



Differential operators on the superline, Berezinians, and Darboux transformations

DOI:

[10.1007/s11005-017-0958-7](https://doi.org/10.1007/s11005-017-0958-7)

Document Version

Accepted author manuscript

[Link to publication record in Manchester Research Explorer](#)

Citation for published version (APA):

Li, S., Shemyakova, E., & Voronov, T. (2017). Differential operators on the superline, Berezinians, and Darboux transformations. *Letters in Mathematical Physics*, 107(9), 1-26. <https://doi.org/10.1007/s11005-017-0958-7>

Published in:

Letters in Mathematical Physics

Citing this paper

Please note that where the full-text provided on Manchester Research Explorer is the Author Accepted Manuscript or Proof version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version.

General rights

Copyright and moral rights for the publications made accessible in the Research Explorer are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Takedown policy

If you believe that this document breaches copyright please refer to the University of Manchester's Takedown Procedures [<http://man.ac.uk/04Y6Bo>] or contact uml.scholarlycommunications@manchester.ac.uk providing relevant details, so we can investigate your claim.



DIFFERENTIAL OPERATORS ON THE SUPERLINE, BEREZINIANS, AND DARBOUX TRANSFORMATIONS

SIMON LI, EKATERINA SHEMYAKOVA, AND THEODORE VORONOV

ABSTRACT. We consider differential operators on a supermanifold of dimension $1|1$. We define non-degenerate operators as those with an invertible top coefficient in the expansion in the “superderivative” D (which is the square root of the shift generator, the partial derivative in an even variable, with the help of an odd indeterminate). They are remarkably similar to ordinary differential operators. We show that every non-degenerate operator can be written in terms of ‘super Wronskians’ (which are certain Berezinians). We apply this to Darboux transformations (DTs), proving that every DT of an arbitrary non-degenerate operator is the composition of elementary first order transformations. Hence every DT corresponds to an invariant subspace of the source operator and, upon a choice of basis in this subspace, is expressed by a super-Wronskian formula. We consider also dressing transformations, i.e., the effect of a DT on the coefficients of the non-degenerate operator. We calculate these transformations in examples and make some general statements.

1. INTRODUCTION

In this paper, we consider differential operators on the superline (by which we mean a $1|1$ -dimensional supermanifold) and their Darboux transformations. All differential operators on the superline can be expressed in terms of the ‘superderivative’ $D = \partial_\xi + \xi\partial_x$, where x and ξ are the even and the odd indeterminates, respectively, and we define non-degenerate operators as those with an invertible top coefficient of their expansion in D . As we observe, such operators, though partial, behave very similarly to ordinary differential operators. Our main results are as follows.

We analyze algebraic properties of non-degenerate differential operators on the superline and deduce, in particular, that every such operator can be expressed in terms of ‘super Wronskians’, which are Berezinians or superdeterminants of ‘super Wronski’ matrices. (Super Wronskians were first introduced in [19].) This is analogous to the classical statement for ordinary differential operators; however, it is trickier because the Berezinian is a more complicated function than the ordinary determinant; in particular, it is a rational function, not a polynomial. Note that one has to use both Ber and Ber^* , where $\text{Ber}^* A := \text{Ber} A^\Pi$ and A^Π denotes the parity-reversed matrix.¹

We establish a complete classification of *Darboux transformations* (DTs) of an arbitrary non-degenerate operator. (We consider monic operators for simplicity, but

Date: 13 (26) March 2017.

2010 Mathematics Subject Classification. Primary: 16S32; secondary: 58C50, 37K35, 37K25 .

Key words and phrases. Darboux transformation, intertwining relation, superline, Berezinian, super Wronskian, dressing transformation.

¹For A invertible, $\text{Ber}^* A = (\text{Ber} A)^{-1}$; however, we need to use non-invertible matrices as well.

this is not essential.) By definition, a Darboux transformation $L \rightarrow L_1$, where L and L_1 are two monic operators of the same order, is given by a monic operator M satisfying the *intertwining relation*

$$ML = L_1M. \quad (1)$$

We show that Darboux transformations of an operator L correspond to invariant subspaces of L (of particular dimensions, such as $s+1|s$ or $s|s$). We conclude, therefore, that *the problem of finding Darboux transformations for a given L can be regarded as a generalization of the eigenvalue problem.*

In the short note [12], we showed that every Darboux transformation is the composition of elementary Darboux transformations of first order. We elaborate the proof here and show in addition that every Darboux transformation can be expressed by a super-Wronskian formula, which corresponds to a choice of basis in an invariant subspace. (The Darboux transformation itself does not depend on this choice.) This extends to differential operators on the superline the classic Wronskian formulas for Darboux transformations of ordinary differential operators, due to Darboux, Crum, and Matveev.

Finally, we consider the effect of a Darboux transformation on the coefficients of an initial operator L ; this effect is sometimes called the ‘dressing’ (or ‘undressing’) transformation. We calculate the dressing transformations for arbitrary operators L of orders $n \leq 4$ in D , in terms of the coefficients of the intertwining operator M . These coefficients can be obtained by expanding super Wronskians. (We include a discussion of expansion of Berezinians, where we obtained some new formulas for ‘Berezinian cofactors’ as a by-product of our study.) There are some general patterns concerning dressing transformations of operators of arbitrary order n that one can notice from these examples. We present the corresponding theorem.

Recall that what is now known as Darboux transformations of differential operators appeared in the studies of Darboux himself [9] and other classical differential geometers such as Moutard and Tzitzeica, who applied them to the geometry of surfaces. Particular examples had been known already to Euler and Laplace (see [24]). These works were almost forgotten until the 1970s. (Exceptions were applications in quantum mechanics, probably found by physicists independently, without knowledge of the prior work of geometers, and the re-discovery and generalization of DTs for the Sturm–Liouville operators by Crum [8].) A real rebirth of Darboux transformations happened with the advent of soliton theory. Wahlquist and Estabrook [34] and Wadati–Sanuki–Konno [33] used transformations of differential operators later recognized as Darboux transformations for constructing analogs of Bäcklund transformations for the famous non-linear KdV equation. Matveev [22] identified these transformations with those of Darboux and Crum, introduced the name ‘Darboux transformations’ and developed them into a powerful tool in integrable system theory [23].

The investigation of Darboux transformations in the supercase was pioneered by Liu [17] and Liu & Mañas [18, 19] for the ‘super Sturm–Liouville operator’

$$L = \partial^2 + \alpha D + u,$$

who introduced transformations built on “seed solutions” and their iterations. In [19], the dressing transformation of the coefficients α , u were expressed in terms of super Wronskians. In [16], Li and Nimmo introduced analogs of elementary first order Darboux transformations in the abstract setting of ‘twisted derivations’ and obtained a formula for the composition in terms of quasideterminants (thus generalizing the results of [11]). From these quasideterminant formulas they deduced the super Wronskian formulas for the super Sturm–Liouville operator obtained in [19] and extended them to the case where such a formula had not been available.

We would like to stress that in the classic monograph [23] and in many other works, Darboux transformations are defined in terms of seed solutions and Wronskian type formulas. However, underlying these specific formulas, there is an algebraic structure that makes it possible to study DTs in a much more general sense. To a certain extent, this structure can be extracted from Darboux’s original works, but it has not become explicit in modern literature until the 1990s.

Loosely speaking, a Darboux transformation should map a differential operator to another operator ‘of the same form’ together with a linear transformation between kernels (or arbitrary eigenspaces). This is achieved if the ‘old’ and ‘new’ operators are connected by an intertwining relation of the form $ML = L_1M$ or $NL = L_1M$. (In modern context, an intertwining relation with M only, without N , appeared, for Sturm–Liouville operators, in [26], [32], and [2]; for the ‘Laplacians’ $-\Delta + u$, in [4, 5]; an intertwining relation with possibly different M and N , appeared explicitly in [31].)

Besides having the advantage of algebraic clarity, the abstract algebraic framework based on intertwining relations allows one to include more general types of Darboux transformations (such as the Laplace transformations of the Schrödinger operator in 2D and their generalizations) and to search for new Darboux transformations. Therefore this framework seems to be more suitable for a new setup, such as the study of Darboux transformations on supermanifolds. We consider here the intertwining relation of the form (1) with a single operator M because we view the superline as a kind of 1D object, an analog of the ordinary line. Rather than depart from a seed solution ansatz and Wronskian type formulas, we obtain them as a result of our classification theorem. (We consider arbitrary non-degenerate differential operators on the superline, so the formulas of [19], [16] for the super Sturm–Liouville operator appear as a special case.)

For comparison, the possibility to factorize a Darboux transformation of the Sturm–Liouville operator on the ordinary line into elementary transformations was established in several steps: in [32, Thm. 5], when the new potential differs from the initial one by a constant; in [27], for transformations of order two; and for the general case, in [2] and the follow-up paper [3], see also [25, §3]. For an arbitrary operator on the line, the factorization theorem was proved only recently: in [1]. The 2D analog of these results is much more complicated. Factorization of Darboux transformations for the 2D Schrödinger operator was established in [30]; this required developing a new algebraic apparatus. We hope to consider the ‘super 2D’ case elsewhere.

2. PROPERTIES OF DIFFERENTIAL OPERATORS ON THE SUPERLINE

Consider a $1|1$ -dimensional supermanifold, which we shortly call the ‘superline’. (Note that it is possible to extend the ordinary line by more than one odd variable, but we do not consider it here.) Its global structure is not important for our purposes. Let x be an even coordinate and ξ be an odd coordinate. Functions on the superline have the form $f(x, \xi) = f^0(x) + \xi f^1(x)$. We should always assume that the coefficients of the expansion in ξ may depend on some unspecified extra odd parameters. Therefore, for an even function $f(x, \xi)$, the coefficients $f^0(x)$ and $f^1(x)$ are even and odd respectively. Similarly, for an odd $f(x, \xi)$, the coefficient $f^0(x)$ is odd and the coefficient $f^1(x)$ is even. (In the absence of extra odd parameters, the odd coefficients become of course zero.) See Remark 2.3 below.

