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# A LARGE CLASS OF SOFIC MONOIDS

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ABSTRACT. We prove that a monoid is sofic, in the sense recently introduced by Ceccherini-Silberstein and Coornaert, whenever the  $\mathcal{J}$ -class of the identity is a sofic group, and the quotients of this group by orbit stabilisers in the rest of the monoid are amenable. In particular, this shows that the following are all sofic: left or right cancellative monoids with amenable group of units; monoids with sofic group of units and finitely many non-units; and monoids with amenable Schützenberger groups and finitely many  $\mathcal{L}$ -classes in each  $\mathcal{D}$ -class. This provides a very wide range of sofic monoids, subsuming most known examples (with the notable exception of locally residually finite monoids). We conclude by discussing some aspects of the definition, and posing some questions for future research.

## 1. INTRODUCTION

*Sofic groups* are a class of groups forming a common generalisation of amenable groups and residually finite groups. They were introduced by Gromov [6], applied to dynamical systems by Weiss [7] and subsequently studied by many authors. For a detailed introduction, see for example [1, Chapter 7].

Ceccherini-Silberstein and Coornaert [2] have recently introduced a definition of a *sofic monoid* (see Section 2 below). Examples of sofic monoids include sofic groups, finite (or more generally, locally residually finite) monoids, commutative monoids, free monoids, and cancellative left or right amenable monoids. The class of sofic monoids is closed under operations including direct products, inductive and projective limits and the taking of submonoids. A notable example of a *non-sofic* monoid is the bicyclic monoid.

It follows from their work and standard results in semigroup theory (see Proposition 3 below for the deduction) that the  $\mathcal{J}$ -class of the identity in a sofic monoid must be a single group, and that this group must be sofic. Another of their results [2, Proposition 4.7], which may be viewed as a partial converse, says that a monoid is sofic provided it has no non-trivial left or right units, which in our language means provided the  $\mathcal{J}$ -class of the identity is a trivial group.

Our main result here (Theorem 3.1 below) can be viewed as a very strong generalisation of this latter statement, providing a broad class of sofic monoids which subsumes most of the known examples listed above (with the notable exception of locally residually finite monoids). Specifically, we show that a monoid in which the  $\mathcal{J}$ -class of the identity is a sofic

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group will be sofic, provided the quotients of this group by the pointwise stabilisers of right translation orbits in the rest of the monoid are amenable groups. Although the latter condition, which we term *local amenability of the action*, is rather technical in full generality, it applies trivially in many cases of interest, such as when the group is amenable, or the rest of the monoid is finite, or the non-unit  $\mathcal{R}$ -classes are finite.

In Section 4 we consider how the local amenability condition interacts with a standard semigroup-theoretic method of using Green's relations to structurally decompose the action of the group of units of the monoid into two parts. We show that the local amenability condition can be deduced from conditions on the Schützenberger groups and on the action of the group of units on the set of  $\mathcal{H}$ -classes. Consequences include the fact that a monoid will be sofic provided the  $\mathcal{J}$ -class of the identity is a sofic group, and each non-unit  $\mathcal{D}$ -class has either finitely many  $\mathcal{L}$ -classes, or finite or abelian Schützenberger groups.

We conclude, in Section 5, with some open questions arising naturally from our results, and some discussion of aspects of the definition of a sofic monoid.

## 2. SOFIC MONOIDS, GREEN'S RELATIONS AND AMENABILITY

In this section we briefly recall some required definitions from classical semigroup theory [3] and from the new theory of sofic monoids [2], as well as proving a preliminary result connecting them. We also recall the notion of *amenability* for groups, which will be required in later sections.

**2.1. Sofic Monoids.** Let  $M$  be a monoid with identity element 1. We say that  $M$  is (*left*) *sofic* if for every finite subset  $K$  of  $M$  and  $\epsilon > 0$  there exists a finite set  $X$  and a map

$$M \times X \rightarrow X, (m, x) \rightarrow m \cdot x$$

such that

- $1 \cdot x = x$  for all  $x \in X$ ;
- for every  $g, h \in K$ , the proportion of elements  $x$  in  $X$  such that  $g \cdot (h \cdot x) = (gh) \cdot x$  is at least  $1 - \epsilon$ ; and
- for every  $g, h \in K$  with  $g \neq h$ , the proportion of elements  $x$  in  $X$  such that  $g \cdot x = h \cdot x$  is at most  $\epsilon$ .

We call a map satisfying these conditions a (*left*)  $(K, \epsilon)$ -*action* of  $M$  on  $X$ .

Sofic monoids were introduced by Ceccherini-Silberstein and Coornaert [2], with a slightly different formal definition using a notion of a  $(K, \epsilon)$ -morphism to a full transformation monoid; our formulation is trivially equivalent, being essentially just a different notation which makes our proofs slightly more concise. For a more leisurely introduction, including a discussion of the basic properties of sofic monoids, see [2].

