_{b_{d,n}}(\mathbb{Z}/N\mathbb{Z})$ -torsor over $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}$.*
- (2) *The stack $\mathcal{C}_{d,n}^{[N]}$ is smooth finite type separated and Deligne-Mumford over $\mathbb{Z}[1/N]$ with an affine coarse moduli space.*
- (3) *If $N \geq 3$ is coprime to $a_{d,n}$, then the stack $\mathcal{C}_{d,n}^{[N]}$ is representable by a smooth affine scheme over $\mathbb{Z}[1/N]$.*

Proof. The structure as a $\text{GL}_{b_{d,n}}(\mathbb{Z}/N\mathbb{Z})$ -torsor is given by the (representable) forgetful morphism $\mathcal{C}_{d,n}^{[N]} \rightarrow \mathcal{C}_{d,n,\mathbb{Z}[1/N]}$. This proves (1).

Since $d \geq 3$, $n \geq 1$ and $(d,n) \neq (3,1)$, the stack $\mathcal{C}_{d,n}$ is a smooth finite type Deligne-Mumford stack over \mathbb{Z} . As $\mathcal{C}_{d,n}^{[N]} \rightarrow \mathcal{C}_{d,n,\mathbb{Z}[1/N]}$ is a representable étale finite type morphism, we may pull-back an étale finite type presentation of $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}$ to $\mathcal{C}_{d,n}^{[N]}$ to find that $\mathcal{C}_{d,n}^{[N]}$ is a smooth finite type Deligne-Mumford stack. Furthermore, since $\mathcal{C}_{d,n}$ is a separated algebraic stack and the morphism $\mathcal{C}_{d,n}^{[N]} \rightarrow \mathcal{C}_{d,n,\mathbb{Z}[1/N]}$ is finite, the stack $\mathcal{C}_{d,n}^{[N]}$ is separated over $\mathbb{Z}[1/N]$ with finite inertia. In particular, by [21], the stack $\mathcal{C}_{d,n}^{[N]}$ has a coarse moduli space, say $\mathcal{C}_{d,n}^{[N],\text{co}}$. Let $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}^{\text{co}}$ be the coarse moduli space of $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}$. Since $\mathcal{C}_{d,n}^{[N]} \rightarrow \mathcal{C}_{d,n,\mathbb{Z}[1/N]}$ is finite, the induced morphism $\mathcal{C}_{d,n}^{[N],\text{co}} \rightarrow \mathcal{C}_{d,n,\mathbb{Z}[1/N]}^{\text{co}}$ is finite. As $\mathcal{C}_{d,n,\mathbb{Z}[1/N]}^{\text{co}}$ is affine, it follows that $\mathcal{C}_{d,n}^{[N],\text{co}}$ is affine [22, Thm. A.2]. This proves (2).

Now, to prove (3), let $N \geq 3$ be coprime to $a_{d,n}$. Note that Corollary 2.2 implies that the geometric points of the stack $\mathcal{C}_{d,n}^{[N]}$ have trivial automorphism groups. In particular, by [15, Cor. 2.2.5.(1)], the stack $\mathcal{C}_{d,n}^{[N]}$ is an algebraic space over $\mathbb{Z}[1/N]$. Thus, the coarse moduli space morphism $\mathcal{C}_{d,n}^{[N]} \rightarrow \mathcal{C}_{d,n}^{[N],\text{co}}$ is an isomorphism. This concludes the proof. \square

Proof of Theorem 1.2. By Theorem 3.2, it suffices to treat the case $(d, n) = (3, 1)$. The stack $\mathcal{C}_{3,1,\mathbb{Z}[1/3]}$ is a smooth finite type separated Deligne-Mumford stack over $\mathbb{Z}[1/3]$ with affine coarse moduli space. Let \mathcal{M} be the stack of elliptic curves over $\mathbb{Z}[1/3]$, and let $\mathcal{E} \rightarrow \mathcal{M}$ be the universal family over \mathcal{M} . Let U be an affine scheme over $\mathbb{Z}[1/3]$ equipped with a finite étale morphism $U \rightarrow \mathcal{M}$ (for instance, we may take $U = \mathcal{Y}(3)$ to be the fine moduli scheme of elliptic curves with full level 3 structure over $\mathbb{Z}[1/3]$; see [20, Cor. 4.7.2]).

