Geometric axioms for differentially closed fields with several commuting derivations

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GEOMETRIC AXIOMS FOR DIFFERENTIALLY CLOSED FIELDS WITH SEVERAL COMMUTING DERIVATIONS

Omar León Sánchez
University of Waterloo, ON, Canada
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Abstract. A geometric first-order axiomatization of differentially closed fields of characteristic zero with several commuting derivations, in the spirit of Pierce-Pillay [13], is formulated in terms of a relative notion of prolongation for Kolchin-closed sets.

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1. Introduction

An ordinary differential field is a field of characteristic zero equipped with a derivation, that is, an additive map \( \delta : K \to K \) such that \( \delta(ab) = (\delta a)b + a(\delta b) \). A differentially closed field is a differential field \((K, \delta)\) such that every system of differential polynomial equations in several variables, with a solution in some differential extension, has a solution in \(K\). An elegant first-order axiomatization of the class of ordinary differentially closed fields was given by Blum in [1]. In [13], Pierce and Pillay give a geometric axiomatization. Their axioms say that \((K, \delta)\) is differentially closed if and only if \(K\) is algebraically closed and whenever \(V\) and \(W\) are irreducible (affine) algebraic varieties with \(W\) contained in the prolongation of \(V\) and projecting dominantly onto \(V\), then there is a \(K\)-point in \(W\) of the form \( (\bar{x}, \delta \bar{x}) \).

Similarly, a field \(K\) of characteristic zero equipped with \(m\) commuting derivations is differentially closed if every system of partial differential polynomial equations in several variables with a solution in some extension has a solution in \(K\). A first-order axiomatization generalizing Blum’s was given by McGrail in [10] (other work along these lines can be found in Tressl [15] and Yaffe [17]). However, the Pierce-Pillay condition mentioned above is no longer true for differentially closed fields with \(m\) commuting derivations (see [12], Counterexample 3.2). Nonetheless, in [12], Pierce does find geometric first-order conditions on a subvariety \(W\) of the \(r\)-th prolongation of affine space that will ensure that its projection on the \((r-1)\)-th prolongation has a \(K\)-point of the form \( (\delta^{r_1} \cdots \delta^{r_m} \bar{x} : r_1 + \cdots + r_m < r) \). His conditions include Pierce-Pillay type conditions, but also a requirement on which of the coordinates form a transcendence basis for the function field \(K(W)\). Pierce shows that these conditions (ranging over all \(r\)) do axiomatize differentially closed fields. However, his axiomatization does not specialize to the Pierce-Pillay axioms and ultimately has a different flavor.

In this paper we take a different approach, establishing an axiomatization of differentially closed fields with \((m+1)\) commuting derivations which is geometric relative to the theory of differentially closed fields with \(m\) derivations. Our axioms
are a precise generalization of the Pierce-Pillay axioms, and can be used in very much the same way. Two complications arise in our setting that do not appear in the ordinary case: one has to do with extending commuting derivations and the other has to do with first-order axiomatizability. Differential-algebraic results due to Kolchin are behind our solutions to both of these problems.

Suppose $\Delta = \{\delta_1, \ldots, \delta_m\}$ are commuting derivations on a field $K$ of characteristic zero and $D : K \to K$ is an additional derivation on $K$ that commutes with $\Delta$. If $V$ is a $\Delta$-closed set defined over the $D$-constants of $K$, then Kolchin constructs a $\Delta$-tangent bundle of $V$ which has $\bar{x} \to (\bar{x}, D\bar{x})$ as a section ([5], Chap. VIII, §2). In general, if $V$ is not necessarily defined over the $D$-constants, then $D$ gives a section of a certain torsor of the $\Delta$-tangent bundle of $V$ that we call the $D/\Delta$-prolongation of $V$ (cf. Definition 3.1). Our axioms will essentially say that $(K, \Delta \cup \{D\})$ is differentially closed if and only if $K$ is algebraically closed and whenever $V$ and $W$ are $\Delta$-closed sets with $W$ contained in the $D/\Delta$-prolongation of $V$ and projecting onto $V$, then there is a $K$-point in $W$ of the form $(\bar{x}, D\bar{x})$. “Essentially”, because in actual fact we also have to consider not just $\Delta$ and $D$ but also various linear combinations of them (cf. Theorem 4.3 below).

Pierce-Pillay type axiomatizations have been obtained in various other contexts: difference fields (Chatzidakis and Hrushovski [3]), difference-differential fields (Bus- tamante [2]), derivations of the Frobenius and commuting Hasse-Schmidt derivations in positive characteristic (Kowalski [6], [7]). However, the techniques used in these works do not seem to translate to our context.

The paper is organized as follows. In Section 2 we establish the differential-algebraic facts that underpin our results. In Section 3 we introduce relative prolongations and prove a geometric characterization of differentially closed fields. Finally, in Section 4, we address the issue of first-order axiomatizability.

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2. Extending $\Delta$-derivations

In this paper the term ring is used for commutative ring with unity and the term field for field of characteristic zero.

Let us first recall some terminology from differential algebra. For details see [4]. Let $R$ be a ring and $S$ a ring extension. An additive map $\delta : R \to S$ is called a derivation if it satisfies the Leibniz rule; i.e., $\delta(ab) = (\delta a)b + a(\delta b)$. A ring $R$ equipped with a set of derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$, $\delta_i : R \to R$, such that the derivations commute with each other is called a $\Delta$-ring. A $\Delta$-ring which is also a field (of characteristic zero) is called a $\Delta$-field.