Denote $D = \partial_\xi + \xi \partial_x$, so $D^2 = \partial$, $\partial = \partial_x$. Hence $\partial_\xi = D - \xi D^2$. The ring of differential operators on the superline will be denoted by $\text{DO}(1|1)$. This ring was probably first considered in [21], where it was embedded into the ring of formal pseudodifferential operators, see below. An arbitrary operator $A \in \text{DO}(1|1)$ can be uniquely written as

$$A = a_0 D^m + a_1 D^{m-1} + \dots + a_m,$$

where the coefficients a_k are functions of x, ξ and may also depend on some ‘external’ even or odd parameters. (See Remark 2.3.) We define *order* of elements of $\text{DO}(1|1)$ by saying that for an operator A as above, $\text{ord } A \leq m$. This differs from the usual notion; e.g., $\text{ord } D = 1$, but $\text{ord } \partial_x = 2$ and $\text{ord } \partial_\xi = 2$.

Remark 2.1. The operator $D = \partial_\xi + \xi \partial_x$ is well-known in physics as the simplest supersymmetry generator, and it goes in the literature under different names such as ‘superderivative’ or ‘covariant derivative’. It has a simple invariant characterization as one of the two normal forms of rectifiable odd vector fields [28] (distinguished by whether the square of the vector field vanishes or not).

Remark 2.2. For ordinary manifolds, a vector field is (locally) rectifiable, i.e., can be written as a partial derivative, if and only if it does not vanish at a point. This is what is usually meant by a “non-degeneracy” of a vector field at a point (and therefore on a neighborhood of this point). The situation for supermanifolds is subtler. In the above-cited seminal work of V. N. Shander [28], it was shown that there are three normal forms for rectifiable vector fields. Namely, for even vector fields it is a partial derivative such as $\partial/\partial x^1$, where x^1 is an even local coordinate (the same as on ordinary manifolds); but for odd vector fields it is either a partial derivative such as $\partial/\partial \xi^1$, where ξ^1 is an odd local coordinate, or the ‘superderivative’ $\partial/\partial \xi^1 + \xi^1 \partial/\partial x^1$. (The latter two cases are distinguished by the square of the odd vector field.) The additional subtlety is in a “non-degeneracy” condition that would guarantee the rectifiability. It was found that the naive extension from ordinary manifolds is not enough. If a vector field on a supermanifold does not vanish at a point (called “weak non-degeneracy” in [28]) and it is even, then it is rectifiable, i.e., takes the form $\partial/\partial x^1$ in suitable local coordinates. So in this case there is no difference with ordinary manifolds. For an odd vector field, it was observed that “weak non-degeneracy” (non-vanishing at a point) does not lead to rectifiability and indeed to any classification. Such a vector field can be brought locally to the form $\partial/\partial \xi^1 + \xi^1 Y$ where Y is an arbitrary even vector field

not subject to any conditions. Two results leading from such a seeming impasse were obtained in [28]. If an odd vector field X is homological, i.e., satisfies $X^2 = 0$, then the weak non-degeneracy gives the rectifiability: locally X takes the form $\partial/\partial\xi^1$. On the other hand, if a stronger “non-degeneracy” condition is imposed, namely, that the vector field as an operator on functions is locally epimorphic, then for even vector fields it is equivalent to weak non-degeneracy (and gives the usual rectifiability), while for odd vector fields it excludes homological vector fields but guarantees the rectifiability with the normal form $\partial/\partial\xi^1 + \xi^1\partial/\partial x^1$. Below we define non-degenerate differential operators on the superline (Definition 2.4). From this viewpoint, they can be connected with the notion of a non-degenerate vector field in the sense of [28]. Another important observation of [28] was that for supermanifolds, the role of ordinary differential equations should be taken by a particular type of partial differential equations, with the “1|1-dimensional time” and the operator $\partial/\partial\tau + \tau\partial/\partial t$ with even t and odd τ replacing the ordinary time derivative d/dt . In the present paper, we obtain in a certain sense “dual” results: as we shall see, the non-degenerate (partial) differential operators on the superline, as we define them, possess features very close to those of ordinary differential operators on the (ordinary) line.

Remark 2.3. We consider all objects parameterized by an unspecified supermanifold playing the role of the base of the family of the considered objects. In other words, the coefficients of functions, differential operators, etc., are taken from a commutative superalgebra which is the algebra of global functions on such a base, so that our objects are ‘defined’ over this algebra. Varying this unspecified base leads to the ‘functor of points’ familiar from algebraic geometry. On these notions in the context of supergeometry see, e.g., [14, 15], [20], and [10].

Definition 2.4. We say that an operator A of order m is *non-degenerate* if the top coefficient a_0 is invertible (in particular, even).

Example 2.5. $\partial_x = D^2$ is non-degenerate.

Example 2.6. $\partial_\xi = D - \xi D^2$ is degenerate (top coefficient ξ).

Non-degenerate operators of even order are even, and of odd order, odd. Unlike arbitrary elements of $\text{DO}(1|1)$, non-degenerate operators cannot be divisors of zero. The set of all non-degenerate operators is multiplicatively closed. If A is non-degenerate, then $A = a_0 \cdot B$, where $B = D^m + b_1 D^{m-1} + \dots + b_m$ is monic. So non-degeneracy corresponds to the classical idea of a differential equation that can be resolved with respect to the highest derivative.

Our key observation is that, although the ring $\text{DO}(1|1)$ contains nilpotents (because the algebra of functions contains them) and cannot be described by nice algebraic words such as ‘Euclidean’, many important properties hold for non-degenerate operators. If N is arbitrary and M is non-degenerate, it is possible to divide by M with a remainder from the left and from the right: that is, there exist unique Q_1, R_1 and Q_2, R_2 such that

$$N = MQ_1 + R_1, \quad N = Q_2M + R_2,$$

where $\text{ord } R_1, \text{ord } R_2 < \text{ord } M$.

The next important fact is that the solution space for a non-degenerate operator on the superline is finite-dimensional (unlike the degenerate case).

Example 2.7. If $D\varphi = 0$, then $\varphi = \text{const}$. (Indeed, for $\varphi = \varphi^0(x) + \xi\varphi^1(x)$, $D\varphi = \varphi^1(x) + \xi\partial_x\varphi^0(x)$, so the equation $D\varphi = 0$ implies $\partial_x\varphi^0 = 0$, $\varphi^1 = 0$.) The constant can be even or odd, but as basis vector in the solution space we may take $\varphi = 1$, so $\dim \text{Ker } D = 1|0$.

(Compare with the equation $\partial_\xi\varphi = 0$ where the solution space consists of all functions of x .)

Lemma 2.8. *For a non-degenerate operator A of order m ,*

$$\begin{aligned} \dim \text{Ker } A &= n + 1|n && \text{if } m = 2n + 1, \text{ and} \\ \dim \text{Ker } A &= n|n && \text{if } m = 2n. \end{aligned}$$

Proof. Consider the equation

$$D^m\varphi + a_1D^{m-1}\varphi + \dots + a_m\varphi = 0. \quad (2)$$

Introduce a column vector φ with the coordinates $\varphi, D\varphi, \dots, D^{m-1}\varphi$. Note that it is written in a non-standard format: we assume that the parities of positions alternate starting from even, so the vector φ is even for an even φ and odd, for an odd φ . (For the notion of a matrix format, see Section 3.) Equation (2) can be re-written in the matrix form as

$$D\varphi = \Gamma\varphi, \quad (3)$$

where

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \dots & -a_1 \end{pmatrix} \quad (4)$$

is an odd matrix. (Note once again that the parities of rows and columns are alternating, starting from even, and the last row is of parity m . It is of course possible to re-write Γ using standard matrix format, but it will not be elucidating.) We may consider a general linear equation of form (3) with an arbitrary odd matrix Γ . Here φ is a vector-function, $\varphi = \varphi(x, \xi)$. Write $\varphi = \varphi_0(x) + \xi\varphi_1(x)$ and $\Gamma = \Gamma_0(x) + \xi\Gamma_1(x)$; then (3) is equivalent to the system

$$\begin{cases} \frac{d\varphi_0}{dx} = (\Gamma_1 + \Gamma_0^2)\varphi_0, \\ \varphi_1 = \Gamma_0\varphi_0. \end{cases} \quad (5)$$

Hence everything reduces to solving an ordinary linear differential equation with the (even) matrix $\Gamma_1 + \Gamma_0^2$. Its solution is uniquely defined by an initial condition $\varphi_0(0)$. Therefore the dimension of the solution space of the above system (5), and hence of the matrix equation (3), is equal to the dimension of the space of initial conditions. In our particular case of (3) arising from (2), it is precisely $n + 1|n$ for $m = 2n + 1$ and $n|n$ for $m = 2n$, as claimed. \square

Remark 2.9. One can extend the ring $\text{DO}(1|1)$ by adjoining the formal inverse D^{-1} . Due to the relation $D^2 = \partial$, we have $D^{-1} = D\partial^{-1} = \partial^{-1}D$ or, explicitly, $D^{-1} = \xi + \partial^{-1}\partial_\xi$. So adjoining D^{-1} or adjoining ∂^{-1} is the same. The resulting rings $\text{DO}(1|1)[D^{-1}]$ and $\text{DO}(1|1)[[\partial^{-1}]]$ of formal pseudodifferential operators were introduced by Manin and Radul [21]. Non-degenerate operators become invertible elements in the ring $\text{DO}(1|1)[[\partial^{-1}]]$.