There is an obvious dual notion of a *right*  $(K, \epsilon)$ -*action*, which leads to a definition of a *right sofic monoid*; see [2, Section 7] for discussion of this distinction. In this paper we consider explicitly only left sofic monoids (which, for conciseness and following [2], we simply term “sofic”), but of course our results have dual statements for right sofic monoids.

**2.2. Green's Relations.** *Green's relations* are five binary relations which can be defined on any monoid  $M$  as follows:

- $x\mathcal{L}y$  if  $Mx = My$ ;
- $x\mathcal{R}y$  if  $xM = yM$ ;
- $x\mathcal{H}y$  if  $Mx = My$  and  $xM = yM$ ;
- $x\mathcal{J}y$  if  $MxM = MyM$ ; and
- $x\mathcal{D}y$  if there exists  $z \in M$  with  $x\mathcal{L}z$  and  $z\mathcal{R}y$ .

All five are equivalence relations on  $M$ ; this is trivial in the first four cases and requires slightly more work to show in the case of  $\mathcal{D}$ . The relations encapsulate the (left, right and two-sided) principal ideal structure of the monoid, and play a key role in most areas of semigroup theory. For a detailed introduction see for example [3, Chapter 2].

**2.3. The  $\mathcal{J}$ -class of 1 in a Sofic Monoid.** The following is a straightforward consequence of the results of [2] together with some foundational results of semigroup theory.

**Proposition 2.1.** *Let  $M$  be a sofic monoid. Then the  $\mathcal{J}$ -class of the identity is equal to the group of units of  $M$ , and this is a sofic group.*

*Proof.* Suppose for a contradiction that  $j$  is in the  $\mathcal{J}$ -class of the identity but is not a unit. Then  $1 = ajb$  for some elements  $a, b \in M$ . Now if  $a$  is not a unit, then it is a left unit which is not a unit. On the other hand, if  $a$  is a unit then conjugating both sides by  $a$  we obtain  $1 = jba$ , in which case  $j$  is a left unit which is not a unit. So in all cases there are elements  $x, y \in M$  such that  $xy = 1$  but  $x$  is not a unit.

We claim that  $x$  and  $y$  generate a submonoid of  $M$  isomorphic to the bicyclic monoid. Indeed, if not, then since they satisfy the defining relation  $xy = 1$ , they must generate a proper quotient of a bicyclic monoid. But the only proper quotients of the bicyclic monoid are cyclic groups [3, Corollary 1.32] and if they generated a group we would also have  $yx = 1$ , contradicting the fact that  $x$  is not a unit.

Thus,  $M$  contains a copy of the bicyclic monoid. By [2, Proposition 3.5], submonoids of sofic monoids are sofic, but by [2, Theorem 5.1] the bicyclic monoid is not sofic, so this gives the required contradiction.

Finally, the group of units is in particular a submonoid of  $M$ , so is sofic as a monoid by [2, Proposition 3.5] and hence also as a group by [2, Proposition 2.4].  $\square$

**2.4. Amenability.** For our main theorem we shall need the notion of an *amenable group*. Recall that a group is called *amenable* if it admits a finitely additive probability measure which is invariant under the (left or right) translation action of the group. Most of the time we shall not make direct use of the definition of amenability, but rather of a well-known combinatorial property of them. A group  $G$  has the *Følner set property* if for every finite subset  $K$  of  $G$  and every  $\epsilon > 0$  there exists a finite subset  $F$  of  $G$ , such that the proportion of elements  $f \in F$  satisfying  $Kf \subseteq F$  is at least  $1 - \epsilon$ . In fact, a group is amenable if and only if it has the Følner set property [5].

For a full introduction to amenable groups, we refer the reader to, for example, [1, Chapter 4]. (The Følner set property has a number of slightly

different but equivalent statements; the exact formulation we use is taken from [7, Section 2] and is not explicitly mentioned in [5] or [1], but it is an easy exercise to deduce its equivalence to the properties defined in [1, Section 4.7] and shown to be equivalent to the standard definition of amenability in [1, Section 4.9].)

### 3. MAIN THEOREM

Let  $G$  be a group acting on a set  $X$ . We say that the *action of  $G$  on  $X$  is locally amenable* if for every orbit in  $X$ , the quotient of  $G$  by the pointwise stabiliser of the orbit is an amenable group. Note that, because quotients of amenable groups are amenable [1, Proposition 4.5.4], every action of an amenable group is locally amenable. In fact it is easy to show that a group is amenable exactly if its translation action on itself (from either side) is locally amenable.

We are now ready to state our main theorem, which gives a very general sufficient condition for a monoid to be sofic.

