GRAPHS ASSOCIATED WITH THE SPORADIC SIMPLE GROUPS Fi_{24} AND BM

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Our aim is to calculate some graphs associated with two of the larger sporadic simple groups, Fi_{24} and the Baby Monster.

Firstly we calculate the point line collinearity graph for a maximal 2-local geometry of Fi_{24}. If Γ is such a geometry, then the point line collinearity graph $\mathcal{G}$ will be the graph whose vertices are the points in Γ, with any two vertices joined by an edge if and only if they are incident with a common line. We found that the graph has diameter 5 and we give its collapsed adjacency matrix.

We also calculate part of the commuting involution graph, $\mathcal{C}$, for the class $2C$ of the Baby Monster, whose vertex set is the conjugacy class $2C$, with any two elements joined by an edge if and only if they commute. We have managed to place all vertices inside $\mathcal{C}$ whose product with a fixed vertex t does not have 2 power order, with all evidence pointing towards $\mathcal{C}$ having diameter 3.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

The classification of finite simple groups was finally completed in 2004, after more than one hundred years of work, involving hundreds of mathematicians and spanning many tens of thousands of journal pages.

Theorem 1.0.1 (The Classification of Finite Simple Groups)**.** *Let G be a finite simple group. Then G belongs to one of the following families of groups:*

1. *Cyclic groups of prime order.*
2. *Alternating groups of degree at least 5.*
3. *Simple groups of Lie type, including*
   - *The classical groups of Lie type, PSL, PSp, PSU and O.*
   - *The exceptional and twisted groups of Lie type, including the Tits group.*
4. *The 26 sporadic simple groups.*

Even though these finite simple groups have been classified, still a lot is not known about them, especially the larger sporadic simple groups.

Finding new ways to study these large sporadic groups, for example the Monster and Baby Monster simple groups, is of upmost importance, as simply studying these groups alone is not feasible. For example Linton et al constructed computationally the Monster group, over $GF(2)$ [36], and although generators are available the group is too large to completely load on a computer, and Wilson [42] constructed computationally the Baby Monster over $GF(2)$, and although it is possible to load the entire group, carrying out anything other than very simple elementary calculations inside it is asking for trouble.

Therefore, studying structures which these groups act on, especially when they involve involutions, which play a very important role in the structure of a simple group, could give us a practical route into studying these massive objects.

This thesis compromises of two main projects, both computational in nature, regarding graph structures associated to Fi_{24} and the Baby Monster, the second and third largest of the sporadic simple groups.

The second chapter is devoted to work carried out by myself in collaboration with my supervisor Prof. Peter Rowley. It is concerned with calculating the point-line collinearity graph for the maximal 2-local geometry for Fischer's largest sporadic simple group, Fi_{24}. The geometry was first introduced by Ronan and Smith [24] in 1980, and calculating the structure of this graph has been an open problem ever since. The work is very computational in nature and although the graph is defined in the language of incidence geometry, we quickly reduced the problem to simple combinatorics to make the computations possible. As this graph is huge, a full description is not given, as this would be impossible, however we do give a 120 by 120 matrix detailing the collapsed adjacency graph. The (i, j)th entry of this matrix gives the number of points in the jth orbit of $\mathcal{G}$ connected to a single point in the ith orbit, as the stabilizer of a point in Γ acts on $\mathcal{G}$.

Chapter three is concerned with the commuting involution graph for the class 2C in the Baby Monster group, the second largest of the sporadic simple groups. Classes $2A$ and $2B$ were completed by Bates, Bundy, Rowley and Perkins [12], and $2C$ is one of the two remaining cases for calculating commuting involution graphs for all the sporadic simple groups - the cases not covered in [12] other than the two remaining Baby Monster cases have been completed by Rowley and P. Taylor [39]. It is the overall aim to compute these commuting involution graphs for all the finite simple groups, a large chunk of which have already been completed. Again this work is rather computational in nature, however due to the restrictions in working inside the 4370 dimensional linear representation (over $GF(2)$) of the Baby Monster, we often had to drop down to more manageable representations for some of the maximal subgroups.

Both chapters are devoted to graph structures associated with finite groups and so have a few shared definitions. In both cases our graph is regular, that is each note x has the same number of edges connected to it. So let $\mathcal{G}$ be a regular graph with vertex set X. Firstly, for $x, y \in X$, we define a distance function on $\mathcal{G}$, $d(x, y)$, in the obvious way. That is $d(x, y)$ is the length of the shortest path connecting x and y. It is clear that if we assume our graph is connected, this distance function follows all the rules expected from a metric. Now for a fixed vertex $t \in X$ we can define the discs of $\mathcal{G}$ as

$$\Delta_i(t) = \{x \in X \mid d(t, x) = i\}$$

for an integer i. In both chapters the structure of these discs will be independent on the choice of t. Finally we define the diameter of $\mathcal{G}$ to be the maximum distance between any two vertices of $\mathcal{G}$.

All the calculations detailed in this thesis were carried out using Magma v2.15, apart from a few which were carried out in Gap v4.4.10. In all cases we used several 3.2GHz machines, each with between 8 and 16GB of RAM, located in the School of Mathematics at The University of Manchester.

One final remark, during this thesis we make great use of the Atlas of Finite Groups [18] and the World Wide Web Atlas of Group Representations [22]. As we refer to these almost every other sentence, we will simply refer to them as the Atlas and The Online Atlas respectively and reference them here.

Chapter 2

The Point-Line Collinearity graph for the Maximal Local 2-Geometry of Fi_{24}

2.1 Introduction and Basic Definitions

Definition 2.1.1. An Incidence Geometry is a 4-tuple $(\Gamma, \star, \Delta, d)$ where Γ is a set, whose elements are called varieties (that is points, lines, planes, hyper-planes, etc), d is a map from Γ to the finite set Δ which gives the type of each element in Γ, that is whether the variety is a point, line, plane, etc, and $\star$ is a binary symmetric and reflexive relation on Γ called the incidence relation, where the above is subject to axioms 1 and 2 given below. For $i \in \Delta$ we denote $d^{-1}(i)$ by Γ_i, and call its elements i-varieties, or simply just points, lines, planes etc. A flag F is a set of pairwise incident varieties. The type of F is the set $d(F) \subset \Delta$ and the rank of F is the size of $d(F)$. The residue $R(F)$ of a flag F is the ordered 4-tuple $(\Gamma', \star', d', \Delta')$ where Γ' is the set of all varieties of Γ of type $i \in \Delta \setminus d(F)$ which are incident to all elements of F, $\star'$ and d' are the restrictions of $\star$ and d to Γ' and $\Delta' = d(\Gamma')$.

Axiom 1 Every maximal flag contains one and only one variety of type i for every $i \in \Delta$ and every non-maximal flag is contained in at least two maximal flags.

Axiom 2 For any distinct $i, j \in \Delta$, $\Gamma_i \cup \Gamma_j$ is connected under $\star$, that is for any two $x, y \in \Gamma_i \cup \Gamma_j$ there exists a chain of elements $x_\alpha \in \Gamma_i \cup \Gamma_j$ where $0 \leq \alpha \leq n$ such that $x_\alpha \star x_{\alpha+1}$ and $x_0 = x$, $x_n = y$, and this property holds in every residue $R(F)$ for a flag F.

Definition 2.1.2. Let Γ be an incidence geometry, then the Point-Line Collinearity Graph, $\mathcal{G}$, for Γ is a graph where the vertices are the points of Γ, with any two vertices joined by an edge if and only if they are incident with a common line.

Now let G be a finite group; we can create an incidence geometry from G by letting $\mathcal{F} = \{G_i\}$ be a family of subgroups of G, and letting the objects of type i be the cosets of G_i in G, with two cosets xG_i, yG_j incident if and only if $xG_i \cap yG_j \neq \emptyset$. Furthermore, if G has even order, we can let $\mathcal{F}$ be the collection of maximal 2-local subgroups of G; then the geometry Γ created from $\mathcal{F}$ is the maximal 2-local geometry for G. These geometries have been extensively studied in the case of groups of Lie type by Tits [41] and Buekenhout [8] and for the sporadic groups by Ronan and Smith [24]. This Chapter will be devoted to studying the point line collinearity graph for the maximal 2-local geometry for Fischer's larger sporadic group Fi_{24}.