Remark 2.10 (not used in the sequel). There is a nice way of expressing symbolically the solution of a linear differential equation (3) with an arbitrary odd matrix $\Gamma = \Gamma(x, \xi)$. By using the notion of ordered exponential (or multiplicative integral), we can write

$$\varphi_0(x) = P \exp \int_{x_0}^x dx (\Gamma_1 + \Gamma_0^2) \cdot \mathbf{c},$$

where \mathbf{c} is a constant vector (initial value). Hence

$$\varphi(x, \xi) = \exp(\xi\Gamma) \cdot P \exp \int_{x_0}^x dx (\Gamma_1 + \Gamma_0^2) \cdot \mathbf{c}.$$

In the scalar or commutative case, $\Gamma_0^2 = 0$ and we would get just the exponential of $\xi\Gamma + \int_{x_0}^x dx \partial_\xi\Gamma = (\xi + \partial^{-1}\partial_\xi)\Gamma = D^{-1}\Gamma$ (if we identify the operator of indefinite integration $\int_{x_0}^x dx$ with ∂^{-1}). Hence in this case the solution would be $\exp(D^{-1}\Gamma) \cdot \mathbf{c}$. In the general case, we write symbolically

$$\varphi(x, \xi) = P \exp(D^{-1}\Gamma) \cdot \mathbf{c},$$

where we set

$$P \exp(D^{-1}\Gamma) := \exp(\xi\Gamma) \cdot P \exp \int_{x_0}^x dx (\Gamma_1 + \Gamma_0^2).$$

Consider a non-degenerate operator L . From Lemma 2.8 it follows that there always exists an even solution of $L\varphi = 0$. Moreover, we can assume it to be invertible (at least if we assume that it is always possible to divide by a non-zero function of x).

Example 2.11 (used in the future). For an invertible function φ , define the operator

$$M_\varphi := D - D \ln \varphi = \varphi \circ D \circ \varphi^{-1}.$$

Then $M_\varphi(\varphi) = 0$ and $\text{Ker } M_\varphi$ is spanned by φ . Every monic first-order operator has the form M_φ for some invertible φ . Indeed, if $L = D + \alpha$, then set $\varphi := \exp(-D^{-1}\alpha)$.

Lemma 2.12 (“Bézout’s theorem”). *Let L be an arbitrary non-degenerate operator of order m and let φ be an even solution of the equation $L\varphi = 0$. Then L is divisible from the right by M_φ , so that $L = L'M_\varphi$, for a non-degenerate operator L' of order $m - 1$.*

Proof. Divide with a remainder from the right: $L = L'M_\varphi + r$, where r is a function. Then $0 = L\varphi = L'M_\varphi(\varphi) + r\varphi = r\varphi$, hence $r = 0$. \square

Lemma 2.13. *Every non-degenerate operator L of order m can be factorized as*

$$L = a_m \cdot M_{\varphi_1} \dots M_{\varphi_m},$$

for some functions $\varphi_1, \dots, \varphi_m$.

Proof. By induction. □

We see that non-degenerate operators on the superline are similar in many aspects to ordinary differential operators. There is an important statement pushing this similarity even further, namely, the expression of a non-degenerate operator in terms of the superanalog of Wronskian. For that, we need to recall some information about Berezinians or superdeterminants.

3. DIGRESSION: BEREZINIANS AND THEIR EXPANSIONS

Information in this section is mostly known or can easily be obtained from what is known. We just present it in a form convenient for our purposes. For the Cramer rule in the supercase, see [7] and also [13]. Cofactors were introduced in [13], but there is no detailed discussion in the literature. Formula (17) for an odd cofactor is new.

Matrices of linear operators acting on vector spaces or free modules over a commutative superalgebra carry an extra piece of information, namely, parities (taking values in \mathbb{Z}_2) of their rows and columns. These are the parities of the corresponding basis vectors. Labeling matrix rows and columns by parities is called a *(super)matrix format* [6, 14, 15, 20]. A matrix labeled in this way is sometimes referred to as a *supermatrix*. We usually suppress the prefix. For an n by m matrix, an introduction of a matrix format means a partition of the set of its n rows into r ‘even’ rows and s ‘odd’ rows, where $n = r + s$, and a partition of the set of the m columns into p ‘even’ columns and q ‘odd’ columns, where $m = p + q$. (We stress that these are but labels attached to the indices and a priori they have nothing to do with the parities of the matrix entries.) The matrix is called *even* if it has even entries in the even-even and odd-odd positions, and odd entries in the even-odd and odd-even positions (and it is called *odd* in the opposite case). It is always possible to (uniquely) transform any matrix into *standard format* where all even rows go first and all odd rows go after them, and the same for columns. Matrices not in standard format naturally occur in practical examples (such as the Wroński matrix in the next section). Bringing the matrix to the standard format does not change a relative order within rows or columns of the same parities.

Berezinian (the super analog of determinant) was discovered by F. A. Berezin and is now named after him. Notation: $\text{Ber } A$, alternative notations: $\text{ber } A$ or $\text{sdet } A$. It is an essentially unique, up to taking powers, multiplicative function on the space of even invertible matrices:

$$\text{Ber}(AB) = \text{Ber } A \cdot \text{Ber } B. \tag{6}$$

It is convenient to define $\text{Ber } A$ axiomatically, by the following properties as a function of rows:

- $\text{Ber } A$ is homogeneous of degree $+1$ in each even row;
- $\text{Ber } A$ is homogeneous of degree -1 in each odd row;
- $\text{Ber } A$ is unchanged under elementary row transformations;
- $\text{Ber } E = 1$ for an identity matrix E .

(A similar characterization is possible in terms of columns. The axioms basically follow from the required multiplicativity property and the condition that Ber should reduce to ordinary determinant in the purely even case.) Here ‘homogeneous of

degree ± 1 ' means that multiplying a row of A by an invertible factor λ results in the multiplication of $\text{Ber } A$ by the factor of $\lambda^{\pm 1}$. An elementary row transformation means replacing a row by the sum with a multiple of another row, $r_i \rightarrow r_i + \lambda r_j$ (here λ does not have to be invertible and its parity is the sum of the parities of r_i and r_j). By Gaussian elimination, one immediately arrives from these axioms at the one of the two equivalent explicit formulas (which are often taken as the definition of Ber):

$$\text{Ber } A = \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det A_{11}} = \frac{\det A_{00}}{\det(A_{11} - A_{10}A_{00}^{-1}A_{01})}. \quad (7)$$

Here the blocks correspond to the subdivision of the rows and columns according to their parities.

Note that Berezinian is a rational function, not a polynomial in matrix entries. That is why it is not defined on arbitrary even matrices. Initially it is required that A be invertible for $\text{Ber } A$ to be defined. One can see that it suffices to require only the block A_{11} be invertible.

The *parity reversion* or Π -*transpose* of a matrix is the change of its format so that all the parities of the rows and columns are replaced by the opposite (nothing happens with the entries themselves). Notation: A^Π . In terms of the block decomposition,

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}^\Pi = \begin{pmatrix} A_{11} & A_{10} \\ A_{01} & A_{00} \end{pmatrix}. \quad (8)$$

If A is an even invertible matrix, both $\text{Ber } A$ and $\text{Ber } A^\Pi$ make sense and

$$\text{Ber } A^\Pi = (\text{Ber } A)^{-1}. \quad (9)$$

Following [7], we define the *inverse Berezinian* $\text{Ber}^* A$ by

$$\text{Ber}^* A := \text{Ber } A^\Pi \quad (10)$$

for an arbitrary even matrix whenever $\text{Ber } A^\Pi$ makes sense. The function $\text{Ber}^* A$ is defined if and only if the block A_{00} is invertible. The rational functions $\text{Ber } A$ and $\text{Ber}^* A$ are mutually reciprocal on invertible even matrices; in general, it may happen that one is defined and the other is not.

The following remarkable property holds: the Berezinian $\text{Ber } A$, as a function of rows, is multilinear in all even rows of A . Likewise, the inverse Berezinian $\text{Ber}^* A$ is multilinear in all odd rows of A . The same holds with respect to columns: $\text{Ber } A$ is multilinear in all even columns of A and $\text{Ber}^* A$ is multilinear in all odd columns of A . (Note that rows are multiplied by scalars on the left and columns, on the right.)

From the linearity it follows that it is possible to extend the definition of $\text{Ber } A$ and consider Berezinian of a 'wrong' in the sense of parity matrix A , namely, a matrix obtained from an even matrix by replacing one (only one!) even row by an odd row-vector (i.e., with odd entries on the even positions and even entries on the odd positions). The same holds for columns (when one even column is replaced by an odd column-vector). In the same way, $\text{Ber}^* A$ makes sense for a 'wrong' matrix where an odd row or an odd column is replaced by an even vector. Note that so defined 'wrong' matrices are neither even nor odd. Berezinians of 'wrong' matrices take odd values. Practically, they are calculated by the same explicit formulas (no ambiguity arises).

Remark 3.1. Warning: one should be careful with the properties of Berezinians for the case of ‘wrong’ arguments. If an odd vector stands in A at an even position, the Berezinian is still invariant under elementary transformations provided multiples of ‘correct’ vectors are added to the ‘wrong’ vector, not the other way round. Otherwise one can run into a contradiction.

Example 3.2. Let x be even, ξ be odd. The square matrix of order $1|1$

$$\begin{pmatrix} \xi & x \\ \xi & x \end{pmatrix}$$

is a ‘wrong’ matrix (the odd vector (ξ, x) stands in an even row; we assume standard format). We have

$$\text{Ber} \begin{pmatrix} \xi & x \\ \xi & x \end{pmatrix} = \frac{\xi - xx^{-1}\xi}{x} = 0.$$

We can subtract the second row from the first, but not the first from the second.

Suppose $A = (a_{ij})$ is an even square matrix of order $n|m$. The *cofactors* $\text{adj}_{ij} A$ and $\text{adj}_{ij}^* A$ of a matrix A are defined as follows (see [13]²). When the index i is even, $\text{adj}_{ij} A$ is defined as the Berezinian of the matrix obtained from A by replacing its i th row by a row of zeros except for the j th place where 1 is inserted. Likewise, $\text{adj}_{ij} A$ is defined when j is even, by replacing the j th column by a column of zeros except for 1 at the i th position. Note that when both i and j are even, these definitions agree. In the same way, one defines $\text{adj}_{ij}^* A$, for the case where i is odd or j is odd and by using Ber^* instead of Ber . By the definition, there are the following expansion formulas: for a fixed even i_0 and a fixed even j_0 ,

$$\text{Ber} A = \sum_j a_{i_0 j} \cdot \text{adj}_{i_0 j} A \quad (\text{row expansion}) \quad (11)$$

$$= \sum_i \text{adj}_{i j_0} A \cdot a_{i j_0} \quad (\text{column expansion}). \quad (12)$$

Similarly, for a fixed odd i_0 and a fixed odd j_0 ,

$$\text{Ber}^* A = \sum_j a_{i_0 j} \cdot \text{adj}_{i_0 j}^* A \quad (\text{row expansion}) \quad (13)$$

$$= \sum_i \text{adj}_{i j_0}^* A \cdot a_{i j_0} \quad (\text{column expansion}). \quad (14)$$

Note that $\text{adj}_{ij} A$ is undefined when i, j are both odd and $\text{adj}_{ij}^* A$ is undefined when i, j are both even.