We fix for the rest of this section a $\Delta$-ring $R$. Let $\Theta$ denote the free commutative monoid generated by $\Delta$; that is,

$$\Theta := \{\delta_{r_m} \cdots \delta_{r_1} : r_m, \ldots, r_1 \geq 0\}.$$ 

The elements of $\Theta$ are called the derivative operators. Let $\bar{x} = (x_1, \ldots, x_n)$ be a family of indeterminates, and define

$$\theta \bar{x} := \{\partial x_j : j = 1, \ldots, n, \partial \in \Theta\}.$$
The \( \Delta \)-ring of \( \Delta \)-polynomials over \( R \) in the differential indeterminates \( \bar{x} \) is \( R\{\bar{x}\} := R[\theta \bar{x}] \); that is, the ring of polynomials in the algebraic indeterminates \( \theta \bar{x} \) with the canonical \( \Delta \)-ring structure given by 
\[
\delta_i(\delta_{m}^{r_m} \cdots \delta_1^{r_1} x_j) = \delta_{m}^{r_m} \cdots \delta_1^{r_1+1} x_j.
\]

We fix an orderly ranking in \( \theta \bar{x} \) by:
\[
\delta_{m}^{r_m} \cdots \delta_1^{r_1} x_i \leq \delta_{m}^{r_m} \cdots \delta_1^{r_1} x_j \iff \left( \sum l_i, i, r_m, \ldots, r_1 \right) \leq \left( \sum r'_l, j, r'_m, \ldots, r'_1 \right)
\]
in the lexicographical order. According to this ranking, we enumerate the algebraic indeterminates by \( \theta \bar{x} = (\theta_1 \bar{x}, \theta_2 \bar{x}, \ldots) \). Therefore, if \( f \in R\{\bar{x}\} \) there is a unique \( \bar{f} \in R[t_1, t_2, \ldots] \) such that \( f(\bar{x}) = \bar{f}(\theta \bar{x}) \).

We will be interested in adding an extra derivation on \( R \).

**Definition 2.1.** Let \( S \) be a \( \Delta \)-ring extension of \( R \). A \( \Delta \)-derivation from \( R \) to \( S \) is a derivation \( D : R \to S \) such that \( D\delta = \delta D \) for all \( \delta \in \Delta \).

Fix a \( \Delta \)-ring extension \( S \) of \( R \) and a \( \Delta \)-derivation \( D : R \to S \). We are interested in the extensions of \( D \) to \( \Delta \)-derivations from finitely generated \( \Delta \)-ring extensions of \( R \) to \( S \). This subject was studied by Kolchin in ([5], Chapter 0, §4). We will need the following terminology to present the main results. If \( f \in R\{\bar{x}\} \), by \( f^D \) we mean the \( \Delta \)-polynomial in \( S \{\bar{x}\} \) obtained by applying \( D \) to the coefficients of \( f \). Note that the map \( f \mapsto f^D \) is itself a \( \Delta \)-derivation from \( R\{\bar{x}\} \) to \( S\{\bar{x}\} \). By the Jacobian of \( f \) we will mean
\[
df(\bar{x}) := \left( \frac{\partial \bar{f}}{\partial \theta_i} (\theta \bar{x}) \right)_{i \in \mathbb{N}}
\]
viewed as an element of \( (R\{\bar{x}\})^N \). Note that \( df \) is finitely supported, in the sense that all but finitely many coordinates are zero.

**Remark 2.2.** Suppose \( \bar{a} \) is a tuple of \( S \) and \( D' : R\{\bar{a}\} \to S \) is a \( \Delta \)-derivation extending \( D \). If \( f \in R\{\bar{x}\} \), then an easy computation shows that
\[
D'f(\bar{a}) = df(\bar{a}) \cdot \theta D'\bar{a} + Df(\bar{a})
\]
Here if \( \bar{a} = (a_1, \ldots, a_n) \) then \( D'\bar{a} = (D'a_1, \ldots, D'a_n) \) and \( \theta D'\bar{a} = (\theta_1 D'a_1, \theta_2 D'a_2, \ldots) \). Note that the dot product is well defined since \( df \) has finite support.

**Definition 2.3.** Let \( f \in R\{\bar{x}\} \). We define the \( \Delta \)-polynomial \( \tau_{D/\Delta}f \in S\{\bar{x}, \bar{y}\} \) by
\[
\tau_{D/\Delta}f(\bar{x}, \bar{y}) := df(\bar{x}) \cdot \theta \bar{y} + f^D(\bar{x}).
\]
When \( \Delta \) and \( D \) are understood we simply write \( \tau f \). If \( \bar{a} \in S \), we write \( \tau(f)_{\bar{a}}(\bar{y}) \) for \( \tau f(\bar{a}, \bar{y}) \in S\{\bar{y}\} \). Note that \( \tau \theta \bar{x} = \theta \bar{y} \) and if \( c \in R \) then \( \tau c = Dc \).

