EVALUATING THE TEACHING AND LEARNING OF FRACTIONS THROUGH MODELLING IN BRUNEI:

MEASUREMENT AND SEMIOTIC ANALYSES

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TOTAL NUMBER OF WORDS: 76,297

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LIST OF ABBREVIATIONS

1//	Question 1 in the Pre-Teaching Fractions test
1/2/	Question 1 and 2 in the Pre- and Post-Teaching Fractions tests, respectively
1/2/3	Question 1, 2 and 3 in the Pre-, Post- and Delayed Post-Teaching Fractions tests, respectively
c	With context

C	Control group
CDP	Control group Delayed Post-Teaching Fractions test
CPDD	Curriculum Planning Development Division
Cpost	Control group Post-Teaching Fractions test
Cpre	Control group Pre-Teaching Fractions test
DP	Delayed Post-Teaching Fraction Test
E1	Experimental 1 group
E2	Experimental 2 group
ES/*d*	Effect size
F(Length)	Fraction in terms of length
F(Part)	Fraction in terms of number of parts
F(Time)	Fraction in terms of time
FA	Fraction Ability
FG	Factual Generalization
ICT	Information Communication Technology
MaLT	Mathematics Assessment for the Learning and Teaching
MIB	*Melayu Islam Beraja* (Malay Islamic Monarchy)
MiC	Britannica Mathematics in Context
NCTM	National Council of Teachers of Mathematics
OBJ	Objectification
PE	Physical Education
PMB	*Penilaian Menengah Bawah* (Lower Secondary Assessment)
PO	Preparatory Objectification

Post	Post-Teaching Fractions test
PPD	Pre-, Post- and Delayed Post-Teaching Fractions Tests
PPO	Pre-Preparatory Objectification
Pre	Pre-Teaching Fractions Test
PSR	*Penilaian Sekolah Rendah* (Primary School Assessment)
QED	Quasi-Experimental Design
RME	Realistic Mathematics Education
RQ/RQs	Research Question(s)
SD	Standard Deviation
SMO	Semiotic Means of Objectification
S-NET	Semiotic Network
Std	Student
T	Test
u	Utterance
wc	Without context

LIST OF TERMINOLOGIES USED IN CHAPTERS 7 – 9

This list is also presented in section 7.3, but it is presented here for quick reference.

Terms	Meanings
Chain of Signification, or Semiotic Chain	". . . a chain of signification is the embedding of signs." (Sáenz-Ludlow, 2003, p. 186) "The basic component of a chain of signification is a sign . . . . A new link in the chain is established when a sign itself becomes the signified for a new signifier." (Gravemeijer et al., 2000, p. 262)
Corresponding amount	The researcher used this term here, in the context of the instruction, as "equivalent" when referring to "time" and "inches". For example, the corresponding amount of 1 inch is 5 minutes (i.e., 1 inch ≡ 5 minutes).
Episode	A group of selected "interactions" of the discourse which the researcher has identified as significant which could contribute to the semiotic analysis. This episode would include all the semiotic means such as language, gestures, rhythm, symbols, and artifacts. For each "interaction" the following information is provided: ▪ The exact time (in minutes) the episode/interaction took place; ▪ The specific lesson in the experimental instruction; ▪ The group number to which the students belong to in E1.
Generalization: ▪ Factual ▪ Contextual ▪ Symbolic	The terms "factual generalization", "contextual generalization" and "symbolic generalization" are used as in Radford (2001, 2003 & 2006). "A factual generalization is a generalization of numerical actions in the form of an *operational scheme* (in a non-

	Piagetian sense) that remains bound to the numerical level, nevertheless allowing the students to virtually tackle any *particular* case successfully." Radford (2001, pp. 82–83) There are two differences between contextual and factual generalization. The first difference is that the operational scheme in contextual generalization is not based on face-to-face interaction, where both rhythm and ostensive gestures have also been excluded. The second difference is that the previously constructed operational scheme is generalized through language. Its generative capacity lies in allowing the emergence of new abstract objects to replace the previously used specific concrete objects (Radford, 2003). In symbolic generalization, the spatial and temporal limitations of the objects of contextual generalization have to be withdrawn. Symbolic mathematical objects (in Radford's case algebraic ones) should become "nonsituated and nontemporal" (Radford, 2003, p. 55).
Interaction	A selected part of the discourse which the researcher felt was significant in order to get an objectification for that particular episode (it could also be considered as a "sub-episode"). The specifics provided are similar to what is explained for "episode".
Intuition	"There is no commonly accepted definition for intuitive knowledge. The term "intuition" is used as one uses mathematical primitive terms like point, line, set, etc" (Fischbein, 1999, p. 12). He further added that intuition suggests self-evidence instead of a logical-analytical undertaking (Fischbein, 1999).

	Fischbein (1999) outlined outlined five general characteristics of intuitive cognitions:- 1. *Direct, self-evident* – cognitions which do not need any proof. 2. *Intrinsic certainty* – does not need external support, but it is usually associated with a feeling of absolute certainty. 3. *Subjective Coerciveness* – rejecting an alternative conception that conflicts with their intuitions, such as, multiplication always makes things bigger. 4. *Extrapolativeness* – ability to conclude beyond any empirical proof, for instance, the acceptance that there will always be a number (whole) after a given number (whole). 5. *Globality* – an intuitive direct response, as opposed to a logical – analytically-based solution.
Gestures to the self (Self-gestures)	Unintentional gestures to the self which are not pre-planned, nor meant to attract attention to the gesturer. These could be as a result (by-product) of listening to what others in the gesturer's immediate surroundings have said or it could be a by-product of what the gesturer has read.
Meaning	"It is the human being, strong of the acquired culture, strong of the specific expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and now that must be coherent with that history." (D'Amore & Pinilla, 2008, p. 20) "Meaning is only one of the zones of sense, the most

	stable and precise zone. A word acquires its sense from the context in which it appears; in different contexts, it changes its sense." (Vygotsky, 1978 in Azarello, 2006, p. 282) Learning mathematics involves taking over the conventional meanings of mathematical signs, but it also depends on switching between different possibilities of interpretation–on "seeing an *A* as a *B*" (Hoffmann, 2006, p. 80). "To *perceive* something means to endow it with meaning." (Sabena, Radford, & Bardini, 2005, p. 129)
Signification, or Meaning-making	According to O'Sullivan, Hartly, Saunders, and Fiske (1983) in Chapman (2003), "Signification, or meaning-making, is "the relationship of a sign or sign system to its referential reality" (p. 130). "To foster a better understanding of how students "connect" both the different fields within mathematics, and "school mathematics with the experiential realities of learners", Presmeg focuses on a "nested model" of "meaning making" to describe learning processes as a step by step development." (Hoffmann, 2006, p. 284) ". . . mathematical meaning-making is social in nature and that any account of mathematical meaning-making should take account of development." (Peirce and Vygotsky, in Vile, 1999, p. 91)
Objectification	". . . refers to an active, creative, imaginative and interpretative social process of gradually becoming aware of something." (Radford, 2003, p. 393)

	". . . the act of representing an abstraction as a physical thing or a concrete representation of an abstract idea or principle." (http://www.thefreedictionary.com)
Part-of-a-whole	The researcher is adopting Orton et al.'s (1996) definition of the types of fraction representation or models: *Region model* (also referred to as "part of a whole" or "part-whole" model), *Discrete model, A position on number line model, Quotient model* and *Ratio model.* The part-whole model as defined by Orton et al., (1996) is associated with the concept of area. However, in this thesis, when using the phrase 'part of a whole' the researcher is usually referring to either a part/portion of the bread in relation to the total number of equal parts/portions, it could also be in terms of time (hours and minutes), or in terms of length (with 12 inches as the unit whole).
Semiotic Contraction	As defined by Radford (2006), the "reduction of signs and concentration of meanings constitutes a *semiotic contraction*" (p. 12). In the context of this study, the semiotic contraction being referred to here is taking place in the process of objectification.
Semiotic (Learning) Trajectory	The term *semiotic learning trajectory* here refers to the semiotic chains that the students actually go through during the experimental instruction.
Semiotic Link	It is a link to connect the physical form of the sign (signifier) to what it actually refers to (signified). It is a "link" because it connects (potentially) to another sign to form a "chain"
Semiotic Means of Objectification	"To make something apparent (which is the etymological sense of objectification) learners and teachers make recourse to signs and artifacts of different sorts (mathematical symbols, graphs, words, gestures,

	calculators, and so on). These artifacts and signs used to objectify knowledge we call *semiotic means of objectification*." (Radford et al., 2006a, p. 685) ". . . we called semiotic means of objectification the whole arsenal of intentional resources that individuals mobilize in the pursuit of their activities and emphasized their social nature. The semiotic means of objectification appear embedded in socio-psycho-semiotic meaning-making processes framed by cultural modes of knowing that encourage and legitimize particular forms of sign and tool use whereas discarding others." (Radford, 2003, p. 44)
Sign	"Signs . . . are entities that serve as vehicles to trigger thought, to facilitate the expression of thought, and to embody original and conventional thought." (Sáenz-Ludlow, 2003, p. 182) "The *sign* is the whole that results from the association of the signifier with the signified (Saussure, 1983, p. 67; Saussure, 1974, p. 67). The relationship between the signifier and the signified is referred to as 'signification . . .' (Chandler , 2009, para. 4)
Signifier and Signified	". . . mental construct (signified) . . ." (Presmeg, 1998, p. 29). ". . . a 'signifier' (*significant*)–the *form* which the sign takes; and the 'signified' (*signifié*)–the *concept* it represents" (Chandler, 2009, para. 3).
Understanding	". . . the transformation or interpretation of a sign into a previous sign . . . for which the individual has attained a more or less stable cultural meaning" (Radford, Bardini,

	Sabena, Diallo, & Simbagoye, 2005, p. 117). Ricardo Neirovsky has suggested that instead of being mere mental processes, understanding and imagination of mathematical concepts are literally embedded in perceptuo-motor action: the “understanding of a mathematical concept spans diverse perceptuo-motor activities” (Nemirovsky 2003, 108), so that in this regard, “understanding is . . . interwoven with motor action” (Nemirovsky, 2003, p. 107, in Radford, 2010, p. 4).

ABSTRACT

This thesis is submitted to the University of Manchester for the degree of Doctor of Philosophy (PhD). This study developed an experimental small group teaching method in the Realistic Mathematics Education tradition for teaching fractions using models and contexts to year 7 children in Brunei (N=89) whose effectiveness was evaluated using a treatment-control design: the E1 group was given the experimental lessons, the E2 group who was given "normal" lessons taught by the experimenter, and a whole class (E3) group which acted as the control group. The experimental teaching was video recorded and subject to semiotic analysis, aiming to describe the objectifications that realized 'learning of fractions' by the groups.

The research addresses two research questions:

1. How effective was the experimental teaching in helping learners make sense of fractions, with respect to equivalence of fractions and flexibility of unitizing?
2. What were the semiotic learning and teaching processes in the experimental group of the RME-like lessons?

This study used a mixed method approach with a quasi-experimental design (QED) for the quantitative side, and a semiotic analysis for the qualitative side. Quantitatively, the experimental teachings proved to be relatively effective with an effect size of 0.6 from the pre- to the delayed post-teaching test, compared to the E2 and the control groups.

The basic findings pertaining to the semiotic analyses were:

a. The mediation of the production of fractions in terms of length, from the production of fractions in terms of the number of parts which led to equivalence of fractions;
b. The use of language and gesture help to objectify the equivalence of fractions and the flexibility of unitizing–in some case it involved gesturing to the self;
c. The role of the Hour-Foot clock (HFC) as a model in a realistic context; and
d. The complexity of the required chains of objectifications reflects the difficulties of the topic.

Submitted electronically on the 2nd Mac 2011 by Hajah Zurina Haji Harun.

DECLARATION

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

COPYRIGHT STATEMENT

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CHAPTER 1

BACKGROUND

1.1 OVERVIEW

In this thesis, I wanted to look at one part of mathematics (i.e., fractions) that has for a long time, regardless of culture and background, presented a problem for students both at primary and secondary levels. Past research (see Chapter 2) had proven this to be the case, where at certain ages students found that of all the topics in mathematics, fractions is the hardest. However, many of the research studies in the area of fractions were done with primary school students. As I am involved with secondary school mathematics, I decided to look at this particular area in the secondary school perspective. This research began with an analysis of relevant documents, such as the mathematics curriculum (see Chapter 3). On top of that, textbooks used by students in their mathematics lessons were also analyzed, by comparing and contrasting them with textbooks that were designed according to the *Realistic Mathematics Education* (hereafter, RME). This was followed by designing a series of RME-like experimental lessons, with the aim of measuring and contrasting the experimental students' learning outcomes with that of a control group. For such measurement and contrasting, a quasi-experimental design was implemented (see Chapter 4), and a new measurement instrument was calibrated using the Rasch methodology (see Chapter 5).

The next step was to conduct semiotic analyses in order to examine and explain the students' semiotic learning processes (see Chapters 7–9). The outcomes of these semiotic analyses highlighted a few issues. First, the students' learning trajectories were identified, and secondly, it clarified how the experimental lessons worked (if and when they did "work"). With regard to the analytical tools for conducting the above semiotic analyses, since the experimental lessons were developed in the RME framework, I decided to use a similar method to that used by Koukkoufis (2008) who developed an analytical tool for conducting a semiotic analysis in the RME framework, drawing on Radford (2002, 2003).

In the final chapter (i.e., Chapter 10) of this thesis, some conclusions and contributions to knowledge are presented. This includes some relevant discussion, limitations encountered in the study and some suggestions for future research.

1.2 RESEARCH PROBLEM

As presented in the overview above, in order to be able to measure the students' learning outcomes and to see whether the RME-like experimental lessons affected their attainment of fractions concepts, in particular for equivalence of fractions and flexibility of unitizing, a quasi-experimental design was needed, which entailed the calibration of a measurement instrument. As for students' semiotic learning processes, a semiotic analysis was required to examine the students' meaning-making processes.

The knowledge gap of the thesis is offered in section 2.7, and the research questions (RQs) presented in section 2.8 are as follows:

RQ.1: *After the students were given lessons on fractions, inspired by RME or RME-like thinking with the introduction of context and models, how effective was it in helping them make sense of fractions, with respect to equivalence of fractions and flexibility of unitizing better.*

RQ.2: *What are the semiotic learning processes in the Experimental 1 group of RME-like lessons?*

1.3 STRUCTURE OF THE THESIS

There are 10 chapters to this thesis. ***Chapter 1*** contains the background information of this thesis. It also gives a brief overview of the investigation that was carried out.

Chapter 2 consists of the literature review, which leads to the knowledge gap dealt with in this thesis, the research framework and also the research questions.

Chapter 3 of this thesis will present the relevant document and textbook analyses. The document analysis looked at the relevant official documents, which defined those aspects of fractions which teachers were expected to teach at different grade level. As for the textbook analysis, a comparative analysis of the *Secondary Mathematics 1A* book and the *Britannica Mathematics in Context* books was done, whereby a detailed account of all the relevant sections, as far as fractions is concerned, is presented.

Chapters 4–6 constitute the quantitative part of the thesis. ***Chapter 4*** presents the experimental lessons and a detailed explanation of the quasi-experimental design.

Chapter 5 consists of the calibration of the fraction ability (FA) scale using the Rasch methodology which was used for the measurement of the learning outcomes of the Experimental 1 group, while in ***Chapter 6*** findings from the measurement pertaining to the learning outcomes of the students are presented.

Chapters 7–9 consist of the semiotic analyses part of this thesis. In chapter 7, the semiotic analysis methodology which was used for the analysis in chapters 8 and 9 is presented. In addition, terminology and coding which were used in the presentation of the selected relevant episodes from the experimental teaching are also presented. These are also used for the analysis done in chapters 8 and 9. Chapter 8 presents the semiotic analysis of the factual generalization of equivalence of fractions, while chapter 9 consists of the semiotic analysis of the generalization of the flexibility of unitizing of fractions, together with the presentation of the semiotic role of the Hour-Foot Clock (hereafter, HFC), and the semiotic network (hereafter, S-NET) of gestures and language.

Chapter 10, which will present the conclusions and relevant discussion of the thesis, also includes the limitations of the study and suggestions for future research.

CHAPTER 2
LITERATURE REVIEW

2.1. INTRODUCTION

This chapter will review the research literature, both local (referring to my home country) and international, relating to the main research questions for this study. Particular attention will be paid to issues associated with the extent and quality of prior knowledge with respect to fractions that students beginning Year 7 (hereafter, Form 1 students) might be expected to possess. In addition, literature on Realistic Mathematics Education will also be highlighted.

In the relevant sections of this chapter, the aims are as follows:

> Sections 2.2 to 2.6 are the sections where relevant literature reviews identifying the knowledge gap dealt with in this thesis are discussed.
>
> Section 2.7 will concentrate on the knowledge gap, and its importance is discussed.
>
> Section 2.8 is where the research framework and research questions of this thesis are outlined.

2.2 WHAT IS A FRACTION?

Orton and Frobisher (1996) presented five types of fraction representation or models: *Region model, Discrete Model, A position on a number line model, Quotient model and Ratio model*.

Region model. This is also referred to as “part of a whole” or “part-whole” model, and it is associated with the concept of area. This aspect of fractions involves subdivision of a well defined “whole” into equal-sized parts and a statement regarding the number of the parts under consideration. Orton and Frobisher (1996) proposed that students find that this is the easiest model for them to understand, but at the same time it also confuses them as the “whole” or the “unit” is not the same in all cases. For example, Figure 2.1

below (Orton and Frobisher, 1996) illustrate “three-quarters”, but all the areas (or regions) that represent the “three-quarters” are different from each other.

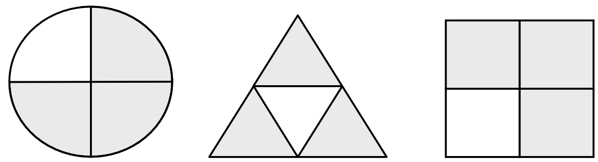


Figure 2.1. Illustrating three-quarters

Orton and Frobisher (1996) commented that

> The fundamental idea behind the concept of fraction is that of a “unit”, or what is often referred to as the “whole”. Without an understanding that a “unit” is forever changing, and that it is a function of the situation and is not absolute, little progress can be made into the understanding of “parts of a unit”. (p. 107)

Discrete model. This model is similar to the *region model* if the parts of a given shape are separated and looked at individually as discrete parts, for example, the square in Figure 2.1 above, if the parts of the square were to be separated to become four smaller squares instead of one big square. In this case the “whole” is made up of four discrete squares instead of one square. It should not then be surprising that a young mind finds it difficult to comprehend the idea that four separate objects (in this case, squares, see Figure 2.2) can be regarded as “one whole” (Clements and Del Campo, 1987). It can be argued that the concept is contradictory to what the students already know about the concept of “one whole”–they would expect to *see* something that is one intact entity, as opposed to having separate objects as a unit.

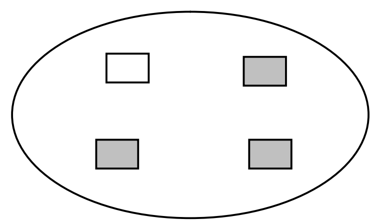


Figure 2.2. A fraction task based on discrete objects

A position on a number line model. Here a fraction is considered as a rational number (i.e., an abstract number) that has a unique position on a number line (see Figure 2.3). This model also does not reflect, or give an impression that a fraction is part of a whole. However, Dickson et al., (1984, cited in Orton & Frobisher, 1996), suggested that fractions on a

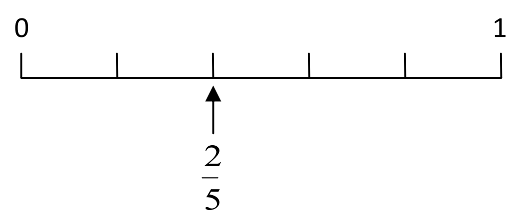


Figure 2.3. A fraction as a position on a number line

number line should only be introduced to students in secondary school. This is because they found that many children were unsure of the position of a fraction as a number-measure which corresponds to a point on the number line. One possible reason might be, in the primary school, fractions are usually first introduced as parts of geometric pictures.

Quotient model. This model involves division of whole numbers that will give a non-whole number, for example, 3 by 4, i.e., 3 ÷ 4, which will not give a whole number answer, implying that three wholes are shared among four. This view of fraction can be represented diagrammatically (see Figure 2.4) and in actions (e.g., cutting three pizzas into four equal parts). From this students are then expected to generalize a "pure number" concept (i.e., $\frac{3}{4}$) from a physical embodiment.

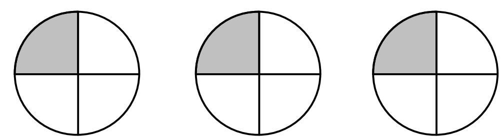


Figure 2.4. The quotient model of fraction

Ratio model. Here "a fraction is the result of comparing the number of objects in two sets or two measurements" (Orton and Frobisher, 1996, p. 111). The symbol $\frac{3}{4}$ is used to represent a ratio context, for example, three pizzas for every four people (see Figure 2.5).

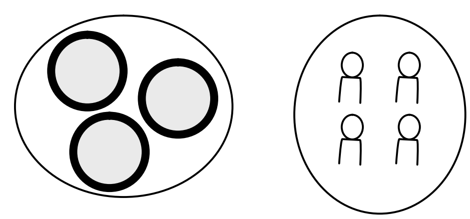


Figure 2.5. The ratio model of fraction: The discrete case

The language associated with this usage can disguise the fact that, essentially, the concept being used is a ratio. For example, a statement saying, "Three-quarters of students in the class have access to a calculator at home" (see Figure 2.6).

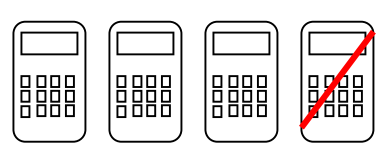


Figure 2.6. The ratio model of fraction: The continuous case

Having presented all these models on fractions, teachers are, however, pressured by the need to prepare students for forthcoming tests and examinations. The most common and quick strategy used is to teach fractions as abstract entities, which are manipulated according to rules which operate almost by "magic" (Selwyn, 1980). Croff (1996), however, argued that fractions taught in this way do not make much sense to the pupils, and are rarely used outside the classroom. Hence the vicious cycle, where students continue to face problems with some aspects of fractions.

Next, some literature highlighting some aspects of fractions that students find difficult is presented and discussed.

2.3 OH DEAR, IT'S FRACTIONS!

Primary school pupils all over the world find some fractions difficult. Indeed, educators and researchers tend to accept the view that of all the topics in the primary school mathematics curriculum, fractions is the hardest (see, e.g., Cramer, Post, and delMas, 2002; Fatimah, 1998; Leong, Fatimah, and Sainah, 1997). For decades there has been a considerable amount of research done into the teaching and learning of fractions to try to understand how children think and perceive fractions, especially in the 1980s and 1990s. For example, Behr, Lesh, Post, and Silver (1983) found that students have difficulties understanding the number line, in particular when the subdivisions of the unit are different from the denominator of the fraction, as compared to when the subdivisions of the unit are equal to the denominator. In the 1990s, for example, Behr, Harel, Post, and Lesh (1992) found that students need to be able to reconceptualize units (i.e., wholes) for them to draw on their partitioning knowledge, in order to develop a better understanding of multiplication of fractions.

Similarly, the involvement of intuition in the learning of fractions was also actively researched around the same time, for example, Carpenter, Fennema, Peterson, Chiang, and Loef (1989) and Mack (1990) found that children used their informal knowledge[1] to develop their understanding of mathematical concepts, symbols and procedures in various mathematical ideas. Sadly, many of these research studies were done with primary school children, with the exception of a few, for example, Lamon (1996) who researched Grade 4 through Grade 8 students on "The Development of Unitizing: Its

[1] According to Mack (1995), researchers have referred to knowledge that students brought with them to school which they constructed themselves based on their everyday experiences as intuitive knowledge, situated knowledge and informal knowledge.

Role in Children's Partitioning Strategies." She emphasized that her study "strongly invites partitioning activities into the middle school curriculum . . . partitioning is more than an introductory fraction activity" (p. 190). Amato (2005), who studied students' understanding of the concept of fractions as numbers involving 11-year olds, found that students benefitted "from the use of multiple representations for fractions equal to one unit (n/n) and mixed numbers" (p. 55). He added that mixed numbers are easier to locate on a number line because it indirectly indicates between which two consecutive whole numbers it is located. He in fact justified Kerslake's (1986) suggestion that geometric part-whole interpretations of fractions hinder students' understanding of fractions as numbers. This is due to the fact that students see fractions as part of a shape or quantity and not as numbers. I believe that only by examining how the older students learn fractions can us as educators pinpoint what is missing, or needs reinforcement at the primary level.

Since I conducted the study in Brunei schools, a brief account of the history of mathematics research that had been done there is provided in section 2.3.1, followed by an account of other research done on fractions, which is presented in section 2.3.2.

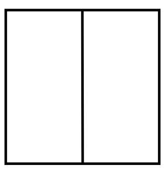
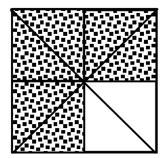
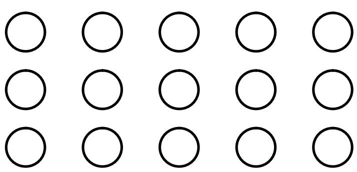
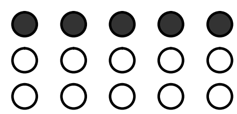
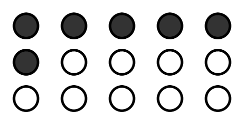
2.3.1 Students in Brunei Darussalam Find Fractions Difficult

Past research revealed that of all topics, many primary school students in Brunei Darussalam find fractions the most difficult (Leong et al., 1997; Samsiah, 2002). For example, Leong et al. (1997) found that students had difficulty in solving real-life tasks which involved fractions concepts. In a question where a set of sweets were to be shared among three people (sweets and people were drawn on the test paper), the students were able to diagrammatically divide the sweets equally, but were not able to write the fraction of the sweets that each person got. This indicates that they were able, from their personal experience, to give each person their equal share, but they had difficulty in converting that knowledge in terms of fractions, and hence were unable to write the fractions symbolically. Therefore, there seemed to be something missing that would have linked the students' informal intuitive knowledge to their formal school mathematics knowledge.

Samsiah (2002) found that most Primary 6 pupils had poorly developed fraction concepts and an inadequate grasp of standard algorithm procedures. A good example of this was when pupils added the numerators and denominators in their attempt to find the sum of two fractions; they found it even harder to deal with multiplication and division of fractions (see examples in Table 2.1). Data such as those summarized in Table 2.1 leave one in little doubt that many upper-primary (11 year-old) students in Brunei Darussalam are struggling with some fractions concepts and skills.

Table 2.1

Common Errors on Fractions Tasks (from Samsiah, 2002)

Question	Number (and %) of Correct Answers	Common Errors	Number (and %) of Pupils Giving This Answer
Shade the correct number of parts to represent the fraction given. $\frac{3}{8}$ 	115 (89.2%)		3 (2.3%)
Shade the number of circles to represent $\frac{1}{5}$ of 15. 	11(8.5%)		33 (25.6%)
			24 (10.9%)
Simplify: $\frac{1}{5} \times 10$	24 (18.6%)	$\frac{10}{50}$	17 (13.2%)
		$\frac{1}{2}$	6 (4.7%)
What is the value of $\frac{1}{2} + \frac{1}{2} - 1$?	13 (10.1%)	$\frac{1}{2}$	23 (17.8%)

A research study into the teaching and learning of fractions in *secondary* schools in Brunei Darussalam, conducted by Suffolk and Clements (2003), investigated the performance of Form 1 (11-year olds) and Form 2 (12-year olds) students on fractions tasks, and the result revealed that at the lower-secondary level, many students are experiencing serious difficulties with primary-level fractions tasks. Thus, for example, less than 50% of Form 1 and 50% of Form 2 students give a correct answer to a question requiring them to find the value of $\frac{1}{2} + \frac{1}{2} - 1$. Similar findings were found in two other research studies, which were conducted to investigate the performances of Form 3 and Form 4 N-level students in Brunei Darussalam (Zurina, 2003a, 2003b), respectively. For example, Zurina (2003b, 2004) reported that Form 4 N-level students in Brunei Darussalam who had been studying aspects of fractions and decimals in Primary 3, 4, 5 and 6, and Form 1 through 3 were still having difficulty with some fractions and decimals tasks. They had a superficial, and often very imperfect, knowledge of fractions and decimals concepts and skills. The associated confidence data revealed that, often, students did not know that they did not know. The analyses of interviews and performance data revealed that none of the students in the sample group had any relational understanding of most of the fractions and decimals concepts. These data also revealed that these students, who had been taught using the traditional approach of drill and practice, were struggling to remember the prescribed "rules" about fractions and decimals that they had learned in previous years. The data also suggested that students responded to fractions and decimals tasks in a totally mechanical or instrumental way, with little or no understanding of why they did what they did.

Important information revealed from the data was that on straightforward money calculation tasks, many of the students did seem to know what they were doing, even though the necessary calculations involved operations on decimal quantities. One reason for this could be that money could be easily remodeled without fractions, and also money was something tangible that they dealt with in their everyday life. This raises the issues of the need to make the curriculum of the primary and secondary schools in Brunei Darussalam more practical in orientation by providing more relevant contexts. As mentioned earlier, since relatively less research has been done for secondary school students, it is imperative that more research be conducted, for Brunei Darussalam's schools at the secondary level, especially involving RME. This constitutes a gap in the research field in Brunei Darussalam that needs to be addressed.

2.3.2 Most Students Find Fractions Difficult

The problems experienced with fractions by primary school pupils in Brunei Darussalam are certainly not confined to that country. Primary school pupils all over the world find some, if not most fractions difficult. As mentioned earlier, educators and researchers believe that fractions is one of the hardest topics in the primary school mathematics curriculum (see, e.g., Cramer, Post, & delMas, 2002)

In this section several relevant aspects will be looked at:

- Linguistic (Conceptual) Complexities;
- Intuitiveness; and
- Students' Informal (Prior) Knowledge.

Linguistic (Conceptual) Complexities. Many mathematics students are the product of their mathematics learning histories. For years they have had to learn school mathematics in classes in which the language of instruction has not been their first language. Those who struggled to learn notoriously difficult topics like fractions in primary school are likely to have acquired many persistent misconceptions. Almost certainly, many of them have some conceptual weakness in lower secondary classes, but often they "do not know that they do not know" (Khoo, 2001; Lim, 2000; Samsiah, 2002, Zurina, 2003a, 2003b, 2004). Language plays a major role in trying to convey knowledge to students, as it can be the most powerful tool to ensure that whatever we wanted to convey reached the receiver in the correct way. On the other hand, it could also become the most destructive tool if it was not used in the same context as the receiver. This is especially true when teachers are trying to explain or convey new knowledge to students.

Vocabulary used in out-of-school contexts generally has a different meaning than in the formal mathematical context. Teachers need to be aware of these differences and should highlight each special meaning in mathematics as soon as it is encountered; and they also need to remind the students from time to time. An example of such a word is "half," in normal everyday conversation it would mean splitting or cutting something into two parts which are not necessarily equal, whereas in mathematical context, the two parts must be equal for it to be called “half” (Backhouse, Haggarty, Pirie, and Stratton, 1992).

Graeber and Tanenhaus (1992) emphasized how confusing the language of fractions can be. Thus, for example, students need to grasp that although the symbols $3\sqrt{12}$, 12÷ 3 and $\frac{12}{3}$ all reduce to the same number (“4”), in English there are numerous verbal expressions which describe the same process. Thus, for example, students are supposed to learn that “Three divided into twelve,” “Three into twelve,” “Twelve divided by three,” “Three goes into twelve,” “Twelve over three,” and even “Twelve, how many threes?” all have the same result. Yet the linguistic structures of these expressions are complex, and different. Not surprisingly, then, learning fractions in English presents special difficulties for students who do not have English as their first language.

In addition, symbols in fractions (e.g., $\frac{1}{3}$) have at least seven different conceptual meanings (Clements and DelCampo, 1990). For example, by “$\frac{1}{3}$” can be meant:

1. Sharing a continuous quantity between three people;
2. Sharing a number of discrete objects equally among three people;
3. Dividing the number 1 by the number 3;
4. A ratio of quantities;
5. A 1 for 3 replacement operator which is the rule which defines a functional relationship;
6. A rational number equal to $\frac{2}{6}, \frac{3}{9}$ and so forth; and
7. A decimal fraction, 0.333…

Lamon (2007) highlighted that when a fraction is considered as a number, it actually is referring to the underlying rational number. She said:

> understanding a fraction as a number entails realizing, for example, that $\frac{1}{4}$ refers to the same relative amount in each of the following pictures [Figure 14.1 in Lamon, 2007]. (p. 635)

From the illustrations (in Lamon 2007, Figure 14.1, p. 636), she stressed that it did not matter what she called the fraction (i.e., either $\frac{1}{4}, \frac{4}{16}, \frac{3}{12}$ or $\frac{2}{8}$) since all refer to the same relationship. However, she added that it will matter, especially for instructional purposes, that “when fractions are connected with pictorial representations, which fraction name you connect with which picture *is* an important issue” (p. 636).

Also, Zurina and Williams (accepted) suggested that one possible reason for students finding fractions difficult, in particular equivalent fractions, is because of the way the words used to describe them were different from the written representation (see also Mamede, Nunes, and Bryant, 2005)–for example, $\frac{1}{3}$ (one-third), $\frac{2}{6}$ (two-sixths) and $\frac{4}{12}$ (four-twelfths).

For this reason, it is no wonder that students in Brunei schools are experiencing difficulty learning fraction concepts in a relational way. However, they are usually taught only a limited number of possible ways of thinking about fraction concepts. Most mathematics lessons do not go beyond simple part-whole shading tasks (Samsiah, 2002). Students are not given experiences comparing a wide range of concrete models for fractions (Cramer, Post, and delMas, 2002). Also, students can easily confuse fraction concepts with whole number concepts. In certain contexts, students’ prior knowledge of whole number operations can have an undesirable effect on their understanding of fractions concepts, and on their ability to carry out fractions operations (Jabaidah & Leong, 2002). That same observation has been made by other researchers (see, e.g., Streefland, 1984).

Mack (2001) suggested that knowledge of partitioning may be used as a stepping stone to the development of students’ understanding of multiplication of fractions, which has been supported by numerous other researchers (see, e.g., Armstrong and Bezuk, 1995; Empson, 1999; Steffe, 1988, Streefland, 1991, 1993). However, this seemed problematic for students, since the concept of a *whole* itself creates an undeniably confusing concept, as only in mathematics does the word *whole* have a different representation for a different situation. In contrast, the everyday *whole* would mean intact, full, in one piece, and so on; this all refers to something that is complete as opposed to partial, which fractions is all about. Then again another source of confusion is when a *whole* is represented by the symbol “1” for fractions greater than or less than 1 (e.g., $1\frac{1}{2}, \frac{3}{4}$).

In every nation, it seems, a significant proportion of upper-primary and lower-secondary students add fractions by “adding the numerators and adding the denominators”–for example, $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ (McIntosh, 2002; Samsiah, 2002). Also, many students believe that addition must "make bigger," and so must multiplication. This is hardly surprising, since most school textbooks introduce multiplication as repeated addition. Thus, students find it difficult to accept a product which is smaller than either factor, for example, $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ (Graeber & Tanenhaus, 1992).

Graeber and Tanenhaus (1988, cited in Graeber & Tanenhaus, 1992) reported that fourth- and fifth-grade students also explain division as "undoes multiplication." From that assumption, many students expect the answer to be smaller than the number divided, and are reluctant to accept an answer which is greater than the number divided–as in the case, for example, of quotients like $2 \div \frac{1}{2}$. This is in line with one of Fischbein's (1999), characteristics of intuitive cognitions. He suggested that students will tend to not accept another conception that diverges from their intuitions, as in the above situation.

Now, a brief review on intuition will be provided as this is one of the important attributes of RME.

<u>*Intuitiveness.*</u> Fischbein (1999) wrote that "there is no commonly accepted definition of intuitive knowledge. The term *intuition* is used as one uses mathematical primitive terms like point, line, set, etc." (p. 12).He further added that intuition suggests self-evidence instead of a logical-analytical undertaking. Some of the definitions used by other researchers, as mentioned by Fischbein (1999), are:

> intuition is presented as the genuine source of true knowledge. (Descartes, 1967; Spinoza, 1967)
>
> intuition as the faculty through which objects are directly known, in distinction to understanding, which leads to indirect conceptual knowledge. (Kant, 1980)
>
> Bergson (1954) distinguishes between intelligence and intuition. Intelligence is the way by which one knows the material world, the world of solids, of space properties of static phenomena. By intuition we reach directly the essence of the spiritual life, the grasp of time phenomena ('la durée'), of motion.

On the other hand, Hans Hahn (1956) and Bunge (1962), as cited by Fischbein (1999), both felt that intuition can be misleading, especially in scientific reasoning. Fischbein (1999) believed that intuitions are concealed in our subconscious mental activity, and trying to decode this is not easy. He further remarked that "Not every direct cognition is an intuition. Perceptions are directly grasped by senses, but they are not intuitions. Intuitions are *intellectual* cognitions–expressing a general conception (a notion, a principle, an interpretation, a prediction, a solution) while perceptions are sensorial cognitions (for instance, I *see* a chair, a triangle, etc.)" (p. 18). Fischbein also felt that counter-intuitive notions will lead to the creation of a new intuitive context, and that intuitive models are not always possible, even though they can be useful. Fischbein (1999) outlined five general characteristics of intuitive cognitions:-

1. *Direct, self-evident* – cognitions which do not need any proof;
2. *Intrinsic certainty* – does not need external support, but it is usually associated with a feeling of absolute certain;
3. *Subjective Coerciveness* – rejecting an alternative conception that conflicts with their intuitions, such as, multiplication always makes bigger;
4. *Extrapolativeness* – ability to conclude beyond any empirical proof, for instance, the acceptance that there will always be a number (whole) after a given number (whole); and
5. *Globality* – an intuitive direct response, as opposed to a logical – analytically-based solution.

An example where intuitive cognitions are used:

Convert $\frac{1}{3}$ into a decimal.

Most students (even college students) would easily give an answer of 0.33333…, but if we were to ask the question in another way, say, "The decimal number 0.33333… is equal to $\frac{1}{3}$ or tends to $\frac{1}{3}$?" The common answer given by students, including college students is it tends to $\frac{1}{3}$. In actual fact, it is equal to $\frac{1}{3}$, because of the symmetrical relation of equality that $\frac{1}{3}$ = 0.33333… since we are dealing with a potential infinity (similar to when a child can accept the fact that a straight line can be extended indefinitely). On the other hand, 0.33333… = $\frac{1}{3}$ somehow is counter-intuitive for the students for the same reason that it is potentially infinite and can never actually reach $\frac{1}{3}$ (Fischbein, 1999).

Fischbein (1999) categorized intuitions into two:

1. *Affirmatory intuitions* – refers to solutions or representations that are acceptable to the individual; and
2. *Anticipatory intuitions* – refers to solutions which are not direct and which the individual has to process in three stages:
    a. Need to understand the question, and recognize what is given and what is required (global representation);
    b. Need to synthesize previously learned knowledge to link between what is given and what is required (feeling of conviction or absolute certainty); and
    c. Once these efforts reach a coherent (according to the individual) stage, the individual will *suddenly* feel that he/she has found the solution, even though any justification may not be found yet.

Students' Informal (Prior) Knowledge. Many scholars believed that prior knowledge is a crucially important factor influencing what people are capable of learning. Ausubel (1968) stated:

> If I had to reduce all educational psychology to just one principle, I would say this: the most important single factor influencing learning is what the learner already knows. Ascertain this and teach accordingly. (p. vi)

All students come to school with different experiences and backgrounds and hence they bring with them some sort of mathematical concepts knowledge as a result of their interaction with their physical environment, social world, or from their previous mathematics lessons. The knowledge they bring with them to school may not be correct, and these misconceptions may be deeply embedded, and may prove difficult to change even after lessons to rectify them have been given (Smith, diSessa, and Roschelle, 1993). They further added that in order to correct the misconceptions, they needed to be replaced by new knowledge which could prove to be very difficult to do. Usually students will resist the changes unless the old knowledge does not work any longer, or unless the teacher could show them exactly where the error lies and replace it with new knowledge.

If learners are to have any chance of breaking out of their erroneous cycles they need to have their errors pinpointed and brought to their attention. They also need to be convinced of the need to make the cognitive effort required to shed their old, fixed, but erroneous ways of thinking about something, and to put in place correct (but often more complex) procedures. They also need assistance and motivation to make the effort to shed the old patterns and take on the new. As stated in the Cockcroft Report (Cockcroft, 1982):

> Even if an answer is incorrect, or is not the one which the teacher was expecting or hoping to receive, it should not be ignored; exploration of a pupil's incorrect or unexpected response can lead to worthwhile discussion and increased awareness for both teacher and pupil of specific misunderstandings or misinterpretations. (p. 72)

Freudenthal (1973) stressed that there should be (referring to mathematics) a relationship with the students' everyday life to give it meaning, as information without meaning would easily be forgotten very quickly.

Mack (2001) reported that it is viable to use students' informal knowledge to develop a more complex mathematical content domain. Some researchers suggested that many students' initial understanding (either intuition, informal knowledge, or other forms of prior knowledge) can, and should, play an important role in the development of students' understanding of complex domains (Pirie and Kieren, 1994).

Irwin (2001) found that the level of understanding that students achieve when taught a body of knowledge is greatly influenced by what their teachers actually teach. In particular, levels of understanding are greatly affected by the extent to which *what* is taught, and *the manner by which it is taught*, connects with their prior knowledge and past experiences as a learner. Students will be more likely to be able to remember and apply their knowledge if the material which is presented to them is *meaningful* to them, in the sense that it is connected to their present knowledge (NCTM, 2000).

The way prior knowledge is structured and organized influences students' learning. If prior knowledge with respect to a topic is richly connected then it will be easier for the students to relate this to new knowledge. Thus, for example, Samsiah (2002) conjectured that because of an over-emphasis on teaching and learning of fraction concepts through area models, primary school students in Brunei Darussalam develop impoverished, overly narrow, fraction concepts. As a result, they struggle to link fraction concepts with fraction-related situations, both in and out of school, which "invite" the application of fraction concepts *unless* those situations are obviously related to area model ideas. As Hiebert and Carpenter (1992) pointed out, well connected prior knowledge is usually well understood knowledge. Well connected prior knowledge provides a much better scaffold for the building of new concepts.

From the above, it seems to be well established, and intuitively obvious, that the “stronger” the prior knowledge, the better chance students will have in assimilating new knowledge meaningfully.

2.4 REALISTIC MATHEMATICS EDUCATION (RME)

2.4.1. What is RME?

Ever since the conception of Realistic Mathematics Education (hereafter, RME), emphasis was already put in place that children should not learn mathematics in isolation, void of activities that are experientially real to them, as information without meaning would be quickly forgotten (Freudenthal, 1973). He added, it is vital to begin organizing activity (or *mathematizing* as he later refer to it) of mathematical and subject matter by mathematizing everyday activities which students are familiar with and which are experientially real to them (i.e., mathematics is seen as a human activity). As Freudenthal (1971) said, “It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter” (p. 413–414). Therefore, we can see that the crux for RME is realistic problems, which are experientially real *to* the students. However, Gravemeijer (1997a) stressed that “realistic” does not imply real-life problems. He said

> Context problems in realistic mathematics education do not necessarily have to deal with authentic everyday-life situations. What is central, is that the context in which a problem is situated is experientially real to students in that they can immediately act intelligently within this context. (p. 31)

This is also highlighted by van den Heuvel-Panhuizen (2003) when she said

> Contexts are not necessarily restricted to real-world situations. The fantasy world of fairy tales and even the formal world of mathematics can be very suitable contexts for problems, as long as they are ‘real’ in the students’ minds. (p. 10)

Treffers (1987) introduced two types of processes in this connection: horizontal mathematization and vertical mathematization. In the former, the students come up with the mathematical tools in order to organize and solve a real-life situation (i.e., interpreting everyday-life contextual problems into mathematical problems/symbols); whereas the latter refers to a process of reorganizing within the mathematical system itself (e.g., making shortcuts and finding connections between concepts, etc.) (see also, van den Heuvel-Panhuizen, 2001, 2003). However, Freudenthal (1991) stressed that there are no clear boundaries to determine what is horizontal, and what is vertical mathematization–the distinction between the two is vague. The bottom line is how we define reality. On the one hand, according to Gravemeijer (1997a), vertical mathematization is at the centre of this "human activity" process, since in vertical mathematizing:

> Problem descriptions can develop into an informal language, which in turn can evolve into a more formal standard-like language, due to a process of simplifying and formalizing. (Gravemeijer, 1997a, pp. 331–332)

On the other hand, "children bring cultural knowledge and language from outside school with them, and that this can help or hinder them in building mathematics" (Linchevski & Williams, 1999, pp. 144–145).

Within the RME approach, one of the pedagogical principles that underpins the RME approach is viewing mathematics as organizing human activity where mathematics becomes an emerging tool in concept formation (Seegers and Gravemeijer, 1997) through models and modeling. Models are used to model the students' informal mathematical activity, which will gradually develop into a model of more formal reasoning. In short the models' role is to bridge the gap between the informal understanding of real-life problems and the understanding of formal mathematics (Cobb, 2002, Gravemeijer, 1997a, 1997b, 2007; van den Heuvel-Panhuizen, 2001, 2003). This will be elaborated further in section 2.4.4. below. Another principle that underpins RME is the belief that mathematics is both a process and a product where students went through a process of guided reinvention which give them opportunity to "think and perform as developers of mathematics instead of consumers of pre-developed rules and principles" (Seegers and Gravemeijer, 1997, p. 258) which is discussed in section 2.4.5 below.

Next, a brief discussion on how this theory differs from Gagné's perspective is presented. The importance of this will become apparent in Chapter 3 where an analysis of textbooks used by students in this study who used textbooks that are in line with Gagné's events of learning, and some textbooks that promotes RME is discussed.

2.4.2 R*ME Versus Gagné's Events Of Learning*

Robert Gagné (1973, 1975, 1985) outlined nine events of instruction (hereafter, EoL) (see Appendix 1). According to him ". . . the most important agent in an educational program is the teacher. It is the teacher's job to see that the various influences surrounding the student are selected and arranged to promote learning" (Gagné, 1975, p. 2). RME, however, would also consider students as a major contributor to learning. A RME lesson starts with mathematizing of everyday activities which are experientially real to the students (Freudenthal, 1973). This is an important feature of a RME lesson, as great emphasis on the relation between mathematics and students' everyday life is crucial to give it meaning. In Gagné's events of learning, even though stimulus inherent to the lesson is expected to be presented to the students, it is not necessarily something that is experientially real. Instead, the emphasis is more on presenting tangible "objects" which might consist of a definition, a group of geometrical shapes, and so forth (see Appendix 1). Also, recalling previously learned knowledge is essential for "*contiguity* in learning" (Gagné, 1973, p. 309); RME would instead use the students' experiences to ensure continuous links for their learning. In short, in constructing knowledge, RME believed that informal knowledge plays a central role, as opposed to prior knowledge. Thus, in a RME lesson it is more of a guided discovery, rather than guided learning, which Gagné is advocating.

The principle of guided reinvention which underpins Freudenthal's (1973) suggestion that mathematics (or learning) is both a process and a product is yet another difference that can be noticed from what Gagné is proposing, who I felt conjectured more on assessing the end-product. This can be seen in the EoL where students are expected to "reproduce" what they had learned by performing similar tasks with feedback given to indicate the correctness of their performance; and usually at the end of instruction of a topic, they are expected to take a "test" to measure their performance.

Another difference between these two concepts is, for RME, students can eventually claim ownership of the knowledge acquired, as students are the experimenters and explorers of knowledge (Gravemeijer, 1997a), which would make the knowledge more meaningful. However, the EoL places more emphasis on mastery learning, where students are expected to be able to apply their newly acquired knowledge to real-life situations. In this sense, the main difference is that RME starts from real-life situations towards students owning what they have discovered, and Gagné conversely is advocating guided learning and applying the end-product to real-life situations.

2.4.3. How is RME different from the traditional approach?

Wilhelm Busch once said:

> *"Wenn alles schläft und einer spricht, den Zustand nennt man Unterricht"* (When everyone is sleeping and one is speaking, this state is called instruction). (as cited in Freudenthal, 1973, p. 155)

A good way to highlight the difference between the RME and traditional methods of teaching is by looking at the distinction that Hiebert, Fennema, Fuson, Human, Murray, Olivier, and Wearne (1996) made between functional understanding and structural understanding. According to them, functional understanding is "participating in a community of people who practice mathematics" (p. 16), whereas structural understanding is "representing and organizing knowledge internally in ways that highlight relationships between pieces of information" (p. 17). I feel that these two views, to a certain extent, represent some similarities to RME and traditional views. Functional understanding, for example, emphasize on classroom activities which encourage students to problematize the situation where the teacher takes an active role in encouraging students to problematize mathematics and collectively search for solutions. This is similar to what RME is advocating. Whereas, structural understanding emphasize on what the students take with them from the classroom. In this sense, it is similar to the traditional approach as it is also focusing on the by-product of the teaching, which the structural view referred to as *residue* (Hiebert et al., 1996). This residue partly depends on prior knowledge which is similar to the traditional method which also used students' previous knowledge (i.e., from previous lesson) upon which to build on the current teaching. However, the similarity of this aspect ends there as structural understanding also emphasize on conceptual understanding whereas traditional method only emphasize on instrumental one. On the other hand, another similarity with the traditional method is “particular procedures . . . for solving particular problem” (Hiebert et al., p. 17). Similar strategy can also be seen happening within the traditional approach, where almost all students will produce the same answer using the same method. It is, however, according to Seegers and Gravemeijer (1997), not sufficient to reject the “teaching by telling” (p. 260) approach; what is more crucial is for teachers to develop strategies which could help students construct mathematical knowledge.

Seegers and Gravemeijer stressed that “guidance in a teaching-learning processes that allows for a maximum of student autonomy is a complex and subtle process” (p. 264). The phrase “student autonomy” here refers to students as active participants and assuming greater responsibilities for, and takes charge of their learning. While Hiebert et al. (1996) felt that in order for students to be able to construct understanding in mathematics, they “should be allowed to make the subject problematic” (p. 12). According to them, the reflective inquiry and problematizing process is not about the task itself, rather it is more about the students and the classroom culture. In reflective inquiry students are active participants in constructing meaning, and Hiebert et al. (1996) stated that it “emphasizes the process of resolving problems and searching for solutions rather than manufacturing a product” (p. 19). However, Dewey (1929, p. 189, cited in Hiebert et al., 1996, p. 15) said that “All reflective inquiry starts from a problematic situation.” Thus, for Hiebert et al. (1996), problematizing process means that students are allowed “to wonder why things are, to inquire, to search for solutions, and to resolve incongruities. It means that both curriculum and instruction should begin with problems, dilemmas, and questions for students” (p. 12).

> Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them. (p. 16)

They also felt that the issue of “real-life versus school” is not important. What is of the utmost importance to them are “(1) has the student made the problem his or her own, and (2) what kind of residue is likely to remain” (p. 19).

As stated in Seegers and Gravemeijer (1997), Brink (1989) and Nelissen (1987) have shown in their studies that a realistic approach in teaching mathematics and in particular fractions produced encouraging results. In the same paper, a similar finding was also reported by Wijnstra (1988) and Bokhove, Schoot, and Eggen (1996), who found that in a large-scale national assessment study on mathematics, the realistic approach did lead to better results. On the other hand, Harskamp and Suhre (1986) and Harskamp (1988) found no distinctive difference between the realistic and the traditional approach (in Seegers and Gravemeijer, 1997).

Gravemeijer (1997a) noted that there is now a shift in Mathematics education, from the traditional teacher as the authority of knowledge, to the student as the experimenter and explorer of knowledge, when he said:

> Globally, we can speak of a shift away from the 'transmission of knowledge' by teachers towards, 'investigation', 'construction', and 'discourse' by students. (p. 13)

In other words, mathematics used to be considered (maybe still is by some) as something that one just does by following the algorithms, but nowadays, with the new reform, the demand is much more than just the ability of the students to perform the algorithms correctly. Gravemeijer (1993) highlighted that:

> (in constructing knowledge) there is a central role for the informal knowledge and strategies developed by the students themselves. It is a process of guided discovery, the students will be supported to mathematize their informal knowledge and strategies in order to build more abstract mathematics. (p. 141)

In addition, there is now more emphasis on students understanding the underlying mathematical concepts, so that they will be able to explain and justify their answers, and not just be able to reach the correct answer (Seegers & Gravemeijer, 1997).

2.4.4. Models and Modelling

In order for students to be able to fully *reinvent* and *reinforce* whatever knowledge they have internalized, in particular in fractions, Streefland (1997) suggested the use of modeling to shift meaning from the context situation to the mathematization stage. The terms *models*, according to Linchevski and Williams (1999):

> sometimes implies a manipulative aid (such as diagram or chart, or the double abacus) and at other refers to some situation in which "intuitive" knowledge can be used (such as balloons with weights attached). In general it does not imply a 'real experience'. (p. 134)

They also added that the intention of using models is to add "obviousness" and "correctness" (p. 134), which is in accordance with Fischbein (1987). Van den Heuvel-Panhuizen (2003) also advocate that "model" should not be taken in a literal way; apart from manipulatives, paradigmatic situations can also serve as models, as in repeated subtraction in long division (see also Gravemeijer, 1997a). As mentioned earlier, emergent models bridge the gap between informal knowledge and formal mathematics. The model "comes to the fore first, as a model that is a model *of* a situation that is familiar to the students" (Gravemeijer, 1997a, p. 29). Gradually, the model "takes a life of its own . . . . and starts to serve as a model *for* more formal, yet personally meaningful, mathematical reasoning" (Gravemeijer, 2007, p. 139). According to van den Heuvel-Panhuizen (2003):

> Students are not handed ready-made models that embody particular mathematical concepts, but they are confronted with context problems, presented in such a way that they elicit modeling activities, which in their turn lead to the emergence of models. (p. 29)

Sfard (1991) and Gravemeijer (1997a) suggested that the process of transforming from "model-of" to "model-for" implies a process of *reification* where "procedures are reinterpreted as objects" (Gravemeijer, 1997a, p. 29). He also added that "what is reified is *the process of acting with the model*, not the means of symbolization itself" (p. 29).

Gravemeijer (1997a) stressed that emergent models are not only about symbolization–though it is important–but the *situatedness* of the activity, which portrays real meaningful experiences which are embedded in the contextual activities that are the core of it all. This is done by allowing the students to imagine the situation, and by imagining dealing with tasks in that situation through symbolizing or modeling. This is when the model is gradually developed, hence the phrase "emergent models". Gravemeijer and Stephan (2002) see modeling as "an organizing activity from which the models emerge. The idea is that subsequent acting with these models will help the students (re)invent the more formal mathematics that is aimed for" (p. 148).

The striking difference between the traditional approach and the RME approach is the way the models "emerge." In the traditional approach, the models are the product of formal mathematics, whereas in the RME they are the result of informal mathematics. He outlined four levels of model development: situational, referential, general and formal. Situational level is where situational knowledge and informal strategies are used to deal with the contextualized situation; the referential level (i.e., model-of) is where concrete models are used to represent the situation; the general level (i.e., model-for) focuses on the strategies used rather than on modeling the context of the problem, and this is where the process of *reification* occurs; the formal level is the stage where their strategies are formalized and they will start to solve the problem using formal procedures and notations. For example, in this study the situation level was the sharing of food, while the referential level was the cutting of a 12-inch loaf of bread and measuring the length of each piece where each portion represented each share. The general level was where each portion can be represented as fractions in terms of the number of parts and in terms of length in inches. Thus, the formal level was where the two fractions produced were formalized to be equivalent fractions.

In essence, what Gravemeijer (1997a) is trying to highlight as the pinnacle of RME is "real-life problems." In addition, he stressed that a contextual problem does not have to be what the students experience in their everyday life, but one that is "experientially real" to them and they can then deal with tasks within the context intelligently.

Freudenthal (1983) and Gravemeijer (1997a) believe that the focal point of students' mathematics learning process is "the phenomena in which operations are embedded rather than the embodiment of them in manipulatives" (Carpenter, 1997, p. 35). He also stated that

> Rather than starting with the formal operations and developing manipulatives to give them meaning, instruction begins with problems that children can solve using intuitive modeling strategies. In this way formal mathematical operations are conceived as means of generalizing and symbolizing the informal models children already use and understand. (Carpenter, 1997, p. 35)

This, according to Carpenter (1997) would allow students to revisit their basic strategies that are related to more advance strategies that they have developed and used for different problems.

2.4.5. Reinvention Process

As mentioned in section 2.5.1. above, Treffers (1987) identifies with Freudenthal's (1973) notion about mathematizing problems as a form of horizontal mathematization, i.e., transforming a contextual problem into a mathematical one, and the notion of mathematizing concepts, notations and problem-solving procedures, on the other hand, is referred to as a form of vertical mathematization. He further added that in order for students to be able to reinvent any mathematical insights, knowledge, and procedures, they need both the horizontal and vertical mathematization, as he believed that the process and the product should be connected (Freudenthal, 1973).

These processes of mathematization imply that direct instruction or teaching by telling is out of the question. A question that may come next, according to Seegers and Gravemeijer (1997a), might be "what type of teacher guidance is left to the teacher" (p. 264). They reported that data from teaching sequences with RME and an RME-like approach in the US indicated that there seemed to be three types of pro-active teacher guidance in play: pre-active, interactive and retro-active.

Pre-active guidance is where teachers provide students with a series of tasks that will allow for the students' own production of strategies to take place, which may then be the starting point for progressive mathematization. Teachers may also, if they have to, introduce conventional (or didactical) symbolizations in a casual and non-committal way, which students may or may not accept. The interactive guidance, as the word interactive implies, literally means that the guidance is through interactions between teacher and students. The teacher will encourage students to discuss and compare different strategies that they can use, which may push their mathematical thinking to move on to the next level. Seegers and Gravemeijer (1997), however, cautioned that

> Students are expected to take the reality, as implied in the contexts into account, however, not all solutions that are possible in reality (like 'go to the shop and buy an extra pizza') are accepted in a mathematics classroom. (p. 265)

The last type of guidance is the retro-active guidance. This is where the teacher can suggest how to use symbolizing or an algorithm that fits with the mathematical conventional procedures. In order to ensure that students' autonomy remains intact, they will be encouraged to analyze and discuss this procedure as they did when they were discussing their colleagues' own inventions.

2.4.6. Progressive Mathematization

Gravemeijer (1997a) proposed that whether a problem or task is routine or not depends on where the students stand at that particular time. If at that point the students no longer have to explain or justify their solution to their class community, the problem or task will have become routine and will later be taken-as-shared by the class community. Freudenthal (1973) treated mathematics both as a process and a product, hence according to him mathematization must be related and linked, and is underpinned with the principle of guided reinvention. Gravemeijer (1997a) stated that:

> According to the reinvention principle a learning route has to be mapped out along which the student could be able to find the result by him- or herself. The emphasis is on the character of the learning process rather than on inventing as such. The idea is to allow learners to come to regard the knowledge they acquire as their own, private knowledge; knowledge for which they themselves are responsible. On the teaching side, students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process. (pp. 21–22)

Freudenthal (1983) suggested that instead of the concretization of concepts through the embodiment of manipulatives, the converse approach should be used. This means that rather than trying to find materials/manipulatives that can concretize a given concept, he is suggesting to start with problems/situations that provide opportunities for the students to mathematize the problem and progressively forming the given concepts. The manipulatives should be utilized so that there will be a stage in the process where all the information would, as Freudenthal described it, *beg* to be organized:

> . . . starting from those phenomenology that beg to be organized and from the starting point teaching the learner to manipulate these means of organizing. (p. 32)

In other words, what Freudenthal (1983) is proposing is the reverse of the traditional approach of teaching the concepts (concrete embodiment) prior to giving the applications.

The underpinning idea of RME, that mathematics is a human activity, continues to hold. This is because at the very beginning students will always use concrete materials (manipulatives) or drawing to represent (or model) the problem, but as they progress the models become more abstract and finally they will not need to represent the problem situation concretely. Hence this can be depicted as the students' progressive mathematization of their intuitive informal modeling strategies (Gravemeijer, 1997a). In the words of Seegers and Gravemeijer (1997):

> The consequence is that learners should be given the opportunity to think and perform as developers of mathematics instead of consumers of pre-developed rules and principles. (p. 258)

2.4.7 Verbalizing Thinking

Many teachers found it hard to implement a problem-oriented approach of teaching mathematics which might be due to the schooling culture and the teachers' own beliefs that might hamper such implementation in the classrooms (McLeod and McLeod, 2002;

Stigler and Hiebert, 1999). Also in my opinion, another possible reason was not that they had not adapted the theory itself, but rather they were *forced* to go back to the traditional approach by students who did not *cooperate*. It appears that students do not like uncertainty, and prefer to be told what to do, hence they ask questions that force teachers to revert from a more involved task to mere one-track exercises (Desforges and Cockburn, 1987). In order to overcome or minimize such dilemmas faced by teachers, it is practical to stress to the students the importance of peer discussion, as Carpenter (1997) found that students sharing their thinking with their peers had a number of advantages:

1. They would use a strategy that they really understood so that they could explain it;
2. Students who had developed a more advanced strategy could share their thinking with others;
3. Not only was the strategy shared between the students, the thinking process that took place also became transparent;
4. The ability to solve the problem and explain it to their peers led them to reflect on their solution; and
5. It signaled to the students that their thinking counted.

For schools that are only starting to change their classroom culture of the teacher as the sole authority of knowledge in the classroom, changing the norm to students as the developers or reinventors of knowledge has to be gradual. The change has to be verbalized through action. Lampert (in Brandt, 1994) said,

> . . . establishing a culture in the classroom. A big piece of teaching for understanding is setting up social norms that promote respect for other people's ideas. You don't get that to happen by telling. You have to change the social norms–which takes time and consistency. (p. 26)

Next, a review on the semiotic theory of objectification is warranted as I am using semiotic analysis to examine the relevant episodes in the study.

2.5. THE SEMIOTIC THEORY OF OBJECTIFICATION

2.5.1. The Idea of Objectification

The theory of objectification hypothesized that students depend on an array of semiotic sources that could help them to see or find something that was unnoticed previously

(Radford, 2001, 2002, 2003, 2006, 2008; Radford, Demers, Guzman, and Cerulli, 2003; Radford, Bardini, & Sabena, 2006a, 2006b). Radford further defined *objectification* as:

> . . . a process aimed at bringing something in front of someone's attention or view. (Radford, 2002, p. 2)

and furthermore,

> To learn to generalize means to "notice" (Mason, 1996) something that goes beyond what is actually seen. Ontogenetically speaking, this act of noticing unfolds in a gradual process in the course of which the object to be seen emerges progressively. This process of noticing we have termed a process of *objectification*. (Radford et al., 2006a, p. 685)

2.5.2 Semiotic Means of Objectification

Frege (1960, p. 144, cited in Radford, 2002) believed that only through symbols can we perceive objects (mathematical objects) that cannot be directly perceived or made apparent. For example he said "when I write 1+2=3 I am putting forward a proposition about the numbers 1, 2 and 3, but it is not those symbols that I am talking about" (cited in Radford, 2002, p. 14). Radford (2002) put forward the idea that the notion of objectification goes beyond symbols, quite the reverse of the thinking of Frege, who had condensed semiotic activity to written symbols. Like many other mathematicians, Frege strongly felt that "our knowledge of mathematics can be derived from a few principles that do not require any *sensible experience*. That what was required was a good symbolic language" (cited in Radford 2002, p. 15). This led to the beginning of Radford's (2002) *semiotic means of objectification*.

> To make something apparent (which is the etymological sense of objectification) learners and teachers make recourse to signs and artifacts of different sorts (mathematical symbols, graphs, words, gestures, calculators, and so on). These artifacts and signs used to objectify knowledge we call *semiotic means of objectification*. (Radford et al., 2006a, p. 685)

This is in line with Vygotsky (1962), Vygotsky and Luria (1994), Zinchenko (1985), Geertz (1973) and Wertsch (1991) who all considered signs as *tools* or *prostheses* of the mind to bring about actions necessitated by the contextual activities the individuals are engaged in, instead of perceiving signs as the reflection of internal cognitive processes (cited in Radford, 2000). Radford (2003) further described *semiotic means of objectification* (SMO) as:

> We called semiotic means of objectification the whole arsenal of intentional resources that individuals mobilize in the pursuit of their activities and emphasized their social nature. The semiotic means of objectification appear embedded in socio-psycho-semiotic meaning-making processes framed by cultural modes of knowing that encourage and legitimize particular forms of sign and tool use whereas discarding others. (p. 44)

Frege (cited in Radford 2002) said that sense and knowledge are deeply interrelated. He used analogies in order to distinguish between the *object,* the subjective *idea* that individuals may have of the object, and the *sense*.

> The following analogy will perhaps clarify these relationships. Somebody observes the Moon through a telescope. I compare the Moon itself to the reference [object–L.R.]; it is the object of the observation, mediated by the real image projected by the object glass in the interior of the telescope, and by the retinal image of the observer. The former I compare to the sense, the latter is like the idea or experience. (Frege, 1960, p. 60, cited in Radford, 2000)

From this analogy, the Moon is the *object*, the retinal image is the subjective individual's *idea*, sense lies between and outside both the *object* and the individual's *idea*, and is embodied *in* the material telescope. So according to Radford (2002) "sense is expressed through a cultural artifact" (p. 16). Hence the telescope, in Radford's terminology, is a means of objectification. Following Frege's attribution, Radford (2002) came up with the Figure 2.7 below, which has been copied from Radford (2002), which shows a diagrammatic representation of the objectification of mathematical knowledge.

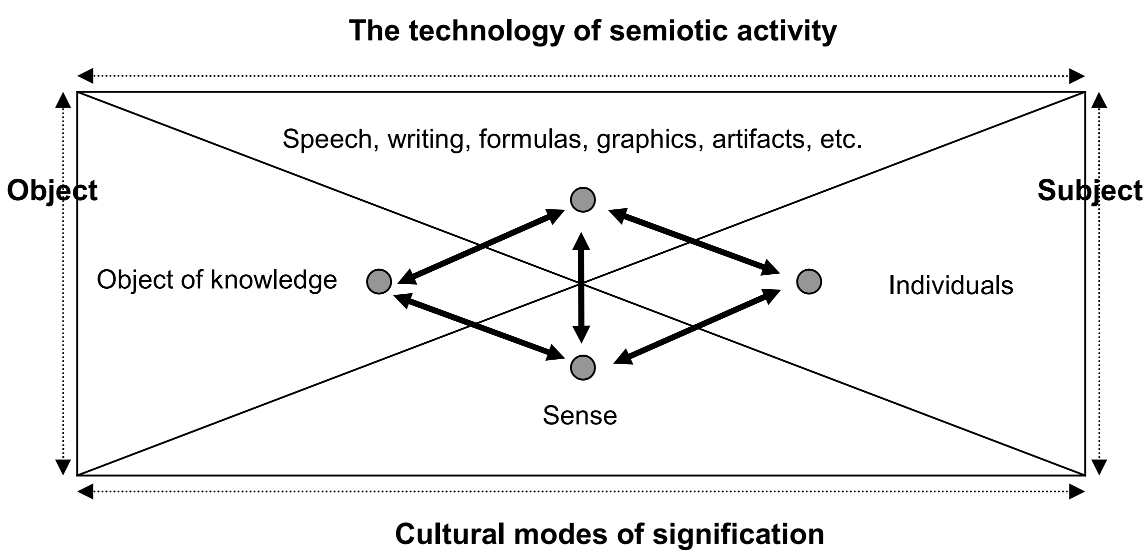


Figure 2.7. The Objectification of Knowledge (Radford, 2002)

In the next section, a discussion on *deixis*, *indexical gestures* and *objectifying deictics* is presented.

2.5.3 Deixis, Indexical Gestures and Objectifying Deictics

Let us first look at the definition of the terms *deixis* and *deictic*. I have presented three definitions below.

> . . . *deixis* (meaning in Greek 'display', 'reference'): That is, in linguistic terms (e.g., *this, that*) related to actions of showing or pointing out something and that, as it turns out, constitute a key element in mathematical discursive meaning production processes. (Radford, 2002, p. 15)

> . . . where deictic words are used to refer to objects in the perceptual field of the speakers. This form of deixis is called *demonstration ad oculos*. The classic example is the word "that" to point to something in our surrounding. (Radford, 2002, pp. 17–18)

> Another kind of deixis occurs when pointing is made to something absent . . . . This type of deixis Bühler called *deixis at phantasma*. A typical example occurs when somebody uses indicative words to describe a path to follow in a city (e.g., from *there*, you walk two blocks down the street). (Radford, 2002, p. 17)

Numerous examples of students' use of the *deictic language* can be found in various excerpts in Radford (2000, 2001, 2002, 2003, 2005a, 2008), Radford et al. (2003), and Radford et al. (2006a, 2006b). Not only were the students in those excerpts using *deictic language*, for example,

> Line 72: (…) you always add 1 <u>to the bottom</u>, right? Then you always add 1 <u>to the top</u>. (Radford, 2000, p. 248)

but they were also making use of *deictic gesture* such as *pointing* at "Figure 1" (Radford, 2006, p. 10) to draw attention to something that was in front of them so that others might also notice it (i.e., to make it stand out). Radford (2002) also termed these (e.g., pointing) as *indexical gestures*. He further defined the linguistic terms used in the process of objectifying deixis (making things apparent) as *objectifying deictics.* In section 2.5.4, *semiotic contraction* and *semiotic narratives* are discussed.

2.5.4 Semiotic Narratives, Semiotic Nodes and Semiotic Contraction

When discussing *semiotic narratives, semiotic nodes* and *semiotic contraction* Radford (2002, 2006, 2008) and Radford et al., (2003) referred to algebra. These may not be directly relevant here but Radford's ideas behind the *semiotic narratives, semiotic* nodes and *semiotic contraction* can also apply to fractions. Hence I am going to discuss these three ideas as Radford presents them. In Radford (2006), he defined *semiotic contraction* as “reduction of signs and concentration of meanings” (Radford 2002). The term *semiotic contraction* is used in a situation where students used *letters to substitute nouns* in order to produce a symbolic expression, which would later lead to *symbolic generalization* (Radford, 2001, 2003, 2006)–which will be further discussed in the subsequent section.

Radford (2006) used the term *semiotic contraction* when he noticed that students started to operate with a “reduced form of expression,” for example, from an expression like “You add it by itself, like. You do 2 plus 2, then after this, plus 1, like. You always do this, right?” [. . .] You would do 3 plus 3 . . . plus 1, 4 plus 4 . . . plus 1, 5 plus 5 . . . plus 1,” to an expression like “You add the figure and the next figure.” In such a situation, students counteract the reduction of *semiotic resources* (words) by using those which have more intensified meanings, as in the above example, where the words “the figure” that the students used is a linguistic generic expression that refers to any term in the sequence, and not just one particular term (Radford, 2006). Prior to achieving the *semiotic contraction*, the students’ argument was just a narrative–telling us to go from one particular figure to the next, you have to add the figure and the next figure–which Radford (2002) termed a *semiotic narrative*. This instance, in which knowledge objectification is achieved, is what Radford et al., (2003), Radford (2005b, 2008), and Radford et al., (2006a) termed *semiotic nodes*: “pieces of the students’ semiotic activity where action, gesture and word work together to achieve knowledge objectification” (Radford et al., 2003, p. 56).

In the subsequent sections, discussion on the types of generalization (Radford, 2001, 2003, 2006) will be presented in order to make objectification clearer. Radford conjectured that there are three forms of *generalizations* with different layers of generality (Radford, 2003, 2006). The most elementary form of mathematical generalization is the *factual generalization* (Radford, 2006), which is achieved through actions. The second form of generalization is called the *contextual generalization,* which is achieved through language, and the third one is called the *symbolic generalization* and this is achieved through symbols (Radford, 2001, 2003, 2006). Discussion of the first type of generalization follows.

2.5.5 Factual Generalization

As mentioned earlier, *factual generalization* is the simplest form of generalization (Radford, 2006). In factual generality, he said:

> Indeterminacy remains unnamed; generality rests on *action* performed on numbers; *actions* are made up here of words, gestures and perceptual activities. (Radford, 2006, p. 16)

He further added that the objectification in this form of generalization will be in the form of *perceptual semiosis*, which he portrayed as:

> . . . a process relying on a use of signs dialectically entangled with the way that concrete objects become perceived by the individuals. In this process, the mathematical structure of the pattern is revealed and ostensively asserted by linguistic key terms in the students' utterances. (Radford, 2001, p. 83)

Consequently, in order for factual generalization to occur, face-to-face interactions are crucial (Radford, 2003). He also added that in order to achieve factual generalization words, gestures, and drawings are used in a co-ordinated manner. There are various means of objectification as illustrated in Radford (2003), and he conjectured that "some maybe powerful enough to reveal the individual's intentions and to carry them through the course of achieving a certain goal" (p. 50), but "some may be inadequate in more complex situations where greater precision is required." (p. 50)

Radford (2003) classified the SMO required to objectify *factual generalization* as follows:

a. Linguistic devices and signs–For students to make apparent their intentions, and to be able to perform their actions, an abstracting scheme can be made by using varied linguistic devices and signs. This can be done by using a co-ordinated use of words, gestures and drawings.
   For example, the term "the next" in the student's remark (Radford, 2003, p. 46), "The *next* figure has two more than . . ." is a central SMO in the student's factual generalization.

b. Linguistic devices, signs and indexical gesture–Similar to the previous ones, but this time it is accompanied by *illustrative concrete actions* (such as pointing gestures), in addition to using the adverb "always", which he reported elsewhere to emphasize the *generative functions of language.* In other words, this made it possible for students to express methods and actions which can be carried out reiteratively. For example, "It's *always* the next. Look! [and pointing to the figures with the pencil] 1 plus 2, 2 plus 3 [. . .]" (Radford, 2003, p. 46). These are what Radford (2003) described as spur-of-the-moment linguistic expressions that put across the notion of the abstracting scheme, which is the core of the generalization of actions.

c. Rhythm and movement–For this case there was no verbalization of the abstracting scheme; rather they depend on "*rhythm* of the utterance, the *movement* during the course of the undertaken numerical actions, and the ostensive correspondence between pronounced words and written signs" (Radford, 2001, p. 83). Rhythm complements gestures and words, and Radford (2006) described the sense of generality as a result of rhythm as:

> Rhythm creates the expectation of a forthcoming event (You, 1994) and constitutes a crucial semiotic means of objectification to make apparent the feeling of an order that goes beyond the particular figures. (p. 11)

For example, "O.K. Anyways. Figure 1 is plus 2. Figure 2 is plus 3. Figure 3 is plus 4. Figure 4 is plus 5 [the student pointed to the figures on the paper as she utters the sentence]" (Radford, 2003, p. 49)

The objectification of the first two types of SMOs mentioned above makes use of the *perceptual semiosis* and, finally, students' construct of meaning is:

> . . . grounded in a type of social understanding based on implicit agreements and mutual comprehension that would be impossible in a non face-to-face interaction. (Radford, 2003, p. 50)

Radford (2006) then asked two questions: "Why did the students gesture? Why did they not limit themselves to talking?" He answered that:

> Gestures helped the students to refine their awareness of the general. These gestures stood for the rows that *could not be seen*. Gestures helped the students *visualize* (Presmeg, 2006) and thereby came to fill the gap left by impossible direct perception. Generally speaking, gestures do not merely carry out intentions or information; they are key elements of the process of knowledge objectification. (p. 16)

Next, discussion on contextual generalization is presented.