2.2 Literature Review

The maximal 2-local geometry for Fi_{24} was first described by Ronan and Smith in [24]. In this paper they gave the diagram geometries for many of the sporadic simple groups, in which the stabilizer of a vertex is a maximal 2-constrained 2-local subgroup. The combinatorial structure of these geometries have been studied by many authors, for example P. Rowley, L. Walker [31],[32],[33], J. Maginnis and S. Onofrei [21], Y. Segev [37] and A. Ivanov [17]. Central to this structure is the point-line colinearity graph $\mathcal{G}$.

The structure of $\mathcal{G}$ for many of these geometries has been calculated and we will outline these results here.

In [28], [29] and [30], Rowley and Walker calculated the point line collinearity graph $\mathcal{G}$ for the maximal 2-local geometry Γ for Janko's largest sporadic simple group

J_4. Throughout the paper they didn't assume that the group G in question was in fact J_4, they only assumed the following geometric data.

Let Γ be a residually connected string geometry, with type set $\{0, 1, 2\}$ and suppose for $x \in \Gamma$, $\Gamma_x = \{y \in \Gamma \mid x \star y\}$. Now let G be a subgroup of $Aut\Gamma$ which satisfies the following properties:

1. For $a \in \Gamma_0$, Γ_a is the rank 2 geometry of trios and sextets (defined on the Steiner system $S(5, 8, 24)$), $G_a/Q(a) \cong M_{24}$ and $Q(a)$ is the 11-dimensional M_{24} Todd module.

2. For $X \in \Gamma_2$, Γ_X is the rank 2 geometry of duads and hexads (defined on the Steiner system $S(3, 6, 22)$), $G_X/Q(X) \cong M_{22} : 2$ and $Q(X) \cong 2^{1+12}.3$ with $O_2(G_X) = O_2(Q(X))$ the extraspecial group of order 2^{13}.

We note that the maximal 2-local geometry for J_4 possesses both of these properties. Now suppose $\mathcal{G}$ is the point line collinearity graph for such a geometry, and hence $\mathcal{G}$ is the point line collinearity graph for the maximal 2-local geometry for J_4; we have the following theorem.

Theorem 2.2.1 (P. Rowley and L. Walker)**.** *Let $\mathcal{G}$ be the point line collinearity graph for the geometry Γ defined above and suppose $a \in \Gamma_0$. Then*

1. $|\Gamma| = 173, 067, 379$.

2. *$\mathcal{G}$ has diameter 3.*

3. *$\mathcal{G}$ consists of seven orbits as G_a acts on Γ_0, labeled a, $\Delta_1(a)$, $\Delta_2^1(a)$, $\Delta_2^2(a)$, $\Delta_2^3(a)$, $\Delta_3^1(a)$ and $\Delta_3^2(a)$.*

4. $|\Delta_1(a)| = 2^2.3.5.11.23$, $|\Delta_1^2(a)| = 2^4.7.11.23$, $|\Delta_2^2(a)| = 2^7.3.5.7.11.23$, $|\Delta_2^3(a)| = 2^{11}.32.7.11.23$, $|\Delta_3^1(a)| = 2^{11}.3.5.7.11.23$ *and* $|\Delta_3^2(a)| = 2^{18}.3^2.5.7$.

In [26] and [27], Rowley and Walker calculated the point line collinearity graph for the maximal 2-local geometry for the Baby Monster BM. As in the J_4 case, they didn't assume the group G was BM, and only assumed that Γ was a rank 4 geometry, with G a subgroup of $Aut(\Gamma)$ with the following properties:

1. Γ is a string geometry.

2. For $l \in \Gamma_1$, $|\Gamma_0(l)| = 3$ and two collinear points in Γ determine a unique line.

3. For $a \in \Gamma_0$ and $X \in \Gamma_3$, Γ_a is isomorphic to the Co_2-minimal parabolic geometry and Γ_X is isomorphic to a projective 3-space geometry (over GF(2)).

4. G acts flag transitively on Γ.

5. For $a \in \Gamma_0$, $G_a \cong 2^{1+22}Co_2$, $Q(a) \cong 2^{1+22} = O_2(G_a)$ and $Z_1(a) = Z(G_a) = Z(Q(a)) = \mathbb{Z}_2$. Moreover $Q(a)/Z(Q(a))$ is isomorphic to the irreducible 22-dimensional $GF(2)$ Co_2 module which occurs as a composition factor in the Leech lattice reduced mod 2.

6. Let $l \in \Gamma_1$ and $X \in \Gamma_3$, then $G_l \cong 2^{2+10+20}(S_3 \times M_{22}.2)$ has a unique minimal normal subgroup of order 2^2 and $G_X \cong 2^{9+16+6+4}L_4(2)$ with $Q(X) = O_2(G_X) \cong 2^{9+16+6+4}$.

Note that all the properties above hold for $G = BM$ and Γ the maximal 2-local geometry for G. They proved the following theorem:

Theorem 2.2.2 (P. Rowley and L. Walker)**.** *Let $\mathcal{G}$ be the point line collinearity graph for the geometry Γ described above and let $a \in \mathcal{G}$. Then*

1. *$\mathcal{G}$ has diameter 4.*

2. *$\Delta_1(a)$ consists of a single G_a orbit, as G_a acts on the vertices of $\mathcal{G}$.*

3. *$\Delta_2(a)$ consists of three G_a orbits.*

4. *$\Delta_3(a)$ consists of four G_a orbits.*

5. *$\Delta_4(a)$ consists of a single G_a orbit.*

In [40], Rowley and Taylor studied the point line collinearity graphs for the minimal parabolic geometries for the sporadic simple groups HN and Th, geometries closely related to the maximal 2-local geometries. These graphs are of interest because they appear in full as subgraphs of the point line collinearity graph for the maximal 2-local geometry of the Monster sporadic simple group. They proved the following two theorems:

Theorem 2.2.3 (P. Rowley and P. Taylor)**.** *Let $\mathcal{G}$ be the point line collinearity graph for the minimal parabolic geometry Γ for the Thompson sporadic simple group Th. Then $\mathcal{G}$ has diameter 5 and for a fixed vertex a, the discs of $\mathcal{G}$ break up into the following orbits, as G_a acts on the vertices of $\mathcal{G}$.*

1. $|\Delta_1(a)| = 270$ *and consists of a single G_a orbit.*
2. $|\Delta_2(a)| = 64,800$ *and consists of two G_a orbits.*
3. $|\Delta_3(a)| = 15,060,480$ *and consists of six G_a orbits.*
4. $|\Delta_4(a)| = 858,497,006$ *and consists of twenty G_a orbits.*
5. $|\Delta_5(a)| = 103,219,200$ *and consists of two G_a orbits.*

Theorem 2.2.4 (P. Rowley and P. Taylor)**.** *Let $\mathcal{G}$ be the point line collinearity graph for the minimal parabolic geometry Γ for the Harada-Norton sporadic simple group HN. Then $\mathcal{G}$ has diameter 5 and for a fixed vertex a, the discs of $\mathcal{G}$ break up into the following orbits, as G_a acts on the vertices of $\mathcal{G}$.*

1. $|\Delta_1(a)| = 150$ *and consists of a single G_a orbit.*
2. $|\Delta_2(a)| = 17,760$ *and consists of three G_a orbits.*
3. $|\Delta_3(a)| = 1,638,400$ *and consists of eight G_a orbits.*
4. $|\Delta_4(a)| = 68,721,664$ *and consists of fifty G_a orbits.*
5. $|\Delta_5(a)| = 3,686,400$ *and consists of three G_a orbits.*

For both of these graphs they translated the geometric definition of $\mathcal{G}$ into a group theoretic definition, then used MAGMA to calculate the graphs. This translation worked in the following way. If x is a point in either of the geometries in question, then it is true that $G_x = C_G(i_x)$ where i_x is an involution in G and $Z(G_x) = \langle i_x \rangle$.

Therefore we may identify Γ_0 with the conjugacy class $X = i_x^G$. Under this translation two points $x, y \in X$ are joined by an edge if and only if $y \in O_2(C_G(x))$, and thus this graph is closely related to the commuting involution graph. After this translation calculation of the structure of $\mathcal{G}$ for both the groups Th and HN using MAGMA was relatively simple as the size of both of the groups in question is relatively small.

Rowley and Walker calculated the point line collinearity graph for the maximal 2-local geometry for Fi_{23}; this was a substantial amount of work and is spread over three papers [31], [32] and [33]. They proved that the graph $\mathcal{G}$ has diameter 4, with the following orbit decomposition with respect to a fixed vertex t.