Note that, under the assumptions of Remark 2.2, for all \( f \) in the prime \( \Delta \)-ideal \( I(\bar{a}/R) : = \{ f \in R\{\bar{x}\} : f(\bar{a}) = 0 \} \) we get
\[
\tau(f)_{\bar{a}}(D'\bar{a}) = D'f(\bar{a}) = 0.
\]
Thus any \( \Delta \)-derivation \( D' \) from \( R\{\bar{a}\} \) to \( S \) extending \( D \) gives a tuple \( D'\bar{a} \) of \( S \) at which \( \tau(f)_{\bar{a}} \) vanishes for all \( f \in I(\bar{a}/R) \). The following proposition is the converse of this implication and gives a criterion for when a \( \Delta \)-derivation can be extended to a finitely generated \( \Delta \)-ring extension. The case when \( \Delta = \emptyset \) can be found in ([8], Chap. 7, §5), and is the main point in the Pierce-Pillay geometric axiomatization of ordinary differentially closed fields.
Proposition 2.4 ([5], Chap. 0, §4). Let $D : R \to S$ be a $\Delta$-derivation and $\bar{a}$ a tuple of $S$. Suppose there is a tuple $\bar{b}$ of $S$ such that

\[(2.1) \quad \tau(f)\bar{a}(\bar{b}) = 0, \text{ for all } f \in I(\bar{a}/R).\]

Then there is a unique $\Delta$-derivation $D' : R\{\bar{a}\} \to S$ extending $D$ such that $D'\bar{a} = \bar{b}$.

Thus if we want to extend $D$ to a $\Delta$-derivation from $R\{\bar{a}\}$ to $S$, we need to find a solution of the system of $\Delta$-equations \{$\tau(f)\bar{a}(\bar{y}) = 0 : f \in I(\bar{a}/R)$\}. In the case when $S$ is a field, Kolchin showed that this system does have a solution in some $\Delta$-field extension of $S$. Indeed he shows ([5], Chap. 0, §4, Proposition 5) that the ideal generated by \{$\tau(f)\bar{a}(\bar{y}) : f \in I(\bar{a}/R)$\} in $S\{\bar{y}\}$ is a prime $\Delta$-ideal. From this and Proposition 2.4 one obtains:

Corollary 2.5 ([5], Chap. 0, §4). Suppose $(K, \Delta)$ is a differentially closed field extending $R$ and $D : R \to K$ a $\Delta$-derivation. Then there is a $\Delta$-derivation $D' : K \to K$ extending $D$.

We will require an improvement on Proposition 2.4. We would like to only have to check condition (2.1) for a set of $\Delta$-polynomials $A \subset R\{\bar{x}\}$ such that \{$A\} = I(\bar{a}/R)$, where $\{A\}$ denotes the radical $\Delta$-ideal generated by $A$. As the reader may expect this will be useful when dealing with issues of first-order axiomatizability (see Proposition 3.2 below).

First we need a lemma. For each $i = 1, 2, \ldots$, let $\bar{x}_i$ be an $n$-tuple of differential indeterminates. Suppose $D : R \to R$ is a $\Delta$-derivation. Then $\tau : R\{\bar{x}_1\} \to R\{\bar{x}_1, \bar{x}_2\}$. Thus we can compose $\tau$ with itself, for each $k \geq 1$ and $f \in R\{\bar{x}_1\}$, $\tau^k f = \tau \cdots \tau f \in R\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^k}\}$. Define $\nabla \bar{x} := (\bar{x}, D\bar{x})$ and note that, for each $k \geq 1$, the composition $\nabla^k \bar{x} = \nabla \cdots \nabla \bar{x}$ is a tuple of length $n2^k$.

Lemma 2.6. Suppose $D : R \to R$ is a $\Delta$-derivation and $f \in R\{\bar{x}_1\}$.

1. If $\bar{a}$ is a tuple of $R$, then for each $k \geq 1$,

\[\tau^k f(\nabla^k \bar{a}) = D^k f(\bar{a})\]

In particular, if $f(\bar{a}) = 0$ then $\tau^k f(\nabla^k \bar{a}) = 0$.

2. For each $k \geq 1$, we have

\[\tau^k f = k!(\tau f)^k + f^p\]

for some $p \in R\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^k}\}$.

Proof. (1) By induction on $k$. Remark 2.2 gives us

\[\tau f(\nabla \bar{a}) = df(\bar{a}) \cdot \theta D\bar{a} + f^D(\bar{a}) = Df(\bar{a}).\]

The induction step follows easily:

\[\tau^{k+1} f(\nabla^{k+1} \bar{a}) = \tau(\tau^k f)(\nabla(\nabla^k \bar{a})) = D\tau^k f(\nabla^k \bar{a}) = D D^k f(\bar{a}) = D^{k+1} f(\bar{a}).\]

(2) We prove that for each $l = 1, \ldots, k$ we have

\[(2.2) \quad \tau^l(f^k) = \frac{k!}{(k-l)!} f^{k-l}(\tau f)^l + f^{k-l+1} p_l\]

for some $p_l \in K\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^k}\}$. From which the results follows when $l = k$. Since $\tau f^k = k f^{k-1} \tau f$, we get (2.2) holds for $l = 1$ with $p_1 = 0$. Assume it holds for
If $1 \leq l < k$, then

\[
\tau^{l+1} f^k = \tau \tau^l f^k = \tau \left( \frac{k!}{(k-l)!} f^{k-l}(\tau f)^l + f^{k-l+1} p_l \right)
\]

\[
= \frac{k!}{(k-l)!} \left( (k-l) f^{k-l-1}(\tau f)^l f + f^{k-l}(\tau f)^l + f^{k-l+1} \right) p_l
\]

where

\[
p_{l+1} = \frac{k!}{(k-l)!} (\tau f)^l \tau f + (k-l+1) (\tau f) p_l + f f p_l.
\]

\[
\square
\]

**Proposition 2.7.** Suppose $R$ is a reduced $\mathbb{Q}$-algebra and $D : R \to R$ is a $\Delta$-derivation. Let $\bar{a}$ a tuple of $R$ and $A \subseteq I(\bar{a}/R)$. Suppose there is a tuple $\bar{b}$ of $R$ such that

\[(2.3) \quad \tau(f)_{\bar{a}}(\bar{b}) = 0, \text{ for all } f \in A.
\]

Then $\tau(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in \{A\}$.