2.5.6 Contextual Generalization

Unlike factual generalization, the operational scheme in contextual generalization is not based on face-to-face interaction, where both rhythm and ostensive gestures have also been excluded (Radford, 2001, 2003, 2006). Following this, Koukkoufis and Williams (2006) claimed that the operational scheme has to be replaced by a new operational scheme generalized solely through language and symbols. There are two elements which differentiate contextual from factual generalization: *the socio-communicative element* and *the mathematical element* (Radford, 2001, 2003).

<u>The socio-communicative element</u>

> Implicit and mutual agreements of face-to-face interaction (e.g., gestures, clue words) need to be replaced by objective elements of social understanding demanding a deeper degree of clarity in the communication. (Radford, 2003, p. 50)

In other words, contextual generalization is not based on face-to-face communication; rather it is done through a written explanation (Radford, 2003).

<u>The mathematical element</u>

This requires students to create procedures that “ask for any although nonspecific figure” (Radford, 2003, p. 51). In contrast to factual generalization, contextual generalization does not operate on the level of concrete numbers; instead, it appears as the abstraction of concrete actions in the form of an operational scheme. For example, the generic expression, “the figure” and “the next figure” enable students to attain a new level of generality, which includes an abstraction from actions and an abstraction from specific figures (like the fifth, sixth, etc) (Radford, 2003). Another way of saying is that contextual generalization is located at a deeper layer of generality (Radford, 2006) which involves the generalization of both actions and objects. Therefore, it can be said that the SMO for this type of generalization is achieved by using a linguistic generic term such as “the figure” to refer to “any given figure” in order to put forward the idea of generality (Radford, 2003). Another linguistic term used by students in Radford (2003) was “the next figure.” This is a locative term which highlights the attributes of the sequence of figures which appear relevant. Both the generic and locative terms behave in similar way to *deictic* (or “demonstrative”) terms, like *this* and *that* when used in everyday speech. However these terms are referring to nonmaterially present objects. He also described it as “nonsymbolically based type of generalizations, performed on conceptual, spatial temporal situated objects” (p. 54). Radford (2001, 2003) then raised the question:

> How then will the students proceed to produce meanings that will not rely on time and space–meanings that, so far, have resulted from their spatial temporal embodied situated experience? (Radford, 2003, p. 55)

In the next section, we will focus our attention on the “cognitive/semiotic dimension” (Radford, 2001) of what Radford (2001, 2003) termed the “*desubjectification* process”, which is a “process that emphasizes changes in the relation between the object of knowledge and the knowing participant” (Radford, 2003, p. 55).

2.5.7 Symbolic Generalization

An example of the *desubjectification* process, as described by Radford (2001), is the incorporation of a speech genre based on the *impersonal voice*, for example, ““Your first figure” in line 6 becomes “n” in line 8 and 9” (p. 85). Another example is the exclusion of “individual owning or acting on the figures” phrases such as “I add”, “you put”, and the general deictic objects, such as “*this figure*”, which are the basis of students’ previous mathematical experiences (Radford, 2001, 2003). However, the *desubjectification* process is harder than it seems because of what Radford (2001, 2003) termed “*the positioning problem”*–“related to the students’ difficulty in symbolizing “the next figure,” something that requires finding a way to forge a symbolic link between the figure and the next figure and their corresponding ranks” (Radford, 2001, p. 86). Based on the students’ discourse analysis, he suggested that:

> . . . to reach desubjectification and to end up with the objective kernel of the algebraic generalization, meaning has to be disembodied and become thus pure mathematical sense. (Radford, 2001, p. 87)

Radford pointed out that the difference in symbolization between symbolizing an unknown in say, a word problem involving a single unknown, and symbolizing the nth term of an elementary pattern, is symbolizing them according to their position (Radford, 2003). With regard to the SMO used in this type of generalization, unlike the two presymbolic generalizations, the linguistic relation between participant and object cannot be sustained. In Radford’s (2003) words:

The dual reference participant-object becomes lost . . . the students are deprived of indexical and deictic spatial temporal terms and have to refer to the objects in a different way. (p. 66)

He also found that, contrary to Kant's (1781/1996) second condition to attain knowledge–". . . the exclusion of time (that he related to logical necessity)"–the SMOs used by students in his study in achieving symbolic generalization were related to actions and movement, and hence to time. This can be seen in "Figure 4" in Radford (2003) (hereafter, Figure 2.8) where according to him ". . . the students' indexical signs are implicated in the numerical actions that they symbolize: that is why the whole structure of the symbolic expression captures and reflects the flow or movement of the numerical actions" (p. 66). In Figure 2.8, the sign *n* is an abbreviation of the generic linguistic term "the figure", which is embedded in the previous stratum. Hence, Radford's (2003) suggestion that, "the sign n can be seen as an index in Peirce's (1955) sense (Radford, 2000b, 2000c)" (p. 61). Radford believed, despite Peirce's terminology, that "letters have to become genuine symbols":

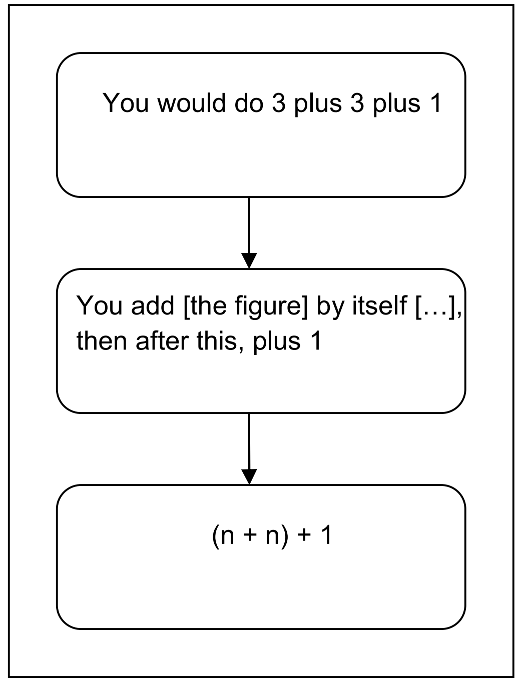


Figure 2.8. The additive numerical action

. . . students formulas often tend to simply narrate events and remain attached to the context. (Radford, 2006, p. 16)

He also strongly believed that participant-object plane is not epistemologically sturdy without the plane of *social interaction*.

This concludes the literature review of this thesis. In the next section an over view of this chapter is presented.

2.6 OVERVIEW AND KNOWLEDGE GAP

The review above indicated that studies carried out in Brunei Darussalam and in other parts of the world point towards three generalizations.

1. All over the world, most primary school pupils find some fractions difficult. If this is the case in the primary schools, then it needs to be recognized that students take with them inadequately developed fraction concepts when they move into secondary schools. In fact the study done by Suffolk and Clements (2003) revealed that even at the lower-secondary level, many students are experiencing serious difficulties with elementary fractions tasks. Thus, for example, less than 50% of Form 1 and 50% of Form 2 students give a correct answer to a pencil-and-paper task requiring them to find the value of $\frac{1}{2} + \frac{1}{2} = 1$.

2. In the teaching of fractions there is an overemphasis on the manipulation of symbols, without adequate attention being paid to conceptual understanding.

3. Although primary teachers tend to use the area model in their introduction to fraction concepts, insufficient attention is paid to other representations of fractions concepts. In particular, models of fractions of discrete sets (e.g., one-third of six objects) receive less attention in classroom teaching than might reasonably be expected.

The review revealed that there can be little doubt that education authorities in Brunei Darussalam need to pay serious attention to issues associated with the teaching and learning of fractions in the nation's schools. The need is magnified by the rather depressing performance data reported in studies reviewed in this chapter.

The end result of the narrow approaches typically taken to the teaching of fractions in primary schools is that upper-primary pupils cannot readily apply fractions concepts and skills to pertinent real-life situations. One may ask whether that situation improves as the pupil moves through into secondary school. This study can be probably be best seen, therefore, as part of an effort which needs to be made to improve the prevailing situation with regard to the learning of fractions in the nation's primary and secondary schools. The need for improvements is urgent.

From the above, I have formulated 3 hypotheses:

1. The first issue is that there is relatively less research being done in the area of fractions at secondary school. As mentioned earlier, more research, specifically in terms of equivalence and flexibility of unitizing, needs to be done for secondary school students.

2. The second issue is that there is an absence of any research done which involves RME or RME-like lessons in Brunei Darussalam. This will provide a platform for a curriculum reform as far as mathematics is concerned.

3. The third issue is that students' semiotic meaning-making processes are inadequately analysed as far as equivalence and flexibility of unitizing in fractions is concerned. The semiotic analyses hopefully will provide an analytical tool for conducting more detailed semiotic analyses on fractions, within the RME framework.

Based on these three issues which formed the knowledge gap in this study, the research framework (Figure 2.9) and research questions are presented in the next section.

2.7 RESEARCH FRAMEWORK AND RESEARCH QUESTIONS

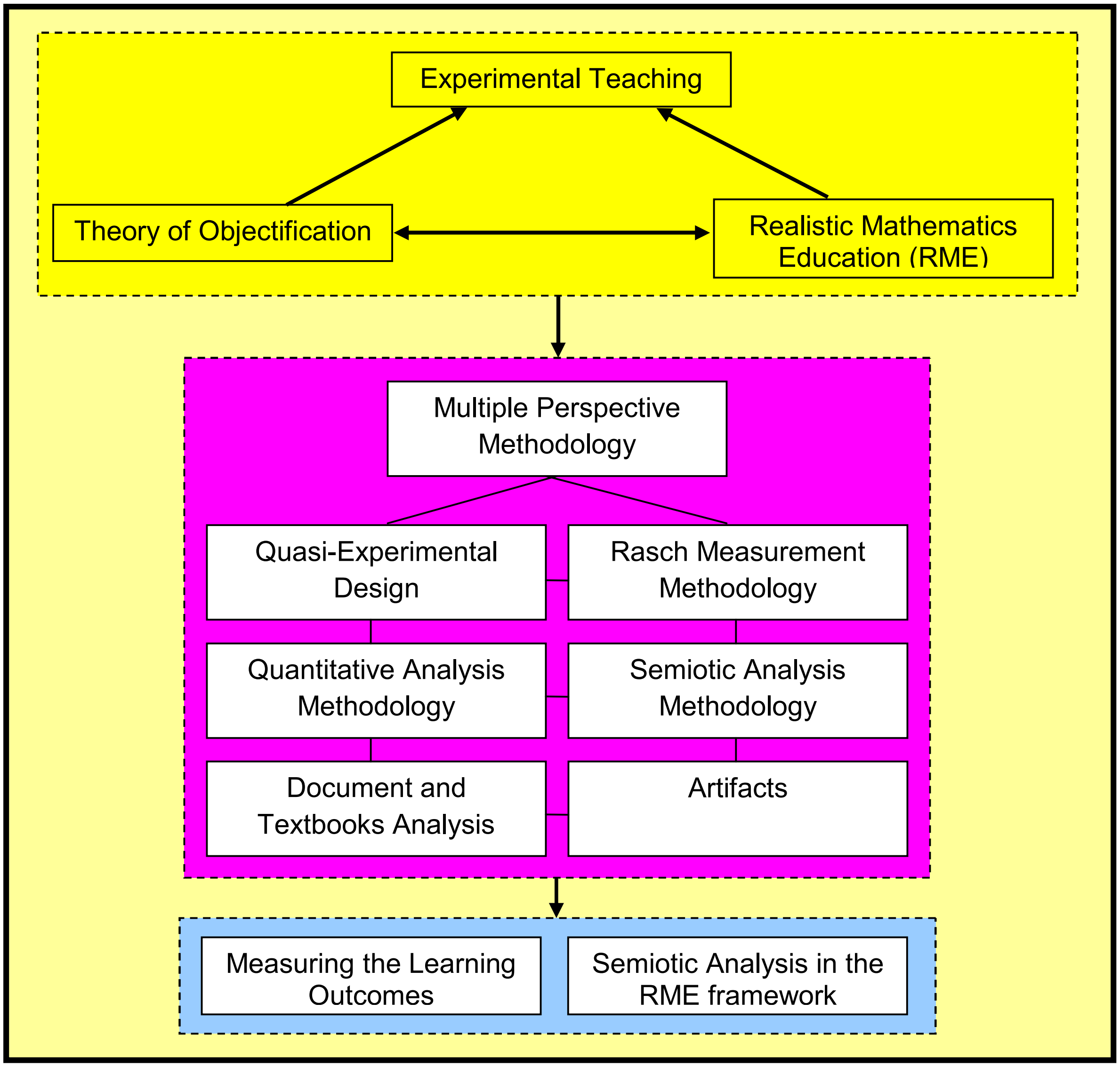


Figure 2.9. Research Framework

With respect to the topic "Fractions," for the Form 1 mathematics classes, the following are the two research questions (RQs) for this thesis:

RQ.1: *After the students were given lessons on fractions, inspired by RME or RME-like thinking with the introduction of context and models, how effective was it in helping them make sense of fraction, with respect to equivalence of fractions and flexibility of unitizing better.*

RQ.2: *What are the semiotic learning processes in the Experimental 1 group of the RME-like lessons?*

CHAPTER 3

DOCUMENTARY AND TEXTBOOK ANALYSIS

3.1 INTRODUCTION

In this chapter, an analysis of the relevant documents to ascertain the extent of fractions knowledge that the students are expected to have in Brunei is presented. I also considered it vital to examine the textbooks they are given to help them achieve these expectations. Additionally, a comparative study of these textbooks with another set of textbooks (the *Britannica Mathematics in Context*, hereafter “MiC”) which adopted the RME context was also done: to evaluate how different/similar the two sets of textbooks are. This will help in designing supplementary teaching inspired by RME OR RME-like thinking with the introduction of context and models. The MiC offers a teaching method that I hypothesize has been “missing” from the experience of the students–many of whom are having difficulty with fractions. Therefore, remedial, supplementary teaching would best be constructed to offer the students some of these learning experiences. However, prior to the presentation of the analyses, a discussion of the difference between RME and Gagné’s (1985) events of learning is presented. Thus, the respective sections are:

3.2 Analysis of Syllabus Documents: Analyses of official documents which defined those aspects of fractions which teachers were expected to teach at different grade levels is provided.

3.3 Textbook Analysis: A comparative analysis of the *Secondary Mathematics 1A* book and the MiC books is presented. A detail account of all the relevant sections, as far as fractions is concerned, is presented.

3.2 ANALYSIS OF SYLLABUS DOCUMENTS

In this section, details relating to the national Mathematics syllabus, for the upper primary (primary 4 to 6), and Form 1, documents in which the *intended* curriculum so far as fractions are concerned can be found.

3.2.1 Fractions in the Upper Primary Mathematics Syllabuses

<u>The Primary 4 syllabus on fractions</u>.

Here, “Fractions” is a major part of the Primary 4 Mathematics syllabus (Curriculum Development Department (hereafter, CDD) (2006), which consists of seven sub-topics. The topic is defined, for Primary 4, in the upper-primary Mathematics syllabus (CDD, 2006) and in the corresponding Teacher’s Guide (CDD, 1997) in the following way:

1. *Concept of fractions.* The objectives here are:
   (a) Pupils will be able to grasp “fraction” as “part of a whole;”
   (b) Pupils will be able to write the fraction for a determined portion of any uniformly divided object or form; and
   (c) Given a uniformly divided diagram and a fraction, pupils will be able to shade the appropriate portion of the diagram to represent the fraction.

2. *Simple fractions with denominators not greater than 100.* The objective is for pupils to be able to state and write fractions with denominators not greater than 100.

3. *Equivalent fractions.* The objectives are:
   (a) Pupils will be able to recognise equivalent fractions visually and numerically; and
   (b) Pupils will be able to convert any fractions into equivalent forms.

4. *Comparison of fractions.* The objectives are:
   (a) Pupils will be able to compare two fractions with denominators not greater than 10 by referring to a fraction chart; and
   (b) Pupils will be able to compare two fractions by the use of equivalent fractions.

5. *Addition and subtraction of fractions with common denominators.* The two objectives are:
   (a) Pupils will be able to add two fractions with a common denominator; and
   (b) Pupils will be able to subtract fractions with a common denominator.

6. *Proper fractions, improper fractions and mixed numbers.* The objectives are:
   (a) Pupils will be able to recognise and identify proper and improper fractions and mixed numbers; and
   (b) Pupils will be able to convert improper fractions into mixed numbers and vice versa.

7. *Simple word problems on fractions.* The objective is for pupils to be able to solve simple word problems on addition and subtraction of fractions [*Mathematics Teacher's Guide, Darjah 4*, CDD, 1997, pp. 14−30].

<u>The Primary 5 syllabus on fractions</u>.

At this level there are only five sub-topics for "Fractions" in the national Primary 5 Mathematics syllabus (CDD, 2006) and in the corresponding *Teacher's Guide* (CDD, 1998b).

The important statements in the documents are as follows:

1. *Addition and subtraction of fractions with different denominators.* The objectives are:
   a. Pupils will be able to convert any fraction into its equivalent forms (Revision);
   b. Pupils will be able to add two fractions with different denominators; and
   c. Pupils will be able to subtract fractions with different denominators.

2. *Simple multiplication and division of fractions.* The objectives are:
   (a) Pupils will be able to multiply a fraction by a whole number;
   (b) Pupils will be able to multiply a fraction by another fraction;
   (c) Pupils will be able to divide a fraction by a whole number; and
   (d) Pupils will be able to divide a fraction by another fraction.

3. *Fraction of a quantity.* The objective here is for pupils to be able to relate fractions of a unit to fractions of quantities.

4. *Expressing one quantity as a fraction of another*. The objectives are:
   (a) Pupils will be able to express one quantity as a fraction of another; and
   (b) Pupils will be able to compute a quantity when the fraction and the overall quantity are known.

5. *Solve word problems on fractions.* The objective here is for pupils to be able to solve word problems on fractions.
   [*Mathematics Teacher's Guide, Darjah 5*, CDD, 1998b, pp. 27−43].

<u>*The Primary 6 syllabus on fractions*</u>.

In the national Primary 6 Mathematics syllabus (CDD, 2006), there is no specific topic called "Fractions". Nevertheless, as specified in the Primary 6 *Teachers' Guide* (CDD, 1995), fractions concepts and skills are to be developed within the topic called "Numbers and the Four Operations." The Primary 6 Mathematics syllabus defines the following fractions-related subtopics and related objectives:

1. *The number line:* Addition and subtraction of numbers, including integers and fractions on the number line. Objectives that are related to fractions are:
   (a) Pupils will be able to carry out addition and subtraction of fractions on the number line; and
   (b) Pupils will be able to carry out addition and subtraction involving integers and fractions (including mixed numbers) on the number line.

2. *The four operations on numbers, fractions and decimals including the use of brackets.* Objectives that are related to fractions are:
   (a) Pupils will be able to carry out the four operations involving fractions as follows: (i) addition of fractions; (ii) subtraction of fractions; (iii) multiplication of fractions; and (iv) division of fractions; and
   (b) Pupils will be able to solve simple word problems involving fractions.
   [*Mathematics Teacher's Guide, Darjah 6*, CDD, 1995, pp. 12−25].

3.2.2 Fractions in the Form 1 Mathematics Syllabuses

It is just a sub-topic in the topic *Real Numbers* (not a topic in its own right), which the teachers need to cover within four weeks (CDD, 1998a). The sub-topics which are included in this topic are:

1. Idea of negative numbers and number lines;
2. Addition and subtraction of integers;
3. Multiplication, division and combined operations of integers;
4. Fractions;
5. Decimals and use of calculator; and
6. Squares, square roots, cubes and cube roots.

The following three headings are defined for sub-topic fractions:

1. Types of fractions;
2. Addition and subtraction; and
3. Multiplication and division.

The scope of the sub-topic fractions, as prescribed in the scheme of work for teachers, states the following:

1. Identify equivalent fractions and obtain a fraction which is equivalent to a given one;
2. Reduce a fraction to its lowest term;
3. Convert an improper fraction to a mixed number and vice versa;
4. Add, subtract, multiply and divide fraction concretely, pictorially and symbolically;
5. Give example of reciprocals and note that $\frac{a}{b} \times \frac{b}{a} = 1$ and that 0 has no reciprocal; and
6. Perform simple mental computation involving fractions such as the following:

$$\frac{1}{2}+\frac{3}{4};\quad 1-\frac{1}{3};\quad \frac{1}{2}+\frac{1}{3};\quad 4\times\frac{3}{4};\quad 3-1\frac{2}{3}.$$

In addition, there are also suggested activities for each sub-topic as stated in the scheme of work for teachers:

1. Use fraction pieces and fraction charts to show operations of fractions; and

2. Recall that to multiply two fractions; we multiply the numerators and the denominators respectively to get the product. To divide a number by a fraction, multiply by its reciprocal.

There are also other Form 1 Mathematics topics which require knowledge and use of fractions. These include (a) decimals and use of calculator (converting a faction into a decimal and vice-versa); (b) angle properties of polygons (exterior and interior angles); (c) perimeter and area (of a triangle, trapezium, etc.); (d) rate (speed); (d) proportions (direct and indirect); (e) percentages; and (f) statistics (mean).

In the next section, a comparative analysis of the textbooks used by the students, as compared with another set of textbooks which used context, is presented.

3.3 TEXTBOOK ANALYSIS

3.3.1 The Secondary Mathematics 1A (CPDD) Textbook

All Form 1 mathematics students in Brunei are provided with two textbooks, *Secondary Mathematics 1A* and *Secondary Mathematics 1B*, both of which are written by the Curriculum Planning and Development Division (CPDD), Ministry of Education, Singapore, and published by the EPB Pan Pacific, Singapore. Both textbooks are written in English, but all the contexts in the books are Singapore contexts. The unit on fractions is in the *Secondary Mathematics 1A* textbook (CPDD, 2007).

The *Secondary Mathematics 1A* textbook (hereafter, CPDD 1A) is part of a 10-textbook package written by CPDD Singapore. Both in Singapore and Brunei Darussalam, each year two of the textbooks are used from Form 1 to Form 5, and the whole series is specifically designed to prepare students for the O-level examinations from Form 1 through Form 5.

The CPDD 1A textbook has 215 pages subdivided into six “units”. The first two units deal with whole numbers and integers, whereas the last three units deal with decimals, measures and money, and an introduction to algebra. The unit on “Fractions” is the third unit in the CPDD 1A textbook which is covered comprehensively in unit 3. The topic occupies 41 pages of the textbook–pages 60 through 109. CPDD 1A has an accompanying workbook (CPDD, 2002) which provides a large amount of guided practice, and every pupil is expected to purchase this workbook (CPDD, 2002), as it will be used side by side with the textbook.

3.3.2 The Britannica Mathematics in Context (MiC) Books

The *Britannica Mathematics in Context* books is part of a three-level set of activity books: Level 1 consists of an 8-activity books package, Level 2 consists of a 9-activity books package, and Level 3 also consists of a 9-activity books package. For each level there is a textbook, which is in fact just the compilation of all the activity books, and with each activity book a teacher's guide book is also available together with a "teaching transparency" folder containing the master copy of the related transparencies. In addition, there are also two additional resources whose aims are to reinforce students' understanding: Number Tools Workbook and Algebra Tools Workbook. However, the workbook is not linked to the textbooks (unlike the CPDD book) per se: rather they are supporting resources to reinforce students' understanding of ratios, fractions, decimals, and percents, and the connections between these representations. For that reason, this book will not be compared with the CPDD books. The *Mathematics in Context* books were developed in collaboration with the Wisconsin Center for Education Research, School of Education, University of Wisconsin-Madison and the Freudenthal Institute at the University of Utrecht, The Netherlands, with the support of the National Science Foundation. They were designed according to the *Realistic Mathematics Education*. According to the "Letter to the Teacher" in the *Teacher's Guide* book,

> The term *realistic* is meant to convey that the contexts and mathematics can be made "real in your mind." Rather than relying on you (teacher) to explain and demonstrate generalized definitions, rules, or algorithms, students investigate questions directly related to a particular context and develop mathematical understanding and meaning from that context. (p. vi)

Unlike the CPDD textbooks, the MiC books do not specifically allocate one book solely for fractions, rather the building up of the concepts and formalizing the operations of fractions are spread out in six different books known as the *Number Strand* which emphasizes number sense, computations with number, and the ability to use number to better understand a situation: Models you can count on, Fraction Times, More or less, Facts and Factors, Ratios and Rates, and Revisiting Numbers. The first three are part of the Level 1 series, the next two in the Level 2 series, and the last book in the Level 3 series. Figure 3.1 (copied from MiC) below explained the sequence of the books and what each book will focus on. For the purpose of this analysis, only the first three books which are at the Level 1 series will be looked at, and compared with the CPDD 1A book, as these are at equivalent levels for students of equivalent ages.

The MiC *Models You Can Count On: Number* (Britannica, 2003b) book has 75 pages subdivided into five sections: The Ratio Table, The Bar Model, The Number Line, The Double Number Line, and Choose Your Model. In this analysis, section B, the bar model, has been compared with the first sub-unit, meaning of fractions, in the CPDD 1A book. Section C (The Number Line) has been compared against the sub-unit called order of fractions, whereas section D (The Double Number Line) has been compared with multiplication and division of fractions.

The MiC *Fraction Times: Number* (Britannica, 2003a) book has 57 pages subdivided into five sections: Survey Results, It Adds Up, Festival and the Decimal Connection, Ratios, Fractions, Decimals and Percents, and Fractional Parts. From this book, section A (Survey Results) has been compared with the order of fractions sub-unit, section B (It Adds Up) with addition and subtraction, and section E (Fractional Parts) was compared with multiplication of fractions.

The MiC *More or Less: Number* (Britannica, 2003c) book has 47 pages subdivided into four sections: produce pricing, discounts, many changes and more or less. There is no specific subunit on fractions in this book, but fractions are discussed together with decimals and percents in the second section. So this section has been compared with mental computation, and estimation in the CPDD 1A book.

Pathways through the Number Strand
[Arrows indicate prerequisite units]

Level 1	**Models You Can Count On**	▪ Builds on students' informal knowledge of ratios, part-whole relationship and benchmark percents. ▪ Informally add, multiply and divide benchmark fractions. ▪ Focus on four models: 1. The Ratio Table 2. The Bar Model 3. The Number Line 4. The Double Number Line. ▪ The *Double Number Line* model allows students to make accurate calculations and estimates for all sorts of ratio problems especially where real numbers are involved.
	Fraction Times	▪ Makes connections and builds on the models, skills and concepts that are developed in the previous unit. ▪ Further develops and extends students' understanding of relationships between fractions, decimals and percents. ▪ Bar models and pie charts are used to make connections between fractions and decimals. ▪ Bar models or ratio tables are used to compare, informally add and subtract, and simplify fractions. ▪ To informally multiply fractions when students calculate a fraction of a fraction by using fractions of whole numbers. ▪ Addition and subtraction of fractions are formalized.
	More or Less	▪ Formalize, connect and expand students' knowledge of fractions, decimals and percents. ▪ Use benchmark fractions to find percents and discounts.
Level 2	**Facts and Factors**	▪ The area model is developed and used to increase students' understanding of how to multiply fractions and mixed numbers.
	Ratios and Rates	▪ Focuses on the connections between rational numbers and percents.
Level 3	**Revisiting Numbers**	▪ Solve context problems where operations, multiplication and division, with fractions and mixed numbers are involved. ▪ Solve these experiences to solve non-contextualized problems by thinking of a context that fits them.

Figure 3.1. Sequence of the *Britannica Mathematics in Context* books

3.3.3 Comparative Analysis of the CPDD and MiC Textbooks

A comparative analysis of the CPDD 1A and the MiC texts were done based a textbook analysis schedule developed by Haggarty and Pepin (2002) (Appendix 2). This schedule was chosen because I felt that it provides a suitable analytical tool to map out any similarities (if any) and differences between the two types of textbooks. The seven aspects which are adopted from this analysis schedule are as follows:-

1. Methodology [Pedagogy];
2. Relevance [Context];
3. Connectedness;
4. Representation [Models];
5. Reinforcement;
6. Known Misconceptions; and
7. Cognitive Level.

This schedule helped map out the type of pedagogy used in the textbooks, for example by asking if more than one method is presented in text, or is the method presented the only method used. The second aspect is of great importance to this study as it help map out how far the activities in each of the textbooks used context which are experientially real in order to help students understand the concepts better. As for the third aspect, it helps to highlight if there are any evidence in the textbooks that the new knowledge are built upon existing knowledge. The next aspect of the schedule helps to map out the type of models used (if any), and if the students identify with the models. This is important as models which are not experientially real for the students might hinder concept development. The way the students' understandings are reinforced is also an important aspect to look at, as it will provide a clear indication to the depth of students' understanding of the concepts tested. In addition, this schedule also helped map out if there is any provision given in the textbooks to avoid common misconceptions. The last aspect is one of the most important as it helps analyse the content knowledge where Bloom's taxonomy of cognitive domain was used to presumably reflect students' behavior. From this classification of level of difficulty, it would be easier to map out the cognitive demand that each textbooks imposed on the students.

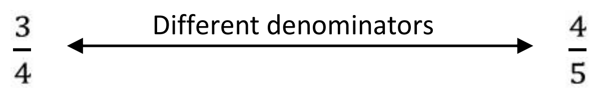
A comparative analysis was done between the CPDD 1A book and the Mathematics in Contexts books based on the above aspects (see Appendix 3 for details). The individual sub-units of the unit on Fractions of the CPDD 1A were compared to a similar sections or sub-sections from one of the MiC books, as fractions were not discussed as a topic on its own exclusively, but as part of other connected knowledge such as decimals, percents and ratios. Unfortunately not all the sub-units of the unit on fractions in CPDD 1A has an equivalent section or sub-section in the MiC books. By the end of the comparison of the two types of texts, four out of ten sub-topics do not have a similar section or sub-section in the Level 1 MiC books: Types of fractions, Reducing a fraction to its lowest terms, Using the calculator, and Word problems.

The following is the summary of the above analysis based on the seven aspects mentioned earlier;

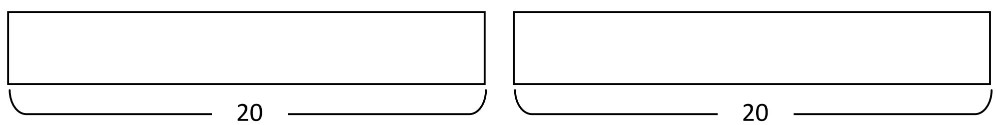
Methodology [Pedagogy]

In the CPDD 1A textbook, all worked examples only presented one solution for each type of task, unless we consider using diagrams and symbolic representations as different methods of representing answers. In the accompanying workbook, however, the last two activities were additional activities that students can do on their own which encouraged students to explore methods to find a fraction between two given fractions, and to express a given non-unit fraction as a sum of different unit fractions. Both of these activities were not even discussed in the textbook. In contrast to CPDD 1A, the MiC books, in the section "Answers to check your work" at the back of the book, emphasized that there are many different strategies to solve the given problems, and what the books have presented are just a sample of the possible strategies that students can use to solve the problem. In addition, the intended concepts were presented through a series of exploratory exercises where students were exploring their informal methods of solving problems. For example, to discuss fractions as equal parts of a whole in the *School Garden* section, a series of exploratory exercises of dividing the plot into equal parts were done in a fair-share context before they were presented with a $\frac{1}{4}$, and a relationship such as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$, as opposed to presenting students with $\frac{5}{8}$ in one paragraph without any supporting activity apart from the given circle with eight equal parts with 5 out of 8 parts shaded.

In terms of commentary on the mathematical content, some explanations were provided alongside the examples, for example, in the "Reducing a fraction to its lowest terms" sub-unit, in one of the examples, to reduce $\frac{54}{72}$ to its lowest terms, the text presented it using a cancellation method, and there was a comment at the side to explain that 54 and 72 have a common factor of 2, and 27 and 36 have a common factor of 9, hence the final answer $\frac{3}{4}$. In the MiC books, comments on the mathematical content were also provided at some of the stages, for example, in the MiC *Fraction Times:* Number book in section B, to compare $\frac{3}{4}$ and $\frac{4}{5}$ the following explanation was presented;



You might draw two bars and imagine that each has 20 segments.



but mostly students' thoughts were directed in the right direction, to question and find out why certain things are the way they are, and it was presented in ways that are familiar to students, not in the standard school mathematics way. For example, students can be purposely asked to confront a common misconception, as in the case in the *produce pricing* sub-unit of the MiC *More or Less: Number* book;

> Paul calculated the price of 0.8kg of Red Delicious apples at Save Supermarket. He used his calculator and made these entries.
>
> 0.8 x $2.40 =
>
> His calculator displayed 1.92 as the total.
>
> Mary disagrees, and said "That can't be right! When you multiply, isn't the answer always larger than the two numbers you started with?"

Students were asked to reflect on this scenario, and they have to decide who is right, Mary or Paul, and why. Even though this example is on decimals, it could easily be translated for fractions also. Another example can also be found in the MiC *Fraction*

Times: Number book, in section A, where students were asked: “Which is more: 7 out of 30 or 11 out of 45? Explain”.

Apart from the title of the unit or sub-unit, students were not informed explicitly at the beginning of the unit what are the objectives of the unit and what they are going to learn in the unit titled “Fractions” in the CPDD 1A. At the beginning of the unit, however, there was supposed to be an introductory activity where students were to view the “ETV” programme entitled “Fractions”. According to the information for the students in the “preface” section, in each unit there will be, apart from other things:

> *An introductory activity which will help you understand why or how the topic is relevant to you.*

Unfortunately, the “ETV programme entitled Fractions” was not available for analysis. On the other hand, in the MiC books, even though the objectives were not stated explicitly, at the beginning of each book it explains what types of knowledge are going to be discussed in the book, and how the knowledge is going to be presented to them, in addition to what is expected from them.

Relevance (Context)

The contexts used in the CPDD 1A book are familiar situations, but then even if the situations are familiar, this does not mean that the students have experienced them before. For example in one of the examples for word problems, the context was about the number of pupils who were absent on a particular day for the whole school. It was a familiar situation, but it would only be experientially “real” for them in the context of their own class, and when it was about the whole school it could create some problems for some students as they would not know intuitively if their answer was correct or not as they never have to know how many pupils were absent for the entire school. Only six sub-units have contextual models, either both in the explanation part and the exercise, the explanation part only, or the exercise only.

- *Meaning of fractions* – for the introductory part a group of children, a collection of coins, and cakes were used, while for the exercise, for three of the questions, the contexts used were a box of doughnuts, a class of pupils and a group of orchid plants.

- *Equivalent fractions* – A fraction chart was used for the activity as part of the explanation part.
- *Order of fractions* – Number lines were used both for the introductory part and the exercises.
- *Estimation* – the context used for the introductory part was a group of students.

From the above we can see that the contexts were mainly limited to the introductory part, and when it came to the exercises, very few contexts were used at all, except for the sub-unit word problems. Some of the contextual models did encourage students to use their intuition, as in the cases of the cakes context. In the text however, for every type of knowledge presented only one solution was presented and there was no mention of the possibility of other solutions, or that the solution presented was just one of the many possible strategies that can be used. All the contexts presented in the textbook were familiar situations, but not all of them may have been directly experienced by the students. Examples of such cases are:-

~ Salary – the students may be familiar with this idea, but they have never actually dealt with it directly, so they will not have any *feeling* whether their answer was right or not; and

~ Spending – even though the concept of spending is familiar to them, asking a student to find the <u>fraction of money</u> left may not make sense to them.

To support the explanation presented in the text, there are six activities altogether in the accompanying workbook, but only four of the activities are part of the introductory activities for sub-units:

1. Meaning of fractions – Activity 1: Dominoes game;
2. Equivalent fractions – Activity 2: Fraction chart;
3. The four operations (Multiplication) – Activity 3: Diagrammatic representation of multiplication; and
4. The four operations (Division) – Activity 4: Reciprocal.

The other two activities are additional activities which students can do on their own. Only the first two activities use contextual models.

By looking at the activities, the assumptions that can be made were that they were designed to help students understand the following concepts better:

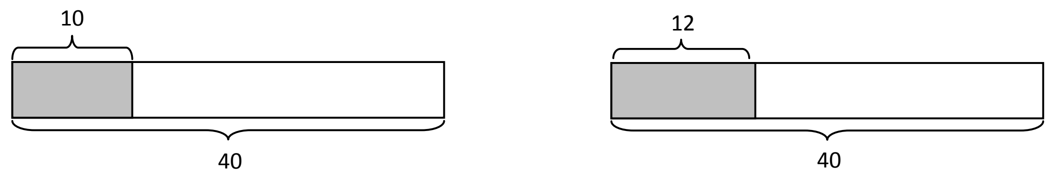
- For the dominoes game (Activity 1), the objective was to reinforce students' understanding of the fraction symbols, word fractions and diagrammatic representations of fractions;
- For Activity 2, the fraction chart, it aimed at illustrating equivalence and the different ways of writing equivalent fractions;
- Activity 3 demonstrated diagrammatically multiplication of two fractions, and hence arrived at the rule of multiplication of fractions, which equals the product of the numerators divided by the product of the denominators; and
- Activity 4 showed that when a number is multiplied by its reciprocal the product is equal to 1.

As mentioned earlier, the two contexts used in the two activities may be familiar to the students, but they are not related to their out of school experiences. Dominoes are not normally played in Brunei by students at home, nor is a fraction chart used in their daily activities. There were no instructions on how to carry out the activities (unless the teacher gave them explicit instructions), except for the dominoes game where 2–4 students can play the game at a time, and for the last two additional activities, where there was instruction that students can work on these activities on their own, this was stated in the textbook, not in the workbook.

In the MiC books, context was used, for example, the "*What teens are interested most in"* problem. The whole context of the problem should be familiar to the student. In the MiC *More or Less: Number* book, for example, at the very beginning of the book, the authors try to connect or relate to students' experiences by posing questions that would set students' minds in the directions they wanted:

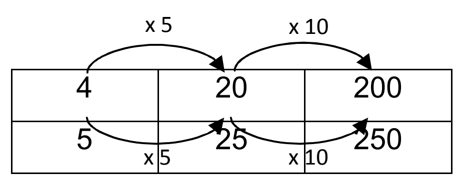
~ Do you purchase items that need to be weighed?
~ Do you buy your favorite items on sale?

The contextual models used in the MiC books are based on real life experiences. Unlike in the CPDD 1A, the context covered the whole section, from the introduction to the questions which are based on the context. In fact, the introductory part was also to introduce the context for the problem to be solved. In addition, the contextual models encourage students to use their intuition, as in one example, in a context of *investigating the amount of real fruit in two brands of apple juice,* in order to move from segmented bar model to non-segmented bar models of equal lengths where estimates are used to shade the bars. For this students know intuitively that 10 should be less than 12, this also leads to comparison of fractions with unlike denominators mentally, and hence to subtraction of fractions.

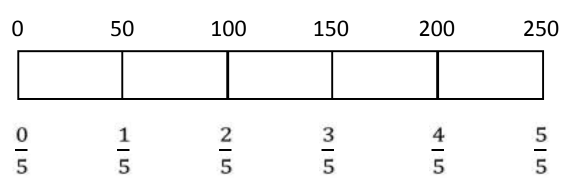


Apart from that, the context used may be the same, but it may be used for different situations and different strategies, for example in the MiC *Fraction Times: Number* book to get the value for $\frac{4}{5}$ of 250 kg, students can explore different strategies (number tools) to deal with such tasks:

- $\frac{4}{5}$ of 250 kg, by making use of the fact that $\frac{1}{5}$ of 250 kg is 50, so $\frac{4}{5}$ of 250 kg will be 4 x 50 which is 200kg, or
- Using a ratio table, or



- Using a fraction bar.



Even though the text does not explicitly encourage students to check whether their answer made sense or not, there is a part where students were asked to reflect on what they have done, or to justify their answer.

In the MiC books, there was only one activity similar to those in the CPDD 1A book, which is presented in the MiC *Fractions Times: Number* book for section A (Survey Results). The activity is called "Favorite colours–From Bar Charts to Pie Charts." The purpose of the activity is to demonstrate how a segmented bar model can be transformed into a pie chart to represent the same data. On the other hand, the tasks described in the books are one big activity by itself which was designed to develop understanding of the concepts; supporting activities using the prepared work sheets were also presented to help the development of the concepts;

- In the MiC *Models You Count On: Number* book for the sections that were looked at, seven sub-questions involved using activity sheets:
  - ~ The Bar Model
    - Activity sheet 3 consisting of drawings of 6 "strips of paper" to be cut, and a diagrammatic representation of the garden plots. Questions 2 and 3 used these strips to divide into the number of equal pieces indicated, and also to label the "garden plots" accordingly. Question 4 used this activity sheet for students to create their own problem and present the answer on the diagram;
    - Activity sheet 4 consisting of drawings of the water gauge on the four different days. Question 8 used this to show the water level indicated for that day, whereas question 11a used it for writing the answer as a fraction next to the shaded area of the gauge; and
    - Activity sheet 5 consisting of diagrams of the gauge for the five different-sized tanks. Question 13 used it for writing the answer as a fraction next to the shaded area of the gauge.
  - ~ The Double Number Line
    - Activity sheet 11 consisting of an empty grid of the city map, and an empty double number line model. Question 8 used this to show the location required.
- In the MiC *Fraction Times: Number* book for the sections that were looked at, seven sub-questions involved using activity sheets:
  - ~ Survey Results
    - Activity sheet 1 consisting of empty segmented bars, and question 3 used this to show survey results;

- Activity sheet 2 consisting of empty segmented bars with more bars, and question 6a used this to represent survey results for different-sized classes; and
- Activity sheet 3 consisting of pie charts, and question 7c used this as a comparison to check students' answers.

~ It Adds Up

- Activity sheet 2 (same as above) was used by questions 3b to show the result of survey in a bar chart, question 4c to determine fraction, question 5a and 6 to summarize the survey information using a bar chart.

▪ In the MiC *More or Less: Number* book for the sections that were looked at, only one sub-question involved using activity sheet:

~ Discounts

- Activity sheet 1 consisting of empty circles and segmented bar model was used by question 2a for displaying survey results using the segmented bars and pie chart.

Connectedness

In both texts, there was no mention at all of what pre-requisite knowledge students need to have before they can really understand the unit. In the CPDD 1A, prior to learning fractions, they were supposed to have covered two major topics; whole numbers and integers, so the assumption was that by the time they are doing fractions, students have learned about the four operations, and properties of the four operations. Unfortunately, as far as justification or proving for the given rules are concerned, apart from justifying the rules using diagrammatic representation, and examples, no other strategies were used. For the entire unit, there were only three instances where existing or previously learned knowledge was explicitly mentioned as the basis of the new knowledge:-

~ In the "Reducing a fraction to its lowest term" sub-unit: "You learnt that $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$ and $\frac{6}{12}$ are equivalent fractions. In what way is $\frac{1}{2}$ different from the other equivalent fractions?"

~ In the "Order of fractions" sub-unit:

- "Recall that we can arrange integers in order with the help of a number line…;" and

- "You know that any integer on the number line is greater than integers to its left. This is also true for fractions…"

In addition, there are only two instances where there was any mention of topics (past or future) that are related to the topic learned, and both referred to the same previously learned topic (i.e., integers).

In the MiC books, as far as justification is concerned, they did not explain specifically the reasons why the rules or strategies work, or not, rather the students were supposed to find out for themselves through the carefully planned and designed contextual activity, for example, in the *discounts* sub-unit, a situation was described to encourage students to use benchmark fractions to do mental calculation:

> Ms. Vander told Mr. Loggen that $33\frac{1}{3}$% of 180 customers wish Save Supermarket would carry a wider variety of apples. Without a calculator, Mr. Loggen quickly figured out that $33\frac{1}{3}$% of 180 customers is 60 customers.

Then, students were asked to answer the following questions:

1. What strategy do you think Mr. Loggen used to find the answer?
2. List percents that are easy to rewrite as fractions. Include the corresponding fractions.

   Fractions like $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{10}$ are often called *benchmark fractions.*
3. Show how you can use benchmark fractions to calculate each of these percent problems.

   a. 25% of 364 d. 5% of 364 g. 20% of 364

   b. 75% of 364 e. 30% of 364 h. 80% of 364

   c. 10% of 364 f. 35% of 364

For the MiC books, there were five phrases that referred to previously learned knowledge that students could use to help them deal with the given task:-

- In the MiC *Models You Can Count On: Number* book,

- ~ "You can use what you know about fraction strips to order different fractions on a number line."

- In the MiC *Fraction Times: Number* book,
  - ~ "Ratio table can help you determine the number of segments to include in the bar;"
  - ~ "To compare these data you can create bar charts that have the same number of segments;"
  - ~ "You can also use ratio tables to compare $\frac{3}{4}$ and $\frac{4}{5}$;"
  - ~ "You can use ratio tables to add fractions with different denominators."

Representation

There were some drawings used to represent fractions that could be found in the CPDD 1A book, for example, drawings (a group of children, some coins on a table, round cakes) were used in the "meaning of fractions" sub-unit to explain the different ways of interpreting fractions. There were also drawings of children "thinking" used in the "mental computation" sub-unit to indicate that they were supposed to do the calculations in their heads instead of using paper and pencil. In order to highlight important points, steps, or statements, they were presented using (rather "within") a diagram of a blackboard, and rules were presented in a blue "cloud". The representations were logically placed, beside the argument or examples, whereas the blue "clouds" were placed before the worked examples, and they were things that were familiar to the students. For the contextualizing models, the models used in the CPDD 1A book were: dominoes game, sharing, number line, fraction chart, partitioning, spending expenditure, and discount. For the non-contextualizing models, the models used were the geometrical shapes (circles, rectangles, triangles, square, hexagon, and parallelogram), with circles and rectangles as the most commonly used models, and also a group of items. The textbook is consistent in its usage of the non-contextualizing models, in the sense that circles were always used first before the rectangular shapes. The models were mostly used at the introductory part of the sub-units to show the number of shaded regions, equivalence, the position of a fraction (number line), and addition of fractions. In the CPDD 1A book, no explicit instruction was given for students to use models in their answers, but generally, the students could if they wished, even though it was not expected of them.

For this unit of fraction, at least, the models used are models that the students can identify with, except for the dominoes game which may not be such a familiar game amongst Brunei children. Also, no alternative strategies were suggested apart from the methods presented in the text.

As expected, the representations in the MiC books are things that are related to the topic, and the placement is also logical. They are mainly represented as photographs, and sometimes drawings of objects being described in the texts. Students were continuously reminded that strategies presented in the text are just a sample of the possible solution, and students may use and choose any valid strategies they wish. For example for the problem mentioned above, there are three different strategies that students can choose from to arrive at the solution;

- Using the fact that $\frac{1}{5}$ of 250 kg is 50, so $\frac{4}{5}$ of 250 kg will be 4 x 50, or
- Using a ratio table, or
- Using a fraction bar.

For the *Number strand* MiC books, the models emphasized are the ratio table, the bar model, the number line, and the double number line.

Reinforcement

In the CPDD 1A book, for the unit on fractions, the only reinforcement activity was routine practice exercises. Since all of them are practice exercises, this drills the students on the procedure of how to answer standard questions, and as for helping them to understand the concept, it was not directly obvious. From the appearance of the exercises, it can be assumed that students are expected to do all the tasks, unless their teachers instructed them otherwise. As for the range of numbers used to illustrate fractions in the CPDD 1A book, the fractions used for the diagrammatic representation were halves, thirds, quarters, sixths, eighths, twelfths and fifteenths. For the number line, the fractions used were between -2 and 2, whereas for other procedural examples, fractions with denominators less than 100 were used. Benchmark fractions were also used at the introductory explanation for all sub-units.

In the MiC books, at the end of each sub-unit, there are two sections: "Check Your Work" and "For Further Reflection"; both are an extension of the activity they did but with more cognitive demand. In addition, there is also a section at the end of the text called "Additional Practice" with more questions for students to try which are related to the context.

Known Misconceptions

Unlike in the MiC books, there were no provisions given to avoid common misconceptions, nor special explanations or attention given to parts that are prone to misconceptions so that they can be minimized. Contrary to the approach in the MiC books, students are trained to be able to take risks, analyze their errors and inadequate strategies. Students are also encouraged to confront their misconceptions, for example, in addition, the common misconception of adding the numerators and the denominators. Instead of re-explaining what it should have been, one method used was to ask students to prove that it is not equal to the sum of the numerators over the sum of the denominators, for example, $\frac{1}{2}+\frac{2}{3}\neq\frac{3}{5}$. Some section closes with a reflection which addresses a common misconception in multiplication of fraction, for example,

> *Your parents ask what you learned today in math class, specifically how you can compare two fractions that have different denominators. Write what you will tell them.*
>
> *"When you multiply two numbers, the product is always larger than either of the factors." Tell if this statement is true or false. Explain your reasoning.*

Other examples were also mentioned earlier under methodology.

Cognitive Level

With reference to Bloom's taxonomy, the questioning technique and their frequencies used in the CPDD 1A are as follows, according to my interpretation:

- <u>Explanation & Examples</u>
  - ~ *Knowledge* – what (3), write (2), find (6), give (1), pick (1), reduce (1), evaluate (7), how much (3), and how many (1);

- ~ *Comprehension* – draw (1), represent (1), indicate (1), arrange (1), simplify (1), is it... (1), and can you find more… (1);
- ~ *Comprehension/Application* – show (1); and
- ~ *Application/Analysis* – compare (1).

- Exercises
  - ~ *Knowledge* – what (12), write (2), find (2), give (1), pick (1), evaluate (4), express (1), state (1), rewrite (1), how much (2), how many (4), and how long (1);
  - ~ *Comprehension* – draw (1), arrange (2), and can you think of… (1); and
  - ~ *Comprehension/Application* – study (1), check (1), and find out if… (1)

- Workbook
  - ~ Exercise
    - ✓ *Knowledge* – what (2), write (3), find (6), pick (3), evaluate (3), complete (5), work out (1), and fill in the blanks (4).
  - ~ Activity
    - ✓ *Knowledge* – write (2), find (3), state (1), and complete (4);
    - ✓ *Comprehension* – compare (2) and what methods (1);
    - ✓ *Compare/Application* – study (1), compare methods (1), and check methods (1);
    - ✓ *Application* – game (1);
    - ✓ *Analysis* – what do you notice (3); and
    - ✓ *Synthesis* – what other methods do you used (1).

Table 3.1 below gives the overall summary of the cognitive levels of the four different class activities that are prescribed in the CPDD 1A textbook, pertaining to fractions.

Table 3.1
Summary of the cognitive level of the CPDD 1A Textbook

	Explanation & Examples	Exercises	Workbook	
			Exercises	Activity
Knowledge	25	35	27	10
Comprehension	7	4	0	3
Application	0	0	0	0
Comprehension/Application	1	3	0	0
Application/Analysis	1	0	0	1
Analysis	0	0	0	3
Synthesis	0	0	0	1
Evaluation	0	0	0	0

Almost all of the performance outcomes expected from the students are the straight forward types that use action verbs such as express, evaluate, write, find, arrange, state, rewrite, give, what fraction, and how much. There was only one instance where students were asked to find other possible answers: "The square below is divided into 4 equal parts (*the square was divided using diagonal lines*). Can you think of any more ways of dividing the square into 4 equal parts?" There were two instances where students were asked to study a given *worked solution* and they had to decide if it is correct or not, and to give the correct solution if it is not.

The only part that had any hint of encouraging students to do a bit of questioning and justification was at the beginning of the unit in the "meaning of fractions" sub-unit, where a square had been divided into four unequal parts with one part shaded, and the students were asked "can we say that $\frac{1}{4}$ of the square is shaded? Why?" In the sub-unit "The four operations", for division, two statements and a question was presented: "I am zero. I have no reciprocal. Why?" to probe students' thinking, but no follow-up was given after that.

In the MiC books, there were no explicit examples of model answers that students were expected to imitate strictly; rather they were guided through the discussion in a smooth logical way while doing certain tasks along the way. The questioning techniques for the different books were in the following categories, based on the author's interpretation:

- In the MiC *Models You Can Count On: Number* book, for parts in sections B, C and D that were looked at,
  - *Comprehension* – explain (1), use (7), label (3), what (1), describe (1), draw (1) and which (1);
  - *Application* – write (1), how many…write (2), what…write (1), and estimate (1);
  - *Analysis* – explain (2), how many / far (5), what…write (1), describe (2), which (2), draw (1), write (1), what (1), how many…show (1), find (1), and use (3);
  - *Synthesis* – choose (1), use…describe (1), show (1), make your own (1) fill in (1), how many…how did (1), and give reasons (1); and
  - *Evaluation* – reflect (1).
- In the MiC *Fraction Times: Number: Number* book, for parts in sections A, B and E that were looked at,
  - *Knowledge* – what types (1), how many (2), use (1), and find (1);
  - *Comprehension* – summarize (3), show (3), whose (1), explain why/how (4), compare (1), complete (1), write (1), how many/much (3), do…explain (1), what (2), choose…make (1), use (1), and describe (1);
  - *Application* – complete (1), draw (3), how many (2), name (2), show (1), use (6), create (1), and find (1);
  - *Analysis* – what (8), write (1), compare…how (3), which…explain (2), which (6), how many (5), make comparison (1), and explain how (1);
  - *Synthesis* – how do…conclusion (1), name (2), how (2), write (2), what (1), and why (1); and
  - *Evaluation* – explain (1), summarize (1), compare (1), and reason (3).
- In the MiC *More or Less: Number* book, for section B only,
  - *Comprehension* –describe (2), complete (1), list (1), what strategy (1), Do…explain (1), and display (1);
  - *Application* – show how (1), and draw (1);
  - *Analysis* – describe (1);
  - *Synthesis* – Which type…explain (1), and can…explain (1); and
  - *Evaluation* – reflect (1).

Table 3.2 below summarizes the cognitive level for relevant sections in the three MiC books that have fractions in them.

Where applicable students were always encouraged to describe any additional strategies they might have. At some stage of each section, there will be a reflection stage where students are asked to justify their answer, reflect on what they have done, and so forth.

Table 3.2
Summary of the cognitive level of the MiC book–Models You Can Count On, Fraction Times: Number, and More or Less

	Models You Can Count On (Section B, C & D only)	*Fraction Times: Number* (Sections A, B & E only)	*More or Less* (Section B only)
Knowledge	2	3	0
Comprehension	15	23	7
Application	5	17	2
Analysis	20	27	1
Synthesis	7	9	2
Evaluation	1	6	1

3.3.4 Overview of the textbook analyses

From the above analyses, in relation to the two prescribed textbooks for Brunei, the situation is undesirable when compared to the MiC textbooks. In addition, the prescribed textbooks provided to schools are written by Singaporean authors, for Singaporean students. The analyses also revealed that the predominantly used model in the prescribed textbooks is the area model of fractions concepts. Except for word problems, very few contextualizing models are used to explain or illustrate fractional concepts to students (see Appendix 4 for samples from CPDD books). Even when or if some kind of model is used, this does not extend much further than simply describing the problem itself, and when model solutions are given symbolical procedures dominated the worked solution. It seems that the theoretical framework behind the design of the CPDD books is in line with Gagné's nine *Events of Instruction* (Gagné, 1985, see Appendix 1) in mind.

Similar to Gagné's theory, it seems that context in this textbook is not the central focal point. In the words of Richey (2000), *transfer of training* from Gagné's point of view is:

> … a function of the extent to which a learner has:
>
> • The required prerequisite knowledge and skills
>
> • The ability to recall prior learning, and
>
> • Developed those cognitive strategies appropriate for the task. (p. 263)

This is evident for the Form 1 Singapore textbooks which are being used. The emphasis seemed to be to *train* students to be automated in responding to problems; a certain prescribed procedure for a certain type of problem, just like a specific medicine for a specific illness. One may, however, argue that even though Gagné's first function of transfer of training did not focus on context as part of the actual instruction, the other two functions are in fact employing contextual elements. Recalling relevant prior knowledge, as is also evident in the CPDD 1A textbook, is what Richey (2000) referred to as *contextual anchoring* to previous lessons which would reinforce meaning in the present instruction. Moreover, it would develop cognitive strategies in the generalization and transfer of knowledge to new problems or situations.

Students will produce the same answer with the same (if not exactly) procedures i.e. closed situation. This is a superficial context that will lead students to produce answers that have little to do with the reality of the situation described i.e. the context seemed to be *lost* within the procedures. In contrast, in the MiC books, even though students are working on the same problem, they will have different strategies and different ways of dealing with the problems i.e. open situation. The context is close to students' experiences, which allow them to come up with a variety of models of the real world. This is what the RME advocates where students are at the helm in the process of acquiring new knowledge, based on their informal knowledge, as they develop the strategies themselves with the support of the teachers to mathematize their informal knowledge (Gravemeijer, 1993).

Generally, the cognitive demands of the two textbooks are also markedly different, the MiC books designed activities which are geared more towards the comprehension, application, and analysis levels of Bloom's taxonomy (see Appendix 5 for samples from

MiC books), whereas the CPDD 1A is oriented more towards a recall type of knowledge. This is reflected in the way the two books are presented; there are not many contextualizing models being use in the CPDD book as compared to the MiC books.

From the above analysis, I conclude that the CPDD 1A textbook is designed to prepare students to transfer skills in a sequential cumulative fashion, where less complex skills can be transferred to more complex skills, which is what Gagné is advocating– generalizing simple skills to other situations. Here the teacher controls the direction and pace of the lessons. The MiC books however are more exploratory, where students are encouraged to investigate, construct, and discuss the possible solution(s) to a problem. In other words, it is what Gravemeijer (1993) described as a process of guided discovery, where the students are more in control of the learning processes. Evidently, there is minimal context presented in the CPDD 1A textbook which might be one of the reasons why Brunei students are finding fractions difficult. Thus, since there is an obvious lack of context in the textbook used by the Brunei students, this motivates me to design activities that include some context, with respect to fractional-equivalence and flexibility of unitizing, which will be discussed in the next chapter.

CHAPTER 4

INTRODUCTION AND QUASI EXPERIMENTAL DESIGN

4.1 INTRODUCTION

In this study, a multiple perspective methodology was adopted, which gathered data from various “vantage points”. These points included pre-, post- and delayed post-teaching mathematics pencil-and-paper performance data, pre- and post-teaching interview data for students and experimental lesson on fractions. Other data obtained were confidence data, students’ artifacts, textbooks analysis, and document analyses (syllabuses). This will be further discussed in section 4.4, *Forms of Data and Data Collection Process*. The study involved gathering data from just one school, and in that sense it is a “single site” study. Furthermore, since in the study the learning of one topic only was investigated, the study was also a “single topic” study.

Different methodologies for the two research questions were used as part of the mixed methods study to accommodate the exploratory needs of each question. Therefore:

RQ.1 – a *quasi-experimental design methodology* was adopted to find out the effectiveness of the RME-like fractions lessons with respect to fractional-equivalence and flexibility of unitizing;

RQ.2 – a *semiotic analysis methodology* was adopted to analyze the learners’ outcome.

This chapter will discuss the following:

4.2 The Experimental Teaching sessions;
4.3 Quasi-Experimental Design;
4.4 Sample and Access;
4.5 Forms of Data and Data Collection Process;
4.6 The Pencil-and-Paper Instruments;
4.7 Validity of the Design;
4.8 Ethical Considerations; and
4.9 Limitation of the Design.

4.2 THE EXPERIMENTAL TEACHING SESSIONS

As can be seen from the textbook analyses in Chapter 3, I hypothesize that context is missing from the textbook used by Brunei students. This motivated me to design an experimental teaching session that included some context, which I taught to a group of students, Experimental 1 (hereafter, E1). At the same time, I needed to ensure that any improvement made by E1 was not just because of my presence. For this I needed to teach "normal" lessons to another group of students, Experimental 2 (hereafter, E2). I also wanted to compare E1's performance with students of similar abilities, who were not given such lessons (i.e., a control group, hereafter, E3). It is for these reasons that I chose the quasi-experimental design (hereafter, QED).

Treatment 1 (hereafter, T1) was given to E1 in the QED. Treatment 2 (hereafter, T2) was given to E2 where *normal* teaching on topics that the mathematics teachers gave was provided, whereas, in treatment 3 (hereafter, T3) for the E3 group, no teaching at all was given to them. The experimental teaching involved small group teachings, where the students were divided into nine groups of similar abilities, and each group were taught separately according to a designated schedule. Each group consisted of four students, and they worked in pairs (or, threes if a student was absent–Figure 4.1) to complete the given activity in order to ensure that all students had a chance to contribute in the activities. There were seven activities in the experimental teaching, and since there were nine groups altogether, I only met each group once a week (i.e., one activity per week). In this section the seven activities for the experimental teaching will be briefly described (refer to Appendix 6 for detailed description). The seven activities are listed below:

1. Activity 1: *Fraction of a foot*;
2. Activity 2: *My Different Looks*;
3. Activity 3: *Hour-Foot*;
4. Activity 4: *Square Me Game*;
5. Activity 5: *Identical but Different*;
6. Activity 6: *Let's Share!*; and
7. Activity 7: *Guess How Much They Get*.

I will now present how the activities were related and what were their possible contributions towards the objectifications of equivalence of fractions and flexibility of unitizing.

Figure 4.1. Students doing one of the activities

The first four activities were designed to afford the objectification of equivalence of fractions, where "measuring of length" was the context used for Activity 1. A 12-inch bread loaf was employed to produce equivalent fractions in terms of the number of equal parts (hereafter, F(Part)), and in terms of the length (hereafter, F(Length)), which led to F(Part)=F(Length). This part-whole activity was chosen in order to give students an activity that they were familiar with, before exploring more "unfamiliar" activities. Activity 2 was an extension of Activity 1, where the students worked with some pre-cut bread of varying lengths to form a 12-inch loaf of bread, hence reinforcing what they had done previously. This indirectly allowed for some addition and subtraction of fractions to occur contextually, though this was not my prime objective, but might have been needed for part of Activity 3. Even though length in inches was a familiar experience to the students, it was not something they dealt with everyday. Hence, I designed another activity that included "measuring time", which was experientially real to them. Coupled with their knowledge from the previous activities, an *Hour-Foot Clock* (hereafter, HFC) was created, to link the measuring of length and time–the 12-inch bread linked to the 12-inch circumference and 12-hour clock. This enabled the production of another fraction (i.e., in terms of time (hereafter, F(Time)), and allowed equivalence of fraction (i.e., F(Part)=F(Length)=F(Time)) to be objectified. These activities were designed in this way because I wanted the students to understand why they got the fractions that they got. According to Nunes & Bryant (2008), ". . . in order to come up with the right fraction, pupils need to understand how situations are represented by fractions" (p. 263). Also, Connell & Peck (1993) found that students prefer to use a bar (in the context of a rectangular cake) as a measuring instrument as a model for the development of a number line, which in my case was for the development of the HFC. In addition, many researchers (e.g., Mack, 1990; Graeber & Tanenhaus, 1993) advocate explicitly building on informal knowledge of students, which is in line with RME. Activity 4 was a board game which was designed to reinforce what the students had learned from the previous three activities.

The next three activities contributed to the objectification of flexibility of unitizing. Activity 5 used "sharing of food" as the context, where each pair of students were to share different lengths of bread–each group ended with $\frac{1}{2}$ as the fraction of each share, but with different lengths–to explore their knowledge of flexibility of unitizing. This gave the students an experience where fractions were seen as numbers, which required them to recognize that $\frac{1}{2}$ referred to the relative amount. In the next activity, again the same context of sharing was employed but with a twist, where after they had cut the given square pizza into four, they needed to cut one of the shares again so that it could be equally shared between three people. They were then to determine the fraction of each of the three little pieces; they were encouraged to use the HFC to reach their answer. This activity allowed them to notice that "the unit" mattered. Activity 7 is a reinforcement activity, where they were able to use their knowledge on flexibility of unitizing to perform it.

To sum this up, Nunes & Bryant (2008) said, "If pupils focus their attention entirely on the signs, they could both miss that two different fractions can be equivalent and also the fact that the same fraction might refer to different quantities" (p. 266).

Next the quasi-experimental design methodology is discussed vis-à-vis the learning outcomes of the RME-like experimental teaching on fractions.

4.3 QUASI-EXPERIMENTAL DESIGN

In order to investigate RQ1, a quasi-experimental design methodology with a pre-teaching test, post-teaching test and delayed post-teaching test was used. The rationale for choosing a quasi-experimental design was because of the non-random distribution of students to the treatment groups (i.e., E1, E2 and E3 group). Campbell and Stanley (1963), as cited in Robson (1993), defined QED as:

> a research design involving an experimental approach but where random assignment to treatment and comparison groups has not been used. (p. 98)

QED (Figure 4.2) may be considered as a "not true experimental design" (Robson, 1993, p. 99), but Burns (2000) stated that "There are many situations in educational research in which it is not possible to conduct a true experiment" (pp. 147–148). The reason for a non-random allocation of the students will be discussed in section 4.4.

There were three groups of students in the design chosen who had taken a pre-teaching test, a post-teaching test and a delayed post-teaching test:

i. ***E1:*** This was the group of students who had been given the seven RME-like fractions lessons;
ii. ***E2:*** This was the group of students that I taught for 7 lessons on topics that the class teacher had given using the *usual* method. This was necessary to cancel out *my effect* on the students; and
iii. ***E3:*** This group did not have any teaching contact with me at all during the study, thus they were used as the reference point for all factors other than the effects of the experimental teaching. This allowed for control on maturation, training effects and other general threats to validity.
iv. ***The pre-teaching test, post-teaching test and delayed post-teaching test:*** These three tests provided fractions performance ability measures for all students immediately before, immediately after and long after the

experimental teaching sessions. They were needed to enable comparison of students' performance in the tests:

a. Immediately before (pre-teaching test) and immediately after (post-teaching test) the fractions lessons;
b. Immediately before (pre-teaching test) and long after (delayed post-teaching test) the fractions lessons; and
c. Immediately after (post-teaching test) and long after (delayed post-teaching test) the fractions lessons.

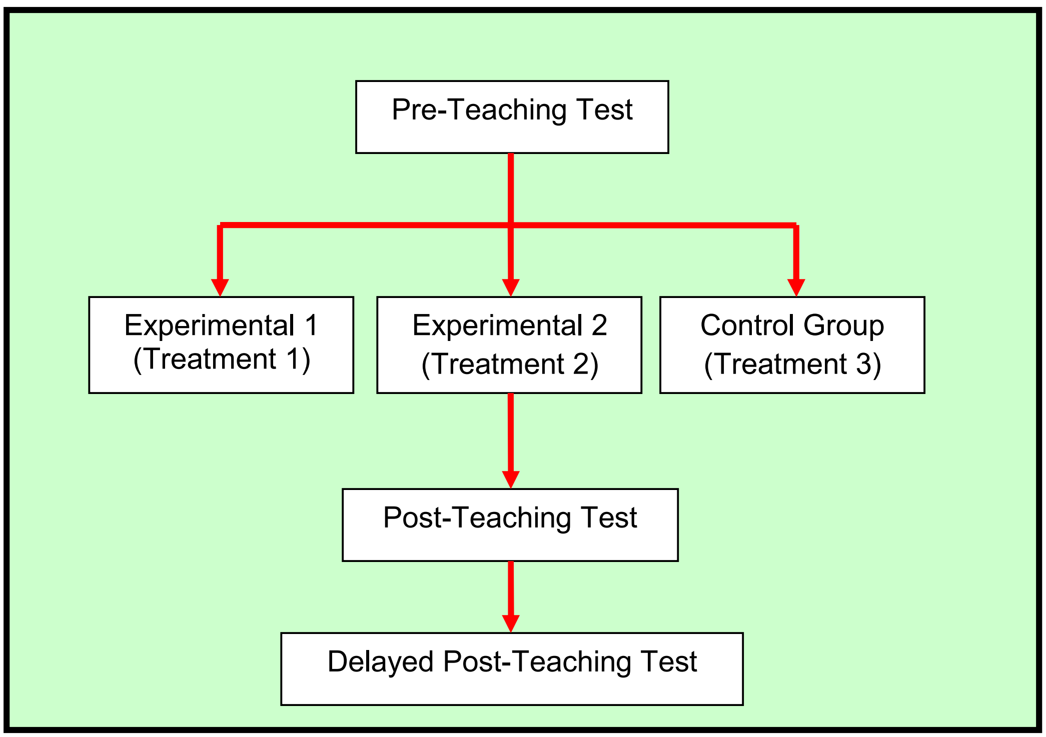


Figure 4.2. The Quasi-Experimental Design

However it should be noted here also that the delayed effect comparison for E1 and E2 cannot be made, because in effect the post-teaching test for the E2 group can also be considered as their *pre-teaching test*. The reason is that the teaching for E2 only occurred after the experimental teaching of the fractions lessons, so in essence the post-teaching test was the *pre-teaching test* for E2. On the other hand, the actual pre-teaching test for the E2 group was not a waste as it can still be taken into account as the base line data of the students' fractions ability prior to the study. This can be seen clearly in Table 4.1 below.

Consequently, while controlling for maturation and training effects, the design also allows for measures of the learning outcomes of the experimental teaching as compared to the E3 group immediately after and long after the experimental teaching is done.

Table 4.1
Summary of the Quasi-Experimental Design Methodology

	E1	E2	E3
Pre-teaching test	✓	✓	✓
Fractions lessons	✓		
Post-teaching test	✓	✓	✓
Non-fraction lessons		✓	
Delayed post-teaching test	✓	✓	✓

4.4 SAMPLE AND ACCESS

The sample of the QED comprised Form 1 students at a secondary school in Negara Brunei Darussalam. The reason for choosing Form 1 students rather than other levels is because these students had just passed their PSR (End-of-Primary 6 Assessment) Examination, including Mathematics as one of the subjects. In addition to that, their Mathematics teachers had just finished teaching the sub-topic on fractions, as stated in the prescribed Mathematics National Syllabus for Negara Brunei Darussalam. The assumption that could be made is that they should have had an adequately resilient fractional concept by the time they reached secondary school, as they had been doing fractions for the past three years (Primary 4, 5, & 6) prior to entering secondary school.

The school selected for this study was situated in the outskirts of Bandar Seri Begawan, the capital of Negara Brunei Darussalam. The school is a public co-ed school operating as a double session school (morning, from 7:15 a.m. to 12:30 p.m., and afternoon, from 12:45 p.m. to 5:30 p.m., sessions). The QED sample was from the afternoon session, as all Form 1 and 2 classes were conducted in the afternoon session. There were six classes for Form 1 and all of them were included in the QED. Based on the public examinations results, the *Penilaian Menengah Bawah* (Lower Secondary Assessment, hereafter, PMB) for Form 3 for 2007 and the GCE Cambridge O level for Form 5, this school was an average school.

The E1 group comprised students who scored from 7 to 16 out of a possible 22 in the pre-teaching test paper. It should also be noted that not all the students who fell into that category were selected, due to the large number of students falling into that category. Hence the final decision on who were to be included in E1 was left to the two mathematics teachers. They chose students who they thought would be able to benefit the most from the experimental teaching and also on the ability of the students to communicate in English.

The E2 group was all students from the same class that the teacher had agreed to let me teach for her on topics that the other students from other classes were also learning. There were six students in that class who were also part of the E1 group, and they were excluded during T2.

It should also be noted that there were four students who joined the school after the pre-teaching test was administered to the students. These students still sat for the post-teaching test and the delayed post-teaching test but they will not be considered as valid cases. On the other hand they will be included as part of the post-teaching test and delayed post-teaching test data analysis. Table 4.2 summarizes the sample size for each group.

From Table 4.2 below, there were 89 students altogether for the six classes in Form 1; 33 students in the E3 group, 36 students in E1 and 20 students in E2. Nevertheless, one parent from E1 did not give permission for his child to participate in the teaching sessions (the child informed me that this was because her father did not want her to miss the *MIB* (i.e., Malay Islamic Monarchy) class, as she was a Chinese girl and her father wanted her to learn more about the subject even though the subject did not contribute to her overall end-of-year achievement). So I asked the teachers to nominate another student to take her place, and the said student agreed to participate, with the parents' consent.

Table 4.2
Sample Sizes in the QED

QED & number of students / Absent for:	E1 (36)	E2 (20)	E3 (33)
Pre only	0	0	0
Pre + Post	0	0	0
Post only	2	1**	0
Post + DP	1*	0	0
DP only	0	1**	0
Valid Cases	**33**	**18**	**33**

*This student transferred to Sports school after the second lesson of the experimental study.
**Another student also transferred to Sports school from this group.

As for the experimental mortality, I have categorized it into two types: students who were absent for one or more of the tests, and students who were repeatedly absent for the lesson and missed all the tests, or who transferred out of the school. The reason for doing this is to differentiate students who were not included in the data due to absenteeism for the tests only, but attended the teaching, and those who were regularly absent but managed somehow to be present during the tests. Although these students were absent for one or two of the tests, they were still included in the data for the tests that they sat. Thus, there were only 35 (33 if students who were absent for one or two of the tests are taken into account) in E1, 20 (18 if students who were absent for one or two of the tests are taken into account) in E2 and 33 valid cases in the E3 group.

Pertaining to acquiring access to the school to conduct the study, the Deputy Principal Academic of the school was contacted through email, to explain the nature of the study and to find out whether the school was interested and willing to give me access to conduct my study. Once consent from the school was obtained, I emailed the Director of

Schools, Ministry of Education of Negara Brunei Darussalam to get the Ministry's approval to conduct the study in the selected school. Only once I had written permission from the Director, did I start to conduct the study.

Upon receiving written approval from the Ministry, I had a meeting with the school's Principal and Deputy Principal Academic, to explain the study to them and its significance to Brunei Darussalam's schools in general. I also informed the administrators that some students would be taken out of their classrooms (for small-group teaching sessions), and that the lessons would be audio-taped and video-taped solely for the purpose of the study. I also informed them that some of the students would be interviewed. Assurances of anonymity and confidentiality were also given to the administrators. Next I met the teachers who were responsible for teaching mathematics to the Form 1 students. In the meeting, the nature of the study was further explained. In addition, the protocol of the administration of the tests and other relevant issues (e.g., who to invigilate the tests, the collection point of the tests papers, etc.) were also discussed. At the end of the meeting, the teachers were given consent forms for parents to ask their permission for their children to participate in the study. Lastly, oral confirmations of the students' consent to participate in the study were also received once I met the students.