By combining the obvious identities $\text{adj}_{ij} A = (-1)^i \partial \text{Ber} A / \partial a_{ij}$ for an even i or even j and $\text{adj}_{ij}^* A = (-1)^{i+1} \partial \text{Ber}^* A / \partial a_{ij}$ for an odd i or odd j with the following formula of differentiation of Berezinian:

$$\delta(\text{Ber} A) = \text{Ber} A \text{ str}(\delta A \cdot A^{-1}) \quad (15)$$

²There is a small difference in notation between the one used here and that in [13].

(see, e.g., [13]; here δ is an arbitrary even derivation and str denotes *supertrace*, $\text{str } B = \text{tr } B_{00} - \text{tr } B_{11}$ for an even matrix B), one deduces the following relations between the cofactors of A and the elements of the inverse matrix $A^{-1} = (a_{ij}^*)$:

$$\frac{\text{adj}_{ij} A}{\text{Ber } A} = a_{ji}^* = \frac{\text{adj}_{ij}^* A}{\text{Ber}^* A} \quad (16)$$

(for all values of indices where the cofactors make sense). (Above, and everywhere we use the notation where tilde over the symbol means the parity of the symbol.)

Example 3.3. For an even square matrix A of size $1|1$, with the self-explanatory notations for the matrix entries, $\text{Ber } A = a_{00}a_{11}^{-1} - a_{01}a_{10}a_{11}^{-2}$, and we have for example $\text{adj}_{00} A = 1/a_{11}$, $\text{adj}_{01} A = -a_{10}/a_{11}^2 = \text{Ber} \begin{pmatrix} 0 & 1 \\ a_{10} & a_{11} \end{pmatrix}$ (a ‘wrong’ matrix).

Remark 3.4. When both i and j are even, the cofactor $\text{adj}_{ij} A$ is up to a sign the Berezinian of the even square matrix of size $n - 1|m - 1$ obtained from A by deleting the i th row and the j th column, i.e., a ‘minor’ of A . The sign is $(-1)^{N(i)+N(j)}$, where $N(\dots)$ is the absolute number of a row or column among the rows or columns of the same parity. This case is completely analogous to the familiar case of ordinary matrices. Unlike that, if i is even and j is odd (or other way round), then the cofactor $\text{adj}_{ij} A$ is the Berezinian of a ‘wrong’ matrix and is odd. The non-unit matrix entries in the j th column cannot be replaced by zeros and the cofactor cannot be reduced to a Berezinian of a smaller size in a naive way; still, there is a (more complicated) expression for it in terms of smaller size Berezinians:

$$\text{adj}_{ij} A = -\frac{\text{Ber}^* d_{jj}(A_{r_i \rightarrow r_j})}{(\text{Ber}^* d_{jj}(A))^2} \quad (17)$$

for $\tilde{i} = 1$ and $\tilde{j} = 0$. Here $r_i \rightarrow r_j$ means that the i th row is replaced by the j th row, and by d_{ij} we denote the operation of deleting the i th row and the j th column in a matrix. Note Ber^* and not Ber in (17), and that the matrix in the numerator is ‘wrong’. For comparison, the formula for the cofactor when both i and j are even, mentioned above, is

$$\text{adj}_{ij} A = (-1)^{N(i)+N(j)} \text{Ber } d_{ij}(A), \quad (18)$$

for $\tilde{i} = \tilde{j} = 0$. Similar expressions, with interchanging Ber and Ber^* , can be written for $\text{adj}_{ij} A$. As explained, formula (18) is obtained directly; but both formulas (18) and (17) can be obtained together by writing down a system of linear equations for the coefficients $c_i = \text{adj}_{ij} A$ (with fixed j , $\tilde{j} = 0$) and solving it by using Cramer’s rule. Then the two different cases come about naturally. (We omit the details.)

Finally, from the expansion formulas (11) and (13) we obtain the following ‘row by row’ rule for differentiation of Berezinian, generalizing the rule for ordinary determinants:

$$\delta(\text{Ber } A) = \sum_{i \text{ even}} \text{Ber} \begin{pmatrix} \dots \\ \delta(A)_i \\ \dots \end{pmatrix} - (\text{Ber } A)^2 \cdot \sum_{i \text{ odd}} \text{Ber}^* \begin{pmatrix} \dots \\ \delta(A)_i \\ \dots \end{pmatrix}. \quad (19)$$

Here δ is some derivation (may be odd), $(A)_i$ denotes the i th row of a matrix, and the matrices in both sums are obtained from A by replacing $(A)_i$ by the row of derivatives

$\delta(A)_i$. A similar formula holds for $\delta(\text{Ber}^* A)$, with Ber and Ber^* interchanged; also with columns instead of rows (in the latter case, with extra signs).

4. EXPRESSION OF A DIFFERENTIAL OPERATOR VIA SUPER WRONSKIANS

In this section, we establish an analog for the superline of the classic relation between ordinary differential operators and Wronskian determinants.

Given a function f of variables x, ξ , we define its *superjet* as the infinite sequence f, Df, D^2f, \dots . If we terminate the sequence at $D^n f$, we speak about the *n-superjet*. If f is even, its n -superjet is a point (an even vector) in the vector *n-superjet space*, which has dimension $k+1|k$ if $n = 2k$ or $k+1|k+1$ if $n = 2k+1$. For an odd function f , its n -superjet is an odd vector in this space (a point or an even vector in the reversed parity space, of dimension $k|k+1$ if $n = 2k$ and $k+1|k+1$ if $n = 2k+1$). Without the name, we used superjets in the proof of Lemma 2.8. If we use the natural numbering of the coordinates of a superjet (corresponding to the successive powers of D), then their parities alternate. If needed, the coordinates can be renumbered so that all even coordinates go first and all odd coordinates go second. Note also that it is natural to write superjets as column vectors (because the multiplication of a function by a constant from the right induces the multiplication of its superjet from the right and column vectors make naturally a right module). For typographic reasons we may still write them horizontally, but use square rather than round brackets.

Example 4.1. For the 2-superjet of an even function φ , we have $[\varphi^0, \varphi^1, \varphi^2] = [\varphi, D\varphi, D^2\varphi]$ in the natural numbering, where at the i th position stands an element of parity $i \bmod 2$. Or we may write $[\varphi, D^2\varphi | D\varphi]$, where the first two coordinates are even and the last coordinate is odd.

Definition 4.2. Consider functions $\varphi_0, \varphi_1, \dots, \varphi_n$, where φ_0 is even, φ_1 is odd, etc. The *Wroński matrix* of such a system, notation:

$$\mathbf{W}(\varphi_0, \varphi_1, \dots, \varphi_n) = [W^i_j(\varphi_0, \varphi_1, \dots, \varphi_n)]_{ij=0\dots n},$$

has entries $W^i_j(\varphi_0, \varphi_1, \dots, \varphi_n) = D^i \varphi_j$. We define the (*super*)*wronskian* of a system $\varphi_0, \varphi_1, \dots, \varphi_n$ as the Berezinian of the Wroński matrix:

$$W(\varphi_0, \varphi_1, \dots, \varphi_n) := \text{Ber } \mathbf{W}(\varphi_0, \varphi_1, \dots, \varphi_n)$$

Define also the *inverse (super)wronskian* of a system $\varphi_0, \varphi_1, \dots, \varphi_n$ by

$$W^*(\varphi_0, \varphi_1, \dots, \varphi_n) := \text{Ber}^* \mathbf{W}(\varphi_0, \varphi_1, \dots, \varphi_n),$$

as the inverse Berezinian of the Wroński matrix.

Superwronskians were introduced by Liu & Mañas [19]. We shall often drop the prefix and refer to them as simply ‘Wronskians’. It is clear that it is not the usual Wronskian of functions of a single variable.

The columns of the matrix $\mathbf{W}(\varphi_0, \varphi_1, \dots, \varphi_n)$ are the n -superjets of functions $\varphi_0, \varphi_1, \dots, \varphi_n$. Note that the Wroński matrix has non-standard format, where the parity of an i th row or column is $i \bmod 2$. Recall from the previous section that the Berezinian of a matrix of a non-standard format is calculated by the usual formula where it is understood that the blocks are obtained by extracting from the matrix the rows and

columns of the required parities with their order preserved (effectively, re-numbering of the rows and columns so to make the format standard, but without changing the order of rows and columns of the same parity).