**Proof.** First we show equation (2.3) holds for all $f$ in $[A]$, where $[A]$ is the $\Delta$-ideal generated by $A$. For each $\partial \in \Theta$, $f \in A$ and $h \in R\{\bar{x}\}$, we have

\[(2.4) \quad \tau(h \partial f)_{\bar{a}}(\bar{b}) = \tau(h)_{\bar{a}}(\bar{b}) \partial f(\bar{a}) + h(\bar{a}) \partial \tau(f)_{\bar{a}}(\bar{b}).
\]

Here we used the fact that $\tau(\partial f)_{\bar{a}}(\bar{b}) = \partial(\tau f)_{\bar{a}}(\bar{b})$ (see [5], Chap. 0, §4, pp.9). By assumption $\tau(f)_{\bar{a}}(\bar{b}) = 0$ and since $f \in A \subseteq I(\bar{a}/R)$ we get $\partial f(\bar{a}) = 0$. Thus (2.4) yields $\tau(h \partial f)_{\bar{a}}(\bar{b}) = 0$. It follows that for each $f \in \{A\}$, $\tau(f)_{\bar{a}}(\bar{b}) = 0$.

Now let $f \in \{A\}$, since $R(\bar{x})$ is also a $\mathbb{Q}$-algebra $\{A\} = \sqrt{\{A\}}$, and so there is $k \geq 1$ such that $f^k \in \{A\}$ and hence $\tau f^k(\bar{a}, \bar{b}) = 0$. By part (1) of Lemma 2.6,

\[
\tau^k f^k(\nabla^{k-1}(\bar{a}, \bar{b})) = \tau^{k-1}(\tau f^k)(\nabla^{k-1}(\bar{a}, \bar{b})) = D^{k-1} \tau f^k(\bar{a}, \bar{b}) = 0.
\]

Thus, by part (2) of Lemma 2.6, we have

\[
kl(\tau f(\bar{a}, \bar{b}))^k + f(\bar{a}) \rho(\nabla^{k-1}(\bar{a}, \bar{b})) = 0,
\]

for some $p \in R\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$. Since $f(\bar{a}) = 0$, we get $k!(\tau f(\bar{a}, \bar{b}))^k = 0$. Thus, since $R$ is a reduced $\mathbb{Q}$-algebra, $\tau(f)_{\bar{a}}(\bar{b}) = \tau f(\bar{a}, \bar{b}) = 0$. \square

**Corollary 2.8.** If $S$ is a field, then Proposition 2.4 holds even if we replace $I(\bar{a}/R)$ for any $A \subseteq R\{\bar{x}\}$ such that $\{A\} = I(\bar{a}/R)$.

**Proof.** Suppose $\{A\} = I(\bar{a}/R)$ and $\tau_{D/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in A$. We need to show that $\tau_{D/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in I(\bar{a}/R)$. Let $(K, \Delta)$ be a differentially closed field extending $S$. By Corollary 2.5, we can extend $D$ to a derivation $D' : K \to K$. Now, by Proposition 2.7, $\tau_{D'/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ holds for all $f \in \{A\}_K$, where $\{A\}_K$ denotes the radical $\Delta$-ideal in $K(\bar{x})$ generated by $A$. But $\{A\} \subseteq \{A\}_K$, so that $\tau_{D/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in I(\bar{a}/R)$, as desired. \square
3. Relative prolongations and a characterization of $DCF_{0,m+1}$

Let us recall the notion of prolongation for ordinary differential fields. Given a $\delta$-field $K$ and $V$ a Zariski-closed set of $K^n$, the prolongation of $V$, $\tau V$, is the Zariski-closed subset of $K^{2n}$ defined by the equations $f(\bar{x}) = 0$ and $\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(\bar{x})y_i + f^a(\bar{x}) = 0$, for each polynomial $f \in K[\bar{x}]$ vanishing on $V$. Note that, in terms of our notation from Definition 2.3, the last equation is just $\tau_{\delta/\emptyset} f(\bar{x}, \bar{y}) = 0$.

Now we fix a differential field $(K, \Delta \cup \{D\})$ with $\Delta = \{\delta_1, \ldots, \delta_m\}$. We introduce a prolongation for $\Delta$-closed sets with respect to $D$.

**Definition 3.1.** Suppose $V \subseteq K^n$ is a $\Delta$-closed set. The $D/\Delta$-prolongation of $V$, $\tau_{D/\Delta} V \subseteq K^{2n}$, is the $\Delta$-closed set defined by

$$f = 0 \text{ and } \tau_{D/\Delta} f = 0, \text{ for all } f \in \mathcal{I}(V/K).$$

Here $\mathcal{I}(V/K) = \{f \in K\{\bar{x}\} : f \text{ vanishes on } V\}$. When $\Delta$ and $D$ are understood, we just write $\tau f$ and $\tau V$. For $\bar{a} \in V$, $\tau(V)_{\bar{a}}$ denotes the fibre of $\tau V$ at $\bar{a}$. Note that when $m = 0$ this is consistent with the ordinary case discussed above.

By Remark 2.2, if $\bar{a}$ is in $V$ then $(\bar{a}, D\bar{a}) \in \tau V$. This implies that the projection $\pi : \tau V \to V$ given by $\pi(\bar{x}, \bar{y}) = \bar{x}$ is surjective and that $\bar{x} \mapsto (\bar{x}, D\bar{x})$ is a section.