4.5 FORMS OF DATA AND DATA COLLECTION PROCESS

In this study, altogether, ten forms of data, representing different vantage points, were obtained in order to construct a composite "whole." Once that construction was achieved I was then well positioned to answer the two research questions. The six vantage points were represented by the following labels, and brief comments on each are now provided.

4.5.1 Forms of Data

The six vantage points that the researcher looked at are as follows:

1. *Document analysis*

   Among the relevant documents which were collected and analysed for details concerning expectations in regard to the development of fractions concepts among Form 1 mathematics students were the following:

- National Ministry of Education curriculum statements; and
- National Form 1 Mathematics syllabus in the record of the selected secondary school;

2. *Textbook analysis*

The sections on fractions in the textbook used by the Form 1 students in Bruneian secondary schools were examined. Attention was especially focused on the following:-

- How the textbook Authors' approached the topic of fraction, and in particular the extent to which they emphasised instrumental as opposed to relational understanding (Skemp, 1976); and
- Identifying the learning gaps in the students' experiences that needed to be filled with RME type experiences.

As this analysis is an important aspect, a whole chapter (Chapter 3) has been dedicated to discussing the findings of this analysis and the document analysis. To remind readers, as discussed in section 3.3.4, the analysis revealed that the textbooks used by the Brunei students predominantly used area models of fractions concepts with very few contextualizing models used. Even if models were used, it did not go beyond simply describing the problem itself. It can then be said that the emphasis of the CPDD textbook is more on instrumental understanding where students are trained to be automated when responding to problems. Most likely, students will produce the same answer using the same procedure. This is in contrast to the MiC textbook which are "loaded" with contextual activities and hence emphasizing on relational understanding. The activities encourage students to explore through investigation, construction and discussion of possible solution(s) to a problem. So, in these situations there will be more than one possible solution to a problem. Hence the analysis shows that Brunei students are missing activities involving models and context which involved problem solving and discussion with multiple methods.

3. *Pre-, post- and delayed post-teaching fractions performance data*

Three pencil-and-paper tests–the *Pre-Teaching Fractions Test*, the *Post-Teaching Fractions Test* and the *Delayed Post-Teaching Fractions Test*–were used to generate the pre-teaching, post-teaching and delayed post-teaching performance data. The tests were almost the same as the one used in the pilot study. The pilot study involved six Form 1 students, chosen by the subject teacher, who were identified as low achievers in mathematics. The students worked in groups of three. Audio and visual recordings of the interactions in each group were made, and were fully transcribed. This is to check the setting for both the audio and visual recordings whether they are audible enough, and also used for a practice analysis of some of the episodes. Prior to the experimental teaching, a pre-test was given to all Form 1 students (n=159, but only 121 students present on the day). A post-test was given to all the Form 1 student (only 142 present on the day) after the experimental teaching took place. The actual teaching only took 4 days to complete, which I felt was too ambitious as there were nine activities to do. From the pilot study, it was found that all the items in the two performance test were not showing any misfits with a reliability of 0.83 and 0.85 for the pre- and post-test, respectively. A semiotic analysis of some of the interesting episodes was also done.

Like in the pilot study, in the *Pre-Teaching Fractions Test*, there were 22 items concerned with operations of fractions. The *Pre-Teaching Fractions Test* had a Cronbach alpha reliability of 0.76. In the *Post-Teaching Fractions Test*, there were 23 items, but 13 of the items were identical to the *Pre-Teaching Fractions Test,* and the other 10 were different questions of similar and higher level of difficulty. The aim was to minimize the effect of practice, so that the test would give a better picture of improvements, if any, made by the students. This test had a Cronbach alpha reliability of 0.81.The third test, the delayed post-teaching test had 24 items and also had 13 items identical to the pre- and post-teaching test, and the rest of the questions were taken from both the pre-teaching test and the post-teaching test. This test had a Cronbach alpha reliability of 0.84. It should be noted that a *Confidence Scale* was linked to the *Pre-Teaching Fractions Test*, the *Post-Teaching Fractions Test* and *Delayed Post-Teaching Fractions Test* respectively. This invited students to indicate their levels of confidence in the correctness, or otherwise, of their answers. The test had a Cronbach alpha reliability of 0.65, 0.71 and 0.73 respectively. Unfortunately, I was not able to analyse this due to time constraints, and I hope to do this as part of my proposed future research. Also, it should be noted here that prior to giving the post-teaching fractions test, all the Form 1 students were given special revision lessons, which included fractions, in preparation for their mid-year examination.

4. *Interview data*

   Seven students of the thirty-six E1 sample students were individually interviewed immediately before the experimental teaching sessions were conducted, but only six of the seven students were interviewed after the experimental teaching took place. This was because the parent of one student did not want his child to participate in the study. The interview was digitally-recorded. An interview schedule, based on the Newman interview protocol, was prepared for the interviews with students (see Appendix 7). The Newman interview technique (Newman, 1983) was used in the interview with students. However, the interview data was not actually analysed, rather this data acted as support data for reference, if the need arose.

5. *Experimental teaching data*

   This consisted of 7 activities which aimed to help provide the Form 1 students in the sample with an experience of learning and exploring fractions using an alternative way, namely an RME-like lesson, other than what they used to have in their Mathematics classroom.

6. *Data arising from artifacts*

   The types of artifacts that were collected were the students' responses to: (a) the pencil-and-paper tests; (b) the activities in the experimental teaching; and (c) the interview written responses.

4.5.2. Data Collection Process

Now details of the administration of the three pencil-and-paper tests and the protocol of the interviews are provided. In this study a pre-teaching test, a post-teaching test and a delayed post-teaching test were administered, and seven out of the thirty-six students were interviewed after the pre-teaching test (*pre-teaching interview*) but only six students were once again interviewed after the post-teaching test (*post-teaching interview*).

The pre-teaching test (see “Pre-Teaching Fractions Test,” Appendix 8) was administered during the test week of the school and it was administered in a whole class setting. A day prior to the test, all the teachers who were supposed to invigilate the test were given a briefing on how to conduct the test (e.g., there should not be any help given by the teachers, no calculators allowed, etc.) and other relevant issues (e.g., where to send the papers once the test is over). Since the pre-teaching test was conducted during the test week, the classrooms were free of any supporting materials. The students were given 70 minutes to complete the test, but this proved to be more than enough, as the students managed to complete the test in about 40 to 50 minutes. The test was administered simultaneously for all the six classes, so it was impossible for the researcher to be in all classes during the test. Thus, in order to ensure everything ran smoothly and to answer any queries, the researcher visited each class as the test progressed to clarify any doubts that arose.

The pre-teaching interviews were done as soon as the test week was over, and it took six school days to interview the seven selected students. The reason was that I could only take the students out of the classroom during certain periods (i.e., MIB and PE) so that the students did not miss their other lessons. Each interview took an average of 48 minutes to complete, and the students were interviewed separately. As mentioned above, the interview followed the Newman interview protocol (see Appendix 9 for detail). During the interview the students were asked to do the test again and were asked to explain their thinking when they answered the questions. This served as a way of getting a glimpse of the students' thinking process when they were dealing with fractional tasks. An interview schedule, based on the Newman interview protocol, was prepared and used (see Appendix 7), and all interviews were tape-recorded. Although interview schedules were prepared, additional questions were asked during interviews, if and when this was deemed to be necessary by the interviewer. In particular, any comments made by interviewees which appeared to reveal important aspects of their attitudes towards fractions were noted and probed by the interviewer.

The experimental teaching required approximately 3 hours of teaching for each of the nine groups. They were conducted in seven sessions, one activity per session per week. In total the whole experimental teaching took seven weeks to complete–for each school day only one or two groups could be given the instruction as the school only allowed the students to be taken out during MIB or PE periods. The thirty-six students were divided into small groups of 4, almost all the time students from the same class were in the same group so that they would feel at ease with each other.

The post-teaching test (see "Post-Teaching Fractions Test," Appendix 10) was administered a day after the experimental teaching was completed. Since the post-teaching test was not conducted during the test week, the researcher and the invigilating teachers made sure that each classroom was free of any supporting materials. The students were given 70 minutes to complete the test, and again this proved to be more than enough, as the students managed to complete the test in about 40 to 50 minutes. The test was administered simultaneously for all the six classes in a class setting, and it was impossible for the researcher to be in all classes during the test. Thus, in order to ensure everything ran smoothly and to answer any queries, again the researcher visited each class as the test progressed to clarify any doubts that arose.

The post-teaching interviews were conducted a day after the post-teaching fractions test was administered so that the test papers of the students involved in the interviews could be marked prior to the interview. The post-teaching interview took six school days to complete. As before, the reason was that the researcher could only take the students out of the classroom during certain periods (i.e., MIB and PE) so that the students did not miss their other lessons. Each interview took an average of 20 minutes to complete, and the students were also interviewed separately. The interview followed the Newman interview protocol. Unlike the pre-teaching interview, the students were asked to do the questions which they got wrong in the post-teaching test and were asked to explain their thinking when they answered the questions. This served as a way of getting a glimpse of the students' thinking process when they were dealing with fractional tasks. An interview schedule, based on the Newman interview protocol, was also prepared and used, and all interviews were tape-recorded. Although interview schedules were prepared, additional questions were asked during interviews, if and when this was deemed to be necessary by the interviewer.

The next step was to teach E2. This group came from the same class, and their original number was actually 26, but six of them were also part of E1. Hence, E2 only consisted of 20 students. In order to avoid contamination by the six E1 group students, while I was teaching the rest of the class on topics that their mathematics teacher assigned for me to teach, the six E1 students were taught exactly the same topics by their mathematics teacher in a separate classroom. In this way, no students in the class were left behind in their school work. The contact time with E2 was also seven lessons, which is two and a half weeks of school days, the same as the amount of contact time for each of the E1 sub-groups (i.e., seven contact lessons per sub-group). While all this was going on, the rest of the Form 1 students were also taught the same topics by their mathematics teachers.

As soon as the teaching for E2 was completed, all the students involved in the QED were administered the delayed post-teaching fractions test. The delayed post-teaching test (see "Delayed Post-Teaching Fractions Test," Appendix 11) was administered a day after the teaching sessions for E2 were completed. The invigilating teachers and I had to make sure that each classroom was free of any supporting materials. The students were given 70 minutes to complete the test, and once again this proved to be more than enough, as the students managed to complete the test in about 40 to 50 minutes. Similarly to the two previous tests, the test was administered simultaneously for all the six classes in a class setting, and it was impossible for the researcher to be in all classes during the test. Hence, again the researcher visited each class as the test progressed to clarify any doubts that arose, to ensure everything ran smoothly and to answer any queries.

That completes the presentation for the data collection process for the QED. In subsequent section details of the data collection are provided.

4.6 THE PENCIL-AND-PAPER INSTRUMENTS

There were six pencil-and-paper data collection instruments and two interviews included in the data collection process. A summary of the six instruments is presented in Table 4.3 below. Further details relating to these six instruments are now provided.

Prior to presenting the details of the six instruments, I would like to highlight that the questions for the three performance tests were taken from a pool of thirty-two sub-questions. Each question on these tests was selected from either Samsiah's (2002) *Fractions Questions*, past PSR (1985 to 2006) Examinations, the MaLT (Mathematics Assessment for the Learning and Teaching) test papers for Year 8 to 13, or questions that the researcher created herself. Thirteen of the questions were common to all the three performance fractions tests, five questions were only asked in the *Pre-Teaching Fractions Test,* four question appeared in both the *Pre-Teaching Fractions Test* and the *Delayed Post-Teaching Fractions Test*, three questions were in the *Post-Teaching Fractions Test* only, whereas seven questions were asked in both the *Post-Teaching Fractions Test* and the *Delayed Post-Teaching Fractions Test*.

The rationale behind this was so that valid comparison could be made for the six tests on how each individual in E1 *evolved* (if any) in their strategies and ways of thinking when they were faced with fractional tasks. Details of all the common items will be presented as part of the *Pre-Teaching Fractions Test* below. For each test, I have further categorized the questions as items *with context* and items *without context*. This is done so that it will be easier to focus on the students' performance on fractional tasks in these two contexts.

Table 4.3
The Pencil-and-Paper Instruments

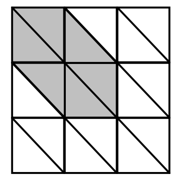
PENCIL-AND-PAPER TESTS	COMMENTS
Pre-Teaching Fractions Test Pre-Teaching Confidence Scale	Completed by all the QED students before the commencement of the experimental teaching. The Confidence scale is linked to the Pre-Teaching Fractions Test.
Post-Teaching Fractions Test Post-Teaching Confidence Scale	Completed by all the QED students after the completion of the experimental teaching for E1, and before the teaching sessions for E2 started. This test also acted as the *pre-teaching test* for E2.
Delayed Post-Teaching Test Delayed Post-Teaching Confidence Test	Completed by all the QED students after the instruction of E2, which was about seven weeks after the experimental teaching for E1 ended. Basically, this acted as the *post-teaching test* for E2. This was purposely done as I was not focusing on the improvement or non-improvement of E2. The sole purpose of that group was to cancel out my effect on the students, to ensure that any improvement that the E1 students made was not due to my presence (Hawthorne Effect), but rather because of the RME-like lessons that had been given to them.

With context refers to questions that have a *situation* that was presented as part of the question, for example:

> *Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has* $\frac{1}{2}$ *loaf of bread and Irah also has* $\frac{1}{2}$ *loaf of bread. How much bread do they have left once they use it to make the pudding?*

Items *without context* are those questions that are typical straightforward "bare numbers" tasks, without any *stories* being merged as part of the question. For example,

1. *Find the value of* $\frac{1}{2} + \frac{1}{2} - 1$
2. *What fraction of this shape is shaded?*



The Pre-Teaching Fractions Test. This test comprised 21 questions (see Appendix 8), but in fact, there were 22 sub-questions altogether. Any response to a sub-question was allocated 1 (for a correct response), 0 (for an incorrect response), or 9 (for no response), and hence the maximum possible score on the *Pre-Teaching Fractions Test* was 22. The descriptions of the test items will be provided as three separate headings: "The Common Items," "The Pre-Teaching Fractions Test Items Only" and "The Pre-Teaching Fractions and the Delayed Post-Teaching Fractions Tests Items." Aspects of fractions covered in the 22 sub-questions on this test are now provided.

1. **The Common Items**

   There are altogether 13 common items which have been categorized as items *with context* and items *without context*, and they cover the following aspects of fractions:

   a. ***With Context***. There are 8 out of 13 sub-questions which I have grouped under this category, and all of them are word problem questions. Although all these items are "story-based" questions, they cover different aspects of the fractions.

      i. Question 7 (see Appendix 8) was concerned with 'what Orton and Frobisher (1996) categorized as "quotient model." This question incorporates the idea of *sharing* involving division of whole numbers that will give a non-whole number;

ii. Question 9 asked students to translate a given measurement (in inches) with respect to another measurement which is the whole in this case (i.e., one foot) in terms of a fraction. Orton and Frobisher (1996) categorized this as the "ratio model;"

iii. 2 of the 13 questions (Questions 17 and 19) were concerned with whether students could *see* the *whole* as a single unit. This is what Orton and Frobisher (1996) categorized as "discrete model." Another question asked students to write a fraction representing a group of items with respect to the total, while the second question required the students to find a whole number answer after a certain fraction of the whole items were removed;

iv. Three of the questions tested the students' ability to *interpret word problems involving a fractional quantity.* One question, Question 13, was actually an exact equivalent question to Question 10 but without any context given to it. It would be interesting, in my opinion, to compare students' performances and responses (strategies) to these two identical questions. Whereas the other two questions (Questions 16 and 20) involved students performing the four operations, one required students to multiply a fraction with a whole number, and the other involved subtraction of two fractions of different denominators; and

v. One question, Question 21, was based on the "part-whole model" (Orton and Frobisher, 1996) which also involved the use of the four operations where students were asked to find the fraction which represented a part of the whole, when two fractions of different denominators were given.

b. ***Without Context.*** 5 of the 13 questions have been put under this category.

i. Three questions were concerned with students' knowledge on the *four operations for fractions.* These questions involved:

- Division of fractions by a whole number (Question 2);
- Multiplication of fractions with a whole number (Question 15); and
- Combinations, involving addition and subtraction in the same question. This question, Question 10, was the equivalent question to Question 13 but without the context being attached to it.

ii. Question 18 asked students to *identify the whole* of the parts of the given diagram; and

iii. One question, Question 14, was the equivalent question to Question 7. This question asked students to *divide a given diagram into equal parts,* to see the strategies that students adopted to arrive at their answer.

2. **The Pre-Teaching Fractions Test Items Only**

   There are only five questions that were solely part of this test, and these have also been categorized as items *with context* and items *without context*, and they cover the following aspects of fractions:

   a. ***With Context.*** Out of the five questions, only one question, Question 5, was, in my opinion, a question where a situation was presented with the question. This question involved *ordering of fractions,* where students were asked to order mixed numbers involving fractions with different denominators from the smallest to the biggest. This question could also give a good indicator of students' knowledge on *equivalent fractions,* as one possible way of approaching this problem would be by changing all the fractions into their equivalent fractions that would have the same denominators.

   b. ***Without Context.*** Four of the questions fell under this category.

      i. 3 of the 4 questions (Questions 1, 3, and 11) were concerned with whether students could *translate fractions into diagrammatic representations, and vice versa*. One question asked students to shade the correct number of parts of a given diagram, in order that the answer would represent a given fraction. One question asked students to choose the fraction and its equivalent fractions that corresponded to the shaded diagrammatic representation of the fraction. The third question asked students to estimate the fraction for the given shaded diagrammatic representation of the fraction; and

      ii. One question, Question 8, was concerned with *equivalent fractions*. In this question students were required to pick a fraction which was equivalent to the given fraction.

3. **The Pre-Teaching Fractions and the Delayed Post-Teaching Fractions Tests Items**

   There are four sub-questions that appeared in both the *Pre-Teaching Fractions Test* and the *Delayed Post-Teaching Fractions Test*. All four of these questions fall under the question without context. One question (Question 4) was concerned about whether students could *translate fractions into diagrammatic representations,* where the question asked students to pick all the correct diagrammatic representation(s) of the given fraction. This was also what Orton and Forbisher (1996) described as the "part-whole model". Two sub-questions (Questions 6a and 6b) involved *equivalent fractions* where the questions asked students to find one missing value of a given pair of equivalent fractions. The fourth question (Question 12) was a question on the *four operations* where students were asked to perform subtraction of two fractions with different denominators.

The Post-Teaching Fractions Test. This test comprised 21 questions (see Appendix 10–Q7 and Q8 in this paper have been exchanged for ease of analysis, so when I refer to Q7 it will be Q8 in the paper, and vice versa), but in fact, there are 23 sub-questions altogether. Any response to a sub-question was allocated 1 (for a correct response), 0 (for an incorrect response), or 9 (for no response), and hence the maximum possible score on the *Post-Teaching Fractions Test* was 23. Similar to the *Pre-Teaching Fractions Test,* the descriptions of the test items will be provided as separate headings - "The Post-Teaching Fractions Test Items Only" and "The Post-Teaching Fractions and the Delayed Post-Teaching Fractions Tests Items." The common items will not be described again as they are the same as mentioned above. Aspects of fractions covered in the rest of the 10 sub-questions on this test are now provided.

1. **The Post-Teaching Fractions Test Items Only**

   There are three questions in this category, Questions 3, 5 and 6. All the three questions in this category were without context. Two questions were concerned with whether students could *translate fractions into diagrammatic representations,* and vice versa. One of the questions invited students to choose the diagrammatic representation(s) that represented the given fraction, while the other one required students to be able to shade the given diagram to correctly represent the given fraction before they could answer the question. Question 5 involved number sequence where some numbers were missing and the students were to find the missing numbers.

2. **The Post-Teaching Fractions and the Delayed Post-Test Fractions Test Items**

   There are seven sub-questions that were common to both the *Post-Teaching Fractions Test* and the *Delayed Post-Teaching Fractions Test*. The seven questions are Questions 1, 4, 8, 11a, 11b, 12a, and 12b. They have also been categorized as items *with context* and items *without context*, and they cover the following aspects of fractions

   a. ***With Context.*** Two sub-questions (Sub-questions 12a and 12b) have been considered as questions with context. These sub-questions involved the *idea of comparing the number of objects in two sets of "objects*," in this case a certain number of loaves of bread that can feed a certain number of animals. In the main part of the question the ratio between the two "objects," A:B was given. Part (a) of the question asked students to calculate the number of animals that could be fed for a different number of loaves of bread, while the second part of the question asked for the number of loaves of bread required to feed a different number of animals. Orton and Forbisher (1996) called this model the "ratio model."

   b. ***Without Context.*** Under this category there are five questions.

      i. Question 1 tested students on their knowledge of *equivalent fractions* where students were asked to select the correct equivalent fraction to the given fraction;

      ii. Question 4 is about the *four operations* where students were to add two fractions with the same denominators;

      iii. Question 8 was concerned with whether students could *translate diagrammatic representations into fractions*; and

      iv. In these sub-questions the students were asked to *find the original whole* where the fraction and the whole number answer was given. Technically, this involved four operations involving division of an integer by a fraction.

The Delayed Post-Teaching Fractions Test. Although this test was made up of 21 main questions (see Appendix 11), in fact, there were 24 sub-questions altogether. Any response to a sub-question was allocated 1 (for a correct response), 0 (for an incorrect response), or 9 (for no response), and hence the maximum possible score on the *Delayed Post-Teaching Fractions Test* was 24. Since all the questions in this paper appeared in one, or both test papers above, no further description of the questions will be given under this heading. A summary of which questions appear in which paper is sufficient, as shown in Table 4.4 below.

Table 4.4
List of Questions for the Delayed Post-Teaching Fractions Test Paper

	Pre & Delayed-Post	Common Items	Post & Delayed-Post
Questions	4, 6a, 6b, 12	2, 7, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21	1, 3, 5, 8a, 8b, 11a, 11b

Note: In the *Post-Teaching Fractions Test* Q5 was Q8, Q8a & 8b were Q11a & 11b, and Q11a & 11b were Q12a & 12b. The rest of the questions had the same number as stated for each paper.

For more details of the distribution of the questions and in which test papers they appeared, see Appendix 12.

4.7 VALIDITY OF THE DESIGN

In this section I am going to look at two types of validity: internal and external threats of validity. Robson (2002) listed 12 internal threats to validity in which, according to Robson, eight were suggested by Campbell and Stanley (1963), and the other four were later developed and extended by Cook and Campbell (1979). As for the external threat to validity, I looked at Mertens (1998) who listed 10 threats. A detailed discussion and examination of all these threats are presented in Appendix 13.

Regarding internal validity, as presented in Appendix 13, the researcher is confident that there are no threats to internal validity that would gravely affect the study. As for external validity, Mertens (1998) considered two types of external validity: population validity and ecological validity. According to Mertens (1998);

> Population validity . . . to whom you can generalize the results based on sampling strategies. (p. 67)
>
> Ecological validity . . . the extent to which the results of an experiment can be generalized from the set of the environment conditions created by the researcher to other environmental condition. (p. 68)

He further discussed how claims of generalizability often came up within a postpositivist paradigm, and seldom in the constructivist paradigm (Martens, 1998). When no claim of generalizability is made the concept of transferability is used (Guba & Lincoln, 1989, cited in Mertens, 1998). Following that, I felt that the concept of transferability is the most appropriate in this study as no claim of generalizability was made. Merten (1998) wrote that "The researcher's task is to provide sufficient "thick description" about the case so that the readers can understand the contextual variables operating in that setting (Guba & Lincoln, 1989)." Hence a detailed description of the study was presented in earlier sections on the experimental study, the quasi-experimental design, the sample and access, the forms of data, the data collection process, and the pencil-and-paper instruments used.

As far as ecological validity is concerned, there are three threats that could be considered as noteworthy for this particular study: *Hawthorne effect, Posttest sensitization* and *Interaction of history and treatment effects*.

> *Hawthorne effect . . .* the idea of receiving special attention, of being singled out to participate in the study, was enough motivation to increase productivity. (Mertens, 1998, p. 68)

The aim of having E2 was to provide evidence that this threat did not affect the design of the study, and from the analysis of the results (see Figure 6.2, p. 153) of the test this has proven to be the case. Then again one cannot be absolutely sure that E1's increase in productivity was because they were receiving special attention.

> *Posttest sensitization.* This is similar to pretest sensitization in that simply taking a posttest can influence a participants' response to the treatment. (Mertens, 1998, p. 68)

Even though I did not inform the students that they would take a post-test, they could have suspected that there would be one, and they could have paid more attention during the instruction. Thus, it is possible that this threat could affect the study. Nevertheless, in order to further minimize the effect of this threat, for all the three pencil-and-paper instruments, there were only thirteen items in common, so that any practice effect could also be reduced. However, post-testing the students was inevitable because I wanted to measure the effect of the treatment.

> *Interaction of history and treatment effects.* An experiment is conducted in a particular time replete with contextual factors that cannot be exactly duplicated in another setting. If specific historical influences are present in a situation (e.g., unusually low morale because of budget cuts), the treatments may not generalize to another situation. (Mertens, 1998, pp. 68–69)

There were only two particular features of the QED that could potentially have posed a threat to the study:

i. The audio taping and the video taping of the experimental lessons. Nevertheless, the students had been given assurance that the audio tape and video tapes were solely for me, to help me review what went on during the lessons. I am quite confident that the presence of these taping machines did not affect the way they acted, and they seemed comfortable enough with the camera; and
ii. The small group teaching instead of whole class teaching.

Of these two special features, only the later special feature affected the QED, and this is discussed in the limitations of this study.

4.8 ETHICAL CONSIDERATIONS

Even though the study did not involve sensitive issues, I adopted BERA Ethical Guidelines For Educational Research (2004) (see Appendix 14) as my guideline. In the BERA (2004) guidelines, there are 3 aspects that were relevant: Responsibilities to participants, Responsibilities to Sponsors of Research, and Responsibilities to the Community of Educational Researchers. For the study, only the first two aspects pertaining to the participants and sponsors were dealt with, even though no adverse consequences were expected to arise:-

Responsibilities to participants

i. *Voluntary Informed Consent*

With regard to Guidelines (10 and 11), prior to conducting the study, I secured formal written consent from the Ministry of Education, Brunei to do a study in a selected government secondary school in the Brunei-Muara District in Brunei Darussalam. The nature of the study was explained in detail in the letter to the Ministry. The nature of the study was also explained to the school, teachers, and parents in writing (written in both English and Bahasa Melayu languages). On top of that, oral informed consent was obtained from the school Principal and the teachers involved. In addition, parents were given consent forms to sign if they agreed to let their children participate in the study (see Appendix 15). I also got oral informed consent from all the students involved in the QED by meeting the students, class by class, and asking for their consent to be part of the study. All students consented to be part of the QED, including the one student whose parent did not give his consent for his child to be in the E1 group, but he did not mind his child being part of the E3 group.

ii. *Deception* (Guideline 12)

Deceit was not used at any stage of the study, and the students in E1 and E2 themselves were asked again individually whether they were willing to participate in the study, and they were also required to sign a consent form as proof of consent to participate. I also informed parents of the E1 group that the lessons would be audio-taped and video-taped.

iii. *Right to Withdraw*

Parents and students were informed that they could ask for their children (or the students themselves could request) to withdraw from the study at any time or stage of the study with or without any reason(s) given (Guideline 13).

iv. *Children, Vulnerable Young People and Vulnerable Adults*

As far as Guidelines 14 and 15 were concerned, the students in the study were encouraged to voice freely any thoughts that they felt were related to the tasks at hand. As for Guideline 16, it was not applicable in this case as there were no special needs children in the school for that year group. As for complying with legal requirements in relation to working with school children or vulnerable young people and adults (Guideline 17), the researcher herself is a trained teacher who has served (and still is serving) the Ministry of Education of Brunei Darussalam as an education officer. Therefore with the consent from the Ministry of Education she is legally authorized to work with, in this case, school children, within her capacity as an Education Officer. As far as loss of individual subject time was concerned, this was an issue that both the school administrators and I were concerned about, especially for the E1 small group teaching sessions. In order to minimize this, the school administrators agreed to let the students be removed from their main class during subjects that the students would not be assessed on (i.e., MIB, ICT, and PE). With the help of the timetabling section of the school, the officers concerned and I managed to arrange a suitable timetable for conducting the small group teaching of the E1 group. Since all the study was conducted during school hours, there was no extra burden required from them (Guideline 18 and 19).

v. *Incentives*

At the end of the study I presented the school with some reading materials (i.e., story books) for their school library and also for the English Resource room that the English Department had kindly allowed me to use for the small group teaching throughout the study. As for the students, no physical incentives were given at any stage of the study, and no economic cost arose in the process of the study either to the school, teachers or students who participated in the study (Guideline 20).

vi. *Detriment Arising from Participation in Research*

Regarding any detriment arising from participation in the study, I do not expect any harm to arise from students' participation in the research. It would be my expectation that the students will gain some beneficial knowledge from the study as it may in the long run open their minds to alternative ways of looking at fractions, even though it may not show (yet) in their results (Guideline 21). As for minimizing the effects of design that advantage or are perceived to advantage the group given the treatment (Guideline 22), upon completion of the study, I shared with the teachers concerned the activities that I had carried out with the

E1 students. It was hoped that they would in turn try these activities with the rest of the students, as I had limited time, and to do the activities with all the six classes would have taken too long.

vii. *Privacy*

Although assurance of confidentiality and anonymity of the school, the principal, the teachers, the parents and the students were given, parents and students were given a choice to waive their right for confidentiality and anonymity if they wished to do so, by indicating in the given consent form (Guideline 23). In this study no personal data, with the exception of the students' names and ages, were collected, therefore Guidelines 24 and 25 were not relevant to this study.

viii. *Disclosure*

At the end of the research the school and parents had access to copies of reports of the study if they wished (anonymity and confidentiality were also maintained in these reports). My email addresses and contact numbers were given to every participant's parents (Guideline 29).

Responsibilities to Sponsors of Research

At the end of the research, copies of the report of findings were made available to the Ministry of Education, Brunei Darussalam, on behalf of the government of Brunei Darussalam, who is the sponsor for the study.

That concludes the ethical issues considerations, and now the limitations of the QED are provided.

4.9 LIMITATIONS

Not all research designs are perfect, and all of them have their own limitations. Similarly, there are several limitations for this study pertaining to the quasi-experimental design.

The first limitation is the small sample size involved in this experimental study. This might have made it more difficult to find and notice statistically significant relationships as compared to in a much larger sample size group. The only thing that I can do to overcome this limitation is to conjecture whether a significant relationship has been found, and to propose further study with a larger sample.

The second limitation is the small-group teaching as opposed to whole class instruction. In this experimental study, the students were grouped into 9 groups of 4 students, and each group was taught during a designated time of the week. The reason for such a special condition was to enable the researcher/teacher to have more individualized interaction with the students; compared to if they were in a whole class setting.

The third limitation is the possibility of the presence of the *Hawthorne effect*. In this case I cannot be completely certain that it did not affect the external validity. If such an effect was in fact present, it might then mean that the result presented in Chapter 6 (see Figure 6.2, p. 153) is inaccurate. However, in order to minimize this effect, if any, I did manage to have access to the E3 group (a whole class) and was able to teach them a topic (i.e., after the experimental study was done) which the subject teacher had assigned to me for the same length of time as E1.

The fourth limitation is regarding the *posttest sensitization*. Like the third limitation, again I cannot be definitely sure that it did not affect the external validity. However, if *posttest sensitization* did appear at all, the most obvious evidence would be students making an extra effort to concentrate more during the lesson, knowing that they would again be tested at a later date. As far as I noticed, this did not happen at all, as the students acted in a similar manner throughout the experimental study.

In addition to these limitations, other limitations related to the Rasch analysis are also discussed in Chapter 5.

This completes the discussion of the QED limitations of this QED methodology chapter. In the next chapter, Chapter 5, the calibration of a fraction ability (FA) scale using the Rasch Model is presented.

CHAPTER 5

A FRACTION ABILITY (FA) SCALE CALIBRATION USING THE RASCH MODEL

5.1 INTRODUCTION

The main aim of chapter 5 is to try to acquire measures of every student's fractions ability (FA) in the quasi-experimental design (QED) at the times of the pre-teaching fractions test, the post-teaching fractions test and the delayed post-teaching fractions test. This in turn, as previously mentioned in Chapter 4, will enable the relevant analyses to be conducted for the QED which will be presented in Chapter 6. In order to remind oneself of the QED for this study, a summary of it can be seen in Figure 4.2, p. 97 (Chapter 4).

In order to obtain an appropriate measure for the students' fractions ability, I needed to calibrate a 'fraction ability scale' for the Form 1 students. The question is "why do we go to all this trouble just to measure students' abilities?"

According to Masters (2001), for the methods' effects to be measured for data collected from the pre-test, the post-test and the delayed post-test, students' abilities measures had to be produced. He added that the measures must be used instead of using students' raw scores since according to him raw scores "do not have the properties of 'measures'" (p. 14). To calculate such measures, the *Rasch model* was employed which will produce *item difficulty* and *person ability* estimates. Bond and Fox (2001, 2007) expressed the relationship for the Rasch model as $P_{ni}(x = 1) = f(B_n - D_i)$ which is the probability (P_n) of person *n* getting a score (*x*) of 1 on a given item *(i)* is a function *(f)* of the difference between a person's ability (B_n) and item's difficulty (D_i) (p. 278). Masters (2001) expressed it as:

$$\beta - \delta = \ln\frac{P_1}{P_0} \qquad (1)$$

where β is the person's ability, δ is the item difficulty, P_1 is the probability of getting a correct answer and P_0 is the probability of getting an incorrect answer in the item (Masters, 2001).

Brief descriptions of the relevant sections in this chapter are now presented which allows for the production of the students' FA measures at the times of the pre-teaching fractions test, the post-teaching fractions test and the delayed post-teaching fractions test:

5.2 A brief literature review on Rasch methodology to give an overview of what was done and why;

5.3 The dataset and item bank used in the calibration of the FA scale;

5.4 The first attempt at analyzing the data (Analysis 1) which was later rejected;

5.5 The second attempt at calibration of the data (Analysis 2) which was finally chosen to be used for the estimation of the FA measures for the students in the QED group;

5.6 The statistical comparison of Analysis 1 and Analysis 2 with particular attention given to the item difficulties and students' abilities on the scales produced by the two analyses;

5.7 A qualitative discussion on the two scales; and

5.8 Conclusion reached about the two analyses.

Next some brief literature review of the Rasch model will be provided, in particular the Rasch assumptions and the postulates of this model.

5.2 THE RASCH MODEL AND ITS RELEVENCE TO THIS STUDY

5.2.1 Dichotomous Rasch Model

According to Wright and Masters (1982), there are five models in the Rasch family: (i) *Dichotomous* or the *Rasch* model; (ii) *Poisson Counts* model; (iii) the *Binomial Trials* model; (iv) the *Rating Scale* model; and (v) the *Partial Credit* model. In fact, they considered the latter four models as an extension of the Dichotomous model. In this study, I chose the Dichotomous model because only "Right" and "Wrong" are recorded for each item (i.e., 0 for wrong, and 1 for right). In this way, no credit will be given for an almost correct answer or an incomplete answer.

Furthermore, according to Wright and Mok (2000):

> . . . in order to construct inference from observation, the measurement model must: (a) produce linear measures, (b) overcome missing data, (c) give estimates of precision, (d) have devices for detecting misfit, and (e) the parameters of the object being measured and of the measurement instrument must be separable. Only the family of Rasch models solves these measurement problems. (pp. 86–87)

Also, according to Panayides, Robinson, and Tymms (2010), “Only the family of Rasch measurement models does this” (p. 624).

5.2.2 Rasch Assumptions

There are three main assumptions in the Rasch family of models which are essential prerequisites of measures that conform to the characteristics of measures mentioned in section 5.1 (Masters, 2001). The three assumptions are as follows: (a) Unidimensionality; (b) Local Independence; and (c) Item Discrimination. Descriptions of each of these assumptions are now presented.

(a) Unidimensionality

The first assumption of Rasch models is *unidimensionality* which Bond and Fox (2007) referred to as the “focus on one attribute or dimension at a time” (p. 24). This assumption, which only relates to a single principal underlying variable or construct (latent trait), is based on the produced scales. This predominant latent trait is assumed to be an *ability* of the person measured against the produced scale (Bond & Fox, 2007) Hence, *person abilities* and *item difficulties* are measured across a common *unidimensional* scale. Wright and Linacre (1989), on the other hand, conjectured that the idea of unidimensionality is a qualitative concept, and they believed that it is virtually impossible to be absolutely certain that this can be fully accomplished in real life. However, they did not totally dismiss it, and they argued that if a generalizable outcome is desired, the idea of unidimensionality needs to be approximated. In this study, the latent trait is an “ability” referring to students’ construct of *fraction*, with conceptions pertinent to the equivalence of fractions and the flexibility of unitization. As mentioned before, I refer to this “ability” as the *fraction ability* (hereafter, FA) and the corresponding scale which will be produced through Rasch methodology as the *FA scale.*

(b) Local Independence

The next assumption of the Rasch models is *local independence,* where there should not be any questions whose answers will depend on answers from other questions (Bond & Fox, 2007; Panayides et al., 2010). With respect to this study, none of the questions depended on the answers from other questions. According to Wright and Linacre (1989), since local independence is always a matter of finding the middle ground, Rasch model fit statistics, namely the infit and outfit, enable the measurement in each case of how far each item fits the model.

(c) Item Discrimination

The last assumption of the Rasch models is *item discrimination,* which refers to the ability of questions to differentiate between low performers and high performers. According to Kelley, Ebel, and Linacre (2002):

> Item discrimination indicates the extent to which success on an item corresponds to success on the whole test. Since all items in a test are intended to cooperate to generate an overall test score, any item with negative or zero discrimination undermines the test. Positive item discrimination is generally productive, unless it is so high that the item merely repeats the information provided by other items on the test. This is the "attenuation paradox." (p. 883)

Panayides et al. (2010), in their response to Dickson and Kohler's (1996) criticism regarding "items to have equal discriminating power" (p. 622), said that:

> The aim of measurement should not be to accommodate the test data but to satisfy the requirements of measurement. The aim is to measure, not to model. The 2-P model, which introduces a discrimination parameter, seeks to fit a model to the data, not vice versa. (p. 623)

5.2.3 Rasch Postulates

Once the above three assumptions are adequately met, *parameter separation* ('item-free' person ability estimates and 'person-free' item difficulty estimates) can be obtained (Bond & Fox, 2007). Wright and Masters (1982) explained that item difficulties are independent of the persons used for their calibration, and similarly, person abilities are also independent of the items that were used for the ability estimation. Bond and Fox (2007) stated that parameter separation is unique to the Rasch model since it "supports direct comparisons of person ability and item difficulty estimates . . . independently of the distribution of those abilities and difficulties" (p. 71). They further summarized it as:

> Rasch measures represent a person's ability as independent of the specific test items, and item difficulty as independent of specific samples within standard error estimates. Parameter separation holds for the entire family of Rasch models. (p. 280)

This unique and fundamental property of the Rasch models allows measurement to be made objectively. This characteristic enables a set of parameters (e.g., the items) to be estimated independently of the values of the other parameters (e.g., the persons) (Bond & Fox, 2007). Therefore, an item's difficulty may be estimated from a selected group of persons and similarly, a person's difficulty may be estimated, based on part of the items calibrated on a scale in the complete dataset. In other words, unbiased estimates of item difficulty parameters can be obtained, regardless of the way in which person measures are distributed.

Consequently, these allow the calibration of the FA scale using an item bank. Item banking is created when items from different instruments are pooled together and they are calibrated based on data collected from representative sample. Masters (2001) defined it as "a set of items calibrated together on a common measurement variable" (p. 32). Rudner (1998) highlighted that among the advantages of item banking is that "new subtests and tests, with predictable characteristics, can be developed by drawing items from the bank" (p. 1). Therefore, for this study it allows different tests to be created to enable a person's FA measurement to be produced.

For this study, multiple linked tests were used, namely three tests with 13 common items linking them. The reason for doing this was because there were too many items in the item bank, and to give all the items in the bank in one test so that they could be calibrated would have posed a problem for the students. Results from such a test might not have been of the expected standard when compared to when the items were split into different shorter tests. This could have been due to students not doing their best when presented with an extremely long test, and on top of that, access to the students would have been difficult because of the length of time required to complete the test. Similar strategy has also been used by other researchers (e.g., Koukkoufis, 2008; Williams & Ryan, 1999).

Hence, based on the dichotomous Rasch model, using the item banking methodology and the linked tests, a Rasch analysis could be produced to calibrate the FA scale.

5.2.4 Validity of Rasch Models

In the context of Rasch methodology, internal validity would refer to the assessment of *fit* to the Rasch model. In other words, fit would mean how accurately the *item difficulties* fit the model and how accurately the *person abilities* fit the model. Keeves and Alagumalai (1999) saw a validity test as a test of one of the Rasch model assumption i.e., unidimensionality. The discussions of *fit statistics* (*item fit* and *person fit*) for Rasch models are now presented.

Item Fit Statistics

Smith, Rush, Fallowfield, Velikova, and Sharpe (2008) reported that:

> There are two commonly used mean square fit statistics, namely the infit mean square (also referred to as the weighted mean square) and outfit (or unweighted) mean square (Rasch Fit Statistics section, para. 2).

Similarly, Bond and Fox (2007) also stated that "Rasch analysis programs usually report fit statistics as chi-square ratios: infit and outfit mean square statistics (Wright, 1984; Wright & Masters, 1981)" (p. 238). Linacre (2010) defined infit and outfit as follows:

> **Infit:** inlier-pattern-sensitive fit statistics…This is more sensitive to unexpected patterns of observations by persons on items that are roughly targeted on them (and vice versa).
> **Outfit:** outlier-sensitive fit statistics. This is more sensitive to unexpected observations by persons on items that are relatively very easy or very hard for them (and vice versa). (p. 488)

From this then, the questions arise, which of these fit statistics is suitable in this case, and what range of values do we consider as fitting the model.

For the analyses in this thesis I will focus on the *infit* statistics for the following reasons:

1. According to Linacre (2010), outfit mean-squares are "less threat to measurement" (p. 490);
2. Linacre (2010) also stressed that "in the Rasch context, outliers are often lucky guesses and careless mistakes, so these outlying characteristics of respondent behavior can make a "good" item look "bad"" (p. 490);

Due to these observations, Wright and Masters (1982) have suggested that infit statistics should be employed instead.

Having said that, a mean-square (infit value) of 1.0 would be the expected value which would reflect a perfect match between the observed responses and the predicted responses by the model. However, such an ideal match would be almost impossible to create. In Bond and Fox (2007), Wright and Linacre (1994b) raised the question of

> when is a mean-square too large or too small? There are no are no hard-and-fast rules. Particular features of a testing situation . . . can produce idiosyncratic mean-square distributions. (p. 241)

Furthermore, Bond and Fox (2007) suggested

> As a rule of thumb, values between 0.70 and 1.30 are generally regarded as acceptable. Values greater than 1.30 are termed misfitting, and those less than 0.70 as overfitting. (p. 310)

However, Adams and Khoo (1993) said that “It is customary for items to be considered to fit the Rasch model if they have *item infit* or *weighted mean square* statistics with the range 0.77 to 1.30” (in Keeves and Alagmumalai ,1999, p. 34). Since I am using Quest for the analyses, I decided to adopt this range of values for the infit statistics in my analyses instead. This would enable me to determine whether the produced scale would have an acceptable overall infit, and also to see if there were any misfitting items that needed to be examined further.

Person Fit Statistics

With regards to item-fit and person-fit, Reise (1990) reported that item-fit analysis is essential in the construction and calibration of instruments where according to him

> IRT (item response theory) requires a formal investigation of the manner in which test items functions as trait measures. Once a formal model has been defined and item response functions (IRFs) have been estimated, studies must be conducted to verify that the observed data conform to the estimated IRFs. Without item-fit analysis, the researcher cannot be confident whether the advantages accured by specifying the formal IRT model can or will be realized. (p. 127)

Regarding person-fit, he states that person-fit analysis is useful “to identify examinees whose response patterns are incongruent with the specified response model” (p. 128).

Since the main focus here was to calibrate a suitable measure for the instruments, the above quotations show that for this study only the item fit statistics needed to be looked at closely. Therefore, the person-fit analysis would not be discussed any further as in this study I am not interested in misfitting persons to an already calibrated instrument. However, in the analyses that follows, the overall infit for persons (cases) will also be presented.

That concludes the discussion of validation of the Rasch models. The next section presents the discussion of the reliability of the Rasch models.

5.2.5 Reliability (Item and Person Separation)

According to Linacre (2010),

> **Reliability (separation index) means "reproducibility of relative measure location".** It does not report on the quality of the data. So "high reliability" (of persons or items) means that there is a high probability that persons (or items) estimated with high measures actually do have higher measures than persons (or items) estimated with low measures. (p. 507)

He also notes that

> If you want high person (test) reliability, you need a person sample with a large ability (or whatever) range and/or an instrument with many items (or long rating scales). If you want high item reliability, you need a test with a large item difficulty range and/or a large sample of persons. (p. 507)

Linacre (2010) also states that, "'person reliability' is equivalent to the traditional 'test' reliability" (p. 508) whereas 'item reliability' has no traditional equivalent" (p. 508). Bond and Fox (2007) considered item *reliability index* or *item reliability* and *person reliability index* or *person reliability* to be analogous to Cronbach's alpha, and they have a lower bound of 0 and an upper bound of 1. The analyses done in this thesis have shown that the person reliabilities are approximately 0.7, which is acceptable, whereas the item reliabilities are around 1 (i.e., 0.97 or 0.99), which is very high. The most likely reason for the not-so-high person reliabilities might be because of the lack of cases with more extreme abilities (high and low) in the three treatment groups. As for the item reliabilities, the high values indicate that in the calibration, the test items did provide a wide item difficulty range.

In addition, an alternative index for both items and persons is the separation index (Bond & Fox, 2007; Linacre, 2010; Wright & Masters, 1982). According to Wright and Masters (1982), both the person separation index and the item separation index are expressed in standard error units (i.e., an estimate of the spread or separation of items (or persons) on the measured variables by dividing the adjusted item (or person) standard deviation by the average measurement error (Bond & Fox, 2007)). Linacre (2010) provided a relationship between separation index and reliability as follows:

$$Reliability = \frac{Separation^2}{(1 + Separation^2)} \quad (3)$$

$$Separation = \sqrt{\frac{Reliability}{(1 - Reliability)}} \quad (4)$$

This enables reliability to be calculated from the separation index and vice versa. Bond and Fox (2001) highlighted that

> Unlike the person separation reliability, the person separation index is not bound by 0 and 1, hence proving more useful for comparing reliabilities across several analyses. (p. 207)

Next, prior to presenting and discussing the analyses, a brief explanation of the relevant dataset and item bank for this study is given below.

5.3 DATASET AND ITEM BANK

In this section, the sample sizes and items administered to each of the treatment groups. The whole dataset comprised of:

1. E1 students (those receiving the fractions instruction);
2. E2 students (those receiving *normal* lesson); and
3. The E3 students (not receiving any treatment at all).

Table 5.1 below presents the number of students in each group, which was also presented in section 4.4 as Table 4.2, p. 100.

Since all the students were tested at three different times (pre-test, post-test and delayed post-test), there are therefore three measures for each student as indicated in the three categories in Table 5.1. All three tests are different, and the complete item bank that made up all three tests can be found in Appendix 16. In order to link the three tests, common items were used to link them, which can be seen in Table 5.2. In the table, items for each test are presented as shaded boxes, with the common items highlighted with the darker shades.

Table 5.1
The Students Involved in the Fraction Ability Scale Calibration

	Pre-Test	Post-Test	Delayed Post-Test	The final number of students in each group
E1	36	33*	35**	33
E2	20	19***	19***	18
E3	33	33	33	33

* = 2 students (Std3 & Std123) were absent and 1 student (Std54) transferred out, so 3 students presented experimental mortality
** = 1 student (Std54) transferred out and presented experimental mortality
*** = 1 student was absent for each test (Std109 for post-test and Std 94 for delayed post-test) and both presented experimental mortality

Based on these collected datasets, I have calibrated the FA scale for all the students, which is presented in the next section.

Note that items 10 to 22 are the common items that linked all the three tests. I will now refer to the three tests as *Test 1* (hereafter, T1) for the pre-teaching fraction test, *Test 2* (hereafter, T2) for the post-teaching fraction test and *Test 3* (hereafter, T3) for the delayed post-teaching fraction test. Using data collected from the above datasets and tests, first I tried to calibrate the FA scale for the individual tests, which is presented in the next section.

Table 5.2

The 3 Linked Tests

Tests	Item 1 – Part-whole model	Item 2 - Equivalent	Item 3 – Translate into fraction	Item 4 - Ordering	Item 5 - Equivalent	Item 6 – Translate into diagram	Item 7 – Equivalent	Item 8 - Equivalent	Item 9 - Subtraction	Item 10 – Division	Item 11 - Sharing	Item 12 – Discrete model	Item 13 – Addition & Subtraction	Item 14 – Addition & Subtraction	Item 15 – Partitioning a diagram	Item 16 - Multiplication	Item 17 - Multiplication	Item 18 – Discrete model	Item 19 – Identify the whole	Item 20 – Discrete model	Item 21 - Subtraction	Item 22 – Part-whole model	Item 23 - Equivalent	Item 24 - Addition	Item 25 – Translate into fraction	Item 26 – Discrete model	Item 27 – Discrete model	Item 28 – Ratio model	Item 29 –Ratio model	Item 30 – Translate into diagram	Item 31 - Sequence	Item 32 – Translate into fraction
Pre																																
Post																																
Del.Post																																

5.4 CALIBRATION OF THE FRACTION ABILITY SCALE FOR ALL THE DATASETS (Analysis 1)

In this calibration process the three tests (T1, T2 and T3) were analyzed as one whole dataset with all the data and students included in the calibration. The purpose of this calibration is to produce a scale suitable for the Form 1 students in the targeted school in Brunei Darussalam. The following steps were taken to produce and interpret the scale for all the 32 items in the dataset whereby the following were evaluated to see if:

i. The item and case (person) estimate are satisfactory for the following;
    a. Overall infit;
    b. Reliabilities; and
    c. Separation indexes.
ii. The student ability and item difficulty estimated are well distributed;
iii. The item difficulty is acceptable; and

iv. There are no misfit items present based on the items' mean square infit (MNSQ Infit) measures.

All the analyses in this study have been conducted using the program *Quest* (Adams & Khoo, 1996). Now, the analysis of the produced scale is presented.

5.4.1 The fraction ability based on all dataset (T1, T2 and T3)

This scale was calibrated using all the items in all the three tests, henceforward the dataset for the combination of these three tests will be referred to as the PPD (i.e., pre-, post- and delayed-test) dataset. Table 5.3 below presents the overall statistics based on all the items in the dataset. The table shows that the scale had a satisfactory 0.99 overall infit item estimates and it also has a satisfactory 0.99 overall infit of case (person) estimates. Table 5.3 also presented an item estimates reliability of 0.98 and a case estimates reliability of 0.77. The separation index is 7.02 for items and 1.71 for persons.

Table 5.3
Overall Statistics for the overall Fractions Test (PPD)

	Items	Cases (Persons)
Overall Infit	0.99	0.99
Reliability Estimates	0.98	0.77
Separation Index	7.02	1.71

As for the item fit, from Figure 5.1 below, we can see that the infit scale only presents one misfit and two overfit items. The item that presents a misfit is item 18/18/18 which is a multiple choice question and asked the students to identify the whole for the parts of the given diagram. Even though the question required the students to be able to understand the written part of the question, I felt that it should not have affected the students' answers as the question is a straightforward question.

Furthermore, prior to their attempting the questions, each question was translated and explained in their mother tongue by the respective invigilators to ensure that they understood what was required from them. After scrutinizing the students' answers closely, I found that the most common error made by the students was to include "D" as one of their answers (for those who tried to identify all the correct shapes), but the majority were able to pick “A” and “E” as part of their choices. On the other hand, I also felt that the question could have confused some of the students, since in the question the word “shape” was used instead of its plural. After looking at the students’ answers, I felt this was one of the contributing factors for some of the students only giving one out of the five given shapes as their answer. In addition the phrase “You can choose more than one answer” could also lead to the same response from the student, since it implied that that they could give more than one shape if they wish, or they could also give just one shape if they want to.

```
Item Fit                                                         29/ 7/ 9 20: 1
all on ppddata (N = 267 L = 32 Probability Level= .50)
------------------------------------------------------------------------------
INFIT
 MNSQ        .63       .71       .83      1.00      1.20      1.40      1.60
--------------+--------+--------+--------+--------+--------+--------+----------
  1 1//                  *   .              |              .
  2 3//                      .         *    |              .
  3 11//                     .              |         *    .
  4 5//                      *              |              .
  5 8//                      .      *       |              .
  6 4//4                     .              |*             .
  7 6a//6a                   .              |    *         .
  8 6b//6b                   .              |*             .
  9 12//12                   .      *       |              .
 10 2/2/2                    .              |          *   .
 11 7/7/7                    .       *      |              .
 12 9/9/9                    .              *              .
 13 10/10/10                 .     *        |              .
 14 13/13/13                 .    *         |              .
 15 14/14/14                 .              | *            .
 16 15/15/15                 .              |    *         .
 17 16/16/16                 .      *       |              .
 18 17/17/17                 .              |*             .
 19 18/18/18                 .              |              .*
 20 19/19/19                 .            * |              .
 21 20/20/20                 .              |   *          .
 22 21/21/21                 .      *       |              .
 23 /1/1                     .              |       *      .
 24 /4/3                     .              |    *         .
 25 /8/5                     .           *  |              .
 26 /11a/8a                  .              |     *        .
 27 /11b/8b                  .       *      |              .
 28 /12a/11a                 .              *              .
 29 /12b/11b                 .             *|              .
 30 /3/                      .              |    *         .
 31 /5/                    * .              |              .
 32 /6/                      .              |     *        .
===============================================================================
```

Figure 5.1. Infit for the FA Scale based on the PPD (All) Test

```
Item Estimates (Thresholds)                                  29/ 7/ 9 20: 1
all on ppddata (N = 267 L = 32 Probability Level= .50)
----------------------------------------------------------
  5.0                                 |
                                      |
                                      |  /5/
                                      |
                                      |
                                      |
  4.0                                 |
                                      |
                                      |
                                      |
                                 X    |
                                      |
  3.0                            X    |  16/16/16
                                 X    |
                                      |  4//4
                                 X    |  21/21/21
                                 X    |  18/18/18
                                      |  /11b/8b
  2.0                            X    |
                             XXXXX    |  /11a/8a  /12b/11b
                                 X    |  19/19/19
                           XXXXXXX    |
                             XXXXX    |
                                 X    |
  1.0                        XXXXX    |  /12a/11a
                              XXXX    |  5//
                   XXXXXXXXXXXXXXX    |  3//
                            XXXXXX    |  9/9/9
              XXXXXXXXXXXXXXXXXXXX    |  7/7/7
                              XXXX    |  17/17/17  20/20/2
           XXXXXXXXXXXXXXXXXXXXXXX    |  8//
   .0                       XXXXXX    |  14/14/14
                     XXXXXXXXXXXXX    |
             XXXXXXXXXXXXXXXXXXXXX    |  11//
                         XXXXXXXXX    |  2/2/2  10/10/10
      XXXXXXXXXXXXXXXXXXXXXXXXXXXX    |  13/13/13
                            XXXXXX    |  12//12
 -1.0                XXXXXXXXXXXXX    |
                              XXXX    |
                           XXXXXXX    |
                    XXXXXXXXXXXXXX    |
                            XXXXXX    |
                               XXX    |
 -2.0                          XXX    |  1// 15/15/15 /3/
                        XXXXXXXXXX    |
                           XXXXXXX    |  /1/1
                                      |
                             XXXXX    |  /6/
                                XX    |  /4/3   /8/5
 -3.0                                 |
                                 X    |
                                      |  6b//6b
                                      |  6a//6a
                                      |
                                 X    |
 -4.0                                 |
----------------------------------------------------------
  Each X represents    1 student
```

/5/(wc) – sequence
16/16/16(c) – multiplication with an integer
4//4(wc) – part-whole model
21/21/21(c) – part-whole model
18/18/18(wc) – part-whole model
/11b/8b(wc) – division of an integer by a fraction
/11a/8a(wc) – division of an integer by a fraction
/12b/11b(c) – ratio model
19/19/19(c) – discrete model
/12a/11a(c) – ratio model
5//(c) – ordering
3//(c) – subtraction with different denominators
9/9/9(wc) – ratio model
7/7/7(c) – quotient model (≡ 14/14/14)
17/17/17(c) – discrete model
20/20/20(c) - subtraction with different denominators
8//(wc) – part-whole model
14/14/14(wc) –quotient model (≡7/7/7)
11//(wc) – part-whole model
2/2/2(wc) – division by an integer
10/10/10(wc) – addition with same denominator followed by subtraction of an integer (≡ 13/13/13)
13/13/13(c) – addition with same denominator followed by subtraction of an integer (≡ 10/10/10)
/12//12(wc) – subtraction with different denominators
1//(wc) – part-whole model
15/15/15(wc) – multiplication with a while number
/3/(wc) – part-whole model
/1/1(wc) – equivalent fraction
/6/(wc) – part-whole model
/4/3(wc) – addition with same denominators
8/5(wc) – part-whole model
6b//6b(wc) – equivalent fraction
6a//6a(wc) – equivalent fraction

Figure 5.2. The FA Scale of the PPD (All) Test Items, T1, T2 & T3

From the above explanation, there is no reason why item 18/18/18 could not be included in the analysis. The FA Scale based on the PPD dataset is presented in Figure 5.2 above, and the figure shows that the logits for the students' abilities is normally distributed with a mean of -0.38 and a standard deviation of 1.22. The most able student has a logit of 3.31 and the least able students have a logit of -3.81. In order to avoid confusion at any later stages about the item names, I decided to stick to the original item names that I have given to each item as some of these items will appear in more than one test. For example, ***1//*** would mean that the item appears as question 1 in the pre-test only, ***4//4*** would mean that the item appears as question 4 in the pre-test and the delayed post-test, ***/4/3*** would mean that the item is question 4 in the post-test but it appears as question 3 in the delayed post-test, whereas ***2/2/2*** would mean that the item appears as question 2 in all the three papers, and hence it is one of the common items.

A brief explanation of each item's groupings is given besides the items' names which are all enclosed in the grey box. For example, 18/18/18(wc) is question 18 in the test data, the letters *wc* given in brackets indicates that the question has no context attached to it, and it is also grouped as a part-whole model question.

As for item difficulties, the spread of the items' difficulties is from -3.56 to 4.74 with a mean of 0 and a standard deviation of 2.09. From Figure 5.2 it can be seen that for the two pairs of identical questions (one given a context attached to it whereas the other one is presented without any context–7/7/7 and 14/14/14, and 10/10/10 and 13/13/13), the way students reacted to the questions was similar, and the scale reflects that the item difficulties for these four items are similar, both for the items with context and those without context. For instance, question 7/7/7 has a logit of 0.35 whereas question 14/14/14 has a logit of -0.07, and question 10/10/10 has a logit of -0.59 while question 13/13/13 has a logit of -0.69. From this analysis, the scale also indicates that the most difficult question for this overall scale is question /5/, where the item analysis data shows that 98.8% of the students were not able to give a correct response to this question.

Upon examining the items closely–moving from easier to more difficult items–students found 1-step solutions easier to handle than items that would require at least 2-step solutions. Students also found questions involving equivalent fractions the easiest. Next, of the four operations involving fractions, additions of fractions with the same denominators was easier for the students, while division of an integer by a fraction was the most difficult question for the students to handle.

5.5 CALIBRATION OF THE FRACTION ABILITY SCALE "ANCHORED ON PRE-TEST" (Analysis 2)

5.5.1 Analysis Rationale

Since one of the aims of the study was to measure whether students' achievement had improved as far as *equivalence* and *flexibility in unitization* in fractions is concerned with the introduction of RME-like instructions, I had to measure the students' achievement results after the instruction. In order to do this I had to calibrate a scale of fraction ability based on the pre-teaching fractions test results (prior to the RME-like instruction). The item difficulties were then exported in an anchor file as soon as the calibration was completed. The next step was to calibrate a second scale, based on all the available data (pre-teaching, post-teaching and delayed post-teaching fractions test) based on the anchored item difficulties of the pre-teaching fractions test. Now I had a calibrated scale, instead of just raw data, to find out whether there were improvements or not made by the students as a result of the RME-like instructions that took place.

5.5.2 The fraction ability scale based on the pre-teaching fractions test

The first scale was calibrated using all the items in the T1 test. Table 5.4 below presents the overall statistics based on all the pre-teaching fractions test items. The table shows that the scale had a satisfactory 1.00 overall infit item estimates and it also has a satisfactory 0.98 overall infit of case (person) estimates. Table 5.4 also presented an item estimates reliability of 0.97 and a case estimates reliability of 0.67. The separation index is 5.79 for items and 1.27 for persons.

Table 5.4
Overall Statistics for the Pre-Teaching Fractions Test

	Items	Cases (Persons)
Overall Infit	1.00	0.98
Reliability Estimates	0.97	0.67
Separation Index	5.79	1.27

As for the item fit, from Figure 5.3, we can see that the scale did present one misfit and one overfit item. The misfitting item is item 9/9/9. This item was purposely included in the item bank to find out whether E1 students in particular actually benefited from the lesson and would be able to respond to the question correctly or not, or does it not made any difference at all in the way they responded to the question. At T1, almost 73% of the E1 students got this question wrong with the majority giving $\frac{12}{9}$ as their answer. Item 9/9/9 was concerned with whether students could see the whole as a single unit, and asked the students to translate a given measurement (in inches) with respect to the whole (i.e., one foot) in terms of fractions.

```
Item Fit                                                           5/ 8/ 9 22: 3
all on predata (N = 89 L = 22 Probability Level= .50)
--------------------------------------------------------------------------------
INFIT
MNSQ        .63       .71       .83      1.00      1.20      1.40      1.60
---------------+--------+--------+--------+--------+--------+--------+----------
1 1//                  *    .              |              .
2 2/2/2                     .          *   |              .
3 3//                       .              |          *   .
4 4//4                      *              |              .
5 5//                       .      *       |              .
6 6a//6a                    .              | *            .
7 6b//6b                    .              |      *       .
8 7/7/7                     .              |*             .
9 8//                       .           *  |              .
10 9/9/9                    .              |              .   *
11 10/10/10                 .      *       |              .
12 11//                     .              |*             .
13 12//12                   .   *          |              .
14 13/13/13                 .          *   |              .
15 14/14/14                 .          *   |              .
16 15/15/15                 .              |        *     .
18 17/17/17                 .              |*             .
19 18/18/18                 .              |            * .
20 19/19/19                 .              |  *           .
21 20/20/20                 .              *              .
22 21/21/21                 .   *          |              .
```

Figure 5.3. Infit for the FA Scale based on the Pre-Teaching Fractions Test, T1

The misfitting could have occurred because the students were just guessing the answers by either giving $\frac{9}{12}$, which is the correct answer, or they could be giving $\frac{12}{3}$ $(= 1\frac{1}{3})$ which many students also gave as their answer. I felt that this question should be retained as it could, as mentioned earlier, give a little insight on whether the students gained some knowledge from the experimental lessons.

The FA Scale is presented in Figure 5.4 below. Figure 5.4 below shows that the logits for the students' abilities are almost normally distributed with a mean of -0.38 and a standard deviation of 1.02.The most able students have a logit of 1.77 and the least able students have a logit of -2.70. Similar to the previous analysis, a brief explanation of each item's groupings is given besides the item's name, which is all enclosed in the grey box.

As for the item difficulties, the spread of the items difficulties is from -3.72 to 2.98 with a mean of 0 and a standard deviation of 1.78. From Figure 5.4 it can be seen that for the two pairs of identical questions (one given a context attached to it whereas the other one is presented without any context–7/7/7 and 14/14/14, and 10/10/10 and 13/13/13), there is a marked difference in how the students reacted to the questions. For instance, question 7/7/7 has a logit of -3.29 whereas question 14/14/14 has a logit of 0.71, which is a huge approximately 4 logits difference in item difficulty. Question 7/7/7 is a question with context attached to it whereas question 14/14/14 is the identical question but without any context given to the question. It seemed that for these particular pairs of questions, the students found it easier to deal with problems with a context than those without any context. A similar scenario occurs with the other pair of identical questions (i.e., 10/10/10 and 13/13/13) where there is about a 1 logit difference.

As for the three questions (6a//6a, 21/21/21 and 19/19/19) which the students found the hardest, it was interesting to note that the students found question 6a//6a the hardest of all, but at the same time, the similar question 6b//6b was found to be one of the easiest questions, alongside question 7/7/7. These two questions are actually the sub-questions of question 6 in the test where it involved equivalent fractions, and the question asked students to find a missing value from a given pair of equivalent fractions. In question 6a//6a, the numerator of the second fraction was missing, and it involved division to find the missing value which the students found difficult. On the other hand, in question 6b//6b the denominator of the first fraction was missing, and I assumed that in this case the students easily saw the connection which involved multiplication. In fact, if question 6a//6a had been written in reverse, it would be almost the same as question 6b//6b. As for questions 21/21/21 and 19/19/19, even though these two questions were presented with a context, in order to arrive at the solution at least two steps were required, unlike the other questions in this test which only required a one step solution, except for question 13/13/13.

5.5.3 The fraction ability scale based on the PPD (All) tests Dataset

The item difficulties in the scale based on T1 were used to anchor the difficulties of the items in the scale based on T1, T2 and T3 (i.e., PPD data). This scale was constructed by including all the items in the item bank. In the analysis, item 2/2/2 presented a slight misfit with an infit mean square of 1.40 (see Figure 5.5 below). I was genuinely surprised that this item presented a misfit (infit mean square of 1.4), because item 2/2/2 (i.e., What is $\frac{1}{3} \div 4$?) was a standard straightforward textbook and examination question that in my opinion should not have posed or presented any problems to the students. Nor did the question have a "language demand" on the students' language abilities. Thus we can rule out that the misfit is due to the possibility that the question is testing other things apart from the students' knowledge of fractions i.e., the assumption on unidimentionality is upheld. It is noteworthy that the most common error made by students in this question, which occurred in all the three tests (T1, T2 and T3), is a classic one. The students tried to remember and apply the "multiply and invert" rule when dealing with this problem. Although they remembered that the division sign needed to be changed to multiplication, they unfortunately forgot to invert the second value. Hence, many ended up with an answer of $\frac{4}{3}$ or $1\frac{1}{3}$.

```
Item Estimates (Thresholds)                                        5/ 8/ 9 22: 3
all on predata (N = 89 L = 22 Probability Level= .50)
-------------------------------------------------------------------------------
  3.0                           | 6a//6a
                                |
                                |
                                |
                                |
                                | 19/19/19   21/21/21
                                |
                                |
  2.0                           |
                                | 18/18/18
                         XXX    |
                                |
                           X    |
                                |
                                |
                        XXXX    | 11//
  1.0                           | 4//4
                      XXXXXX    | 2/2/2
                                | 10/10/10   14/14/14
                    XXXXXXXX    |
                           X    |
                                | 17/17/17
                    XXXXXXXX    | 5//    20/20/20
                                |
   .0                   XXXX    |
                                | 3//
                   XXXXXXXXX    | 12//12
                                |
                           X    | 13/13/13
         XXXXXXXXXXXXXXXXXXX    |
                                |
 -1.0                XXXXXXX    |
                                | 8//    9/9/9
                                |
                      XXXXXX    |
                                |
                                |
                        XXXX    |
                                |
 -2.0                      X    | 1//
                           X    |
                        XXXX    | 15/15/15
                                |
                                |
                           X    |
                           X    |
                                |
 -3.0                           |
                                |
                                | 7/7/7
                                |
                                |
                                |
                                | 6b//6b
                                |
 -4.0                           |
----------------------------------------------------------------------------------
-- Each X represents    1 student
```

6a//6a(wc) – equivalent fractions

19/19/19(c) – discrete model

21/21/21(c) – part-whole model

18/18/18(wc) – part-whole model

11//(wc) – part-whole model

4//4(wc) – part-whole model

2/2/2(wc) – division by a whole number

10/10/10(wc) – addition with same the denominator followed by subtraction of a whole number (≡ 13/13/13)

14/14/14(wc) –quotient model (≡ 7/7/7)

17/17/17(c) – discrete model

5//(c) – sequence

20/20/20(c) - subtraction with different denominators

3//(wc) – part-whole model

12//12(wc) – subtraction with different denominators

13/13/13(c) – addition with same denominator followed by subtraction of a whole number (≡ 14/14/14)

8//(wc) – equivalent fractions

9/9/9(wc) – ratio model

1//(wc) – part-whole

15/15/15(wc) – multiplication with a whole number

7/7/7(c) – quotient model (≡ 14/14/14)

6b//6b(wc) – equivalent fractions

Figure 5.4. The FA Scale of the Pre-Teaching Fractions Test Items, T1

Table 5.5 below presents the overall statistics based on all the items in the dataset. The table shows that the scale had a satisfactory 1.01 overall infit item estimates and it also has a satisfactory 1.03 overall infit of case (person) estimates. Table 5.5 also presented an item estimates reliability of 0.99 and a case estimates reliability of 0.77. The separation index is 7.02 for items and 1.71 for persons.

```
Item Fit                                                         29/ 7/ 9 20: 0
all on all (N = 267 L = 32 Probability Level= .50)
-------------------------------------------------------------------------------
INFIT
 MNSQ        .63       .71       .83      1.00      1.20      1.40      1.60
--------------+--------+--------+--------+--------+--------+--------+----------
  1 1//              *       .              |              .
  2 3//                      .         *    |              .
  3 11//                     .              |         *    .
  4 5//                     *.              |              .
  5 8//                      .      *       |              .
  6 4//4                     .              |     *        .
  7 6a//6a                   .              |         *    .
  8 6b//6b                   .              |*             .
  9 12//12                   .         *    |              .
 10 2/2/2                    .              |              .    *
 11 7/7/7                    .           *  |              .
 12 9/9/9                    .              |      *       .
 13 10/10/10                 .      *       |              .
 14 13/13/13                 .    *         |              .
 15 14/14/14                 .              |            * .
 16 15/15/15                 .              |         *    .
 17 16/16/16                 .      *       |              .
 18 17/17/17                 .              |*             .
 19 18/18/18                 .              *              .
 20 19/19/19                 .              |            * .
 21 20/20/20                 .              |  *           .
 22 21/21/21           *     .              |              .
 23 /1/1                     .              |        *     .
 24 /4/3                     .              |    *         .
 25 /8/5                     .           *  |              .
 26 /11a/8a                  .              |     *        .
 27 /11b/8b                  .       *      |              .
 28 /12a/11a                 .              *              .
 29 /12b/11b                 .             *|              .
 30 /3/                      .              |     *        .
 31 /5/                     *.              |              .
 32 /6/                      .              |      *       .
```

Figure 5.5. Infit for the FA Scale based on the PPD (All) (Anchored) Test

The FA Scale is presented in Figure 5.6 below. Figure 5.6 shows that the logits for the students' abilities is almost normally distributed with a mean of -0.28 and a standard deviation of 1.23.The most able students have a logit of 3.33 and the least able students have a logit of -3.76. Similar to the previous analysis, a brief explanation of each item's groupings is given besides the item's name which is all enclosed in the grey box.

Table 5.5

Overall Statistics for the overall Fractions Test (PPD anchored to PRE)

	Items	Cases (Persons)
Overall Infit	1.01	1.03
Reliability Estimates	0.99	0.77
Separation Index	7.02	1.71

As for the item difficulties, the spread of the items difficulties is from -3.72 to 4.55 with a mean of 0.09 and a standard deviation of 2.07. From Figure 5.6 it can be seen that for the two pairs of identical questions (7/7/7 and 14/14/14; 10/10/10 and 13/13/13), the way students reacted to the questions is similar, and the FA scale for these four items shows that the item difficulties are not much different, for both the items with context and those without context. For instance, question 7/7/7 has a logit of 0.69 whereas question 14/14/14 has a logit of 0.71, and question 10/10/10 has a logit of -0.33 while question 13/13/13 has a logit of -0.58. From this analysis, the scale also indicates that the most difficult question for this overall scale is question /5/ with item difficulty of 4.55, and the item analysis data shows that only 1.2% of the students were able to give a correct response to this question.

```
Item Estimates (Thresholds)                                       29/ 7/ 9 20: 0
all on all (N = 267 L = 32 Probability Level= .50)
------------------------------------------------------------
  5.0                          |
                               |
                               |
                               | /5/
                               |
                               |
  4.0                          |
                               |
                               |
                               |
                        X      |
                        X      |
  3.0                   X      | 4//4
                               | 16/16/16
                        X      |
                        X      |
                               | 19/19/19   21/21/21
                               | /11b/8b
  2.0                 XXX      |
                               | 18/18/18 /11a/8a /12b/11b
                     XXXX      |
                               |
                      XXX      |
                        X      | /12a/11a
  1.0                XXXX      | 5//    9/9/9
                 XXXXXXXX      |
                      XXX      | 3//  7/7/7  14/14/14
                   XXXXXX      |
                   XXXXXX      |
             XXXXXXXXXXXX      | 8// 17/17/17 20/20/20
                        X      |
   .0           XXXXXXXXX      |
                       XX      | 11//
            XXXXXXXXXXXXX      | 10/10/10
                        X      | 13/13/13
        XXXXXXXXXXXXXXXXX      |
 -1.0           XXXXXXXXX      | 12//12   2/2/2
                               |
               XXXXXXXXXX      |
                        X      |
                     XXXX      |
                       XX      | /3/
 -2.0                   X      | 1//
                XXXXXXXXX      | 15/15/15   /1/1
                               | /6/
                               | /4/3
                     XXXX      | /8/5
                               |
 -3.0                          |
                        X      |
                               | 6b//6b
                               |
                               | 6a//6a
                        X      |
 -4.0                          |
--------------------------------------------------------------------------------
 Each X represents    2 students
```

/5/(wc) – sequence
4//4(wc) – part-whole model
16/16/16(c) – multiplication with an integer
19/19/19(c) – discrete model
21/21/21(c) – part-whole model
/11b/8b(wc) – division of an integer by a fraction
18/18/18(wc) – part-whole model
/11a/8a(wc) – division of an integer by a fraction
/12b/11b(c) – ratio model
/12a/11a(c) – ratio model
5//(c) – ordering
9/9/9(wc) – ratio model
3//(c) – subtraction with different denominators
7/7/7(c) – quotient model (≡ 4/14/14)
14/14/14(wc) –quotient model (≡7/7/7)
8//(wc) – equivalent fractions
17/17/17(c) – discrete model
20/20/20(c) - subtraction with different denominators
11//(wc) – part-whole model
10/10/10(wc) – addition with same denominator followed by subtraction of an integer (≡13/13/13)
13/13/13(c) – addition with same denominator followed by subtraction of an integer (≡10/10/10)
12//12(wc) – subtraction with different denominators
2/2/2(wc) – division by an integer
/3/(wc) – part-whole model
/1/1(wc) – equivalent fractions
15/15/15(wc) – multiplication with a
1//(wc) – part-whole model while number
/6/(wc) – part-whole model
/4/3(wc) – addition with same denominators
/8/5(wc) – part-whole model
6b//6b(wc) – equivalent fractions
6a//6a(wc) – equivalent fractions

Figure 5.6. The FA Scale of the PPD (All) (Anchored) Test Items

5.6 ANALYSIS 1 & ANALYSIS 2 STATISTICAL COMPARISON

In order to find out whether or not the two analyses above generated significantly different estimations of item difficulties and student abilities, another statistical comparison was conducted. The table below presents a summary of analysis 1 and analysis 2.