Example 4.3. Below we use horizontal and vertical lines to show partition of matrices into blocks according to parity.

$$\begin{aligned}
W(\varphi_0) &= \varphi_0, \\
W(\varphi_0, \varphi_1) &= \text{Ber} \begin{pmatrix} \varphi_0 & \varphi_1 \\ D\varphi_0 & D\varphi_1 \end{pmatrix} = \text{Ber} \left(\begin{array}{c|c} \varphi_0 & \varphi_1 \\ \hline D\varphi_0 & D\varphi_1 \end{array} \right), \\
W(\varphi_0, \varphi_1, \varphi_2) &= \text{Ber} \begin{pmatrix} \varphi_0 & \varphi_1 & \varphi_2 \\ D\varphi_0 & D\varphi_1 & D\varphi_2 \\ D^2\varphi_0 & D^2\varphi_1 & D^2\varphi_2 \end{pmatrix} = \text{Ber} \left(\begin{array}{cc|c} \varphi_0 & \varphi_2 & \varphi_1 \\ \hline D^2\varphi_0 & D^2\varphi_2 & D^2\varphi_1 \\ D\varphi_0 & D\varphi_2 & D\varphi_1 \end{array} \right), \\
W(\varphi_0, \varphi_1, \varphi_2, \varphi_3) &= \text{Ber} \begin{pmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 \\ D\varphi_0 & D\varphi_1 & D\varphi_2 & D\varphi_3 \\ D^2\varphi_0 & D^2\varphi_1 & D^2\varphi_2 & D^2\varphi_3 \\ D^3\varphi_0 & D^3\varphi_1 & D^3\varphi_2 & D^3\varphi_3 \end{pmatrix} = \\
&= \text{Ber} \left(\begin{array}{cc|cc} \varphi_0 & \varphi_2 & \varphi_1 & \varphi_3 \\ \hline D^2\varphi_0 & D^2\varphi_2 & D^2\varphi_1 & D^2\varphi_3 \\ D\varphi_0 & D\varphi_2 & D\varphi_1 & D^2\psi_2 \\ D^3\varphi_0 & D^3\varphi_2 & D^3\varphi_1 & D^3\varphi_3 \end{array} \right).
\end{aligned}$$

Example 4.4.

$$W^*(\varphi_0, \varphi_1) = \text{Ber}^* \begin{pmatrix} \varphi_0 & \varphi_1 \\ D\varphi_0 & D\varphi_1 \end{pmatrix} = \text{Ber} \left(\begin{array}{c|c} D\varphi_1 & D\varphi_0 \\ \hline \varphi_1 & \varphi_0 \end{array} \right).$$

From the properties of Berezinians it follows that *the superwronskian of a system of functions is linear in each even function, and the inverse superwronskian is linear in each odd function.*

From the invariance of Berezinians w.r.t. elementary transformations of columns, it also follows that *if an even function in the argument of a superwronskian is replaced by a linear combination (with constant coefficients of required parities) of the remaining functions, then the superwronskian vanishes; and the same holds for inverse superwronskians and odd functions.*

Now the main statement of this section.

Theorem 4.5. *A monic differential operator on the superline*

$$M = D^n + a_1 D^{n-1} + \dots + a_n$$

is completely defined by its kernel (or solution space) $\text{Ker } M$. Namely, if $\varphi_0, \dots, \varphi_{n-1}$ is a basis in $\text{Ker } M$, where $\tilde{\varphi}_i = i \bmod 2$, then for $n = 2k + 1$ the action of M on arbitrary odd function ψ is given by

$$M\psi = \frac{W^*(\varphi_0, \dots, \varphi_{n-1}, \psi)}{W^*(\varphi_0, \dots, \varphi_{n-1})}, \quad (20)$$

and for $n = 2k$ the action of M on arbitrary even function φ is given by

$$M\varphi = \frac{W(\varphi_0, \dots, \varphi_{n-1}, \varphi)}{W(\varphi_0, \dots, \varphi_{n-1})}. \quad (21)$$

In each case, to functions of the different parity, M is extended by linearity.

(This obviously generalizes to arbitrary non-degenerate operators. Note the emergence of inverse Wronskians for operators of odd order.)

Proof. Suppose $\varphi_0, \dots, \varphi_{n-1}$ is a basis in $\text{Ker } M$ as stated. Then

$$\begin{aligned} a_n\varphi_0 + a_{n-1}D\varphi_0 + \dots + a_1D^{n-1}\varphi_0 + D^n\varphi_0 &= 0, \\ a_n\varphi_1 + a_{n-1}D\varphi_1 + \dots + a_1D^{n-1}\varphi_1 + D^n\varphi_1 &= 0, \\ &\dots \\ a_n\varphi_{n-1} + a_{n-1}D\varphi_{n-1} + \dots + a_1D^{n-1}\varphi_{n-1} + D^n\varphi_{n-1} &= 0, \end{aligned}$$

which gives a system of linear equations for determining the coefficients a_n, a_{n-1}, \dots, a_1 with the square matrix $\mathbf{W}(\varphi_0, \varphi_1, \dots, \varphi_{n-1})$ and the right-hand sides $D^n\varphi_0, \dots, D^n\varphi_{n-1}$. Hence the operator M is uniquely defined by $\varphi_0, \dots, \varphi_{n-1}$. (Explicit expressions for the coefficients can be obtained, e.g., by the ‘super Cramer formulas’ [7], [13], which use the notion of cofactors introduced in the previous section.) To deduce equalities (20) and (21), consider the linear operators defined by the right-hand sides of (20) and (21). They are of order n . Note that, by the properties of super Wronskians, if a linear combination of $\varphi_0, \dots, \varphi_{n-1}$ of the required parity (the coefficients being treated as independent parameters some of which are odd) is substituted for ψ in the right-hand side of (20) or ϕ in the right-hand side of (21), then the right-hand sides of (20) and (21) vanish. By differentiating w. r. t. the parameters, it follows that all the functions $\varphi_0, \dots, \varphi_{n-1}$ are in the kernels of these operators. To identify them finally with M , it is sufficient to see that these operators are monic. This amounts to calculating the coefficient of $D^{2k+1}\psi$ in the expansion of

$$W^*(\varphi_0, \dots, \varphi_{2k}, \psi) = \text{Ber}^* \begin{pmatrix} \varphi_0 & \varphi_1 & \dots & \varphi_{2k} & \psi \\ D\varphi_0 & D\varphi_1 & \dots & D\varphi_{2k} & D\psi \\ \dots & \dots & \dots & \dots & \dots \\ D^{2k}\varphi_0 & D^{2k}\varphi_1 & \dots & D^{2k}\varphi_{2k} & D^{2k}\psi \\ D^{2k+1}\varphi_0 & D^{2k+1}\varphi_1 & \dots & D^{2k+1}\varphi_{2k} & D^{2k+1}\psi \end{pmatrix},$$

or, similarly, of $D^{2k}\varphi$ in the expansion of $W(\varphi_0, \dots, \varphi_{2k-1}, \varphi)$. This is exactly the inverse superwronskian $W^*(\varphi_0, \dots, \varphi_{n-1})$ or the superwronskian $W(\varphi_0, \dots, \varphi_{n-1})$, respectively. \square

From the properties of Berezinians, the right hand sides of (20) and (21) are invariant under an arbitrary non-singular transformation of the basis $\varphi_0, \dots, \varphi_{n-1}$, so do not depend on a choice of basis in $\text{Ker } M$.

Example 4.6. Let $M = D + \mu$ be a first-order operator. Its solution space is one-dimensional. Suppose φ is an invertible even function such that $M\varphi = 0$. For an odd

function ψ ,

$$M\psi = \frac{W^*(\varphi, \psi)}{W^*(\varphi)} = \frac{\text{Ber}^* \begin{pmatrix} \varphi & \psi \\ D\varphi & D\psi \end{pmatrix}}{\text{Ber}^*(\varphi)} = \varphi \text{Ber} \left(\begin{array}{c|c} D\psi & D\varphi \\ \psi & \varphi \end{array} \right) = \varphi \frac{D\psi - D\varphi \cdot \varphi^{-1} \psi}{\varphi} = (D - D\varphi \cdot \varphi^{-1}) \psi.$$

The final expression is manifestly linear in ψ and does not depend on the parity of ψ ; the same formula defines the action of M on functions of arbitrary parities.

To obtain the coefficients of an operator

$$M = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$$

explicitly, it is possible to use directly the Cramer rule of [7] and [13] (as noted in the proof of Theorem 4.5) or apply the formulas for the cofactors from the previous section. This gives an alternative description of an operator in terms of its fundamental solutions.

Theorem 4.7. *If $n = 2m + 1$,*

$$a_1 = -\frac{1}{W(\varphi_0, \dots, \varphi_{2m})} \text{Ber} \begin{pmatrix} \varphi_0 & \varphi_1 & \dots & \varphi_{2m} \\ D\varphi_0 & D\varphi_1 & \dots & D\varphi_{2m} \\ \dots & \dots & \dots & \dots \\ D^{2m-1}\varphi_0 & D^{2m-1}\varphi_1 & \dots & D^{2m-1}\varphi_{2m} \\ D^{2m+1}\varphi_0 & D^{2m+1}\varphi_1 & \dots & D^{2m+1}\varphi_{2m} \end{pmatrix}, \quad (22)$$

$$a_2 = -\frac{1}{W^*(\varphi_0, \dots, \varphi_{2m})} \text{Ber}^* \begin{pmatrix} \varphi_0 & \varphi_1 & \dots & \varphi_{2m} \\ D\varphi_0 & D\varphi_1 & \dots & D\varphi_{2m} \\ \dots & \dots & \dots & \dots \\ D^{2m+1}\varphi_0 & D^{2m+1}\varphi_1 & \dots & D^{2m+1}\varphi_{2m} \\ D^{2m}\varphi_0 & D^{2m}\varphi_1 & \dots & D^{2m}\varphi_{2m} \end{pmatrix}, \quad (23)$$

and so on,

$$a_{2m+1} = -\frac{1}{W(\varphi_0, \dots, \varphi_{2m})} \text{Ber} \begin{pmatrix} D^{2m+1}\varphi_0 & D^{2m+1}\varphi_1 & \dots & D^{2m+1}\varphi_{2m} \\ D\varphi_0 & D\varphi_1 & \dots & D\varphi_{2m} \\ \dots & \dots & \dots & \dots \\ D^{2m-1}\varphi_0 & D^{2m-1}\varphi_1 & \dots & D^{2m-1}\varphi_{2m} \\ D^{2m}\varphi_0 & D^{2m}\varphi_1 & \dots & D^{2m}\varphi_{2m} \end{pmatrix}. \quad (24)$$

In words: we replace one row of the Wronski matrix by the row-vector

$$(-D^n \varphi_0, \dots, -D^n \varphi_{n-1})$$

and apply, alternating, Ber and Ber. Similarly for $n = 2m$, where we have to start from Ber* for the coefficient a_1 .*

Proof. Cramer's rule in the form given in [13]. □

Note also the following useful formulas for differentiating superwronskians.