Using this relative prolongation we will extend the Pierce-Pillay axioms of ordinary differentially closed fields ([13], §2), to the case of several commuting derivations. In the language of differential rings $\mathcal{L}_m = \{0, 1, +, -, \times, \delta_1, \ldots, \delta_m\}$, we denote by $DCF_{0,m}$ the theory of differential fields of characteristic zero with $m$ commuting derivations, and by $DCF_{0,m}$ its model-completion, the theory of differentially closed fields.

The following consequence of Proposition 2.7 implies that the $D/\Delta$-prolongation varies uniformly with $V$.

**Proposition 3.2.** Suppose $(K, \Delta) \models DCF_{0,m}$. If $V = V(f_1, \ldots, f_s) := \{\bar{a} \in K^n : f_i(\bar{a}) = 0, i = 1, \ldots, s\}$, then $\tau V = V(f_i, \tau f_i : i = 1, \ldots, s)$.

**Proof.** Clearly $\tau V \subseteq V(f_1, \tau f_i : i = 1, \ldots, s)$. Let $(\bar{a}, \bar{b}) \in V(f_i, \tau f_i : i = 1, \ldots, s)$. By Proposition 2.7, $\tau f(\bar{a}, \bar{b}) = 0$ for all $f \in \{f_1, \ldots, f_s\}$. Since $(K, \Delta) \models DCF_{0,m}$, we have $\{f_1, \ldots, f_s\} = \mathcal{I}(V(f_1, \ldots, f_s)) = \mathcal{I}(V)$. Hence, $(\bar{a}, \bar{b}) \in \tau V$. $\square$

**Remark 3.3.**

1. Suppose $(K, \Delta) \models DCF_{0,m}$. If $V$ is defined over the $D$-constants, that is, $V = V(f_1, \ldots, f_s)$ where $f_i \in C_D(\bar{x})$, then $\tau V$ is just Kolchin’s $\Delta$-tangent bundle of $V$. Indeed, by Proposition 3.2, the equations defining $\tau V$ become $f_i(\bar{x}) = 0$ and $\tau f_i(\bar{x}, \bar{y}) = df_i(\bar{x}) \cdot \theta \bar{y} = 0$, $i = 1, \ldots, s$. These are exactly the equations for Kolchin’s $\Delta$-tangent bundle $T_{\Delta} V$ ([5], Chap.VIII, §2).

2. In general, $\tau V$ is a torsor under $T_{\Delta} V$. Indeed, from the equations one sees that $\tau(V)_{\bar{a}}$ is a translate of $T_{\Delta}(V)_{\bar{a}}$, and so the map $T_{\Delta} V \times_V \tau V \to \tau V$ given by $((\bar{a}, \bar{b}), (\bar{a}, \bar{c})) \mapsto (\bar{a}, \bar{b} + \bar{c})$ is a regular action of $T_{\Delta} V$ on $\tau V$ over $V$.

Note that in case $\Delta = \emptyset$, part (2) of Remark 3.3 reduces to the fact that the prolongation of a Zariski-closed set is a torsor under its tangent bundle.

The following characterization of $DCF_{0,m+1}$ will be used in the next section to obtain a geometric first-order axiomatization.

**Theorem 3.4.** Suppose $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. Then $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$ if and only if
(1) \((K, \Delta) \models DCF_{0,m}\)

(2) For each pair of irreducible \(\Delta\)-closed sets \(V \subseteq K^n\), \(W \subseteq \tau V\) such that \(W\) projects \(\Delta\)-dominantly onto \(V\). If \(O_V\) and \(O_W\) are nonempty \(\Delta\)-open subsets of \(V\) and \(W\) respectively, then there exists \(\bar{a} \in O_V\) such that \((\bar{a}, D\bar{a}) \in O_W\).

As we will see in the proof, it would have been equivalent in condition (2) to take \(O_V = V\) and \(O_W = W\). Also note that, under the convention that \(DCF_{0,0}\) is the theory of algebraically closed fields of characteristic zero \(ACF_0\), when \(m = 0\) this is exactly the Pierce-Pillay axioms.

**Proof.** Suppose \((K, \Delta \cup \{D\}) \models DCF_{0,m+1}\), and \(V, W, O_V\) and \(O_W\) are as in condition (2). Let \((\bar{U}, \bar{\Delta})\) be a large differentially closed field; i.e., a universal domain for \(\Delta\)-algebraic geometry. If \(X\) is an \((\mathcal{L}_m)\)-definable subset of \(K^n\), by \(X(\bar{U})\) we mean the interpretation of \(X\) in \(\bar{U}^n\). Let \((\bar{a}, \bar{b}) \in \bar{U}^n\) be a \(\Delta\)-generic point of \(W\) over \(K\); that is, \(I(\bar{a}, \bar{b}/K) = I(W(\bar{U})/K)\). Then \((\bar{a}, \bar{b}) \in O_W(\bar{U})\). Since \((\bar{a}, \bar{b}) \in \tau V(\bar{U})\) we have that \(\tau(f)_{\bar{a}}(b) = 0\) for all \(f \in I(V(\bar{U})/K)\). The fact that \(W\) projects \(\Delta\)-dominantly onto \(V\) implies that \(\bar{a}\) is a \(\Delta\)-generic point of \(V\) over \(K\), so \(\bar{a} \in O_V(\bar{U})\) and \(I(\bar{a}/K) = I(V(\bar{U})/K)\). Hence, \(\tau(f)_{\bar{a}}(b) = 0\) for all \(f \in I(\bar{a}/K)\).