**Theorem 3.1.** *Let  $M$  be a monoid such that the  $\mathcal{J}$ -class of the identity is a sofic group  $G$ , and the right translation action of  $G$  on  $M \setminus G$  is locally amenable. Then  $M$  is sofic.*

See Section 5 below for a discussion of the extent to which the hypotheses of Theorem 3.1 are necessary, as well as sufficient, conditions for soficity.

The proof is based on a fairly elementary, but quite technical, combinatorial construction, partly inspired by the proof of [2, Proposition 4.7]:

*Proof.* Let  $K$  be a finite subset of  $M$  and  $\epsilon > 0$ . Let  $G$  be the group of units of  $M$ , and  $S = M \setminus G$  the set of non-units. Note that since  $G$  is the entire  $\mathcal{J}$ -class of 1,  $S$  is an ideal of  $M$ . In particular,  $SG \subseteq S$ , so we may consider the action by right translation of  $G$  on  $S$ .

For each  $k \in (K \cup K^2) \cap S$ , let  $H_k \trianglelefteq G$  be the pointwise stabiliser of the orbit of  $k$ , and let  $G_k = G/H_k$ . Consider the induced morphism from  $G$  to the direct product of the  $G_k$ 's. Let  $\overline{G}$  be the image of this map. For each  $g \in G$  write  $\overline{g}$  for its image in  $\overline{G}$ .

Note that the kernel of this map is the intersection of the subgroups  $H_k$  for  $k \in K \cap S$ . It follows that the right translation action of  $G$  on  $S$  induces a well-defined right action of  $\overline{G}$  on each orbit of an element of  $(K \cup K^2) \cap S$ . Indeed, if  $g, h \in G$  are such that  $\overline{g} = \overline{h}$  and  $s$  is in the orbit of  $k \in (K \cup K^2) \cap S$  then  $gh^{-1} \in H_k$  which by definition means  $sg h^{-1} = s$ , so  $sg = sh$ . For clarity we denote this action by  $*$ , so  $s * \overline{g} = sg$ .

Let

$$\overline{K} = \{\overline{k} \mid k \in K \cap G\} \subseteq \overline{G}.$$

Choose any  $\delta > 0$  sufficiently small that  $(1 - \delta)^3 > 1 - \epsilon$ . By assumption the groups  $G_k$  are all amenable; the group  $\overline{G}$ , being a subgroup of a finite direct product of them, is therefore also amenable by [1, Proposition 4.5.1 and Corollary 4.5.6]. Thus, we may choose a finite subset  $F \subseteq \overline{G}$  such that the proportion of elements  $f \in F$  satisfying  $\overline{K}f \subseteq F$  exceeds  $1 - \delta$ .

Also, recalling that  $G$  is by assumption sofic, we may choose a finite set  $P$  and a  $(K \cap G, \delta)$ -action of  $G$  on  $P$ .

Now let  $Z$  be a large finite set, let

$$Y = (K \cap S) \cup [(K \cap S) * F] \cup [(K^2 \cap S) * F] \subseteq S$$

and define

$$X = Y \cup (Z \times F \times P) \cup \{\perp\}$$

where  $\perp$  is a new symbol not in any of the previous sets.

Let

$$H = \{(z, f, p) \in Z \times F \times P \mid \overline{K}f \subseteq F\}.$$

Clearly by choosing  $Z$  large enough, we can ensure that the proportion of elements of  $X$  which come from  $Z \times F \times P$  exceeds  $1 - \delta$ , while by the definition of  $F$  the proportion of elements of  $Z \times F \times P$  which lie in  $H$  also exceeds  $1 - \delta$ . Thus, the proportion of elements of  $X$  which lie in  $H$  exceeds  $(1 - \delta)^2$ .

We define a map

$$M \times X \rightarrow X, (m, x) \mapsto m \cdot x$$

as follows:

- $m \cdot n = \begin{cases} \text{the } M\text{-product } mn & \text{if } n \in Y \text{ and } mn \in Y \\ \perp & \text{if } n \in Y \text{ and } mn \notin Y. \end{cases}$
- $m \cdot (z, f, p) = \begin{cases} (z, \overline{m}f, m \cdot p) & \text{if } m \in G \text{ and } \overline{m}f \in F \\ m * f & \text{if } m * f \text{ is defined} \\ \perp & \text{otherwise.} \end{cases}$
- $m \cdot \perp = \perp$ .

(Recall that  $m * f$  is defined precisely when  $m$  lies in the orbit of an element of  $(K \cup K^2) \cap S$  under the right translation action of  $G$ .)

Our aim is to show that this map is a  $(K, \epsilon)$ -action of  $M$  on  $X$ . We begin by recording an elementary consequence of the definition, for ease of reference later in the proof:

- (F) If  $m \in K \cap G$ , then for any  $(z, f, p) \in H$  we have  $\overline{m}f \in F$  (by the definition of  $H$ ), and hence  $m \cdot (z, f, p) = (z, \overline{m}f, m \cdot p)$ .