Table 5.6
Summary of Analysis 1 and Analysis 2

Analyses	Item Reliability	Case Reliability	Separation Index	Misfit (Infit MNSQ)
Analysis 1 - Unanchored	0.98	0.77	7.02	1818/18 (1.31)
Analysis 2 - Anchored to Pre	0.99	0.77	7.02	2/2/2 (1.40)

The presence of the moderate misfits for the two analyses was already discussed above. In addition, I would like to add that the two respective items presenting misfits could also be due to the small sample size in the study.

Next, comparisons of the items difficulties and students abilities generated by the two analyses were also done to see if they produced different results. They were contrasted by using scatter plot graphs and also the use of trend lines. The 95% Confidence Intervals (hereafter, CI) bands for the trend lines will also be constructed to help in the comparison. From these, we will be able to get the coefficient of the independent variable and the R^2 value for the trend line. The degree to which the two FA measures differ can also be seen from the positions of the points with respect to the CI bands.

The analysis is conducted by using SPSS 16.0 software. I am going to start with the comparison of the item difficulties, followed by the students' abilities.

i. Item Difficulties

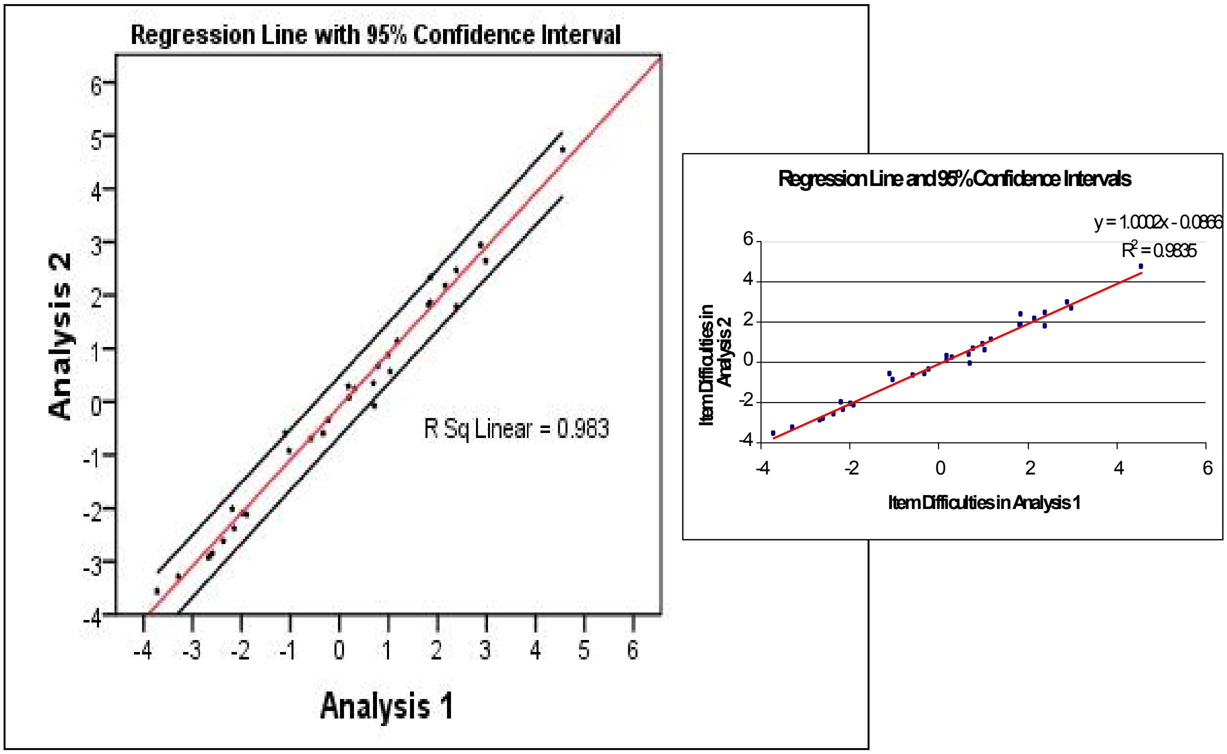


Figure 5.7. Regression Graph for Item Difficulties

The item difficulties in the two analyses are compared in Figure 5.7 above. A linear regression line (or, trend line) for the two analyses is also produced, which is represented by the red straight line in the middle. The other two lines on either side of the trend line are the lines that represent the 95% CI for the trend line.

From the above figure, it can be seen that all, except for one, of the points are either inside the confidence interval bands or are on the boundaries themselves, and even the only point that is outside is just barely outside the boundary line. In addition, the equation of the line indicates that the coefficient shows that, on the whole, the item difficulties for the two analyses are almost equal. This is further confirmed by the R^2 value which is 0.9835, which again is evidence that there is a really strong positive correlation between the FA estimates produced by Analysis 1 and Analysis 2.

ii. *Student Abilities*

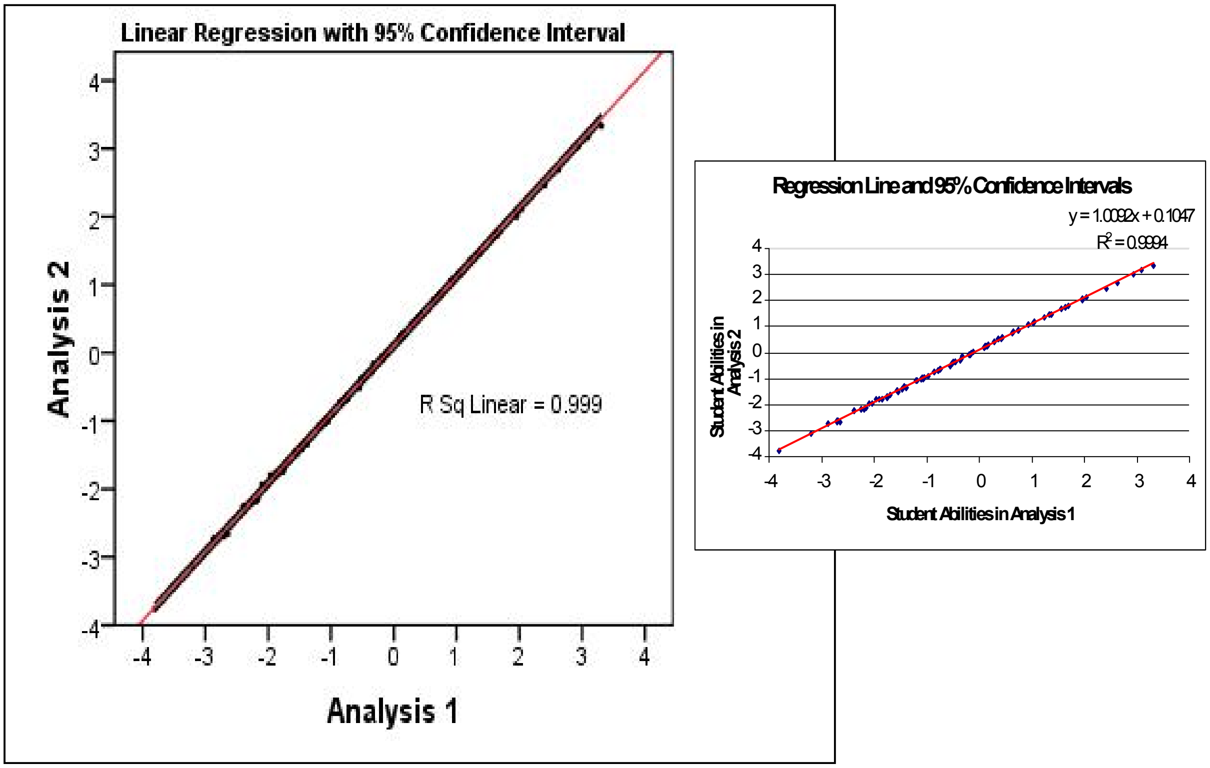


Figure 5.8. Regression Graph for Student Abilities

The student abilities in the two analyses are contrasted in Figure 5.8 above. As with the item difficulties, a trend line for the two analyses was produced which is represented by the red straight line in the middle. The other two lines on either side of the trend line are the lines that represent the 95% CI for the trend line. Data from all the students in the QED group were used.

From the above figure, it can be seen that the confidence interval is even narrower than that for the item difficulties estimates, which gives an indication that there is an even smaller standard error for student ability estimates. In addition, all of the points are either inside the confidence interval bands or are on the boundaries themselves. Furthermore, the equation of the line indicates that the coefficient (i.e., 1.0092) shows that, on the whole, the student abilities for the two analyses are almost equal. This is further confirmed by the R^2 value which is 0.9994, which again is evidence that there is an extremely strong positive correlation between the FA estimates produced by the two analyses.

From the result of the statistical comparison presented above, it can be concluded that the two analyses generate almost equal estimates, and either one of the measures produced by the respective analysis can be chosen. Therefore, based on the following qualitative discussions and the theoretical discussions presented earlier, the selection of the most appropriate FA measure scales will be made.

5.7 QUALITATIVE DISCUSSION ON THE TWO SCALES

As presented in section 5.4 and section 5.5.1 earlier, the rationales or purposes for the two analyses differ slightly. The first was more to examine the general improvement (if any) made by the students throughout the three different tests, whereas the second analysis aimed specifically to measure the improvement of the students' abilities in relation to the pre-test instrument.

In the first analysis (analysis 1), the process is a simple straightforward analysis, where in this calibration process the three tests (T1, T2 and T3) were analyzed as one whole dataset with all the data and students included in the calibration. The main purpose was to produce a scale suitable for students in Form 1 classes in the targeted school in Brunei Darussalam. This analysis produced an appropriate FA scale for the Form 1 students, where their ability in the three tests was measured independently from each other. Since one of my main interests in this study was to find out how much improvement they had made with the introduction of RME-like instructions, I was not completely confident that the scale produced would provide a trustworthy calibration of the item difficulties and student abilities for the Form 1 students in the QED group. Therefore, to overcome this I had generated another calibration that I felt would be able to measure the students' improvement (if any) in order to develop a better scale which would suit the purpose of the scale.

In my opinion, Analysis 2 produced a scale which does just that. In Analysis 2, the analysis construct compels the calibration of all the items to be compared to the pre-test because of the anchoring method used. As mentioned earlier, since I wanted to measure students' abilities, I had to calibrate a scale of fraction ability based on the pre-teaching fractions test results (prior to the RME-like instructions) by only anchoring 22 out of the

32 items independently. Through this anchoring procedure, a more reliable and suitable FA scale was produced.

As seen from the previous section, section 5.6, the overall statistics for the two analyses are almost the same. The relative positions of the item difficulties for all the items are also almost the same, and the unanchored items that the students found to be easier for the two analyses are the same. Based on these, at this stage, I cannot infer that there is any conclusive evidence yet that there has been any improvement of the students' fractional ability from the pre-teaching fractions test to the post-teaching fractions test.

5.8 CONCLUDING THE RASCH ANALYSIS

Based on the discussion presented in section 5.7, I have decided to choose the FA measures from the second analysis. I felt that it was the most suitable scale to use as it was generated based on the scale from the pre-teaching fractions test data which would, in my opinion, give a better comparison and indication of the students' performance to the subsequent two tests. On the other hand, I need to highlight once again the limitation of this analysis that students found half of the unanchored items hard and also the misfit presented by item 2/2/2 which could affect the effect sizes for the experimental group of the study.

From all the above discussion points, I can confidently say that an appropriate FA scale has been constructed through analysis 2, which has been anchored on the pre-teaching fractions test. From this scale it would now be possible to measure the E1 students' abilities. Hence, based on Rasch methodology assumptions, the produced FA scale can now be use to measure the fractions ability of Form 1 students in the targeted secondary school in Brunei Darussalam.

Chapter 5 is now concluded after a FA scale has been calibrated which provided FA measures for the students in the QED group at the pre-teaching fractions test, the post-teaching fractions test, and the delayed post-teaching fractions test. Next, following the calibration of the FA measures, statistical analyses of these measures will be presented to find out the outcomes of the QED. The produced FA measures are presented in Table 6.1 and Table 6.2 in the next chapter.

CHAPTER 6
EXPERIMENTAL RESULTS

6.1 INTRODUCTION

Having presented the Quasi-Experimental Design (hereafter, QED) methodology in Chapter 4, in this chapter the result of the QED groups is presented after conducting the statistical analyses based on the calibration of a FA scale in Chapter 5 which afforded a FA measure for the QED students for the pre-teaching fractions test, the post-teaching fractions test and the delayed post-teaching fractions test. These analyses demonstrated the learning outcomes of the three groups - the E1 group, compared to the E2 and the E3 groups.

An overview of what is presented in this chapter is as follows:

i. The FA measures produced for the QED in Chapter 5;
ii. Statistical analyses of the results generated by (a) analyzing the sample frequency distribution; (b) looking at the ability changes of the students through the use of medians and boxplots; and (c) calculating the effect sizes (Becker, 2000; Cohen, 1988) for each QED treatment; and
iii. The conclusions regarding the outcomes of the experimental methods from the quantitative analyses carried out in this thesis.

Hence, chapter 6 will have the following structure:

6.2 FA Measures Produced for the QED;
6.3 Descriptive Statistics; and
6.4 Evaluation of the QED Outcomes.

Next the FA measures produced using the Rasch calibration for the QED are presented.

6.2 FA MEASURES PRODUCED FOR THE QED

The valid cases in the QED after the FA scale calibration in Chapter 5 were:

- E3: 33 students;
- E1: 33 students; and
- E2: 18 students.

The reasons behind how the above sample size numbers were reached were already discussed in Chapter 4 (refer to Table 4.2, p. 100) i.e. after elimination from the Form 1 QED group (n=89) students who presented experimental mortality and the extrapolation for the zero score estimate. In Table 6.1 and Table 6.2 below, the FA measures which were produced based on the pre-teaching fractions test, the post-teaching fractions test and the delayed post-teaching fractions test for the valid cases are presented. Suitable abbreviations are used in the table below to present the different tests: "Pre," "Post," and "DP" for "Pre-teaching fractions test," "Post-teaching fractions test," and "Delayed post-teaching fractions test" respectively.

Table 6.1
QED Students' FA Measures (Logits) for the E3 Group (33 valid cases)

E3 Group					
Pre	Post	DP	Pre	Post	DP
0.78	0.08	0.94	-0.99	-0.86	-2.37
0.22	0.37	0.94	-0.07	0.00	-1.18
-0.07	3.39	3.21	0.52	0.95	-0.52
0.50	0.95	0.65	-0.36	-2.42	-1.95
-0.36	0.08	0.28	-0.99	0.66	0.94
0.50	0.08	0.65	-0.66	-1.20	-1.18
0.50	0.66	0.08	-0.28	0.15	-0.22
-0.36	1.54	0.08	-0.66	-1.20	-1.43
-0.66	-0.54	-0.22	-0.66	-0.54	-0.12
0.76	0.66	0.94	-0.66	0.59	-0.52
1.07	1.85	1.51	-0.66	-1.20	-0.84
0.50	0.37	-0.24	-0.66	0.37	-0.22
0.78	0.66	3.08	-0.99	-1.20	-0.84
-0.99	0.08	-0.22	-0.66	-1.20	-0.52
-0.66	0.54	-0.22	-0.66	-0.85	-0.52
0.78	1.25	0.94	1.70	-0.23	0.94
-0.66	0.08	-0.22			

Note: Yellow boxes = logits of at least 1.0

Table 6.2

QED Students' FA Measures for the E1 (33 valid cases) and E2 (18 valid cases) Groups

E1			E2		
Pre	Post	DP	Pre	Post	DP
1.07	2.76	2.11	-2.66	-0.70	-1.18
0.78	0.95	1.80	-1.34	-1.96	-0.52
1.07	2.18	2.11	-2.70	-1.57	-1.55
0.50	2.53	2.11	-2.15	-1.30	-1.95
1.07	0.42	0.94	-1.73	-1.96	-1.18
1.37	1.25	1.80	-2.17	-0.54	-0.84
1.70	1.54	1.80	-1.00	-1.96	-1.95
0.22	0.66	1.22	-1.73	-2.42	-3.54
0.50	-0.20	-0.52	-1.34	-1.96	-1.95
0.50	0.64	0.65	-1.96	-1.04	-0.70
0.22	-0.23	0.94	-2.17	-1.96	-2.39
0.42	0.37	1.51	-1.73	-1.20	0.08
0.36	0.37	0.94	-1.34	-1.96	-0.52
0.22	1.54	1.80	-1.73	-0.23	-1.95
0.22	0.66	2.11	-1.34	-1.96	-2.90
0.22	-0.23	-0.22	-1.34	-0.86	-0.52
-0.07	0.37	0.05	-2.17	-1.20	-1.18
0.22	0.95	1.22	-1.34	-0.54	-1.55
-0.36	-0.23	0.37			
-0.36	0.37	-0.84			
0.22	0.08	0.08			
-0.66	-0.23	-1.27			
-0.66	0.37	-0.22			
-0.66	-1.33	-0.52			
-0.66	-0.85	0.08			
-0.07	0.08	-0.22			
-0.66	-0.54	0.37			
0.50	0.08	1.22			
-0.99	0.37	0.37			
-0.66	0.66	0.94			
-0.36	-0.23	0.37			
-0.66	0.66	0.94			
-0.99	-1.84	-2.39			

Note: Yellow boxes = logits of at least 1.0

Comparing the students' FA measures in Tables 6.1 and 6.2, it can be seen that there are more high numbers among students in the E1 group at the post-teaching fractions test and at the delayed post-teaching fractions test, as compared to the E3 and the E2 groups. In order to verify this further, I decided to concentrate on the students' FA measures of at least 1.0 logit – which are coloured yellow - as an initial indication of the students' fractions ability for each groups. From Table 6.1 and Table 6.2, it can be clearly seen that after the pre-teaching fractions test, there was one student who had a logit of more than one in the E3 group, there were two students in E1 and there was none for E2. As for the post-teaching fractions test, the number of cases with greater than 1.0 logit was 4 cases for the E3 group, 6 cases for E1, and none for E2; while at the delayed post-teaching fractions test, there were only 3 cases for the E3 group, twelve cases for E1 and none for E2. This gave an early indication that there were more cases of higher abilities in E1. From this preliminary observation, a more structured analysis is needed.

Based on the FA measures for the QED students in Table 6.1 and Table 6.2, in the following section, a descriptive data display technique will be used to examine the outcomes of the E1 group as compared to the E3 group and the E2 group.

6.3 DESCRIPTIVE STATISTICS

In this section, the three sample groups were further examined using visual displays based on the measures in Table 6.1 and 6.2 above. The means, medians and frequencies of the three sample groups will be presented using tables and graphs.

<u>Analyzing the Samples' Frequency Distributions and Means for Each Treatment</u>

The data from the three sample groups were first analysed using the sample frequency distributions according to their treatment and type of test (pre-teaching fractions test, post-teaching fractions test and delayed post-teaching fractions test), as shown in the following Figure 6.1. In the presentation of the tables and graphs from this point onwards, the following abbreviations will be used: "C" for the E3 Group; "E1" for the E1 Group; "E2" for the E2 Group; "Pre" for the Pre-Teaching Fractions Test; "Post" for the Post-Teaching Fractions Test; "DP" for the Delayed Post-Teaching Fractions Test. From Figure 6.1, most of the frequency distributions of these three samples for the three different types of tests do not fit very well with the normal distribution, with the exception of the "E1 Post" (i.e., post test for E1) which shows an almost normal distribution. On the other hand, the histograms also demonstrated that there was a wide range of abilities among the samples, which is what would be expected. From the frequency distribution graphs, it can be seen that there is a shift to the right from the pre test to the delayed post test for all the three different groups; indicating an improvement (although very slight). However, of the three groups, students from E1 showed a slightly better improvement than the other two groups.

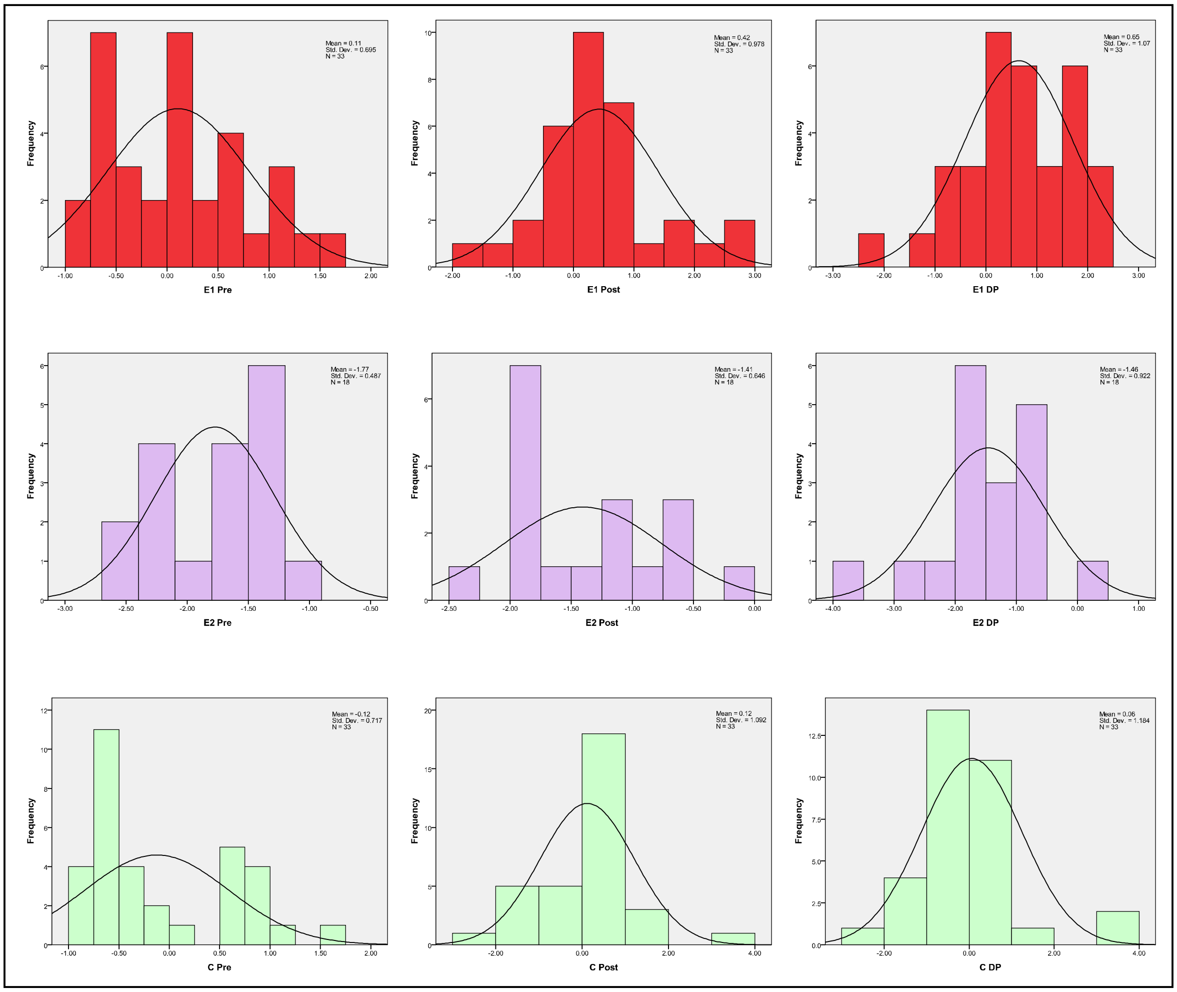
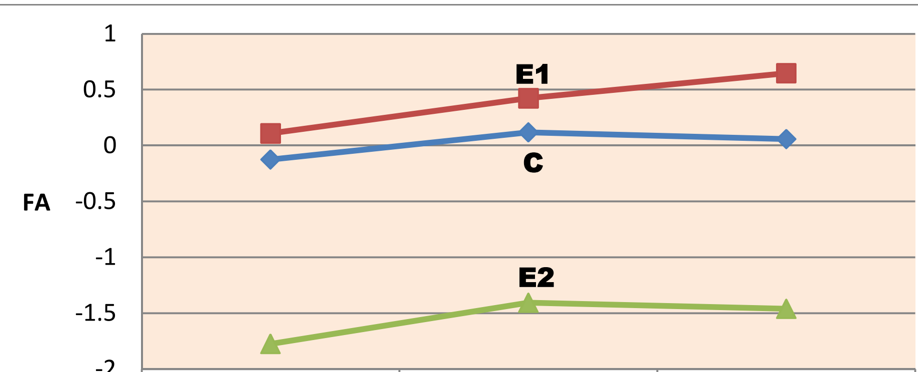


Figure 6.1. The Sample's Fraction Abilities (FA) Frequency Distributions

Figure 6.2 below presents a comparison of the sample means for each different type of treatment. At the pre-teaching fractions test, E1 has the highest mean, while E2 has the lowest mean. This pattern remains unchanged for the three different tests administered to them. It can also be noticed that there is an increase in the means for all the three groups from the pre-teaching test to the post-teaching test. The only explanation that I can think of for such a similar increase in the means is that prior to the time the post-teaching fractions test was administered all the students were given revisions for all their previous topics with their respective mathematics teachers in preparation for the upcoming mid-year examination which all of them have to sit for. It could be argued that the same could also be the case for the improvement seen for E1. Therefore, in this case I cannot be absolutely sure that any improvement made by E1 is directly as a result of the experimental teaching given to them prior to the post-teaching fractions test. However, from Figure 6.2 below, it can be seen that there is also an almost similar increase in the means for E1 at the delayed post-teaching fractions test, whereas there is none for E2 and the E3 group (in fact both show a slight decrease in the means). In view of this, I assume that the increase in the means for E1 could be a result of two factors, namely, any revisions done with their respective mathematics teachers, and also from the experimental teaching. Furthermore, the almost constant increase of the mean for E1 indicates they have made further improvements and that whatever knowledge they have acquired was enhanced and retained, unlike for the E3 and the E2 groups. In addition to that, for E2, the decrease in the means also indicates to me that whatever improvement that E1 has achieved is not just due to the researcher-teacher's presence in the classroom. Therefore, I am confident that the *Hawthorne effect* (see section 4.7) is almost non-existent.



	Pre	Post	DP
C	-0.12455	0.118788	0.059091
E1	0.107879	0.422727	0.646667
E2	-1.77444	-1.40556	-1.46056

Types of Test

Figure 6.2. The Samples Groups' Means for the different tests

Uniformity of Changes in Abilities Among the Different Treatments

In order to further see whether the students in the experimental study have acquired some knowledge from the experimental teaching, individual changes in each student's abilities were looked at to see whether there is a uniform pattern in terms of the increase or decrease of their abilities for each of the different treatment groups between the three tests. This is presented in Table 6.3 below whereby the individual students' FA measures are put under three categories: (i) constant, indicated by a horizontal double-arrow (↔), (ii) an increase, indicated by an arrow upwards (↑), and (c) a decrease, indicated by an arrow downwards (↓). For each treatment group, the mean changes in each student's FA measures are summarized for the different tests, namely, from pre-teaching fractions test to post-teaching fractions test (pre → post), from post-teaching fractions test to delayed post-teaching fractions test (post → DP), and also from pre-teaching fractions test to delayed post-teaching fractions test (pre → DP).

Table 6.3

Frequencies of Mean changes for individual students for the different treatments and tests

QED Treatments	E3 Group			E1 Group			E2 Group		
Type of Changes / Mean Change	↔	↑	↓	↔	↑	↓	↔	↑	↓
Pre → Post	0	20	13	0	21	12	0	11	7
Post → DP	0	16	17	2	21	10	0	10	8
Pre → DP	0	24	9	0	25	8	0	11	7

For the E3 group, the number of increases and decreases in the students' abilities were almost the same between the pre-teaching fractions test and the post-teaching fractions test, and between the pre-teaching fractions test and the delayed post-teaching fractions test; whereas between the post-teaching fractions test and the delayed post-teaching fractions test there were almost as many increases compared to the decreases. These outcomes are in line with the previous results (see Figure 6.2) where, as stated earlier, the considerable increase from the pre-teaching fractions test to the post-teaching fractions test could be due to the effect of the revision sessions given to the students. However, between the post-teaching fractions test and the delayed post-teaching fractions test, the data indicates that the number of students who have improved from the post-teaching fractions test to the delayed post-teaching fractions test is only about half. This suggested that whatever knowledge the students acquired during the revision sessions was not retained by some of the students. Hence it could hardly be said that the improvement seen between the pre- and the post-teaching fractions test was because they had understood what they had "learnt", rather I conjectured that it could be because they were just learning the materials by rote for examination purposes. This agrees with the table above which shows that the number of students whose FA abilities improved between the post- and the delayed-teaching fractions test has dropped. Even though this resulted in slightly more increases between the pre-teaching test and the delayed post-teaching test, which agrees also with Figure 6.2 above, at the same time, Figure 6.2 indicates that there was also a slight drop in the mean between these two tests. For this reason, the effect between the pre-teaching test and the delayed post-teaching test becomes almost negligible, and I believe this is an acceptable comparison for the other two groups.

In the E1 group, the effect between the pre-teaching fractions test and both the post-teaching fractions test and the delayed post-fractions tests are almost the same (see Figure 6.2). On the other hand, Table 6.3 indicates the number of increases and decreases are almost the same between the pre-teaching test and the post-teaching fractions test, and also between the delayed post-teaching fractions test. The outcomes indicate that most of the students benefited from the experimental teaching between the pre-teaching fractions test and the post-teaching fractions test, and also between the pre-teaching fractions test and the delayed post-teaching fractions test. This data agrees with Figure 6.2 above, which also shows an increase in the means from the pre-teaching test to the post-teaching test, and also to the delayed post-teaching test.

As for E2, when compared to the pre-teaching fractions test, both at the post-teaching fractions test and the delayed post-teaching test, the number of increases and decreases is almost the same. This effect (see Figure 6.2) is in fact a similar pattern to the E3 group; the only difference is that the average mean for E2 is lower than that of the E3 group. This again confirms my proposal of the non-existence of the *Hawthorne effect* in the experimental study.

Next, the median, distribution and extreme case examinations were done for the QED treatments' sample, using boxplots clustered by test type and by treatments, as shown in Figure 6.3 and Figure 6.4 below.

From Figures 6.3 and 6.4 below, for the E3 group, the median for the sample moved somewhat at the post-teaching fractions test, but decreased again to almost the same logit as in the pre-teaching fractions test at the delayed post-teaching fractions test. So it can be said that there is no significant increase in the median between the pre-teaching fractions test and the delayed post-teaching fractions test. This indicates that there is a slight regression effect among the higher achievers. With regards to the variance for the E3 group, this remained the same at the post-teaching test and increased slightly at the delayed post-teaching test. Hence, I conjectured that whatever knowledge the students had *temporarily gained* at the post-teaching fractions test as a result of the revision session given by their respective mathematics teachers was not retained at the delayed post-fractions test. I assumed one possible reason for this was that they were learning the material by rote without much understanding involved.

As for E1, the median for the group kept on increasing from the pre-teaching fractions test to the delayed post-teaching fractions test (E1pre < E1post < E1dp). This agrees with earlier findings (see Figure 6.2) where the mean increases almost equally between the measurements. Even though the means do not improve by much between the measures, the outcome from Figure 6.4 shows that the variance in the delayed post-teaching fractions test increased, with the upper limit of the box moved upwards in the scale. However, the lower boundary also moved downwards at both the post-teaching and delayed post-teaching fractions tests. This indicates that most of the high-ability students maintained their post-teaching test abilities and benefited from the experimental teaching, hence managing to maintain or improve their FA abilities. At the same time however, there were also student who were showing a regression effect. I assumed that this may have been because they were reverting back to their *normal* way of doing fractions, which may have caused them some confusion. From these indications, I conjectured that, in general, some of the E1 sample managed to retain their recently acquired skills and knowledge, which is indicated by the increase in both the median and the mean between all the measures (i.e., pre → post, post → dp, and pre → dp).

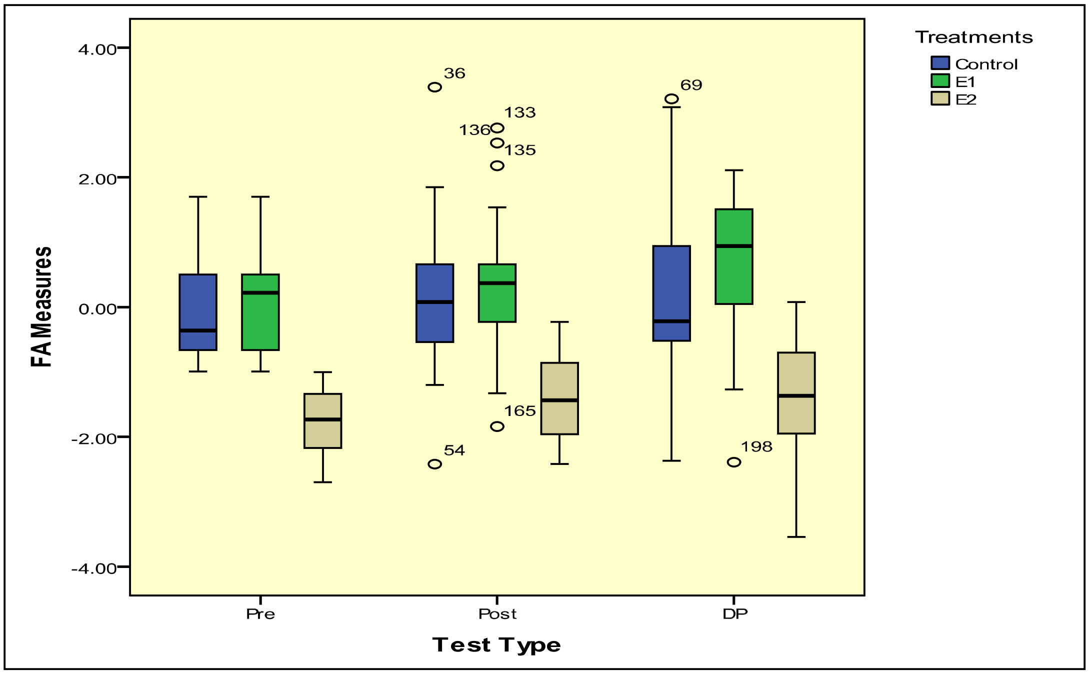


Figure 6.3. QED students' boxplots clustered by treatments

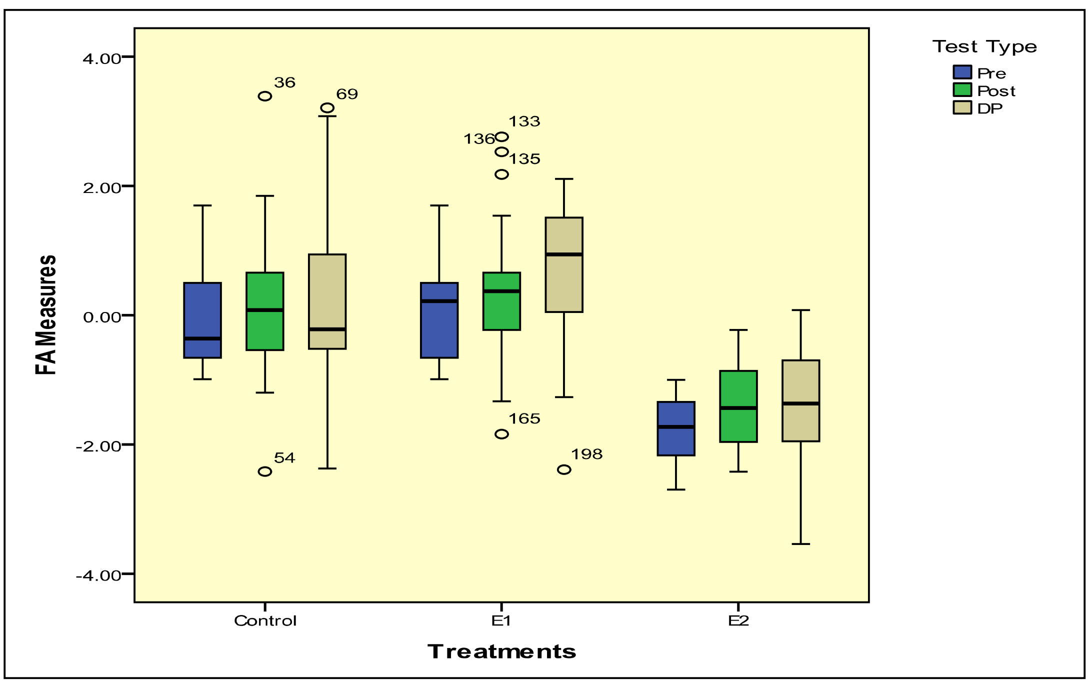


Figure 6.4. QED students' boxplots clustered by test type

In E2, the median for the sample increased slightly at both the post-teaching fractions test and the delayed post-teaching test; while the variance increased at the post-teaching test and remained almost unchanged at the delayed post-teaching test. Although both the median and variance increased between the measures, Figure 6.2 and Table 6.3 above indicates that the means decreased between the measures and also half of the students decreased in their abilities.

Following the presentation of the graphical statistical measures, the effect sizes of the changes in the means of the different treatment groups will now be calculated in order to see the magnitude of the change in each treatment.

Effect Sizes for each treatment

According to Becker (2000) in the overview of his article entitled “Effect Size (ES)”:

> Effect size (ES) is a name given to a family of indices that measures the magnitude of a treatment effect. Unlike significance tests, these indices are independent of sample size. ES measures are the common currency of meta-analysis studies that summarize the findings from a specific area of research. (para. 1)

Becker (2000) also pointed out that there are two methods of measuring effect size, namely, “a) as the standardized difference between two means, or b) as the correlation between the independent variable classification and the individual scores on the dependent variable. This correlation is called the “effect size correlation” (Rosnow & Rosenthal, 1996, p. 333).

The standard difference between two means “is a scale-free measure of the separation between two group means” (Valentine and Cooper, 2003, p. 3). This is also referred to as *d* (Cohen, 1988), which means a descriptive measure. Cohen (1988) defined *d* as the difference between the means, divided by the standard deviation of either group. He argued that when the variances of the two groups are homogeneous, then the standard deviation of either group could be used. Therefore, Cohen (1988) has suggested as a measure of effect size the statistic *d*, where:

$$\boldsymbol{d = \frac{M_1 - M_2}{\sigma}} \qquad (1)$$

The assumption for the above formula is that the standard deviations (i.e., σ_1 and σ_2) are the same, but in reality such cases would very unlikely to occur. To overcome this, the pooled standard deviation, σ_{pooled}, is usually used (Rosnow & Rosenthal, 1996). Becker (2000) stated in his paper that

> The pooled standard deviation is found as the root mean square of the two standard deviations (Cohen, 1988, p. 44). That is, the pooled standard deviation is the square root of the average of the squared standard deviations. When the two standard deviations are similar the root mean square will not differ much from the simple average of the two variances. (Standardized difference between two groups section, para. 4)

Thus, the modified formulae for calculating the effect sizes are as follows:

$$d = \frac{M_1 - M_2}{\sigma_{pooled}} \quad (2)$$

$$\sigma_{pooled} = \sqrt{\frac{(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 + \sigma_7^2 + \sigma_8^2 + \sigma_9^2)}{9}} \quad (3)$$

Becker (2000) also highlighted a possible disagreement about the computation of effect sizes in repeated measure designs in his statement,

> There is some controversy about how to compute effect sizes when the two groups are dependent, e.g., when you have matched groups or repeated measures. (Effect size measures for two dependent groups, para. 1)

Here he was referring to *correlated designs,* where participants are randomly assigned to one of two treatments (i.e., experimental and control). However, in this experimental study the samples were not randomly assigned to treatments. In addition, since in this experimental study the means (*M*) and standard deviations (σ) of the pre-teaching tests treatment cannot be assumed to be equal for all the treatment groups, it would be possible to calculate the effect sizes using one treatment only at each time. Therefore, for each treatment group, three separate comparisons were made: (i) the pre-teaching test with the post-teaching test; (ii) the post-teaching test with the delayed post-teaching test; and (iii) the pre-teaching test and the delayed post-teaching test.

Cohen (1988) has also suggested a categorization of effect size (Table 6.4), while also stressing that:

> . . . there is a certain risk inherent in offering conventional operational definitions for these terms for use in power analysis in as diverse a field of inquiry as behavioral sciences. This risk is nevertheless accepted in the belief that more is to be gained than lost by supplying a common conventional frame of reference

> which is recommended for use when no better basis for estimating the ES index is available. (p. 25)

Table 6.4
Cohen's categorization of effect size

Effect size (*d*)	Size of Effect
$0.2 \le d < 0.5$	Small
$0.5 < d < 0.8$	Medium
$d \ge 0.8$	Large

It is imperative to bear in mind that these effect size measures are not by any means a definitive indication on the relative efficacy of any interventions or treatments. The interpretation of effect sizes depends on the circumstances of the study in question. Table 6.5 below lists some effect sizes from previous researches in the field of education (Lipsey and Wilson, 1993).

Thus, for the QED in this thesis, since the d-values might be affected by the Rasch calibration that underestimated the difficulty of some items in the post-teaching test and the delayed post-teaching test, and due to the fact that the sample size is small, following Cohen's *d* measures interpretation might underestimate the real effect of the experimental lesson on the students' learning. On the other hand, due to the lack of an alternative accepted interpretation to Cohen's, I decided to adopt Cohen's (1988) categorization of effect sizes.

Table 6.6 below presents the means (*M*) and the standard deviations (σ) of each of the treatment groups at the pre-teaching test, the post-teaching test and the delayed post-teaching test.

The effect sizes were calculated using formulae (2) and (3) above based on the means (*M*) and standard deviations (σ) presented in Table 6.6. The calculated effect sizes for each treatment are presented in Table 6.6 below.

Table 6.5
Some examples of effect sizes from educational research studies

Intervention	Outcome	Effect Size	Source
Modern ("new") mathematics curricula vs. traditional instruction	Achievement	0.24	Athappily, Smidchens, and Kofel (1983)
Individualized mathematics instruction for elementary and secondary students	Achievement	0.29	Hartley (1977)
Instructional simulation games vs. Conventional instruction	Cognitive learning	0.33	Szczurek (1982)
Small class size vs. large class size, all grade levels	Achievement	0.20	Hedges & Stock (1983)
Computer assisted instruction vs. conventional methods in secondary school classrooms	Achievement	0.32	Kulik, Bangert, and Williams (1983)

The effect sizes were calculated using formulae (2) and (3) above based on the means (*M*) and standard deviations (*σ*) presented in Table 6.6. The calculated effect sizes for each treatment are presented in Table 6.6 below.

Table 6.6
Sample's Mean and Standard Deviations (SD)

Test Type		C	E1	E2
Pre	Mean	-0.12	0.11	-1.77
	SD	0.717	0.695	0.487
Post	Mean	0.12	0.42	-1.41
	SD	1.092	0.978	0.646
DP	Mean	0.06	0.65	-1.46
	SD	1.184	1.07	0.922

The effect sizes in Table 6.7, following to Cohen's interpretation, indicate that the effect size for E1 between the pre-teaching test and the delayed post-teaching test represents a *medium effect*, whereas the rest represent a *small effect*.

With that the analyses for the QED are completed. Next is the presentation of the evaluation of the findings of the QED.

Table 6.7

Samples' Effect Sizes

	E3	E1	E2
Pre → Post	0.268	0.347	0.403
Post → DP	-0.067	0.257	-0.056
Pre → DP	0.201	0.604	0.347

6.4 EVALUATION OF THE QED OUTCOMES

Before we go further, let me just remind readers, as mentioned earlier (see p. 152), prior to the post-teaching test, all the students were given special revision lessons, including fractions, in preparation for their upcoming examination. This could explain why both the E3 and E2 groups showing positive effect sizes. In addition, I presumed that since the E2 students is from a less-able class, the teacher could have given them extra revision lessons compared to the other two groups of students, hence why they have the highest effect size between the pre- and post-teaching tests. On the other hand, even though E2 presented the highest effect size (i.e., $ES_C < ES_{E1} < ES_{E2}$) between the pre-teaching test and the post-teaching test, the effect sizes between the post-teaching test and the delayed post-teaching test for both the E2 and E3 treatment groups show negative effect sizes which indicate that a regression effect is taking place between these two tests. However, there is a positive "small" improvement between the pre-teaching test and the delayed post-teaching test for both these groups, though it was only a "temporary" effect which shows that they were learning by rote. In contrast, the E1 group presented a positive effect size, indicating improvement (i.e., better learning outcomes), throughout. The repeated positive increase in the effect size indicates that E1 students were able to retain the newly acquired knowledge between the post-teaching test and the delayed post-teaching test, and between the pre-teaching test and the delayed post-teaching test better. These outcomes are also supported by the data presented in Figure 6.2 and Table 6.3. It was also shown in Figures 6.3 and 6.4 that most of the high-ability students in E1 benefitted from the experimental teaching and were able to improve their learning outcomes. In short, even though all the students had been "crashed prepared" for an exam and had positive effect sizes between the pre- and post-teaching tests, only the E1 group sustained that throughout the post- and delayed-teaching tests.

One might ask about the possibility that the increase in effect size shown by E1 might be due to “practice” effect (with 13 items of the three tests being used for all administrations), or student maturation, or some unknown factor (such as some students receiving external tuition), or some combination of all of these. Can this possibility be ruled out? The answer is “Yes,” provided we assume that such factors would have been equally likely to influence the students in all the three treatment groups. Assuming that such factors did apply equally to all groups, the main distinguishing factor for E1 was that it received specific instruction on fractions.

At this point, based on the above analyses, I can conclude that E1 did have relatively better learning outcomes than the other two treatment groups (i.e., E2 and E3). For the rest of the thesis, students’ semiotic meaning-making processes for E1 will be examined. This will begin with the presentation of the semiotic analysis methodology (Chapter 7).

CHAPTER 7
INTRODUCTION AND SEMIOTIC METHODOLOGY

7.1 INTRODUCTION

The semiotic analysis of this study is presented in chapters 7–9. These chapters are related to the second research question (see section 2.8). The semiotic analysis of the RME-like lessons in this study is based on the issue that students' semiotic meaning-making processes are inadequately analysed as far as equivalence and flexibility of unitizing in fractions is concerned. The semiotic analyses will hopefully provide a design for an analytical tool for conducting a more detailed semiotic analysis on fractions, within the RME framework. However, since there is a limited time span to conduct the analyses, the learning trajectory for both the equivalence of fractions and the flexibility of unitization will be based on episodes which I considered to be "typical" for the majority of the groups. This would mean that it is not possible to generalize this trajectory to other topics.

Since there is no separate chapter on methodology, and this is the first chapter which concerns semiotic analysis, this chapter will first tackle the following before dealing with the above issue:-

- The methodology used for the semiotic analysis; and
- With respect to all the semiotic analyses in this thesis, all the relevant semiotic terminology that is used in this semiotic analysis and the necessary coding used in presenting the relevant episodes.

The semiotic investigation of the RME-like lessons for E1 is addressed separately in chapters 8 and 9 with the titles:

CHAPTER 8 – Semiotic Analysis of the Factual Generalization of Equivalence of Fraction

CHAPTER 9 – Semiotic Analysis of the Generalization of the Flexibility of Unitizing of Fraction, the Semiotic Role of the *Hour-Foot* Clock and the Semiotic Network (S-NET) of Gestures and Language

The content of the two chapters above touches various aspects. Chapter 8 contains the following:

- The Semiotic Analysis of the Equivalence of Fraction
- Chain of Signification of the Equivalence of Fraction

Chapter 9 contains the following:

- Semiotic Objectifications of the Flexibility of Unitizing
- Chain of Signification of the Flexibility of Unitizing
- The Semiotic Role of the *Hour-Foot* Clock Model
- Semiotic Network (S-NET) of Gestures and Language
- Concluding the Contribution of Chapters 7–9

The next section discusses the semiotic analysis methodology.

7.2 SEMIOTIC ANALYSIS METHODOLOGY

7.2.1 Introduction

There are five sub-headings in this section which are presented concerning semiotic analyses which are used in chapters 7 to 9. These are as follows:

7.2.2 Semiotic Analysis Methodology;

7.2.3 Data Collection, Instruments and Selection of the Episodes;

7.2.4 Sample and Access to the Data; and

7.2.5 Ethical Considerations.

Next, the semiotic analysis methodology is presented.

7.2.2 Semiotic Analysis Methodology

Unlike the QED methodology, for which the methodology was presented at the beginning of the section, the semiotic analysis methodology could not be presented in such a straightforward manner. Rather it is more fitting to present the appropriate methodology in a step-by-step manner of what kinds of analysis are presented in each chapter, while simultaneously making connections to previous research. This is because semiotics in mathematics education is “still a recent event” (Radford, 2003, p. 39).However, at this stage it is sufficient to say, in the meaning-making process of the experimental fractions instructions, I examined closely the different semiotic means involved in students’ processes and attempted to identify their individual and/or combined contribution. As I will be mainly following in Radford’s footsteps in these analyses, the semiotic means that lead to *objectification* (Radford, 2002, 2003), namely, the *semiotic means of objectification* (hereafter, SMO) (Radford, 2002, 2003) will be of particular importance. There are four SMOs that are examined in this study, namely, gestures (including self gestures–a gesture which has yet to be explored in great detail and discussed in great length), language, symbols and physical pedagogical artifacts (i.e., the HFC model). The semiotic analysis in chapters 7–9 is done in three steps, which I have adapted from Koukkoufis’ (2008) semiotic analysis methodology, which looks at the semiotic activity from different points of view. However, in this semiotic analysis, I will also try to look at another angle of the analysis which has not been done in great detail before. What I am referring to here is that I will attempt to look not only at gestures which are intended for others, but also at those gestures which are not intentional and are not specifically intended to be addressed to others in the immediate surroundings of the gesturer (i.e., self-gesture).

In step 1, the objective is to concentrate on relevant episodes which I felt were significant moments in the experimental instructions, and a close-up microanalysis is conducted for each episode or sub-episode. The analysis carried out is based on Radford’s theory of objectification (Radford, 2002, 2003), and the methodology used is similar to that used by Radford (2003), whereby he talked about the objectifications, how the semiotic means of objectifications played a role in bringing about the objectifications and also the type of generalizations they achieved (i.e., factual, contextual or symbolic). It should be noted that the meanings of the terms “objectification”, “semiotic means of objectification”, or any other terms which are related to them, such as “generalization”, "semiotic contraction" will not be discussed here again, as they have been explained in Chapter 2. By making the most of these theoretic constructs, in order to be able to analyze the students' contributions which relate to their mathematical objectifications and then making apparent their semiotic meaning-making processes, it is imperative to have a systematic and reliable methodology to carry out an interpretive analysis which will involve looking at all the semiotic means that can be found in the students' discourse. In the experimental fractions instructional activities, the students were in groups of 4. All the instructional activities were videotaped and audio taped with the audio tape being used as a backup resource–in fact two audio tapes were used for every lesson, with one audio tape placed between two students–just in case the sound quality of the camera was of poor quality, as the camera was placed at a certain distance from the students. The audiovisual transcripts were then analyzed by using situated discourse analysis, whereby sentences or utterances in significant segments were taken as the point of departure, then moving up to conversation or discourse, or moving down to parts of an utterance or sentence. As in Radford's (2003) work, students' repetitions will be excluded.

> *situated discourse analysis*, which provides an organization of students' utterances in "salient segments," omitting (when necessary) students' repetitions. (Radford, 2003, p. 45)

According to Radford (2000), attaining these significant or salient segments would involve what he called a "three-step" process, which should not be confused with the three steps of the analysis in this chapter. The three-step process entails (i) treating all utterances equally, (ii) categorizing and refining the utterances into salient segments, and contextualizing them by adding interpretative comment (in italics) in the dialogue, and (iii) inserting the tempo of the dialogue by indicating pauses and verbal hesitations (detailed in Radford (2000), p. 244). A similar process was also adopted in the analysis done in this study. Through the analysis done in this step, students' semiosis for each separate objectification would be highlighted, but at this stage, since the objectifications were identified separately, how these objectifications work together in the larger standpoint of the instructions will not be apparent just yet.

After objectifications were identified from the analysis in step 1, based on research studies done by Cobb, Gravemeijer, Yackel, McClain, and Whiteneck (1997) and also

Gravemeijer, Cobb, Bowers, and Whiteneck (2000) on chains of signification in the Realistic Mathematics Education (RME) framework, signs are examined as *signifier-signified* links, thus making up chains of signification. This follows the idea of Arzarello and Edwards (2005), and Koukkoufis (2008), where they proposed that the signifier-signified links are produced through objectifications. In this step (i.e., step 2), based on the objectifications in step 1, chains of significations are produced for equivalence of fractions and flexibility of unitizing. For these, there are eight chains of signification that have been identified and objectified for equivalence of fraction, while eleven chains have been identified and objectified for flexibility of unitizing.

In step 3, based on the chains of signification in step 2 above, how these chains are connected and work together was revealed. This also highlighted how these chains of signification, identified in step 2, made up the semiotic network (S-NET), which consists of gestures and language, which in turn are also creating sub-networks depending on the needs of the development of each aim. Significantly, at certain points there is more than one chain happening at the same time, which enables the formation of new chains.

7.2.3 Data Collection, Instruments and Selection of Episodes

For this study the data was collected through videotaping and audio taping of the small group instructions (presented in section 4.2). A wide range of evidence of the semiotic means that were found in the teachings was captured and recorded. Parental and school permission was obtained prior to videotaping and audio taping the lessons (see section 7.2.4 below).

With regard to the selection of the episodes for the semiotic analysis of this study, I looked at all the data collected through the videotaping for all the instructions in order to identify those episodes that according to my point of view presented the most 'representative' example of the students' learning processes in the experimental fractions instructions. Hence, after examining all the groups involved in the study, I have identified what seemed to be the typical *semiotic learning trajectories* (see section 7.3 for definition) in the students' semiotic activity within the group, and thus the trajectories that could possibly give an explanation of students' semiotic learning processes for the whole group in the study.

7.2.4 Sample and Access to the Data

Similar to the sample used in the QED, the sample involved in the semiotic analyses in this study came from the same group of students (i.e., E1), and the partially randomly selected (see section 7.2.3) students were then interviewed twice, before the experimental instructions, and after the completion of the post-teaching fractions test. For this reason, it is unnecessary to outline in detail again the characteristics of the sample, the teaching processes that took place and any other issues pertaining to the access of the data. In the matter of permission to videotape and audio tape the lessons, prior permission was sought from the Ministry of Education, Brunei when I was seeking permission to conduct a study at the school concerned, and they in turn wrote to the school to inform them of the matter. This was followed up through personal communication, when I discussed the details of the study with the school administrators (principal and deputy principal academic). The school administrators then informed the parents about this through the class teachers. In addition, the researcher also gave out a consent form for the parents to sign to give their consent for their children to participate in the study.

7.2.5 Ethical Considerations

For chapters 7 to 9, there were two sensitive issues in particular which resulted from the way the research (experimental fractions instruction in small groups) was conducted:

i. The use of video-taping and audio-taping of the instructions, which might possibly have made some students feel uncomfortable and unable to act as they would normally do; and
ii. An even more sensitive issue which involved anonymity of the students involved, as this research involved videotaping the students.

With regard to the first issue of videotaping and audio taping the lesson, as explained earlier, the school's consent was acquired through the consent from the Ministry of Education, Brunei, and from that the class teachers obtained the parents' consent, in addition to the consent form that the parents have signed for the participation of their children. On top of that, in order to ensure that the students understood what was involved, prior to beginning the first lesson for each group, I informed them that the entire lesson would be videotaped and audio taped and asked whether they were comfortable with that, and they responded that they did not mind. Even though some of them felt a little bit awkward at first, they soon settled in and were not even aware of the presence of the equipment. As for the second issue, I assured the students that in order to ensure anonymity, particularly in this thesis, or in any other research reports, articles (conference or journal), and any kind of presentations based on this study, the students' names, teachers and school had been and would be replaced with pseudonyms. Furthermore, students' facial characteristics would also be obscured in any video or photo representations.

Next, terminology used in the semiotic analyses presented in chapters 7–9 is presented and discussed. This is followed by the presentation of the relevant coding and the corresponding meaning of each code.

7.3 USE OF TERMINOLOGY AND INTERPRETATION OF CODING IN EPISODES

A list of key terms used in the semiotic analyses and for coding used in the presentation of relevant episodes in chapters 7 to 9 is presented and discussed in this section. The first table, Table 7.1 provides a list of all the key terms and their equivalent meanings associated with the semiotic analyses.

Table 7.1
Key terms (in chronological order) and their equivalent meanings in relation to the semiotic analyses

Terms	Meanings
Chain of Signification, or Semiotic Chain	". . . a chain of signification is the embedding of signs." (Sáenz-Ludlow, 2003, p. 186) "The basic component of a chain of signification is a sign . . . . A new link in the chain is established when a sign itself becomes the signified for a new signifier." (Gravemeijer et al., 2000, p. 262)

Corresponding amount	The researcher used this term here, in the context of the instruction, as "equivalent" when referring to "time" and "inches". For example, the corresponding amount of 1 inch is 5 minutes (i.e., 1 inch $\equiv$ 5 minutes).
Episode	A group of selected "interactions" of the discourse which the researcher has identified as significant which could contribute to the semiotic analysis. This episode would include all the semiotic means such as language, gestures, rhythm, symbols, and artifacts. For each "interaction" the following information is provided: ▪ The exact time (in minutes) the episode/interaction took place; ▪ The specific lesson in the experimental instruction; ▪ The group number to which the students belong to in E1.
Generalization: ▪ Factual ▪ Contextual ▪ Symbolic	The terms "factual generalization", "contextual generalization" and "symbolic generalization" are used as in Radford (2001, 2003 & 2006). "A factual generalization is a generalization of numerical actions in the form of an *operational scheme* (in a non-Piagetian sense) that remains bound to the numerical level, nevertheless allowing the students to virtually tackle any *particular* case successfully." Radford (2001, pp. 82–83) There are two differences between contextual and factual generalization. The first difference is that the operational scheme in contextual generalization is not based on face-to-face interaction, where both rhythm and ostensive gestures have also been excluded. The second difference is that the previously constructed operational scheme is generalized through language. Its generative capacity lies

	in allowing the emergence of new abstract objects to replace the previously used specific concrete objects (Radford, 2003). In symbolic generalization, the spatial and temporal limitations of the objects of contextual generalization have to be withdrawn. Symbolic mathematical objects (in Radford's case algebraic ones) should become "nonsituated and nontemporal" (Radford, 2003, p. 55).
Interaction	A selected part of the discourse which the researcher felt was significant in order to get an objectification for that particular episode (it could also be considered as a "sub-episode"). The specifics provided are similar to what is explained for "episode".
Intuition	"There is no commonly accepted definition for intuitive knowledge. The term "intuition" is used as one uses mathematical primitive terms like point, line, set, etc" (Fischbein, 1999, p. 12). He further added that intuition suggests self-evidence instead of a logical-analytical undertaking (Fischbein, 1999). Fischbein (1999) outlined outlined five general characteristics of intuitive cognitions:- 6. *Direct, self-evident* – cognitions which do not need any proof. 7. *Intrinsic certainty* – does not need external support, but it is usually associated with a feeling of absolute certainty. 8. *Subjective Coerciveness* – rejecting an alternative conception that conflicts with their intuitions, such as, multiplication always makes things bigger.

	9. *Extrapolativeness* – ability to conclude beyond any empirical proof, for instance, the acceptance that there will always be a number (whole) after a given number (whole). 10. *Globality* – an intuitive direct response, as opposed to a logical – analytically-based solution.
Gestures to the self (Self-gestures)	Unintentional gestures to the self which are not pre-planned, nor meant to attract attention to the gesturer. These could be as a result (by-product) of listening to what others in the gesturer's immediate surroundings have said or it could be a by-product of what the gesturer has read.
Meaning	"It is the human being, strong of the acquired culture, strong of the specific expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and now that must be coherent with that history." (D'Amore & Pinilla, 2008, p. 20) "Meaning is only one of the zones of sense, the most stable and precise zone. A word acquires its sense from the context in which it appears; in different contexts, it changes its sense." (Vygotsky, 1978 in Azarello, 2006, p. 282) Learning mathematics involves taking over the conventional meanings of mathematical signs, but it also depends on switching between different possibilities of interpretation–on "seeing an *A* as a *B*" (Hoffmann, 2006, p. 80).

	"To *perceive* something means to endow it with meaning." (Sabena, Radford, & Bardini, 2005, p. 129)
Signification, or Meaning-making	According to O'Sullivan, Hartly, Saunders, and Fiske (1983) in Chapman (2003), "Signification, or meaning-making, is "the relationship of a sign or sign system to its referential reality" (p. 130). "To foster a better understanding of how students "connect" both the different fields within mathematics, and "school mathematics with the experiential realities of learners", Presmeg focuses on a "nested model" of "meaning making" to describe learning processes as a step by step development." (Hoffmann, 2006, p. 284) ". . . mathematical meaning-making is social in nature and that any account of mathematical meaning-making should take account of development." (Peirce and Vygotsky, in Vile, 1999, p. 91)
Objectification	". . . refers to an active, creative, imaginative and interpretative social process of gradually becoming aware of something." (Radford, 2003, p. 393) ". . . the act of representing an abstraction as a physical thing or a concrete representation of an abstract idea or principle." (http://www.thefreedictionary.com)
Part-of-a-whole	The researcher is adopting Orton et al.'s (1996) definition of the types of fraction representation or models: *Region model* (also referred to as "part of a whole" or "part-whole" model), *Discrete model, A position on number line model, Quotient model* and *Ratio model.* The part-whole model as defined by Orton et al., (1996) is associated with the concept of area. However, in this thesis, when using the

	phrase 'part of a whole' the researcher is usually referring to either a part/portion of the bread in relation to the total number of equal parts/portions, it could also be in terms of time (hours and minutes), or in terms of length (with 12 inches as the unit whole).
Semiotic Contraction	As defined by Radford (2006), the "reduction of signs and concentration of meanings constitutes a *semiotic contraction*" (p. 12). In the context of this study, the semiotic contraction being referred to here is taking place in the process of objectification.
Semiotic (Learning) Trajectory	The term *semiotic learning trajectory* here refers to the semiotic chains that the students actually go through during the experimental instruction.
Semiotic Link	It is a link to connect the physical form of the sign (signifier) to what it actually refers to (signified). It is a "link" because it connects (potentially) to another sign to form a "chain"
Semiotic Means of Objectification	"To make something apparent (which is the etymological sense of objectification) learners and teachers make recourse to signs and artifacts of different sorts (mathematical symbols, graphs, words, gestures, calculators, and so on). These artifacts and signs used to objectify knowledge we call *semiotic means of objectification*." (Radford et al., 2006a, p. 685) ". . . we called semiotic means of objectification the whole arsenal of intentional resources that individuals mobilize in the pursuit of their activities and emphasized their social nature. The semiotic means of objectification appear embedded in socio-psycho-semiotic meaning-making processes framed by cultural modes of knowing

	that encourage and legitimize particular forms of sign and tool use whereas discarding others." (Radford, 2003, p. 44)
Sign	"Signs . . . are entities that serve as vehicles to trigger thought, to facilitate the expression of thought, and to embody original and conventional thought." (Sáenz-Ludlow, 2003, p. 182) "The *sign* is the whole that results from the association of the signifier with the signified (Saussure, 1983, p. 67; Saussure, 1974, p. 67). The relationship between the signifier and the signified is referred to as 'signification . . ." (Chandler , 2009, para. 4)
Signifier and Signified	". . . mental construct (signified) . . ." (Presmeg, 1998, p. 29). ". . . a 'signifier' (*significant*)–the *form* which the sign takes; and the 'signified' (*signifié*)–the *concept* it represents" (Chandler, 2009, para. 3).
Understanding	". . . the transformation or interpretation of a sign into a previous sign . . . for which the individual has attained a more or less stable cultural meaning" (Radford, Bardini, Sabena, Diallo, & Simbagoye, 2005, p. 117). Ricardo Neirovsky has suggested that instead of being mere mental processes, understanding and imagination of mathematical concepts are literally embedded in perceptuo-motor action: the "understanding of a mathematical concept spans diverse perceptuo-motor activities" (Nemirovsky 2003, 108), so that in this regard, "understanding is . . . interwoven with motor action" (Nemirocsky, 2003, p. 107, in Radford, 2010, p. 4).

Table 7.1 presents a discussion of the key terminology used in the semiotic analyses presented in Chapters 7–9. Next is the presentation of the coding used in the presentation of the relevant episodes in these chapters. This includes information on:

- The kind of details provided at the beginning of each episode;
- How the relevant semiotic means of objectifications (words, gestures, rhythm and artifacts) are referred to (signified);
- How the coordinated use of semiotic means is signified;
- How the researcher's own commentary are brought in during the episodes.

Table 7.2
Coding used and their equivalent meanings

Codes	Corresponding Meanings
Episode/Interaction (minutes/lesson/group)	General descriptor about the episode or interaction: ▪ Episode number; ▪ The exact time (in minutes) the episode/interaction took place; ▪ The group number to which the students belong to in E1.
Name in left column	Participating students' pseudonyms in the episode or interaction.
Number before name	The utterance number which is used to refer to a particular dialogue in an episode.
Obj.	Refers to objectification.
Regular text	The section of the discourse as spoken by the students in the group.
Regular text in round brackets	Refers to gestures that took place in the discourse that may contribute to the semiotic analysis.
Regular text in square brackets	This refers to part of the discourse that includes a semiotic role that contributes in the analysis.
Regular text in curly brackets	This is the researcher's English translation of the students' discourse that has been spoken in the students' mother tongue.
SMO	Refers to semiotic means of objectification.
Italics text	*The necessary explanation or description presented by*

	the researcher who acts as the reporter of the episode or interaction.
Italics text in round brackets	*Refers to students' gestures/actions/outcomes that took place in the discourse that do not contribute to the analysis* (e.g., *(laugh), (chorused answer)*).
…	Short pause or an interruption of a student's discourse by another student.
[…]	Irrelevant discourse that has been purposely left out.
[inaudible]	Discourse that cannot be heard properly due to poor sound quality (e.g., whispering, or due to noises outside the class).

CHAPTER 8

SEMIOTIC ANALYSIS OF THE FACTUAL GENERALIZATION OF EQUIVALENCE OF FRACTION

8.1 ANALYSIS CONTRIBUTION

In this chapter, episodes from E1 for *Activity 1: Fraction of a Foot,* and *Activity 3: Hour-Foot* are presented in which I investigated the factual generalization of equivalence of fractions. It should be noted the two activities mentioned above were designed to involve and include students' everyday life experiences in order to give meaning as stressed by Freudenthal (1973). However, due to cultural influences (where students were always expected to be at the receiver-end, and not to question), data revealed from the two activities only managed to provoke factual generalization of equivalence of fractions. This was evident from the interactions (between the teacher/researcher and students) presented below, where the semiotic means of objectification drawn on (by the teacher and students) to achieve the objectifications mainly involved linguistic devices, which in some cases are followed by illustrative concrete actions (e.g., pointing gestures).

The episodes presented here will be analyzed in two steps:-

i. Radford's (2002, 2003) *objectifications* will be scrutinized while focusing on the ways the range of different *semiotic means of objectification* (SMO) (Radford, 2002, 2003) interact with each other. These SMOs could include
    a. Language: deictic (e.g., I, you, this, that);
    b. Artifacts: the HFC model as an *emergent model* (Gravemeijer, 1997a, 1997b), which acts as a tool to help visualize the sectors that can be formed by the hour and minute hands and also to demonstrate how the different fractions (based on a foot and sixty minutes) are related. *Activity 1* and *Activity 2* served as refresher activities that will provide and remind the students of what they were supposed to have learnt before in their primary education, and also the twelve inches (a foot) used is designed to focus their attention on the numbers one to twelve which are the numbers commonly used to tell time in the 12-hour clock. Production of a clock-

face (*Activity 3: Hour-Foot*) will invoke a range of equivalent fractions–such as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$–due to the base of 60 involved, and will also provide a visual aid for finding the size of a sector formed by the area between the hour and minute hands; and

c. Gestures: in most cases gestures signify objects by the artifact's mediation, so the sign is the gesture mediated by the artifact signified.

ii. These semiotic objectifications, from the previous step, will then be organized and linked into semiotic chains to provide a picture of the semiosis involved in the formation of the equivalence of fraction, similar to the mode of analysis in those studies conducted by Koukkoufis (2008), whereby he examined chains of signification (Cobb et al., 1997; Gravemeijer et al., 2000), signifier-signified links and objectifications (Radford, 2002, 2003).

The benefit of this two-step analysis according to Koukkoufis (2008) is that "it reveals the semiotic activity enabling (and resulting in) the factual generalization . . ." (p. 159). He also believed that

> . . . it pushes forward the theory of objectification (Radford, 2002b, 2003) as well as symbolizing in Realistic Mathematics Education (RME) (Freudenthal, 1973, 1983) . . . through providing some new tools of analysis for the investigation of the meaning making processes involved in factual generalization in the context of RME. (p. 159)

In section 8.2, four episodes are discussed vis-à-vis the production and objectification of equivalence of fractions.

8.2 STEP 1 – SEMIOTIC ANALYSIS OF THE EQUIVALENCE OF FRACTION

8.2.1 Adaptations to Radford's Theory of Objectification in the Factual Generalization of the Equivalence of Fraction

Radford (2002, 2003) originally developed the *theory of objectification* for topics in algebra, which is a different context from what I am about to do. For that reason, I am going to adopt Koukkoufis's (2008) *preparation* idea which he described as ". . . the objectification of the necessary semiotic links in the relevant chain of signification. As these preparatory objectifications were indispensable for the achievement of the targeted . . . objectification" (p. 160). He also added that it is through a combination of these objectifications that the factual generalization can be achieved and not just through the last objectification. Therefore, factual generalization as in his case, and also in my case here, can only be achieved through a series of linked objectifications.

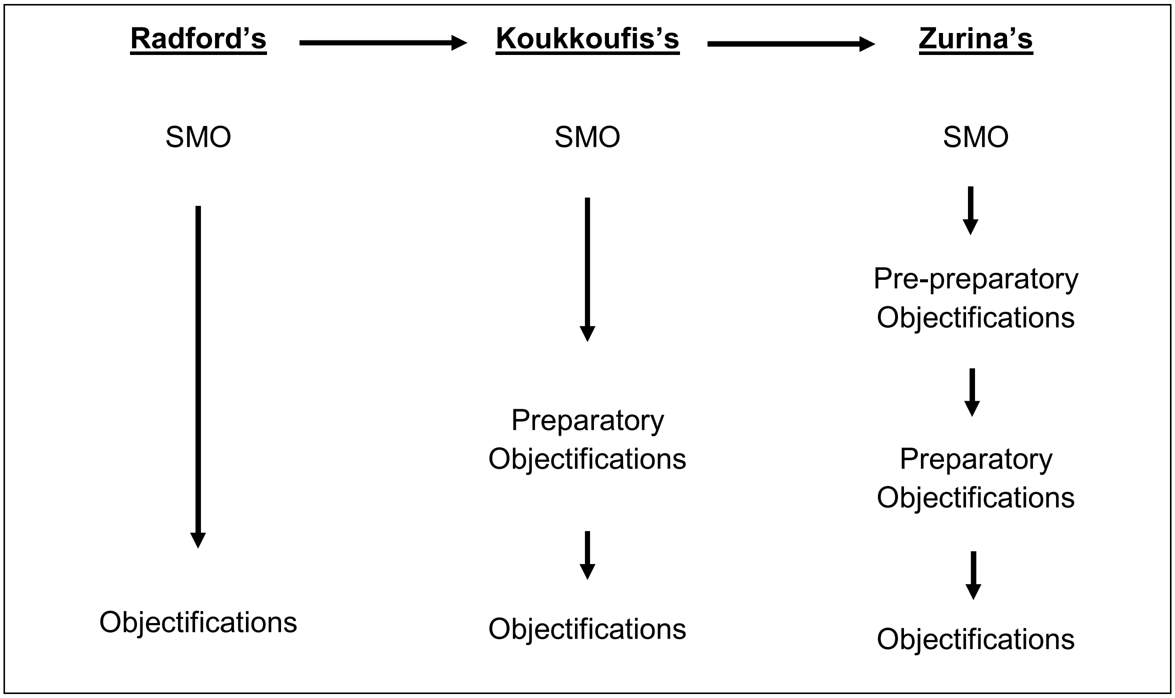


Figure 8.1. Evolution of Differences and Adaptations to Radford's Theory of Objectification

In this study however, it is not as straight forward as in Koukkoufis. Here each objectification also involved a series of *pre-preparatory objectifications*. These differences and adaptations are necessary because of the complexity of the fraction concepts which I felt made them difficult to learn. Presented above (Figure 8.1) is a summary of the differences and adaptations made to Radford's Theory of Objectification.

Before going on further, I wish to explain briefly my rationale for having these pre-preparatory objectifications (which are presented in each episode or sub/episode) when readers might have noticed (after reading further) that factual generalization of equivalence of fractions was achieved in some of the earlier sub-episodes. The experimental lessons were basically designed to incorporate their everyday experiences in the activities (e.g., the HFC), so that they could relate to them and hence reduce the likelihood of them being forgotten easily (Freudenthal, 1973). However, in order to achieve this I had to plan it in stages, from what they were *normally* used to in the classroom and only progressively introducing the HFC. On the whole, there are 12 pre-preparatory objectifications (hereafter, PPO), two preparatory objectifications (hereafter, PO) and finally the desired objectification which is linked to all the pre-preparatory and preparatory objectifications. More explanation will be provided throughout the episodes which ought to make it clearer as we progress. Moreover, readers will notice in subsequent episodes that almost all of the students' responses consisted of only a few words which might be due to their cultural background (i.e., shy, speaks only when spoken to and with minimal number of words). Due to this, the researcher-teacher also had to have a leading role in the discussion.

8.2.2 Episode 1: Objectification of "part of a whole" (objectification 1)

The first episode occurred during Activity 1 which was the first lesson for each group. In this episode the students were given a 12-inch loaf of cheese bread, and they were asked to share the loaf of cheese bread between the two members of each team (in each group the students were usually working in groups of two, so for every session there were two teams working except on occasions when a student was absent, when they worked as one team only. However, these teams were not competing against each other, the intention was first to maximize opportunities for the students to contribute in the activity). Activity 1 requires the students to express each piece of the cut cheese bread in terms of fractions, and finally in terms of the whole, which in this case is twelve inches in length (*Figure 8.2*. Students cutting the cheese bread).