By Proposition 2.4, there is a unique \(\Delta\)-derivation \(D' : K\{\bar{a}\} \rightarrow \bar{U}\) extending \(D\) such that \(D'\bar{a} = \bar{b}\). By Corollary 2.5, we can extend \(D'\) to all of \(\bar{U}\), call it \(D''\). Hence, \(\bar{U}\) becomes a \(\Delta \cup \{D''\}\)-field extending the \(\Delta \cup \{D\}\)-closed field \(K\). Since \(\bar{a} \in O_V(\bar{U})\), \((\bar{a}, \bar{b}) \in O_W(\bar{U})\) and \(D''\bar{a} = \bar{b}\), we get a point \((\bar{a}', \bar{b}')\) in \(K\) such that \(\bar{a}' \in O_V\), \((\bar{a}', \bar{b}') \in O_W\) and \(D\bar{a}' = \bar{b}'\).

The converse is essentially as in [13]. For the sake of completeness we give the details. Let \(\phi(\bar{x})\) be a conjunction of atomic \(\mathcal{L}_{m+1}\)-formulas over \(K\). Suppose \(\phi\) has a realisation \(\bar{a}\) in some \((F, \Delta \cup \{D\}) \models DCF_{0,m+1}\) extending of \((K, \Delta \cup \{D\})\). Let

\[
\phi(\bar{x}) = \psi(\bar{x}, \delta_{m+1}\bar{x}, \ldots, \delta_{m+1}\bar{x})
\]

where \(\psi\) is a conjunction of atomic \(\mathcal{L}_m\)-formulas over \(K\) and \(r > 0\). Let \(\bar{c} = (\bar{a}, D\bar{a}, \ldots, D^{r-1}\bar{a})\) and \(X \subseteq F^{nr}\) be the \(\Delta\)-locus of \(\bar{c}\) over \(K\). Let \(Y \subseteq F^{2nr}\) be the \(\Delta\)-locus of \((\bar{c}, D\bar{c})\) over \(K\). Let

\[
\chi(\bar{x}_0, \ldots, \bar{x}_{r-1}, \bar{y}_0, \ldots, \bar{y}_{r-1}) := \psi(\bar{x}_0, \ldots, \bar{x}_{r-1}, \bar{y}_{r-1}) \land (\bigwedge_{i=1}^{r-1} \bar{x}_i = \bar{y}_{i-1})
\]

then \(\chi\) is realised by \((\bar{c}, D\bar{c})\). Since \((\bar{c}, D\bar{c})\) is a \(\Delta\)-generic point of \(Y\) over \(K\) and its projection \(\bar{c}\) is a \(\Delta\)-generic point of \(X\) over \(K\), we have that \(Y\) projects \(\Delta\)-dominantly onto \(X\) over \(K\). Thus, since \((K, \Delta) \models DCF_{0,m}\), \(Y(K)\) projects \(\Delta\)-dominantly onto \(X(K)\). Also, since \((\bar{c}, D\bar{c}) \in \tau X\), we have \(Y(K) \subseteq \tau(X(K))\).

Applying (2) with \(V = O_V = X(K)\) and \(W = O_W = Y(K)\), there is \(\bar{d}\) in \(V\) such that \((\bar{d}, D\bar{d}) \in W\). Let \(d = (d_0, \ldots, d_{r-1})\) then \((d_0, \ldots, d_{r-1}, Dd_0, \ldots, Dd_{r-1})\) realises \(\chi\). Thus, \((d_0, Dd_0, \ldots, D^r d_0)\) realises \(\psi\). Hence, \(d_0\) is a tuple of \(K\) realising \(\phi\). This proves that \((K, \Delta \cup \{D\}) \models DCF_{0,m+1}\).

4. Geometric first-order axioms

The Pierce-Pillay characterization of \(DCF_0\), that is Theorem 3.4 when \(m = 0\), is indeed first-order. Expressing irreducibility of a Zariski-closed set as a definable condition on the parameters uses the existence of bounds to check primality of ideals in polynomial rings in finitely many variables [16]. Also, if the field is
Proposition 4.2 (4], Chap. II, the elements of ∆+···
that, writing DCF
The idea, of course, was that
K
tions on
K
transformations. Our modification of Theorem 3.4 will therefore need to refer to such
4.2 below), this can always be achieved if we allow
K
onto
V
where the third equality holds because ¯K
Let us note here that when ∆ =
∆-dominantly onto another ∆-closed set as a definable condition.
It follows that if M ⊆ Vj, for j = i, we can pick a gj ∈ I(Vj/K) such that gj(¯a) ≠ 0. Then, if g = ∏j gj, we get fg ∈ I(V/K) and so
0 = τ(fg)¯a(¯b) = τ(f)¯a(¯b)g(¯a) + f(¯a)τ(g)¯a(¯b) = τ(f)¯a(¯b)g(¯a)
where the third equality holds because ¯a ∈ Vj. Since g(¯a) ≠ 0, we have τ(f)¯a(¯b) = 0, and so ̅b ∈ τ(Vi)¯a.

It follows that if W ⊆ τV projects ∆-dominantly onto V and Vi is a K-irreducible component of V, then a K-irreducible component of W ∩ τV projects ∆-dominantly onto Vi.