Next, note that  $1 \cdot x = x$  for all  $x \in X$ ; indeed, we have

- $1 \cdot n = 1n = n$  for all  $n \in Y$ ;
- $1 \cdot (z, f, p) = (z, \overline{1}f, 1 \cdot p) = (z, f, p)$  for all  $(z, f, p) \in Z \times F \times P$ ; and
- $1 \cdot \perp = \perp$ .

Now suppose  $s, t \in K$ . We claim first that the proportion of elements in  $(z, f, p) \in H$  which satisfy

$$s \cdot [t \cdot (z, f, p)] = (st) \cdot (z, f, p).$$

is at least  $1 - \delta$ . We prove the claim by analysing a number of cases, depending on whether  $s$  and  $t$  are in  $G$ :

- If  $s, t \in G$  then also  $st \in G$ . Now since we are using a  $(K, \delta)$ -action of  $G$  on  $P$ , the proportion of elements in  $P$  satisfying  $s \cdot (t \cdot p) = (st) \cdot p$  is at least  $1 - \delta$ . Since  $H$  is defined as a subset of  $Z \times F \times P$  by placing a restriction only on  $F$ , it follows that the proportion of elements

$(z, f, p) \in H$  such that  $s \cdot (t \cdot p) = (st) \cdot p$  is also at least  $1 - \delta$ . For these elements, using fact (F) gives

$$s \cdot [t \cdot (z, f, p)] = s \cdot (z, \bar{t}f, t \cdot p).$$

Now if  $(\overline{st})f = \overline{s}(\bar{t}f) \in F$  then we have

$$s \cdot (z, \bar{t}f, t \cdot p) = (z, \overline{st}f, s \cdot (t \cdot p)) = (z, \overline{st}f, (st) \cdot p) = (st) \cdot (z, f, p),$$

while if  $(\overline{st})f = \overline{s}(\bar{t}f) \notin F$  then we have

$$s \cdot (z, \bar{t}f, t \cdot p) = \perp = (st) \cdot (z, f, p),$$

in both cases establishing the required equation.

- If  $s \notin G$  and  $t \in G$  then first note that we have  $st \notin G$ , since  $S = M \setminus G$  is an ideal. Hence,  $st \in K^2 \cap S$  which means  $(st) * f$  is defined. Also,  $s \in K \cap S$ , so  $s * (\bar{t}f)$  is also defined. Moreover, if we write  $f = \bar{g}$  for some  $g \in G$  then we have

$$(st) * f = (st)g = s(tg) = s * \bar{t}g = s * (\bar{t}f).$$

Thus,

$$\begin{aligned} s \cdot [t \cdot (z, f, p)] &= s \cdot (z, \bar{t}f, t \cdot p) && \text{by (F), since } t \in K \cap G, (z, f, p) \in H \\ &= s * (\bar{t}f) && \text{by definition, since } s \in K \cap S \\ &= (st) * f && \text{by the preceding argument} \\ &= (st) \cdot (z, f, p) && \text{by definition.} \end{aligned}$$

so all elements of  $H$  satisfy the required equation.

- Finally, if  $t \notin G$  (whether or not  $s \in G$ ) we again have  $st \notin G$ . As in the previous case, we have  $(st) * f$  defined, and this time  $t * f$  is also defined. Again writing  $f = \bar{g}$  for some  $g \in G$ , we have

$$s \cdot (t * f) = s(t * f) = s(t\bar{g}) = (st)\bar{g} = (st) * f,$$

and hence,

$$\begin{aligned} s \cdot [t \cdot (z, f, p)] &= s \cdot (t * f) && \text{by definition} \\ &= (st) * f && \text{by the preceding argument} \\ &= (st) \cdot (z, f, p) && \text{by definition} \end{aligned}$$

so again all elements of  $H$  satisfy the required equation.

This completes the proof of the first claim.

Next let  $s, t \in K$  with  $s \neq t$ . We claim that the proportion of elements  $(z, f, p) \in H$  satisfying  $s \cdot (z, f, p) \neq t \cdot (z, f, p)$  is at least  $1 - \delta$ . Again, we consider a number of cases:

- If  $s, t \in G$  then, again using the  $(K \cap G, \delta)$ -action of  $G$  on  $P$ , the proportion of elements  $p \in P$  such that  $s \cdot p \neq t \cdot p$  is at least  $1 - \delta$  and it again follows that the proportion of elements in  $(z, f, p) \in H$  for which  $s \cdot p \neq t \cdot p$  is at least  $1 - \delta$ . For such elements, by fact (F), we have

$$s \cdot (z, f, p) = (z, \bar{s}f, s \cdot p) \neq (s, \bar{t}f, t \cdot p) = t \cdot (z, f, p).$$

- If exactly one of  $s$  and  $t$  lies in  $G$  then using fact (F) again, exactly one of  $s \cdot (z, f, p)$  and  $t \cdot (z, f, p)$  lies in  $Z \times F \times P$ , so they cannot be equal for any  $(z, f, p) \in H$ .

