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# GOOD REDUCTION OF ALGEBRAIC GROUPS AND FLAG VARIETIES

A. JAVANPEYKAR AND D. LOUGHRAN

ABSTRACT. In 1983, Faltings proved that there are only finitely many abelian varieties over a number field of fixed dimension and with good reduction outside a given set of places. In this paper, we consider the analogous problem for other algebraic groups and their homogeneous spaces, such as flag varieties.

## 1. INTRODUCTION

An important classical result in algebraic number theory is the theorem of Hermite and Minkowski: there are only finitely many number fields of bounded degree over  $\mathbb{Q}$  which are unramified outside a given finite set  $S$  of primes of  $\mathbb{Q}$ . It was first noticed by Shafarevich [16] that such finiteness statements occur elsewhere in number theory. In particular, he conjectured an analogous statement for curves over number fields. In his famous paper on Mordell's conjecture [8], Faltings proved Shafarevich's conjecture, and also showed a corresponding finiteness statement for abelian varieties. Namely that, for  $K$  a number field,  $g$  an integer and  $S$  a finite set of finite places of  $K$ , the set of  $K$ -isomorphism classes of  $g$ -dimensional abelian varieties over  $K$  with good reduction outside  $S$  is finite.

It is natural to ask whether the analogue of Faltings's result holds for other algebraic groups, and more generally for their homogeneous spaces (e.g. torsors). This brings us to the first result of this paper.

**Theorem 1.1.** *Let  $K$  be a number field, let  $n \in \mathbb{N}$  and let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme. Then the set of  $B$ -isomorphism classes of pairs  $(G, E)$ , where  $G$  is an  $n$ -dimensional reductive group scheme over  $B$  and  $E$  is a  $G$ -torsor, is finite.*

Here we say that  $(G_1, E_1)$  is  $B$ -isomorphic to  $(G_2, E_2)$  if  $G_1 \cong G_2$  as group schemes over  $B$  and there exists a  $B$ -isomorphism  $E_1 \cong E_2$  which respects the action of  $G_1$  and  $G_2$ .

In this paper all reductive group schemes are connected (see Definition 2.3). The analogous statement of Theorem 1.1 *fails* for disconnected groups; for example the constant group scheme  $\mathbb{Z}/2\mathbb{Z}$  over  $\mathbb{Z}$  admits infinitely many non-isomorphic subgroup schemes, given by simply deleting the

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non-identity component above each prime  $p$ . Taking  $E = G$  in Theorem 1.1, we obtain the following immediate corollary.

**Corollary 1.2.** *Let  $K$  be a number field, let  $n \in \mathbb{N}$  and let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme. Then the set of  $B$ -isomorphism classes of  $n$ -dimensional reductive group schemes over  $B$  is finite.*

Corollary 1.2 can be used to obtain a result analogous to Faltings's theorem. Namely, let  $S$  be a finite set of finite places of a number field  $K$ . We say that a reductive algebraic group  $G$  over  $K$  has good reduction outside  $S$  if there exists a reductive group scheme over  $\mathcal{O}_K[S^{-1}]$  whose generic fibre is  $K$ -isomorphic to  $G$ .

**Corollary 1.3.** *Let  $n \in \mathbb{N}$  and let  $S$  be a finite set of finite places of a number field  $K$ . Then the set of  $K$ -isomorphism classes of  $n$ -dimensional reductive algebraic groups over  $K$  with good reduction outside  $S$  is finite.*

For motivation, let us briefly explain some special cases of our results. Firstly, for each field extension  $K \subset L$  of degree  $n$  of a number field  $K$ , there is an  $n$ -dimensional algebraic torus  $R_{L/K}(\mathbb{G}_{m,L})$  over  $K$  given by the Weil restriction of  $\mathbb{G}_{m,L}$ . Here for such a torus to have good reduction outside  $S$ , the corresponding field  $L$  must be unramified outside  $S$ . By Hermite-Minkowski there are only finitely many such fields, in particular, there are indeed only finitely many such tori. Secondly, for a central simple algebra  $D$  of dimension  $n^2$  over  $K$ , we may consider the corresponding special linear group  $\text{SL}(D)$ . If  $\text{SL}(D)$  has good reduction outside  $S$ , then the central simple algebra  $D$  must be unramified outside  $S$ . However by class field theory, there are only finitely many such central simple algebras of fixed dimension, as expected.