The second issue, that of ∆-dominant projections, is more difficult to deal with. Let us note here that when ∆ = ∅, that is, in the case of DCFω, one can just replace dominant projections by surjective projections in the Pierce-Pillay axiomatization. Indeed this reformulation is stated in [14]. We will not give a proof here as it will follow from Theorem 4.3 below. However, what makes this work, in the case of a single derivation D, is the fact that if a is D-algebraic over K, then Dk+1a ∈ K(a, Da, . . . , Dka) for some k. In several derivations it is not necessarily the case that if a is ∆ ⊆ {D}-algebraic over K, then Dk+1a is in the ∆-field generated by a, Da, . . . , Dka over K, for some k. However, by a theorem of Kolchin (Proposition 4.2 below), this can always be achieved if we allow Z-linear transformations of the derivations. Our modification of Theorem 3.4 will therefore need to refer to such transformations.

For every M = (ci, j) ∈ SLm+1(Z), let ∆′ = {δ1, . . . , δm} and D′ be the derivations on K defined by δi = ci,1δ1 + · · · + ci,mδm + ci,m+1D and D′ = cm+1,1δ1 + · · · + cm+1,mδm + cm+1,m+1D. In this case we write (∆′, D′) = M(∆, D). Clearly, the elements of ∆′ ⊆ {D′} are also commuting derivations on K.

Proposition 4.2 ([4], Chap. II, §11). Let (K, ∆ ⊆ {D}) |= DCFω,m+1. Let ̅a = (a1, . . . , an) be a tuple of a ∆ ⊆ {D}-field extension of K. Suppose all the ai’s are ∆∪{D}-algebraic over K, then there exists k > 0 and a matrix M ∈ SLm+1(Z) such that, writing (∆′, D′) = M(∆, D), we have that D′a is in the ∆′-field generated by ̅a, D′a, . . . , Dk′a over K, for all ℓ > k.

Theorem 3.4 characterizes DCFω,m+1 in terms of the geometry of DCFω,m. The idea, of course, was that DCFω,m has a similar characterization relative to
$DCF_{0,m-1}$, and so on. In Theorem 4.3 we will implement this recursion and give a geometric first-order axiomatization of $DCF_{0,m+1}$ for all $m \geq 0$, that refers only to the base theory $ACF_0$.

**Theorem 4.3.** Suppose $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. Then $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$ if and only if

1. $K \models ACF_0$
2. Suppose $M \in SL_{m+1}(\mathbb{Z})$, $(\Delta', D') := M(\Delta, D)$, $V = \mathcal{V}(f_1, \ldots, f_s) \subseteq K^n$ is a nonempty $\Delta'$-closed set, and

$$W \subseteq \mathcal{V}(f_1, \ldots, f_s, \tau_{D'/\Delta} f_1, \ldots, \tau_{D'/\Delta} f_s) \subseteq K^{2n}$$

is a $\Delta'$-closed set that projects onto $V$. Then there is $\bar{a} \in V$ such that $(\bar{a}, D'\bar{a}) \in W$.

**Proof.** Suppose $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. Clearly $K \models ACF_0$. Suppose $M$, $\Delta'$, $V$ and $W$ are as in condition (2). Clearly $(K, \Delta' \cup \{D'\}) \models DCF_{0,m+1}$, so by Proposition 3.2 we have that $\mathcal{V}(f_i, \tau_{D'/\Delta} f_i : i = 1, \ldots, s) = \tau_{D'/\Delta} V$. Let $V_i$ be an irreducible component of $V$ and $W = W' \cap \tau_{D'/\Delta} V_i$. By Lemma 4.1, we can find an irreducible component of $W'$ projecting $\Delta'$-dominantly onto $V_i$. Now just apply Theorem 3.4 (with $\Delta' \cup \{D'\}$ rather than $\Delta \cup \{D\}$) to get the desired point.

For the converse, we assume conditions (1) and (2) and prove that $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. Given $r = 1, \ldots, m+1$ and $N \in SL_{m+1}(\mathbb{Z})$, let $K_{r,N} = (K, \Delta_{r-1} \cup \{D\})$ where $(\Delta, D) = N(\Delta, D)$ and $\Delta_{r-1} = \{\delta_1, \ldots, \delta_{r-1}\}$. Set $K_{0,N}$ to be the pure algebraic field $K$. We show by induction that for each $r = 0, \ldots, m+1$, $K_{r,N} \models DCF_{0,r+1}$ for all $N \in SL_{m+1}(\mathbb{Z})$. The result will then follow by setting $r = m+1$ and $N = Id$. The case of $r = 0$ is just assumption (1). We assume $0 \leq r \leq m$, $N \in SL_{m+1}(\mathbb{Z})$, and we show that $K_{r+1,N} = (K, \Delta_r \cup \{D\}) \models DCF_{0,r+1}$.

Suppose $\phi(\bar{x})$ is a conjunction of atomic $L_{r+1}$-formulas over $K$, with a realisation $\bar{a} = (a_1, \ldots, a_n)$ in some $\Delta_r \cup \{D\}$-field $F$ extending $K_{r+1,N}$. We need to find a realisation of $\phi$ in $K_{r+1,N}$. We may assume that each $a_i$ is $\Delta_r \cup \{D\}$-algebraic over $K$ (this can be seen algebraically or one can use the existence of prime models of $DCF_{0,r+1}$ over $K$, see §3.2 of [10]).