- If  $s, t \notin G$  then for any  $(z, f, p) \in H$ , by the definition of  $\cdot$  we have we have  $s \cdot (z, f, p) = s * f$  and  $t \cdot (z, f, p) = t * f$ . These cannot be equal, or we would have  $s = (s * f) * f^{-1} = (t * f) * f^{-1} = t$  where  $f^{-1}$  is the inverse of  $f$  in the group  $\overline{G}$ .

This completes the proof of the second claim.

Now since the proportion of elements in  $X$  which lie in  $H$  exceeds  $(1 - \delta)^2$ , for any  $s, t \in K$  the proportion of elements  $x \in X$  satisfying  $s \cdot (t \cdot x) = (st) \cdot x$  is at least  $(1 - \delta)^3$ , which by the definition of  $\delta$  is at least  $1 - \epsilon$ . Similarly, for any  $s, t \in K$  with  $s \neq t$ , the proportion of elements of  $X$  satisfying  $s \cdot x \neq t \cdot x$  is at least  $1 - \epsilon$ . Thus, we have defined a  $(K, \epsilon)$ -action of  $M$  on the finite set  $X$ , and so  $M$  is sofic.  $\square$

**Corollary 3.2.** *Let  $M$  be a monoid such that the  $\mathcal{J}$ -class of the identity is a sofic group  $G$ . If any of the following conditions hold, then  $M$  is sofic:*

- (i)  $G$  is amenable;
- (ii)  $M \setminus G$  is finite; or
- (iii)  $M$  has finite  $\mathcal{R}$ -classes outside the group of units;

*Proof.*

- (i) If  $G$  is amenable then all of its quotients are amenable [1, Proposition 4.5.4], so the action is locally amenable and the theorem applies.
- (ii) If  $M \setminus G$  is finite then orbits under the translation action are finite, so their pointwise stabilisers must have finite index. Thus the relevant quotients are finite, and hence by [1, Proposition 4.4.6] amenable, so the action is locally amenable and the theorem applies.
- (iii) It is easily seen that each orbit under the right translation action is contained in an  $\mathcal{R}$ -class, so if the latter are all finite then orbits must be finite and the same argument as in case (ii) applies.  $\square$

We also have the following corollary:

**Corollary 3.3.** *Every left or right cancellative monoid with amenable group of units is sofic.*

*Proof.* Let  $C$  be a left or right cancellative monoid with amenable group of units. By the same argument as in the proof of Proposition 2.1 above, the  $\mathcal{J}$ -class of the identity must be the group of units. Indeed, if it wasn't then it would contain a copy of the bicyclic monoid, which is neither left nor right cancellative, giving a contradiction. The result now follows from condition (i) in Corollary 3.2.  $\square$

It follows from a recent result of Donnelly [4, Theorem 5] that a cancellative left or right amenable monoid has amenable group of units. Hence, Corollary 3.3 may be viewed as a generalisation of [2, Proposition 4.6], which says that cancellative one-sided amenable monoids are sofic.

The corollaries above apply Theorem 3.1 in rather restricted cases, although this still suffices to give many new examples of sofic monoids. The following example comes closer to applying Theorem 3.1 in its full generality.

**Example 3.4.** *Let  $G$  be any sofic group, and let  $S$  be any set of normal subgroups of  $G$  with amenable quotients (for example, the set of all finite*



index normal subgroups). Let  $M$  be the monoid generated by  $S$  together with the singleton subsets of  $G$  (which we view as cosets of the trivial subgroup) under setwise multiplication.

Then  $M$  is a monoid of cosets of normal subgroups of  $G$ . The identity element is the trivial subgroup  $\{1\}$ . The units are the singleton subsets, which we can identify with their single elements so that the group of units is just  $G$  itself. There are no left or right units apart from the singletons, so the  $\mathcal{J}$ -class of the identity is the group  $G$ .

The right translation action of the group of units is just the natural right translation action of  $G$  on the set of cosets. Each orbit under this action is the set of all cosets of some normal subgroup  $H$  of  $G$ . The pointwise stabiliser of this orbit in  $G$  is  $H$  itself, so the quotient by the stabiliser is  $G/H$ . Assuming the orbit is not the group of units, it follows from the definition of  $M$  that  $H$  is a product of subgroups from  $S$ , and in particular must contain some non-trivial normal subgroup  $K$  from  $S$ . Now  $G/H$  is a quotient of  $G/K$ , which by assumption is amenable, so by [1, Proposition 4.5.4],  $G/H$  is amenable. Hence,  $M$  satisfies the conditions of Theorem 3.1 and is sofic.