These discussions should make clear that our proofs will require certain finiteness results for the cohomology of group schemes. These will be provided to us by a theorem of Gille and Moret-Bailly [9] (which in turn builds upon work of Borel and Serre [2]). This result encapsulates for example the finiteness of Tate-Shafarevich groups of linear algebraic groups.

The assumption that the group be *reductive* in our results is crucial; in §4 we shall construct explicit counter-examples which show that the analogues here for unipotent group schemes fail.

Reductive algebraic groups admit other natural homogeneous spaces (besides torsors), namely flag varieties. As an application of our methods we prove the following, which was in fact the original motivation for this paper.

**Theorem 1.4.** *Let  $n \in \mathbb{N}$  and let  $S$  be a finite set of finite places of a number field  $K$ . Then the set of  $K$ -isomorphism classes of flag varieties over  $K$  of dimension  $n$  with good reduction outside  $S$  is finite.*

Here by *good reduction*, we mean good reduction as a flag variety (see Definition 3.5). Theorem 1.4 generalises some already known results. Namely, the analogue of Theorem 1.4 was already known for Brauer-Severi varieties

[17, Thm. 5.3] and quadric hypersurfaces in projective space [10]. New special cases include, for example, finiteness results for twists of Grassmannians and twists of products of projective spaces.

The authors have recently initiated a program of study into the “Shafarevich conjecture for Fano varieties”. The case of Fano varieties of dimension 1 (i.e. conics) is classical and the case of dimension 2 is dealt with in [17]. The case of flag varieties, handled in Theorem 1.4, is in some respects the next easiest case in this programme. In [10] the authors have proven other cases of this for certain complete intersections in projective space. These papers should all be viewed as certain aspects of the same research programme.

Let  $n \in \mathbb{N}$ . Most of our results generalise from algebraic groups of dimension  $n$  over number fields to arbitrary global fields, provided that one avoids certain small characteristics depending on  $n$ . For example, take  $K = \mathbb{F}_p(x)$  for some prime  $p$ . Then  $K$  admits infinitely many Artin-Schreier extensions  $L$  of degree  $p$  ramified only at  $\infty$  (see e.g. [19, §6.4]). Moreover one can check that the  $p$ -dimensional tori  $R_{L/K}(\mathbb{G}_{m,L})$  give rise to infinitely many  $K$ -isomorphism classes with good reduction outside of  $\infty$  (see [3, Ex. 7.2.8] for the case of semi-simple groups).

There is no common generalisation of our results and the results of Faltings, applying to groups which are built out of abelian varieties and reductive groups. For example, there are infinitely many semi-abelian schemes over  $\mathbb{Z}$ , up to  $\mathbb{Z}$ -isomorphism. Explicitly, such schemes occur as the identity components of the Néron models of the elliptic curves

$$y^2 + xy = x^3 + n, \quad n \in \mathbb{Z} \setminus \{0\},$$

since these curves are semi-stable over  $\mathbb{Z}$ . Moreover, the analogue of Theorem 1.1 is in fact *false* for abelian varieties (see [11, p. 241]). Namely, there exists a non-empty finite set of places  $S$  of  $\mathbb{Q}$ , such that the set of  $\mathbb{Q}$ -isomorphism classes of genus one curves over  $\mathbb{Q}$  with good reduction outside  $S$  is infinite (note that such curves are torsors for their Jacobians). This is due to the fact that, for an elliptic curve  $E$  over  $\mathbb{Q}$ , the “ $S$ -Tate-Shafarevich group”

$$\text{III}_S(E) = \ker \left( H^1(\mathbb{Q}, E) \rightarrow \prod_{p \notin S} H^1(\mathbb{Q}_p, E) \right)$$

can be infinite. In particular, our results for reductive groups are actually *stronger* than the corresponding results for abelian varieties.

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## 2. REDUCTIVE GROUPS AND TORSORS

The aim of this section is to prove Theorem 1.1.