1. $\Delta_1(t)$ has size 506 and consists of a single G_t orbit.
2. $\Delta_2(t)$ has size 141,680 and consists of two G_t orbits.
3. $\Delta_3(t)$ has size 29,233,920 and consists of six G_t orbits.
4. $\Delta_4(t)$ has size 166,371,328 and consists of six G_t orbits.

These calculations were obtained purely by hand, and no machine calculations were used. They quickly proved that the number of points incident with a fixed line was 3, and any two of these 3 points uniquely determine the line. They studied the graph in a similar way that we will study the Fi_{24} graph, by letting G_{tx} act on the set of lines incident with a vertex x, and taking representatives from each of these line orbits. They then calculated the two other points incident with these line orbit representatives, to get a full list of representatives. As there was only a small number of G_t orbits, this was possible to do by hand.

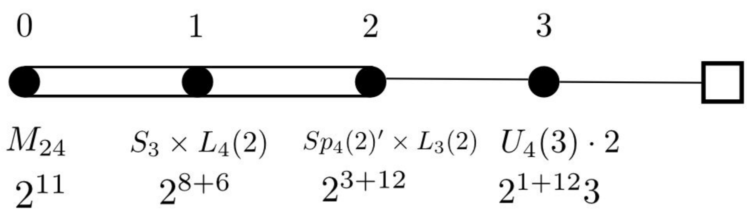
In [35], Rowley and Walker calculated the first three discs of the point line collinearity graph for the maximal 2-local geometry for Fi_{24}. This was relatively straight forward, as the Fi_{23} graph embeds itself into the Fi_{24} graph, with only two new G_t orbits found in the Fi_{24} case. These calculations were carried out entirely by hand.

2.3 Fi_{24}

Let G be Fischer's largest sporadic group Fi_{24}. We first note that Fi_{24} is itself not simple, however its derived group Fi'_{24} is, with Fi_{24} its automorphism group. We will let F denote the derived group Fi'_{24}, and thus F is simple. The group G contains four classes of involutions, in Atlas notation denoted $2A$, $2B$ $2C$ and $2D$. The class $2C$ generate G, and are the so called 3-transpositions, that is the product of any two involutions in $2C$ either is 1 if they are the same, an involution if they commute, or an element of order 3. We will call a maximal set of mutually commuting involutions from $2A$ a base. It is a fact that for a base $\mathcal{B}$, $|\mathcal{B}| = 24$, with any two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ conjugate in G (see the Atlas for these details).

As defined in section 2.1, we can define the maximal 2-local geometry for G, which we will call Γ. The diagram for this geometry is given in Figure 2.1.

Figure 2.1: The maximal 2-local geometry for Fi'_{24}



In Figure 2.1, the number given above each vertex is its type, that is point, line, plane, hyper plane, etc and the groups given below are the stabilizers of such a point, line, plane, etc in Fi'_{24}. Note that because this diagram gives the stabilizers inside of Fi'_{24}, the stabilizer of a point in G has shape $2^{12}.M_{24}$.

Now the stabilizer in G of a base $\mathcal{B}$ is isomorphic to $2^{12}.M_{24}$, and since G only contains one conjugacy class of groups of this shape we may identity points of Γ with bases of G. Since G acting by conjugation on $2C$ has permutation degree 306,936 when studying $\mathcal{G}$ we may work inside $Sym(306936)$ to make calculations possible. In preparation for this task we prepare a representation for G inside $Sym(306936)$ using the following code

```
F<a,b1,c1,d1,e1,f1,b2,c2,d2,e2,b3,c3> := FreeGroup(12);
```

```
   Rels:={a^2=Id(F),b1^2=Id(F),c1^2=Id(F),d1^2=Id(F),e1^2=Id(F),
       f1^2=Id(F),b2^2=Id(F),c2^2=Id(F),d2^2=Id(F),e2^2=Id(F),
       b3^2=Id(F),c3^2=Id(F),
       (a*b1)^3=Id(F),(a*c1)^2=Id(F),(a*d1)^2=Id(F),(a*e1)^2=Id(F),
       (a*b2)^3=Id(F),(a*c2)^2=Id(F),(a*d2)^2=Id(F),(a*e2)^2=Id(F),
       (a*b3)^3=Id(F),(a*c3)^2=Id(F),(b1*c1)^3=Id(F),(b1*d1)^2=Id(F),
       (b1*e1)^2=Id(F),(b1*b2)^2=Id(F),(b1*c2)^2=Id(F),(b1*d2)^2=Id(F),
       (b1*e2)^2=Id(F),(b1*b3)^2=Id(F),(b1*c3)^2=Id(F),(c1*d1)^3=Id(F),
       (c1*e1)^2=Id(F),(c1*b2)^2=Id(F),(c1*c2)^2=Id(F),(c1*d2)^2=Id(F),
       (c1*e2)^2=Id(F),(c1*b3)^2=Id(F),(c1*c3)^2=Id(F),(d1*e1)^3=Id(F),
       (d1*b2)^2=Id(F),(d1*c2)^2=Id(F),(d1*d2)^2=Id(F),(d1*e2)^2=Id(F),
       (d1*b3)^2=Id(F),(d1*c3)^2=Id(F),(e1*b2)^2=Id(F),(e1*c2)^2=Id(F),
       (e1*d2)^2=Id(F),(e1*e2)^2=Id(F),(e1*b3)^2=Id(F),(e1*c3)^2=Id(F),
       (b2*c2)^3=Id(F),(b2*d2)^2=Id(F),(b2*e2)^2=Id(F),(b2*b3)^2=Id(F),
       (b2*c3)^2=Id(F),(c2*d2)^3=Id(F),(c2*e2)^2=Id(F),(c2*b3)^2=Id(F),
       (c2*c3)^2=Id(F),(d2*e2)^3=Id(F),(d2*b3)^2=Id(F),(d2*c3)^2=Id(F),
       (e2*b3)^2=Id(F),(e2*c3)^2=Id(F),(b3*c3)^3=Id(F),
       (a*b1*c1*a*b2*c2*a*b3*c3)^10=Id(F),
       (f1*e1)^3=Id(F),(f1*d1)^2=Id(F),(f1*c1)^2=Id(F),(f1*b1)^2=Id(F),
       (f1*a)^2=Id(F),(f1*b2)^2=Id(F),(f1*c2)^2=Id(F),(f1*d2)^2=Id(F),
       (f1*e2)^2=Id(F),(f1*b3)^2=Id(F),(f1*c3)^2=Id(F),
       f1=(a*b1*c1*d1*b2*c2*b3)^9,f1=(a*b1*c1*d1*b2*b3*c3)^9};

Y442 := quo<Fr|Rels>;

S:={a,b1,c1,d1,e1,f1,b2,c2,d2,b3,c3,
   (a*b1*c1*d1*e1*f1*a*b2*c2*d2*e2*a*b3*c3)^17};
H:=sub<Y442|S>;

m, G := CosetAction(Y442,H);
```

```
g1 := m(f1);
g2 := m((f1*d1)^e1);
g3 := m((d1*b1)^c1);
g4 := m((b1*b2)^a);
g5 := m((b2*d2)^c2);
g6 := m((d2*f2)^e2);
g7 := m((b1*b3)^a);
g8 := m((b2*b3)^a);
g9 := m((b1*a*b2*b3*c3)^4);
```

This presentation is based on a Y-type diagram given in the ATLAS, we recall that $Y_{542} = Y_{442} \cong 3^{\cdot}Fi_{24}$. We note that $G \cong Fi_{24}$ is generated by 12 permutations, which we will call $a1, \dots a12$. For ease of use later on we will save these permutations in a file *Fi24perms.m* and let G be the subgroup of $Sym(306936)$ generated by them. The elements $g1, \dots, g9$ generate a subgroup of shape $2^{12}M_{24}$ which will play the part of G_a in our calculations.

Now let $x \in \Gamma_0$, that is x is a point of Γ, then by our previous observation, we may identify x with a base of G, which we will denote Ω_x. So in particular $|\Omega_x| = 24$ and G_x, the stabilizer of Ω_x in G has shape $2^{12}.M_{24}$. More importantly G_x acts on Ω_x, with the induced action being the standard action of M_{24} on a 24 point set. Therefore when studying Γ we may use the powerful machinery of Curtis's Miracle Octad Generator (the MOG) [14]. From this point of view the lines of Γ incident with x can be identified with the octads of Ω_x. If we consult the ATLAS, we see that the octads of Ω_x are precisely the subsets of Ω_x of size 8 which product to 1 in G (recall that all involutions in Ω_x commute). As we are considering the standard action of M_{24} on a 24 point set, there are 759 such octads for each base x. Therefore we can now describe $\mathcal{G}$ in a more accessible way. Indeed, the vertices of $\mathcal{G}$ are the bases of

G, with two vertices Ω_x and Ω_y joined by an edge if and only if $\Omega_x \cap \Omega_y$ is an octad of either Ω_x or Ω_y. We now note that G acts transitively on the set of bases of G, therefore if $\Omega_x \cap \Omega_y$ is an octad of Ω_x then it is also an octad of Ω_y and vice versa.