Figure 8.2. Students cutting the cheese bread (Activity 1: Fraction of a foot)

This episode consists of three separate pre-preparatory objectifications from group 1. For each of these in order to make it easier for the readers to comprehend what I am trying to convey I have presented the selected mini-episodes in table form so that what is happening at each utterance of the mini-episodes can be captured easily. The following series of students' pre-preparatory objectifications leads to the objectification of "part of a whole."

<u>Sub-episode 1.1 (minutes 02:40-03:22 /Activity 1)</u>

The researcher-teacher was trying to explain what they were supposed to do in Activity 1. In this case I indicated through my questioning that they needed to measure the length of the original cheese bread first, once they have cut it into two parts they were to determine the length of one piece of the cut cheese bread.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
1	Res:	[…] Can you measure (points to the loaf of cheese bread) [how long, how many inches] is that cheese stick?	Placed a cheese stick in front of the group prior to this utterance. **SMO1** – pointing to a loaf of bread **SMO2 –** linguistic means (e.g., how many inches is that?)
2	Halim:	[Twelve] (took a ruler and measured the loaf of cheese bread.)	**OBJ1a** – twelve
3	Res:	Twelve, so everybody agreed that [it's twelve inches]?	Ensuring that all have the same length of cheese bread **OBJ1a** – it's twelve inches
4	Students:	(*All nodded*)	
5	Res:	[…] [it's one foot, twelve inches is one foot]. Right, so now if you are to [share] this bread, [how much] will each get?	Note that one loaf of cheese bread was given to every two students to share. Here the first

			emerging signs of equation of fractions–embedded in the equations of measures: $\frac{1}{1}$ = $\frac{12}{12}$ – occurred. **OBJ1b** – how much
6	Students:	[Six] (*chorused answer*)	Most probably they had mentally divided twelve by two since the original is twelve inches long and they are to share one loaf between the two of them. **OBJ1b** – six
7	Res:	How did you get [six]?	
8	Halim:	[Divide by two]	
9	Fah:	[Divide by two]	She might only be echoing Halim's answer at this stage.

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; {regular text} – Researcher's translation from the mother tongue; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means; OBJ – objectification.

In this sub-episode, it was important to bring to the students' attention the length of each piece of the cut cheese bread so that they can see it in relation to the original 12-inch long original length. My pointing gesture (**SMO 1**) and instruction (utterance 1) indicated that they have to measure the length of the loaf of cheese bread which has been placed in front of them earlier, hence allowing me to put forward the idea that they need to determine the length of the original first **(SMO 2)**. In this particular instance, my gesture and verbalization supplement each other. Then Halim (utterance 2) immediately took a ruler and measured the loaf of cheese bread in front of him, and gave an answer of twelve. In order to ensure that everybody had the same length of cheese bread, I (utterance 3) repeated what Halim said, and also emphasizing that "it's twelve inches" long (**Objectification 1a**). Prior to the students cutting the loaf of cheese bread into two parts, in utterance 5, I implied that $\frac{1}{1} = \frac{12}{12}$, the first nascent sign of equation of fractions which is embedded in the equation of measures, when I said "it's one foot, twelve inches is a foot," also I used the word "share" signifying that each of them might get an *equal share* and the phrase "how much" indicating that they should measure the piece that they have cut. In utterance 6, all the students shouted out the answer in a chorus of "six" without having measured the cut piece, and thus they had objectified the length of each piece of the cut bread (**Objectification 1b**). Further Halim and Fah (utterances 8 and 9) have introduced "two" which presumably points to the two parts (or shares).

Sub-episode 1.2 (minutes 04:40-05:38/Activity 1)

In this sub-episode the researcher-teacher was trying to get the students to describe their share of the cheese bread in terms of fractions, both in terms of the number of parts they have and also in terms of the total length of the cheese bread.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
10	Res:	[…] How would you describe [this portion] (pointed to one of the cut pieces of bread) […]?	The phrase "this portion" was intentionally used to refer to one of the cut pieces. **SMO3** – linguistic mean (e.g., this portion) and pointing.
11	Students:	(*Silent*)	The silence indicated to me that they might not understand what is required of them.
12	Res:	Okay let me ask in another way, how much is [this] (pointed to one cut piece) in [fraction]?	At this point I decided to use the word "fraction" to get across what was needed from them. **SMO4** – deictic word (i.e., this) and indexical gesture of pointing **OBJ1c** - fraction

13	Halim, Yusof & Fah:	[One over two] (*chorused answer*)	The word "fraction" seemed to be more meaningful to them as they responded immediately by giving the fraction. **OBJ1c** – one over two
14	Res:	Why?	
15	Fah:	[Divide it into two]	Similar to utterance 9, she *introduced* "two".
16	Res:	[You] divided into two, okay, one over two. There are [two portions] (pointed to the two portions), so [one portion] (pointed to one of the portions) [must be one over two] (pointed to one of the portion followed by to the two portions), because there are two portions. Can you give me [another] fraction in terms of [its length]? (Refer to Figure 8.3)	Pulling everything together by summarizing what had been said. I also tried to indicate to them the existence of other fractions which are related to its length, hence the pieces of bread here becomes signs for fractions, numbers and lengths–losing the concrete reference in the verbalization. **SMO5** – pointing to the pieces of bread and linguistic means (i.e., you, one portion, another) **OBJ1c** – must be one over two
17	Students:	(*Silent*)	
18	Res:	[…] [How long] is [this] (pointed to one of the cut piece) just now?	Taking a step back (utterance 6) to reflect on what they did before, thus providing

			them with the first link in this chain. **SMO6** – demonstrative deictic word (i.e., this) and pointing
19	Students:	[Six] (*chorus answer*)	
20	Res:	[Six inches], so can you tell me what's the [fraction in terms of inches]?	**OBJ1d** – fraction in terms of inches
21	Halim & Yusof:	[Six over twelve]	**OBJ1d** – six over twelve
22	Res:	[Six over twelve]. How did you get that?	Trying to establish a link of the length of the one part with the whole.
23	Fah:	Because [it's twelve inches]	
24	Halim:	(Gestured with his hands indicating the whole loaf (i.e., 12 inches), starting from the centre of the bread outwards) (Refer to Figures 8.4 and 8.5)	Gesturing to himself instead of verbalizing it. **SMO7** – gesture for oneself (self-gesture)
25	Res:	Because [it's twelve inches] long, the total and [this portion] is [six], so [it's six over twelve]?	The second critical link in this chain. The word/phrase "it's" and "six over twelve" which refers to the one portion of bread and this brought everything together by indicating that the fraction for that portion is six over twelve. **OBJ1d** – it's six over twelve

26 Students: Yes (*chorus answer*)

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; {regular text} – Researcher's translation from the mother tongue; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means; OBJ – objectification.

In utterance 10, I wanted the students to describe the cut piece in relation to the total number of pieces or the total length of the cheese bread, without having to introduce the word "fraction" first by making use of the deictic word "this" to indicate the one portion and at the same time pointing to one of the cut pieces of bread (**SMO 3**). However, since there was no response (utterance 11) from the students, I assumed that they did not understand what I wanted them to do. Thus, I chose to rephrase the question (utterance 12) and began using the word "fraction," together with the use of the deictic word "this" and the indexical gesture of pointing (**SMO 4**). This initiated the objectification process, and almost instantaneously, Halim, Yusof and Fah (utterance 13) gave a correct response of "one over two" (**Objectification 1c**). In utterance 15, once again Fah suggested that they got the answer "one over two" because they "divide it into two" (see also utterance 9).

Picture 1

Picture 2

Figure 8.3. The teacher gesturing "1 out of 2"–in picture 1 holding one of the two cut bread; and in picture 2 counting the two cut pieces

In utterance 16, I repeated what Fah had said and also rephrased it in terms of portions (or parts) to direct the students' attention that we were also dealing with discrete parts. This was to highlight what they had done physically when they cut the loaf of cheese bread, and to direct their attention to each part as having a length which can be compared to its original length–not just dealing with division of *decontextual* numbers. Further, I also used the word "another" suggesting that there is more than one fraction that can describe the same piece of cheese bread. In addition, my reference to "its length" further signified to the students that at least one more fraction can be written down for the cut piece in relation to its length. Unfortunately, I once again had to assume, from their silence (utterance 17), that they did not understand the question. I then chose to take one step back (utterance 18) to get them to think in terms of the length of the cut piece, which they immediately responded to (utterance 19). From here, I emphasized that it is "six inches" long, and moved one step forward again by asking them the fraction of the cut piece in terms of inches (utterance 20) which initiated the objectification process. To this Halim and Yusof (utterance 21) responded correctly when they gave an answer of "six over twelve" (**Objectification 1d**). Here, most probably Fah could have followed Halim's and Yusof's line of thought at this stage, when she responded (utterance 23) that "it's twelve inches." At the same time also in utterance 24, Halim who was sitting on my left (see Figures 8.3 and 8.4) gestured to himself (**SMO7**), while I was listening to Fah's explanation. His hands' movements which started from the centre of the loaf of cheese bread and moved outwards give an indication to me that he was also probably reaffirming to himself, or he could also be answering the question without actually verbalizing it. This could be unconscious behaviour on his part where, while listening to the discussion, some internal reflection could have taken place. So instead of verbalizing it, this could have been externalized through him gesturing to himself. One may also argue that this could be termed a *deictic gesture* of gesturing to the whole loaf of bread to indicate the twelve inches, but instead of doing so with the intention of directing the attention of the other students and me to the length of the loaf of bread, he was just doing it to himself while listening to the discourse that was going on around him. I am going to call this kind of gesture a *listening gesture* from this point onwards. In utterances 25 to 27, once again I (utterance 25) reiterated what the students said to confirm what they meant (**Objectification 1d and 1e**).

Figure 8.4. Halim gesturing the 12 inches

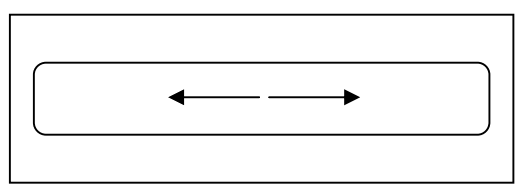


Figure 8.5. Diagrammatic representation of Halim's hands movement–from the centre outwards

Picture 1 Picture 2 Picture 3 Picture 4

Figure 8.6. Teacher gesturing “1 over 2” (pictures 1 & 2) and “6 over 12” (pictures 3 & 4)

Sub-episode 1.3 (minutes 08:13-09:08/Activity 1)

I wanted to establish a relationship between the number of parts and the total number of parts, and also between the lengths of each part to the total length of the cheese bread (12-inch).

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
27	Res:	[…] You said just now, [this] (points to one of the cut portion) is half, [it] is also one over two (*mistakenly said 1 over 2. I meant six over twelve*). My question is […] [one over two, […] and six over twelve the same? Are they equal]? (refer to Figure 8.6)	Bringing together the two fractions they had given earlier. **SMO8** – deictic words (i.e., this, it) and indexical gesture of pointing **OBJ1e** – are they equal?
28	Fah:	[Yes]	**OBJ1e** - Equal
29	Res:	Why is it equal? One over two and six over twelve, [why] are those two fractions equal?	This is to confirm whether she was actually thinking of the cut portion of bread itself (i.e., conceptually), or was she thinking algorithmically.

30	Fah:	[inaudible]	
31	Res:	Sorry? What did you say Fah?	
32	Fah:	Changed into [lowest term]	**OBJ1e** – equal because of the lowest term of the other **FG1** – F(Parts) & F(Lengths)

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; {regular text} – Researcher's translation from the mother tongue; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means; OBJ – objectification.

In this sub-episode, I recapped in utterance 27 what the students had said in utterance 13 and utterance 21 about the fraction for each of their shares, and also brought to the students' attention the possibility of them being equal, which acted as the initial point for achieving objectification 1e. Fah in utterance 28 responded immediately with a "yes." In fact, here Fah as such had attained objectification 1e with an indexical-associative meaning: this *indicates* that one over two and six over twelve are equal, and already a factual generalization of equivalence of fractions (**FG1**) had taken place, but I would like to remind readers that, as mentioned earlier, the aim was to achieve a generalization of equivalence of fractions which incorporated students' experience which they practically used everyday (i.e., time). At the moment, however, we have not reached there yet, and prior to achieving the targeted objectification, there might be more generalizations occurring. Optimistically, this might also be seen as an advantage as the students would get repeated practice. On the other hand, as before, context (i.e., inches) was not present in the students' verbal feedback even though I tried to include it in the discourse (see for example utterances 19 to 21). I presumed that this was a new thing for them which I could not force upon them, and I could only conclude that they needed more practice than what they had in the limited time they were with me. At this stage also, I was not sure whether she said "yes" because the two fractions were referring to the same piece of cheese bread, so the two fractions (i.e., one over two, and six over twelve) must be equal, or whether she was dealing with this problem algorithmically. Utterance 32, where I assumed she equated the two fractions and noticed that one is the lowest term of the other, confirmed that she was after all dealing with this algorithmically instead of conceptually. Even though there was no reference to context in any of her answers, algorithmically she saw that they were equal, and so nonetheless, she had attained objectification 1e. Furthermore, I argue that this could be an example of a case where "losing the context" did not necessarily mean "losing the concept". In this case, one object (i.e., one of the two cut pieces of bread) was signified by two different fractions; we can conclude these are equivalent fractions.

From the above semiotic analysis, of the attainment of objectification 1 (e.g., one part out of two "is changed to" six inches out of twelve inches) has occurred, as this new object (i.e., lowest term) has entered the discourse. However, I was not fully convinced yet, at that stage, that the students appreciated the significance of this as far as fraction is concerned. Even though the students managed to give the answers in terms of a part of a whole, there was no clear indication of any intuitive responses coming into play. In other words, the students may have provided some of the links, but it was I who constructed the chain and, in this analysis, had objectified some of the pre-preparatory objectifications. On the other hand, Fah in utterance 32 may have attained objectification 1 as such. It could just be that they were remembering bits and pieces of mathematical knowledge that they had learned instrumentally in their mathematics lessons in the past three years of their primary education.

Next, in order to provide the students with more alternative methods (e.g., Activity 3: *Hour-Foot*), I further tried to associate the 12-inch with time which they might be more familiar with from their everyday experiences. I then explained to the students that they were to create a clock-face which would have a 12-inch circumference. There were also some specific features of the clock that they needed to have clearly labelled on the clock-face;

- a 1-inch interval labelling around the clock-face to indicate the number of inches, and
- the number of minutes that has passed for every inch.

8.2.3 Episode 2: Objectification of the "corresponding value" (objectification 2) (e.g., 1 inch ≡ 5 minutes)

Unlike episode 1, this objectification only comprised of a single episode. In this episode the objectification of the "corresponding value" occurs. The students first had to construct (Figure 8.7) the HFC which was actually a clock face with a 12-inch circumference. Once they had that, they had to mark the circumference at every one inch interval (hence producing a clock face with numbers from one to twelve around the clock face). Then they were also to label the number of minutes that passed for every inch (thus having numbers from five to sixty at every one inch interval alongside the numbers of one to twelve) (Figure 8.8).

Picture 1 Picture 2 Picture 3

Figure 8.7. Students constructing the HFC (Activity 3)
Picture 1 – measuring and marking every inch on a plastic strip
Picture 2 – making a circle for the clock face
Picture 3 – the clock face with the 12-inch circumference

In this episode I tried to bring the students a step further by introducing another familiar measurement scale to them, which is time. This was done in Activity 3 where in this activity the students were to draw a clock face with a 12-inch circumference. They were then asked to label the clock face from 1 to 12, like most clock faces, but in this case they knew that the distance between each number is one inch.

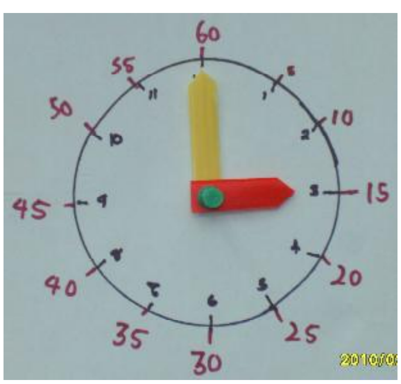


Figure 8.8. HFC made by students

<u>Episode 2 (minutes 15:44-16:12/Activity 3)</u>

Here I set up a situation where the students were able to see and connect the relationships of the two measures (namely, length and time) on their clock faces.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
34	Res:	Yes, now let's think about it in terms of [time]. For [one whole circle], for [one hour], how many [minutes] is it?	Introducing a new whole to them (i.e., 60 minutes) **OBJ2a** – whole
35	Halim, Yusof & Fah:	[Sixty minutes]	**OBJ2a** – sixty minutes
36	Res:	[…] Now if [it] moved like just now, [one inch] (moved one of the clock hands to 1), [so] [how many minutes] has it moved?	The clock has a circumference of 12 inches. **SMO9** – deictic words (i.e., it, so) **OBJ2b** – one inch
37	Halim & Fah:	[Five]	**OBJ2c** – five (minutes)
38	Res:	Five minutes […]	

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; {regular text} – Researcher's translation from the mother tongue; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means; OBJ – objectification.

Prior to this sub-episode, I had already established with the students that if one of the hands has moved to 1, the hand had moved one inch and they also had given a response of one over twelve as the fraction. In utterances 34 and 35, I then tried to establish that for a whole hour (or minutes) there are sixty minutes (or seconds) (**Objectification 2a),** and at the same time this also served as the starting point for the next chain of objectification. Following that, in utterance 36 I tried to establish that for every inch that passed a certain number of minutes had passed, by making use of the deictic word "it" to refer to the new position of the clock hand and saying "one inch" and "how many minutes?" Thus, this signifies to the students the relationship between the twelve inches and the sixty minutes (**Objectification 2b**). Halim and Fah (utterance 37) responded "five" (**Objectification 2c**) without hesitation; however Yusof and Era did not give any input at this stage. In order to re-emphasize again that the centre of discussion at that point is in terms of time, I, in utterance 34, repeated Halim's and Fah's response (utterance 37) and also adding the relevant unit by saying "five minutes" (utterance 38). Again, as before I included the relevant context (i.e., minutes) in the discourse (utterance 36), but again it seemed to be dropped from the students' verbal response after that. However, I argue that even though no relevant contexts were included in the students' responses, this might be because I included it in my discourse, hence it might be an implicit acceptance that their responses were also in the same context. As such, in these responses, Halim and Fah, in particular, objectified the "corresponding value" when they answered "five" as corresponding to "one inch." In this case, Halim and Fah together had attained objectification 2 when they said "five" in utterance 37. Era, on the other hand, did not give any input in this part of the discussion.

Next, I used the HFC to get the students to name fractions in terms of twelfths, and also to urge them to name other fractions, namely in terms of sixty minutes.

8.2.4 Episode 3: Objectification of "fractions in terms of inches and minutes" (objectification 3) (e.g., 6 inches out of 12 inches = 30 minutes out of 60 minutes)

In episode 3, objectification 3 was accomplished in the following single interaction, which in fact overlaps (utterances 46 to 49 below) with the discourse above (i.e., episode 2), where the objectification happened immediately after the second objectification was objectified. With the aim of putting the episode into perspective for the readers, I decided to include some discourse prior to episode 2 also which I felt is relevant for this analysis.

Episode 3 (minutes 14:45-16:27 / Activity 3)

Using the HFC I created a situation by positioning one of the clock hands at the number 1. The students were to find the fraction the hand has moved in terms of inches. Upon getting a fraction, I then urged them to come up with another equivalent fraction in terms of time.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
39	Res:	[…] one of the (*see figure 8.8)* hands moved to one (moved the clock hand to 1), [how many inches] has [it] moved?	**SMO10** – "it" referring to the clock hand.

40	Halim & Fah:	[One]	
41	Res:	[One inch], so it has move [one inch]. Now, can you tell me, in terms of fractions, what is the [fraction it has moved]?	Purposely repeated their response with the unit attach to it. **OBJ3a** – fraction in terms of twelve inches.
42	Fah:	[One over twelve]	**OBJ3a** – one over twelve
43	Halim:	[One over twelve]	
44	Res:	Yes, now what if [it] has moved there (moved the clock hand to 3), [three inches]?	**SMO11** – linguistic mean (i.e., it)
45	Halim & Fah:	[Three over twelve]	
46	Res:	Yes, now let's think about it [in terms of time]. For [one whole circle], for [one hour], [how many minutes] is it?	
47	Halim, Yusof & Fah:	[Sixty minutes].	
48	Res:	[…] [one inch] (moved one of the hands to one), [so] [how many minutes] has [it] moved?	**SMO12** – linguistic means (i.e., so, it) **OBJ2b** – one inch
49	Halim & Fah:	[Five]	**OBJ2c** – five (minutes)
50	Res:	[Five minutes]. So in terms of [fractions], what is it?	**OBJ3b** – fraction in terms of sixty minutes
51	Fah:	[Five over sixty]	**OBJ3b** – five over sixty
52	Halim:	[Five over sixty]	
53	Res:	So [is one over twelve and five over sixty the same]?	**OBJ3c** – one over twelve equals to five

			over sixty
54	Fah:	[Yes]	**OBJ3c** – Yes (i.e., one over twelve equals to five over sixty)
55	Yusof:	[Yes]	

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; {regular text} – Researcher's translation from the mother tongue; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means; OBJ – objectification.

I (utterance 39) set up a situation where one of the clock hands was moved to one and used the phrase "how many inches" instead of "how far" the clock hand had moved. This focused the students' attention to the distance, in terms of inches the clock hand had moved for the section between the hour and minute hands, and thus started the objectification process. Halim and Fah (utterance 40) responded immediately, which I repeated (utterance 41) by adding the unit to the value, "one inch" instead of just "one" in order for it to be a sign to the students that we were talking about inches. In the same utterance I also requested from them the fraction the clock hand had moved with respect to the whole circle. Again in utterance 42 and utterance 43, Fah and Halim gave the corresponding fraction in terms of 12 inches (**Objectification 3a**). I then continued to request for another fraction by moving one of the clock hands to a new position (utterance 44). Fah and Halim (utterance 45) once again responded by giving the correct corresponding fraction. Having established that the students were able to connect between the distance the clock hand had moved and the fraction it represented, next, I (utterance 46) tried to direct the students' attention to time instead and also to focus on the whole circle as the new unit whole when I said "one whole circle," "one hour," and "how many minutes" and hence signifying time. This time, Yusof joined Halim and Fah in giving a response by giving "sixty minutes" as the answer (utterance 47). From this, as mentioned in the previous section (for objectification 2), by moving the clock hand to one I had created a connection between the 12-inch and the sixty minutes by saying (utterance 48), "one minute" and "how many minutes," hence signifying to the students that there is a relationship between the two units in the HFC. As before Halim and Fah gave an immediate response (utterance 49). Again as explained above, I (utterance 50) repeated Halim's and Fah's response and also adding the relevant unit by saying "five minutes." I also in the same utterance, tried to direct the students attention back to fraction, when I said "in terms of fraction, what is it?" (**Objectification 3b**). To this, again only Halim and Fah responded instantaneously by giving an answer of “five over sixty” (utterances 51 and 52). Again to focus the students’ attention to the relationship of the two measures, and also to bring everything in front of them, in utterance 53, I tried to get confirmation from the students the status of the two fractions (i.e., whether one over twelve equals five over sixty) (**Objectification 3c**). This time Fah and Yusof both responded immediately by saying “yes” (utterances 54 and 55). At this stage one could argue that “yes” is ambiguous, but the discourse in the next episode proved that they did know that the two fractions are equivalent.

From the above semiotic analysis, objectification 3 has been achieved collectively through my line of questionings and the students’ responses. Admittedly, I have used *closed* questions as opposed to *open* questions. The reason is to avoid “silent” responses (see for examples in utterances 11 and 17 above). Again, as before I was the one who constructed the chain and the students provided some of the links, and unfortunately again, the context still seemed to disappear from the students’ verbal responses which, as mentioned in section 8.2.3, might be because the measure was explicit in my discourse so it tended to be implicit in theirs. As fraction is a difficult concept to learn due to its complexity, I had to play a leading role in constructing the chains, which is in line with Vygotsky’s (1978) *zone of proximal development* (ZPD), which advocated that children develop the ability to do certain tasks through examples which will encourage and advance their learning. Hence, I argue that the collective production of objectification is justified.

8.2.5 Episode 4: Objectification of equivalent fractions (objectification 4) (e.g., 1 part out of 2 parts = 6 inches out of 12 inches = 30 minutes out of 60 minutes)

In this episode, objectification 4 was objectified in the following discourse which also happened in Activity 3.

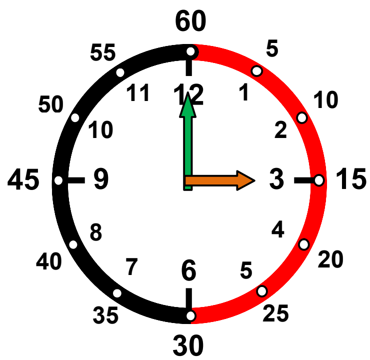


Figure 8.9. Hour hand at 3

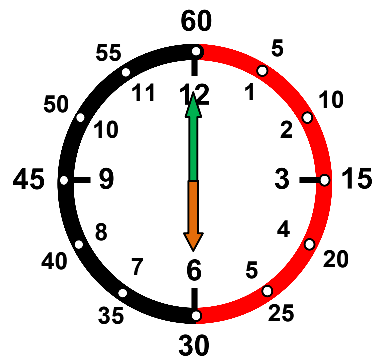


Figure 8.10. Hour hand at 6

<u>Episode 4 (minutes 16:36-17:06 / Activity 3)</u>

Here I moved one of the clock hands to two different positions to encourage the students to come up with an additional fraction apart from the ones in terms of twelve inches and sixty minutes.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
56	Res:	Now go back to [this] (moved the hour hand back to 3), [when [it] is at three], give me [a fraction]]. (Refer to Figure 8.9)	**SMO13** – deictic words (i.e., this, it) **OBJ4a(i)** – a fraction
57	Fah:	[One over four]	**OBJ4a(i)** – one over four
58	Res:	[…] [and…]	Indicative of more fractions. **SMO14** – deictic word (i.e., and) **OBJ4b(i)** – the second fraction
59	Halim & Fah:	[Fifteen over sixty]	**OBJ4b(i)** – fifteen over sixty
60	Res:	Yes, how about if I have [this]? (moved the hour hand to 6) (Refer to Figure 8.10)	I chose to move the clock hand to what I felt might be an "easier" position. **SMO15** – deictic

			word (i.e., this) **OBJ4a(ii)** – the first fraction
61	Fah:	[One over two]	**OBJ4a(ii)** – one over two
62	Res:	[Another one]?	Signifying other equivalent fraction. **SMO16** – deictic word (i.e., another) **OBJ4b(ii)** – the second fraction
63	Halim:	[Thirty over sixty]	**OBJ4b(ii)** – thirty over sixty
64	Res:	[Any more?]	**SMO17** – deictic words/phrase (i.e., any more) **OBJ4c** – the third fraction
65	Halim & Yusof:	[Six over twelve]	**OBJ4c** – six over twelve

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; {regular text} – Researcher's translation from the mother tongue; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means; OBJ – objectification.

I (utterance 56) used *deictic language* of the words "this," and also moving the clockhand, to focus the students' attention to the positions of the hour hand, which was at 3, and the minute hand which was positioned at 12 (**SMO13**). I also, with the phrase "when it is at three," emphasized further that the hour hand is at 3, and also urged the students to provide one of the possible fractions which correspond to that position when I said "a fraction." The use of the word "a" instead of "the" signifies to the students that there is more than one possible way of presenting the corresponding fraction. In utterances 56 and 57, it can be seen that both Fah and I had started the objectification process for the "equivalent fractions" when I asked for "a fraction" and Fah's response of "one over four" (**Objectification 4a(i)**). In the next utterance when I said "and" (utterance

58), it has an indexical-associative meaning indicating that there is a possibility of getting more fractions, and to this Halim and Fah (utterance 59) responded by giving a fraction in terms of time (i.e., “fifteen over sixty”). This shows that Halim, Fah and I had all objectified the second pre-preparatory objectification of the “equivalent fractions (**Objectification 4b(i)**). In order to get responses from the other two students (i.e., Yusof and Era), I decided to move the hour hand to 6 which might be considerably easier than the previous one (i.e., at position 3), and did not request any more fractions from them which would have objectified objectification 4c. Instead I chose to move the clock hand to a different position in order, as mentioned earlier, to elicit responses from the other two students. Again I used *deictic language* in the word “this” (**SMO15**) to direct the students’ attention to the new position of the hour clock (utterance 60). In Fah’s response (utterance 61), once again she attained the first pre-preparatory objectification of the “equivalent fractions” (**Objectification 4a(ii)**) (i.e., “one over two”). In utterance 62, I urged the students by saying “another one,” indicating a second fraction to which Halim (utterance 63) managed to give another correct fraction and hence objectifying objectification 4b(ii) of the “equivalent fractions” (i.e., “thirty over sixty”). Up to this point Yusof and Era have yet to make any attempt to contribute to the discourse. In utterance 64, I once again requested for another fraction by asking “Any more?” In Halim’s and Yusof’s responses (utterance 65) both of them and I in utterance 64 attained the third pre-preparatory objectification of the “equivalent fractions” (**Objectification 4c**) (i.e., “six over twelve”). In this particular objectification, I did not have to explain my intentions again to them, it was sufficient just to use a connective word such as “and,” or short phrases such as “another one” and “anymore” for them to give the relevant equivalent fractions–the last three SMOs were verbal. In a way, the semiotics had been contracted, if not fully contracted.

Section 8.2 above showed how the group (students and researcher-teacher) together achieved the four preparatory objectifications, with Halim’s and Yusof’s objectification (**Objectification 4c**) as the last of the objectification of equivalent fractions. In the next section, the outcome of the above four objectification is presented as the chain of signification of the “equivalence of fractions.”

8.3 STEP 2 - CHAIN OF SIGNIFICATION OF THE EQUIVALENCE OF FRACTIONS

Having presented the semiotic analysis of the equivalence of fractions in step 1 in the previous section, the objectifications in episodes 1–4, which afforded the factual generalization of the equivalence of fractions are now linked by what Cobb et al., (1997) and Gravemeijer et al., (2000) called *chains of signification,* and so Radford's (2002, 2003) *signifier-signified* links (signs) and *objectifications*. Following Koukkoufis's (2008) claim that "objectifications construct *signifier-signified* links to be made meaningful inside *chains of signification*" (p. 157), I will also try to construct the *signifier-signified* links inside *chains of signification* for equivalence fractions for the E1 group involved in this study.

Figures 8.11–8.13 below show semiotic chains where according to Cobb et al., (1997) and Gravemeijer et al., (2000) the existing lower level signifier "slides under" the new signifier. To be precise the existing lower level signifier in a signifier-signified link acts as the signified of a new higher level signifier (Cobb et al., (1997); Gravemeijer et al., (2000). These semiotic chains can be observed in the two activities described earlier, and before I present the chain of significations I will first provide schematic representations (Figures 8.11 and 8.12) on what these two activities entailed and how they were connected, where as mentioned earlier the aim was to achieve factual generalization of equivalence of fractions which would involve students' everyday experiences, which in this case was the use of time.

Figure 8.11 below shows how in the bread cutting task the students were given a foot long loaf of cheese bread. They were then asked to cut the loaf of bread into the required number of equal pieces followed by establishing the fraction each piece represented in terms of the number of parts (hereafter, F(Parts)). Once that had been ascertained, they proceeded to measure the length of one of the equally cut pieces in inches, and from this they once again progressed to establish a second fraction for each piece of cut bread in terms of its length. The next step was to establish that the two fractions were the same, which led to the first factual generalization in this schema (i.e., F(Parts) = F(Length)).

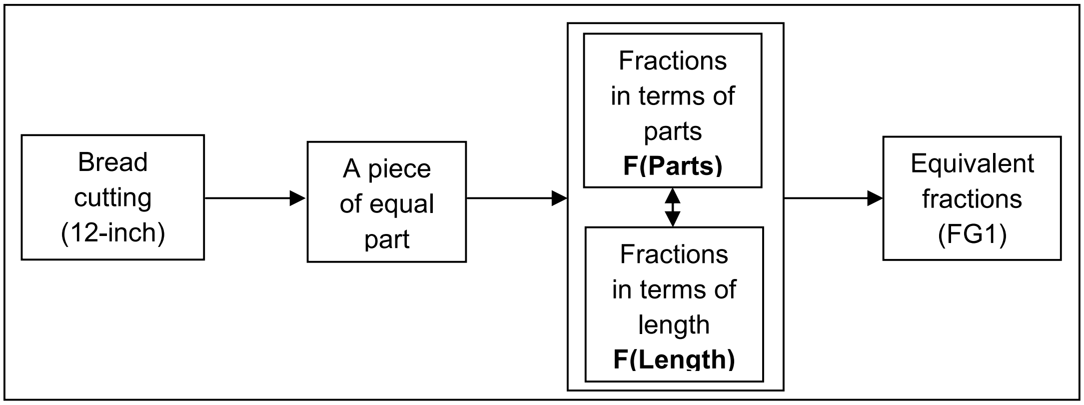


Figure 8.11. Semiotic chains of signification of “bread cutting” (Activity 1)

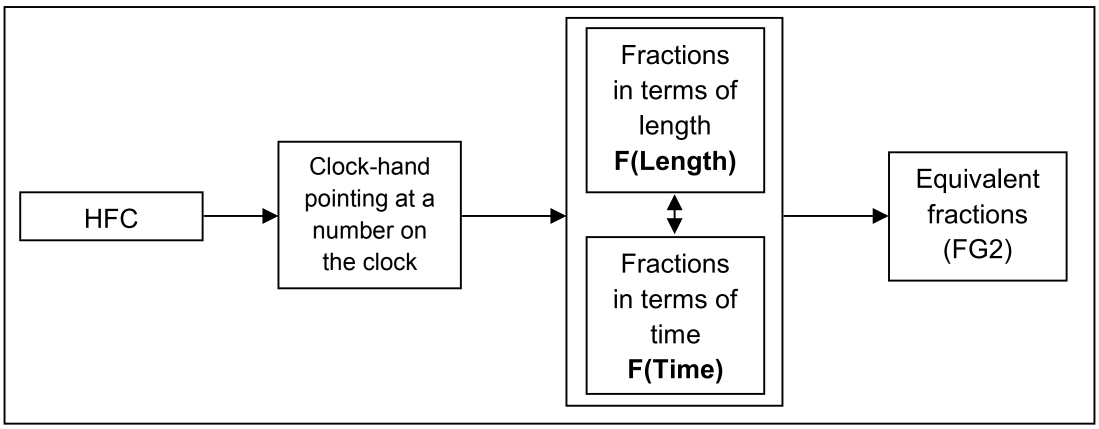


Figure 8.12. Semiotic chains of signification of “*Hour-Foot* Clock” (Activity 3)

Figure 8.12 above shows how, from the HFC, the second factual generalization of equivalence of fractions was achieved. In this activity the students made a 12-inch circumference face clock which they then labeled (i.e., 1 to 12) as they would for any normal clock representing both the time in a 12-hour clock as well as the circumference of the clock face (Figure 8.13). The unique feature of this clock is that the distance along the circumference between each of the twelve numbers is one inch. They then established the fraction for each inch that the clock hand moved in terms of twelve inches (i.e., F(Length)). Next they labeled the number of minutes that passed for each inch (Figure 8.13), and determined the fraction in terms of sixty minutes. The subsequent step was ascertaining that the two fractions

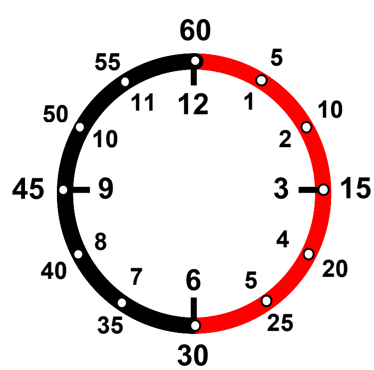


Figure 8.13. Hour-foot clock

were equivalent which is the second factual generalization of equivalence in the schema (i.e., F(length) = F(time)). Hence achieving the desired objectification and also establishing that at least three fractions (i.e., F(parts), F(length) and F(Time)) could be determined from the HFC.

I have summarized the overall connection and association of Activity 1 and Activity 3 in Figure 8.14 below. It shows how these two chains of significations are related to each other. The semiotic chains afforded by the two activities are indicated with different colours (Activity 1–pink, and Activity 2–enveloped in the grey area).

From Figures 8.11, 8.12 and 8.14 it can be seen that there are obviously some overlapping links (i.e., there are semiotic links between the two chains). For the readers' information, chain 1 occurred prior to chain 2. In my opinion, it was necessary for chain 1 to happen first in order to establish the sequence of ideas in order to achieve the desired objectification of equivalence of fractions. The sequence of ideas that the research referred to in this particular case was to start with the 12-inch loaf of bread and cut it into equal parts which will produce F(Part) (e.g., $\frac{1}{2}$) and then linking it to its length in inches, hence F(Length) (e.g., $\frac{6}{12}$). From the loaf of bread, the same idea is used to create the HFC with its 12-inch circumference, which led to the production of F(Part) (e.g., $\frac{1}{2}$) and F(Length) (e.g., $\frac{6}{12}$), and finally linking them to time, and hence F(Time) (e.g., $\frac{30}{60}$), as shown in the following schematic representation (Figure 8.15). Hence linking all the three fractions (i.e., F(Part)=F(Length)=F(Time)) illustrates that the objectification of equivalence of fractions occurred where the unit is the same object. Thus, varied contexts offers varied rational expressions for the same amount of quantity in relation to the same unit

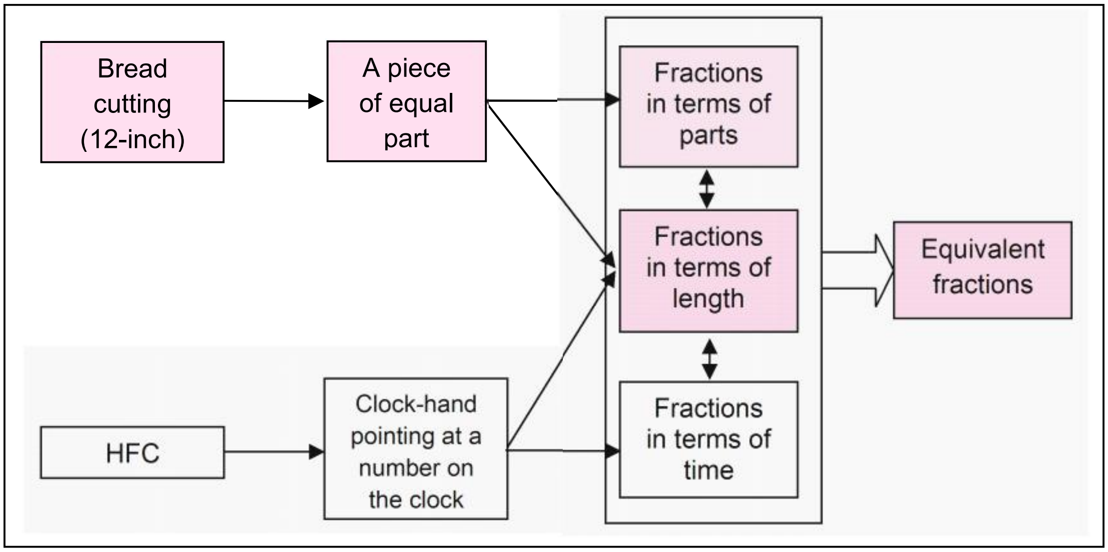


Figure 8.14. Relationship between the semiotic chains of signification of "bread cutting" and "Hour-Foot Clock"

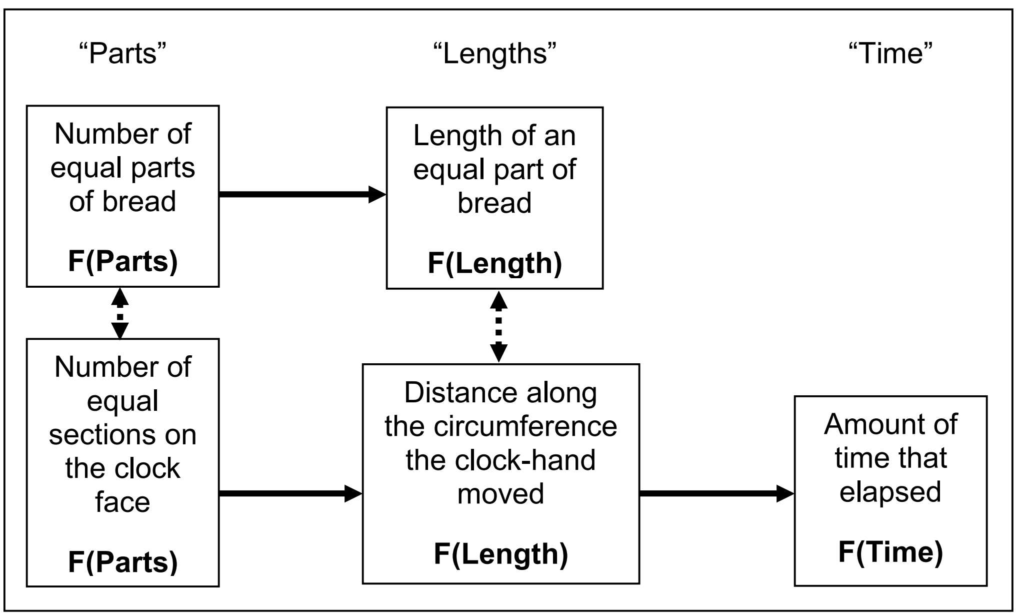
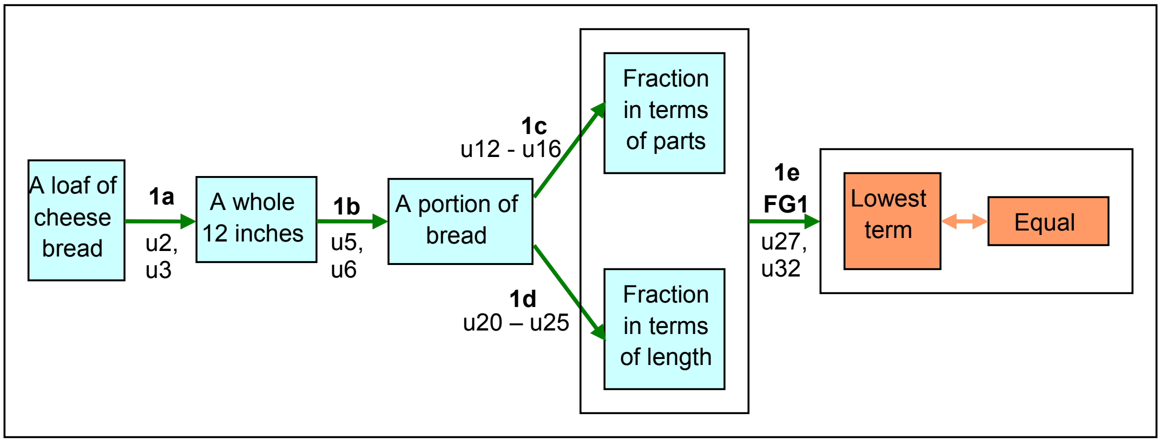


Figure 8.15. Schematic representation

As for the chain of significations, there are in total seven chains of significations and they can be observed and illustrated in the following figures:

(i) The chain of signification of the "part-of-a-whole" (Figure 8.16 represents chains 1 to 3)–a piece of equal part of the bread becomes a fraction in terms of a foot (F(Length) (i.e., 12 inches being the length of the loaf of bread, and the unit for the fraction) and also a fraction in terms of the number of parts (F(Part)) with the number of equal parts as the unit. These two then become equivalent fractions, which as mentioned in section 8.2.2, was where the first factual generalization occurred;



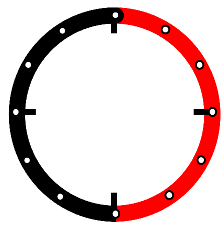
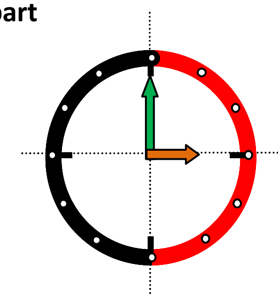
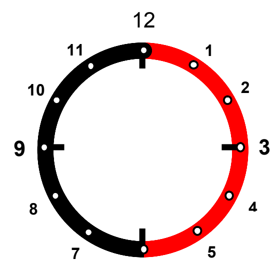
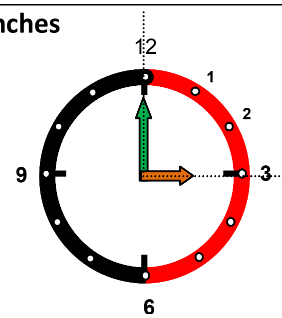
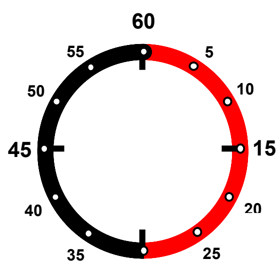
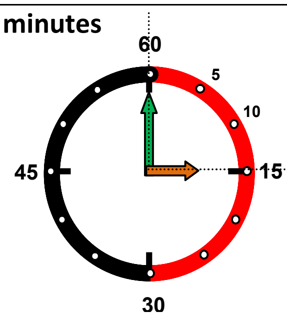
Note: The number followed by a letter (e.g., 2a) indicate the objectification number, whereas the letter "u" followed by a number (e.g., u32) is the utterance number (e.g., u32 means utterance 32 in the episode)

Figure 8.16. Semiotic chains of signification for objectifications 1 [Chains 1–3]

The next four chains of significations were linked to the HFC. Before going further I would like to present Table 8.1 below, which illustrates an example of the relevant units at the different stages in the HFC (i.e., from in terms of the number of parts to the length in inches, and finally in terms of time in minutes) to clarify the unit for the different expressions produced (i.e., F(Part), F(Length), F(Time)). However, I have to remind readers that the sequence of events in the discourse might not be exactly the same as presented in Table 8.1; it is presented as such to simplify presentation. In this example, the minute hand is at 12, and the hour hand is at 3. For the production of F(Part), the unit is the whole circle which for this case can be viewed as having four equal parts, hence $\frac{1}{4}$. Prior to this, for chains 1–3, the unit was a 12-inch loaf of bread (see part (i) above). Similar to the loaf of bread, next was the 12-inch circumference as the new unit for the production of F(Length); the hour hand indicates that it had moved 3 inches, hence $\frac{3}{12}$. Finally, for the production of F(Time), with 60 minutes as the new unit, the hour hand indicates that it had moved 15 minutes, hence $\frac{15}{60}$.

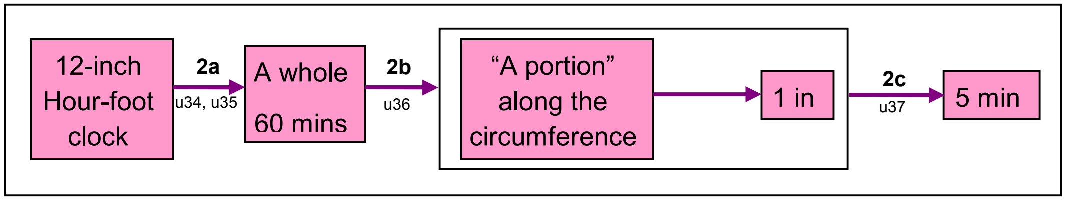
Table 8.1

Sample of units in each representation

Unit	Part	Fractions
 **1 whole circle**	**1 part** 	$\frac{1}{4}$
 **12 inches**	**3 inches** 	$\frac{3}{12}$
 **60 minutes**	**15 minutes** 	$\frac{15}{60}$

One could argue that this is what the Freudenthal Institute had put forward as a *double number line* (DNL). I agree that it does have some similarities, but I argue that it also has some unique features–(a) It has context which is related to the students' everyday experiences (i.e., context of practice); measuring times in hours/minutes and also length in inches, plus it might also be considered in the context of sharing of food; and (b) it is a finite unit (e.g., 12 inches), which is crucial in the production of fractions. I have to stress that I am not claiming that this is a better model over the DNL, however, I felt that it is a useful model to use when first introducing fractions. In other words, this model could be use as a pre-requisite to the introduction of DNL for the production of more complex fractions. Having said that, the four said chains of significations are now presented.

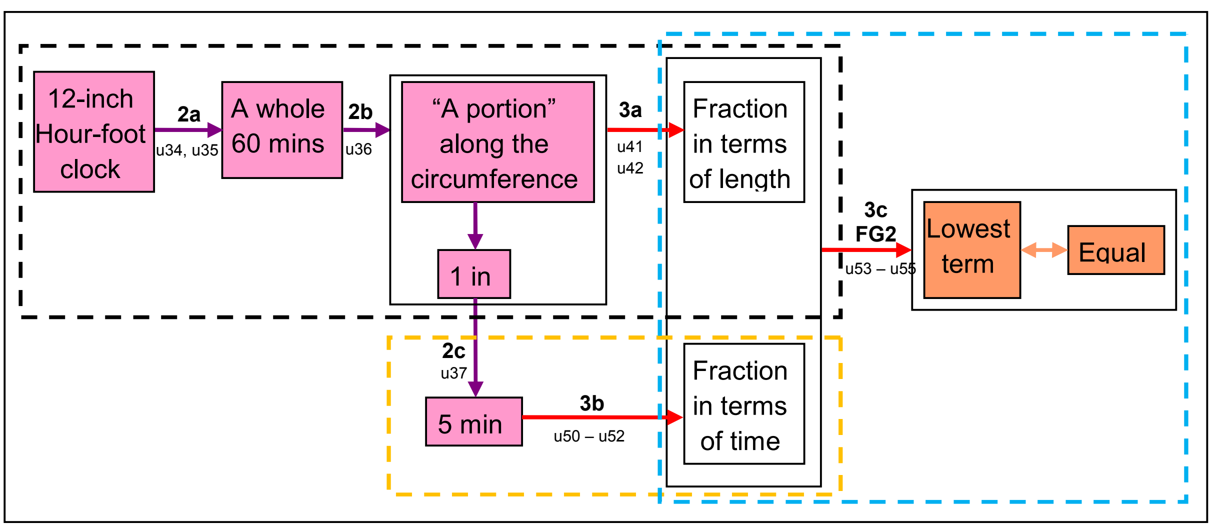
(ii) The chain of signification of the "corresponding value" (Figure 8.17 represents chain 2)–the new whole (unit) (i.e., sixty minutes) was established and attention was focused on the position of the clock hand if it had moved one inch (Figure 8.13) along the circumference which was then associated with the number of minutes that had passed for that distance;



Note: The number followed by a letter (e.g., 2a) indicate the objectification number, whereas the letter "u" followed by a number (e.g., u32) is the utterance number (e.g., u32 means utterance 32 in the episode)

Figure 8.17. Semiotic chain of signification for objectification 2 [Chain 4]

(iii) The chain of signification of the "fractions in terms of inches and minutes" (Figure 8.18 represents chains 5, 6 and 7)–the clock-hand pointing at a number on the clock-face becomes a fraction in terms of inches (with the 12-inch circumference as the unit) which can also be a fraction in terms of minutes (i.e., F(Time)). These fractions then became equivalent fractions which gave rise to the second factual generalization;

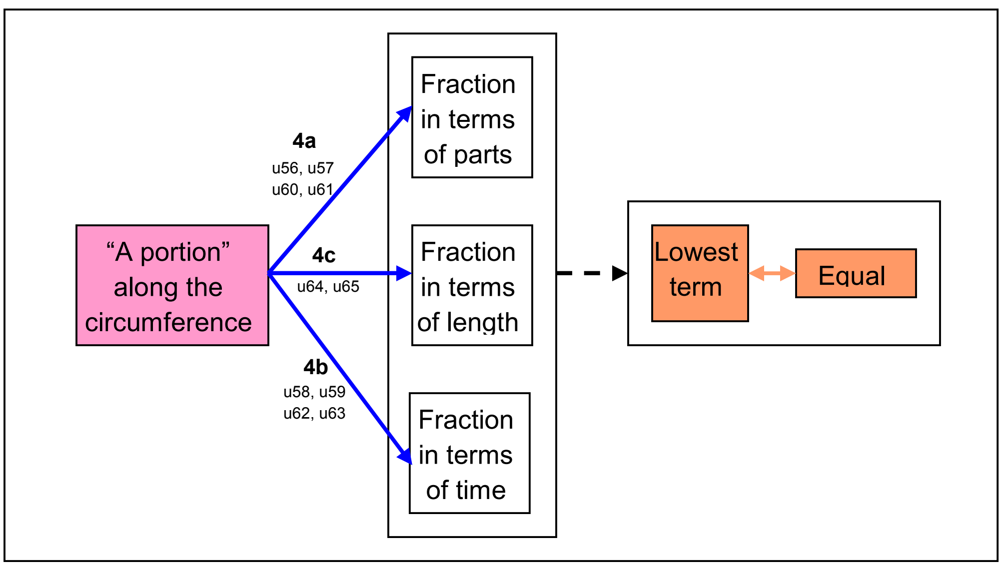


Note: The number followed by a letter (e.g., 2a) indicates the objectification number, whereas the letter "u" followed by a number (e.g., u32) is the utterance number (e.g., u32 means utterance 32 in the episode)

Figure 8.18. Semiotic chain of signification for objectification 3 [Chains 5–7]

(iv) The chain of signification of the HFC which finally afforded the desired equivalent fractions (Figure 8.19 represents chain 8)–as in the previous two chains, the clock hand pointing to a number on the clock face became a fraction in terms of the number of parts (rather, in this case it will be in terms of sections) which became a fraction in terms of sixty minutes and also a fraction in terms of a foot (as the circumference of the clock face is 12 inches), which then becomes equivalent fractions as a result of the association from chains 1 to 3, and chains 5 to 6.

Once chains 1 to 3 had been developed by making use of a 12-inch loaf of cheese bread which the students were asked to cut into a certain number of equal parts and also measuring the length of each part in terms of inches, chains 4 to 8 were then developed by making use of the HFC. However, instead of focusing on equal number of parts, the focus was on the number of equal sections, and also instead of length, circumference was used. It was then linked to time, hence the production of F(Part), F(Length) and F(Time).



Note: The number followed by a letter (e.g., 2a) indicates the objectification number, whereas the letter "u" followed by a number (e.g., u32) is the utterance number (e.g., u32 means utterance 32 in the episode)

Figure 8.19. Semiotic chain of signification for objectification 4 [Chain 8]

From Figure 8.14 above, I have purposely put them in such a way that they looked as if they have been developed in parallel in order to demonstrate the relationships of the two chains and their semiotic links better. Next, I am going to present and discuss semiotically the objectifications of the chains of significations of equivalent fractions.

Following the detailed analysis of the semiotic means in the four objectifications presented in step 1, I made use of the Saussurean dyadic semiotic tradition to discuss the structure of the four objectifications. This semiotic analysis tradition offered a "dyadic" model of the sign which is made up of a "signifier" (i.e., the *form* which the sign takes), and the "signifier" (i.e., the *concept* it represents) (Chandler, 2009). This is followed by a step-by-step re-enactment of the objectifications that took place in the chain of signification of equivalence of fractions.

From the detailed examination done, the following structure, based on the dyadic semiotic tradition, was noted. First, the existence of signs or groups of signs prior to the objectification process beginning–for example, in objectification 1 there was "this portion" referring to the actual piece of equal part of the cut bread (utterance 10), and also in objectification 3 there was "one of the hands" referring to the clock hand on the HFC (utterance 39).

Second, there was a new object introduced after an objectification which was presented with a signifier in the form of a word or a phrase–for example, in objectifications 2 and 3, the new object in the form of the distance (in inches), the number of minutes that had passed and the fraction it represented had entered the discourse through the signifiers "one inch," "five inches," and "five over sixty" (utterances 48 to 51). For convenience, I will now refer to the first two signifiers as "corresponding value" and the second one as the "fraction in terms of time".

Third, a collection of linguistic expressions which pointed to the new signifier and through it the newly introduced signifier-signified link (sign) is established–for example, in this case phrases like "one whole circle . . . one hour" (Researcher) and "sixty minutes" (students) were used to refer to the signifier of "corresponding value." It should be noted that similar terms or phrases may also be used by the students and researcher-students discourse which may also refer to the signifier.

Lastly, a semiotic node was formed (Radford et al., 2003), based on the existing signs, which entailed the objectification of the new object. Like Radford et al., 2003), Koukkoufis (2008) called this "semiotic node along with the existing lower level sign" (p.

169) the “semiotic scheme of objectification” (p. 169). An example of this was in objectification 1 where the semiotic scheme of objectification was “measuring the total length of the loaf of cheese bread so that the length of one equal portion of the cheese bread can be determined,” which afforded the introduction of the phrase “this portion” to refer to one equal part of the cut bread.

From Radford’s (e.g., 2006) semiotic scheme of objectification–the production of a new object (signifier) as a result of the formation of a “semiotic node” enacted on the existing lower level signifier(s)-signified(s) (signs)–Table 8.2 below presents a list of the linguistic expressions signifying the new signifier, the signifier to which the object was connected, the semiotic node of the semiotic scheme of the objectification, and the existing sign(s) involved in the semiotic scheme.

In the bread cutting activity (Activity 1: Fraction of a foot), it started with the students measuring the loaf of cheese bread to establish the length of the whole loaf which was signified by twelve inches. This was then shared equally and each portion of bread became the first signifier (obj. 1a & 1b, sign 1, link 1 in Figure 8.20) of episode 1 and hence the first semiotic link. This is followed with the introduction of a new object into the discourse (i.e., fraction in terms of parts) and thus objectifying (objectification 1c) the second semiotic link (sign 2, link 2 in Figure 8.20). From the first signifier, a third semiotic link (sign 3, link 3 in Figure 8.20) was also objectified (objectification 1d). The final objectification (sign 4, link 4 in Figure 8.20) in episode 1 (objectification 1e) occurred when a new object was introduced into the discourse (i.e., equal).

Before proceeding further, I wish to clarify to readers that objectifications 1a–1e occurred in the bread cutting activity indicated by the green arrows in Figure 8.20. Objectifications 2a–2c, 3a–3c and 4a–4c happened in Activity 3 (*Hour-Foot*), and they are represented by the purple, red and blue arrows respectively in Figure 8.20.

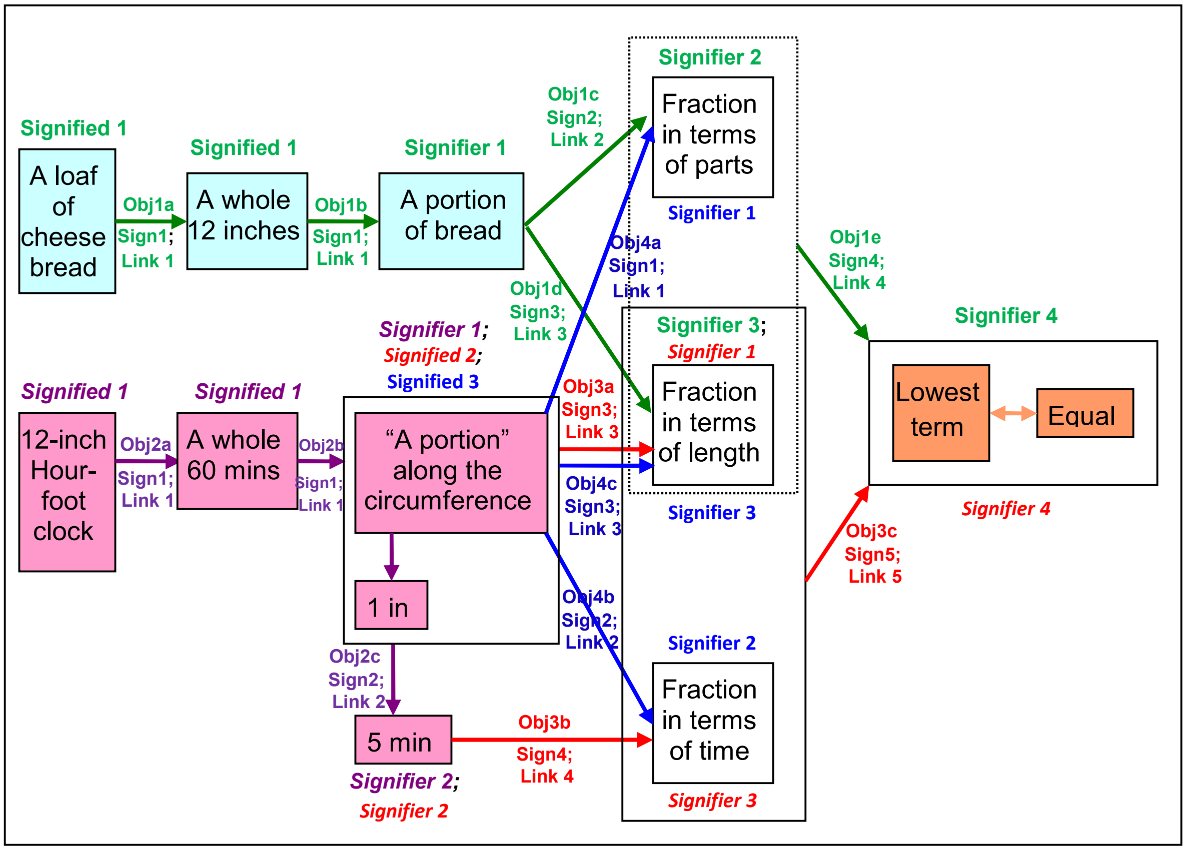


Figure 8.20. All-in-one semiotic chains of signification [Chains 1 to 8]

Table 8.2

The structure of the Four objectifications

<table>
<tr><th rowspan="2">Obj</th><th rowspan="2">Linguistic Expressions Signifying the New Signifier</th><th rowspan="2">Signifier to which the Object was connected</th><th colspan="2">Semiotic Scheme of the Objectification</th></tr>
<tr><th>Semiotic Node</th><th>Existing Sign(s)</th></tr>
<tr><td>1a
1b
1c</td><td>1a: “twelve” (Halim) + “it’s twelve inches” (Res)

1b:“six” (students)

1c: “ fraction” (Res) + “one over two (Halim, Yusof & Fah)</td><td>“Fraction in terms of parts”</td><td>Measuring the total length of the loaf of cheese bread so that the length of one equal portion of the cheese bread can be determined.
1a: Halim taking a ruler and measuring the bread, and the other students agreeing with him.

1b: “share . . . how much will each get” (Researcher), “six” (students), “Divide by two” (Halim, Fah).

1c: “this portion” (Researcher), “fraction” (Researcher), “one over two” (Halim, Yusof, Fah), “Divide it into two” (Fah), “two portions” (Researcher), “so” (Researcher), “must be one over two” (Researcher).</td><td rowspan="2">Divide by the number of equal parts
+
This portion
+
Fraction in terms of the number of parts</td></tr>
<tr><td>1d
1e</td><td>1d: “fraction in terms of inches” (Res) + “six over twelve” (Halim & Yusof)

1e: “Yes” (Fah) + “lowest term” (Fah)</td><td>“Fraction in terms of length”</td><td>1d: “how long” (Researcher), “six” (students), “six over twelve” (Halim, Yusof, Researcher), “it’s twelve inches” (Fah, Researcher), “this portion” (Researcher), “six” (Researcher), “it’s six over twelve” (Researcher).

1e: “this” (Researcher), “one over two . . . and six over twelve the same? Are they</td></tr>
</table>

			equal?" (Researcher), "Yes" (Fah), "why" (Researcher), "lowest term" (Fah).	
2a 2b 2c	2a: "sixty minutes" (Halim, Yusof & Fah) 2b: ""one inch" (Res) 2c: "five" (Halim & Fah) + "five minutes" (Res)	"corresponding value"	Focusing on one equal part to establish the number of inches it corresponds to, followed by focusing on the whole circle of the clock face and representing it in terms of the number of minutes and also in terms of the circumference of the circle which in this case is twelve inches. This is followed by focusing on the one inch to establish the number of minutes it corresponds to. 2a: "time" (Researcher), "one whole circle . . . one hour (Researcher), "sixty minutes" (Halim, Yusof, Fah), "Five" (Halim, Yusof). 2b: "one inch…how many minutes" (Researcher). 2c: "Five" (Halim & Fah), "Five minutes" (Researcher).	The number of minutes for every inch

3a	3a: "one over twelve" (Fah, Halim)	"fractions in terms of length"	Setting up a situation where fractions in terms of the circumference of the clock face could be obtained (e.g., one over twelve). 3a: "how many inches" (Researcher), "one" (Halim, Fah), "one inch" (Researcher), "fraction it has moved" (Researcher), "one over twelve" (Fah, Halim).	Fraction in terms of twelve inches
3b 3c	3b: "five minutes" (Res) + "five over sixty" (Halim & Fah) 3c: "one over twelve and five over sixty" (Res)	"fractions in terms of time"	3b: "Five minutes" (Researcher), "fraction" (Researcher), "five over sixty" (Fah, Halim). 3c: ". . . is one over twelve and five over sixty the same?" (Researcher), "Yes" (Fah and Yusof).	Fraction in terms of sixty minutes
4a 4b 4c	4a(i): "a fraction" (Res) + "one over four" (Fah) 4a(ii): "this" (Res) + "one over two" (Fah)	"Equivalent fractions"	Setting up a situation, and indicating verbally that there is more than one fraction by just making use of connective words such as "and", and also short phrases such as "another one" and "anymore". I did not even have to say in terms of what the fraction should be presented. 4a(i): ". . . when it is at three, give me a fraction" (Researcher), "one over four" (Fah). 4a(ii): "this" (Researcher), "one over two" (Fah).	Fractions in terms of: ~ the number of equal parts; ~ time (i.e., 60 minutes); ~ it's length (i.e., 12 inches).

	4b(i): "and" (Res) + "fifteen over sixty" (Halim & Fah) 4b(ii) "Another one?" (Res) + "Thirty over sixty" (Halim) 4c: "Anymore" (Res) + "six over twelve" (Halim & Yusof)		4b(i): "and . . ." (Researcher), "Fifteen over sixty" (Halim, Fah). 4b(ii): "Another one?" (Researcher), "Thirty over sixty" (Halim). 4c: "Any more" (Researcher), "Six over twelve (Halim, Yusof).	

As mentioned earlier (see section 8.2.1), some of these episodes consisted of sub-episodes which allow for the objectification for that particular episode to happen. As for objectification 2, unlike for the first objectification, it happened in one episode only, but as before it also consisted of a few preparatory objectifications. It occurred in Activity 3 where a clock-face with a 12-inch circumference was constructed, and this was signified by sixty minutes (i.e., the whole). In fact, the first two of these preparatory objectifications were establishing that for a whole hour there is sixty minutes and the distance between each consecutive pair of numbers on the clock face was one inch (obj. 2a & 2b, sign 1, link 1 in Figure 8.20). This was followed by the next objectification which thus objectified objectification 2 and another new object was introduced in the discourse: the corresponding value of one inch in terms of time in minutes. As a result, another new object (e.g., 1 inch≡5 minutes) was introduced in the discourse (obj. 2c, sign 2, link 2 in Figure 8.20).

Third, the sign that I have analysed as "corresponding value" had already been objectified at the end of episode 2, and in episode 3 objectification 3 took place in which semiotic activity enacted on the existing sign of "corresponding value" introduced the new semiotic scheme for this new objectification. In other words, the signifier from the previous chain (i.e., chain 4) became the new signified for these new chains (i.e., chains 5 to 7). From this, a new object (e.g., one over twelve) was introduced and the new signifier “fractions in terms of length” was connected to it (obj. 3a, sign 3, link 3 in Figure 8.20). Another new object and signifier was also introduced in this episode when a fraction in terms of time was established (e.g., five over sixty–obj. 3b, sign 4, link 4 in Figure 8.20). Finally, these two preparatory objectifications gave rise to a third objectification when it was ascertained that the two fractions were equal (obj. 3c, sign 5, link 5 in Figure 8.20). Hence, objectifying objectification 3.

For the fourth and final one, the signs I have analysed as “fraction in terms of parts,” “fraction in terms of length,” “corresponding value,” and “fraction in terms of time” had already been objectified prior to episode 4. In this episode, objectification 4 took place in which semiotic activity enacted on the existing four signs of “fraction on terms of part,” “fraction in terms of length,” “corresponding value,” and “fraction in terms of time” attained objectification 4 from their previous association (i.e., objectification 1e and objectification 3e). The signified was again the signifier 1 of chain 4, and from this three links were made (i.e., obj. 4a, obj. 4b, obj. 4c–refer to Figure 8.20). This thus completes the chains of objectification of equivalence of fraction in terms of the number of parts, length and time.

The semiotic chains of objectifications above are summarized in the schematic diagram above (Figure 8.20) which concludes the presentation of the semiotic chains of signification found in the semiotic chains that objectified these chains for the factual generalization of the equivalence of fractions and hence concludes the analysis of the equivalence fractions for E1.

However, before I end this chapter I wish to remind readers again, with the risk of repeating myself, some of the considerations that I had taken for choosing what I had done in this analysis and the structure of the lessons as a whole.

As mentioned earlier, I wanted to achieve a generalization of equivalence of fractions which revolved around the students’ everyday experiences which I felt they could relate to quite easily, hence my choice of using time to achieve this goal. Nevertheless, in order to reach the desired goal I felt it would need to be done in a few stages which would involve fractions in terms of the number of parts, length and finally time (i.e., F(part)–F(length)–F(time)). Prior to attaining the targeted generalization, along the way, some generalizations had to take place. In the following paragraphs I am presenting the decisions that I made and the considerations made for each decisions.

Before I could introduce fractions in terms of time, I needed to introduce the numbers one to twelve as most timepieces would show and use these numbers. Since I had chosen to do my research in the framework of RME, I needed "context of practice" to introduce the numbers so that it is more meaningful. For this I had chosen to use the context of sharing of food and measuring lengths (i.e., a 12-inch loaf of cheese bread) as the entry point of the lessons. Still I did not want to introduce fractions in terms of length at that point as the students had not had much practice in their mathematics lessons in dealing with such fractions (refer to Chapter 3, section 3.3) with the exception of a few examples which involved unlike items, such as boys and girls, and a collection of coins of different denominations. Instead I decided to start by objectifying fractions in terms of parts as this is what they are most familiar with, both in their classroom mathematics and examinations (either school-based or public) (Samsiah, 2002). Once that had been established, it was only then followed by ascertaining the fraction in terms of the length of the bread. By using the loaf of bread, a few things were established; the numbers one to twelve, one equal portion, F(part), F(length), F(part)=F(length) for the same amount of bread. As mentioned earlier, this led to the occurrence of factual generalization of equivalence of fractions (i.e., F(part) = F(length) for the same portion of bread). However, even though it is a factual generalization of equivalence of fraction, it was not the targeted generalization.

Even though the students were familiar with inches, they don't really use them frequently, so I chose to relate it to another experience/practice that they went through everyday, which is reading the time. Thus, I introduced the HFC. The HFC is a clock with a 12-inch circumference; the distance along the circumference between the numbers on the clock face was one inch. It was further established that the time elapsed between the numbers is five minutes, which led to a crucial link for the overall generalization to occur as it linked the three aspects involved–number of parts, length (distance) and time. As a result of this, factual generalization of equivalence fractions that involve their daily experience was achieved.

On the other hand, one can argue that instead of using a 12-inch long loaf of cheese bread, a circular item, such as cake or pizza, with a 12-inch circumference could be used so that factual generalization could happen immediately. True, that would solve the problem of multiple factual generalizations occurring, but I felt that to cut a circular pizza, for example, into twelve approximately equal slices is something that is not necessarily easy to do with acceptable accuracy. The issue is not getting the twelve slices themselves, but to get them approximately equal would be a daunting task; an obvious problem that I anticipated would be the cut not passing the centre of the circle, which would hence not produce approximately equal shares. Another issue is that it is much easier to cut a 12-inch long item than a 12-inch circular object (imagine trying to divide or cut a 12-inch pizza into twelve equal slices with the curved side only 1-inch long). It is for these considerations that I chose to do what I did.

Radford's (2001, 2003) *theory of objectification* considered factual generalization as the simplest of the three types of generalizations, and is a "generalization of actions" (Radford, 2003, p. 47). From the examples he presented, I felt that he is suggesting it to be a simple objectification process, a "single-step objectification process" (Koukkoufis, 2008, p. 158). The above semiotic analyses, however, have shown that it is not necessarily so, instead it shows how equivalence of fractions can come about as the completion of a series of objectifications in pupil-teacher group work involving the HFC. In this case, in order to achieve the desired objectification, the lesson went through a process of objectification which consisted of pre-preparatory objectifications which made up the preparatory objectification and finally those become the targeted objectification. In other words, no one single pre-preparatory or preparatory objectifications alone could achieve the targeted objectification, rather all of them as a whole contributed to the final objectification of equivalence of fractions.

Figure 8.20 also demonstrates the complexity faced by the students when dealing with fraction concepts, where to prove equivalence they needed to be able to produce different representations of fractions, based on the unit used, which gave rise to different expressions. The HFC model for this case is useful as it provides all three different fraction expressions (i.e., F(Part), F(Length), F(Time)). I suggested that this model might be beneficial as an introductory activity prior to introducing a double-number line. I felt that the duality of the HFC works because of the familiarity of the units, and the separation of the two representations due to the “hour-hand” and “minute-hand”. In addition, I am also advocating that, due to the complexity of the objectification process here, a signifier for one chain can become the signified for another chain (for example, signified 2 for the HFC). This is especially the case for an intricate and overlapping chain–chopping it into shorter chains makes them more manageable to analyse.

In the next chapter, I will present the semiotic chains of signification for flexibility of unitization.

CHAPTER 9

SEMIOTIC ANALYSIS OF THE GENERALIZATION OF THE FLEXIBILITY OF UNITIZING OF FRACTION, THE SEMIOTIC ROLE OF THE HOUR-FOOT CLOCK AND THE SEMIOTIC NETWORK (S-NET) OF GESTURES AND LANGUAGE.

9.1 ANALYSIS CONTRIBUTION

In Chapter 9, the semiotic analysis of the generalization of the flexibility of unitizing is presented. This chapter contains the following sections:

9.2. Step 1 – Semiotic Objectifications of the Flexibility of Unitizing;

9.3. Step 2 – Chain of Signification of the Flexibility of Unitizing;

9.4. The Semiotic Role of the Hour-Foot Clock Model;

9.5. Semiotic Network (S-NET) of Gestures and Language; and

9.6. Concluding the contribution of Chapters 7–9

In section 9.2, similar to the semiotic analysis done in Chapter 8, which was done in two steps, a similar semiotic analysis is also done for the generalization of the flexibility of unitizing. As presented in Chapter 8, in this chapter I will also adopt any fitting adaptations to the theory of objectification.