**2.1. Finiteness results in étale cohomology.** In this paper we shall achieve our finiteness results by using a relationship between good reduction and étale cohomology. We gather here the finiteness results we shall require. Some of these results hold in greater generality than stated, but for simplicity we only work in the generality which we require.

Let  $B$  be a dense open subscheme of  $\text{Spec } \mathcal{O}_K$ , where  $K$  is a number field and  $\mathcal{O}_K$  is its ring of integers. Note that there exists a finite set of finite places  $S$  of  $K$  such that  $B = \text{Spec } \mathcal{O}_K[S^{-1}]$ . If  $G$  is a smooth separated group scheme over  $B$ , we denote by  $\check{H}^1(B, G)$  the Čech cohomology set of  $B$  with coefficients in  $G$  with respect to the étale topology [13, §III.2]. Recall that a faithfully flat and locally of finite type  $B$ -scheme  $E$  is a  $G$ -torsor if it is endowed with a left action of  $G$  such that the morphism

$$E \times_B G \rightarrow E \times_B E, \quad (x, g) \mapsto (x, x \cdot g)$$

is an isomorphism. By [13, Thm. III.4.3], [13, Prop. III.4.6] and [15, Thm. XI.3.1]), the pointed set  $\check{H}^1(B, G)$  classifies  $G$ -torsors over  $B$ . Here we have the following finiteness theorem due to Gille and Moret-Bailly.

**Lemma 2.1.** *Let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme and let  $G$  be a smooth affine group scheme of finite type over  $B$ . Then  $\check{H}^1(B, G)$  is finite.*

*Proof.* This is a special case of [9, Prop. 5.1]. □

If  $G_1$  and  $G_2$  are group schemes over  $B$ , we shall say that  $G_2$  is a *twist* of  $G_1$  if  $G_2$  is  $B$ -isomorphic to  $G_1$  locally for the étale topology on  $B$ . If  $\text{Aut}_B G_1$  is representable by a smooth separated group scheme, then  $\check{H}^1(B, \text{Aut}_B G_1)$  classifies the twists of  $G_1$  (see [13, §III.4]).

We shall require an analogue of Lemma 2.1 for some constant group schemes which are not of finite type. For an (abstract) group  $G$ , we let  $G_B$  denote the constant group scheme over  $B$  associated to  $G$ . As a scheme this is a disjoint union  $\coprod_{g \in G} B$ , and the group scheme structure is defined in the obvious way. Note that  $G_B$  is faithfully flat and locally of finite type over  $B$ , but it is of finite type over  $B$  if and only if  $G$  is finite.

**Lemma 2.2.** *Let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme and let  $n$  be a positive integer. Then  $\check{H}^1(B, \text{GL}_n(\mathbb{Z})_B)$  is finite.*

*Proof.* As  $\text{GL}_n(\mathbb{Z})$  is an arithmetic group, it is finitely presented and contains only finitely many finite subgroups, up to conjugacy; see [14, Thm. 4.2] and [14, Thm. 4.3]. It is therefore “decent” in the sense of [12, §16], hence the result follows from [12, Lem. (a), §16] (this being a simple application of Hermite-Minkowski). □

**2.2. Reductive algebraic groups.** We assume that the reader is familiar with the basic theory of reductive group schemes, as found in [3] or [7].

**Definition 2.3.** Let  $B$  be a scheme. A group scheme  $G$  over  $B$  is *reductive* if the geometric fibres of  $G \rightarrow B$  are smooth connected affine with trivial unipotent radical.

If  $B$  is integral, we define the *rank* of  $G$  to be the dimension of the maximal torus of the generic fibre (note that we do not consider only split maximal tori).

**2.3. Tori and semi-simple groups.** In order to prove Theorem 1.1, we first handle the cases of algebraic tori and semi-simple groups.