We will now introduce an important tool when studying this graph, that of the transposition profile. For $a \in \Gamma_0$, we can let G_a act on the set of 3-transpositions for G. In our setup this corresponds to letting G_a act on the set $\Omega = \{1 \ldots 306936\}$ with the standard permutation action. Then Ω splits into 3 orbits of sizes 24, 24,288 and 282,624 (see the Atlas for these details). The first orbit corresponds to the base Ω_a, the second we will call the cctadic transpositions and denote $\mathcal{O}_a$, and third the duadic transpositions, denoted $\mathcal{D}_a$. So for a base Ω_y of G, we assign $l_1 = |\Omega_y \cap \Omega_a|$, $l_2 = |\Omega_y \cap \mathcal{O}_a|$ and $l_3 = |\Omega_y \cap \mathcal{D}_a|$. Then $l_1|l_2|l_3$ will be referred to as the transposition profile for Ω_y (with respect to Ω_a). Clearly if two bases Ω_x and Ω_y are in the same G_a orbit then they will have the same transposition profile. Therefore this gives us a useful and easily calculated G_a orbit invariant. However the opposite is far from true, for example the orbits $\Delta_3^9(a)$ and $\Delta_4^7(a)$ both have transposition profile $1|1|22$ with respect to Ω_a.

The main results from this investigation are given in the following two theorems. We first remark that as G acts on the set of bases of G, G induces graph automorphisms on $\mathcal{G}$. As this action is transative the disc structure of $\mathcal{G}$ will not depend on the original choice of Ω_a. We also note that G_a acts on the vertices of $\mathcal{G}$ and for any two vertices x and y in the same G_a orbit, $d(a, x) = d(a, y)$. Therefore, for a G_a orbit X, if $x \in X$ belongs to $\Delta_i(a)$ then $X \subseteq \Delta_i(a)$. Thus we will break down the discs of $\mathcal{G}$ into their constituent G_a orbits. Details of these orbits are given in Theorem 2.3.2. By using Gap and the class structure constants for G, S. Linton [20] calculated the permutation rank, that is the number of orbits as G_a acts on the vertices of $\mathcal{G}$ to be 120.

Theorem 2.3.1 (P. Rowley and B. Wright). *Let $\mathcal{G}$ be the point line collinearity graph for the maximal 2-local geometry for Fi_{24}. Then*

(i) The diameter of $\mathcal{G}$ is 5.

(ii) $|\Delta_1(a)| = 1518$ *and* $\Delta_1(a)$ *is a* G_a *orbit.*

(iii) $|\Delta_2(a)| = 1,560,504$ *and* $\Delta_2(a)$ *consists of three* G_a *orbits.*

(iv) $|\Delta_3(a)| = 1,400,874,432$ *and* $\Delta_3(a)$ *consists of ten* G_a *orbits.*

(v) $|\Delta_4(a)| = 656,569,113,600$ *and* $\Delta_4(a)$ *consists of 46* G_a *orbits.*

(vi) $|\Delta_5(a)| = 1,845,442,396,160$ *and* $\Delta_5(a)$ *consists of 59* G_a *orbits.*

Note that the number of orbits in each disc add up to 119 which with the vertex a stabilized by G_a, make up the 120 orbits calculated by S. Linton. The next theorem gives more details about each G_a orbit. For a representative x of each G_a orbit, we present the structure of G_{ax}, that is the stabilizer of x in G_a. For these groups we mostly use notation from the ATLAS, apart from using $Sym(n)$, $Alt(n)$ and $Dih(n)$ for the symmetric, alternating and dihedral groups. For a vertex x of $\mathcal{G}$, recall that G_a has shape $2^{12}.M_{24}$, and F_a has shape $2^{11}.M_{24}$. We use Q_x to denote the largest normal 2-group of F_x, so Q_x is elementary abelian of order 2^{11}. The final column of the table below lists the sizes of the sets $F_{ax} \cap Q_x$. Finally all transposition profiles given in the table below are with respect to a.

Theorem 2.3.2 (P. Rowley and B. Wright)**.** *For* $i = 1, \ldots, 5$, $\Delta_i(a)$ *is the union of the* F_a*-orbits* $\Delta_i^j(a)$ *as detailed in the table below.*

Table 2.1: The Orbits of $\mathcal{G}$

$\Delta_i^j(a)$	$\lvert\Delta_i^j(a)\rvert$	*Transposition Profile*	*Structure of* F_{ax}	$\lvert F_{ax} \cap Q_x\rvert$
$\Delta_0^1(a)$	*1*	24|0|0	$2^{11}.M_{24}$	*2048*
$\Delta_1^1(a)$	*1518*	8|16|0	$2^{10}.2^4.Alt(8)$	*1024*
$\Delta_2^1(a)$	*30360*	0|24|0	$2^9.2^6.(L_3(2) \times 3)$	*512*
$\Delta_2^2(a)$	*170016*	4|20|0	$2^7.2^6.3.Sym(5)$	*128*
$\Delta_2^3(a)$	*1360128*	2|6|16	$2^5.2^4.Sym(6)$	*32*
$\Delta_3^1(a)$	*282624*	2|0|22	$2.M_{22}.2$	*2*
$\Delta_3^2(a)$	*566720*	0|24|0	$2^7.2^6.3.3^2.4$	*128*

$\Delta_3^3(a)$	*1036288*	3|21|0	$2^2.L_3(4).Sym(3)$	*4*
$\Delta_3^4(a)$	*11658240*	2|14|8	$2^4.2^3.(L_3(2)\times 2)$	*16*
$\Delta_3^5(a)$	*21762048*	2|16|6	$2.2^4.Sym(6)$	*2*
$\Delta_3^6(a)$	*40803840*	0|8|16	$2^3.2^2.2^4.Sym(4)$	*8*
$\Delta_3^7(a)$	*40803840*	0|8|16	$2^4.2^2.2^3.Sym(4)$	*16*
$\Delta_3^8(a)$	*108810240*	1|7|16	$2^2.2^2.2^2.3.Sym(4)$	*4*
$\Delta_3^9(a)$	*522289152*	1|1|22	$2^4.Alt(5)$	*1*
$\Delta_3^{10}(a)$	*652861440*	0|2|22	$2.2.2^3.Sym(4)$	*2*
$\Delta_4^1(a)$	*11658240*	0|16|8	$2^4.2^4.L_3(2)$	*16*
$\Delta_4^2(a)$	*11658240*	0|16|8	$2^4.2^4.L_3(2)$	*16*
$\Delta_4^3(a)$	*24870912*	1|15|8	$Alt(8)$	*1*
$\Delta_4^4(a)$	*65286144*	0|0|24	$2.2^6.Alt(5)$	*2*
$\Delta_4^5(a)$	*93265920*	0|2|22	$2.2^4.L_3(2)$	*2*
$\Delta_4^6(a)$	*93265920*	0|2|22	$2.2^4.L_3(2)$	*2*
$\Delta_4^7(a)$	*198967296*	1|1|22	$Alt(7)$	*1*
$\Delta_4^8(a)$	*217620480*	0|8|16	$2^6.(Sym(3)\times Sym(3))$	*1*
$\Delta_4^9(a)$	*217620480*	0|8|16	$2^6.(Sym(3)\times Sym(3))$	*1*
$\Delta_4^{10}(a)$	*217620480*	0|8|16	$2^2.2^4.(Sym(3)\times Sym(3))$	*4*
$\Delta_4^{11}(a)$	*217620480*	0|8|16	$2^2.2^4.(Sym(3)\times Sym(3))$	*4*
$\Delta_4^{12}(a)$	*244823040*	0|8|16	$2^3.2^2.2^3.2^3$	*8*
$\Delta_4^{13}(a)$	*326430720*	0|0|24	$2.2^4.2^4.3$	*2*
$\Delta_4^{14}(a)$	*652861440*	0|10|14	$2.2^2.2^4.Sym(3)$	*2*
$\Delta_4^{15}(a)$	*652861440*	0|10|14	$2.2^2.2^4.Sym(3)$	*2*
$\Delta_4^{16}(a)$	*746127360*	1|9|14	$2.L_3(2).2$	*2*
$\Delta_4^{17}(a)$	*759693312*	1|11|12	$L_2(11)$	*1*
$\Delta_4^{18}(a)$	*870481920*	1|3|20	$2^6.3^2$	*1*
$\Delta_4^{19}(a)$	*1305722880*	0|6|18	$2.2^5.Sym(3)$	*2*
$\Delta_4^{20}(a)$	*1305722880*	0|6|18	$2.2^5.Sym(3)$	*2*
$\Delta_4^{21}(a)$	*1392771072*	1|5|18	$(3\times Alt(5)).2$	*1*
$\Delta_4^{22}(a)$	*2611445760*	0|4|20	$2^2.2^3.Sym(3)$	*1*