#### 4. $\mathcal{H}$ -CLASSES AND THE ACTION OF THE GROUP OF UNITS

In this section we consider from a structural perspective how the group of units of a monoid acts by translation on the rest of the monoid. Specifically, we use a semigroup-theoretic approach of breaking the action down into two parts using Green's  $\mathcal{H}$ -relation — an action *on each*  $\mathcal{H}$ -class, and an action *on the set of*  $\mathcal{H}$ -classes — and study how this deconstruction relates to the local amenability property of the action, which played such an important role in Section 3 above.

Let  $H$  be an  $\mathcal{H}$ -class of a monoid  $M$ , and let  $\Sigma_H$  denote the symmetric group on the set  $H$ , acting on  $H$  from the right and with composition therefore from left to right. Consider the set:

$$\{\sigma \in \Sigma_H \mid \text{there exists } m \in M \text{ with } h\sigma = hm \text{ for all } h \in H\}.$$

of all permutations of  $H$  which are realised by the right translation action of  $M$  on itself. In fact this set is a subgroup [3, Theorem 2.22] of  $\Sigma_H$ , called the (*right*) *Schützenberger group* of  $H$ . Note that if  $H$  happens to be a subgroup (in particular, if  $H$  is the group of units) then the Schützenberger group is isomorphic to the group  $H$  acting on itself by right translation. Schützenberger groups of  $\mathcal{H}$ -classes are a powerful tool for understanding the structure of semigroups – see [3, Section 2.4] for a full introduction.

Now let  $G$  be the group of units of the monoid  $M$ , and consider the action of  $G$  on  $M$  by right translation. It is easily seen that if  $g \in G$  and  $m \in M$  then  $mg\mathcal{R}m$ ; in other words, the right translation action of  $G$  preserves  $\mathcal{R}$ -class. Moreover, if  $m, n \in M$  with  $m\mathcal{L}n$  then clearly  $mg\mathcal{L}ng$  so the action preserves the  $\mathcal{L}$ -relation. It follows that the action preserves the  $\mathcal{H}$ -relation, and so induces a well-defined action on the set of  $\mathcal{H}$ -classes; we denote this action by  $\circ$ .

**Theorem 4.1.** *Let  $M$  be a monoid where the  $\mathcal{J}$ -class of the identity is a sofic group  $G$ , and suppose that for every non-unit  $\mathcal{D}$ -class  $D$  of  $M$ , either or both of the following conditions hold:*

- (i) *there are finitely many  $\mathcal{L}$ -classes in  $D$ , and the Schützenberger group of  $D$  is amenable; or*
- (ii) *the Schützenberger group of  $D$  is finite or abelian, and the  $\circ$ -action of  $G$  on the set of  $\mathcal{H}$ -classes in  $D$  is locally amenable.*

*Then  $M$  is sofic.*

*Proof.* We shall show that the right translation action of  $G$  on  $M \setminus G$  is locally amenable, so that the result follows from Theorem 3.1. Let  $x$  be an element of  $M \setminus G$ , and let  $X$  be its orbit under the right translation action of  $G$ . We wish to show that the quotient of  $G$  by the pointwise stabiliser of  $X$  is amenable.

Let  $Z$  be the union of the  $\mathcal{H}$ -classes of elements in  $X$ . Notice that because the right translation action of  $G$  preserves  $\mathcal{R}$ -class,  $X$  is contained in an  $\mathcal{R}$ -class. Since each  $\mathcal{R}$ -class is a union of  $\mathcal{H}$ -classes,  $Z$  is also contained in a single  $\mathcal{R}$ -class, and hence also a single  $\mathcal{D}$ -class.

Now since the right translation action of  $G$  preserves the  $\mathcal{H}$ -relation,  $Z$  is also a union of orbits under this action. It follows that the pointwise stabiliser of  $Z$  under this action is a normal subgroup of  $G$ ; call it  $Q$ .

Now  $X$  is contained in  $Z$ , so the pointwise stabiliser of  $X$  contains that of  $Z$ , so the quotient of  $G$  by the pointwise stabiliser of  $X$  is a quotient of  $G/Q$ . Since a quotient of an amenable group is amenable [1, Proposition 4.5.4], it will suffice to show that  $G/Q$  is amenable.

Let

$$P = \{g \in G \mid hg\mathcal{H}h \text{ for all } h \in Z\}.$$

It is immediate from the definition that  $P$  is the pointwise stabiliser of the set of  $\mathcal{H}$ -classes in  $Z$ , under the  $\circ$ -action of  $G$ . It follows from the definition of  $Z$  that this set is an orbit under the  $\circ$ -action. Hence,  $P$  is a normal subgroup of  $G$ .

Moreover, any element of  $G$  stabilising  $Z$  pointwise under the right translation action must also stabilise the  $\mathcal{H}$ -classes in  $Z$  under the  $\circ$ -action, which means  $Q \leq P \leq G$ . It follows that  $G/Q$  is an extension of the quotient  $G/P$  by the subgroup  $P/Q$ . Since an extension of an amenable group by an amenable group is amenable [1, Proposition 4.5.5], it will thus suffice for the theorem to show that  $G/P$  and  $P/Q$  are both amenable.