Lemma 4.8.

$$DW(\varphi_0, \dots, \varphi_{2s}) = \text{Ber} \begin{pmatrix} \varphi_0 & \dots & \varphi_{2s} \\ D\varphi_0 & \dots & D\varphi_{2s} \\ \dots & \dots & \dots \\ D^{2s-1}\varphi_0 & \dots & D^{2s-1}\varphi_{2s} \\ D^{2s+1}\varphi_0 & \dots & D^{2s+1}\varphi_{2s} \end{pmatrix} \quad (25)$$

$$DW(\varphi_0, \dots, \varphi_{2s+1}) = -W(\varphi_0, \dots, \varphi_{2s+1})^2 \text{Ber}^* \begin{pmatrix} \varphi_0 & \dots & \varphi_{2s+1} \\ D\varphi_0 & \dots & D\varphi_{2s+1} \\ \dots & \dots & \dots \\ D^{2s}\varphi_0 & \dots & D^{2s}\varphi_{2s+1} \\ D^{2s+2}\varphi_0 & \dots & D^{2s+2}\varphi_{2s+1} \end{pmatrix} \quad (26)$$

(Note ‘wrong parity’ last rows.) *Similar formulas hold for $W^*(\varphi_0, \dots, \varphi_r)$, with the cases of $r = 2s$ and $r = 2s + 1$ interchanged.*

Proof. Use the formulas for differentiation of Berezinian ((19) and its analog for Ber^*). Then apply the invariance of the Berezinian of a ‘wrong’ matrix under elementary transformation of rows. This makes all terms in the sums zero except for one. \square

Corollary. For $n = 2m + 1$, $a_1 = DW(\varphi_0, \dots, a_{2m})$ (the 1st coefficient of M).

5. DARBOUX TRANSFORMATIONS. CLASSIFICATION THEOREM

Consider non-degenerate operators of order m with the some fixed principal symbol (i.e., the top coefficient a_0). For simplicity let the operators be monic, i.e., $a_0 = 1$; the general case is similar.

Definition 5.1. For two monic operators L_0 and L_1 of the same order, a *Darboux transformation* $L_0 \rightarrow L_1$ is given by a monic differential operator M of an arbitrary order r such that it satisfies the *intertwining relation*

$$ML_0 = L_1M. \quad (27)$$

Remark 5.2. There is a more general notion based on intertwining relations of the form $NL_0 = L_1M$ with possibly different N and M , see [29]. We do not consider it here.

By definition, the order of an operator M is called the *order* of the Darboux transformation. Note that L_1 , if exists, is defined uniquely by L_0 and M . However, not every operator M can give a Darboux transformation. As we shall see, equation (27) is, in a sense, an overdetermined system and M must satisfy compatibility conditions. It will become clear from considerations in this section, as well as from particular examples considered in Section 6, that the problem of finding all M defining Darboux transformations $L_0 \rightarrow L_1$ for a given operator L_0 can be viewed as a generalization of the eigenvalue problem (the problem of finding eigenvalues and eigenvectors of a given operator).

We write $L_0 \xrightarrow{M} L_1$ to denote the Darboux transformation defined by an operator M . Darboux transformations can be composed and form a category: if $L_0 \xrightarrow{M_{10}} L_1$ and $L_1 \xrightarrow{M_{21}} L_2$, then $L_0 \xrightarrow{M_{20}} L_2$ where $M_{20} := M_{21}M_{10}$.

Lemma 5.3. *Every first-order Darboux transformation $L_0 \rightarrow L_1$ is given by an operator $M = M_\varphi$, where φ is an even eigenfunction of L_0 with some eigenvalue λ . (And conversely.)*

Proof. Let $M_\varphi L_0 = L_1 M_\varphi$ for some M_φ . Divide L_0 by M_φ from the right, so that $L_0 = Q M_\varphi + f$ with some function f . Hence $M_\varphi \circ (Q M_\varphi + f) = L_1 M_\varphi$. By applying both sides to φ , we obtain $M_\varphi(f\varphi) = 0$; hence $f = \lambda = \text{const}$. (Note that λ may be even or odd depending on m .) Therefore $L_0\varphi = \lambda\varphi$. Conversely, if $L_0\varphi = \lambda\varphi$, we similarly deduce that $L_0 = Q M_\varphi + \lambda$. Hence $M_\varphi L_0 = L_1 M_\varphi$ for $L_1 := M_\varphi Q + \lambda$. \square

First-order Darboux transformations defined by operators $M = M_\varphi$, where $M_\varphi = D - D \ln \varphi$, are called *elementary* Darboux transformations on the superline. (They are analogous to transformations on the ordinary line given by $\partial - \partial \ln \varphi$.) As we observed in the course of the proof, all such Darboux transformations correspond to changing order in an incomplete factorization of L_0 ,

$$L_0 = Q M_\varphi + \lambda \rightarrow L_1 = M_\varphi Q + \lambda.$$

Remark 5.4. By requiring that the intertwining operator M be monic, we exclude non-identical Darboux transformations of order zero. It is convenient to treat the corresponding notion independently. For an invertible function g , operators A and B are related by a *gauge transformation* generated by g if $gB = Ag$ (as composition of operators), i.e., $B = A^g := g^{-1}Ag$. Gauge transformations commute with Darboux transformations in the sense that if $L_0 \xrightarrow{M} L_1$, then $L_0^g \xrightarrow{M^g} L_1^g$. Darboux transformations themselves can be interpreted as ‘gauge’ or ‘similarity’ transformations in the ring $\text{DO}(1|1)[[D^{-1}]]$ of formal pseudodifferential operators with the restriction that the image of L_0 is a differential operator.

Theorem 5.5. *Every Darboux transformation $L_0 \xrightarrow{M} L_r$ of order r is the composition of r elementary first-order transformations:*

$$L_0 \xrightarrow{M_{\varphi_1}} L_1 \xrightarrow{M_{\varphi_2}} L_2 \xrightarrow{M_{\varphi_3}} \dots \xrightarrow{M_{\varphi_r}} L_r. \quad (28)$$

Proof. Suppose $M L_0 = L_r M$ for an operator M of order r . It can be decomposed into first-order factors, $M = M_1 \cdot \dots \cdot M_r$, where $M_i = M_{\varphi_i}$ for some functions φ_i . Note that this does not suffice *per se*, because we would also need to find the intermediate operators L_1, \dots, L_{r-1} related by the corresponding Darboux transformations. We proceed by induction. Suppose $r > 1$. From the intertwining relation, $L_0(\text{Ker } M) \subset \text{Ker } M$. Recall that $\dim \text{Ker } M = s|s$ or $s+1|s$, if $r = 2s$ or $r = 2s+1$. Take an invertible eigenfunction φ of L_0 in $\text{Ker } M$. Then $M = M' M_\varphi$, where M' is a non-degenerate operator of order $r-1$, and M_φ defines a Darboux transformation $L_0 \rightarrow L_1$. We shall show that M' defines a Darboux transformation $L_1 \rightarrow L_r$. Indeed, from $M' M_\varphi L_0 = L_r M' M_\varphi$ and $M_\varphi L_0 = L_1 M_\varphi$, we obtain $M' L_1 M_\varphi = L_r M' M_\varphi$. This implies $M' L_1 = L_r M'$, as M_φ is a non-zero-divisor. Thus $L_0 \xrightarrow{M} L_r$ is factorized into $L_0 \xrightarrow{M_\varphi} L_1 \xrightarrow{M'} L_r$ where $\text{ord } M' = r-1$, and this completes the inductive step. \square

Remark 5.6. An analog of Theorem 5.5 holds true for arbitrary operators on the ordinary line, where elementary transformations are specified by operators of the form

$\partial - \partial \ln \varphi$. A proof can be found in [1]. Before that, it took some effort to establish this result for the particular case of Sturm–Liouville operators [32], [2], [25]. Note that in the fundamental monograph [23], Darboux transformations are introduced by definition as iterations of elementary transformations.

It is possible to give a closed form for the composition of elementary Darboux transformations in Theorem 5.5, as follows. As we know (Theorem 4.5), the r th order operator M specifying a Darboux transformation $L_0 \rightarrow L_r$ can be reconstructed from its kernel. To this end, we need to find a basis in $\text{Ker } M$. For convenience, let us re-write the factorization given by Theorem 5.5 as

$$M = M_{\phi_{r-1}} M_{\phi_{r-2}} \dots M_{\phi_0}$$

(we have changed the notation), where all functions ϕ_i are even. We shall construct a basis $\varphi_0, \varphi_1, \dots, \varphi_{r-1}$ in $\text{Ker } M$, where $\tilde{\varphi}_i = i \bmod 2$, as follows³. We already have $\phi_0 \in \text{Ker } M$; we set $\varphi_0 := \phi_0$. Set φ_1 to be an arbitrary element in the preimage of ϕ_1 under the operator M_{ϕ_0} ; it is defined up to a multiple of ϕ_0 . Since ϕ_1 is even, the function φ_1 has to be odd. By construction, the function φ_1 is annihilated by the operator $M_{\phi_1} M_{\phi_0}$ and hence by M . On the other hand, ϕ_1 is an eigenfunction of the operator L_1 with some eigenvalue λ_1 , so we have $L_1 \phi_1 = \lambda_1 \phi_1$; and from the intertwining relation $M_{\phi_0} L_0 = L_1 M_{\phi_0}$ we have

$$M_{\phi_0} (L_0 \varphi_1 - \lambda_1 \varphi_1) = M_{\phi_0} L_0 \varphi_1 - \lambda_1 M_{\phi_0} \varphi_1 = L_1 M_{\phi_0} \varphi_1 - \lambda_1 M_{\phi_0} \varphi_1 = L_1 \phi_1 - \lambda_1 \phi_1 = 0.$$

(An extra sign can emerge for an odd L_0 .) That means that $L_0 \varphi_1 = \lambda_1 \varphi_1 \pmod{\varphi_0}$. Similarly we can introduce an even function φ_2 as an element in the preimage of ϕ_2 under the operator $M_{\phi_1} M_{\phi_0}$, which is defined up to a linear combination of φ_0 and φ_1 . It will be annihilated by $M_{\phi_2} M_{\phi_1} M_{\phi_0}$ and hence by M . Also, φ_2 will be an ‘eigenfunction for L_0 up to’ the linear span of φ_0 and φ_1 , with the eigenvalue λ_2 (the eigenvalue of L_2 corresponding to the eigenfunction ϕ_2). And so on. In this way we construct functions $\varphi_0, \varphi_1, \dots, \varphi_{r-1}$ of alternating parities which make a basis of $\text{Ker } M$ and such that the restriction of L_0 on its invariant subspace $\text{Ker } M$ is triangular in this basis. Note that $\varphi_0, \varphi_1, \dots, \varphi_{r-1}$ are not necessarily eigenfunctions of L_0 .