$\Delta_4^{23}(a)$	*2611445760*	0|4|20	$2^5.Sym(3)$	*1*
$\Delta_4^{24}(a)$	*2611445760*	0|4|20	$2.2^4.Sym(3)$	*2*
$\Delta_4^{25}(a)$	*3917168640*	0|0|24	$2^2.2^5$	*1*
$\Delta_4^{26}(a)$	*3917168640*	0|6|18	$2.2^3.2^3$	*2*
$\Delta_4^{27}(a)$	*5222891520*	0|2|22	$2.2^3.Sym(3)$	*1*
$\Delta_4^{28}(a)$	*5222891520*	0|6|18	$2^4.Sym(3)$	*1*
$\Delta_4^{29}(a)$	*5222891520*	0|6|18	$2^4.Sym(3)$	*1*
$\Delta_4^{30}(a)$	*5222891520*	0|6|18	$2.2^3.Sym(3)$	*2*
$\Delta_4^{31}(a)$	*5222891520*	0|6|18	$2.2^3.Sym(3)$	*2*
$\Delta_4^{32}(a)$	*6963855360*	0|0|24	$2^2.(3 \times 3).2$	*1*
$\Delta_4^{33}(a)$	*6963855360*	0|0|24	$2^2.(3 \times 3).2$	*1*
$\Delta_4^{34}(a)$	*10445783040*	0|2|22	$2^3.Sym(3)$	*1*
$\Delta_4^{35}(a)$	*10445783040*	0|2|22	$2^3.Sym(3)$	*1*
$\Delta_4^{36}(a)$	*10445783040*	0|2|22	$2^3.Sym(3)$	*1*
$\Delta_4^{37}(a)$	*10445783040*	0|2|22	$2^3.Sym(3)$	*1*
$\Delta_4^{38}(a)$	*15668674560*	0|2|22	$2^3.2^2$	*1*
$\Delta_4^{39}(a)$	*15668674560*	0|2|22	$2^3.2^2$	*1*
$\Delta_4^{40}(a)$	*41783132160*	0|2|22	$Dih(12)$	*1*
$\Delta_4^{41}(a)$	*50139758592*	0|1|23	$Dih(10)$	*1*
$\Delta_4^{42}(a)$	*50139758592*	0|1|23	$Dih(10)$	*1*
$\Delta_4^{43}(a)$	*62674698240*	0|2|22	2^3	*1*
$\Delta_4^{44}(a)$	*62674698240*	0|2|22	2^3	*1*
$\Delta_4^{45}(a)$	*125349396480*	0|1|23	2^2	*1*
$\Delta_4^{46}(a)$	*125349396480*	0|1|23	2^2	*1*
$\Delta_5^{1}(a)$	*24870912*	0|16|8	$Alt(8)$	*1*
$\Delta_5^{2}(a)$	*24870912*	0|16|8	$Alt(8)$	*1*
$\Delta_5^{3}(a)$	*232128512*	0|6|18	$3.Sym(6)$	*1*
$\Delta_5^{4}(a)$	*232128512*	0|6|18	$3.Sym(6)$	*1*
$\Delta_5^{5}(a)$	*870481920*	0|4|20	$2^4.(Sym(3) \times Sym(3))$	*1*
$\Delta_5^{6}(a)$	*870481920*	0|4|20	$2^4.(Sym(3) \times Sym(3))$	*1*

$\Delta_5^7(a)$	*2611445760*	0|4|20	$2^5.Sym(3)$	*1*
$\Delta_5^8(a)$	*2611445760*	0|4|20	$2^5.Sym(3)$	*1*
$\Delta_5^9(a)$	*2611445760*	0|4|20	$2^5.Sym(3)$	*1*
$\Delta_5^{10}(a)$	*2611445760*	0|4|20	$2^5.Sym(3)$	*1*
$\Delta_5^{11}(a)$	*2984509440*	0|2|22	$L_3(2)$	*1*
$\Delta_5^{12}(a)$	*2984509440*	0|2|22	$L_3(2)$	*1*
$\Delta_5^{13}(a)$	*3481927680*	0|2|22	$2^2.(Sym(3) \times Sym(3))$	*1*
$\Delta_5^{14}(a)$	*3481927680*	0|2|22	$2^2.(Sym(3) \times Sym(3))$	*1*
$\Delta_5^{15}(a)$	*3917168640*	0|0|24	$2^2.2^5$	*1*
$\Delta_5^{16}(a)$	*4642570240*	0|3|21	$3_+^{1+2}.2^2$	*1*
$\Delta_5^{17}(a)$	*4642570240*	0|3|21	$3_+^{1+2}.2^2$	*1*
$\Delta_5^{18}(a)$	*4642570240*	0|9|15	$3_+^{1+2}.2^2$	*1*
$\Delta_5^{19}(a)$	*4642570240*	0|9|15	$3_+^{1+2}.2^2$	*1*
$\Delta_5^{20}(a)$	*7958691840*	0|3|21	3.7.3	*1*
$\Delta_5^{21}(a)$	*8356626432*	0|6|18	$Alt(5)$	*1*
$\Delta_5^{22}(a)$	*8356626432*	0|6|18	$Alt(5)$	*1*
$\Delta_5^{23}(a)$	*10445783040*	0|6|18	$2^3.Sym(3)$	*1*
$\Delta_5^{24}(a)$	*10445783040*	0|6|18	$2^3.Sym(3)$	*1*
$\Delta_5^{25}(a)$	*10445783040*	0|2|22	$2^3.Sym(3)$	*1*
$\Delta_5^{26}(a)$	*10445783040*	0|2|22	$2^3.Sym(3)$	*1*
$\Delta_5^{27}(a)$	*13927710720*	0|4|20	$Sym(3) \times Sym(3)$	*1*
$\Delta_5^{28}(a)$	*13927710720*	0|4|20	$Sym(3) \times Sym(3)$	*1*
$\Delta_5^{29}(a)$	*13927710720*	0|7|17	$Sym(3) \times Sym(3)$	*1*
$\Delta_5^{30}(a)$	*15668674560*	0|4|20	2^{1+4}	*1*
$\Delta_5^{31}(a)$	*15668674560*	0|2|22	$2^3.2^2$	*1*
$\Delta_5^{32}(a)$	*15668674560*	0|2|22	$2^3.2^2$	*1*
$\Delta_5^{33}(a)$	*20891566080*	0|2|22	$Sym(4)$	*1*
$\Delta_5^{34}(a)$	*20891566080*	0|2|22	$Sym(4)$	*1*
$\Delta_5^{35}(a)$	*25069879296*	0|0|24	$Dih(20)$	*1*
$\Delta_5^{36}(a)$	*41783132160*	0|0|24	$Dih(12)$	*1*