Let $M' \in SL_{r+1}(\mathbb{Z})$ and $k > 0$ be the matrix and natural number given by Proposition 4.2. Let $M \in SL_{m+1}(\mathbb{Z})$ be

$$M = E \begin{pmatrix} M' & 0 \\ 0 & I \end{pmatrix}$$

where $E$ is the elementary matrix of size $(m+1)$ that interchanges row $(r+1)$ with row $(m+1)$ and $I$ is the identity matrix of size $(m-r)$. Then, setting $(\Delta', D') = M(\Delta, D)$, we get

$$D'^{k+1} \bar{a} = \frac{f(\bar{a}, D'\bar{a}, \ldots, D^k \bar{a})}{g(\bar{a}, D'\bar{a}, \ldots, D^k \bar{a})}$$

for some $f, g \in (K\{\bar{x}_0, \ldots, \bar{x}_k\}_{\Delta_r})^n$. Here $\Delta_r' = \{\delta_1', \ldots, \delta_s\}$ and $K\{\bar{x}\}_{\Delta_r'}$ denotes the $\Delta_r'$-ring of $\Delta_r'$-polynomials over $K$. Let

$$\bar{c} = \left( \bar{a}, D'\bar{a}, \ldots, D^k \bar{a}, \frac{1}{g(\bar{a}, D'\bar{a}, \ldots, D^k \bar{a})} \right).$$

Let $X \subseteq F^{n(k+2)}$ be the $\Delta_r'$-locus of $\bar{c}$ over $K$ and $Y \subseteq F^{2n(k+2)}$ the $\Delta_r'$-locus of $(\bar{c}, D'\bar{c})$ over $K$. 

Claim. $Y$ projects onto $X$.

Consider the $\Delta'_r$-polynomial map $s(\bar{x}_0, \ldots, \bar{x}_{k+1}): X \to F^{n(k+2)}$ given by

$$s = (\bar{x}_1, \ldots, \bar{x}_k, f^{2}_{k+1} - \bar{x}_{k+1}^2 t_{D'/\Delta'_r} g(\bar{x}_0, \ldots, \bar{x}_k, \bar{x}_1, \ldots, \bar{x}_k, f^{2}_{k+1}))$$

where any product between tuples is computed coordinatewise. Using (4.1), an easy computation shows $s(\bar{e}) = D'\bar{e}$. Given $\bar{b} \in X$, we note that $(\bar{b}, s(\bar{b})) \in Y$. Indeed, if $h$ is a $\Delta'_r$-polynomial over $K$ vanishing at $(\bar{e}, D'\bar{e})$, then $h(\cdot, s(\cdot))$ vanishes at $\bar{e}$ and hence on all of $X$. So $(\bar{b}, s(\bar{b}))$ is in the $\Delta'_r$-locus of $(\bar{e}, D'\bar{e})$ over $K$. That is, $(\bar{b}, s(\bar{b})) \in Y$. As this point projects onto $\bar{b}$ we have proven the claim.

Now, by induction, $(K, \Delta'_r) \models DCF_{0,r}$. Indeed, $(K, \Delta'_r) = K_{r,N'}$ where $N'$ is obtained from $M$ by interchanging rows $r$ and $(m + 1)$. Hence, the claim implies that $Y(K)$ projects onto $X(K)$. Also, if $X(K) = V(f_1, \ldots, f_s)$ where each $f_i$ is a $\Delta'_r$-polynomial, then clearly $Y(K) \subseteq V(f_i, t_{D'/\Delta'_r} f_i : i = 1, \ldots, s)$. Hence, by condition (2), there is $\bar{d} \in X(K)$ such that $(\bar{d}, D'\bar{d}) \in Y(K)$.

Now, let $\rho(\bar{x})$ be the $L_{r+1}$-formula over $K$ obtained from $\phi$ by replacing each $\delta_1, \ldots, \delta_{r+1}$ for $d_1, 1_d + \cdots + d_{r+1}, \delta_{r+1}$, where $(d_i, j) \in SL_{r+1}(\mathbb{Z})$ is the inverse matrix of $M'$. By construction, $\phi(K, \Delta'_r \cup \{D'\}) = \rho(K, \Delta'_r \cup \{D'\})$. Thus it suffices to find a realisation of $\rho$ in $(K, \Delta'_r \cup \{D'\})$. We may assume that the $k$ of (4.1) is large enough so that we can write

$$\rho(\bar{x}) = \psi(\bar{x}, \delta_{r+1} \bar{x}, \ldots, \delta_k \bar{x})$$

where $\psi$ is a conjunction of atomic $L_r$-formulas over $K$. Let

$$\chi(\bar{x}_0, \ldots, \bar{x}_{k+1}, \bar{y}_0, \ldots, \bar{y}_{k+1}) := \psi(\bar{x}_0, \ldots, \bar{x}_k) \wedge (\land_{i=1}^{k+1} x_i = y_{i-1}) .$$

Then $(F, \Delta'_r) \models \chi(\bar{e}, D'\bar{e})$, and so, as $(\bar{d}, D'\bar{d})$ is in the $\Delta'_r$-locus of $(\bar{e}, D'\bar{e})$ over $K$, we have that $(F, \Delta'_r) \models \chi(\bar{d}, D'\bar{d})$. But since $\bar{d}$ is a $K$-point, we get $(K, \Delta'_r) \models \chi(\bar{d}, D'\bar{d})$. Writing the tuple $\bar{d}$ as $(\bar{d}_0, \ldots, \bar{d}_{r+1})$, we see that $\bar{d}_0$ is a realisation of $\rho$ in $(K, \Delta'_r \cup \{D'\})$. This completes the proof.

Remark 4.4.

(1) Condition (2) of Theorem 4.3 is indeed first-order; expressible by an infinite collection of $L_{m+1}$-sentences, one for each fixed choice of $M$, $f_1, \ldots, f_s$ and "shape" of $W$.

(2) In condition (2) we can strengthen the conclusion to ask for $\{\bar{a} \in V : (\bar{a}, D'\bar{a}) \in W\}$ to be $\Delta'$-dense in $V$.