9.2 STEP 1 – SEMIOTIC OBJECTIFICATIONS OF THE FLEXIBILITY OF UNITIZING

9.2.1 Adaptations to the theory of objectification in the Factual Generalization of the Flexibility of Unitizing

In this section, the factual generalization of the flexibility of unitizing is discussed, similar to the discussion of the equivalence of fractions in Chapter 8, which was based on Radford's theory of objectification (2002, 2003). This theory has proved to be helpful in shedding light on students' semiotic meaning-making processes in these particular activities (i.e., Activity 1 and 3). However, since this theory was originally developed by Radford for a different situation (i.e., algebra), it was expected that adaptations, similar to the ones made in Chapter 8, might have to be made to the theory.

Unlike in Radford's theory of objectification of factual objectification, which is essentially a one-step objectification process, the factual objectification of the flexibility of unitizing required five preparatory objectifications before this objectification can be achieved. As before each preparatory objectification was also made up of some pre-preparatory objectifications. In other words, this particular factual generalization is a six-step semiotic process. Similar to the objectification of the equivalence of fractions in Chapter 8, these preparatory objectifications are very important for the attainment of the sixth objectification which objectified the flexibility of unitizing. Therefore, all these six objectifications as a whole add up to the factual generalization in this case, and not just the sixth objectification alone, as without any one of the five preparatory objectifications, the sixth one will probably be rather difficult to objectify, especially for this particular experimental group of students. Thus, as before, the factual generalization here entailed the involvement of a string of objectifications.

The six-step semiotic process is presented in section 9.2.2 in episodes 5–9. All these episodes occurred in Activity 6 (which I called "Let's share!"), where the students were asked to share a 12-inch square pizza between the four of them, and when they had divided the pizza equally among themselves, someone in the group decided not to have it, and they ended up having to divide that person's share again so that the rest of them would have another equal share. In this activity they were to find how much each of them got initially, then they were to decide how much of that portion each of them received after the second division, and finally how much each of them got at the end. A detailed explanation of this activity can be found in section 4.2 in Chapter 4.

9.2.2 Factual Generalization of the Flexibility of Unitizing

9.2.2.1 Episode 5: Objectifications of "original amount" (objectification 1) and "shared between" (objectification 2)

Episode 5 occurred at the beginning of activity 6 when I was introducing the lesson. I was informing the students of the amount of material, in this case a square pizza, which was available to them. In other words, I was indicating to them that the whole in this case is a square pizza. As before, there were four students in the group and they were to work together to do this activity.

Episode 5 (minutes 00:16-00:23 /Activity 6)

The researcher-teacher was trying to explain what they were supposed to do in Activity 6. In this case I informed the students what was available for them and what they were supposed to do.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
1	Res:	Now I have [this (points to the pizza) square pizza], and I want you to [share between the four of you]. How would you do it? How would you divide it? How would you cut it?	Presenting what is available (whole) and what to do with it. **SMO1** – deictic word (i.e. this) **SMO2 –** indexical gesture (i.e., pointing) **OBJ1** – this square pizza **OBJ2** – share between

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means of objectifications; OBJ – objectification.

From the first utterance alone, two preparatory objectifications happened consecutively. This occurred when I used a deictic word (i.e., this) (**SMO1**) and also an indexical gesture of pointing (**SMO2**) to bring to the students' attention to the square pizza, thus this brought about objectification of the "original amount" which is the first preparatory objectification (**Objectification 1**). This was followed by the attainment of objectification 2 (utterance 1) of "share between" when I said that "I want you to share between the four of you," which indicated that the square pizza should be shared equally and they needed to divide/cut it equally into four shares.

9.2.2.2 Episode 6: Objectification of "each share" (objectification 3)

This episode occurred after the students had cut the square pizza into four equal parts (see Figure 9.1 and Figure 9.2). Figure 9.1 shows the students made sure that they cut the pizza into equal parts by taking note of the length of one of the sides (picture 1) and

Hazim (picture 2) indicating the halfway point of the side. Whereas in Figure 9.2, Fah gestured how to cut the whole pizza in order to get four parts.

Picture 1 Picture 2

Figure 9.1. Picture 1–Fah and Era (1st and 2nd from the right) measuring the whole pizza before cutting; Picture 2–Halim (1st from the left) pointing to the middle

Picture 1 Picture 2

Figure 9.2. Picture 1–Fah (right) gesturing with her index finger a cut from left to right; Picture 2–Fah gesturing with her index finger a cutting motion from top to bottom

<u>Episode 6 (minutes 04:05-04:21 /Activity 6)</u>

In this episode the researcher-teacher was trying to establish the fraction for one equal share of the cut pizza.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
2	Res:	Alright, you have [this] (points to the square pizza) [one whole] pizza and you have [cut it into four], [one for each one of you]. [How much] will each of you be getting?	Highlighting and reiterating what the students had done and paving the way to objectify the fraction through

			questioning. **SMO3** – indexical gesture (i.e., pointing) **OBJ3a** –cut it into four, how much
3	Fah:	[One over four]	**OBJ3b** – one over four
4	Res:	One over four. Exactly, yes, there are [four parts and there are four of you] […]	Summarizing and reemphasizing what had been said earlier **OBJ3b** – four parts and there are four of you

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means of objectifications; OBJ – objectification.

In episode 6, the third pre-preparatory objectification was jointly attained by Fah and me. This occurred when I indexically gestured (**SMO3**) towards the square pizza followed by me asking "how much will each of you be getting?" (utterance 2) which was meant to direct the students' attention to the amount, specifically the fraction, of each share (**Objectification 3a**). Fah immediately provided with the fraction of each share in utterance 3 (**Objectification 3b**), and I rephrased what she said by saying "there are four parts and there are four of you" to indicate that each of the four students will get one share (**Objectification 3b**), the same as what Fah had said when she gave a fraction "one over four." With this, objectification 3 occurred.

9.2.2.3 Episode 7: Objectification of "new shared between" (objectification 4)

Utterance 5 in this episode is actually part of utterance 4 in episode 5. For simplicity of analysis, I decided to split it into two separate utterances, as it involves two separate pre-preparatory objectifications.

Episode 7 (minutes 04:22-04:52 /Activity 6)

The researcher-teacher was putting forward another situation where they have to deal with a different number of shares.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
5	Res:	[…] What if Halim does not want his share as he is fasting, and he wishes to [give his share] to all of you. What are you going to do with his share (points to one of the four one over four parts)?	Created a new situation where there was a new whole and the number of people to share it also changed. **SMO4** – pointing gesture **OBJ4a** – give his share
6	Halim:	[Divide into three]	**OBJ4b** – three (shares)

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means of objectifications; OBJ – objectification.

In this episode I created a situation where Halim decided that he did not want his share and wanted to give it to the rest. When I said that "he wishes to give his share to all of you" (**objectification 4a**) and also indicating with my indexical gesture (**SMO4**) by pointing to one of the four "one over four" parts, I had intentionally tried to focus the students' to a possible new whole (i.e., unit). To this, Halim in utterance 6 immediately responded by saying "Divide into three"–which indicates the number of people to share the "one over four" part–and hence objectifying objectification 4 (i.e., new shared between). Following that Fah as shown in Figure 9.3 cut one of the four "one over four" parts into three equal pieces.

Figure 9.3. Fah (right) cutting one of the one over four part into three equal parts

9.2.2.4 Episode 8: Objectification of "new original" (i.e. unit) (objectification 5)

Episode 8 happened as soon as Fah (Figure 9.3) cut one of the "one over four" parts into three equal parts–they measured the side of the pizza before cutting to ensure equal division. Prior to this episode I pointed to the "one over four" part that had been cut and asked them "how many pieces now?" and Yusof responded "Three" to point out that the "one over four" part is now cut into three pieces.

In utterance 7, I wanted to ascertain the new whole (i.e., unit) when I said "what's the original?" and in order not to confuse them with the *original* original (i.e., the whole square pizza), I continued to add by asking them the fraction for one of the new whole (i.e., unit) share which they had established in utterance 3. This was done through using deictic language (i.e., "this" and "just now") (**SMO5**). The first deictic language was intended to indicate that I was referring to the "one over four" part, and the phrase "just now" was to remind them that this had already been established earlier. From this the first pre-preparatory objectification was achieved where it brought to the students' attention that the new whole is a fraction (**Objectification 5a**).

However, Fah, in utterance 8, sounded unsure when she began to say "One . . ." and did not continue. In utterance 9, Halim immediately interrupted and gave an answer of "one over four" (**Objectification 5b**) and hence achieved the second pre-preparatory objectification. It was not clear, however, whether Halim really knew that the fraction for the new whole was one over four, or whether he was just following Fah's cue, and completing her sentence. In utterance 10, I completed the objectification by making use of a deictic word (i.e., “that”) to signify the fraction (**SMO6**) that Halim had given as “the” unit in this case, hence, objectifying the “new original” (i.e., objectification 5).

Episode 8 (minutes 07:55-08:10/Activity 6)

The researcher-teacher wanted to establish the new whole for this new situation.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
7	Res:	[…] [What’s the original]? What was [this] (points to one of the four original shares) [fraction] [just now]?	Brought to light the new original (i.e., unit) through questioning. **SMO5** – deictic languages (i.e., this, just now) and indexical gesture (i.e., pointing) **OBJ5a** – this fraction (piece of pizza)
8	Fah:	One…	Sounded unsure of herself
9	Halim:	[One over four]	**OBJ5b** – one over four
10	Res:	Yes Halim. [That] was the original […]	**SMO6** – deictic language (i.e., that) **OBJ5c** – that was the original

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher’s description; SMO – semiotic means of objectifications; OBJ – objectification.

9.2.2.5 Episode 9: Objectification of “new share” (objectification 6)

Utterance 11 in episode 9 is in fact part of utterance 10 from the previous episode. As before, it had been split to simplify the analysis. After establishing the new original, I proceeded to focus their attention to the three newly cut pieces when I said “you have cut it into three” (**Objectification 6a**) which was followed by directing their attention to one of the three cut pieces when I said “this” and pointing to it at the same time (utterance 11) (**SMO7**). Fah responded by saying “one over three?” (utterance 12) (**Objectification 6a**). Her response sounded more like a question than a statement, and she was quite hesitant saying it. I assumed that Fah was giving the fraction for the little piece of fraction per se and maybe “forgetting” the unit *now* is one over four of the square pizza (i.e., the original whole). In order to confirm this, I probed further when I said “out of what? Is it one over three of the whole?” (utterance 13). Here I was trying to bring to their attention that the unit matters. In utterance 14, Yusof who was listening to this discourse shook his head, possibly to show his disagreement, but when I asked him “one over three of what” (utterance 15) he just smiled (utterance 16). Most probably he was not sure either but he probably knew that it was not “one over three of the whole” as the three pieces were cut from one of the “one over four” parts. In utterance 17, I said “what’s the original again?” emphasizing the word “again” and also pointing to one of the “one over four” parts (**SMO8, Objectification 6b**). Halim and Fah (utterance 18) immediately responded by giving an answer of “one over four” (**Objectification 6b**). I emphasized this again in utterance 19 using the word “this” to indicate one of the three cut pieces and also pointing to the “one over four” part with the three cut pieces (pictures 1 and 2, Figure 9.4) (**Objectification 6c**).

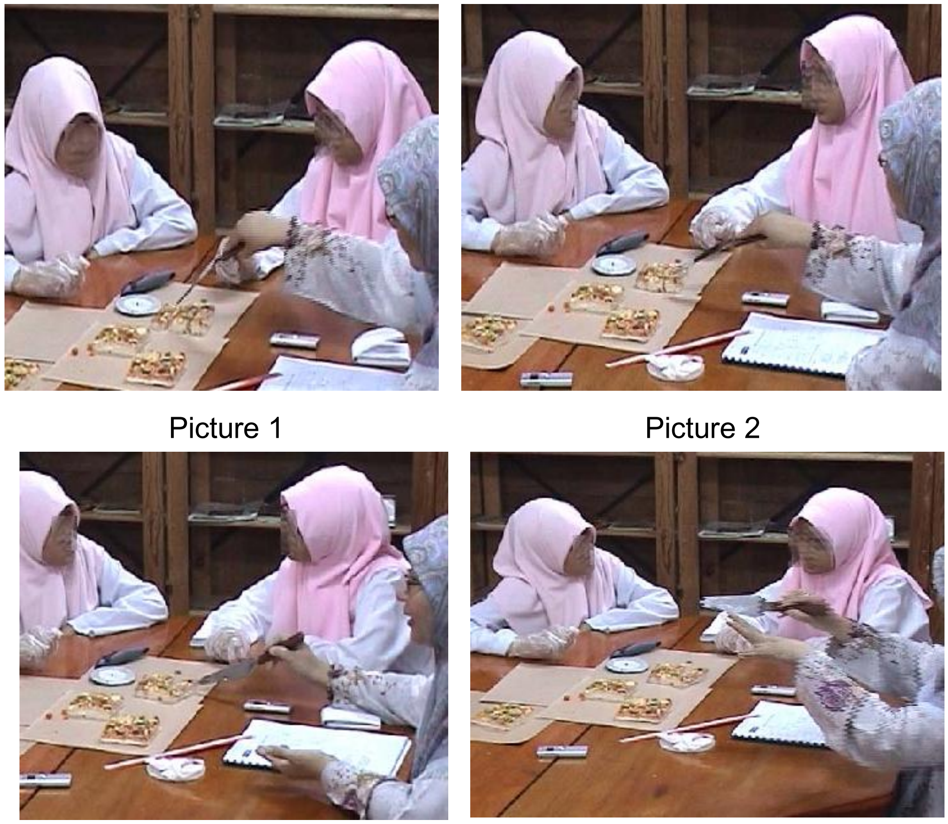
Episode 9 (minutes 08:12-09:01)/Activity 6)

Following on from the previous objectification, the researcher-teacher wanted to ascertain that the fraction for the “new” share is of the fraction of the “old” share.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
11	Res:	[…] and now [you have cut it into three] so each of [this (points to one of the three little pieces) is]?	**SMO7** - deictic languages (i.e., this) and indexical gesture (i.e., pointing) **OBJ6a** – cut into three
12	Fah:	[One over three?]	Again, sounded quite hesitant and responding with a question instead of a statement. **OBJ6a** – one over three
13	Res:	Out of what? Is it one over three of the whole?	

14	Yusof:	(Shook his head)	
15	Res:	No? Yusof, you are shaking your head. [One over three of what] Yusof?	
16	Yusof:	(*Smiled*)	
17	Res:	What's the [original] [again]? (points to one of the remaining uncut one over four parts)	**SMO8** – verbal and pointing gesture **OBJ6b** – original
18	Halim & Fah:	[One over four]	**OBJ6b** – one over four
19	Res:	So [this] (points to one of the three newly cut pieces (*picture 1 in Figure 9.4)* is [one over three of one over four] (points to the one of the remaining one over four parts *(picture 2 in Figure 9.4)*, and not one over three of the whole (gestures a circular motion with both hands *(picture 4 in Figure 9.4)* pizza just now […]	**OBJ6c** – one over three of one over four

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means of objectifications; OBJ – objectification.



Picture 1 Picture 2

Picture 3 Picture 4

Figure 9.4. The teacher pointing the bread to indicate “one over three of one over four” and “one over three of the whole”: Picture 1–pointing to one of the three pieces; Picture 2–pointing to the all the three pieces (one over four parts); Picture 3–pointing to one of the three pieces; Picture 4–gesturing a circular motion with both her hands to indicate a whole

In conclusion, then, the episodes 5–9 of the factual generalization of the flexibility of unitization afforded the different stages involved in achieving the final objectification. In episode 5, the first two preparatory objectifications were achieved right-away. I had introduced the original whole (i.e., the square pizza) and also the number of people to share it using a deictic word (i.e., this) to indicate the whole, which was accompanied by pointing to the pizza. In episode 6, Fah objectified the fraction for each share that each recipient would get, thus introducing a new object. I then rephrased it further with an indexical-associative meaning by saying “there are four parts and there are four of you,” indicating that each of the four students would receive one of the four parts. In Episode 7, however, a new scenario was introduced, where Halim did not want his share and gave it to the rest of his friends. With this, a new object (i.e., a new number of people to share Halim’s one over four of the pizza) was introduced. Halim presented the new object when he put forward an indexically-associate phrase of “divide into three” which indicates that “3” is the new number of recipients. In episode 8, Halim and I jointly presented one over four (i.e., Halim’s share) as the “new original” (i.e., new unit). In the last episode, episode 9, Fah presented a new object (i.e., one over three) which was the fraction for one part of the cut piece. However, from the tone of her utterance she was probably not sure of her response, or it might have been indicative that she was experiencing conflict, as the “new original” was one over four and now she was presenting one over three as the fraction of the new cut piece. In pulling everything together I made the final objectification that the new share was one over three of one over four. The semiotic analysis for the objectification of the flexibility of unitization continues in the following three sections.

9.2.3 Contextual Generalization of the Flexibility of Unitizing

For all the groups, contextual generalization did not happen. This may have been because of the innately straightforward nature of the activity. If any contextual generalization were to occur, it would have been as follows:

> If one of the three pieces is one over twelve, then two pieces will be two over twelve, or if one of the three pieces is five minutes, two pieces will be ten minutes.

The analysis of the semiotic objectifications of the flexibility of unitization continues in the following sections with the symbolic generalization of the flexibility of unitization.

9.2.4 Adaptations to the theory of objectification in the Symbolic Generalization of the Flexibility of Unitizing

The symbolic generalization that is examined in section 9.2.5 is associated with the transition from a linguistic description of a share of the square pizza, to the number of equivalent minutes it represents, to the fraction in terms of minutes, and finally to the amount of total share, in fraction, each student gets after a further sharing of one of the original shares, which then later translates to the production of equivalent fractions. Here, the symbolic generalization would involve the replacement of descriptive words like “big one” and “big piece” by a numerical representation of the share (i.e., in this case in terms of minutes or inches) and then by a fraction that represents it and the final total amount, in fractions, of the share. The main difference that can be noted in this symbolic generalization is the bypassing of the contextual generalization, which did not hinder symbolic generalization.

9.2.5 Symbolic Generalization of the Flexibility of Unitizing

9.2.5.1 Episode 10: Objectification of "Total share" (objectification 7)

Prior to this episode, it was already established (i.e., objectified) that one "big" piece is equivalent to fifteen minutes, at which point I then proceeded to ask them the equivalent number of minutes for each of the three little pieces that they had previously cut from the "big" piece. As mentioned in earlier section, the HFC was designed to aid students in solving some fractional problems using items that they are familiar with (i.e., a clock). Since we were dealing with one over four I purposely asked for the equivalent number of minutes, which indirectly checks that they know the whole is sixty minutes. Halim responded instantaneously with an answer of "five" as the number of minutes representing one of the three little pieces. However, when I asked him how he got five, he said "sixty divided by twelve," which indicated that he was "remembering" our discussion earlier (which is not presented in this analysis) of how for the whole square pizza there are twelve equal parts if each of the "one over four" parts is divided into three. In all probability he overlooked the illustration that the new original (i.e., unit) is now fifteen minutes so each of the three little pieces would represent five minutes. So in order to direct them to the new original, in utterance 26, I once again asked for the fraction, but this time I had also used the phrase "this original" followed by "this big one" so as to highlight to them that there was now a new original (**objectification 7a**). Fah and Halim (utterance 27) gave "one over four" (**objectification 7b**), and when I asked for the equivalent number of minutes (utterance 28), all the students responded "fifteen" (utterance 29, **objectification 7c**), hence objectifying it for the second time.

Sub-episode 10.1 (minutes 13:30-14:17 /Activity 6)

The researcher-teacher summarized the whole discussion in order to determine the fraction for the new share and noticed that it was also linked to time.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
26	Res:	What was [this original] (points to one of the one over four part)? What's the [fraction] of [this big	Referring to the new whole (i.e., one over four)

		one]?	**SMO9** – verbal and indexical pointing **OBJ7a** – this original, this big one
27	Fah & Halim:	[One over four]	**OBJ7b** – one over four
28	Res:	One over four. How many [minutes]?	
29	Students:	[Fifteen]	**OBJ7c** - fifteen
30	Res:	Fifteen minutes. Now, you [divided it into three] you will get [five minutes] (points to one of the three little pieces) […] you know the [big piece, that's fifteen minutes] (points to one of the three remaining uncut one over four parts), and I know [this] (points to one of the three little pieces) is [one third of one quarter (referring to the one over four part)], I know [one quarter (referring to one over four) is fifteen minutes], so [this] (points to one of the three little pieces) must be [five]. Five over?	Summarizing what had been discussed **SMO10** – indexical pointing **OBJ7d** – divide it into three **OBJ7d** – five minutes
31	Fah:	Five over…[five over sixty]	**OBJ7e** – five over sixty

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means of objectifications; OBJ – objectification.

In utterance 30, I summarized again what had been discussed to bring to their attention that they had fifteen minutes (i.e., the new original), and divided it into three (i.e., the new share out) to give them five minutes (**objectification 7d**); this was also accompanied by my indexical pointing gestures to one of remaining three uncut larger pieces and also one of the three little pieces (**SMO10)**. Fah, in utterance 31, gave a fraction in terms of minutes (i.e., five over sixty) for one of the three new shares (**objectification 7e**). In fact, in a subsequent utterance Fah had also managed to give "one over twelve" as the equivalent fraction for the new share. With that they now had objectified the two fractions (i.e., one over four and one over twelve) separately, each of which had a different whole/unit. Nevertheless, to be really sure that they understood this, they were asked to translate these fractions in terms of minutes based on the HFC, hence producing the total share for each student.

Sub-episode 10.2 (minutes 15:23-16:26 /Activity 6)

The researcher-teacher summarized the whole discussion in order to determine the fraction for the new share.

Utterance No.	Uttered by	Utterance (Spoken text/ gestures/actions)	Comment (including interpretations and SMO codes)
32	Res:	[…] how much will each of you get? At [first you have one quarter] (points to one of three remaining the uncut larger pieces), then H doesn't want his, so each of you gets a third (points to one of the three little pieces) of his share each, and now you [have an extra of one over twelve]. So now you have [one over four plus one over twelve]. How much is that? […] [in terms of time]	**SMO11** – indexical pointing **OBJ7f** – one over four plus one over twelve **OBJ7g** – time
33	Halim:	[Twenty]	**OBJ7g** – twenty
34	Res:	Twenty over what?	
35	Fah:	[Over sixty]	**OBJ7h** – over sixty
36	Res:	Yes Fah, continue…	
37	Fah:	[Four over twelve]	**OBJ7h** – four over twelve
38	Res:	[And?]	
39	Fah:	[One over three]	**OBJ7h** – one over three

Note: […] – irrelevant; (regular text) – gestures/smos; [regular text] – contributes to analysis; (*italics*) – gestures/actions that do not contribute to analysis; … - pause/interruption; *italics* – researcher's description; SMO – semiotic means of objectifications; OBJ – objectification.

In sub-episode 10.2, I asked them to work out the fraction for their new total share in terms of time (utterance 32) by also pointing to one of the four cut pieces (i.e., their original share) and to one of the three little pieces (i.e., their new share from Halim). Halim in utterance 33, instantaneously gave an answer of "twenty" and when prompted for a fraction, when I said "twenty over what?" (utterance 34), Fah responded in utterance 35 with "over sixty" (**objectification 7h**). Fah subsequently gave "four over twelve" and "one over three" as the equivalent fraction in utterances 37 and 39.

Radford (2006) stated that "In symbolic generalizations, the students' formulae often tend to simply narrate events and remain attached to the context" (p. 16). Admittedly, in the episodes presented, there was no "formula" as such produced by the students due to the nature of the activity, which was "context in practice" and only involved them finding a new share–no general formula was required. Instead, fractional value(s) were needed; this I argue might be considered as an indexical symbolic generalization of the flexibility of unitizing, since each of the three fractions produced indirectly indicated where they originated (from time, length or number of parts), hence might also be considered as having context. Fah managed to produce three fractions with changing units (i.e., from a whole pizza to one over four). The "original" for the fraction at the different stages determined the outcome of the fraction. For the three fractions, she presumably knew, based on the HFC model, that (i) $\frac{1}{4} = \frac{3}{12}$; (ii) $\frac{1}{4}$ = 15mins, and $\frac{1}{12}$ = 5mins; (iii) $\frac{4}{12} = \frac{1}{3}$. At this point, I argue that the dual reference participant-object (e.g. $\frac{1}{4}$ is no longer someone's share but now it represents 3 inches or 15 minutes) diminished. Thus she was able to translate or represent the fraction in terms of minutes in order to produce a fraction at a more advanced stage, in this case getting the total share at the end of the activity (i.e., addition of two fractions with different denominators). This completed the symbolic generalization of the flexibility of unitizing.

9.3 STEP 2 – CHAIN OF SIGNIFICATION OF THE FLEXIBILITY OF UNITIZING

As discussed in Chapter 8 earlier, the same argument concerning the objectification as the instrument for the starting point of signs and chains of signification is also behind the structure of the objectification and also chains of signification of the flexibility of unitization presented in this section.

The objectification of the flexibility of unitization, as mentioned prior to this section, occurred through factual generalization and symbolic generalization only, without going through the contextual generalization. The factual generalization was completed in six steps, and there were no semiotic contractions found in the generalization. The symbolic generalization consisted of only one step but with several pre-preparatory objectifications, and likewise did not contain any semiotic contractions. Figures 9.5 and 9.6 presented below show the six-steps objectifications which are made up of eight chains of signification.

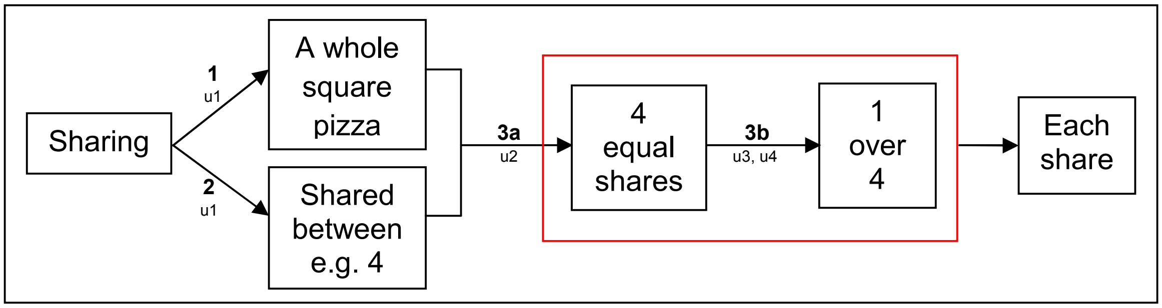


Figure 9.5. Semiotic chains of signification for objectifications 1 to 3 [Chains 1–3]

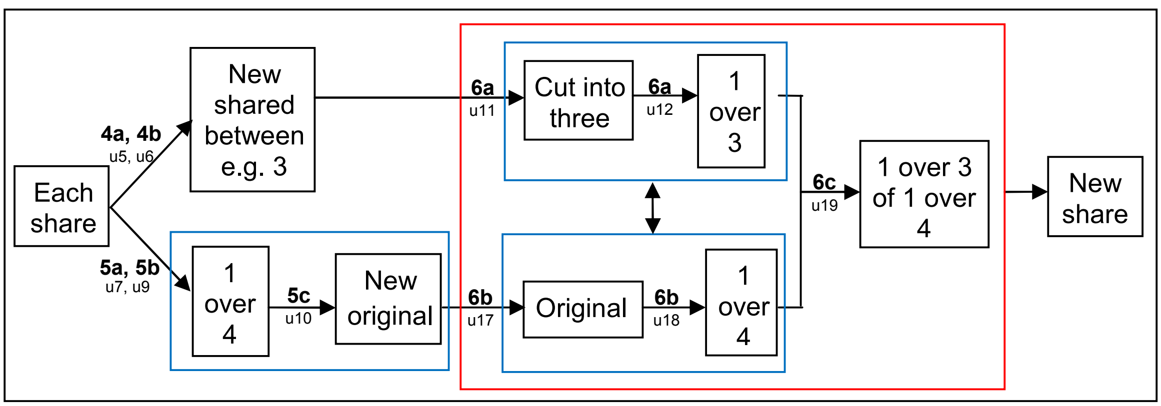


Figure 9.6. Semiotic chains of signification for objectifications 4 to 6 [Chains 4–8]

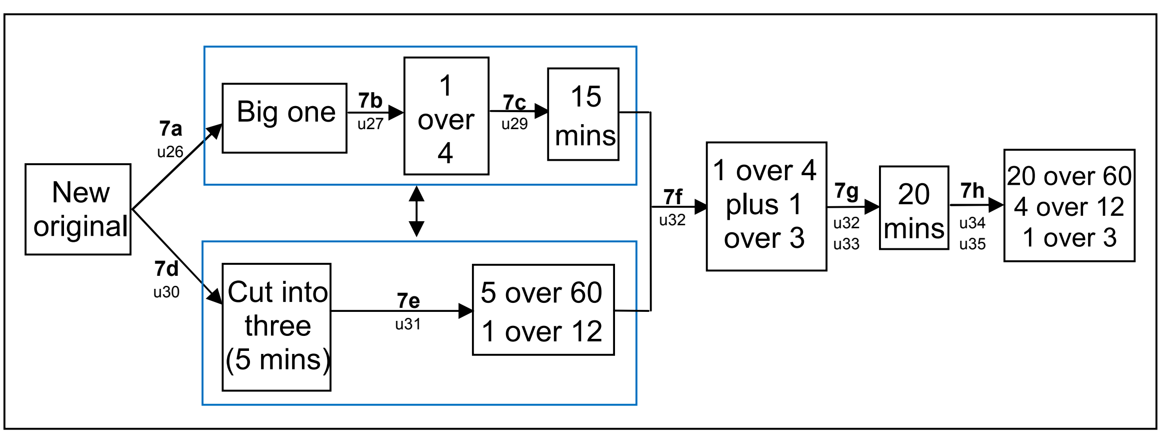


Figure 9.7. Semiotic chains of signification for objectification 7 [Chains 9–11]

The symbolic generalization which consisted of only one step with three chains is schematized in Figure 9.7 above. Based on these factual generalizations and symbolic generalizations, the structure of the objectifications is presented in Table 9.1 below.

Table 9.1
The structure of the seven objectifications

Obj	Linguistic Expressions Signifying the New Signifier	Signifier to which the Object was connected	Semiotic Scheme of the Objectification	
			Semiotic Node	Existing Sign(s)
1	"this square pizza" (Res)	Original amount	The students were presented with a square pizza when I said "I have this square pizza" which they were asked to share among themselves.	Square pizza
2	"share between the four of you" (Res)	Shared between	Since there were four students in the group, the pizza was to be shared equally between the four of them–"I want you to share between the four of you" (Researcher).	Number of students
3a 3b	3a: "How much" (Res) 3b: "one over four" (Fah) + "four parts and there are four of	Each share	Cutting the pizza into four equal portions so that every student in the group has one equal share. This is followed by me (teacher) describing again in order to bring forward to the students' attention what they had done, and at the same time pointing out that each piece is for each student. 3a: "one whole" (Researcher), "cut it into four" (Researcher), "one for each of you" (Researcher). 3b: "one over four" (Fah), "four parts…four of you" (Researcher).	Each piece

	you" (Res)			
4a 4b	4a:"give his share" (Res) 4b:"Divide into three" (Halim)	New shared between	After a fraction for each share has been established, a new situation was presented where one of the students gave up his share, hence presenting a new number of students. When I asked "what are you going to do with his share?" Halim responded by saying "Divide into three."	Number of students
5a 5b 5c	5a: "this (pointing to one of the four one over four parts) fraction" (Res) 5b:"one over four" (Halim) 5c: "that was the original" (Res)	New original	The unit has changed where the share of pizza that each of the students get now becomes the new unit when I asked "what's the original? What was this (pointing to one of the four one over four parts) fraction just now?" and to which Halim answered "one over four."	Each share
	6a: "cut it into three" (Res) + "one over three?"	New share	After they had cut one of the four one over four part into three equal parts, I tried to focus their thought on the fraction for one of the three cut pieces. I had to help them focus on the new original when Fah gave a respond of "one over three" by asking them again what the original was. 6a: "you have cut it into three…this (points to one of the three little pieces) is? (Researcher), "one over three?" (Fah).	Little piece

6a 6b 6c	(Fah) 6b: “original …?” (Res) + “one over four” (Halim & Fah) 6c: “one over three of one over four” (Res)		6b: “one over three of what?” (Researcher), “original again?” (Researcher), “one over four” (Halim & Fah). 6c: “this (points to one of the three newly cut pieces)” (Researcher).	
7a 7b 7c 7d 7e 7f 7g	7a: “this original, this big one” (Res) 7b: “one over four” (Fah & Halim) 7c: “Fifteen” (Students) 7d: “divide it into three” + “five minutes” (Res) 7e: “five over sixty” (Fah) 7f: “one over four plus one over three” (Res) 7g: “time” (Res)	New share	Using the HFC to determine the fraction for the new total share. 7a: “this original” + “fraction of this big one” (Researcher). 7b: “one over four” (Fah and Halim). 7c: “fifteen” (Students). 7d: “divide it into three” + “five minutes” + “big piece, that’s fifteen minutes” + “this is one third of one quarter” + “this must be five” (Researcher). 7e: “five over sixty” (Fah). 7f: “first you have one quarter” + “have an extra of one over twelve” + “one over four plus one over twelve” + “time” (Researcher). 7g: “in terms of time” (Researcher)	Big piece plus little piece
7h	7h: “Twenty” (Halim) +		7h: “Twenty” (Halim) + “over sixty” (Fah) + “four over	

	"over sixty" (Fah) + "four over twelve" (Fah) + "one over three" (Fah)		twelve" (Fah) + "And?" (Researcher) + "one over three" (Fah).	

Figure 9.8 below presents the overall schematized representation of the relationship of all the seven objectifications presented above. Figure 9.9 shows the basic schematized diagram of flexibility of unitizing in the sharing context.

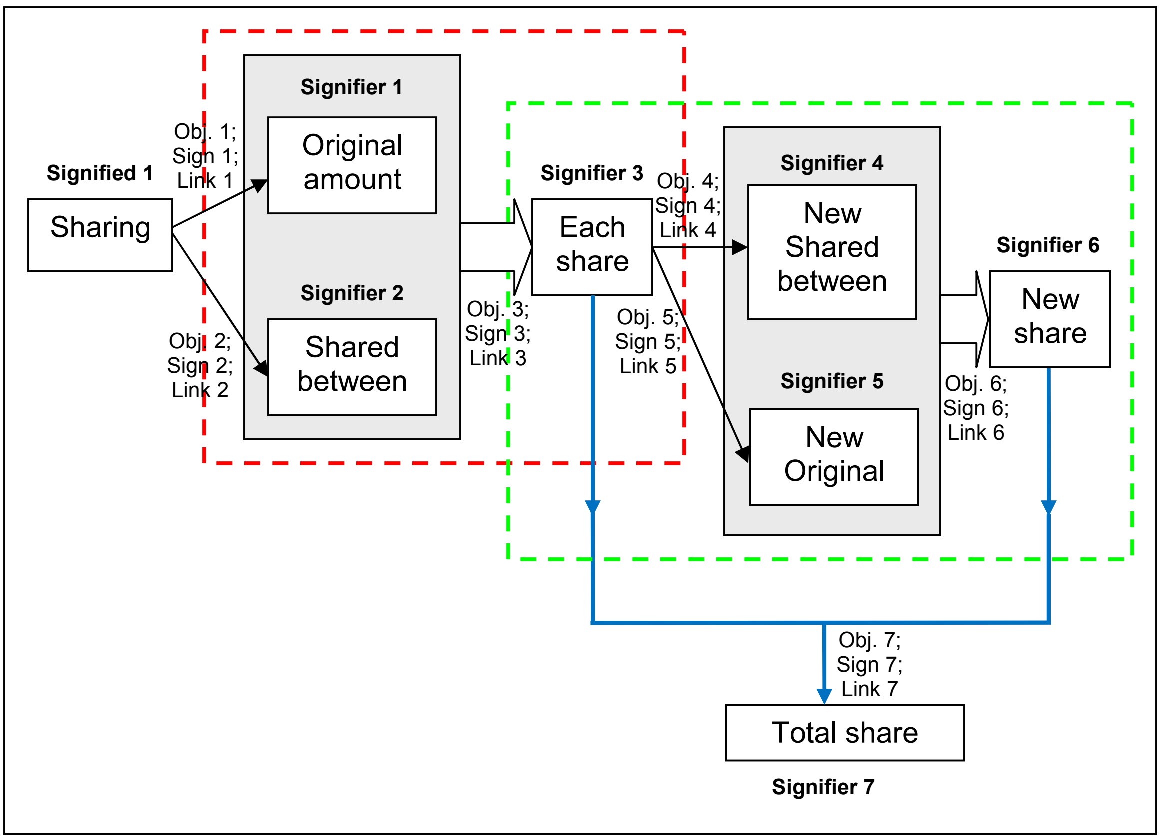
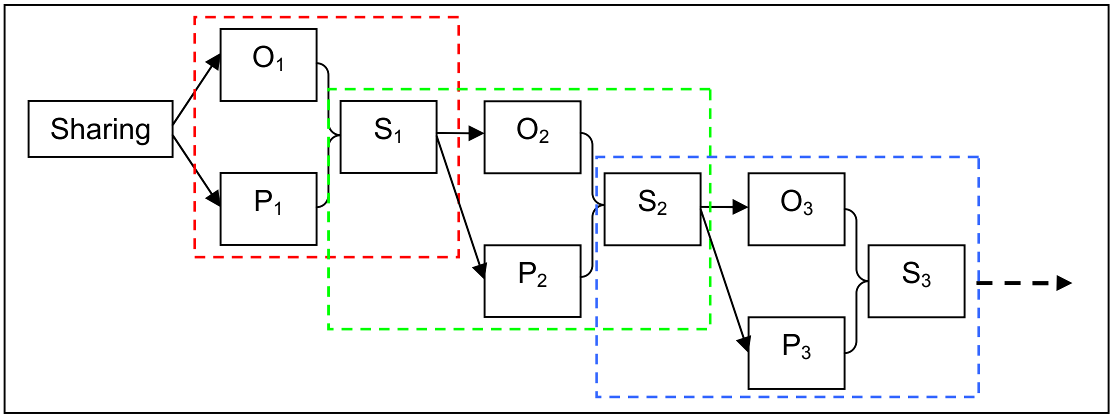


Figure 9.8. All-in-one chains of signification of the flexibility of unitizing



Note: O_1, O_2, O_3 – represents the original amount of pizza available for sharing
P_1, P_2, P_3 – represents the number of people who have an equal share of the pizza
S_1, S_2, S_3 – represents each equal share

Figure 9.9. Schematic representation of flexibility of unitizing in sharing context

With this, the semiotic analysis of the generalization of the flexibility of unitization in sharing has been completed with the objectification of the chain of signification of the "total share." The completion of the chain of signification of the flexibility of unitization not only helped the students to find equivalent fractions (discussed in Chapter 8) and determine the fraction of a share from the "changing" originals, it also allowed students to perform simple addition and subtraction of fractions based on the HFC.

In the next section, discussion on the semiotic role of the HFC model is presented.

9.4 THE SEMIOTIC ROLE OF THE HOUR-FOOT CLOCK MODEL

In discussing the equivalence of fractions and the flexibility of unitizing in Chapters 8 and 9, to a certain extent the HFC has been presented in the analyses in the context of its role in the semiotic objectification processes. In this section, the semiotic role of the HFC is examined as a whole instead of separately. In the discussion below, three main ways in which the HFC contributed to students' semiotic processes are presented:

i. As a tool for getting the required fraction–mediation of semiotic activity;
ii. As an emergent model in aiding thought; and
iii. Bypassing the contextual generalization.

The HFC was not immediately put forward in the activities as one specific item *per se*, rather it was gradually developed from the first activity where the use of a 12-inch loaf of bread was concretized or objectified. This was necessary as the HFC would have a 12-inch circumference. Therefore it was crucial that the students' thoughts were focused on the twelve inches and able to express fractions in terms of the number of inches. An example of semiotic activity showing this can be seen in episode 1.2 (section 8.2.2), Fah's and Halim's objectification, in particular Halim's gesturing (Figure 8.4, p. 189). His self gesturing alone, I think, is a good indication that the 12-inch measurement, which will eventually be part of the HFC, contributes for the first objectification of equivalence of fraction to occur. As Radford (2008) said "mathematical cognition is not only mediated by written symbols, but (that) it is also mediated, in a *genuine sense*, by actions, gestures and other types of signs" (p. 112). Halim's self gesturing was also small and diminutive and was performed within his immediate space, where his hand and finger movements were also diminutive.

The next step in the creation of the HFC was to create a clockface with a 12-inch circumference. An example of semiotic activity which illustrates the role of the hour-clock in such activity can be found in episode 2 (section 8.2.2). Here, once again Halim and Fah, had objectified that if the clock hand moves one inch, it is equivalent to five minutes, hence objectifying the "corresponding value." In my opinion, this is a crucial stage, which concretized the fact that if the clock hand moved to one, which is in fact one inch in this case, they could immediately *see* that this meant five minutes had passed, hence bringing to their attention that one inch is equivalent to five minutes in the HFC. If the clock had had a different circumference, not twelve inches, it would probably have been difficult to establish and concretize the "corresponding value." As a result of this objectification, subsequent objectifications could occur based on the HFC, for example, the objectification of fractions in terms of inches and times (e.g., 6 over 12 = 30 over 60). Finally, following that, equivalence of fractions was objectified. However, unlike in the objectification of the equivalence of fractions, in the flexibility of unitizing, the role of the HFC was less evident. In this case, the semiotic role of the HFC was more evident towards the end of the objectification, when the students were calculating the final fraction (i.e., the total share) in sub-episodes 10.1 and 10.2 (section 9.2.5.1). This is important, as it helped the students in determining the total fraction (i.e., $\frac{1}{4} + \frac{1}{3}$) without having to revert to their usual classroom method of adding fractions which would involve Lowest Common Multiples (hereafter, LCM), and so forth.

From the above examples, the mediating role of the HFC can be seen in the semiotic activity to determine the required fractions. As mentioned previously, it was not a particular item that was used immediately in the activities, rather the HFC mediated the semiotic activity throughout, and after its creation. Important objectifications occurred from the moment the 12-inch loaf of bread was introduced, during the creation of the HFC itself and finally after it was created. The students externalized their thoughts either through self gesturing or gestures on either the 12-inch loaf of bread or the HFC itself. This self gesturing or gestures for the public was important in the formation of semiotic nodes which resulted in objectifications. Hence, it can be seen that the HFC became a model that the students used in aiding their thoughts to find the required fraction. In addition, I also claim that even though the HFC was not always placed physically in front of them, as in the case of the flexibility of unitizing, they were still able to make use of it, by presumably visualizing it in their heads and mentally pointing to the appropriate part of the HFC, as it was something similar to what they encountered everyday when reading the time. As Zurina and Williams (accepted) believed, people will use some imagery "pointers" like imaginary fingers to help them mentally point to something that they may visualize in their heads.

The HFC also allows for the bypassing of the contextual generalization, as shown in episode 11, where the "big one" became " one over four" and "fifteen (minutes)," and "this (points to one of the three little pieces)" became "one over three" and "five minutes." This direct shift from factual to symbolic generalization is feasible because of the construction of a chain of signification (Gravemeijer et al., 2000) in the form of a shift from fractions in terms of the number of parts, to fractions in terms of inches, to fractions in terms of minutes, and from fractions in terms of the number of parts to fractions in terms of the number of minutes. Because of this, the mathematical symbols for fractions were already embedded in the operational scheme of the equivalence of fractions, which was objectified through the factual generalization. Therefore, since the mathematical symbols are embedded in the operational scheme of the factual generalization of the equivalence of fraction using the HFC, a direct link from factual to symbolic generalization in the flexibility of unitizing is to be expected.

This completes steps 1 and 2 of the semiotic analysis methodology of Chapters 7 to 9. In the next section, a brief discussion of how the chains of signification work together (i.e., step 3) is presented for both the equivalence of fractions and the flexibility of unitizing.

9.5 SEMIOTIC NETWORK (S-NET) OF GESTURES AND LANGUAGE

In this section all the eleven chains of significations (i.e., four from the objectification of equivalence of fractions and seven from the objectification of flexibility of unitizing) will be discussed as part of a semiotic network (S-NET), as described by Koukouffis (2008), "In the S-NET these chains, rather than working in isolation, are almost always semiotically coordinated . . ." (p. 211). The focal point of the discussion will be on the objectification and how the chains interact in the S-NET.

As can be seen in Figure 9.10 below, these chains, in most cases, are semiotically coordinated in order to achieve the two main objectifications (i.e., equivalence of fraction and flexibility of unitizing). The chains would act as the starting point of new chains, whenever necessary, throughout the development of the process of achieving objectifications, which includes the pre-preparatory and preparatory objectifications. In fact the initiation of the objectification of the new chains occurred within the first three minutes of Activity 1, as soon as the students completed constructing the HFC in Activity 3, and at the very beginning of Activity 6. As can be seen in Chapters 8 and 9, it is at these instances that new links in one or more chains of signification are being objectified.

For the objectifications of the equivalence of fractions and flexibility of unitizing to be achieved, the chains in the S-NET needed to be activated, and coordinated. The activation enables the development of subsequent chains. For example, in equivalence of fractions, only if the number of parts is known can the fraction in terms of the numbers of parts be determined (see Figure 8.16, p. 206, objectification 1c), whereas in the case of the flexibility of unitizing, in order to determine the fraction for one share of a whole, it is necessary to know the number of people who share it (see Figure 9.5, p. 237, objectification 2). This means that the chain was activated to enable the objectifications to occur–at least two fractions (i.e., F(Parts), F(Length) or F(Time)) for the same object must be known to prove equivalence. In addition, the use of the chains is also coordinated, for example, in equivalence of fraction, it is crucial that at least two of the three fractions (i.e., F(parts), F(length) and F(time)) are known/found before it can be decided/shown whether one is the lowest term of the other, or one is a multiple of the other (see Figure 8.16, p. 206, objectification 1e). An example from flexibility of unitizing would be, in the case where a share from the whole needed to be divided again, it is important to know the number of divisions to be made, so that the fraction in terms of the new original (i.e., new whole/unit) can be determined (see Figure 9.6, p. 237). A new chain in the S-NET can also develop from an existing chain, for example, in the case of equivalence of fractions, once a distance along the HFC is determined, it is possible to get all the three fractions (i.e., F(parts), F(length) and F(time)) (see Figure 8.19, p. 209, objectification 4).

In order to activate and coordinate these chains of signification, *semiotic regulators,* which consist of certain gestures and/or linguistic terms, will mediate the way students adjust and conform to the chains. Next, a brief examination and the roles of these regulators are presented. Seven semiotic regulators have been identified to influence the regulation of the chains of signification for the objectifications of equivalence of fractions and flexibility of unitizing.

First, words like “share,” “portions,” and so forth activate the chain of signification of “part-of-a-whole” objectification. These regulators serve to coordinate the chain of signification of one portion of bread after it had been cut to give a number of equal parts/shares. Thus, the role of these regulators is to enable the formulation of a fraction in terms of the number of parts, which is one of the pre-preparatory objectifications of “part-of-a-whole.”

Second, gesture affording to further objectify the length of the whole (i.e., 12 inches)–self gesturing using two index fingers starting from the centre outwards as performed by Halim–and phrases like “inches,” “divide,” “how long,” and so forth. In this episode (see section 8.2.2, sub-episode 1.1), the length of the bread was in fact first objectified at the beginning of the lesson when Halim took a ruler, measured the loaf of bread and said “twelve.” This was followed by them cutting/sharing the loaf of bread between them–may I remind readers that students were working in pairs, and each pair had a loaf of bread to work with. After this point, they managed to produce a fraction in terms of the number of parts instead, which is a natural thing for them to produce as this is what they are being taught in their formal classroom mathematics lessons, as far as fractions is concerned. In this sense, these regulators serve to coordinate and activate the chain of signification of representing the same portion of bread in terms of another unit. Therefore, the role and consequently the importance of these regulators is that they allow the formulation of another fraction (i.e., in terms of its length) and in doing so serve to enable the students to see that there is more than one way to represent the same piece of bread in terms of fractions.

Third, phrases like "moved" and "one inch" during and after the creation of the HFC. These regulators serve to activate the chain of signification of the relationship between lengths, in this case the twelve inches, and also time, in this case, sixty minutes. The role and semiotic significance of these regulators is that they introduce a visual relationship between the 12 inches and 60 minutes. It was easy for the students to see the actual relationship of the two different units, as the circumference of the clock-face had intentionally been made 12 inches long, which is the same as the numbering in most clock faces. Hence, it was visually logical for them to see and relate that if the clock-hand moved to, for example "1," the students would know that it had moved one inch.

Fourth, phrases like "minutes" and "one inch" during and after the creation of the HFC. As with the previous regulators, these also serve to active, and also coordinate the chain of signification of the relationship between inches and minutes. These regulators coordinate the length (i.e., inches) and time (i.e., minutes) further by focusing on the connection between the distance along the circumference of the HFC that the clock-hand has moved, to the number of minutes that has lapsed. Thus, the main role of these regulators is to enable the students to find the equivalent number of minutes for a given numbers of inches along the circumference, hence completing the second objectification which is the "corresponding value." This objectification also enables the formulation of fractions in terms of length with their equivalent fractions in terms of time. This finally led to the completion of the objectification of equivalence of fractions.

Fifth, phrases like "one whole," "cut into four," and the whole square pizza as the unit. The whole pizza acts as the original amount in this sharing activity, which also serves as a regulator to the ensuing chain of signification. Together with the known number of people to share it, they bring together how the fraction is going to be presented. So, the main role of this regulator is to act as the crucial element in the formulation of the fraction for each share/division from the whole pizza.

Sixth, phrases like “give his share,” “divide into three,” and so forth, and each share as the new unit. The introduction of a new situation where Halim wishes to give his share from the original whole cake to the rest of his friends (i.e., three friends) became a semiotic regulator, which activated the chain of signification for a similar sharing problem to the previous one. However, in this scenario, the way the problem is presented may be the same, but then the crucial elements in producing a fraction to represent the new smaller share (i.e., the original amount) and the number of shares to be made, change. Thus, the role of the regulator here is to enable the students to see that the share that they get from the original whole now becomes the new original with the number of divisions now reduced by one (i.e., excluding Halim). This therefore enables them to see that the fraction for the new share is in terms of the new original (i.e., new unit), and not the original whole.

Lastly, the number of minutes representing the fractions, and phrases like “big one,” “minutes,” “divide into three,” “how much,” and so forth. The number of equivalent minutes for each fraction serves as the regulator which enables the students to determine the total amount of bread each of the remaining three of them are getting. It thus activates the chain of signification for finding the number of equivalent minutes for the two fractions they found earlier–for the original share and the new share. In other words, the regulators allow the students to find the fraction of their total share. These regulators also coordinate the use of the phrase “big one,” which refers to the original share from the original whole, to become “fifteen” which is the equivalent number of minutes for that share. From there, because of the further division of one of the shares, it became "five minutes” and the production of a fraction in terms of time. Consequently, the total number of minutes for the total share was found, which later translates into a fraction in terms of time and further reduces to its equivalent fraction. Hence, this objectifies the symbolic generalization of flexibility of unitizing.

Figure 9.10 below represents the schematized diagram of the relationship between the regulators and the chains of signification. As mentioned earlier, Figure 9.10 shows chains of signification for both the equivalence of fractions and flexibility of unitizing. The orange boxes are designated for the chains of signification or part of them, and the regulators are indicated by the white boxes, while the arrows show the direction of the chains that are introduced and developed and the position where each chain or part chain is placed.

However, there are three issues that needed to be clarified in order to avoid possible confusion. Despite the fact that the development of the eleven chains in chapters 8 and 9 in most cases is unidirectional, in the analysis there are at least three cases in the chains where the students had to "progress" temporarily in the opposite direction (i.e., temporary regression). This is found in the following cases:

(i) A chain within a chain

Section 8.2.4, episode 3 where a chain is embedded in another chain. In this chain involved the development from the objectification of fractions in terms of inches to the objectification of fraction in terms of minutes. However, prior to the objectification of fractions in terms of minutes occurring, a link linking inches to minutes must be made, which is in fact a previously objectified chain (i.e., "corresponding value"). It is necessary to consider this as a separate chain, highlighted prior to this episode, in order to simplify the analysis, since this is a crucial element which links the two different unit measures.

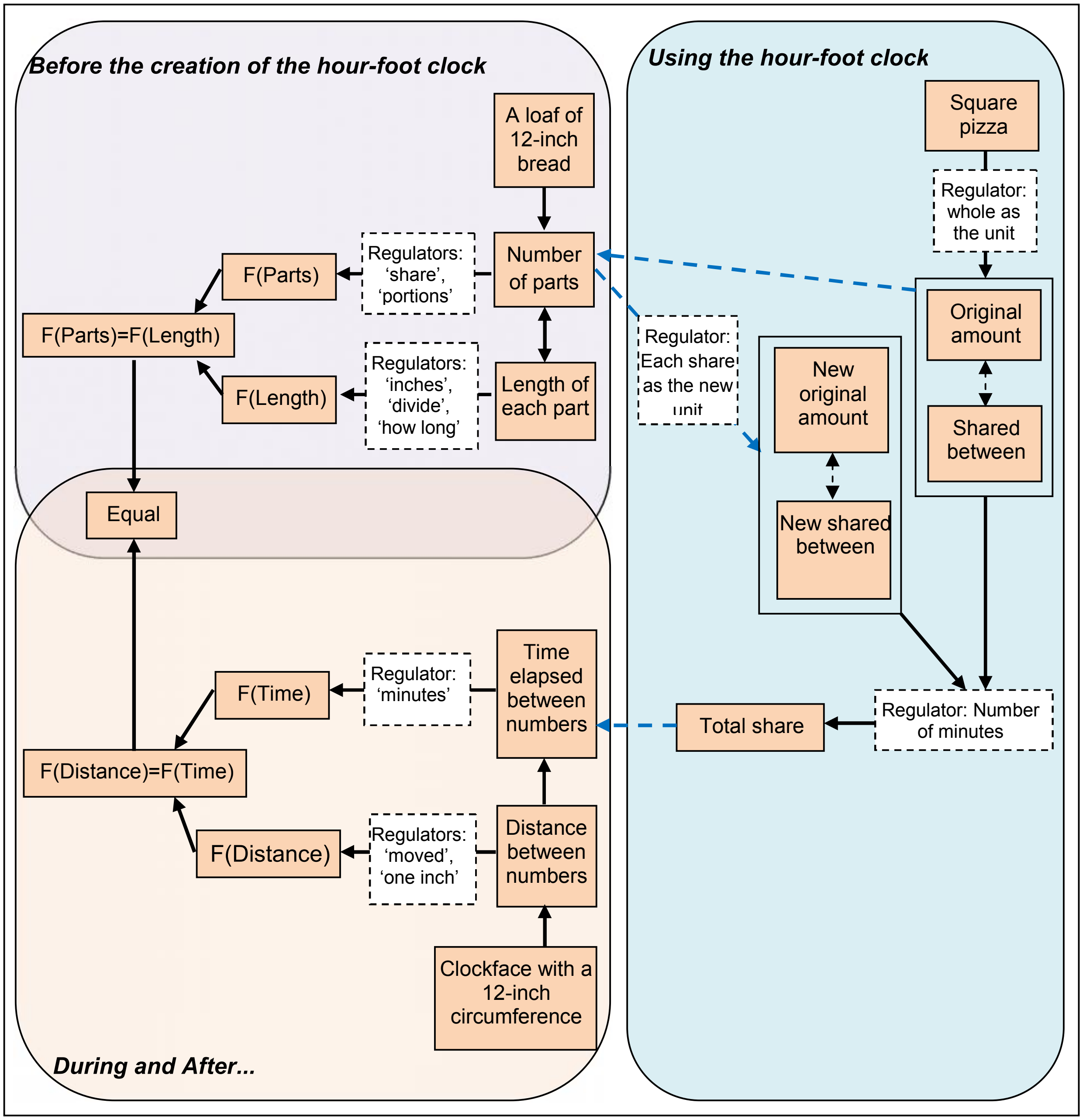


Figure 9.10. The semiotic regulators of the eleven chains of signification in equivalence of fractions and flexibility of unitizing in relation to the HFC

(ii) Temporary Regression (i.e. moving backward instead of forward)

a. In section 8.2.4, episode 3 also, temporary regression occurred when a new unit of measure was first introduced, for example, in utterance 42, a fraction for one inch in terms of inches was already objectified , but in utterance 48, the discourse moved back to one inch during the development of the equivalent fraction in terms of minutes.

b. Section 8.2.4, episode 4 there were two students who I felt did not yet understand the discussion. In such a situation I had to temporarily move backwards by creating an easier example so that they could follow the discussion. Another such example happened in section 9.2.5.1, episode 10 where objectification of one over four being equivalent to fifteen minutes had already occurred, but because Halim overlooked or probably did not realize that fifteen minutes is the new whole, I had to temporarily regress in order to bring to his attention again that fifteen minutes (i.e., one over four) is the new whole, hence objectifying it twice.
c. An episode which is not included in this analysis for the simple reason that it is not a typical representation of the whole group. In this episode a student, Zak, was confused for a moment with a fraction for 2 out of 6 parts with each part measuring two inches each. He knew that the fraction in terms of parts is two over six, but was confused when the fraction was expressed in terms of its length. Another student, Hiz, had to objectify it again by stressing the word “inches,” and with this Zak was back on track with the rest of the students (Zurina & Williams, accepted).

9.6 CONCLUDING THE CONTRIBUTION OF CHAPTERS 7–9

The analysis of the semiotic processes involved in the objectifications “equivalence of fractions” and “flexibility of unitizing” is now completed. It has shown the development of objectification from a semiotic standpoint which finally attained the two intended objectifications with the aid of the hour-clock as the model. Throughout this, the following aspects were particularly focused on:

(i) The gestures and language which made up the arsenal of the semiotic means of objectifications, which mediated the objectification in each of the signifier-signified links and their function;
(ii) The way the theory of objectification was used and the adaptations made in the RME instructional framework before, during and after the construction of the HFC, in particular the pre-preparatory objectifications, the bypassing of contextual generalization, and also the significance of gesturing for the self in the students’ quest for objectification;

(iii) The initiation of the signifier-signified link through the introduction of objectification, which resulted in the development of the chains of signification;

(iv) The HFC became an important part in the semiotic process as the whole clock can be translated as (a) the part-of-a-whole; (b) the length in terms of a foot; and (c) the number of minutes in terms of sixty minutes, and in addition the HFC enables the students to “convert” from one form of representation to another as far as fraction is concerned. As a result these different representations serve to direct and mediate the semiotic activity which draws attention to the function of models and modelling in semiotic processes; and

(v) The establishment of an S-NET of gestures and language which gives way to the objectification of a new chain based on previously objectified chains.

CHAPTER 10

CONCLUSIONS AND DISCUSSION

10.1 THE CONTRIBUTION TO KNOWLEDGE

This first part of the chapter offers a description of the thesis' contributions to knowledge. I believe that this thesis may lead to at least four publications with one already submitted and accepted. The thesis contributions are as follows:

1. <u>Comparative Textbooks Analysis</u>

   The first contribution to knowledge is the comparative textbook analysis between the Britannica Mathematics in Context books and the Secondary Mathematics 1A (1997) textbooks. This comparative textbook analysis specifically concentrated on topics pertaining to fractions. The textbook analysis focuses generally on the methodology (pedagogy), relevance (context), connectedness, representation (models), reinforcement, known misconceptions and cognitive levels which are based on a textbook analysis schedule developed by Haggarty and Pepin (2002). The analysis revealed that the CPDD textbook used very few contextualizing models, except in word problems, to explain or illustrate fractional concepts – even if, and when models are used, it only serves to describe the problems only. Thus, it could be said that context is not the main focal point. This is evident in the examples presented which are dominated by symbolic procedures. I felt that the focus of the textbook is more on training students to be automated in responding to problems where students will come up with the same answer with almost the same procedures. In short, the CPDD textbook is oriented more towards a recall type of knowledge. On the other hand, the MiC textbook uses context which are experientially real to the students. In this textbook, the students are encouraged to produce strategies and ways of dealing with the same problem. It was found that the activities in this book are geared more towards the comprehension, application and analysis levels of the Bloom's taxonomy.

2. <u>The Supplementary Lessons</u>

In this study I have designed some RME-like “supplementary teaching” which draws from the RME sources because I had good reason to believe that the current standard approach that fails many learners “misses” or is lacking essential activities that focus on comprehension, models and modeling, and so forth. From the analyses presented in Chapter 5, it can be seen that the students in the E1 group benefitted from this supplementary teaching–a benefit that has an effect size of 0.6 (even though initially it seemed to be not very significant, probably due to the small sample size)–compared to the control group.

3. <u>Fraction Ability (FA) Scale</u>

The third contribution to knowledge of this thesis is the production and calibration of the fraction ability (FA) scale which was suitable for Form 1 students in Negara Brunei Darussalam. The FA scale was based on the 32 items from the three different tests, with 13 of the 32 items the same in order to link the three tests. This method is an extension of the methodology used by Koukkoufis (2008), when he calibrated an integer ability (IA) scale for use with year 5 and 6 students, which involved the use of higher year group students. As presented in chapter 6, this scale was also used to measure the effectiveness of the experimental study and it too was used to compare and contrast E1 with the control group in the study.

4. <u>The Semiotic Analyses of the Equivalence of Fractions and the Flexibility of Unitization</u>

The fourth contribution to knowledge is the discussion and presentation (Chapters 7–9) of the semiotic analyses methods of the equivalence of fractions and the flexibility of unitization. The analyses show “how” some of these supplementary teaching episodes afforded the “objectifications” that might *explain* their meaning (drawing from Radford’s and Koukkoufis’s approach). The basic findings pertaining to this analysis are presented next. The first finding is the mediation of the production of fractions in terms of length, from the production of fractions in terms of the number of parts for a loaf of bread, and also in the HFC, the production of the fraction in terms of length was only obvious to the students after the equivalence of 1 inch to 5 minutes (i.e., every inch the clock-hand moved is equivalent to 5 minutes that had passed) was established. The second finding is the use of both language and gestures to objectify the equivalence of fractions and the flexibility of unitizing in the semiotic nodes. These were achieved through the objectifications of new signs, thereby objectifying eight chains of signification for the equivalence of fractions, and eleven chains of significations for the flexibility of unitizing. These described how the students and teacher/researcher jointly achieved the objectifications, and hence objectified the chains of significations. The next finding is the role of the HFC as a model in a realistic context, as it represented the time on most clock-faces, in hours and minutes, and at the same time, since it was specifically made to be 12 inches in circumference, it also represented the distance, in terms of inches, the end of the clock-hand had moved at any particular time (e.g., if 15 minutes had lapsed it indicated that the clock-hand had moved 3 inches along the circumference), which afforded the role of models and modeling in semiotic processes. Lastly, it also emerged that in most cases prior to achieving an objectification, pre-preparatory objectifications followed by preparatory objectifications (see Figure 8.1) were found to have taken place. The complexity of the required chains of objectification, I think, reflects the difficulties of the topic.

5. Gesturing for Oneself

The last contribution to knowledge of this work is on gesturing to oneself which highlighted moments in the experimental study where students were found to gesture to themselves in order to achieve objectification, in particular to themselves (i.e., gesturing for oneself). This I believe could partly be as a result of the students' cultural background (i.e., they would only speak if spoken to, particularly to elders), which could also indicate that they were not at the sharing stage of the discussion yet and were just communicating with themselves what they thought was the required answer/response at that particular stage. This gesturing to the self is a new feature that I think has not been looked at closely before. A paper on this had in fact been submitted and accepted. The paper presented two episodes where students were gesturing to themselves when they were faced with a difficult task. The gestures that are focused on are those that are not meant as communication with others. However, they presumably are accompanying inner speech or thought, as when one would silently count by touching one's fingers while thinking or communicating with oneself. These gestures are usually diminutive, where the eye-gaze is averted from others present–either turned inwards or towards artifacts present. These gestures for oneself are in fact the “residue of gestures” that were to accompany speech but had not quite “gone underground” yet. What this adds to this complex phenomenon is that the student allows us to see in novel ways this engagement with oneself: more specifically, the embodied (or better sensuous), socially shaped manner in which one engages with oneself in the public space of interaction in the difficult moments where knowledge is objectified (Zurina and Williams, accepted).

10.2 DISCUSSION

Motivation

I have been involved in the education sector for the past 17 odds years, and have been a secondary school mathematics teacher 13 of those years. At the beginning of my teaching career I was shocked to find that many students made errors on what I regarded as straightforward no nonsense questions which are associated with vulgar fractions (hereafter termed simply “fractions”). What astounded me then was that some Form 4 students were giving incorrect answers to fractional tasks which were set out in the national Primary 4 and 5 syllabuses for Brunei Darussalam (Curriculum Development Department, 2006). Not to my surprise, however, I soon found out that there are many primary school students, lower secondary students (Form 1 to 3), and even upper secondary students (Form 4 to 6) in Brunei who do not comprehend many fractions concepts, and still experience significant difficulty using the four operations with fractions. Due to this, it would be unfair to expect them to be able to follow topics–like gradients of lines, algebraic fractions, and the set of rational numbers–in which knowledge of fractions and decimals operations is assumed (but not impossible). In addition, their ability to survive with dignity in a metricated society, in which a basic knowledge of fractions and decimals is taken for granted, would also be affected.

Overview of the Contribution to Knowledge

To summarize, the first two contributions to knowledge above addresses the second issue raised in Chapter 2. The textbook analysis gave supporting evidence of the need, specifically for Brunei, to re-evaluate the type of textbooks that are used in the classroom, as far as Mathematics is concerned. It might be time for a change in direction in the ways teaching is done and the type of textbook that complement it. This analysis might be helpful for the education authority in designing textbooks by using the MiC textbooks as models. As for the supplementary lessons, as seen from the effect sizes presented in Table 6.7 (p. 162), the series of lessons itself is a contribution to knowledge as it proved to be beneficial to the kind of students that I was targeting, and that the supplementary teaching maybe not just effective for understanding, but effective in terms of learning mathematics of that sort. In addition, for each of these two contributions to knowledge, education papers might be produced and published in international journals. To measure the effectiveness of the supplementary lessons involved carefully designing and calibrating a real measure of fractional ability scale, which is another contribution to knowledge. The final two parts of the contributions to knowledge are attempts to explain why these supplementary lessons might have helped develop the two key understanding which are about equivalence of fractions and flexibility of unitization. The semiotic analysis is an attempt to reveal how some of these episodes might have helped conceptual understanding which is called meaning in the analyses of the students. I am suggesting that the key model is having context, and models allows gestures and informal deictic language, and the perception of two things being equal. This is plausibly supportive of the sort of design with this new element that in my cultural case, the students' gestures appears to be for themselves. For these reasons, the semiotic analysis conducted in this study provided an analytical tool for analyzing more detailed semiotic analyses of fractions within the RME framework, whereas the gesturing for oneself is also an important contribution as they contributed to the objectification processes, and so far these phenomena of gesturing to the self have not been looked at closely. In fact, a paper about gestures for oneself had been produced and accepted in an international journal.

<u>Two sides of the same coin</u>

As presented in this thesis the research was done using a mixed method of quantitative and qualitative approach. The rationale for using the QED with a pre-, post-, and delayed post-test was because of the lack of random selection of students in the sample as discussed in section 4.4. In addition, this design enabled for measuring the effectiveness of students' understanding on fractions in the experimental teachings involving RME or RME-like lessons to be done by testing the students' performance at different stages of the study (i.e., pre, post and delayed). Also, in this design a control group was used to compare with the experimental group. From the result of the experimental teachings discussed in Chapter 5, it was found that the E1 as a group managed to improve their means between the measurements (see Figure 6.2). Although the means did not increased by much, Figure 6.4 indicates that most of the high-ability students in E1 managed to improve their FA abilities (i.e., the variance increased with the upper limit of the box moved upwards in the scale), and hence benefitted from the experimental teaching. This is further confirmed by the effect sizes presented in Table 6.7. This provides some hard evidence that there was an overall measurable effect on attainment of fractions that implies some different "learning" took place in the experiment, which then allows me to infer that there might be something qualitatively important, and thus justifies the qualitative pursuit of *how* the learning took place. Hence, the two sides of the analysis add together to "support" context. In other words, a quantitative analysis without the qualitative analysis would indicate that there is an effect, but there would not be any evidence on how it had been done. Also, a qualitative analysis without the quantitative analysis would provide a description of the learning-teaching process, but would not give evidence of its overall effect on attainment. The latter may be very important for generalizing to policy, while the former is vital to understanding practice. Thus, I conjecture that these two analyses complement each other (i.e., one showing its effectiveness while the other showing the process that took place), and are in fact two sides of the same coin.

Students' Educational Culture

I believe that the way we do things within our culture are bound to be reflected in the way we think, act and respond to situations, either within or outside our community. One might then expect that to include in the ways students behave in the classroom. This is evident in the episodes presented in this thesis, where the students involved in this study behaved in ways that are culturally influenced, such as shyness, which are not a reflection of their intellectual abilities. The students' fragmented answers (i.e., "reluctance" to produce answers in "proper" sentences) are not indicative of their inabilities to produce good responses; rather it is a reflection of the Bruneian culture. In addition, the teaching in Brunei is a very formal one, which I believe also precipitated from the culture where the elders in a family, say, are the authority. I believe that this is what is happening in most classrooms in Brunei, and also in the episodes presented. This Brunei teaching-learning/culture is different from those in which previous research has been conducted, and so this might have led to particular features in the semiotic chains (e.g., the prominence of the teacher's objectifications, the significance of the "relatively passive" objectifications by the students which might have led to their "silent gestures" or gestures while listening). Arguably, these kinds of gestures would not be so noticeable or significant in other cultural contexts where children may be more vocal. Also, I believe that because of this cultural difference, there was a tension between what the RME approach is advocating (i.e., students are at the helm of their learning process), and what was happening in the episodes presented. Even though in this study RME-like lessons were used in the activities, the students' educational culture did influence the way the lessons unfolded. This was evident in the way they responded in the episodes where they seemed to be reluctant to voice their opinions publicly, which may have led to the production of gesturing to the self. Additionally instead of the students creating a workable model themselves, the students and I collectively achieved that, which one might argue is a product of teacher-led discussions. In this study, even though at the beginning, I tried to pose more open questions in order to incite more responses from the students, the plan backfired as I ended-up with no response from the students. I presume that they are not used to such approach yet. In addition, the teacher in me, who also had been exposed to the traditional pedagogy throughout my life, kicked in the automated reflex of teaching by leading, hence the teacher-led discussion through questioning presented in the episodes. Thus, I conjecture that, even in such cultural situations–where the teacher plays a leading role–this still did not prevent the students benefitting from the RME-like lessons in this study (see Table 6.7, p. 162), because the contexts presented were experientially real to them. Consequently, in such cultural contexts, the *unspoken* agreement between the students and the teacher within this RME-like approach did not hinder in the students' meaning-making process pertaining to fractions.

<u>Limitations of the QED design</u>

Next, I am going to revisit some of these issues, at the same time also looking at the effects they have on the interpretation of the conclusions made in this thesis, and what could have been done differently.

Let's start with chapter 6, the experimental results of the QED groups. As presented at the end of this chapter, there were four limitations, and they are:

i. the small sample size;
ii. the small group teaching versus a whole class teaching;
iii. the possibility of the presence of the Hawthorne effect threat to external validity; and
iv. the possibility of the presence of the posttest sensitization threat to external validity.

The first limitation is the small sample size, which might have hindered some statistically significant differences for the treatment groups from being more evident. However, this does not mean that because of this limitation there are statistically significant differences that were not found, it is just that we cannot be certain that such differences existed. The only thing that I can do to overcome this limitation is to conjecture whether a significant relationship was found. Since sample sizes affect the statistical power by somewhat reducing it from its true effect, the fact that with such a small sample in this experimental study, a statistically significant difference was found only strengthens my conjecture that the experimental study did benefit the students.

With regard to the second limitation of the small-group teaching outside the classroom versus whole class instruction where students would work in small groups, one might argue that there will be more researcher/teacher-students' interaction taking place in small-group teaching as opposed to whole-class teaching. This might raise questions whether this might have had some effect on the improvement shown by the students. For this, unfortunately, I was not able to teach the experimental lessons to a whole class due to time limitations, as all the students were going to sit for yet another assessment test, and the teachers wanted to prepare them for it. I felt that it would be unfair to deprive some of the students of their revision sessions with the teachers. Ideally, it would have been good to know for sure whether the small-group teaching did or did not have an effect on the students' learning outcomes. However, from past research, for example, Bangert, Kulik, and Kulik (1983) (cited in Lipsey and Wilson, 1993) who studied the effect of achievement in individualized instruction, it was found that the effect size was only 0.10. Another example is by Smith and Glass (1980) (cited in Lipsey &

Wilson, 1993) who looked at the attitudes of students and teachers for small (<30) class size versus large class size, and they found that the effect size for the students was 0.47 and the teachers was 1.03. Therefore, I conjectured that the small-group teaching would not adversely affect the students' learning outcomes in this experimental study. However, this would be an area that I would also be interested to look into more closely in the future.

As for the third limitation of the Hawthorne effect, I cannot be completely certain that it does not affect the external validity. If such an effect was in fact present, it might then mean that the result presented in Chapter 6 (see Figure 6.2, p. 153) is inaccurate. However, in order to minimize this effect, if any, I did manage to have access to the E2 group (a whole class) and was able to teach them a topic (i.e., after the experimental study was done) which the subject teacher had assigned to me for the same length of time as I spent with the E1 students.

Regarding the fourth limitation of the posttest sensitization, like the third limitation, again I cannot be definitely sure that it did not affect the external validity. However, if posttest sensitization did appear at all, the most obvious evidence would be students making an extra effort to concentrate more during the lesson, knowing that they would again be tested at a later date. As far as I noticed, this did not happen at all, as the students acted in a similar manner throughout the experimental study. Therefore, there was no evidence suggesting that this threat affected the experimental study.

Next, let us look at the two limitations which I have identified on the semiotic analyses in Chapters 7–9. The first limitation is the generalizability of the results to other teaching, and the second limitation is the issue of the teacher-led discussion during the lessons.

Regarding the first issue of the generalizability of the analyses to other teachings, as discussed in Chapter 7, care had been taken to ensure that only groups and episodes which "typically" represented the whole E1 group involved in this study were chosen for analysis. However, this was just my own subjective judgment and others might not agree with these choices. Instead of analyzing all the processes from all the three different groups in this study, only episodes/moments from groups which were considered to be representative of all the groups were analyzed. Ideally, analyzing all the processes from all the groups would be better, however, not only would it be exceedingly time consuming, and most probably would not provide any significant theoretical contribution, the facts still stands that the result would still not be generalizable to other teachings. This is also linked to the previously mentioned limitations of the small sample size and also the small group instruction. Due to these two limitations also, even if the results were able to be generalized to other teaching, it would have to be under similar special conditions as the ones in this study. Having said that, the purpose of the analysis was not for its generalizability to other teaching, rather in its contribution to analytical generalization (i.e., generalizations to the theory), and also as a tool to examine the students' semiotic meaning-making processes. Thus, this analysis shows that the semiotic theory of objectification is flexible enough for it to be adapted to different topics in mathematics even though it was originally designed to analyse lessons in algebra. However, the complexity of each analyses might vary as can be seen from this thesis.