$\Delta_5^{37}(a)$	*41783132160*	0|3|21	$Dih(12)$	*1*
$\Delta_5^{38}(a)$	*41783132160*	0|3|21	$Dih(12)$	*1*
$\Delta_5^{39}(a)$	*41783132160*	0|3|21	$Dih(12)$	*1*
$\Delta_5^{40}(a)$	*41783132160*	0|3|21	$Dih(12)$	*1*
$\Delta_5^{41}(a)$	*41783132160*	0|3|21	$Dih(12)$	*1*
$\Delta_5^{42}(a)$	*41783132160*	0|3|21	$Dih(12)$	*1*
$\Delta_5^{43}(a)$	*41783132160*	0|1|23	$Dih(12)$	*1*
$\Delta_5^{44}(a)$	*41783132160*	0|1|23	$Dih(12)$	*1*
$\Delta_5^{45}(a)$	*41783132160*	0|1|23	$Dih(12)$	*1*
$\Delta_5^{46}(a)$	*41783132160*	0|1|23	$Dih(12)$	*1*
$\Delta_5^{47}(a)$	*50139758592*	0|1|23	$Dih(10)$	*1*
$\Delta_5^{48}(a)$	*62674698240*	0|0|24	2×4	*1*
$\Delta_5^{49}(a)$	*62674698240*	0|4|20	2×4	*1*
$\Delta_5^{50}(a)$	*62674698240*	0|4|20	2×4	*1*
$\Delta_5^{51}(a)$	*62674698240*	0|2|22	2×4	*1*
$\Delta_5^{52}(a)$	*62674698240*	0|2|22	2×4	*1*
$\Delta_5^{53}(a)$	*62674698240*	0|2|22	2×4	*1*
$\Delta_5^{54}(a)$	*62674698240*	0|2|22	2×4	*1*
$\Delta_5^{55}(a)$	*83566264320*	0|0|24	6	*1*
$\Delta_5^{56}(a)$	*83566264320*	0|1|23	$Sym(3)$	*1*
$\Delta_5^{57}(a)$	*83566264320*	0|1|23	$Sym(3)$	*1*
$\Delta_5^{58}(a)$	*125349396480*	0|3|21	2^2	*1*
$\Delta_5^{59}(a)$	*250698792960*	0|1|23	2	*1*

The final result of this chapter is the collapsed adjacency matrix for $\mathcal{G}$. This is a 120 by 120 matrix with entries a_{ij} detailing the number of elements in the jth orbit which are connected to a single fixed element in the ith orbit. Since this is a rather unwieldy beast it has been demoted to the end of the chapter, however a more usable electronic version will also be included.

2.4 Calculating the Discs

During these calculations we will work inside the 306,936 degree permutation representation of G, as G acts on its 3-transpositions. Obviously the set $\Omega = \{1 \dots 306936\}$ represents the actual transpositions with the bases of G being certain subsets of Ω of size 24. We firstly run the following code to get our hands on a copy of $2^{12}.M_{24}$ inside G, which we will call G_a.

```
Ga := sub<G|g1,g2,g3,g4,g5,g6,g7,g8,g9>;
a := Orbit(Ga,1);
b := a^G.10;
```

As G only has one conjugacy class of subgroups of the shape $2^{12}.M_{24}$, the group G_a must be the stabilizer of some base in G. By asking MAGMA for the orbits as G_a acts on Ω we can recover the base Ω_a which we will assume to be the centre of our graph, that is the point from which each disc of $\mathcal{G}$ will be measured, as well as $\mathcal{O}_a$ and $\mathcal{D}_a$, the octadic and duadic transpositions.

Within our representation we have an element called a_{10}, the 10th generator of Fi_{24}, which takes the base Ω_a to Ω_b, where a and b are adjacent in $\mathcal{G}$. Now for any vertex x of $\mathcal{G}$ and octad X of Ω_x there are two vertices y_1 and y_2 such that the bases Ω_{y_1} and Ω_{y_2} intersect Ω_x in X. In fact the octad X corresponds to a line l in Γ, with the three points x, y_1, y_2 incident with l, with two of x, y_1, y_2 determining l uniquely. With this in mind, let a, b, b' be the three points incident with the line determined by a and b. Let $O = \Omega_a \cap \Omega_b (= \Omega_a \cap \Omega_{b'} = \Omega_b \cap \Omega_{b'})$ and l be the corresponding line in Γ then

```
twiddle := g1^(g2*g3*g4*g5);
```

is an element of G which interchanges b and b' and stabilizes a. The following code also defines a subgroup of shape $2^{12}.2^4.Alt(8)$

```
Gal := sub<F|g1,g2,g3,g4,g6,g7,g8,g9,g1^g5,g2^g5,g3^g5,g7^g5,
g1^(g2*g5),g1^(g2*g3*g5),g1^(g3*g5),g1^(g4,g5),g1^(g2*g4*g5),
g1^(g2*g3*g4*g5),g1^(g3*g4*g5)>;
```

This subgroup, named G_{al}, is the stabilizer of both the base Ω_a and the octad O. We have also created an array $Tran$ which contains a transversal for G_{al} in G_a of size 759. At this point we remark on the way that we store elements of G. As we wish to store quite a few elements of G, we thought it best not to store them as actual permutations as this would require a lot of memory. So instead we store an element x of G as an array $[g_{i_1}, \ldots g_{i_n}]$ representing a word for x in the generators $g_1 \ldots g_9$. We have created functions called `MultiplyRandomWord` and `RandomWord` used to create and convert these arrays and the use of these functions will be explained in Section 2.5. Using the array $Tran$ and our original octad O, we can now create all the octads of Ω_a, which we will call $Octadsa$, as well as the first disc of $\mathcal{G}$, $\Delta_1(a)$. Indeed, all the octads of the base Ω_a are given by

$$Octadsa = \{O^t \mid t = Tran[i], 1 \leq i \leq 759\}.$$

For the octad O^t where $t = Tran[i]$, we will refer to i as the octad number for O^t. We also have

$$\begin{aligned}\Delta_1(a) &= \{\Omega_b^h \mid h = Tran[i], 1 \leq i \leq 759\} \cup \{\Omega_b^{(twiddle*h)} \mid h = Tran[i], 1 \leq i \leq 759\} \\ &= \{\Omega_a^{(a_{10}*h)} \mid h = Tran[i], 1 \leq i \leq 759\} \cup \{\Omega_a^{(a_{10}*twiddle*h)} \mid h = Tran[i], 1 \leq i \leq 759\}.\end{aligned}$$

Now as G acting on the vertices of $\mathcal{G}$ acts as a graph automorphism, $\mathcal{G}$ must be a regular graph. The calculation above shows that the valency of $\mathcal{G}$ is 1518, a remarkably low number, which makes these calculations possible. Another useful observation is that we may swing around $Octadsa$ and $\Delta_1(a)$ to get the octads and neighbours for any other vertex x. Indeed if $\Omega_a^g = \Omega_x$ for some $g \in G$, then if we call the octads of x, $octadsx$ we have

$$\begin{aligned}Octadsx &= Octadsa^g \text{ and} \\ \Delta_1(x) &= \Delta_1(a)^g.\end{aligned}$$

As we create new G_a orbits we wish to store a representative Ω_x, so instead of storing the base Ω_x we felt it was more useful to store a group element g which takes us from our fixed base Ω_a to Ω_x. As commented on before, instead of actually storing the element g, as we have 120 of these to store, we will instead store a word in the generators of G for g. From work done by hand in [35], we know that $\Delta_1(a)$ consists of a single G_a orbit, thus we will store the word $[a10]$, the group element which takes us from Ω_a to Ω_b.

In [35], the authors fully determined the first three discs of $\mathcal{G}$ by hand, so we will proceed as follows to calculate the second and third discs. From [35] we know that $\Delta_2(a)$ consists of three G_a orbits and $\Delta_3(a)$ consists of ten G_a orbits. Now $\Delta_1(b)$ as calculated before, gives all 1518 neighbours of the vertex b. In [35] the transposition profiles for representatives in the three orbits of $\Delta_2(a)$ were calculated and known to be different from a and b, hence we can easily pluck out representatives for the three orbits of $\Delta_2(a)$, using the transposition profile as an orbit invariant. We then repeat this procedure on each of these representatives from $\Delta_2(a)$ and pluck out the ten representatives for $\Delta_3(a)$. However in this case we have a small problem, as two of the orbits in $\Delta_3(a)$ have the same transposition profile (both $\Delta_3^6(a)$ and $\Delta_3^7(a)$ have the profile $0|8|16$), and $\Delta_3^2(a)$ shares its transposition profile with an orbit from the second disc. The latter is easily solved as we can tell if a point is in $\Delta_2(a)$ by checking if it is a neighbor of $\Delta_1(a)$, and since $\Delta_1(a)$ is small this is computationally easy. To differentiate between the two orbits in $\Delta_3(a)$ with profile $0|8|16$, we use the fact that for $x_1 \in \Delta_3^6(a)$ and $x_2 \in \Delta_3^7(a)$ there exists an $x_3 \in \Delta_1(a)$ such that $|\Omega_{x_1} \cap \Omega_{x_3}| = 2$ and $|\Omega_{x_2} \cap \Omega_{x_3}| = 4$. We should also now note that there is some discrepancy between the orbits named here and those in [34] and [35]. It was decided from an early stage that the orbits of $\mathcal{G}$ should be named in order of stabilizer (in G_a) size, starting with the smallest from each disc. This is untrue in [34] and [35], and hence the orbits are labeled slightly differently. To compensate for this we have included a listing in Appendix 6 on how to map orbits of $\mathcal{G}$ in this thesis to orbits in [34] and [35].