By [8, Prop. 6.1 and Prop. 6.4], the morphism of stacks $\mathcal{C}_{3,1,\mathbb{Z}[1/3]} \rightarrow \mathcal{M}$ induced by the Jacobian is a neutral gerbe for the finite étale group scheme $\mathcal{E}[3]$ of 3-torsion points of $\mathcal{E} \rightarrow \mathcal{M}$ over $\mathbb{Z}[1/3]$, i.e. $\mathcal{C}_{3,1,\mathbb{Z}[1/3]} \cong B(\mathcal{E}[3])$ as stacks over \mathcal{M} . By pull-back, we find that the stack $\mathcal{C}_{3,1,\mathbb{Z}[1/3]}$ is uniformisable by the classifying stack $B(\mathcal{E}[3]_U)$ over U . We summarise these maps in the following diagram

$$\begin{array}{ccc} B(\mathcal{E}[3]_U) & \xrightarrow{\text{finite étale}} & \mathcal{C}_{3,1,\mathbb{Z}[1/3]} \\ \text{neutral } \mathcal{E}[3]\text{-gerbe} \downarrow & & \downarrow \text{neutral } \mathcal{E}[3]\text{-gerbe} \\ U & \xrightarrow{\text{finite étale}} & \mathcal{M}. \end{array}$$

Since $\mathcal{E}[3]_U \rightarrow U$ is a finite étale group scheme, there is a natural finite étale morphism $U \rightarrow B(\mathcal{E}[3]_U)$. In particular, the composition $U \rightarrow B(\mathcal{E}[3]_U) \rightarrow \mathcal{C}_{3,1,\mathbb{Z}[1/3]}$ is finite étale, and hence $\mathcal{C}_{3,1,\mathbb{Z}[1/3]}$ is uniformisable by U . Thus, for all $N \geq 1$, the stack $\mathcal{C}_{3,1,\mathbb{Z}[1/(3N)]}$ is uniformisable by a smooth affine scheme over $\mathbb{Z}[1/3N]$. As $a_{3,1}$ is divisible by 6, this concludes the proof. \square

Remark 3.3. Note that there are one-dimensional smooth finite type separated Deligne-Mumford stacks over \mathbb{C} which are not uniformisable; see [5].

4. A TORELLI THEOREM

We now prove Theorem 1.3. By [2, Thm. B], the intermediate Jacobian gives rise to a morphism of stacks

$$J : \mathcal{C}_{3,3,\mathbb{Z}[1/2]} \rightarrow \mathcal{A}_{5,\mathbb{Z}[1/2]},$$

where \mathcal{A}_5 denotes the stack of principally polarised abelian fivefolds. Theorem 1.3 follows immediately from the following more general result.

Theorem 4.1. *The morphism of stacks*

$$J : \mathcal{C}_{3,3,\mathbb{Z}[1/2]} \rightarrow \mathcal{A}_{5,\mathbb{Z}[1/2]},$$

is separated, representable by schemes, and universally injective.

Proof. The proof is similar to the proof of [19, Prop. 3.2]. The separatedness of J follows from a simple application of [7, Thm. 1.7] (cf. [19, Lem. 2.6]). Let k be a field of characteristic not equal to 2 and let $\ell \in k^*$. The intermediate Jacobian of a cubic threefold X over k comes with a canonical isomorphism $H^3(X_{\bar{k},\text{ét}}, \mathbb{Q}_\ell(1)) \cong H^1(J(X)_{\bar{k},\text{ét}}, \mathbb{Q}_\ell)$ [2, Prop. 3.6(a)]. As the construction of the intermediate Jacobian is functorial and $\text{Aut } J(X)$ acts faithfully on $H^1(J(X)_{\bar{k},\text{ét}}, \mathbb{Q}_\ell)$, we deduce from [27, Thm. 1.2] that the natural map $\text{Aut } X \rightarrow \text{Aut } J(X)$ is injective. As in the proof

of [19, Prop. 3.1], we then use [34, Tag 04Y5] and [15, Cor. 2.2.7] to see that J is representable by algebraic spaces. However, Beauville's Torelli theorem [3] implies that J is injective on geometric points. That J is representable by schemes now follows from [22, Thm. A.2]. Finally, as J is injective on geometric points and representable by schemes, it follows from [34, Tag 03MU] that J itself is universally injective. \square

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