Consider first  $G/P$ . By assumption, the  $\mathcal{D}$ -class containing  $Z$  satisfies either condition (i) or condition (ii) from the statement of the theorem. In case (i), since  $Z$  is contained in an  $\mathcal{R}$ -class, and each  $\mathcal{H}$ -class is the intersection of an  $\mathcal{R}$ -class and an  $\mathcal{L}$ -class,  $Z$  can contain only finitely many  $\mathcal{H}$ -classes. It is immediate from the definition of  $P$  that  $G/P$  acts faithfully on the set of such, so it must be finite and hence amenable by [1, Proposition 4.4.6]. In case (ii), the  $\circ$ -action is locally amenable so  $G/P$ , being a quotient by the pointwise stabiliser of an orbit, is amenable.

Next, we show that  $P/Q$  is amenable. For each  $\mathcal{H}$ -class  $H$  in  $Z$ , and each  $p \in P$ , let  $p_H \in \Sigma_H$  denote the permutation of  $H$  induced by the right translation action of  $p$ . Since  $p \in G \subseteq M$ , it is immediate from the definition of the Schützenberger group that  $p_H$  lies in the Schützenberger group of  $H$ . Thus the map  $p \mapsto p_H$  gives a morphism from  $P$  to the Schützenberger group. Together, these maps induce a morphism from  $P$  to the direct product of all the Schützenberger groups of  $\mathcal{H}$ -classes in  $Z$ . In

fact, these Schützenberger groups are all isomorphic [3, Theorem 2.25], so it may be viewed as a morphism from  $P$  to the direct power  $S^n$ , where  $S$  is isomorphic to the Schützenberger groups and  $n$  is the (possibly infinite) cardinality of the set of  $\mathcal{H}$ -classes in  $Z$ .

Notice that this map sends  $p \in P$  to the identity if and only if  $p$  fixes every element of every  $\mathcal{H}$ -class in  $Z$ , that is, if  $p$  fixes every element of  $Z$ . In other words, the kernel of this map is exactly the subgroup  $Q$ . So the map induces an embedding of the quotient  $P/Q$  into  $S^n$ .

Again, either condition (i) or condition (ii) from the statement of the theorem applies to the  $\mathcal{D}$ -class containing  $Z$ . In case (i),  $n$  is finite and  $S$  is amenable, and so by [1, Corollary 4.5.6],  $S^n$  is amenable. In case (ii), the assumptions on  $S$  are sufficient to ensure that  $S^n$  is amenable even if  $n$  is infinite. Indeed, if  $S$  is abelian then  $S^n$  is abelian, and hence amenable by [1, Theorem 4.6.1]. If  $S$  is finite then  $S^n$  is locally finite, and hence amenable by [1, Corollary 4.5.12]. In all cases, it follows that  $P/Q$ , being isomorphic to a subgroup of  $S^n$ , is amenable by [1, Proposition 4.5.1].  $\square$

A case of particular interest is where condition (i) applies to *every*  $\mathcal{D}$ -class, that is, where each  $\mathcal{D}$ -class contains only finitely many  $\mathcal{L}$ -classes. As well as ensuring that the action of  $G$  on the set of  $\mathcal{H}$ -classes is locally amenable, this condition also automatically rules out the presence of a bicyclic submonoid, yielding a simpler statement:

**Corollary 4.2.** *Let  $M$  be a monoid with sofic group of units, all non-unit Schützenberger groups amenable and finitely many  $\mathcal{L}$ -classes in each  $\mathcal{D}$ -class. Then  $M$  is sofic.*

*Proof.* To apply Theorem 4.1, all we need is to show that the  $\mathcal{J}$ -class of the identity is the group of units. Suppose not. Then by the argument in the proof of Proposition 2.1,  $M$  admits a submonoid isomorphic to the bicyclic monoid, say generated by elements  $p$  and  $q$  with  $pq = 1$ . Since each  $\mathcal{D}$ -class has finitely many  $\mathcal{L}$ -classes and elements of the bicyclic monoid are certainly  $\mathcal{D}$ -related, by the pigeon-hole principle there must exist distinct natural numbers  $i < j$  with  $p^i \mathcal{L} p^j$ . Since  $\mathcal{L}$  is a right congruence, we get

$$1 = p^i q^i \mathcal{L} p^j q^i = p^{j-i}.$$

Thus, there is an element  $x \in M$  with  $xp^{j-i} = 1$ , in other words,  $p$  is a right unit. But this contradicts the fact that  $p$  is not cancellable on the left in the bicyclic monoid.  $\square$