Moving out from $\Delta_3(a)$ to $\Delta_4(a)$ we use the combinatorial data from [34] in the following way. For a representative Ω_x from one of the ten G_a orbits in $\Delta_3(a)$, we let

G_{ax}, the stabilizer of x in G_a act on $Octadsx$, the octads of the base Ω_x. If we take an octad orbit representative X, then there exists two vertices y and y' such that

$$\Omega_x \cap \Omega_y = \Omega_x \cap \Omega_{y'} = \Omega_y \cap \Omega_{y'} = X.$$

Now suppose that g is the group element that takes us from our fixed vertex a to x and suppose the octad number for X is i, that is X is the ith member of the array $Octadsa^g$. Then

$$\begin{aligned} \Omega_y &= \Omega_a^{(a_{10}*h*g)} \text{ and} \\ \Omega_y' &= \Omega_a^{(a_{10}*twiddle*h*g)} \end{aligned}$$

where $h = Tran[i]$, the element of the array $Tran$ corresponding to the octad number i. Now as we run through all G_a orbit representatives x for $\Delta_3(a)$ and all G_{ax} orbit representatives as G_{ax} acts on the octads of Ω_x we will pick up a G_a orbit representative for all the orbits of $\mathcal{G}$ which are distance 1 away from some point in $\Delta_3(a)$. As expected some of these points will be in either $\Delta_2(a)$ or $\Delta_3(a)$. From [35], we know that up to a few easy exceptions that have already been dealt with, the transposition profiles in the first three discs of $\Delta_3(a)$ are unique, hence these extra representatives in $\Delta_2(a) \cup \Delta_3(a)$ can be quickly crossed off our list. Out of the remaining representatives, it is highly possible that many of these are in the same G_a orbit. To deal with this, we first use the transposition profiles as an initial sieve, grouping the remaining representatives into sets with the same transposition profile, then using the MAGMA command `IsConjugate` inside these subsets to settle matters. As the size of G_a is computationally fairly small, we find that `IsConjugate` takes around 7 seconds on a 3.2GHz machine with 8GB of memory. By removing duplicates in this way we are able to give a full list of the G_a orbit representatives for $\Delta_4(a)$, we found there were 46 of them.

We would quickly like to remark on how we gained the representatives X for the octad orbits, as G_{ax} acts on the octads of a base Ω_x. In [34], the authors give combinatorial data in the form of the MOG for a representative of each of the octad orbits for a representative x from the G_a orbits in the first discs of $\mathcal{G}$. We converted this data in these tables into their corresponding octad numbers by first fixing an octad of x, usually the first one, then running through all the possible octads for x, asking which intersected our fixed octad in a particular number of points, this information being given in the MOG tables. As the size of G_{ax} is also reasonably small, using the `Stablizer` command in MAGMA is possible, so we could also use the stabilizer size for a possible octad orbit representative to distinguish between particular octad orbits. We will now give the octad numbers for each octad orbit for each representative x for G_a orbits in the first three discs. At this point we would like to stress that the names given here are those quoted in [34], and not the names in this thesis. To convert between the two you can use the table in Appendix 6.

$\Delta_1^1(a)$, $L = Stab_G\{\Lambda_1\}$ where

$\Lambda_1 = \{6032, 6158, 6734, 22973, 22975, 22977, 38858, 83012\}$.

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
α_8	1	1	α_2	448	62
α_0	30	248	α_4	280	2

$\Delta_2^1(a)$, $L = Stab_G\{\Lambda_1\}$ where

Λ_1 is the partition given by

$\{\{540, 573, 583, 586, 590, 1177, 1192, 1200\}$,

$\{306821, 306823, 306922, 306923, 306925, 306927, 306935, 306936\}$,

$\{2, 43, 183, 792, 948, 970, 1080, 17319\}\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
α_{80^2}	3	1	α_{42^2}	672	100
α_{4^2}	84	2			

$\Delta_2^2(a)$, $L = Stab_G\{\Lambda_1\}$ where

$\Lambda_1 = \{22973, 22977, 38858, 83012\}$, and Λ_2 is the sextet given by
$\{\{4, 20, 77, 349\}, \{6393, 21350, 49646, 61991\},$
$\{2951, 3008, 3320, 12882\}, \{948, 970, 1080, 17319\},$
$\{17400, 21982, 22598, 62004\}, \{22973, 22977, 38858, 83012\}\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{4,4^2}$	5	1	$\alpha_{2,2^4}$	240	3
$\alpha_{0,4^2}$	10	101	$\alpha_{0,2^4}$	120	344
$\alpha_{1,31^5}$	320	59	$\alpha_{3,31^5}$	64	5

$\Delta_2^3(a)$, $L = Stab_G\{\Lambda_1\}$ where

$\Lambda_1 = \{2, 43, 948, 16365, 17319, 22977, 29733, 83012\}$ and
$\Lambda_2 = \{22977, 83012\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,2}$	1	1	$\alpha_{2,1}$	192	62
$\alpha_{2,2}$	16	111	$\alpha_{4,0}$	60	55
$\alpha_{4,2}$	60	2	$\alpha_{2,0}$	240	176
$\alpha_{4,1}$	160	6	$\alpha_{0,0}$	30	248

$\Delta_3^1(a)$, $L = Stab_G\{\Lambda_1\}$ where

$\Lambda_1 = \{22977, 83012\}$.

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
α_2	77	5	α_0	330	55
α_1	352	6			

$\Delta_3^2(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$ where

Λ_1 is the sextet whose tetrads are $\{540, 573, 583, 590\}$, $\{300337, 301248, 301594, 305089\}$, $\{300364, 300688, 301606, 305099\}$, $\{948, 970, 1080, 17319\}$, $\{1749, 1850, 1883, 1896\}$, $\{2951, 3008, 3320, 12882\}$.

$\Lambda_2 = \{540, 573, 583, 590, 300337, 300364, 300688, 301248, 301594, 301606, 305089, 305099\}$

$\Lambda_3 = \{948, 970, 1080, 1749, 1850, 1883, 1896, 2951, 3008, 3320, 12882, 17319\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{4^2,8,0}$	3	751	$\alpha_{2^4,2,6}$	72	3
$\alpha_{4^2,0,8}$	3	1	$\alpha_{2^4,4,4}$	216	100
$\alpha_{4^2,4,4}$	9	723	$\alpha_{31^5,5,3}$	192	114
$\alpha_{2^4,6,2}$	72	214	$\alpha_{31^5,3,5}$	192	5

$\Delta_3^3(a)$, $L = Stab_G\{\Lambda_1\}$ where

$\Lambda_1 = \{22973, 22977, 83012\}$.

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
α_3	21	1	α_1	360	6
α_2	168	3	α_0	210	101

$\Delta_3^4(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2\}$ where

$\Lambda_1 = \{37797, 38920, 60738, 61698, 62101, 62131, 62135, 62140\}$

$\Lambda_2 = \{22977, 83012\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,0}$	1	759	$\alpha_{2,2}$	56	8
$\alpha_{0,2}$	7	1	$\alpha_{4,1}$	112	146
$\alpha_{0,0}$	7	26	$\alpha_{4,0}^{(2)}$	112	744
$\alpha_{4,2}$	14	136	$\alpha_{2,0}$	168	49
$\alpha_{0,1}$	16	3	$\alpha_{2,1}$	224	5
$\alpha_{4,0}^{(1)}$	42	745			

$\Delta_3^5(a)$, $L = Stab_G\{\Lambda_1\}$ where

$\Lambda_1 = \{479, 1125, 1151, 2252, 1151, 2252, 6955, 16379, 22977, 83012\}$ and

$\Lambda_2 = \{22977, 83012\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,2}$	1	1	$\alpha_{2,1}$	192	62
$\alpha_{2,2}$	16	100	$\alpha_{4,0}$	60	13
$\alpha_{4,2}$	60	2	$\alpha_{2,0}$	240	87
$\alpha_{4,1}$	160	3	$\alpha_{0,0}$	30	248

$\Delta_3^6(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$ where

$\Lambda_1 = \{4, 349, 970, 3320, 12882, 17319, 49646, 61991\}$

$\Lambda_2 = \{11170, 12411, 12416, 12422, 20545, 20551, 20560, 22613\}$

Λ_3 is the partition of Λ_1 given by $\{4, 349\}, \{970, 17319\}, \{3320, 12882\}, \{49646, 61991\}$.