**Proposition 2.4.** *Let  $n \in \mathbb{N}$  and let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme. Then the set of  $B$ -isomorphism classes of  $n$ -dimensional tori over  $B$  is finite.*

*Proof.* Let  $T$  be an  $n$ -dimensional torus over  $B$ . Then by definition,  $T$  is a twist of  $\mathbb{G}_{m,B}^n$ . Such twists are classified by  $\check{H}^1(B, \text{Aut}_B \mathbb{G}_{m,B}^n)$ . However  $\text{Aut}_B \mathbb{G}_{m,B}^n \cong \text{GL}_n(\mathbb{Z})_B$ , hence the result follows from Lemma 2.2.  $\square$

We next treat the case of semi-simple groups which are twists of a fixed *split* semi-simple group.

**Lemma 2.5.** *Let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme and let  $G_K$  be a split semi-simple group over  $K$ . Then the set of  $B$ -isomorphism classes of semi-simple group schemes  $G'$  over  $B$  such that  $G'_K$  is a twist of  $G_K$  is finite.*

*Proof.* Note that as  $G_K$  is split, by the existence theorem [7, Cor. XXV.1.3] there exists a smooth semi-simple group scheme  $G$  over  $B$  such that  $G_K$  is isomorphic to  $G \times_B K$ . Let  $G'$  be a semi-simple group scheme over  $B$  whose generic fibre is a twist of  $G_K$  over  $K$ .

Locally for the étale topology on  $B$ , by [7, Cor. XXII.2.3] we see that  $G$  and  $G'$  are  $B$ -split and have isomorphic generic fibres over  $K$ . Therefore, by the isomorphism theorem [7, Thm. XX.4.1],  $G'$  is a twist of  $G$ , hence corresponds to a some element of  $\check{H}^1(B, \text{Aut}_B G)$ . However the scheme  $\text{Aut}_B G$  is smooth affine of finite type over  $B$  [7, Thm. XXIV.1.3], hence the result follows from Lemma 2.1.  $\square$

To continue, we need to know that there are only finitely many *split* semi-simple algebraic groups of fixed dimension over a given field. This follows from the classification of semi-simple groups, as we now explain. We also prove the analogous result for semi-simple algebraic groups of fixed rank, as it will be used later on when we handle flag varieties.

**Lemma 2.6.** *Let  $n \in \mathbb{N}$  and let  $K$  be a field. Then the set of  $K$ -isomorphism classes of split semi-simple group schemes over  $K$  of dimension  $n$  (or rank  $n$ ) is finite.*

*Proof. The simply connected case.* Any split simply connected semi-simple algebraic group  $G$  may be uniquely written as a finite direct product of *simple* split simply connected semi-simple algebraic groups over  $K$ . It therefore suffices to consider the case where  $G$  is simple. Such groups are classified by irreducible Dynkin diagrams. There are five exceptional diagrams, plus four infinite series  $A_\ell, B_\ell, C_\ell$  and  $D_\ell$ , whose corresponding algebraic groups have dimensions  $\ell^2 + 2\ell, \ell(2\ell + 1), \ell(2\ell + 1)$  and  $\ell(2\ell - 1)$  and ranks  $\ell$ , respectively (these are the dimensions and ranks of  $\mathrm{SL}_{\ell+1}, \mathrm{SO}_{2\ell+1}, \mathrm{SP}_{2\ell}$  and  $\mathrm{SO}_{2\ell}$ , respectively). From this classification, it is clear that there are only finitely many simple split simply connected semi-simple algebraic groups of fixed dimension (or fixed rank).

*The general case.* Any semi-simple algebraic group  $G$  fits into an exact sequence of algebraic groups

$$0 \rightarrow \pi_1(G) \rightarrow \tilde{G} \rightarrow G \rightarrow 0.$$

Here  $\tilde{G}$  denotes the universal cover of  $G$  and  $\pi_1(G)$  the algebraic fundamental group of  $G$ , which is finite and contained in the centre of  $\tilde{G}$ . However the centre of a semi-simple algebraic group is finite, and so the result follows from the simply connected case.  $\square$

**Proposition 2.7.** *Let  $n \in \mathbb{N}$  and let  $B \subset \mathrm{Spec} \mathcal{O}_K$  be a dense open subscheme. Then the set of  $B$ -isomorphism classes of semi-simple group schemes over  $B$  of dimension  $n$  (or rank  $n$ ) is finite.*

*Proof.* This follows immediately from Lemma 2.5 and Lemma 2.6.  $\square$

**2.4. Proof of Theorem 1.1.** We now come to the proof of Theorem 1.1. We begin by handling torsors under a fixed reductive group scheme.