We have established the following theorem.

Theorem 5.7. *Every Darboux transformation of order r between non-degenerate differential operators on the superline $L_0 \rightarrow L_r$ is specified by an invariant subspace of L_0 of dimension $s+1|s$ if $r = 2s+1$ or $s|s$ if $r = 2s$, to which uniquely corresponds a monic operator M so that the intertwining relation $ML_0 = L_r M$ holds; if $\varphi_0, \dots, \varphi_{r-1}$ is a basis of this subspace consisting of functions of parities $\tilde{\varphi}_i = i \bmod 2$, then M is given by the super Wronskian formula*

$$M\psi = \frac{W^*(\varphi_0, \dots, \varphi_{r-1}, \psi)}{W^*(\varphi_0, \dots, \varphi_{r-1})}, \quad (29)$$

for odd functions ψ , if $r = 2s + 1$, or

$$M\varphi = \frac{W(\varphi_0, \dots, \varphi_{r-1}, \varphi)}{W(\varphi_0, \dots, \varphi_{r-1})}, \quad (30)$$

³We use different lettershapes ϕ and φ of the same Greek letter phi. (A delight for Greek-lovers.)

for even functions φ , if $r = 2s$, and on functions of the other parities, the operator M is extended by linearity.

Note that we have a one-to-one correspondence between Darboux transformations and invariant subspaces. A one-dimensional invariant subspace is an eigenspace and corresponds to an elementary Darboux transformation. Theorem 5.5 guarantees that every Darboux transformation factorizes into elementary. Suppose, however, we did not know that; then looking for Darboux transformations directly, by solving the intertwining relation (27) with respect to L_1 , would lead us to a generalization of the eigenvalue problem. This is seen in the examples in the next section.

6. EXAMPLES OF DRESSING TRANSFORMATIONS

Let $L_0 \rightarrow L_1$ be a Darboux transformation given by an intertwining operator M . The expression of (the coefficients of) the operator L_1 in terms of given L_0 and M is often called the *dressing* (sometimes ‘undressing’) *transformation* for L_0 . We shall calculate dressing transformations explicitly for arbitrary (monic) operators on the superline of orders ≤ 4 . To work them out in these concrete examples, nothing is required in principle but patience. But it is elucidating to see all the theoretical considerations above become in calculations completely manifest. Indeed, suppose L_0 is of order n ,

$$L_0 = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n, \quad (31)$$

and we write L_1 as

$$L_1 = D^n + b_1 D^{n-1} + b_2 D^{n-2} + \dots + b_{n-1} D + b_n. \quad (32)$$

Suppose the intertwining operator is of order r :

$$M = D^r + c_1 D^{r-1} + c_2 D^{r-2} + \dots + c_{r-1} D + c_r. \quad (33)$$

(In the chosen notation, the parity of each coefficient such as a_k is $k \pmod{2}$, which is convenient for calculations.) Practically, one needs to find the coefficients b_k in terms of a_i and c_j . These are the desired dressing formulas. In principle, all that one has to do to find the dressing formulas, is to write down the relation (27), substitute into it the expansions (31), (33) and (32), calculate the compositions and compare the coefficients. This would give the expression for b_k . It is worth noting, however, that there are many more equations than we need (the system is overdetermined). As we shall see, the first n equations (linear in b_k) allow to determine the coefficients b_k , where $k = 1, 2, \dots, n$, recursively, where the remaining equations give non-linear compatibility conditions for the coefficients c_i . (For $r = 1$, one obtains a Riccati-type equation, which by a logarithmic derivative substitution becomes an equation for eigenfunctions.) Since the coefficients of M must satisfy compatibility conditions, they are not independent. Therefore dressing formulas are non-unique. However, the described procedure gives them probably in the simplest form.

After expansion, the intertwining relation $ML_0 = L_1M$ takes the form

$$(D^r + c_1 D^{r-1} + \dots + c_{r-1} D + c_r) \circ (D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) = \\ (D^n + b_1 D^{n-1} + \dots + b_{n-1} D + b_n) \circ (D^r + c_1 D^{r-1} + \dots + c_{r-1} D + c_r). \quad (34)$$

To open the brackets in (34), one needs to be able to calculate the compositions of the form $D^k \circ f$, where f is a function. One should use the formula (the Leibniz formula for the superline):

$$D^k(fg) = \sum_{p+q=k} \begin{bmatrix} k \\ p \end{bmatrix} (-1)^{q\bar{f}} D^p f \cdot D^q g. \quad (35)$$

Here $\begin{bmatrix} k \\ p \end{bmatrix}$ is the *superbinomial coefficient* introduced by Manin and Radul [21]. Formula (35) defines a diagonal (coproduct) on operators with constant coefficients on the superline, which form a Milnor–Hopf algebra with respect to the ordinary composition and this coproduct. It is easy to find that

$$\begin{bmatrix} 2s+1 \\ 2t+1 \end{bmatrix} = \begin{bmatrix} 2s+1 \\ 2t \end{bmatrix} = \binom{s}{t}, \quad \begin{bmatrix} 2s \\ 2t+1 \end{bmatrix} = 0, \quad \begin{bmatrix} 2s \\ 2t \end{bmatrix} = \binom{s}{t}. \quad (36)$$

In particular, $\begin{bmatrix} 2s+1 \\ 1 \end{bmatrix} = 1$ and $\begin{bmatrix} 2s \\ 1 \end{bmatrix} = 0$. Also,

$$\begin{bmatrix} k \\ p \end{bmatrix} = \begin{bmatrix} k \\ k-p \end{bmatrix},$$

as for ordinary binomial coefficients. To find the dressing transformations from (34), one needs to calculate the products in the both sides modulo operators of order $\leq r-1$. This would give exactly n equations defining the n coefficients b_1, \dots, b_n . The lower order terms give compatibility conditions for c_i .

Remark 6.1. For operators on the ordinary line, $L = \partial^n + a_1 \partial^{n-1} + \dots$, it is well known that the first coefficient a_1 is a ‘pure gauge’, i.e., by a gauge transformation can be made zero. This is not the case for operators on the superline. If $n = 2m$, the coefficient a_1 in $L = D^{2m} + a_1 D^{2m-1} + \dots$ is gauge-invariant. (The explanation is in the equality $\begin{bmatrix} 2s \\ 1 \end{bmatrix} = 0$.)

Example 6.2. Consider the operator

$$L_0 = D + a_1.$$

We can still learn something from this seemingly trivial example. By formally calculating the dressing transformation, we obtain

$$b_1 = (-1)^r a_1 + 2c_1.$$

To understand it better, consider an elementary transformation with $M = D + \mu$. By fully expanding $(D + \mu)(D + a_1) = (D + b_1)(D + \mu)$, we obtain $b_1 = -a_1 + 2\mu$, which is a particular case of the above, and also $D\mu = Da_1$, which is the compatibility condition. It gives $\mu = a_1 + \lambda$, where λ is an odd constant. That means that $M = D + a_1 + \lambda$ indeed corresponds to an eigenfunction of L_0 (with an odd eigenvalue), as it should be according to general theory. As for the dressing transformation, we have $b_1 = -a_1 + 2(a_1 + \lambda) = a_1 + 2\lambda$, i.e., it is simply a constant shift.

Example 6.3. Consider the operator

$$L_0 = D^2 + a_1 D + a_2 = \partial + a_1 D + a_2.$$

The dressing transformation is

$$\begin{cases} b_1 = (-1)^r a_1, \\ b_2 = a_2 - (-1)^r \begin{bmatrix} r \\ 1 \end{bmatrix} D a_1 - 2(-1)^r c_1 a_1. \end{cases}$$

Hence for odd $r = 2s + 1$,

$$\begin{cases} b_1 = -a_1, \\ b_2 = a_2 + D a_1 + 2c_1 a_1, \end{cases}$$

and for even $r = 2s$,

$$\begin{cases} b_1 = a_1, \\ b_2 = a_2 - 2c_1 a_1. \end{cases}$$

If $a_1 = 0$, $L_0 = \partial + a_2$ is an ‘ordinary’ operator (of first order in the usual sense). We observe that the condition $a_1 = 0$ is preserved by Darboux transformations; moreover, in this case, $b_2 = a_2$, i.e., the operator $L_0 = \partial + a_2$ does not have non-trivial dressing transformations. (All Darboux transformations are “autotransformations” $L_0 \rightarrow L_0$.)