With regard to the second limitation of the teacher-led discussion in the lessons, due to the fact the these students had been exposed to rote learning methodology throughout their school lives, coupled with the cultural background of the community of "listen to thy elders", "silent is good", "only speak when spoken to" and so forth, I had to play a leading role throughout most of the experimental lessons. In short, these two main factors were the main motivations for most of the discourse being a teacher-led discussion. It could be said I initiated the discourse through questioning, which was followed by the students giving their responses, which were usually brief, and in turn I commented on their answers–initiation-response-evaluation (Mehan, 1979). In this experimental study, even though I tried to move away from the standard approach that the students were used to, according to Lefstein (2008), "routine genres are embodied as habitus" and he gave an example of "just as we may feel awkward or even bump into people when walking in a foreign city, so we may experience unfamiliar classroom interactional genres as confusing or unnatural" (p. 709). Also, according to Macbeth (2003), "the problem is not that we would not find structure, or power, in the classroom. For most children schooling is a mandate, and classrooms have their impositional orders" (p. 244).

<u>Some suggestions for Future Research</u>

1. Conduct a whole-class teaching of the experimental lesson.

2. Conduct a similar experimental study involving a larger sample size. Does this produce a significantly different result, or does it not matter at all?
3. Conduct a comparative study on achievement between Bruneian students who only had rote learning experience with those from international schools.
4. Investigate the students' confidence level in answering questions pertaining to equivalence of fraction and flexibility of unitizing before and after the experimental lessons (in this study).
5. Investigation on the role of self-gesturing in learning mathematics. Does this improve achievement?
6. Investigation of the role of self-gestures to meaning-making. Do they help the students to understand concepts better?
7. Investigation of what the students are really thinking about when they gesture to themselves.
8. Investigation on how cultures in the Brunei context influence the way students learn.

Finally, to conclude this thesis, a new instrument has been calibrated and a semiotic investigation of the objectifications of equivalence of fractions and flexibility of unitization has been conducted to demonstrate the students' learning processes. These broadly supported previous approaches in other contexts, while adding some important new features such as (i) the complexity of the required chains of objectifications which reflected the difficulties of the topic, and (ii) the role of self-gestures that had not been noted before.

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APPENDIX 1: Gagne's Nine Events of Instruction

Sources: http://www.tip.psychology.org/gagne.html
http://www.my-ecoach.com/idtimeline/theory/gagne.html

Event of Instruction	Lesson Example	Rationale
1. Gaining Attention	Show variety of computer generated triangles	The use of computer gets the students' attention.
2. Identify Objective	Pose question: "What is an equilateral triangle?"	Make known the expectation of the lesson to make sure they are aware.
3. Recall of Prior Learning	Review definitions of triangles	When learning something new, accessing prior knowledge is a major factor in the process of acquiring new information.
4. Present Stimulus	Give definition of equilateral triangle	The goal is information acquisition; therefore, the stimulus employed is verbal/written content.
5. Guide Learning	Show example of how to create equilateral	This enables the teacher to show how a correct equilateral triangle can be constructed.
6. Elicit Performance	Ask students to create 5 different examples	Requiring the students to produce based on what has been taught/shown enables them to confirm their learning.
7. Provide Feedback	Check all examples are correct/incorrect	Regular feedback enhances learning.
8. Assess Performance	Provide scores and remediation	Independent practice forces students to use what they learned and apply it. Assessing such gives instructors a means of testing student learning outcomes.
9. Enhancing Retention and Transfer	Show pictures of objects and ask students to identify equilaterals	Applying learning in real-life situations is a step towards Mastery Learning.

APPENDIX 2

ANALYTICAL FRAMEWORK FOR THE TEXTBOOK ANALYSIS FOR THE CHAPTER ON FRACTIONS		
1. Methodology [Pedagogy]		
▪ Was the method presented ***the*** only method to solve the problem, or was there other methods used, and were students allowed to use a different valid method which is not mentioned in the textbook?		
▪ Were there any comments given to explain the argument presented?		
▪ Were the objectives of each sub-unit stated?		
▪ Were the students aware of what they are going to learn in the topic?		
▪ Were students aware what pre-requisite knowledge they must have?		
2. Relevance (Context)		
▪ Were there any introductory activities given for the intended concept?		
▪ If there were, how far were the activities designed to help students understand the concept better?		
▪ Were the knowledge presented in ways that are relevant to students' experiences?		
▪ What sorts of context were used to present the knowledge? Was it only restricted to the examples, or were they extended to the exercises?		
▪ Did the contextual models encourage students to use their intuition (to make sense of their		

answer)?		
▪ Were students given samples of different context for the same idea?		
▪ Were they encouraged to create their own context within the same problem?		
3. Connectedness		
▪ Were the pre-requisite knowledge mentioned explicitly prior to introducing a new knowledge?		
▪ Were there any references made to previously learned topic / knowledge?		
▪ Were rules given justified, with connections made to previously acquired knowledge?		
▪ Was there evidence in the textbook that the new knowledge was built upon the existing knowledge?		
▪ Were there any mentioned of topics (past or future) that were related to the topic learned?		
4. Representation [Models]		
▪ What types of models were used to present the knowledge?		
▪ How were the models utilized to convey the intended knowledge? For example, were they presented in the beginning only, as part of the explanation process, to illustrate a number, etc.		
▪ Were there indications that different models can be used to convey the same knowledge?		
▪ Did the students identify with the representation?		
▪ Were students expected to use models when they solve a problem?		

5. Reinforcement		
▪ What type of reinforcement activities were used to reinforced what the students have learned? For example, projects, exercises, etc.		
▪ In what ways were the exercises after each sub-unit helped the students to understand the concepts taught?		
▪ What types of fractions were used?		
▪ Were students expected to do all the tasks, or were they given a choice to skip tasks?		
6. Known misconceptions		
▪ Were there any provision given to avoid common misconceptions?		
▪ For parts that *are* known to be prone to misconceptions, were there any special attention given to explain them in order to minimize misconceptions?		
7. Cognitive level		
▪ What kind of questioning techniques were used? [Recall, …]		
▪ What kinds of performance outcome were expected from the students? [draw, sketch, explain, justify, show, etc.]		
▪ Were students encouraged to come up with a different strategy of their own?		
▪ Did the text encourage students to think logically, ask questions, reason, etc?		

APPENDIX 3: Detailed Comparative Analysis Between CPDD 1A and MiC Books

<table>
<tr><th colspan="3">COMPARATIVE ANALYIS</th></tr>
<tr><th colspan="2">CPDD 1A, Unit 3: Fractions</th><th rowspan="2">Mathematics in Context (MiC)</th></tr>
<tr><td colspan="2">UNIT 3: FRACTIONS

1. Meaning of fractions [p.69 - 72]</td></tr>
<tr><td>1.1 Fractions as equal parts of a whole</td><td>▪ The point of entry is a circle with eight equal parts with 5 of 8 parts shaded, which is followed by a rectangle with 7 of 12 equal parts of the rectangle shaded.
▪ Within one paragraph, five important information are presented:
i. 5 of 8 equal parts of a whole are shaded.
ii. Five-eighths of the circle is shaded.
iii. This is written as $\frac{5}{8}$.
iv. The number $\frac{5}{8}$ is called a fraction in which 8 is the denominator.
v. 5 is the numerator.
▪ Workbook.</td><td rowspan="3">▪ In section B (The Bar Model) of the MiC Models You Can Count On: Number book, the focus is on fractions and the relationship between fractions and percents using fraction bar and percent bar model as the visual aid to support students' understanding. For this analysis, only the parts that involved fractions will be discussed.
▪ The point of entry of the section is fair sharing strategies between four students for a school garden plot.
▪ The methodology used to develop the intended focus are as follows:
~ In order to demonstrate that fractions are the outcomes of fair sharing, an exploratory exercise (<u>School Garden</u>) using strings of dividing a garden plot into four equal parts was presented which results in fractions, in this case $\frac{1}{4}$.
~ This is followed by more exploratory exercises of dividing strip of papers into equal parts, which is extended further where students have to create their own problem, and solve them accordingly.
~ Based on the answers to the previous questions, students were asked to describe</td></tr>
<tr><td>1.2 Fractions as equal parts of a set</td><td>▪ The point of entry is a picture of a group of 10 children which describes the group in terms of fractions as equal parts of a set, e.g. 6 out of 10 children are girls or $\frac{6}{10}$ of the 10 children are girls.
▪ A similar example using 9 coins on a table was also presented.</td></tr>
<tr><td>1.3 Fractions as answers to division of whole numbers</td><td>▪ The point of entry is fair sharing, where a whole cake is to be shared equally between four boys.
▪ A diagrammatic representation of the division of the cake was given, and phrase such as "divide the whole cake into 4 equal parts, and then give 1 part to each boy" was used to describe the process followed by the symbolic representation "$1 \div 4 = \frac{1}{4}$"</td></tr>
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- A similar context using 2 whole cakes was also presented whereby the two cakes were to be shared by three boys.
- The operation shown was division, i.e., 2 ÷ 3 = $\frac{2}{3}$.
- Exercises given for the whole of section 1.
- **Workbook.**

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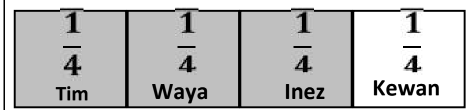
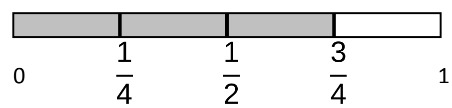
fraction relationships, for example,



$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

~ In the next step, a transition from using a measuring strip to a fraction bar was made whereby the text explained that measuring strips can be used to find parts of a whole, as in the above example, three parts out of four can be expressed as the fraction $\frac{3}{4}$ on a fraction bar. Hence a new model was introduced.



~ This context ended with a discussion of the similarities and differences between a measuring strip and a fraction bar.

~ To reinforce students skills in using fraction bars, a different context problem (**<u>Water Tanks</u>**) was presented, and to make it easier to handle, benchmark fractions were used ($\frac{1}{2}, \frac{3}{4}, \frac{1}{8}, \frac{1}{5}$).

~ Given the maximum capacity of the tank and the amount of water it holds each day, students were asked to use fractions to label the level of water for each day. This is followed by students creating their own problem and describing it with a fraction.

~ The problem was extended where students were to

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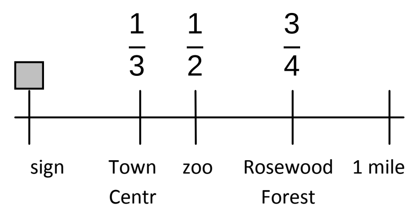
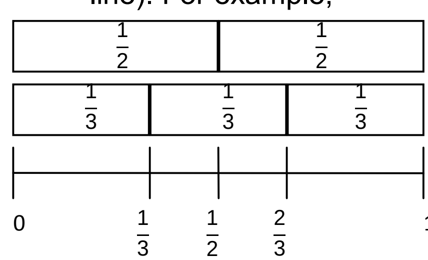
		compare the amount of water in two different-sized tanks, and describing the amount of water in each tanks using fractions. ~ In order to reinforce their skill, three other different-sized tanks were presented and they also have to describe the amount of water in each tank using fractions. ~ The final stage is all the five different-sized tanks were presented with different amount of water in them, and students have to compare and write the fractions representing the amount of water in each tank, and finally choosing the tank with the most water. For this students have to apply their knowledge on fractions. ~ The next two contexts are dealing with percents which will not be discussed here. ▪ A summary of the important points in the section was presented in the *Summary* part. More exercises were given in the *Check Your Work* part for students to try to find out whether they have understood the section or not. ▪ The frequency of the types of questions used for section B (The Bar Model), for the sub-section dealing with fractions only, based on Bloom's Taxonomy is as follows: ✓ Knowledge – 0 ✓ Comprehension – 7 ✓ Application – 4 ✓ Analysis – 4 ✓ Synthesis – 4 ✓ Evaluation – 2
2. Types of fractions		

3. Equivalent fractions	▪ The *point of entry* is 4 circles: $\frac{1}{2}$ shaded, $\frac{2}{4}$ shaded, $\frac{3}{6}$ shaded, $\frac{4}{8}$ shaded. Followed by 3 rectangles: $\frac{1}{5}$ shaded, $\frac{2}{10}$ shaded, $\frac{3}{15}$ shaded. ▪ **Workbook** - an activity using a fraction chart. ▪ Worked examples given. ▪ **Workbook.**	▪ REFER TO THE FOUR OPERATIONS
4. Reducing a fraction to its lowest terms		
5. Order of fractions [p.78 - 82]	▪ Recalling arranging integers on a number line is the *point of entry* of discussion to the sub-unit. ▪ Used number line as a model to order fractions. ▪ A rule on how to compare two fractions was given: ○ *To compare two fractions,* ~ *rewrite the fractions so that they have the same denominator,* ~ *compare the numerators.* ▪ Worked examples given. ▪ **Workbook.** ▪ Exercises given.	▪ For comparing and ordering of fractions, there are two places where it was discussed, in the MiC *Models You Can Count On: Number* book, and the MiC *Fraction Times: Number* book. Each of these books will be looked at and discussed below. ▪ In the MiC *Models You Can Count On: Number* book, Section C (The Number Line) introduce number line as a model to locate fraction and decimals in the context of distance on a map. For the purpose of this analysis only the part dealing with fractions will be discussed. ▪ The *point of entry* of the introduction the number line is using information on a signpost (**<u>Distances</u>**). From the context of the problem, the assumption here is that students already know and understand the concept of fractions, especially the benchmark fractions.

- The methodology used to develop the intended content of the section is based on the same context of distances on a map.
  - ~ Using information on a signpost, students were asked to estimate the position of the fractions on a number line.



  - ~ Based on the same context, the problem was extended to further reinforce students' knowledge on fractions by having them find or determine some missing fractions based of some given information.
  - ~ To formalize number line, fraction strips are used to compare and order fractions on a number line (i.e., find their exact locations on a number line). For example,



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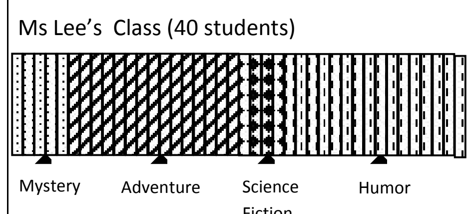
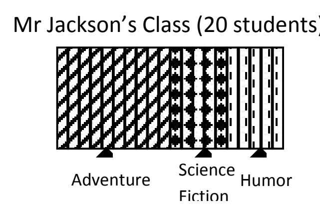
  - ~ Another contextual problem (Biking Trail) was also presented which also used a number line as the model to help students develop a conceptual understanding of fractions

		as numbers. ▪ The next four sub-sections deals with decimal and will not be looked at. ▪ In the *Check Your Work* part two out of 6 questions were reinforcement exercises on number line as a model to find the answer to a problem. ▪ In the *For Further Reflection* part, only decimal problems were summarized and discussed. ▪ The frequency of the types of questions used for section C (The Number Line) based on Bloom's Taxonomy is as follows: ✓ Knowledge – 0 ✓ Comprehension – 2 ✓ Application – 0 ✓ Analysis – 7 ✓ Synthesis – 1 ✓ Evaluation – 0 ▪ In the MiC *Fraction Times: Number* book, comparing fractions with unlike denominators using segmented bars is the main focus for the first section of the book. ▪ The *point of entry* to the discussion is the use of charts and graphs with their articles to help readers understand the information they presented. ▪ The methodology used is not straight forward by just giving students standard textbook methods of ordering fractions, instead it is presented as a series of exploratory activities which leads to the development of the intended focus of the section. ~ Showing an **article** where

the reporters for the newspaper *Fraction Times* used pie charts and bar models to represent data as the starting point for discussion.

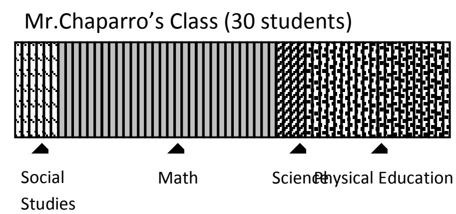
~ Simple class activity (**<u>favourite colours</u>**) of data collection, representing the data on the segmented bars, and finally drawing a pie chart based on the bar model.
~ Giving another set of similar data from a different class, and students are to write the corresponding fractions of the pie charts.
~ Using segmented bars to compare the result of the simple survey (**<u>what teens are most interested in</u>**) for two classes.

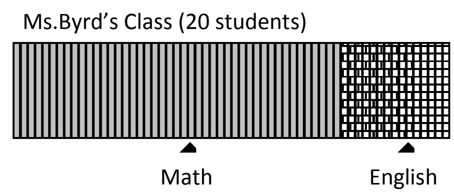




~ Comparing fractions for the segmented bars with the different number of segments (ie: fractions with different denominators).
~ Exploring how the same data can be represented as pie charts.
~ Exploring how a similar set of data (**<u>favourite school subject</u>**) can be

presented using a bar model with equal number of segments, which helps them to compare and compute with fractions with the aid of the visual models.





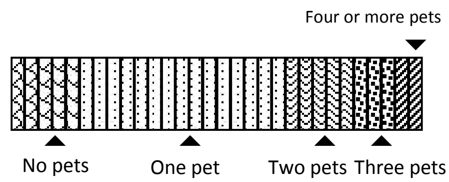
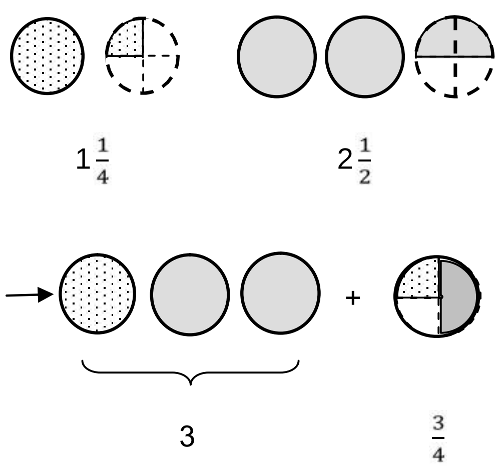
~ Introduces common multiple by using ratio table, for example to compare a class of 25 students to a class of 20 students, to determine the number of segments needed for fractions with different number of denominators.

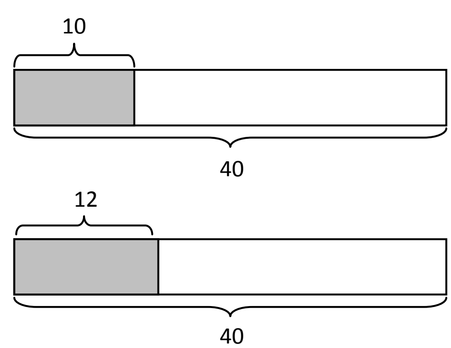
Number of Segments per Student	1	2	3	4
Total Number of Segments	25	50	75	100

Number of Segments per Student	1	2	3	4	5
Total Number of Segments	20	40	60	80	100

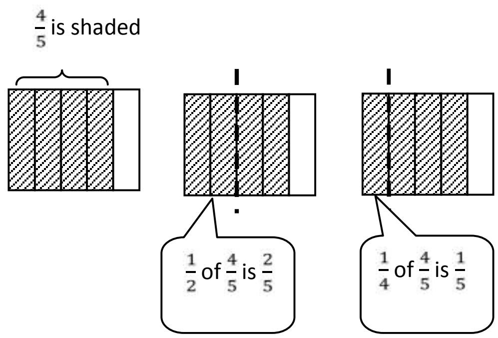
		~ The ratio table indicate that the first table shows that each students will be represented by 4 segments in the bar model, whereas in the second table, each student will be represented by 5 segments. Hence the concept of common multiple is formalized, and comparison of fractions would be easier for them. This was not done as part of student class activity, rather it was presented as a reflection. ~ The last exercise which uses a survey of teens' favourite foods, tries to highlight the difference between relative (as in pie chart) and absolute (as in the bar model with equal number of segments) comparison so that students will not be led to believe that even though they may be represented using the same fraction, it does not mean that they represent a complete picture of the actual data. For example, in Mr.Clune's class there are 6 out of 36 students who liked chicken, whereas in Ms.Grath's class there is only 4 out of 24 students who liked chicken, but both can be represented as $\frac{1}{6}$. ~ A summary of the whole section was also given. ~ This is followed by the *Check Your Work* part which also acts as a reinforcement exercise. ~ The section ended with a

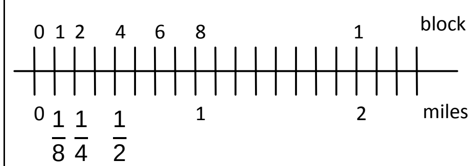
		For Further Reflection part which requires students to reflect back on what they have learned to explain why they think that it is easier to read a bar chart or a pie chart. ~ The frequency of the types of questions used for section A (Survey Results) based on Bloom's Taxonomy is as follows: ✓ Knowledge – 5 ✓ Comprehension – 10 ✓ Application – 5 ✓ Analysis – 6 ✓ Synthesis – 1 ✓ Evaluation - 4
6. The four operations		
6.1 Addition and subtraction 6.1.1 Addition and subtraction of proper fractions. [p.82–84]	▪ The *point of entry* here is by using a circle with 8 equal parts. Three parts have been shaded blue with the corresponding fraction given, and two parts have been shaded grey also with the corresponding fraction given. Using these fraction, addition and subtraction of fractions of the **same denominators** were demonstrated procedurally, followed by stating the rule for the procedure: ▪ $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ ▪ $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$ ▪ **Workbook.** ▪ This is followed by giving some examples of addition and subtraction of fractions with **different denominators**, which	▪ In the MiC *Fraction Times: Number* book, section B, comparing and adding fractions with unlike denominators using segmented and non-segmented bars, and ratio tables as the visual models to represent and solve problems is the main focus. ▪ Benchmark fractions are used as the *point of entry* to the discussion. Bar charts are used as the thinking model. ▪ The methodology used in order to develop the intended concept is a series of activities: ~ Using a **pet survey** as the context, an exploratory exercise using non-segmented bar models and pie charts are used to represent the same set of data. ~ The same context was used to find a fraction of a number, and the text does

	are also done procedurally. ▪ **Workbook.** ▪ Exercises given. ▪ Other than the circle presented at the beginning of the sub-unit, no other context or models were used to present these to the students. ▪ The examples were all procedural algorithms that the students have to follow or imitate in order to find the solutions to the given problems.	not specify which methods to use, so students can choose from ratio table, segmented bars, division or even multiplication. ~ The data was then to be represented using segmented bars, and addition of fractions was introduced: the number of segments to indicate the denominator and the shaded part as the numerator. 
6.1.2 Addition and subtraction of mixed numbers and improper fractions. [p. 85-87]	▪ Like the previous sub-unit, the *point of entry* is using circles to demonstrate addition of mixed numbers, followed by the procedural method.  ▪ A similar procedural method is also used to present the subtraction of mixed numbers. The text also informed readers that: "The addition and subtraction involving improper fractions are	~ The context was further extended where students have to apply previously learned knowledge on how to determine the number of segments needed in order to draw the bar model for fractions with different denominators. ~ A similar problem with a different context (**<u>airplane survey</u>**) was also used which also leads to addition of fractions with different denominators by using segmented bars. ~ Using the same context, based on different newspaper's headlines, an exercise was presented where students have to compare fractions with ratios and percents, and to focus the relationships between them. ~ Another comparison exercise using bar model was also presented as "**<u>comparing the Computer Division and</u>**

	often done by first expressing the improper fractions as mixed numbers." ▪ Worked examples given ▪ **Workbook.** ▪ Exercises given.	**<u>the Communication Division at Bulk Electronics Company in the business section for *Fraction Times*</u>**" to be the context of the problem. ~ This is followed by an exercise where students were to explain their work without drawing and shading bars based on their knowledge of fractions. ~ A different context (**<u>investigating the amount of real fruit in two brands of apple juice</u>**) was used in order to move from segmented bar model to non-segmented bar models of equal lengths where estimates is used to shade the bars. This also leads to comparison of fractions with unlike denominators mentally, and hence to subtraction of fractions.  ▪ ~ Yet another context was used (**<u>comparing favourite fruits of Canadian and U.S. middle school students</u>**) to find the difference of two fractions, and was also extended to addition of fractions using bar models. ~ Problem in the next part of the exercises was presented ***without any***

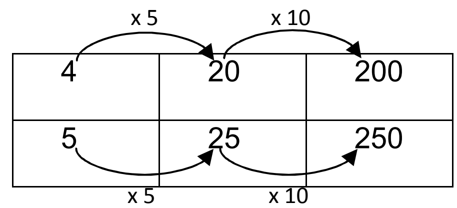
		context where non-segmented bar models were used to compare, subtract and add fractions. ~ An alternative method (ration table) was also presented, and students may choose any strategy they prefer to solve related problems. ~ No formal procedural methods were used throughout the section, but a variety of strategies were discussed for students to choose from. ~ The summary part reviews how to add, subtract, and compare fractions with like and unlike denominators. ~ The *Check Your Work* part exercises are for students to check whether they have understood the lessons or not i.e., self-assessment exercises. ~ Another form of checking students understanding is through the *For Further Reflection* part where students were asked to write what they have learned in their journal. ~ The frequency of the types of questions used for section B (It Adds Up) based on Bloom's Taxonomy is as follows: ✓ Knowledge – 0 ✓ Comprehension – 11 ✓ Application – 11 ✓ Analysis – 14 ✓ Synthesis – 5 ✓ Evaluation - 5
6.2 Multiplication [p.87–90]	▪ The *point of entry* for this sub-unit is an activity in the **workbook** that illustrates multiplication of fractions diagrammatically.	▪ To explain the concept of faction multiplication, it was discussed in two separate books; the MiC *Models You Can Count On: Number* book,

	 ▪ This is followed by stating the rule for multiplication of fractions: ▪ $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ ▪ Worked examples given – proper and improper fractions. ▪ **Workbook.** ▪ Worked examples given involving mixed numbers. ▪ There is also a note in the text to remind students that: "Before multiplying a mixed number, rewrite it as an improper fraction" ▪ Exercises given.	and the MiC *Fraction Times: Number* book. Each of these books will be looked at and discussed below. ▪ In the MiC *Models You Can Count On: Number* book, Section D, double number line is used as visual model for students to make sense of calculations involving fractions and decimals, and in addition, informal strategies will be used based on the double number line to informally multiply and divide fractions. On top of that, the double number line can also be used to demonstrate equivalence. ▪ The *point of entry* for this section is the **double scale line** used in maps to find relationships between miles and kilometres. ▪ The methodology used to develop the intended concept was: ~ Based on the **double scale line** exercise students are to describe how a double scale line can be useful and to write some relationships between miles and kilometers. Exercises on making estimates of distance in miles and kilometers were also given. Within the same context, the problem was extended further to distance table where students were to find some missing values based on the table. ~ Using another context (**City Blocks**), a number line can be used to solve problem involving distances. At this point students can use various strategies that they have learned to find the answer
6.3 Division	▪ The *point of entry* is an exploratory activity in the **workbook** on reciprocal of fractions. ▪ Worked examples. ▪ It was presented using reciprocal of a number as the focal point of the argument behind is to explain why the "invert and multiply" method works. $4 \div \frac{1}{2} = \frac{4 \times 2}{\frac{1}{2} \times 2} = \frac{4 \times 2}{1} = 4 \times 2 = 8$ ▪ General rule: $a \div \frac{b}{c} = a \times \frac{c}{b}$ ▪ More worked examples given. ▪ **Workbook.**	

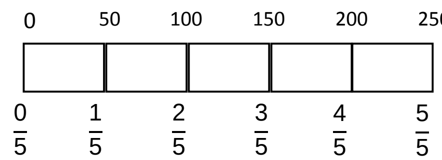
	▪ Exercises given.	such as by using repeated addition, or they can multiply. ~ The next step of the same problem was where a double number line was introduced which is used to solve problems involving the number of minutes of walking, and miles. A double number line was given with some of the values given.  ~ Students were presented with problems where they used the given double number line as the starting point to find the required solution. ~ The problem was further extended where students were asked to draw a similar double number line to solve similar problems to reinforce what they have learned, and they were to compare the method used with a ratio table. ▪ The nest two sub-section deals with decimal which will not be discussed here. ▪ In the *Summary* part of the section, both the double number line and the ratio table models were summarized and compared. ▪ In the *Check Your Work* part, another problem using a double number line was given as an additional exercise. ▪ In order to reflect what they have learned in the section, students were asked to reflect on the similarities and

		differences between a double number line and a ratio table, and which do they find easier to use. ▪ The frequency of the types of questions used for section D (The Double Number Line) based on Bloom's Taxonomy is as follows: ✓ Knowledge – 0 ✓ Comprehension – 6 ✓ Application – 1 ✓ Analysis – 10 ✓ Synthesis – 2 ✓ Evaluation - 0 ▪ In the MiC *Fraction Times: Number* book, section E, the main focus is multiplication of fractions using informal strategies. ▪ The *point of entry* for the development of this section an article (**Recycled Fractions**) about the number of aluminum cans thrown away by park visitors per year and how much is recycled. ▪ The methodology used to develop the intended concept was by using a series of activities: ~ Base on the article on recycling, multiplication of fractions was introduced informally within this context which involved finding a fraction of a number and students no specific strategy was required so they can choose any strategy they wish. This is followed by finding a fraction of a fraction, again students can use their preferred strategy. This can be seen from the context of the problem, where it was estimated that visitors

		throw away 250 kilograms of can per year, at least $\frac{4}{5}$ of the used cans will be put into recycling bins and only $\frac{1}{10}$ of the cans will be lost on the way to the recycling center. This leads to a problem: "Enrique does not know how to calculate $\frac{9}{10}\ of\ \frac{4}{5}$." He decides to calculate the quantity that is actually recycled first. So the problem has been split into four steps to enable Enrique to get a solution for $\frac{9}{10}\ of\ \frac{4}{5}$. 1. The weight of cans expected to be put in the recycling bin. 2. The weight of cans expected to be recycled. 3. The ratio of the weight of the cans recycled to the weight of cans park visitors used each year. 4. The fraction of can that will probably be recycled. ~ Students may choose any strategy they prefer, some may use: ○ $\frac{4}{5}\ of$ 250 kg by making use of the fact that $\frac{1}{5}\ of$ 250 kg is 50, so $\frac{4}{5}\ of$ 250 kg will be 4 x 50 which is 200kg, or ○ Using a ratio table, or



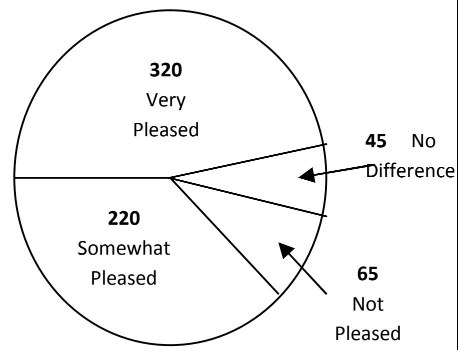
~ Using a fraction bar.



- ~ Students can also use similar strategies for all the different questions above.
- ~ The same problem was also presented using a different context (**What Fraction Saved?**) based on another article about an oil tanker accident to reinforce what they have done in the previous exercise.
- ~ The next step of the development of multiplication of fraction is presented through an extension of the oil tanker accident article whereby students were given with an exploratory exercise of finding a missing value or number on which they can operate the fraction.
- ~ The last part of the section is where students are expected to find a fraction of a fraction in a problem without any context where students are to choose a number on which to operate to find an answer to the product.
- ~ The section closes with a reflection which addresses

<table>
<tr><td></td><td></td><td>a common misconception in multiplication of fraction:
~ "When you multiply two numbers, the product is always larger than either of the factors." Tell if this statement is true or false. Explain your reasoning.
~ The frequency of the types of questions used for section E (Fractional Parts) based on Bloom's Taxonomy is as follows:
✓ Knowledge – 0
✓ Comprehension – 7
✓ Application – 0
✓ Analysis – 9
✓ Synthesis – 4
✓ Evaluation - 0</td></tr>
<tr><td>6.4 Combined operations</td><td>▪ This has been presented as standard school procedural mathematics which does not encourage students to create their own strategies.
▪ Worked examples followed by exercises given.</td><td>▪ This is not "taught" as a separate "thing", rather it was presented alongside all the other fractional concepts.
▪ Even though the combined operations may not be obvious in the problems given, the strategies used by the students to solve them would involve at least multiplication, division and addition.</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>7. Mental computation [p. 95 - 99]</td><td>▪ The methods given for the mental computation have many rules that needed to be remembered.
i. Both numerators = 1
$$Answer = \frac{sum\ of\ denominators}{product\ of\ denominators}$$
ii. Both fractional parts = $\frac{1}{2}$ and sum of whole number parts = even number.
$$Answer = \frac{2nd\ denominator - 1st\ denominator}{product\ of\ denominators}$$
iii. Both fractional parts = $\frac{1}{2}$ and sum</td><td>▪ In the MiC More or Less: Number book, section B, in the Percents and Fractions sub-section, the relationships between benchmark fractions and percent are used to help solve percent problems.
▪ The point of entry is based on the result of a survey where the produce manager were able to figure out quickly that $33\frac{1}{3}\%$ of 180 customers is 60 customers.
▪ The methodology used to develop the intended concept was:
~ "<u>benchmark fractions</u>" are</td></tr>
</table>

	of whole number parts = odd number. iv. Sum of fractional parts = 1 and whole number parts are equal. v. Sum of fractional parts = 1 and whole number parts differ by 1.	used to do mental calculation, but it started as real problem which relates to percents and its equivalent fractions, to some easier problems using benchmarks fractions, then some mental calculation problems which students can do starting from those they find easier. Once students have done the problems, they are to <u>describe</u> three of the strategies they used, and to <u>identify</u> which of the problems are related and how are they related. The final part is students were to <u>reflect</u> on which problems were easier to do mentally, and which they would rather use a calculator. ▪ The frequency of the types of questions used for section B (Percents and Fractions) based on Bloom's Taxonomy is as follows: ✓ Knowledge – 0 ✓ Comprehension – 3 ✓ Application – 1 ✓ Analysis – 1 ✓ Synthesis – 1 ✓ Evaluation - 1
8. Using the calculator		
9. Estimation	▪ In CPDD 1A, estimation is presented as a straight forward estimation of fraction itself in obtaining an answer which is the best estimate of the exact answer so that students will have a rough idea whether their answer is correct or not. For example, ▪ $\frac{48}{95} \approx \frac{50}{100} = \frac{1}{2}$ ▪ $\frac{41}{69} + \frac{69}{79} + \frac{12}{83} + \frac{11}{81}$	▪ In the MiC *More or Less: Number* book, section B, in the *Surveys* sub-section, interpreting survey data using fractions and percents is aim of the sub-section. ▪ The *point of entry* is a survey done by the store manager about customers' opinion of the new scales which are presented using pie charts.

	$\approx \frac{40}{70} + \frac{70}{80} + \frac{12}{84} + \frac{10}{80}$ $= \frac{4}{7} + \frac{7}{8} + \frac{1}{7} + \frac{1}{8}$ $= \frac{4}{7} + \frac{1}{7} + \frac{7}{8} + \frac{1}{8}$ $= \frac{5}{7} + 1$ $= 2$	▪ The methodology used to develop the intended concept was: ~ Based on the pie chart, students were to describe the results of the survey using fractions. For example,  Approximately one-half of the customers were very pleased, about one-third were somewhat pleased, one-tenth were not pleased and about one-fifteenth did not notice the difference. ~ A similar problem was also presented using bar chart, and students were asked to explain how charts help them find the fractions that were required. ▪ The frequency of the types of questions used for section B (Surveys) based on Bloom's Taxonomy is as follows: ✓ Knowledge – 0 ✓ Comprehension – 4 ✓ Application – 1 ✓ Analysis – 1 ✓ Synthesis – 2 ✓ Evaluation - 1
10. Word problems	▪ Given after at the <u>end</u> of a chapter when all the operations and	▪ Given at the <u>start</u> as a basis for construction of knowledge.

[p. 104–106]	procedures have been taught. ▪ The problems given have <u>no meaning</u> for the students as they are not realistic problems which students can identify with. The problems are just standard exam-type problem that only requires standard answers / procedures, and do not support and encourages investigation and inquiry. ▪ Students will produce the same answer with the same (if not exactly) procedures i.e., <u>closed situation</u>. ▪ <u>Superficial context</u> that will lead students to produce answers that have little to do with the reality of the situation described i.e., the context seemed to be *lost* within the procedures.	▪ The problems given to them are problems that students can <u>identify with</u>. Students were able model the context in ways that made sense to them. It encourages the students to explore strategies that they have to justify to every member of the group. ▪ Even though they are working on the same problem, they will have different strategies and different ways of dealing with the problems i.e., <u>open situation</u>. ▪ The <u>context is close to students' experiences</u> that allow them to come up with a variety of models of the real world.

APPENDIX 4: Sample from CPDD 1A textbook and 1A workbook

1.2 Fractions as Equal Parts of a Set

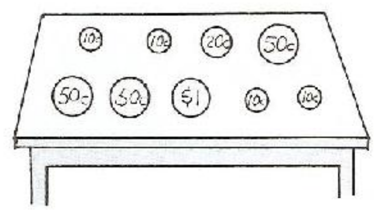
Look at the picture showing a group of 10 children.
We say that:

6 out of 10 are girls or $\frac{6}{10}$ of the 10 children are girls

and 4 out of 10 are boys or $\frac{4}{10}$ of the 10 children are boys.

There are 9 coins on the table.
We say that:

4 out of 9 are ten-cent coins or $\frac{4}{9}$ of the coins are ten-cent coins, and 1 out of 9 are twenty-cent coins or $\frac{1}{9}$ of the coins are twenty-cent coins.



Can you tell what fraction of the coins are:

fifty-cent coins? one-dollar coins?

You would have found that

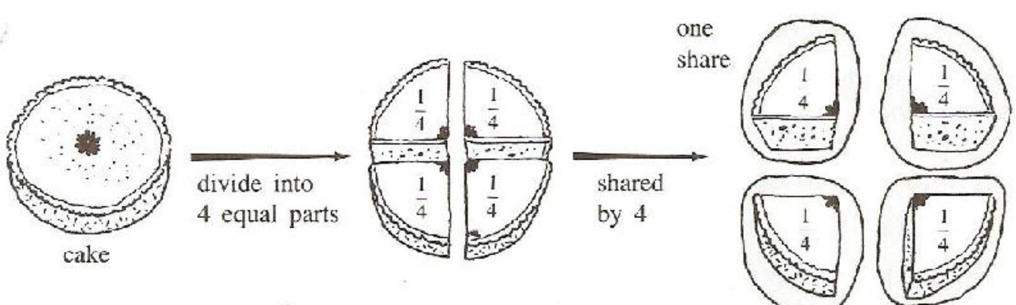
$\frac{3}{9}$ of the coins are fifty-cent coins

and $\frac{1}{9}$ of the coins are one-dollar coins.

1.3 Fractions as Answers to Division of Whole Numbers

If four boys are to share a cake equally, we have to divide the whole cake into 4 equal parts, and then give 1 part to each boy.

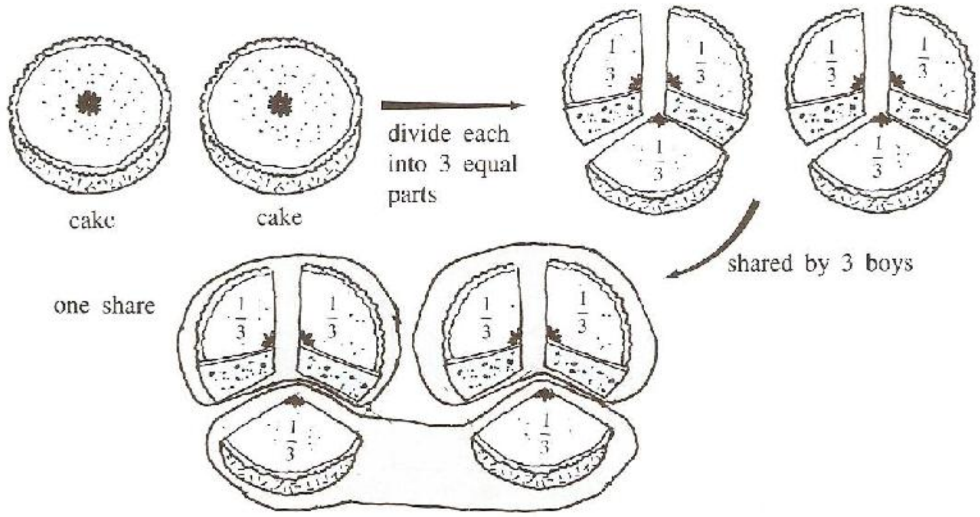
When a cake is divided into 4 equal parts, each part is one-quarter of the cake.



So, each boy will get $\frac{1}{4}$ of a cake. We therefore have

$$1 \div 4 = \frac{1}{4}$$

If two such cakes are to be shared equally among three boys, i.e. $2 \div 3$, then:



We see that each boy will get 2 parts out of 3 equal parts of a cake, i.e. $\frac{2}{3}$ of a cake. Therefore, we have

$$2 \div 3 = \frac{2}{3}.$$

Similarly, we have:

$$1 \div 8 = \frac{1}{8}$$

$$2 \div 7 = \frac{2}{7}$$

$$3 \div 4 = \frac{3}{4}$$

Go To Workbook Exercise W3A

EXERCISE 3A

1. Write each of the following in the form $\frac{a}{b}$:

(a) one-third
(b) three-quarters
(c) two-fifths
(d) eight-ninths
(e) one-tenth
(f) five-sixths
(g) four-sevenths
(h) seven-elevenths

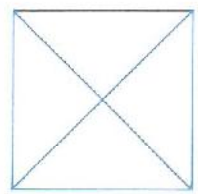
2. Write each fraction in words.

(a) $\frac{2}{3}$ (b) $\frac{1}{12}$ (c) $\frac{1}{4}$

(d) $\frac{3}{11}$ (e) $\frac{2}{7}$ (f) $\frac{5}{6}$

(g) $\frac{3}{8}$ (h) $\frac{7}{9}$

3. The square below is divided into 4 equal parts. Can you think of any more ways of dividing the square into 4 equal parts?



4. A box contains 12 doughnuts. If 4 doughnuts are eaten, what fraction of the original number of doughnuts
(a) has been eaten, (b) is left?

5. There are 40 pupils in a class. 32 are girls. What fraction of the class are
(a) girls, (b) boys?

6. A gardener grew 64 pots of orchids but 16 of them died. What fraction of the plants
(a) died, (b) grew?

2 *Types of fractions*

Here are two groups of fractions. In what ways are these two groups different?

(a) $\frac{1}{3}, \frac{1}{2}, \frac{5}{6}$ and $\frac{12}{17}$

(b) $\frac{7}{5}, \frac{9}{4}, \frac{11}{8}, \frac{3}{3}$ and $\frac{15}{15}$

In (a), the numerators are smaller than the respective denominators of the fractions. We call these fractions **proper fractions**. In (b), the numerators are either greater than or equal to the respective denominators of the fractions. We call these fractions **improper fractions**.

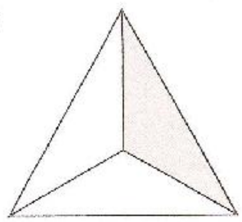
UNIT 3

FRACTIONS

EXERCISE W3A

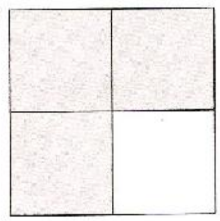
1. What fraction of each figure is shaded?

(a)



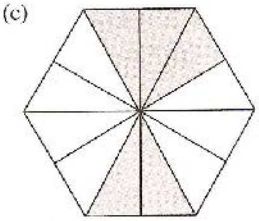
Ans: ________

(b)



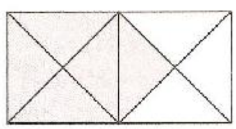
Ans: ________

(c)



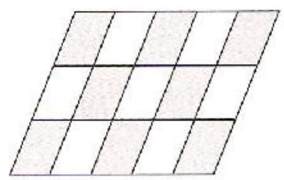
Ans: ________

(d)



Ans: ________

(e)



Ans: ________

2. Write each of the following as a fraction:

(a) 3 out of 5 is ________.

(b) 6 out of 15 is ________.

(c) 5 out of 8 is ________.

(d) 10 out of 14 is ________.

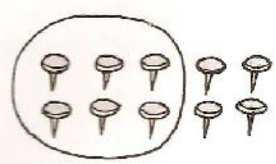
(e) 7 out of 19 is ________.

3. What fraction of each group of objects has been circled?

(a)

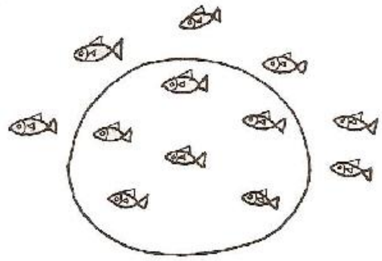
Ans: ________

(b)



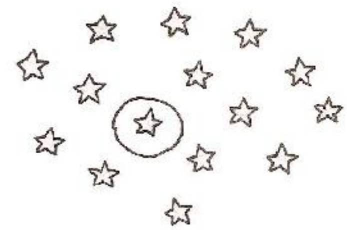
Ans: ________

(c)



Ans: ________

(d)



Ans: ________

4. Write each of the following as a fraction.

(a) $3 \div 5 =$ ________

(b) $2 \div 7 =$ ________

(c) $5 \div 8 =$ ________

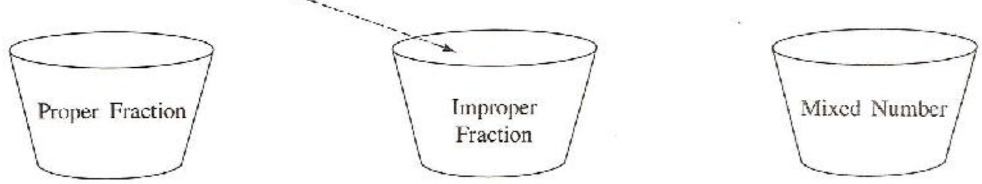
(d) $11 \div 15 =$ ________

Go To Textbook Exercise 3A

EXERCISE W3B

Put each of the numbers into its correct basket. One has been done for you.

• $\frac{9}{7}$ • $\frac{2}{3}$ • $1\frac{5}{8}$ • $\frac{10}{3}$ • $\frac{25}{25}$ • $\frac{2}{5}$ • $\frac{5}{4}$ • $\frac{13}{13}$ • $3\frac{9}{12}$



Go To Textbook Exercise 3B

EXERCISE W3C

1. Fill in the blank(s) in each of the following:

(a) $\frac{2}{15} = \frac{2 \times \square}{15 \times \square} = \frac{4}{\square}$

(b) $\frac{9}{10} = \frac{9 \times \square}{10 \times \square} = \frac{\square}{70}$

(c) $\frac{15}{18} = \frac{15 \div \square}{18 \div \square} = \frac{\square}{6}$

(d) $\frac{12}{16} = \frac{12 \div \square}{16 \div \square} = \frac{3}{\square}$

(e) $\frac{1}{3} = \frac{\square}{9}$

(f) $\frac{7}{8} = \frac{\square}{24}$

APPENDIX 5: Samples of pages from the Mic books

A Survey Results

The Newspaper

Reporters for the newspaper *Fraction Times* use charts and graphs with their articles to help readers understand the information. Here is the front page of *Fraction Times*.

Inside: Bar Graphs and Pie Charts

Fraction Times

Weather: Hot and sunny, high in the mid 80s

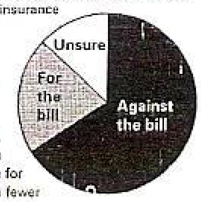
ESTABLISHED 1990 | MONDAY, AUGUST 28, 2003 | 50 CENTS

Jada Washington | Beatrice Flores | Enrique Caston | Mike Johnson | Lauren Cook

Two-Thirds Surveyed Against Health Bill

By Enrique Caston

A national survey was conducted last week asking whether voters were in favor of Health Bill 407. The bill goes in front of the house later this week. It states that health insurance should be paid for by the employer.



About two-thirds of the people polled were against the proposed health bill. Only a small fraction of respondents were for the bill, leaving even fewer people unsure.

Music Downloading Hurts Local Sales of Music DVDs

By Lauren Cook

DVD City, a local DVD Sales store, recently reported that its sales have decreased this year because so many people are downloading music from the Internet. "This year's sales of music DVDs are down 23% from last year's sales," said store owner Jim Roberts. An increase of 37% in DVD movie purchases has allowed the store to remain profitable, however.

1. a. What types of charts do you see on the front page of *Fraction Times?*

   b. Without reading the articles, summarize the information in each chart.

Activity

Favorite Colors—From Bar Charts to Pie Charts

For this activity, you need:

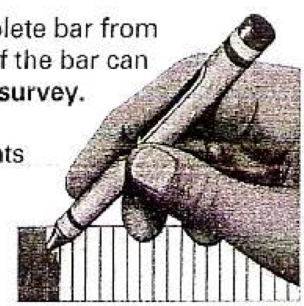
- **Student Activity Sheet 1**
- markers or crayons
- scissors
- tape

Select your favorite color.

Red HHT II

Ask everyone in your class to choose a favorite color. List the favorite colors chosen in your class and tally the number of students who choose that color. Don't forget to include yourself!

To represent the results, cut out one complete bar from **Student Activity Sheet 1**. Each segment of the bar can represent a classmate participating in the **survey.**



Color the bar to show the number of students who like each color. For example, if seven students chose red as their favorite color, color seven consecutive segments of the bar red. Do this for all of the colors chosen by the students in your class.

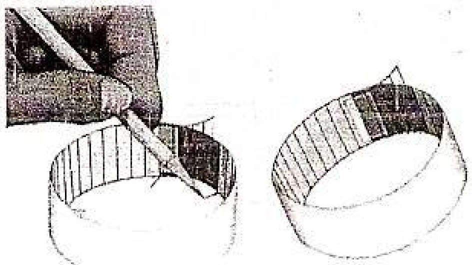
- How many students participated in the survey?
- How many students chose each color? Write the fraction for each color.

Now you can use the **bar chart** to make a **pie chart**.

Cut off the segments you didn't color.

Form a ring with the colors facing inside. Tape both ends of the bar together to form the ring.

Place the ring on a sheet of paper and draw a circle by tracing around the ring's inside edge.



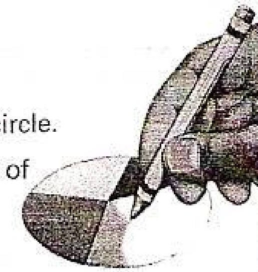
On your paper, mark inside the circle where the different colors begin and end.

Finally, remove the ring to complete the pie chart.

Now mark the color sections in the circle.

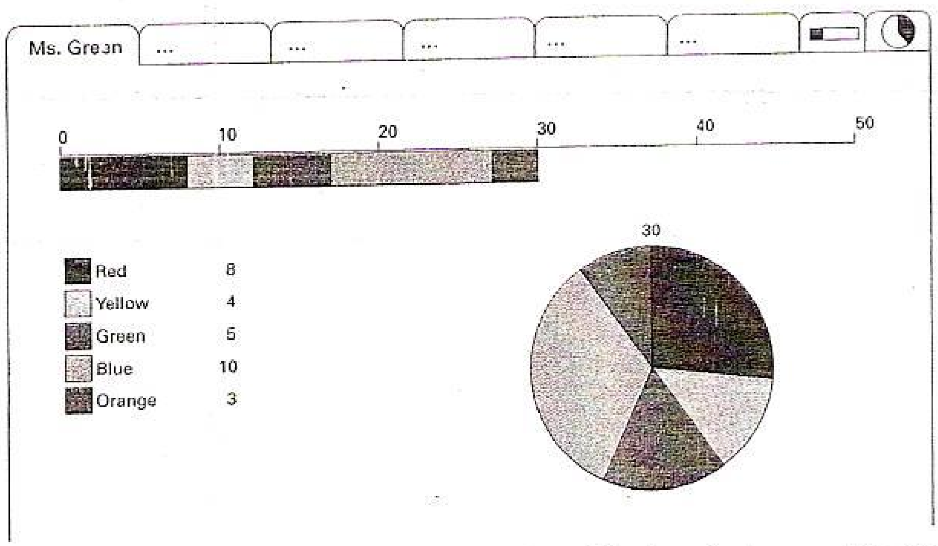
Estimate the location of the center of the circle.

Connect the marks you made on the edge of the circle with the center of the circle. Each "pie piece" is a section of the circle.



Color each section to correspond to the color on the tape.

Ms. Green's class did the same Favorite Color Activity on the computer. Here are the results.



2. Look at the survey results from Ms. Green's class and the survey results from your class. How do the results compare? Write four conclusions. For each conclusion, write what you used as a source: the numbers, the pie chart, or the bar chart.

Just for Teens

Fraction Times plans to launch a new section called "Just for Teens." Each Saturday, *Fraction Times* will feature an article on young people's books, movies, music, and food. Surveys will be used to investigate what teens are most interested in.

Enrique Caston
***Fraction Times* Reporter**

Enrique Caston is the book reviewer. He asked two teachers to conduct surveys about students' favorite types of books. Here is what he found.

Mr. Jackson's class (20 students)	
Mystery	0
Adventure	10
Science Fiction	5
Biography	0
Humor	5

Ms. Lee's class (40 students)	
Mystery	5
Adventure	15
Science Fiction	4
Biography	0
Humor	16

3 a. Use two bars from **Student Activity Sheet 1** to show each class's results.

b. Whose class prefers adventure books?

c. Explain why it will be easier for Enrique to compare the data if the bars have the same number of segments.

d. Use two new bars from **Student Activity Sheet 1** to show each class's results so that both bars have the same number of colored segments. Do not paste the bar charts in your notebook yet. You will need them in problem 5.

e. Compare the survey results.

Enrique wants to see how pie charts show the same survey information as the color bars. He begins with Mr. Jackson's class.

He thinks, "A pie chart for these results is easy to make, because 10 out of 20 is half the class, and 5 out of 20... ."

4. a. Complete Enrique's thoughts.

b. In your notebook, draw a circle and use this drawing to make a pie chart for Mr. Jackson's class. Be sure to include a chart key.

5. a. Make a pie chart for Ms. Lee's class results using the bar chart you made in problem 3d.

   b. On your paper, show a bar chart and pie chart for each class. Write the fraction of the class choosing each category.

   c. What is obvious in a pie chart that is not as obvious from a bar chart?

The "Just for Teens" staff is writing a weekly education column. They ask several classes, "What is your favorite school subject?" The survey results from two sixth-grade classes are shown here.

Ms. Byrd's class (20 students)	
Social Studies	0
Math	15
English	5
Science	0
Physical Education	0

Mr. Chaparro's class (30 students)	
Social Studies	3
Math	15
English	0
Science	2
Physical Education	10

Mr. Chaparro's class is larger than Ms. Byrd's class. This makes it more difficult to compare the results than it would be if the classes were the same size.

6. a. Use **Student Activity Sheet 2** to cut out two bars. Even though the class sizes differ, be creative and show the data using bars that have the same number of segments. Keep these bars handy because you will use them again in problem 7.

   b. Write a fraction to represent the number of students in each class who prefer each subject.

Name ____________________

Student Activity Sheet 1

Use with *Fraction Times*, pages 2, 4, and 13.

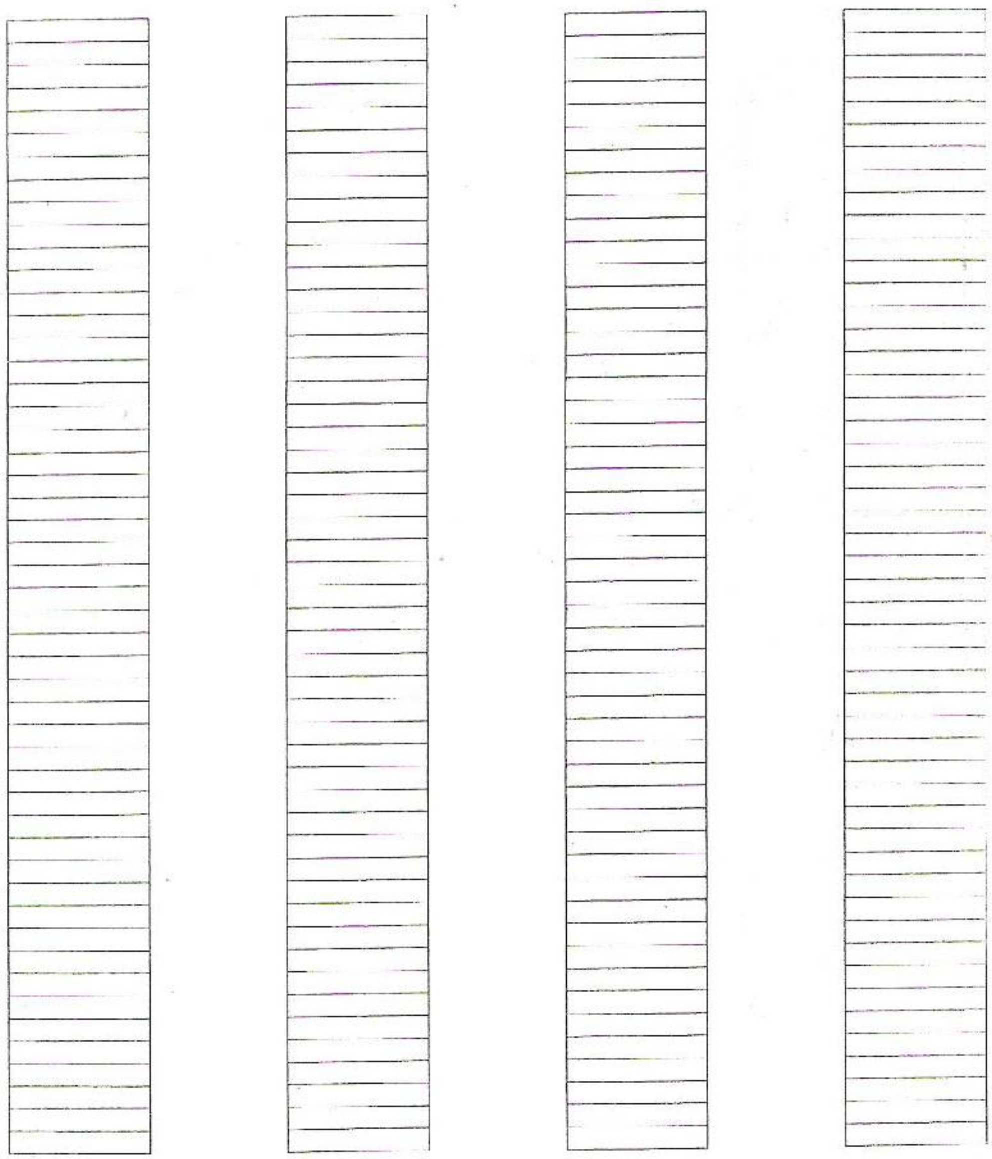
To be used with Section A.

Page 2, Activity (one bar);

Page 4, problem 3 (four bars);

page 11, problem 1 (two bars)

Page 13, Check Your Work, problem 3 (one bar).



Samples from the *Number Tools* MiC book

Name ______________________ Date ____________ Class __________

Recipes (page 1)

Banana Shake

4 servings:

2 bananas

$\frac{1}{3}$ cup lemon juice

$\frac{1}{4}$ cup granulated sugar

$\frac{1}{2}$ quart milk

1 cup vanilla ice cream

Blend until smooth.

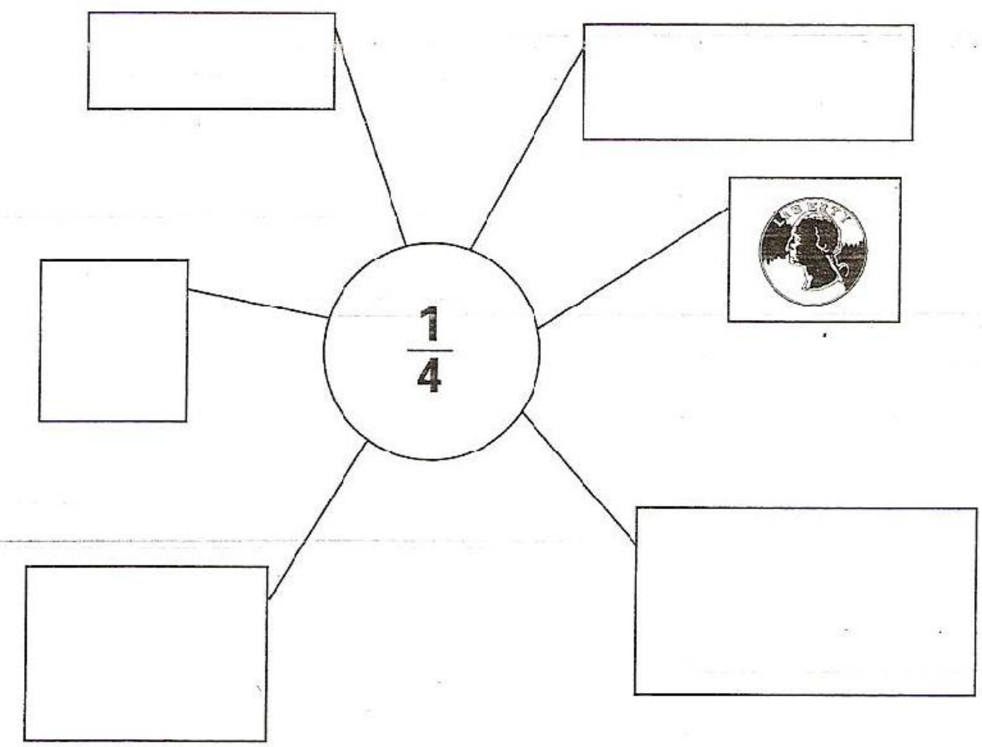
Tanisha's favorite summertime drink is a banana shake. She started the following chart so she would know how to make the shake when different numbers of guests join her. Help Tanisha by filling in her chart.

Servings	4	8		6			
Bananas	2		1		6		
Lemon Juice (cups)	$\frac{1}{3}$						$\frac{1}{12}$
Sugar (cups)	$\frac{1}{4}$						
Milk (quarts)	$\frac{1}{2}$					2	
Ice Cream (cups)	1		$\frac{1}{2}$		3		$\frac{1}{4}$

Name ______________________ Date ____________ Class _____

Map It Out!

What comes to mind when you see $\frac{1}{4}$? Make a mental map of the many different meanings for $\frac{1}{4}$ by filling in the boxes around $\frac{1}{4}$. You may draw additional boxes if you need them. After you finish, work with a partner to make a poster for the mental map of $\frac{1}{4}$.



APPENDIX 6: The Seven Activities in the Experimental Teaching

1. Activity 1: *Fraction of a foot*

   In this activity each group of students were given a ruler, a bread knife, a piece of oil paper and a foot long cheese stick bread (hereafter, cheese bread). The objectives of this activity was to find out whether the students were able to express each piece of the cut bread in terms of fractions and also in terms of the whole (in this case, 12 inches). This would also give evidence whether the students were able to derive some equivalent fractions from this simple activity, and how well they will be able to connect between, in terms of the number of parts, and the whole length of the original.

2. Activity 2: *My Different Looks*

   This aim of this activity was to encourage students to relate to the idea that there is more than one combination of fractions to get a whole (in this case 12 inches). In addition it also reinforces what they did in *Fraction of a foot* activity in the previous session where they associate a fraction with a certain length in inches, for example, $\frac{1}{2}$ with 6 inches. Here the students were given with a fraction chart, cheese bread which has been cut into 6 inches, 4 inches, 2 inches, $1\frac{1}{2}$ inches and 1 inch long. Two sets of three equations (see Appendix 6A), all have been purposely are made to equal to 1 were given to each pair of students. The students were then asked to find out how much does each equation came to by using the pre-cut bread provided. In order to be able to select the correct bread to represent each fraction, they need to decide which length of bread is for which fractions. A fraction chart is provided to help them if they forgot what they did in the first activity. Once they are done with the equations, the researcher ask the students to create their own fraction (which does not have to equal to 1) involving addition and subtraction of fractions to reinforce what they have just done.

3. <u>Activity 3: *Hour Foot*</u>

Having established the association of length in terms of 12 inches and fraction, for example, 6 inches is $\frac{1}{2}$, 4 inches is $\frac{1}{3}$, and so on, this activity aimed to connect it further with another unit which they are most familiar with, namely, time. For this activity I have asked the students to construct a clock face with a circumference of 12 inches. Prior to doing this, the researcher had to make sure that the students understood what circumference meant. Once that have been dealt with, and the 12 inch-circumference circle have been drawn, the students were asked to mark the circumference of the clock at one inch interval, and to label it as a clock should be. This was followed by asking the students to find the number of minutes that had passed for every inch that they had marked, and to write it on the clock also. So they would have something like the following:

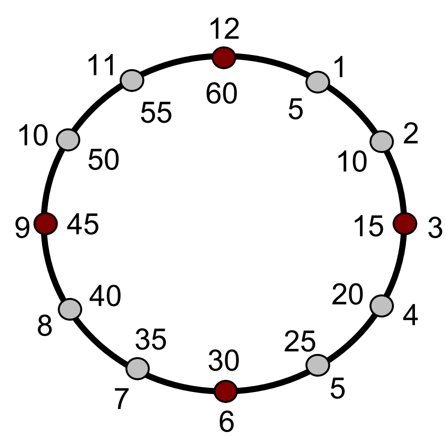


Figure 6.1A. The *Hour-Foot* Clock

The students' *hour foot* clock was complete when they pin the hour and minute hands at the centre of the *hour foot* clock. The researcher then turned the hour and minute hands to show different times, say 3 o'clock. Discussion generated from that where the students came up with at least two equivalent fractions to represent the sector between the hour and minute hand (similar to a piece of a round pizza). For example, if the *hour foot* clock showed 3 o'clock, fractions in terms of 12 inches will be $\frac{3}{12}$, and in terms of 60 minutes will be $\frac{15}{60}$ emerged from that.

4. <u>Activity 4: *Square Me Game*</u>

This is a game the researcher has created herself. This game are concerned with addition and subtraction of fractions, and to see how far the students were able to apply their previously acquired knowledge based on the *Hour Foot* clock in adding and subtracting the fractions. It is a board game with a minimum of one player, if more than one person plays the game they have to agree who will be responsible to act as the banker (where all the extra cards are kept, and to act as a referee). They could agree to be all jointly responsible, or they could invite another person to do the task. The player(s) can play as many (or as few) rounds as they wish, but they need to agree right at the beginning how many rounds they are going to play for fairness to all players. This is a fraction game where seven different sizes and colored cards were used to play the game. The seven cards represent different fractions with the game board as the unit whole: Blue ($\frac{1}{2}$), Pink ($\frac{1}{3}$), Green ($\frac{1}{4}$), Red ($\frac{1}{5}$), Yellow ($\frac{1}{6}$), White ($\frac{1}{10}$) and Black ($\frac{1}{12}$). In addition to the cards and game board, two packs of cards (*Chance* pack and *?* pack) were also provided. The *Chance* pack shows one of the colours on one side of it, and it also has an extra type of card which I called the *Wild* card. This card means that it can be any of the seven colours that students wished it to be. This pack contains three *Wilds*, two blues, two pinks, three greens, three reds, three yellows, six whites and six blacks. The *?* pack has similar cards but it only contains one of each cards in the pack.

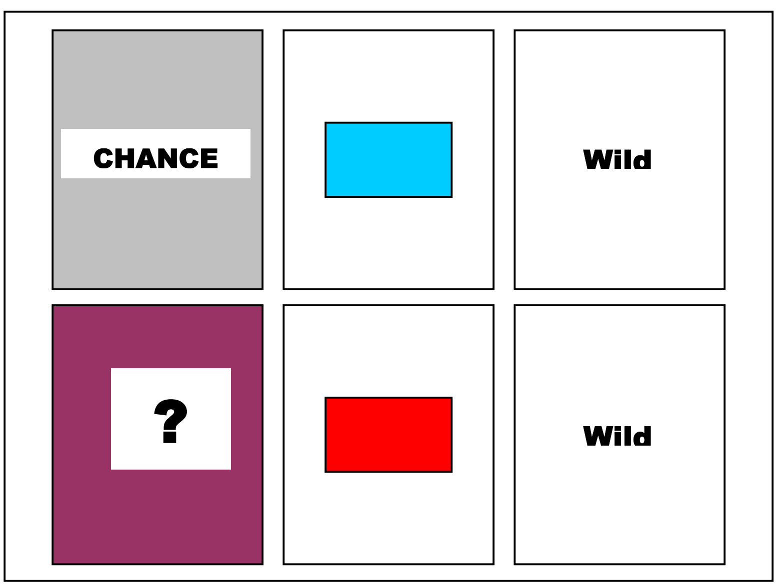


Figure 6.2A. Sample of the *Chance* and *?* Pack Cards

In order to keep scores, a score card is given to each student (see Appendix 6B). The objective of the game is to fill up the square shaped game board with the coloured cards until no more cards can be fitted in any longer on the board without overlapping the cards. The winner of the game is the person with the smallest total fraction at the end of the game. The game is played as follows:

a. At the beginning of each round, each player were given nine cards: 1 Blue; 1 Pink; 1 Green; 1 Red; 1 Yellow; 2 White; 2 Black.
b. Each player threw a dice to decide the order of play.
c. The round started by turning over a card from the **?** pack, and take a card of the displayed colour from the bank as the starting piece to be placed on the game board.
d. The player then place one of his/her card adjacent to the first card on the board, and they can only place one card at a time.
e. After the player have placed a card, he/she then has to take a card from the CHANCE pile, and take whatever colour is shown from the BANK. If a WILD is picked, the player can take any colours of his/her choice from the bank.
f. Once a square has been completed, *OR* a move cannot be made anymore the game for that round ended and all players must record the number of coloured cards they have left onto their SCORE CARDs.
g. Compare their SCORE CARD with their opponent(s).
h. The same colour from different players cancels each other, for example, Player A has 2 Blues and Player B has 1 Blue, so after comparing Player A will only have 1 Blue and Player B has none (see Appendix 6C for further details). This is done for all the colours.
i. The game continues until all the cards from the **?** pile have been used up.
j. Players will keep on recording and comparing the colours they have left at the end of each game.
k. BONUS – The player who will be able to give the correct fraction of the area that is NOT covered by the cards was allowed to return one of his/her card back to the bank, and deduct it from her total. Players must check their score card when they want to return a card to the bank so as not to return a card with a ‘0’ total on their score card.
l. To begin again a new round for that game, repeat step c.

m. At the end of the agreed number of rounds, all players will total up their cards, and will calculate their total fractions.

n. The WINNER of the game is the one with the SMALLEST total fraction at the end of the game.

For more advance players, there are more complex strategies by using their knowledge on fractions they could employ to play this game, such as:

i. A player can trade his/her card(s) with an equivalent card(s) with the bank or other players, but they can only do this ONCE for every turn so that they can get rid of their cards faster. For examples,

2 Greens ($\frac{1}{4}$) = 1 Blue ($\frac{1}{2}$);

2 Blacks ($\frac{1}{12}$) + 1 Yellow ($\frac{1}{6}$) = 1 Pink ($\frac{1}{3}$).

ii. At the comparing stage, if two or more colours are equivalent to another colour, they also cancel each other, for example, 2 Greens ($\frac{1}{4}$) cancels 1 Blue ($\frac{1}{2}$).

5. <u>Activity 5: *Identical but Different*</u>

In activity 4, the researcher tried to probe the students understanding of their knowledge on the flexibility of unitizing. In this activity one pair of students was given a 6-inch cheese bread, and the other pair was given an 8-inch cheese bread. The researcher then asked them to cut their bread into two equal parts. From here discussion on the fraction representing each part were generated, to find out whether the students can say confidently that both of their parts were equal to half. Students were also asked to justify their answer, and they were further probed whether their halves (for the different pair of students) are equal or not.

6. <u>Activity 6: *Let's Share!*</u>

For this activity the researcher had introduced the concept of sharing to the students which they are familiar with in everyday situations. Here the students were given a square pizza. Their task was to share the pizza among themselves equally, and to say the fraction of the pizza each of them received. After they have cut the pizza into four equal parts, they were then told that one of them did not want his/her share.

They were then to share that extra piece among the three of them, and they were requested to figure out how much in terms of fraction, was the new piece of pizza that each of them got. This tested their knowledge on flexibility of unitizing also as they needed to realize that they were now dividing a quarter into three parts, and the new unit is a quarter. After that they were asked to figure out how much of the whole pizza did each of them receive in total.

7. <u>Activity 7: *Guess How Much They Get?*</u>

The idea behind this game was concerned with flexibility of unitizing of fractions through sharing. In this activity the students were requested to say and record how much in fraction is one share, if they were given a certain amount of cake(s) and to be shared among a certain number of people. This activity can be played individually, or with one or more people. The students were provided with one recording sheet to be share by all, a pack of blue cards that says "If I have this number of cake(s)" a pack of yellow cards that says "I want to divide them equally between this numbers of children" and some plastic overlays (see Appendix 10) showing the different size of cake(s) and the possible cuts that could make to help them if they wish. The possible number of cake(s) in this activity was 1 whole cake, 2 whole cakes, $\frac{1}{2}$ of a cake, $\frac{1}{3}$ of a cake and $\frac{1}{4}$ of a cake, and the possible number of people to share the cake(s) was 2, 3 or 4.

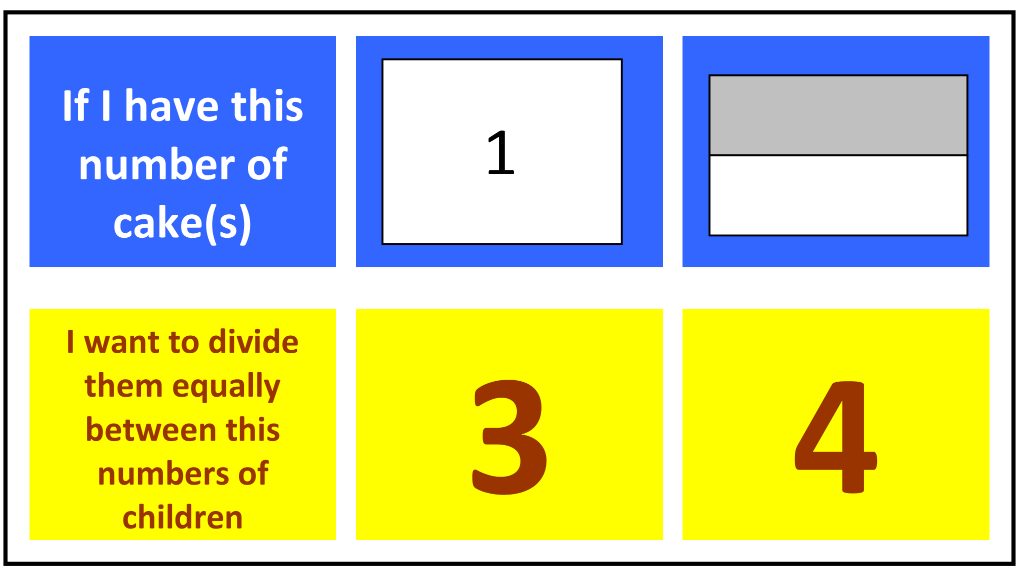


Figure 6.3A. Samples of the Blue and Yellow Packs Cards

The activity is played as follows:

a. Each player will take turn flipping over one Blue card for the number of cake(s) to be shared and one Yellow card for the number of people to share it with.
b. The numbers shown on both cards were recorded on the recording sheet (see Appendix 6D).
c. The player must figure out the amount of cake that each share will have in terms of fractions, and write it on the recording sheet (the plastic overlays can be use to help in this process).
d. Once that's done, pass the recording sheet to the next player and repeat steps a–c.