**Lemma 2.8.** *Let  $B \subset \mathrm{Spec} \mathcal{O}_K$  be a dense open subscheme and let  $G$  be a reductive group scheme over  $B$ . Then the set of  $B$ -isomorphism classes of  $G$ -torsors over  $B$  is finite.*

*Proof.* As explained in §2.1, such torsors are classified by  $\check{H}^1(B, G)$ . This set is finite by Lemma 2.1.  $\square$

Therefore to prove Theorem 1.1, it suffices to show that the set of  $B$ -isomorphism classes of reductive group schemes over  $B$  of dimension  $n$  is finite. To do so, let  $G$  be a reductive group scheme over  $B$  and let  $\mathcal{R}(G)$  and  $\mathcal{D}(G)$  denote the radical and the derived subgroup scheme of  $G$ , respectively (see [7, Exp. XXII]). By [7, Exp. XXII, 6.2.3], we have the following short exact sequence

$$0 \rightarrow \mathcal{R}(G) \cap \mathcal{D}(G) \rightarrow \mathcal{R}(G) \times_B \mathcal{D}(G) \rightarrow G \rightarrow 0$$

of group schemes over  $B$ . Note that  $\mathcal{R}(G) \cap \mathcal{D}(G)$  is a central finite flat subgroup scheme of  $G$ . As  $\mathcal{R}(G)$  is a torus and  $\mathcal{D}(G)$  is semi-simple, it follows from Proposition 2.4 and Proposition 2.7 that, as  $G$  runs over all reductive group schemes over  $B$  of fixed dimension, there are only finitely

many choices for  $\mathcal{R}(G)$  and  $\mathcal{D}(G)$ . Thus, to prove Theorem 1.1, it suffices to show the following.

**Lemma 2.9.** *Let  $T$  be an algebraic torus and let  $D$  be a semi-simple algebraic group scheme, both over  $B$ . Then there are only finitely many central finite flat subgroups  $F \subset T \times_B D$  such that the projections*

$$F \rightarrow T, \quad F \rightarrow D,$$

*are closed immersions.*

*Proof.* As the centre of  $D$  is a finite flat group scheme over  $B$ , the degree of  $F$  over  $B$  is bounded. The result therefore follows from the fact that  $T$  contains only finitely many finite flat subgroup schemes of bounded order, since  $T[n]$  is finite flat for each  $n$ .  $\square$

This completes the proof of Theorem 1.1.  $\square$

### 3. FLAG VARIETIES

The aim of this section is to prove Theorem 1.4.

**3.1. Preliminaries.** We begin with some preliminaries on flag varieties.

**Definition 3.1.** A smooth proper variety over a field  $K$  is a (*twisted*) *flag variety over  $K$*  if it is a homogeneous space for some reductive algebraic group scheme over  $K$ , and the stabiliser of each point is smooth. A smooth proper scheme over a scheme  $B$  is a *flag scheme over  $B$*  if its geometric fibres are flag varieties.

Note that a flag variety may be a homogeneous space for many reductive groups, e.g. the projective line  $\mathbb{P}^1$  is a homogeneous space for both  $\mathrm{SL}_2$  and  $\mathrm{PGL}_2$ . Demazure proved that flag schemes are in fact homogeneous spaces for their automorphism group schemes.

**Lemma 3.2.** *Let  $X$  be a flag scheme over a scheme  $B$ . Then  $\mathrm{Aut}_B(X)$  is representable by a smooth affine group scheme whose neutral component  $\mathrm{Aut}_B(X)^0$  is semi-simple. Moreover  $X$  is a homogeneous space for  $\mathrm{Aut}_B(X)^0$ .*

*Proof.* This is an application of [5, Prop. 6.4].  $\square$

Let  $G$  be a semi-simple group scheme over a scheme  $B$ . Recall that a smooth  $B$ -subgroup scheme  $P \subset G$  is called *parabolic* if the quotient  $G/P$  is a proper  $B$ -scheme. In this case  $G/P$  is obviously a flag scheme. Over an algebraically closed field every flag variety is of this form (one simply takes  $P$  to be the stabiliser of a rational point), but this does not have to be the case over more general bases.