Example 6.4. Consider the operator

$$L_0 = D^3 + a_1 D^2 + a_2 D + a_3 = D\partial + a_1 \partial + a_2 D + a_3.$$

The dressing transformation is

$$\begin{cases} b_1 = (-1)^r a_1 + 2c_1, \\ b_2 = a_2 - (-1)^r \begin{bmatrix} r \\ 1 \end{bmatrix} D a_1 - D c_1, \\ b_3 = (-1)^r a_3 + \begin{bmatrix} r \\ 1 \end{bmatrix} D a_2 + (-1)^r \begin{bmatrix} r \\ 2 \end{bmatrix} \partial a_1 + 2c_1 a_2 + \\ D a_1 c_1 + (-1)^r c_2 a_1 + \partial c_1 - D c_2 + 2c_3 - D c_1 c_1. \end{cases}$$

Example 6.5. Consider the operator

$$L_0 = D^4 + a_1 D^3 + a_2 D^2 + a_3 D + a_4 = \partial^2 + a_1 D\partial + a_2 \partial + a_3 D + a_4.$$

When $a_1 = 0 = a_2$, this is the super Sturm–Liouville operator. Let us first analyze the general case. For simplification, consider an elementary Darboux transformation given by $M = D + \mu$. The corresponding dressing transformation is

$$\begin{cases} b_1 = -a_1, \\ b_2 = a_2 + D a_1 + 2\mu a_1, \\ b_3 = -a_3 + D a_2 + D(\mu a_1) - 2\partial \mu, \\ b_4 = a_4 + D a_3 + 2\mu a_3 - \mu D a_2 + \partial \mu a_1 - (D\mu\mu)a_1 - 2\partial \mu \mu. \end{cases}$$

Notice the following. If $a_1 = 0$, this condition is preserved by Darboux transformations. If $a_1 = 0$ holds, then a_2 is preserved by Darboux transformations. The term $a_2 \partial$ is a ‘magnetic’ term, and

$$L_0 = \partial^2 + a_2 \partial + a_3 D + a_4$$

is the ‘super Sturm–Liouville operator in magnetic field’. The magnetic term is however a pure gauge: it can be made zero by a gauge transformation $L \rightarrow g^{-1}Lg$ with a suitable g . (Recall that gauge transformations commute with Darboux transformations.)

Example 6.6. The previous analysis justifies consideration of the ‘pure’ super Sturm–Liouville operator (without magnetic term)

$$L_0 = \partial^2 + \alpha D + u$$

and explains why its form is invariant under Darboux transformations. Let

$$L_1 = \partial^2 + \beta D + w.$$

The dressing transformation (for a general intertwining operator M of order r) is given by

$$\begin{cases} \beta = (-1)^r \alpha + 2\partial c_1, \\ w = u - (-1)^r \begin{bmatrix} r \\ 1 \end{bmatrix} D\alpha - 2(-1)^r c_1 \alpha - 2\partial c_1 c_1 - 2\partial c_2. \end{cases}$$

Darboux transformations for the super Sturm–Liouville operator were studied in [17], [18, 19], and the most complete formulas were obtained in [16]; by comparing notations, one may conclude that the formulas above are the same as those in [16].

The coefficients c_k in the dressing transformations can be expressed in terms of super Wronskians as in Section 4.

Examples above show some patterns in dressing transformations. They can be summarized in the following simple theorem.

Theorem 6.7. *Consider a general monic operator L_0 of order n given by equation (31). The following holds for its dressing transformations (we show formulas for elementary transformations):*

- *If $n = 2m + 1$, there are simple dressing formulas for the first two coefficients:*

$$\begin{cases} b_1 = -a_1 + 2\mu, \\ b_2 = a_2 + Da_1 - D\mu; \end{cases}$$

- *If $n = 2m$, then under dressing transformations*

$$\begin{cases} b_1 = -a_1, \\ b_2 = a_2 + Da_1 - 2\mu a_1, \end{cases}$$

hence the condition $a_1 = 0$ is invariant and, if it holds, the coefficient a_2 is preserved.

We skip the proof. The dressing formulas are deduced exactly in the same way as in the examples above.

REFERENCES

- [1] Adler, V. È., Marikhin, V. G., and Shabat, A. B.: Lagrangian lattices and canonical Bäcklund transformations, *Theoret. and Math. Phys.* **129**(2):1448–1465 (2001)
- [2] Bagrov, V. G. and Samsonov, B. F.: Darboux transformation, factorization and supersymmetry in one-dimensional quantum mechanics, *Theoret. and Math. Phys.* **104**(2):1051–1060 (1995)
- [3] Bagrov, V. G. and Samsonov, B. F.: Darboux transformation of the Schrödinger equation, *Phys. Part. Nucl.* **28**(4):374–397 (1997)
- [4] Berest, Yu. Yu. and Veselov, A. P.: Singularities of the potentials of exactly solvable Schrödinger equations, and the Hadamard problem, *Russian Math. Surveys* **53**(1(319)):208–209 (1998)
- [5] Berest, Yu. Yu. and Veselov, A. P.: On the structure of singularities of integrable Schrödinger operators, *Lett. Math. Phys.* **52**(2):103–111 (2000)
- [6] Berezin, F. A.: *Introduction to superanalysis*. D. Reidel Publishing Co., Dordrecht (1987)
- [7] Bergvelt, M. J. and Rabin, J. M.: Supercurves, their Jacobians, and super KP equations, *Duke Math. J.* **98**(1):1–57 (1999)
- [8] Crum, M. M.: Associated Sturm-Liouville systems, *Quart. J. Math. Oxford (2)* **6**:121–127 (1955)
- [9] Darboux, G.: *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, volume 2, Gauthier-Villars, Paris (1889)
- [10] Deligne, P. and Morgan, J. W.: Notes on supersymmetry (following Joseph Bernstein). In: *Quantum fields and strings: a course for mathematicians, Vol. 1 (Princeton, NJ, 1996/1997)*, pages 41–97. Amer. Math. Soc., Providence, RI (1999)
- [11] Etingof, P., Gelfand, I. and Retakh, V.: Factorization of differential operators, quasideterminants, and nonabelian Toda field equations, *Math. Res. Lett.* **4**(2-3):413–425 (1997)
- [12] Hill, S., Shemyakova, E. S., and Voronov, Th. Th.: Darboux transformations for differential operators on the superline, *Russian Math. Surveys* **70**(6):1173–1175 (2015)
- [13] Khudaverdian, H. M. and Voronov, Th. Th.: Berezinians, exterior powers and recurrent sequences, *Lett. Math. Phys.* **74**(2):201–228 (2005)
- [14] Leites, D. A.: Introduction to the theory of supermanifolds, *Russian Math. Surveys* **35**(1):13–64 (1980)
- [15] Leites, D. A.: *Supermanifolds theory*, Karelian branch of the USSR Acad. Sci., Petrozavodsk (1984). Russian. (See also: <http://staff.math.su.se/mleites/books.html>)
- [16] Li, C. X. and Nimmo, J. J. C.: Darboux transformations for a twisted derivation and quasideterminant solutions to the super KdV equation, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **466**(2120):2471–2493 (2010)
- [17] Liu, Q. P.: Darboux transformations for supersymmetric Korteweg-de Vries equations, *Lett. Math. Phys.* **35**(2):115–122 (1995)
- [18] Liu, Q. P. and Mañas, M.: Darboux transformation for the Manin-Radul supersymmetric KdV equation, *Phys. Lett. B* **394**(3-4):337–342 (1997)
- [19] Liu, Q. P. and Mañas, M.: Crum transformation and Wronskian type solutions for supersymmetric KdV equation, *Phys. Lett. B* **396**(1-4):133–140 (1997)
- [20] Manin, Yu. I.: *Gauge field theory and complex geometry*, Springer-Verlag, Berlin (1997)
- [21] Manin, Yu. I. and Radul, A. O.: A supersymmetric extension of the Kadomtsev-Petviashvili hierarchy, *Comm. Math. Phys.* **98**(1):65–77 (1985)
- [22] Matveev, V. B.: Darboux transformation and explicit solutions of the Kadomtsev-Petviashvili equation, depending on functional parameters, *Lett. Math. Phys.* **3**(3):213–216 (1979)
- [23] Matveev, V. B. and Salle, M. A.: *Darboux transformations and solitons*, Springer-Verlag, Berlin (1991)
- [24] Novikov, S. P. and Dynnikov, I. A.: Discrete spectral symmetries of small-dimensional differential operators and difference operators on regular lattices and two-dimensional manifolds, *Russian Math. Surveys* **52**(5(317)):1057–1116 (1997)
- [25] Samsonov, B. F.: On the N -th order Darboux transformation, *Russian Math. (Iz. VUZ)* **43**(6):62–65 (1999)

- [26] Shabat, A. B.: The infinite-dimensional dressing dynamical system, *Inverse Probl.* **8**(2):303–308 (1992)
- [27] Shabat, A. B.: On the theory of Laplace–Darboux transformations, *Theoret. and Math. Phys.* **103**(1):482–485 (1995)
- [28] Shander, V. N.: Vector fields and differential equations on supermanifolds, *Funct. Anal. Appl.* **14**(2):160–162 (1980)
- [29] Shemyakova, E. S.: Proof of the completeness of Darboux Wronskian formulae for order two, *Canad. J. Math.* **65**(3):655–674 (2013)
- [30] Shemyakova, E. S.: Factorization of Darboux transformations of arbitrary order for $2D$ Schrödinger type operators, [arXiv:1304.7063 \[math-ph\]](https://arxiv.org/abs/1304.7063) (2013)
- [31] Tsarev, S. P. and Shemyakova, E. S.: Differential transformations of parabolic second-order operators in the plane, *Proc. of the Steklov Institute of Mathematics* **266**(1):219–227 (2009)
- [32] Veselov, A. P. and Shabat, A. B.: A dressing chain and the spectral theory of the Schrödinger operator, *Funct. Anal. Appl.* **27**(2):81–96 (1993)
- [33] Wadati, M., Sanuki, H., and Konno, K.: Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws, *Progr. Theoret. Phys.* **53**:419–436 (1975)
- [34] Wahlquist, H. D. and Estabrook, F. B.: Bäcklund transformation for solutions of the Korteweg-de Vries equation, *Phys. Rev. Lett.* **31**:1386–1390 (1973)

DEPARTMENT OF MATHEMATICS, SUNY NEW PALTZ, NEW PALTZ, NY 12561-2443, USA

DEPARTMENT OF MATHEMATICS, SUNY NEW PALTZ, NEW PALTZ, NY 12561-2443, USA

E-mail address: shemyake@newpaltz.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF MANCHESTER, MANCHESTER, M60 1QD, UK,

DEPT. OF QUANTUM FIELD THEORY, TOMSK STATE UNIVERSITY, TOMSK, 634050, RUSSIA

E-mail address: theodore.voronov@manchester.ac.uk