Recall that a monoid  $M$  is called *regular* if for every  $x \in M$  there exists an element  $y \in M$  with  $xyx = x$ . The class of regular monoids includes in particular all inverse monoids. In a regular monoid every Schützenberger group arises as a (maximal) subgroup around some idempotent (this follows from [3, Theorems 2.22 and 2.24] and the elementary fact that every  $\mathcal{D}$ -class of a regular semigroup contains an idempotent, and hence a maximal subgroup). Thus, Theorem 4.1 has the following immediate corollary in this case:

**Corollary 4.3.** *Let  $M$  be a regular monoid with sofic group of units, all non-unit subgroups amenable and finitely many  $\mathcal{L}$ -classes in each  $\mathcal{D}$ -class. Then  $M$  is sofic.*

## 5. REMARKS AND OPEN QUESTIONS

We consider the extent to which the hypotheses in Theorem 3.1 are necessary. The requirement that the  $\mathcal{J}$ -class of the identity be a sofic group is necessary, by Proposition 2.1. However, it is unclear to what extent the remaining hypotheses are essential. The conditions certainly do not give an exact characterisation of sofic monoids, since the hypotheses do not apply in general to residually finite monoids.

**Example 5.1.** *Let  $F$  be any group which is residually finite but not amenable (for example, a free group of rank 2). Let  $S = \{0, 1\}$  be the 2-element semilattice (the monoid with multiplication given by  $11 = 1$  and  $10 = 01 = 00 = 0$ ). Then the direct product  $F \times S$  is easily seen to be a residually finite monoid, and hence by [2, Corollary 4.2] a sofic monoid. However, the (non-amenable) group of units  $F \times \{1\}$ , which is isomorphic to  $F$ , acts faithfully and transitively on  $F \times \{0\}$  (the rest of the monoid). In other words,  $F \times \{0\}$  is a single orbit with trivial pointwise stabiliser, so the corresponding quotient is again isomorphic to  $F$  and hence non-amenable. Thus, the hypotheses of Theorem 3.1 are not satisfied.*

Sofic groups themselves having been introduced as a generalisation of amenable groups, it is natural to ask if the hypothesis of amenability in Theorem 3.1 can be replaced with the weaker hypothesis of soficity.

**Question 5.2.** *Let  $M$  be a monoid in which the  $\mathcal{J}$ -class of the identity is a sofic group  $G$ , and such that quotients of  $G$  by orbit stabilisers under the right translation action on  $M$  are all sofic. Is  $M$  necessarily sofic?*

Indeed, it could even be that soficity of  $M$  results from soficity of  $G$ , without the necessity of considering the action on  $M \setminus G$ .

**Question 5.3.** *Let  $M$  be a monoid in which the  $\mathcal{J}$ -class of the identity is a sofic group. Is  $M$  necessarily sofic?*

A positive answer to this question would, combined with Proposition 2.1 above, completely describe sofic monoids modulo the case of sofic groups.

A natural first step to answering either of the above questions is to consider the case where group of units and/or its quotients by orbit stabilisers are residually finite but not amenable.

An interesting feature of Theorem 3.1 (presaged by [2, Proposition 4.7]) is that the hypotheses concern only the group of units and its right translation action on the monoid: neither the left translation action of the group of units nor the internal multiplication of the non-unit elements plays any role. It is natural to ask if this is only a feature of the theorem, or if it is an inherent property of sofic monoids:

**Question 5.4.** *Let  $M$  and  $N$  be monoids, and assume that the  $\mathcal{J}$ -class of the identity in each is a group. Suppose there is a bijection  $\rho : M \rightarrow N$  which restricts to an isomorphism between the groups of units, and such that  $\rho(st) = \rho(s)\rho(t)$  whenever  $t$  is a unit in  $M$ . If  $M$  is sofic, must  $N$  also be sofic?*

Note that a positive answer to Question 5.2 would also entail a positive answer to Question 5.4. A positive answer to Question 5.3 would imply the

even stronger statement that soficity for monoids only ever depends on the structure of the  $\mathcal{J}$ -class of the identity, and not even on its action on the rest of the monoid.

Irrespective of the answer to Question 5.4, [2, Proposition 4.7] (and also condition (i) of our Corollary 3.2) implies that the soficity condition imposes no restriction whatsoever on the internal complexity of the monoid outside the group of units. This suggests that soficity of monoids, as studied here, is only likely to be of interest in applications where the group of units plays a fundamental role.

If seeking applications in semigroup theory more widely, one is drawn to ask if there is an alternative, probably stronger, definition of a sofic monoid which also generalises sofic groups but exerts more control on the internal structure of the rest of the monoid. A natural test of whether a definition is satisfactory in this respect would be whether the resulting class is closed under the taking of monoid subsemigroups (that is, subsets closed under the multiplication and with an identity element which is not necessarily the identity of the containing monoid). Such a definition, if found, is also likely to extend naturally to semigroups without an identity element.

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