**Lemma 3.3.** *Let  $G$  be a semi-simple group scheme over a Noetherian scheme  $B$ . Then the set of  $B$ -isomorphism classes of flag schemes which are a homogeneous space for  $G$  is finite.*



*Proof.* Define the functor  $\underline{\text{Par}}_G$  from the category of schemes over  $B$  to the category of sets by

$$\underline{\text{Par}}_G : S \mapsto \{\text{Parabolic subgroups of } G_S\}.$$

This functor is representable by a smooth projective scheme over  $B$ , which we denote by  $\text{Par}_G$  (see [3, Cor. 5.29] or [7, Cor. XXVI.3.5]). Note that  $G$  acts on  $\text{Par}_G$  by conjugation. In particular, the irreducible components of  $\text{Par}_G$  are flag schemes for  $G$ . Now let  $X$  be a flag scheme for  $G$ . We define a morphism from  $X$  to  $\text{Par}_G$  by

$$\begin{aligned} X(S) &\rightarrow \text{Par}_G(S) \\ x &\mapsto \text{stabiliser of } x, \end{aligned}$$

for any  $B$ -scheme  $S$ . As this morphism is  $G$ -equivariant, it identifies  $X$  with an irreducible component of  $\text{Par}_G$ . As  $\text{Par}_G$  is Noetherian, the result is proved.  $\square$

One also has control over the dimension of a flag variety in terms of the rank of its automorphism group.

**Lemma 3.4.** *Let  $X$  be a flag variety over a field  $K$ . Then*

$$\dim X \geq \text{rank Aut}(X)^0.$$

*Proof.* We may assume that  $K$  is algebraically closed. Take  $G = \text{Aut}(X)^0$  and let  $T \subset G$  be a maximal torus. The action of  $G$  on  $X$  is faithful, hence  $T$  also acts faithfully on  $X$ .

Borel's fixed point theorem [1, Thm. II.10.4] implies that  $T$  acts on  $X$  with a fixed point  $x \in X(K)$ . We claim that the action of  $T$  on the tangent space  $\text{Tan}_x(X)$  of  $x$  is faithful. Indeed, let  $H = \ker(T \rightarrow \text{GL}(\text{Tan}_x(X)))$ . By [4, Prop. A.8.10], the fixed locus  $X^H$  is a smooth closed subscheme of  $X$  and  $\text{Tan}_x(X)^H = \text{Tan}_x(X^H)$ . As  $H$  acts trivially on  $\text{Tan}_x(X)$  we find that  $\dim X^H = \dim X$ , hence  $X^H = X$ . However the action of  $T$  on  $X$  is faithful, thus  $H$  is trivial. This proves the claim.

On diagonalising the action of  $T$  on  $\text{Tan}_x(X)$ , we therefore obtain

$$\dim X = \dim \text{Tan}_x(X) \geq \dim T = \text{rank } G,$$

as required.  $\square$

For flag varieties, we use the following notion of good reduction.

**Definition 3.5.** Let  $K$  be a number field and let  $S$  be a finite set of finite places of  $K$ . A flag variety  $X$  over  $K$  has *good reduction outside  $S$*  if there exists a flag scheme  $\mathcal{X}$  over  $\mathcal{O}_K[S^{-1}]$  whose generic fibre is isomorphic to  $X$ .