There is a non-threatening activity where there is no winner or loser.

APPENDIX 6A: Equations for *My Different Looks* Activity (Activity 2)

<u>Set A</u>

1. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

2. $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12}$

3. $\frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6}$

<u>Set B</u>

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{4}$

2. $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} = \frac{1}{12}$

3. $\frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6}$

APPENDIX 6B: *Square Me* Game Score Card

Player's Name:	1		2		3		4		5		6		TOTAL FRACTION
BLUE (1/2)													
PINK (1/3)													
GREEN (1/4)													
RED (1/5)													
YELLOW (1/6)													
WHITE (1/10)													
BLACK (1/12)													
												FINAL TOTAL	

APPENDIX 6C: Sample of *Square Me* Game Canceling Techniques

Player A	1	
BLUE (1/2)	~~2~~	1
PINK (1/3)	~~1~~	0
GREEN (1/4)	~~0~~	0
RED (1/5)	~~2~~	1
YELLOW (1/6)	~~1~~	1
WHITE (1/10)	~~2~~	0
BLACK (1/12)	~~2~~	0

Player B	1	
BLUE (1/2)	~~1~~	0
PINK (1/3)	~~1~~	0
GREEN (1/4)	~~1~~	1
RED (1/5)	~~1~~	0
YELLOW (1/6)	~~2~~	2
WHITE (1/10)	~~2~~	0
BLACK (1/12)	~~2~~	0

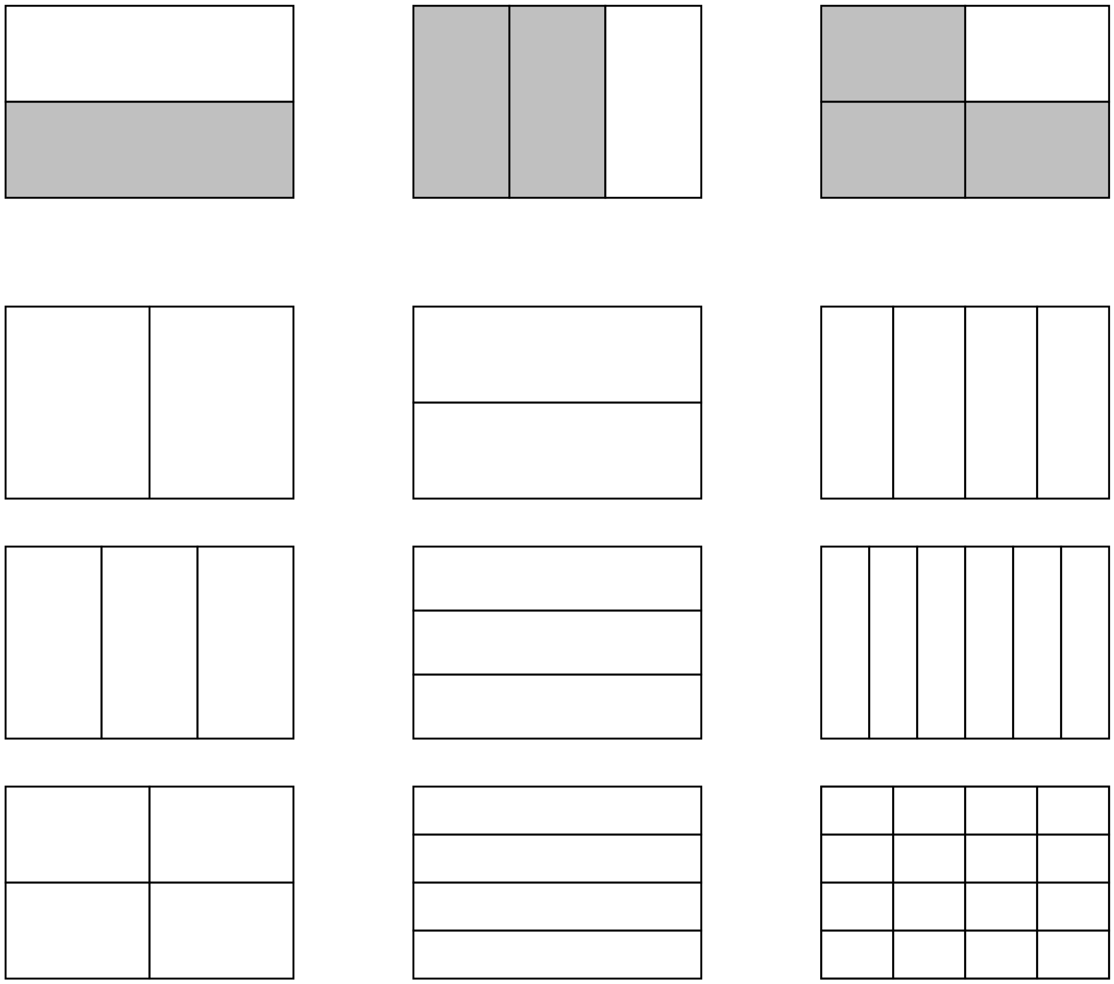
Player C	1	
BLUE (1/2)	~~1~~	0
PINK (1/3)	~~1~~	0
GREEN (1/4)	~~1~~	1
RED (1/5)	~~2~~	2
YELLOW (1/6)	~~0~~	0
WHITE (1/10)	~~2~~	0
BLACK (1/12)	~~2~~	0

All players have at least I Blue to contribute so their number of blue cards is reduced by 1

Player A does not have any Green card to contribute so their number of green cards remains the same

All players have the same number of white and black cards to contribute so all cancels out

APPENDIX 6D: Diagrammatic representation of Activity 7's Plastic Overlays



APPENDIX 7: Newman's Interview Schedule

INTERVIEW SCHEDULE
POST-TEACHING

<table>
<tr><td>Student's Name</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td rowspan="5">Number of Errors

Reading [R]…………………………
Comprehension [C]…………………..
Transformation [T]…………………..
Process Skills [P]……………………..
Encoding [E]…………………………..
Careless [X]………………………..

$\overline{}$
23</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>9</td><td>10</td><td>11(a)</td><td>11(b)</td><td>12(a)</td><td>12(b)</td><td>13</td><td>14</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>15</td><td>16</td><td>17</td><td>18</td><td>19</td><td>20</td><td>21</td><td></td></tr>
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</table>

APPENDIX 8: Pre-Teaching Fractions Test

MAIN STUDY

PRE-TEST

PAPER AND PENCIL TEST

ON

FRACTIONS

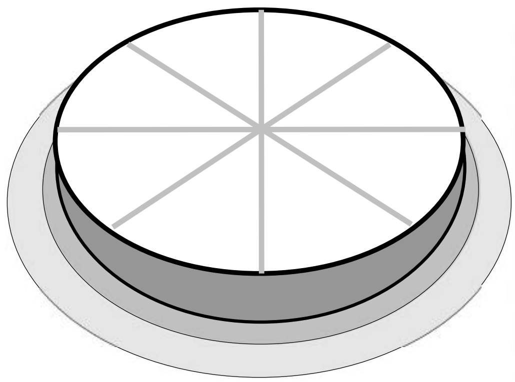
DURATION: 70 minutes

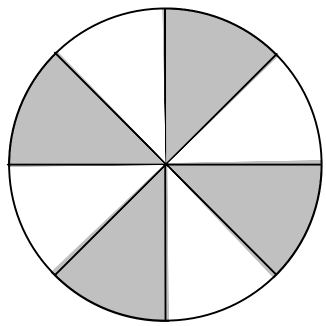
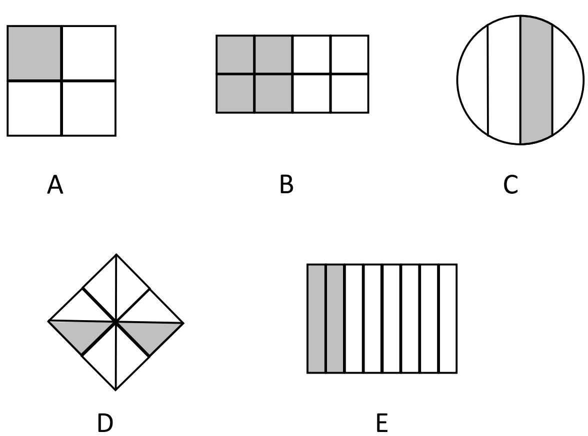
INSTRUCTIONS to students

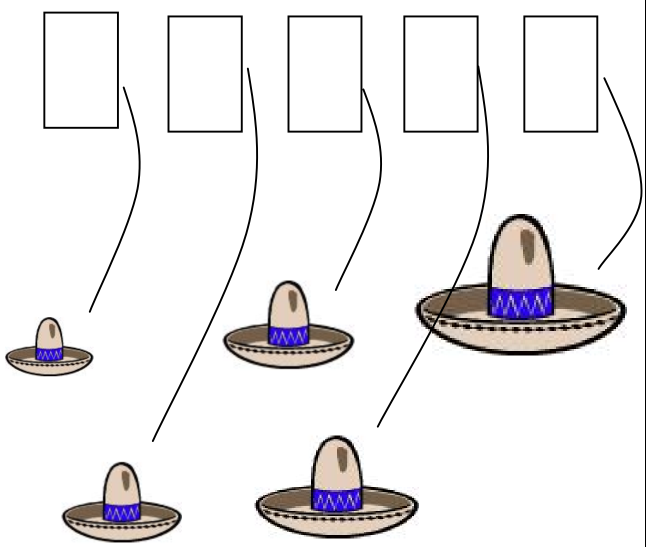
1. Work out your answers in the space underneath each question.
2. Write your answer either on the question itself or in the column marked “Answer”.
3. For every question, you must put a tick in one column, to show how you feel about your answer.

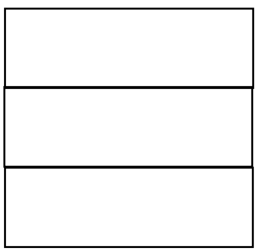
Name ____

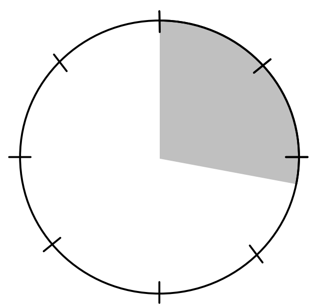
Age: _____ Class: _____________ Date: __________

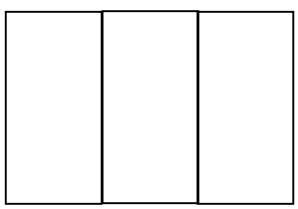
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
Example: $1 + \frac{1}{2}$	$1\frac{1}{2}$	✓				
1. Shade **one-quarter** ($\frac{1}{4}$) of the cake. 						
2. What is $\frac{1}{3} \div 4$?						

	Answer	I'm certai n I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
3. Look at this circle:  Tick (✓) **all** of the statements that are true: A. $\frac{1}{2}$ of the circle is shaded B. $\frac{4}{4}$ of the circle is shaded C. $\frac{2}{4}$ of the circle is shaded D. $\frac{1}{4}$ of the circle is shaded E. $\frac{4}{8}$ of the circle is shaded						
4. Circle **all** the shapes which are **one-quarter** ($\frac{1}{4}$) shaded.  						

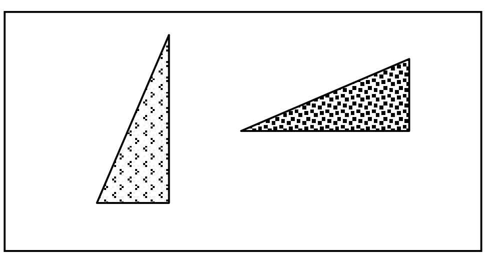
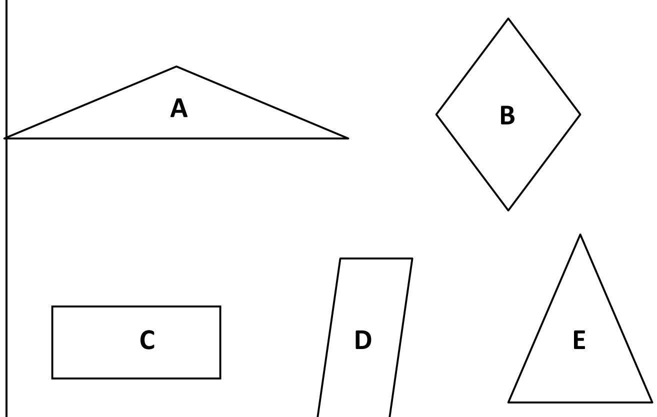
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
5. Mrs Smith sells hats in five different sizes. Put these hats in order, starting with the smallest size. $7\frac{1}{8}$ $7\frac{3}{4}$ $7\frac{5}{8}$ $7\frac{1}{2}$ $7\frac{1}{4}$ 						
6. Fill in the missing numbers in the boxes. (a) $\frac{4}{6} = \frac{\square}{3}$ (b) $\frac{1}{\square} = \frac{5}{25}$	(a) (b)					

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
7. Mahfuzah bought a square-shaped cake for Wa'ie's birthday. She cut the cake into 3 equal parts, each for Wa'ie and her other two sisters. Then she realized that she has not any share of the cake yet. How can she cut the cake now so that she would also have an equal share of the cake? (You can present your answer anyway you like) 						
8. Circle the fraction that is equivalent to $\frac{12}{20}$. $\frac{10}{18}$ $\frac{3}{4}$ $\frac{20}{12}$ $\frac{3}{5}$						
9. 1 foot is equal to 12 inches. What fraction of a foot is 9 inches?						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
10. Find the value of $\frac{1}{2} + \frac{1}{2} - 1$						
11. Approximately how much of the circle is shaded? 						
12. Calculate: $\frac{3}{8} - \frac{1}{4}$						

	Answer	I’m certain I’m right	I think I’m right	I’ve got 50-50 chance of being right	I think I’m wrong	I’m certain I’m wrong
13. Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has $\frac{1}{2}$ loaf of bread and Irah also has $\frac{1}{2}$ loaf of bread. How much bread do they have left once they use it to make the pudding?						
14. Draw as many different ways of dividing the shape below into 4 equal parts 						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
15. Evaluate $\frac{1}{4} \times 12$						
16. Fitri's height is 192 cm. Jaidi's height is $\frac{2}{5}$ of Fitri's height. Find their total height.						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
17. Abdul bought a box of marbles. When he opened it he found that he has 12 marbles, 3 of them are blue, 5 are green, and 4 are red. He decided to give the blue marbles to Amin, what fraction of his marble did he give away?						
18. Which shape can be fully covered by the two shaded triangles? You can choose more than one answer.  						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
19. Abdullah had 12 apples. He gave $\frac{2}{3}$ of them to his mother. How many apples did Abdullah still have left?						
20. Zainal bought $\frac{1}{2}$ kg of seedless grapes, then he gave $\frac{1}{8}$ kg of it to his son, Saiful. What weight of grapes did Zainal have left?						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
21. In the mobile library, $\frac{2}{9}$ of the books are for primary students and $\frac{1}{6}$ are for secondary students. The rest are for kindergartens. What fractions of the books are for kindergartens?						

END OF PAPER

APPENDIX 9: NEWMAN'S REQUEST

The Newman (1983) interview protocol for mathematics word problems was followed. She argued that children needed to go through five stages when solving word problems.

1. *Reading*

   For Newman, reading skill (which she also referred to as "Decoding") is the first stage in problem solving. The fundamental aspect of this skill is that the student should be able to decode words, numerals and symbols in the given questions. If he/she did not recognize them then the information gained may be incorrect for the problems being solved (Newman, 1983).

2. *Comprehension*

   Once the student managed to read the question correctly, he/she then needs to understand the question as a whole. This step is more complicated than decoding. According to Newmam 1983), the student needs to interpret the meaning of the terms in the text and the general meaning of the given question. For example,

   > Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has $\frac{1}{2}$ loaf of bread and Irah also has $\frac{1}{2}$ loaf of bread. How much bread do they have left once they use it to make the pudding?

   Reading the question, the students should be able to understand the relations of words, numbers and symbols in this context. The relations of "$\frac{1}{2}$, $\frac{1}{2}$ and 1" and the meaning of the words "has" and "use" must be understood.

3. *Transformation*

   Subsequently, the student then has to choose the appropriate mathematical process to calculate the answer. He/she has then to interpret the meaning that is gained from the written question into a mathematical form that he/she can identify with and use (Newman, 1983). In the above example, the student has to choose the correct symbols for the word “has” and “use”. Once they have chosen the correct operations, the student has to translate successfully from the language of the task so that a correct answer can be calculated.

4. *Process Skills*

   Next, the student needs to be able to carry out the operation.

5. *Encoding*

   Once the student has gone through all these stages, then the final task is for him/her to state the correct answer. In a test or examination, the student must satisfy the examiner by writing the answer in the correct way.

In an interview situation, each interviewee attempted a question and then the interviewer made the following requests of the student with respect to that question:

1. “Please read the question to me”— the Newman “Decoding” request;
2. “What is the question asking you to find out?”— the Newman “Comprehension” request;
3. “Please explain the method you used when answering the question”— the Newman “Transformation” request;
4. “Talk aloud to me, and tell me what you actually did when you were answering the question”— the Newman “Process Skills” request; and
5. “Please write down your answer to the question”— the Newman “Encoding” request.

APPENDIX 10: Post-Teaching Fractions Test

MAIN STUDY

POST-TEST

PAPER AND PENCIL TEST

ON

FRACTIONS

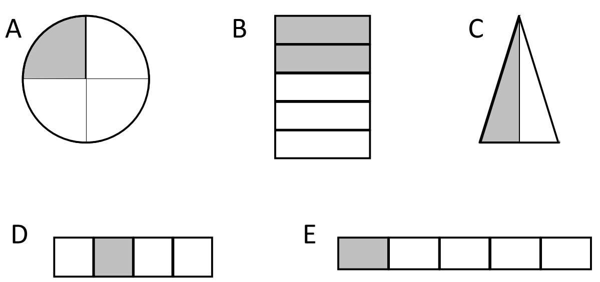
DURATION: 70 minutes

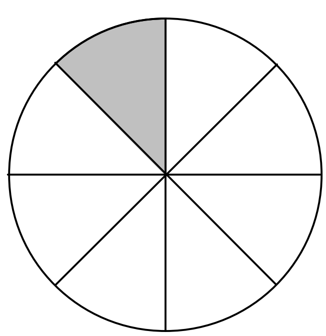
INSTRUCTIONS to students

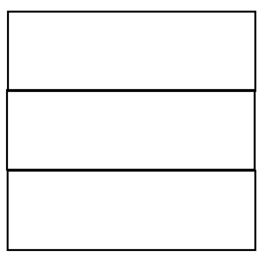
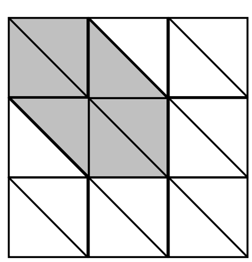
1. Work out your answers in the space underneath each question.
2. Write your answer either on the question itself or in the column marked "Answer".
3. For every question, you must put a tick in one column, to show how you feel about your answer.

Name: ____

Age: __________ Class: ______________ Date: __________

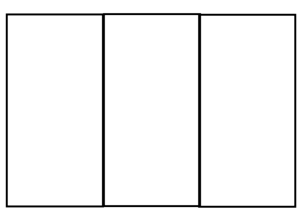
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
Example: $1 + \frac{1}{2}$	$1\frac{1}{2}$	✓				
1. Circle the fraction that has the **same** value as $\frac{2}{3}$: $\frac{3}{4}$ $\frac{1}{2}$ $\frac{3}{2}$ $\frac{4}{6}$ $\frac{1}{3}$						
2. What is $\frac{1}{3} \div 4$?						
3. Tick (✓) the shape or shapes that have **one-quarter** ($\frac{1}{4}$) shaded. 						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
4. Calculate: $\frac{3}{8} + \frac{2}{8}$						
5. In this sequence of numbers, the next number is twice the previous number. Fill in the three missing numbers. □ □ 1 2 4 8 □						
6. How many **extra** parts do you need to shade so that exactly **half** of the circle is shaded? 						

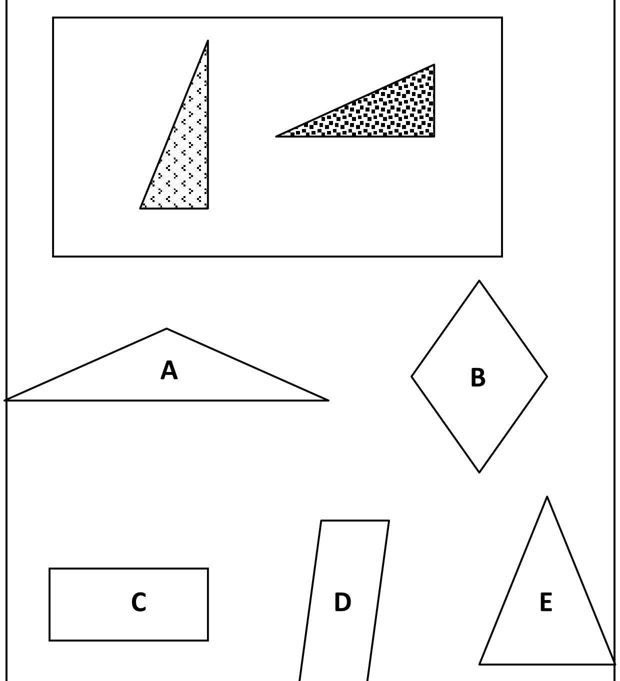
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
7. Mahfuzah bought a square-shaped cake for Wa'ie's birthday. She cut the cake into 3 equal parts, each for Wa'ie and her other two sisters. Then she realized that she has not any share of the cake yet. How can she cut the cake now so that she would also have an equal share of the cake? (You can present your answer anyway you like) 						
8. What fraction of this shape is shaded? 						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
9. 1 foot is equal to 12 inches. What fraction of a foot is 9 inches?						
10. Find the value of $\frac{1}{2} + \frac{1}{2} - 1$						
11. Fill in the missing numbers. (a) 50% of ☐ = 16 (b) A quarter of ☐ = 16						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
12. 4 loaves of bread are used to feed 6 animals at the zoo. (a) How many animals can you feed with 6 loaves? (b) How many loaves of bread are needed to feed 15 animals?						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
13. Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has $\frac{1}{2}$ loaf of bread and Irah also has $\frac{1}{2}$ loaf of bread. How much bread do they have left once they use it to make the pudding?						
14. Draw as many different ways of dividing the shape below into 4 equal parts 						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
15. Evaluate $\frac{1}{4} \times 12$						
16. Fitri's height is 192 cm. Jaidi's height is $\frac{2}{5}$ of Fitri's height. Find their total height.						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
17. Abdul bought a box of marbles. When he opened it he found that he has 12 marbles, 3 of them are blue, 5 are green, and 4 are red. He decided to give the blue marbles to Amin, what fraction of his marble did he give away?						
18. Which shape can be fully covered by the two shaded triangles? You can choose more than one answer. 						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
19. Abdullah had 12 apples. He gave $\frac{2}{3}$ of them to his mother. How many apples did Abdullah still have left?						
20. Zainal bought $\frac{1}{2}$ kg of seedless grapes, then he gave $\frac{1}{8}$ kg of it to his son, Saiful. What weight of grapes did Zainal have left?						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
21. In the mobile library, $\frac{2}{9}$ of the books are for primary students and $\frac{1}{6}$ are for secondary students. The rest are for kindergartens. What fractions of the books are for kindergartens?						

END OF PAPER

APPENDIX 11: Delayed Post-Teaching Fractions Test

MAIN STUDY

DELAYED POST-TEST

PAPER AND PENCIL TEST

ON

FRACTIONS

DURATION: 70 minutes

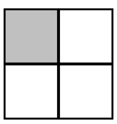
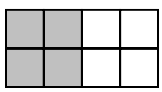
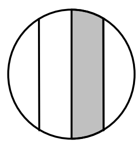
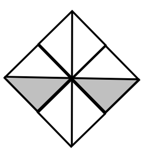
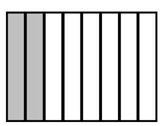
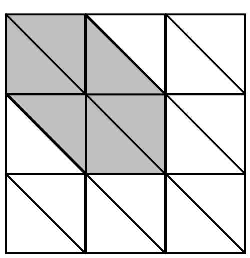
INSTRUCTIONS to students

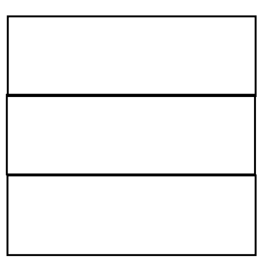
1. Work out your answers in the space underneath each question.
2. Write your answer either on the question itself or in the column marked "Answer".
3. For every question, you must put a tick in one column, to show how you feel about your answer.

Name: ____

Age: __________ Class: ______________ Date: __________

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
Example: $1 + \frac{1}{2}$	$1\frac{1}{2}$	✓				
1. Circle the fraction that has the **same** value as $\frac{2}{3}$: $\frac{3}{4}$ $\frac{1}{2}$ $\frac{3}{2}$ $\frac{4}{6}$ $\frac{1}{3}$						
2. What is $\frac{1}{3} \div 4$?						

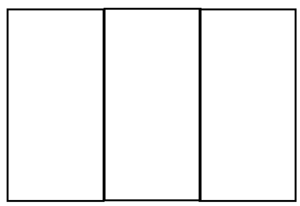
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
3. Calculate: $\frac{3}{8} + \frac{2}{8}$						
4. Circle **all** the shapes which are **one-quarter** ($\frac{1}{4}$) shaded.  A  B  C  D  E						
5. What fraction of this shape is shaded? 						

	Answer	I’m certain I’m right	I think I’m right	I’ve got 50-50 chance of being right	I think I’m wrong	I’m certain I’m wrong
6. Fill in the missing numbers in the boxes. (a) $\frac{4}{6} = \frac{\square}{3}$ (b) $\frac{1}{\square} = \frac{5}{25}$	(a) (b)					
7. Mahfuzah bought a square-shaped cake for Wa’ie’s birthday. She cut the cake into 3 equal parts, each for Wa’ie and her other two sisters. Then she realized that she has not any share of the cake yet. How can she cut the cake now so that she would also have an equal share of the cake? (You can present your answer anyway you like) 						

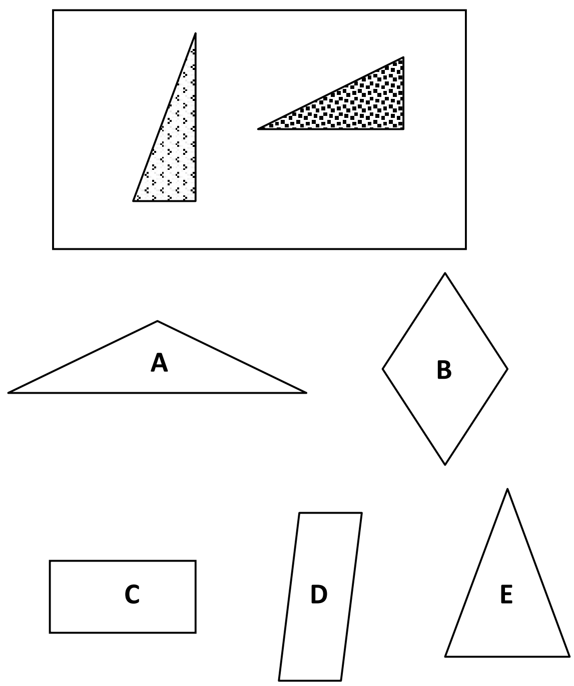
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
8. Fill in the missing numbers. (a) 50% of ☐ = 16 (b) A quarter of ☐ = 16						
9. 1 foot is equal to 12 inches. What fraction of a foot is 9 inches?						
10. Find the value of $\frac{1}{2} + \frac{1}{2} - 1$						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
11. 4 loaves of bread are used to feed 6 animals at the zoo. (a) How many animals can you feed with 6 loaves? (b) How many loaves of bread are needed to feed 15 animals?						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
12. Calculate: $\frac{3}{8} - \frac{1}{4}$						
13. Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has $\frac{1}{2}$ loaf of bread and Irah also has $\frac{1}{2}$ loaf of bread. How much bread do they have left once they use it to make the pudding?						

	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
14. Draw as many different ways of dividing the shape below into 4 equal parts 						
15. Evaluate $\frac{1}{4} \times 12$						

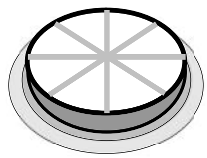
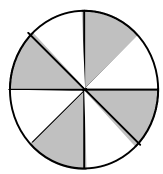
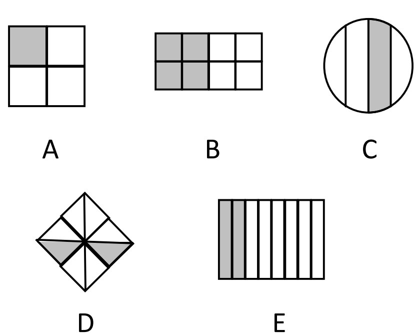
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
16. Fitri's height is 192 cm. Jaidi's height is $\frac{2}{5}$ of Fitri's height. Find their total height.						
17. Abdul bought a box of marbles. When he opened it he found that he has 12 marbles, 3 of them are blue, 5 are green, and 4 are red. He decided to give the blue marbles to Amin, what fraction of his marble did he give away?						

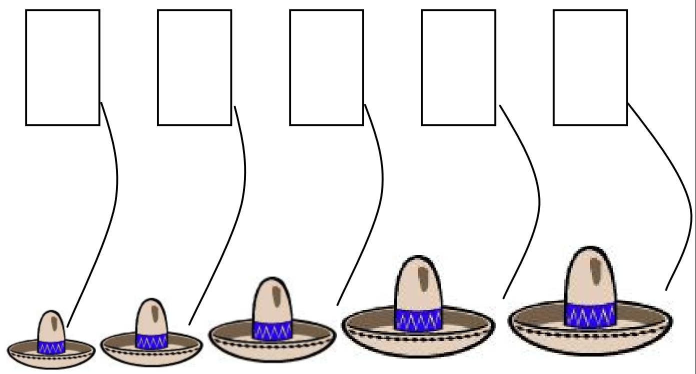
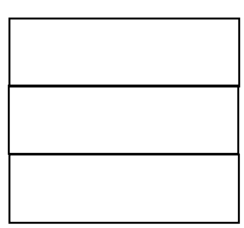
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
18. Which shape can be fully covered by the two shaded triangles? You can choose more than one answer. 						
19. Abdullah had 12 apples. He gave $\frac{2}{3}$ of them to his mother. How many apples did Abdullah still have left?						

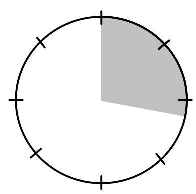
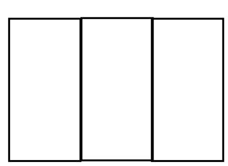
	Answer	I'm certain I'm right	I think I'm right	I've got 50-50 chance of being right	I think I'm wrong	I'm certain I'm wrong
20. Zainal bought $\frac{1}{2}$ kg of seedless grapes, then he gave $\frac{1}{8}$ kg of it to his son, Saiful. What weight of grapes did Zainal have left?						
21. In the mobile library, $\frac{2}{9}$ of the books are for primary students and $\frac{1}{6}$ are for secondary students. The rest are for kindergartens. What fractions of the books are for kindergartens?						

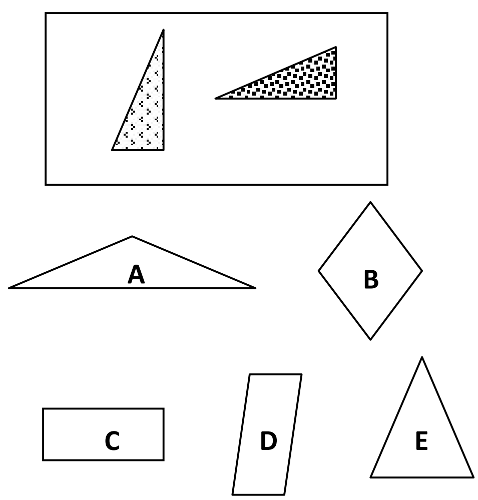
END OF PAPER

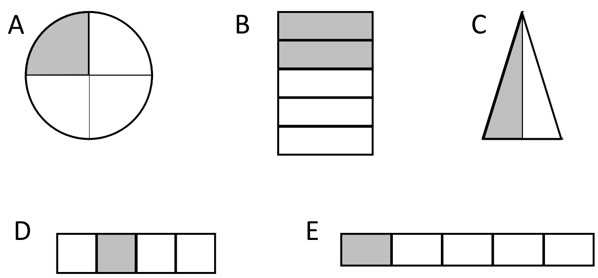
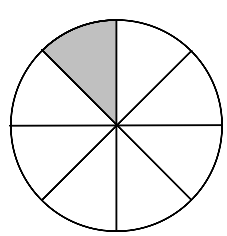
APPENDIX 12: Distribution of Questions in the Three Performance Test Papers

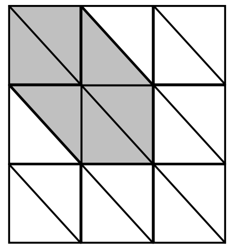
Questions	PRE	POST	DELAYED POST
Shade **one-quarter** ($\frac{1}{4}$) of the cake. 	1		
What is $\frac{1}{3} \div 4$?	2	2	2
Look at this circle:  Tick (✓) **all** of the statements that are true: A. $\frac{1}{2}$ of the circle is shaded B. $\frac{4}{4}$ of the circle is shaded C. $\frac{2}{4}$ of the circle is shaded D. $\frac{1}{4}$ of the circle is shaded E. $\frac{4}{8}$ of the circle is shaded	3		
Circle **all** the shapes which are **one-quarter** ($\frac{1}{4}$) shaded. 	4		4

Questions	PRE	POST	DELAYED POST
Mrs Smith sells hats in five different sizes. Put these hats in order, starting with the smallest size. $7\frac{1}{8}$ $7\frac{3}{4}$ $7\frac{5}{8}$ $7\frac{1}{2}$ $7\frac{1}{4}$ 	5		
Fill in the missing numbers in the boxes. (a) $\frac{4}{6} = \frac{\square}{3}$ (b) $\frac{1}{\square} = \frac{5}{25}$	6a 6b		6a 6b
Mahfuzah bought a square-shaped cake for Wa'ie's birthday. She cut the cake into 3 equal parts, each for Wa'ie and her other two sisters. Then she realized that she has not any share of the cake yet. How can she cut the cake now so that she would also have an equal share of the cake? (You can present your answer anyway you like) 	7	7	7

Questions	PRE	POST	DELAYED POST
Circle the fraction that is equivalent to $\frac{12}{20}$. $\frac{10}{18}$ $\frac{3}{4}$ $\frac{20}{12}$ $\frac{3}{5}$	8		
1 foot is equal to 12 inches. What fraction of a foot is 9 inches?	9	9	9
Find the value of $\frac{1}{2} + \frac{1}{2} - 1$	10	10	10
Approximately how much of the circle is shaded? 	11		
Calculate: $\frac{3}{8} - \frac{1}{4}$	12		12
Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has $\frac{1}{2}$ loaf of bread and Irah also has $\frac{1}{2}$ loaf of bread. How much bread do they have left once they use it to make the pudding?	13	13	13
Draw as many different ways of dividing the shape below into 4 equal parts 	14	14	14
Evaluate $\frac{1}{4} \times 12$	15	15	15

Questions	PRE	POST	DELAYED POST
Fitri's height is 192 cm. Jaidi's height is $\frac{2}{5}$ of Fitri's height. Find their total height.	16	16	16
Abdul bought a box of marbles. When he opened it he found that he has 12 marbles, 3 of them are blue, 5 are green, and 4 are red. He decided to give the blue marbles to Amin, what fraction of his marble did he give away?	17	17	17
Which shape can be fully covered by the two shaded triangles? You can choose more than one answer. 	18	18	18
Abdullah had 12 apples. He gave $\frac{2}{3}$ of them to his mother. How many apples did Abdullah still have left?	19	19	19
Zainal bought $\frac{1}{2}$ kg of seedless grapes, then he gave $\frac{1}{8}$ kg of it to his son, Saiful. What weight of grapes did Zainal have left?	20	20	20
In the mobile library, $\frac{2}{9}$ of the books are for primary students and $\frac{1}{6}$ are for secondary students. The rest are for kindergartens. What fractions of the books are for kindergartens?	21	21	21

Questions	PRE	POST	DELAYED POST
Circle the fraction that has the **same** value as $\frac{2}{3}$: $\frac{3}{4}$ $\frac{1}{2}$ $\frac{3}{2}$ $\frac{4}{6}$ $\frac{1}{3}$		1	1
Tick (✓) the shape or shapes that have **one-quarter** ($\frac{1}{4}$) shaded.  		3	
Calculate: $\frac{3}{8} + \frac{2}{8}$		4	3
In this sequence of numbers, the next number is twice the previous number. Fill in the three missing numbers. ☐ ☐ 1 2 4 8 ☐		5	
How many **extra** parts do you need to shade so that exactly **half** of the circle is shaded? 		6	

Questions	PRE	POST	DELAYED POST
What fraction of this shape is shaded? 		8	5
Fill in the missing numbers. (a) 50% of ☐ = 16 (b) A quarter of ☐ = 16		11a 11b	8a 8b
4 loaves of bread are used to feed 6 animals at the zoo. (a) How many animals can you feed with 6 loaves? (b) How many loaves of bread are needed to feed 15 animals?		12a 12b	11a 11b

NOTE: In the *Post-Teaching Fractions Test* paper, Q7 and Q8 (see Appendix 5) have been swapped

APPENDIX 13: Threats to Internal and External Validity of the QED

The threats to internal and external validity, based on the threats listed by Robson (2002) and Mertens (1998) are discussed separately in this appendix. Both the internal and external validity will be individually examined closely to establish whether it affects the design or not, and in what way does it affects it if any.

a. Internal Validity

The following are the 12 threats as cited in Robson (2002), each one is examined and discussed:

> 1. *History.* Things that have changed in the participants' environments other than those forming a direct part of the enquiry (e.g., occurrence of major air disaster during study of effectiveness of desensitization programme on persons with fear of air travel). (Robson, 2002, p. 70)

Mertens (1998) argued that "history can be controlled by having a control group that is exposed to the same events during the study as the experimental group, with the exception of the treatment," hence for that reason this threat is not present in this study because of the presence of a control group.

> 2. *Testing.* Changes occurring as a result of practice and experience gained by participants on any pre-tests (e.g., asking opinions about factory farming of animals pre some intervention may lead respondents to think about the issues and develop more negative attitudes). (Robson, 2002, p. 70)

This threat was not an issue in this study since all the students took both the pre-test and the post-tests. The time delay between the two tests was about seven weeks. In addition, Mertens (1998) believed that "all the children in the study took both pre- and posttest, so the effects should balance out."

> 3. *Instrumentation.* Some aspect(s) of the way participants were measured changed between pre-test and post-test (e.g., raters in observational study using a wider or narrower definition of a particular behaviour as they get more familiar with the situation). (Robson, 2002, p. 70)

> 4. *Regression*. If participants are chosen because they are unusual or atypical (e.g., high scorers), later testing will tend to give less unusual scores ('regression to the mean'); e.g., an intervention programme with pupils with learning difficulties where ten highest-scoring pupils in a special unit are matched with ten of the lowest-scoring pupils in a mainstream school–

> regression effects will tend to show the former performing relatively worse on a subsequent test. (Robson, 2002, p. 70)

Again this is not a threat in the study as the pre-test was not used to select participants who are atypical (lowest or highest ends of the normal curve).

> 5. *Mortality*. Participants dropping out of the study (e.g., in study of adult literacy programme–selective drop-out of those who are making little progress). (Robson, 2002, p. 70)

From Table 3.2, three students presented experimental mortality for the E3 group, one for the Experimental 1 group, and none for the Experimental 2 group. As mentioned before the reason for the experimental mortality was either because they have transferred out of the school, or have been repeatedly absent from school and have missed all the tests. So as the mortality is very low, it can be argued that this is not a threat to the QED.

> 6. *Maturation*. Growth, change or development in participants unrelated to the treatment in the enquiry (e.g. evaluating extended athletics training programme with teenagers–intervening changes in height, weight and general maturity). (Robson, 2002, p. 70)

Similar to the first threat, this threat is also excluded because of the use of a control group.

> 7. *Selection*. Initial differences between groups prior to involvement in the enquiry (e.g., through use of arbitrary non-random rule to produce two groups: ensure they differ in one respect which may correlate with others). (Robson, 2002, p.70)

In terms, year group, school, race and so on, there was no difference among the QED participants. So in that aspect this is also not a threat to the study.

> 8. *Selection by maturation interaction*. Predisposition of groups to grow apart (or together if initially different); for example, use of groups of boys and girls initially matched on physical strength in a study of a fitness programme. (Robson, 2002, p. 71)

Again, the use of the control group controlled any differences in maturation among the students, thus this did not posed a threat to the QED.

> 9. *Ambiguity about causal direction*. Does A cause B, B cause A? (e.g., in any correlational study, unless it is known that A precedes B, or vice versa–or some other logical analysis is possible). (Robson, 2002, p. 71)

10. *Diffusion of treatments*. When one group learns information or otherwise inadvertently receives aspects of a treatment intended only for a second group (e.g., in a quasi-experimental study of two classes in the same school). (Robson, 2002, p. 71)

Although in the study there are two experimental groups and a control group, this threat was minimal and did not affect the QED for the reasons provided below:

- The students were not taught any topics involving fractions while the study was in progress;
- Therefore, because of the above the researcher believed that there was no reason for the students to discuss about fractions, even during their mathematics lessons;
- The activities for the experimental instructions were confusing enough for the Experimental 1 students themselves, so again I do not see any reason why they would want to talk about that with their peers; and
- Finally, the only other time all the students meet with their friends from other classes is during break time, which is only 20 minutes, and I am confident that they have other interesting things to talk about with their friends, other than fractions, within that short time.

For those reasons, I am confident that this threat is not a threat to the QED.

11. *Compensatory equalization of treatments*. If one group receives 'special' treatment there will be organizational and other pressures for other control group to receive it (e.g., nurses in a hospital study may improve the treatment of a group on grounds of fairness). (Robson, 2002, p. 71)

This threat is excluded because neither the school administrators nor the teachers gave any extra notes or resources to the control group.

12. *Compensatory rivalry*. As above but an effect on the participants themselves (referred to as 'John Henry' effect after the steel worker who killed himself through over-exertion to prove his superiority to the new steam drill); e.g., when a group in an organization sees itself under threat from a planned change in another part of the organization and improves performance). (Robson, 2002, p. 71)

Even though the participants in the Experimental 1 group came from different classes, there was no evidence to show that this is a threat. This is because the researcher did not see any evidence, or feel at any point that the different groups are competing against each other.

b. **External Validity**

The following are the 10 threats as listed by Mertens (1998), each one is examined and discussed:

1. *Explicit description of the experimental treatment.* The independent variable must be sufficiently described so that the reader can reproduce it. (Mertens, 1998, p. 68)

This external validity does not affect the external validity of the QED as a detailed description of the activities in the experimental teaching has been presented in section 4.2.

2. *Multiple-treatment interference.* If participants receive more than one treatment, it is not possible to say which of the treatments, or which combinations of the treatments is necessary to bring about the desired result. (Mertens, 1998, p. 68)

All the participants receive only one treatment, except for the control group who did not received any treatment at all. Hence this is not a threat.

3. *Hawthorne effect.* . . . the idea of receiving special attention, of being singled out to participate in the study, was enough motivation to increase productivity. (Mertens, 1998, p. 68)

Experimental 2 group treatment is in place was to provide evidence that this threat is not affecting the design of the study, but again one cannot be absolutely sure that the Experimental 1 groups' increase in productivity is because they were receiving special attention.

4. *Novelty and disruption effects.* A new treatment may produce positive results simply because it is novel, or the opposite may be true. A new treatment may not be effective initially because it causes a disruption in normal activities, but once it is assimilated into the system, it could become quite effective. (Mertens, 1998, p. 68)

There was no reason to suspect that novelty and disruption effects occurred in the study.

5. *Experimenter effect.* The effectiveness of a treatment may depend on the specific individual who administers it. (Mertens, 1998, p. 68)

This treat was not an issue here because the researcher herself administered all the treatments in the QED.

6. *Pretest sensitization. Participants who take a pretest may be more sensitized to the treatment than individuals who experience the treatment without taking a pretest. This is especially true for pretests that ask the participants to reflect on and express their attitudes toward a phenomena.* (Mertens, 1998, p. 68)

For this threat the only question that could become a threat is Question 9, which is one of the thirteen common questions, but I argue there is no evidence that it did pose a threat to the QED because the nature of the knowledge in the question is not the main core of the treatment. Regarding participants being asked to reflect their attitudes towards a phenomena, again I believed that this is also not a threat here even though there were confidence scales linked to all the three performance tests, but they are only asking students to respond on how confident are they of the answer they have given at that moment.

7. *Posttest sensitization.* This is similar to pretest sensitization in that simply taking a posttest can influence a participants' response to the treatment. (Mertens, 1998, p. 68)

Again, Question 9 but I did not think this is a threat because of the same reason as above. Even though I did not inform the students that they will be a post-test, they could have suspected that there will be one, and they could have paid more attention during the instruction. Thus, it is possible that this threat could affect the study. Nevertheless, in order to further minimize the effect of this threat, for all the three pencil-and-paper instruments, there are only thirteen items which are common so that any practice effect could also be reduced. However, this is unavoidable because I wanted to measure the effect of the treatment.

8. *Interaction of history and treatment effects.* An experiment is conducted in a particular time replete with contextual factors that cannot be exactly duplicated in another setting. If specific historical influences are present in a situation (e.g., unusually low morale because of budget cuts), the treatments may not generalize to another situation. (Mertens, 1998, p. 68-69)

There are only two particular features of the QED that could potentially pose a threat to the study:

i. The audio taping and the video taping of the experimental lessons. Nevertheless, the students have been given assurance that the audio tape and video tapes are solely for me to help me review what went on during the lessons. I am quite confident that the presence of these taping machines did not affect the way they acted, and they seemed comfortable enough with the camera.

ii. The small group teaching instead of a whole class teaching.

Of these two special features, only the later special feature affected the QED which is discussed in the limitations of this study.

> 9. *Measurement of the dependent variable.* The effectiveness of the program may depend on the type of measurement used in the study. For example, one study of the effects of mainstreaming might use multiple choice tests and conclude that mainstreaming doesn't work; another study uses teachers' perceptions of behavior change might conclude that it is effective. (Mertens, 1998, p. 69)

With regard to the type of measurement used, each item in all the measurement instruments have been presented and discussed in section 3.5, whereas in measurement of the dependent variable in all the tests only the *usual* types of items that the students are familiar with (in term of content, language, symbols, etc) are selected and used. Thus, this is not a threat in the study.

> 10. *Interaction of time of measurement and treatment effects*. The timing of the administration of the posttest may influence the results. For example, different results may be obtained if the posttest is given immediately after the treatment as opposed to a week or a month afterward. (Mertens, 1998, p. 69)

This threat is not affecting the QED because the post-test was administered one day after the completion of the experimental teaching, and all QED participants sat for the test at the same time. As for the delayed post-test, again all the students sat for it about seven and a half weeks after the completion of the experimental teaching.

APPENDIX 14: BERA Ethical Guidelines for Educational Research (2004)

Here I am only reproducing the guidelines that are relevant to the participants only.

GUIDELINES

Responsibilities to Participants

1. The participants in research may be the active or passive subjects of such processes as observation, inquiry, experiment or test. They may be collaborators or colleagues in the research process or they may simply be part of the context e.g. where students are part of the context but not the subjects of a teacher's research into his or her own professional practice.

2. The Association considers that educational researchers should operate within an ethic of respect for any persons involved directly or indirectly in the research they are undertaking, regardless of age, sex, race, religion, political beliefs and lifestyle or any other significant difference between such persons and the researchers themselves or other participants in the research. This ethic of respect implies the following responsibilities on the part of the researchers.

Voluntary Informed Consent

3. The Association takes voluntary informed consent to be the condition in which participants understand and agree to their participation without any duress, prior to the research getting underway.

4. Researchers must take the steps necessary to ensure that all participants in the research understand the process in which they are to be engaged, including why their participation is necessary, how it will be used and how and to whom it will be reported. Researchers engaged in action research must consider the extent to which their own reflective research impinges on others, for example in the case of the dual role of teacher and researcher and the impact on students and colleagues. Dual role may also introduce explicit tensions in areas such as confidentiality and must be addressed accordingly.

Deception

5. The securing of participants' voluntary informed consent, before research gets underway, is considered the norm for the conduct of research. Researchers must therefore avoid deception or subterfuge unless their research design specifically requires it to ensure that the appropriate data is collected or that the welfare of the researchers is not put in jeopardy. Decisions to use deception or subterfuge in research must be the subject of full deliberation and subsequent disclosure in reporting. The Association recommends that approval for this course of action should be obtained from a local or institution ethics committee. In any event, if it

possible to do so, researchers must seek consent on a post-hoc basis in cases where it was not desirable to seek it before undertaking the research.

Right to Withdraw

6. Researchers must recognize the right of any participant to withdraw from the research for any or no reason, and at any time, and they must inform them of this right. In all such circumstances researchers must examine their own actions to assess whether they have contributed to the decision to withdraw and whether a change of approach might persuade the participants to re-engage. In most cases the appropriate course of action will be for the researchers to accept the participants' decision to withdraw. Decisions to persuade them to re-engage must be taken with care. Researchers must not use coercion or duress of any form to persuade participants to re-engage with the work. In cases where participants are required by a contractual obligation to participate e.g. when mandated as part of their employment to facilitate an evaluation study, researchers may, however, have proper recourse to a third party (e.g. the employing authority) to request compliance with a contract.

Children, Vulnerable Young People and Vulnerable Adults

7. The Association requires researchers to comply with Article 3 and 12 of the United Nations Convention on the Rights of the Child. Article 3 requires that in all actions concerning children, the best interests of the child must be the primary consideration. Article 12 requires that children who are capable of forming their own views should be granted the right to express their views freely in all matters affecting them, commensurate with their age and maturity. Children should therefore be facilitated to give fully informed consent.

8. The Association considers that the spirit of Articles 3 and 12 above should also apply in research contexts involving young people and vulnerable adults.

9. In the case of participants whose age, intellectual capability or other vulnerable circumstance may limit the extent to which they can be expected to understand or agree voluntarily to undertake their role, researchers must fully explore alternative ways in which they can be enabled to make authentic responses. In such circumstances, researchers must also seek collaboration and approval of those who act in guardianship (e.g. parents) or as 'responsible others' (i.e. those who have responsibility for the welfare and well-being of the participants e.g. social workers).

10. Researchers must ensure that they themselves, and any collaborators or research assistants and students under their supervision, comply with legal requirements in relation to working with school children or vulnerable young people and adults.

11. Researchers must recognize that participants may experience distress or discomfort in the research process and must take all necessary steps to reduce the sense of intrusion and to put them at their ease. They must desist immediately from any actions, ensuing from the research process, that cause emotional or other harm.

12. Researchers must recognize concerns relating to the 'bureaucratic burden' of much research, especially survey research, and must seek to minimize the impact of their research on the normal working and workloads of participants.

Incentives

13. Researchers' use of incentives to encourage participation must be commensurate with good sense and must avoid choices which in themselves have undesirable effects (e.g. the health aspects of offering cigarettes to young offenders or sweets to school-children). They must also acknowledge that the use of incentives in the design and reporting to the research may be problematic; for example where their use has the potential to create a bias in sampling or in participant responses.

Detriment Arising from Participation in Research

14. Researchers must make known to the participants (or their guardians or responsible others) any predictable detriment arising from the process or findings of the research. Any unexpected detriment to participants, which arises during the research, must be brought immediately to their attention or to the attention of their guardians or responsible others as appropriate.

15. Researchers must take steps to minimize the effects of designs that advantage or are perceived to advantage one group of participants over others e.g. in an experimental or quasi-experimental study in which the treatment is viewed as a desirable intervention and which by definition is not available to the control or comparison group respectively.

16. The confidential and anonymous treatment of participants' data is considered the norm for the conduct of research. Researchers must recognize the participants' entitlement to privacy and must accord them their rights to confidentiality and anonymity, unless they or their guardians or responsible others, specifically and willingly waive that right. In such circumstances it is in the researchers' interest to have such a waiver in writing. Conversely, researchers must also recognize participants' rights to be identified with any publication of their original works or other inputs, if they so wish. In some contexts it will be the expectation of participants to be so identified.

17. Researchers must comply with the legal requirements in relation to the storage and use of personal data as set down by the Data Protection Act (1998) and any subsequent similar acts. In essence people are entitled to know how and why their personal data is being stored, to what uses it is being put and to whom it may be made available. Researchers must have participants' permission to disclose personal information to third parties and are required to ensure that such parties are permitted to have access to the information. They are also required independently to confirm the identity of such persons and must keep a record of any disclosures. Disclosure may be written, electronic, verbal or any visual means.

18. The Data Protection Act also confers the right to private citizens to have access to any personal data that is stored in relation to them. Researchers seeking to exploit legal exclusions to these rights must have a clear justification for so doing.

19. Researchers must ensure that data is kept securely and that the form of any publication, including publication on the Internet, does not directly or indirectly lead to a breach of agreed confidentiality and anonymity.

Disclosure

20. Researchers who judge that the effect of the agreements they have made with participants, on confidentiality and anonymity, will allow the continuation of illegal behaviour, which has come to light in the course of the research, must carefully consider making disclosure to the appropriate authorities. If the behaviour is likely to be harmful to the participants or to others, the researches must also consider disclosure. Insofar as it does not undermine or obviate the disclosure, researchers must apprise the participants or their guardians or responsible others of their intentions and reasons for disclosure.

21. At all times the decision to override agreements on confidentiality and anonymity must be taken after careful and thorough deliberation. In such circumstances it is in the researchers' interests to make contemporaneous notes on decisions and the reasoning behind them, in case a misconduct complaint or other serious consequences arises.

22. The Association considers it good practice for researchers to debrief participants at the conclusion of the research and to provide them with copies of any reports or other publications arising from their participation. Where the scale of the research makes such a consideration impractical, alternative means such as a website should be used to ensure participants are informed of the outcomes.

APPENDIX 15: Parent's Consent Form

MANCHESTER 1824
The University of Manchester

PhD RESEARCH STUDY

To all parents concerned,

I am a second year PhD student at the School of Education of the University of Manchester, and have developed a research project on the teaching and learning of fractions - this involves some innovative teaching and learning Mathematical activities which would involve a group of about six 11- or 12-year-olds students working in small groups.

The aim of the activities is to help students within that age group to develop some important – and typically problematic - fraction concepts that would hopefully help them to be able to deal with fractional tasks better such as fraction-equivalence and flexibility-of-unitising: it is an essential part of the design that this research aims to improve learners' understanding, especially for those who have had 'trouble' with fractions. The work is designed ethically from this point of view: **we seek to help**.

I would appreciate it if you could allow your children to participate in the study as your child will benefit from the alternative ways solving fractions and hopefully will improve their abilities to process mathematical problems, particularly in fractions. If, at any stage of the research your child wish to withdraw from the research he/she may do so. At the end of the research, your children's name will not be disclose in the final report, unless you waive that right (Please tick the box below if you wish to waive that right).

☐ I wish to waive the confidentility and anonymity right, and would like my child's name to be acknowledge in the report.

If there are any queries and questions that you would like to ask me, my contact numbers and email address are stated below.

Thank you.

HAJAH ZURINA HAJI HARUN
University of Manchester
United Kingdom.

Contact Number: 6738780236 / 07933410103 (UK)
Email Addresses: Hajah.Haji-harun@postgrad.manchester.ac.uk
zurinaharun@brunet.bn
zurinaharun@hotmail.com
zurinaharun@gmail.com

PENYELIDIKAN PhD

Kepada ibubapa yang berkenaan,

Saya adalah pelajar PhD tahun 2 di *School of Education, University of Manchester*, dan telah menyusun projek penyelidikan pengajaran dan pembelajaran “fractions” yang melibatkan pelbagai pengajaran dan aktiviti inovatif yang melibatkan sekumpulan pelajar dalam lingkungan 11 atau 12 tahun yang belajar secara berkumpulan.

Tujuan aktiviti ini adalah untuk membantu pelajar dalam lingkungan umur tersebut untuk membentuk konsep penting – yang selalunya mendatangkan masalah- “fractions” yang diharapkan akan dapat membantu mereka mengendalikan perkara ynag berkaitan dengan “fractions” dengan lebih baik. Membantu pelajar yang menghadapi masalah “fractions” untuk memahami konsep ini dengan lebih baik adalah perkara yang ditekankan dalam program ini. Program ini disusun dengan hasrat: **Kami ingin membantu.**

Besarlah harapan saya supaya biskita dapat membenarkan anak/anak dibawah jagaan biskita untuk terlibat di dalam penyelidikan ini kerana mereka akan berkesempatan untuk mempelajari cara alternatif untuk menyelesaikan masaalah *fractions* dan diharapkan mereka akan dapat meningkatkan kebolehan mereka memproses permasalahan matematik, terutamanya *fractions*. Jika anak biskita ingin menarik diri dari penyelidikan ini mereka bolehlah berbuat demikian. Di akhir penyelidikan ini nanti nama-nama anak biskita akan di rahasiakan kecuali biskita mengkehendaki nama anak biskita dimasukkan di dalam lapuran tersebut (Sila tandakan kotak di bawah sekiranya biskita mahu nama anak biskita dimasukkan ke dalam lapuran itu).

☐ Saya bersetuju untuk membenarkan nama anak saya dimasukkan ke dalam lapuran penyelidikan ini.

Sekiranya biskita ada pertanyaan biskita bolehlah menghubungi saya melalui talian talipon di bawah, atau melalui email.

Terima kasih.

HAJAH ZURINA HAJI HARUN
University of Manchester
United Kingdom.

Contact Number: 6738780236 / 07933410103 (UK)
Email Addresses: Hajah.Haji-harun@postgrad.manchester.ac.uk
zurinaharun@brunet.bn
zurinaharun@hotmail.com
zurinaharun@gmail.com

The University of Manchester

MANCHESTER 1824

CONSENT FORM

Student's Name
Nama Penuntut : ______________________________

Class
Kelas : ______________________________

School
Sekolah : ______________________________

I, ______________________________ (Parent's / Guardian's Name*) willingly allow my (son/daughter/ward*) whose name is as stated above to be involved in the research study conducted by the researcher named below.

Saya, ______________________________ (Nama ibu-bapa / Penjaga) membenarkan (anak /anak di bawah jagaan*) saya seperti nama yang tercatat di atas untuk terlibat dalam program penyelidikan yang dijalankan oleh penyelidik seperti nama di bawah.*

Parent's / Guardian's Signature
Tandatangan Ibu-bapa / Penjaga : ______________________________

Researcher's Name: HAJAH ZURINA HAJI HARUN
University of Manchester,
United Kingdom

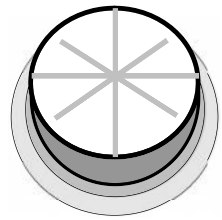
Research Supervisors: Professor Julian Williams & Dr Laura Black

Contact Number: 6738780236 / 07933410103 (UK)
Email Addresses: Hajah.Haji-harun@postgrad.manchester.ac.uk
zurinaharun@brunet.bn
zurinaharun@hotmail.com
zurinaharun@gmail.com

APPENDIX 16: Item Bank

Item 1 (WC): 1//

Shade **one-quarter** ($\frac{1}{4}$) of the cake.



Concerned with whether students could *translate fractions into diagrammatic representations*. This question asked students to shade the correct number of parts of a given diagram in order that the answer would represent a given fraction. This is also what Orton and Forbisher (1996) described as the "part-whole model."

Item 2 (WC): 3//

Look at this circle: Tick (✓) **all** of the statements that are true:

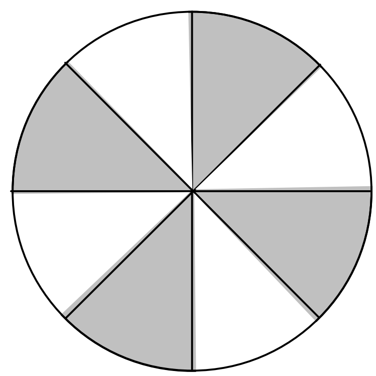
A. $\frac{1}{2}$ of the circle is shaded

B. $\frac{4}{4}$ of the circle is shaded

C. $\frac{2}{4}$ of the circle is shaded

D. $\frac{1}{4}$ of the circle is shaded

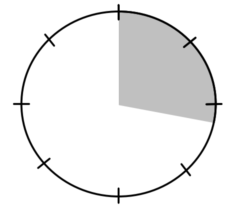
E. $\frac{4}{8}$ of the circle is shaded



Concerned with whether students could *translate diagrammatic representations into fractions*. This question asked students to choose the fraction and its equivalent fractions that correspond to the shaded diagrammatic representation of the fraction. This is also what Orton and Forbisher (1996) described as the "part-whole model."

Item 3 (WC): 11//

Approximately how much of the circle is shaded?

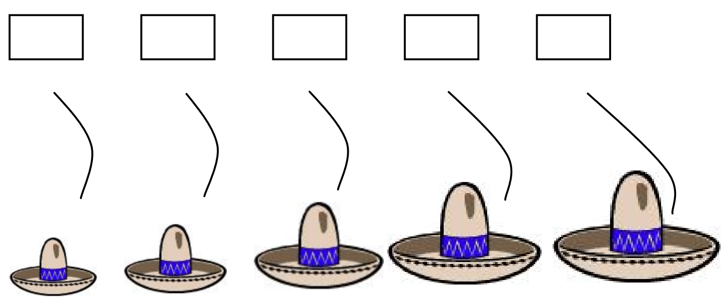


Concerned with whether students could *translate and estimate a given diagrammatic representations into a fraction*. This question asked students to estimate the fraction for the given shaded diagrammatic representation of the fraction. This is also what Orton and Forbisher (1996) described as the "part-whole model."

Item 4 (C): 5//

Mrs Smith sells hats in five different sizes. Put these hats in order, starting with the smallest size.

$7\frac{1}{8}$ $7\frac{3}{4}$ $7\frac{5}{8}$ $7\frac{1}{2}$ $7\frac{1}{4}$



This question involved *ordering of fractions,* where student were asked to order mixed numbers involving fractions with different denominators from the smallest to the biggest. This question could also give a good indicator of students' knowledge on *equivalent fractions* as one possible way of approaching this problem will be by changing all the fractions into their equivalent fractions that would have the same denominators.

Item 5 (WC): 8//

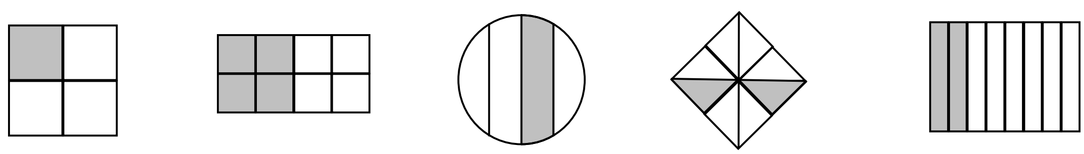
Circle the fraction that is equivalent to $\frac{12}{20}$.

$$\frac{10}{18} \qquad \frac{3}{4} \qquad \frac{20}{12} \qquad \frac{3}{5}$$

Concerned with *equivalent fractions*. In this question students were required to pick a fraction which is equivalent to the given fraction.

Item 6 (WC): 4//4

Circle **all** the shapes which are **one-quarter** ($\frac{1}{4}$) shaded.



A B C D E

Concerned about whether students could *translate fractions into diagrammatic representations* where the question asked students to pick all the correct diagrammatic representation(s) of the given fraction. This is also what Orton and Forbisher (1996) described as the "part-whole model."

Item 7 & 8 (WC): 6a//6a & 6b//6b

Fill in the missing numbers in the boxes.

(a) $\frac{4}{6} = \frac{\square}{3}$ (b) $\frac{1}{\square} = \frac{5}{25}$

This question involved *equivalent fractions* where the questions asked students to find one missing value of a given pair of equivalent fractions.

Item 9 (WC): 12//12

Calculate: $\frac{3}{8} - \frac{1}{4}$

This is a question on the *four operations* where students were asked to perform subtraction of two fractions with different denominators.

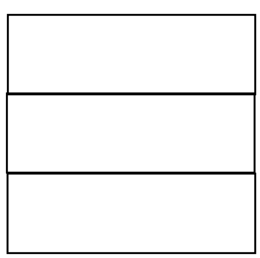
Item 10 (WC): 2/2/2

What is $\frac{1}{3} \div 4$?

Concerned with students' knowledge on the *four operations for fractions,* which involved division of fractions by a whole number

Item 11 (C): 7/7/7

Mahfuzah bought a square-shaped cake for Wa'ie's birthday. She cut the cake into 3 equal parts, each for Wa'ie and her other two sisters. Then she realized that she has not any share of the cake yet. How can she cut the cake now so that she would also have an equal share of the cake? (You can present your answer anyway you like)



Concerned with *sharing* involving division of whole numbers that will give a non-whole number which Orton and Frobisher (1996) categorized as the "quotient model."

Item 12 (C): 9/9/9

1 foot is equal to 12 inches. What fraction of a foot is 9 inches?

Concerned with whether students could *see* the *whole* as a single unit. This is what Orton and Frobisher (1996) categorized as "part-whole model." This question asked students to translate a given measurement (in inches) with respect to the whole (i.e., one foot) in terms of a fraction.

Item 13 (WC): 10/10/10

Find the value of $\frac{1}{2} + \frac{1}{2} - 1$

Concerned with students' knowledge on the *four operations for fractions,* which involved combinations of the four operations, involving addition and subtraction in the same question. This question is the equivalent question to item 14: 13/13 13 but without the context being attached to it.

Item 14 (C): 13/13/13

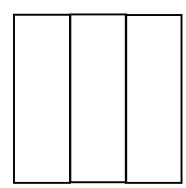
Mumtazah and Irah wanted to make a bread pudding, and they needed 1 loaf of bread. Mumtazah only has $\frac{1}{2}$ loaf of bread and Irah also has $\frac{1}{2}$ loaf of bread. How much bread do they have left once they use it to make the pudding?

Test the students' ability to *interpret word problems involving a fractional quantity,* involving the four operations. This question is actually an exact equivalent question to item 13: 10/10/10 but without any context given to it. It would be interesting, in the researcher's opinion, to compare students' performances and responses (strategies) to these two, exactly the same, questions.

Item 15 (WC):14/14/14

Draw as many different ways of dividing the shape below into 4 equal parts

Equivalent question to item 11: 7/7/7. This question asked students to *divide a given diagram into equal parts* to see strategies that students adopted to arrive at their answer. This is categorized by Orton and Frobisher (1996) as the "quotient model."



Item 16 (WC): 15/15/15

Evaluate $\frac{1}{4} \times 12$

Concerned with students' knowledge on the *four operations for fractions,* which involved multiplication of fractions with a whole number.

Item 17 (C): 16/16/16

Fitri's height is 192 cm. Jaidi's height is $\frac{2}{5}$ of Fitri's height. Find their total height.

Test the students' ability to *interpret word problems involving a fractional quantity*. It involved students to perform the four operations, where they are required to multiply a fraction with a whole number,

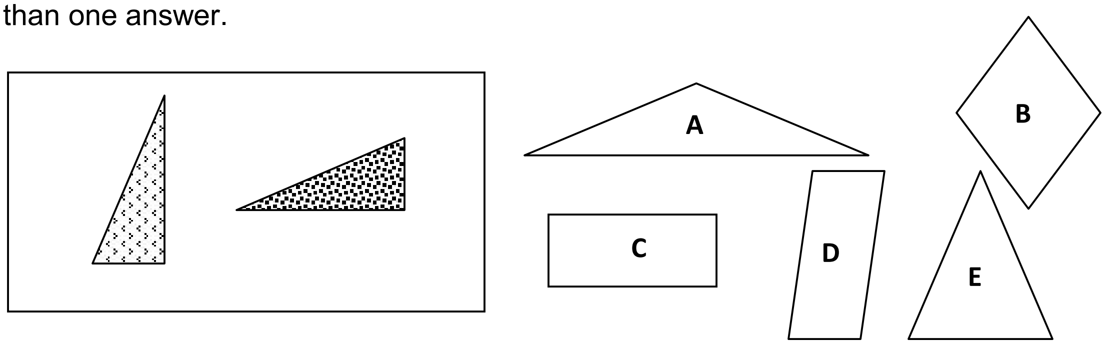
Item 18 (C): 17/17/17

Abdul bought a box of marbles. When he opened it he found that he has 12 marbles, 3 of them are blue, 5 are green, and 4 are red. He decided to give the blue marbles to Amin, what fraction of his marble did he give away?

Concerned with whether students could *see* the *whole* as a single unit. This is what Orton and Frobisher (1996) categorized as "discrete model." This question asked students to write a fraction representing a group of items with respect to the total.

Item 19 (WC): 18/18/18

Which shape can be fully covered by the two shaded triangles? You can choose more than one answer.



Asked students to *identify the whole* of the parts of the given diagram, this, the researcher felt would be an alternative way to find out the students' *view* of a whole. Based on Orton and Frobisher (1996), I would also categorized this as the "part-whole model."

Item 20 (C): 19/19/19

Abdullah had 12 apples. He gave $\frac{2}{3}$ of them to his mother. How many apples did Abdullah still have left?

> Concerned with whether students could *see* the *whole* as a single unit. This is what Orton and Frobisher (1996) categorized as "discrete model." This question required the students to find a whole number answer after a certain fraction of the whole items were removed.

Item 21 (C): 20/20/20

Zainal bought $\frac{1}{2}$ kg of seedless grapes, then he gave $\frac{1}{8}$ kg of it to his son, Saiful. What weight of grapes did Zainal have left?

> Test the students' ability to *interpret word problems involving a fractional quantity*. It involved students to perform the four operations, where it involved subtraction of two fractions of different denominators.

Item 22 (C): 21/21/21

In the mobile library, $\frac{2}{9}$ of the books are for primary students and $\frac{1}{6}$ are for secondary students. The rest are for kindergartens. What fractions of the books are for kindergartens?

> This question is based on the "part-whole model" (Orton and Frobisher, 1996) which also involved the use of the four operations where students were asked to find the fraction which represents a part of the whole, with two fractions of different denominators was given.

Item 23 (WC): /1/1

Circle the fraction that has the **same** value as $\frac{2}{3}$:

$$\frac{3}{4} \quad \frac{1}{2} \quad \frac{3}{2} \quad \frac{4}{6} \quad \frac{1}{3}$$

Test students on their knowledge of *equivalent fractions* where students were asked to select the correct equivalent fraction to the given fraction.

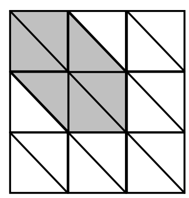
Item 24 (WC): /4/3

Calculate: $\frac{3}{8} + \frac{2}{8}$

This question is about the *four operations* where students were to add two fractions with the same denominators.

Item 25 (WC): /8/5

What fraction of this shape is shaded?



This question is concerned with whether the students could *translate diagrammatic representations into fractions*. This is also what Orton and Forbisher (1996) described as the "part-whole model."

Item 26 & 27 (WC): /11a/8a & /11b/8b

Fill in the missing numbers.

(a) 50% of ☐ = 16 (b) A quarter of ☐ = 16

In these sub-questions the students were asked to *find the original whole* where the fraction and the whole number answer was given. Technically, this involved the four operations involving division of an integer by a fraction, but if the students were able to think of this problem holistically, they would be able to visualized that, for example, half (50%) of what number would give them 16, say.

Item 28 & 29 (C): /12a/11a & /12b/11b

4 loaves of bread are used to feed 6 animals at the zoo.

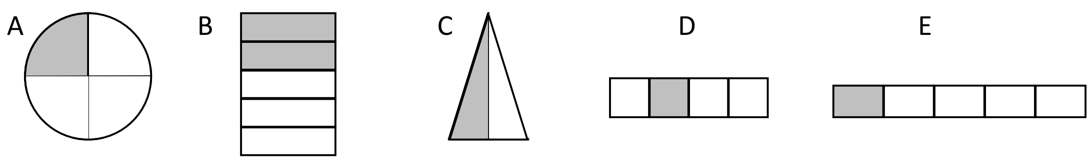
(a) How many animals can you feed with 6 loaves?

(b) How many loaves of bread are needed to feed 15 animals?

These sub-questions involved the *idea of comparing the number of objects in two sets of "objects,"* in this case a certain number of loaves of bread that can feed a certain number of animals. In the main part of the question the ratio between the two "objects," A:B is given. Part (a) of the question asked students to calculate the number of animals that can be fed for a different number of loaves of bread, while the second part of the question asked for the number of loaves of bread required to feed a different number of animals. Orton and Forbisher (1996) called this model as the "ratio model."

Item 30 (WC): /3/

Tick (✓) the shape or shapes that have **one-quarter** ($\frac{1}{4}$) shaded.



Concerned whether students could *translate fractions into diagrammatic representations.* This is what Orton and Forbisher (1996) described as the "part-whole model." This question invite students to choose the diagrammatic representation(s) that represents the given fraction.

Item 31 (WC): /5/

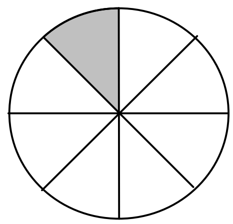
In this sequence of numbers, the next number is twice the previous number.

Fill in the three missing numbers.

This question involved a number sequence where three of the numbers are missing and the students were to find the missing numbers. The researcher has purposely chosen a sequence with integers as the given numbers so that the students could confidently *see* the patterns in the sequence.

Item 32 (WC): /6/

How many **extra** parts do you need to shade so that exactly **half** of the circle is shaded?



Concerned with whether students could *translate fractions into diagrammatic representations.* This question required students to be able to shade the given diagram to correctly represent the given fraction before they could answer the question. This is also what Orton and Forbisher (1996) described as the "part-whole model."