**3.2. Proof of Theorem 1.4.** In order to prove Theorem 1.4, it suffices to show the following.

**Theorem 3.6.** *Let  $K$  be a number field, let  $n \in \mathbb{N}$  and let  $B \subset \text{Spec } \mathcal{O}_K$  be a dense open subscheme. Then the set of  $B$ -isomorphism classes of  $n$ -dimensional flag schemes over  $B$  is finite.*

*Proof.* Let  $X$  be a flag scheme over  $B$  of dimension  $n$ . By Lemma 3.2, we know that  $\mathrm{Aut}_B(X)^0$  is a semi-simple group scheme over  $B$  and that  $X$  is a flag scheme for  $\mathrm{Aut}_B(X)^0$ . Moreover by Lemma 3.4, we have

$$\mathrm{rank} \mathrm{Aut}_B(X)^0 \leq n.$$

Hence by Proposition 2.7 there are only finitely many choices for  $\mathrm{Aut}_B(X)^0$ . Moreover, by Lemma 3.3, there are only finitely many flag schemes over  $B$  associated to  $\mathrm{Aut}_B(X)^0$ . This proves the result.  $\square$

This completes the proof of Theorem 1.4.  $\square$

#### 4. UNIPOTENT GROUPS

We now explain how the analogues of our main results do not hold for unipotent groups.

**4.1. Unipotent group schemes.** We first show that the analogue of Corollary 1.2 fails for unipotent groups of dimension at least two. Let  $p$  be a prime and let  $W_p$  be the additive group of Witt vectors of length 2 over  $\mathbb{Z}$ . Explicitly, this is a smooth commutative unipotent scheme over  $\mathbb{Z}$  with group law

$$\begin{aligned} W_p \times W_p &\rightarrow W_p \\ (x, y) \times (x', y') &\mapsto \left( x + x', y + y' + \frac{(x + x')^p - x^p - x'^p}{p} \right). \end{aligned}$$

For a ring  $R$ , we have  $W_p \times R \cong \mathbb{G}_{a,R}^2$  if and only if  $p$  is invertible in  $R$ . Hence if  $p$  and  $q$  are distinct prime numbers, then  $W_p$  is not isomorphic to  $W_q$ . We therefore see that there are infinitely many smooth commutative unipotent group schemes of dimension 2 over  $\mathbb{Z}$ , up to isomorphism.

For completeness let us note that the analogue of Corollary 1.2 does indeed hold in the one-dimensional case. Namely, let  $B$  be a dense open subscheme of  $\mathrm{Spec} \mathcal{O}_K$ . Then by [20, Cor. 3.8], the set of  $B$ -isomorphism classes of one-dimensional unipotent groups over  $B$  is in bijection with  $\mathrm{Pic}(B)$ . In particular, this set is finite.

**4.2. Good reduction of unipotent groups.** We now show that the analogue of Corollary 1.3 fails for unipotent groups. Here we say that a unipotent group  $U$  over  $K$  has good reduction outside a finite set of finite places  $S$  of  $K$ , if there exists a smooth unipotent group scheme over  $\mathcal{O}_K[S^{-1}]$  whose generic fibre is isomorphic to  $U$ . We construct counter-examples by exploiting a relationship between nilpotent Lie algebras and unipotent algebraic groups. For each  $\lambda \in \mathbb{Z}$ , we define a 7-dimensional nilpotent Lie algebra  $\mathfrak{u}_\lambda$  over  $\mathbb{Z}$  as follows. As a  $\mathbb{Z}$ -module  $\mathfrak{u}_\lambda$  has a basis  $x_1, \dots, x_7$ , and the Lie bracket is given by

$$\begin{aligned} [x_1, x_i] &= x_{i+1} \quad (2 \leq i \leq 6), & [x_2, x_3] &= x_5, & [x_2, x_4] &= x_6, \\ [x_2, x_5] &= \lambda x_7, & [x_3, x_4] &= (1 - \lambda)x_7. \end{aligned}$$

These are the Lie algebras with label  $123457_I$  in [18]. We chose these Lie algebras as their generic fibres are pairwise non-isomorphic over  $\overline{\mathbb{Q}}$ .

A standard argument using the Baker-Campbell-Hausdorff formula (see [6, §IV.2.4]) yields a smooth unipotent group scheme  $U_\lambda$  over  $\mathbb{Z}[1/7!]$  whose Lie algebra is  $\mathfrak{u}_\lambda \otimes \mathbb{Z}[1/7!]$ . In particular, we obtain infinitely many pairwise non- $\overline{\mathbb{Q}}$ -isomorphic unipotent algebraic groups over  $\mathbb{Q}$  of dimension 7 which have good reduction outside  $\{2, 3, 5, 7\}$ .

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