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,8,2^4}$	1	1	$\alpha_{4,0,2^2}$	12	400
$\alpha_{0,8,0^4}$	1	595	$\alpha_{4,2,1^4}$	32	44
$\alpha_{0,0,0^4}$	1	635	$\alpha_{4,2,21^2}$	192	2
$\alpha^{(1)}_{0,4,0^4}$	12	730	$\alpha_{2,2,2}$	32	261
$\alpha^{(2)}_{0,4,0^4}$	16	504	$\alpha_{2,4,2}$	32	510
$\alpha_{4,4,1^4}$	16	24	$\alpha_{2,2,1^2}$	192	408
$\alpha_{4,0,1^4}$	16	56	$\alpha_{2,4,1^4}$	192	406
$\alpha_{4,4,2^2}$	12	113			

$\Delta^7_3(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$ where

$\Lambda_1 = \{43, 948, 17319, 29733\}$

$\Lambda_2 = \{158373, 169472\}$

$\Lambda_3 = \{182449, 194482\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{1,1,0}$	128	653	$\alpha_{3,0,1}$	32	5
$\alpha_{1,0,1}$	128	657	$\alpha_{4,2,2}$	1	136
$\alpha_{1,1,2}$	32	649	$\alpha_{4,0,0}$	4	662
$\alpha_{1,2,1}$	32	292	$\alpha^{(1)}_{0,0,0}$	6	101
$\alpha_{2,2,0}$	24	77	$\alpha^{(2)}_{0,0,0}$	24	607
$\alpha_{2,0,2}$	24	14	$\alpha_{0,2,2}$	4	1
$\alpha_{2,1,1}$	96	24	$\alpha_{0,2,0}$	16	519
$\alpha_{2,0,0}$	96	3	$\alpha_{0,0,2}$	16	511
$\alpha_{3,1,0}$	32	10	$\alpha_{0,1,1}$	64	386

$\Delta_3^8(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$ where

$\Lambda_1 = \{4, 970, 1080, 12882, 17319, 21350, 22598, 83012\}$

$\Lambda_2 = \{970, 1080, 17319, 83012\}$

$\Lambda_3 = \{83012\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,4,1}$	1	1	$\alpha_{4,2,1}$	72	2
$\alpha_{0,0,0}^{(1)}$	6	248	$\alpha_{2,0,0}$	96	491
$\alpha_{0,0,0}^{(2)}$	24	504	$\alpha_{2,1,0}$	192	195
$\alpha_{4,4,1}$	4	15	$\alpha_{2,2,0}$	48	226
$\alpha_{4,1,1}$	16	21	$\alpha_{4,0,0}$	4	102
$\alpha_{2,2,1}$	48	213	$\alpha_{4,1,0}$	48	10
$\alpha_{4,3,1}$	48	17	$\alpha_{4,2,0}$	72	6
$\alpha_{2,1,1}$	64	150	$\alpha_{4,3,0}$	16	65

$\Delta_3^9(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3\}$ where

$\Lambda_1 = \{445, 452, 1059, 1125, 16105, 17319, 28307, 83012\}$

$\Lambda_2 = \{17319\}$

$\Lambda_3 = \{83012\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,1,1}$	1	1	$\alpha^{(2)}_{4,0,1}$	40	23
$\alpha^{(1)}_{0,0,0}$	10	617	$\alpha_{4,1,1}$	60	2
$\alpha_{2,1,1}$	16	111	$\alpha_{4,0,0}$	60	55
$\alpha^{(2)}_{0,0,0}$	20	248	$\alpha_{2,1,0}$	96	100
$\alpha^{(1)}_{4,1,0}$	40	11	$\alpha_{2,0,1}$	96	300
$\alpha^{(2)}_{4,1,0}$	40	81	$\alpha_{2,0,0}$	240	176
$\alpha^{(1)}_{4,0,1}$	40	13			

$\Delta_3^{10}(a)$, $L = Stab_G\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ where

$\Lambda_1 = \{2, 445, 452, 948, 1059, 1151, 16105, 16379\}$

$\Lambda_2 = \{30887, 34121, 52240, 57768, 102195, 142053, 273221, 297652\}$

$\Lambda_3 = \{34642, 51319, 56950, 79889, 102237, 142051, 302809, 302904\}$

$\Lambda_4 = \{2, 948\}$

L - Orbit	Size	Octad Number	L - Orbit	Size	Octad Number
$\alpha_{8,0,0,2}$	1	1	$\alpha_{4,4,0,0}$	6	105
$\alpha_{0,8,0,0}$	1	741	$\alpha_{4,0,4,0}$	6	81
$\alpha_{0,0,8,0}$	1	594	$\alpha_{4,2,2,0}$	48	11
$\alpha^{(1)}_{0,4,4,0}$	12	368	$\alpha_{2,2,4,2}$	8	116
$\alpha^{(2)}_{0,4,4,0}$	16	248	$\alpha_{2,4,2,2}$	8	188
$\alpha_{4,4,0,2}$	6	94	$\alpha_{2,2,4,1}$	96	62
$\alpha_{4,0,4,2}$	6	23	$\alpha_{2,4,2,1}$	96	108
$\alpha_{4,2,2,2}$	48	3	$\alpha^{(1)}_{2,4,2,0}$	24	235
$\alpha_{4,4,0,1}$	16	18	$\alpha^{(2)}_{2,4,2,0}$	96	100
$\alpha_{4,0,4,1}$	16	38	$\alpha^{(1)}_{2,2,4,0}$	24	253
$\alpha^{(1)}_{4,2,2,1}$	96	2	$\alpha^{(2)}_{2,2,4,0}$	96	150
$\alpha^{(2)}_{4,2,2,1}$	32	5			

Moving out from $\Delta_4(a)$ to $\Delta_5(a)$ is far more complicated problem. We now have many more G_a orbits to deal with, and for each representative x from the G_a orbits in $\Delta_4(a)$ we have many more G_{ax} octad orbits, as G_{ax} acts on the octads of Ω_x. Thus working out the octad orbit representatives by hand, as was done for the first three discs, would be impractical. Therefore we use the following routine in Magma to calculate all the octad numbers for y a representative of $\Delta_4^j(a)$, as G_{ay} acts on the octads of Ω_y.

1. For a representative $y \in \Delta_4^j(a)$ calculate $\mathcal{O}_y$ the octads of Ω_y.
2. Choose an octad $O \in \mathcal{O}_y$, note its octad number and calculate $H = Stab_{G_{ay}}(O)$.
3. Calculate T, a transversal for H in $Stab_{ay}$, then $\{O^t \mid t \in T\}$ will be an octad orbit as G_{ay} acts on $\mathcal{O}_y$.
4. Let $\mathcal{O}_y = \mathcal{O}_y \setminus \{O^t \mid t \in T\}$ and go back to 2, until $\mathcal{O}_y = \emptyset$.

This will give us a full list of all the octad numbers for a representative from each of the octad orbits for each representative y of the 46 G_a orbits in $\Delta_4(a)$. Repeating the process as before we then calculate

$$\begin{aligned}\Omega_y &= \Omega_a^{(a_{10}*h*g)} \text{ and} \\ \Omega_y' &= \Omega_a^{(a_{10}*twiddle*h*g)}\end{aligned}$$

where $h = Tran[i]$ corresponding to the octad number i in question and h is the group element that takes us from a to our representative y in $\Delta_4(a)$. As before we now need to cross off anything in the third and fourth discs, again using the transposition profiles as an initial sieve, which will settle matters for elements in $\Delta_3(a)$, and then using `IsConjuagte` to finish things off. After carrying this out, we will have a list of vertices which are in $\Delta_5(a)$ and include a representative for each of the G_a orbits. However as in the $\Delta_4(a)$ case we will have many repetitions which need to be dealt with. Repeating the process as before, we can deal with these repetitions by using transposition profiles and `IsConjuagte`. We would like to point out that this took a considerable amount of time, in the region of a week on a 3.2GHz machine running MAGMA V2.11-15. Luckily we found there were 59 G_a orbits in $\Delta_5(a)$, giving us a total number of G_a orbits found as 120, the number calculated by S. Linton, proving $\mathcal{G}$ has diameter 5.

At this point we created a Magma command `WhereAmI`, which takes as input any base Ω_x of G and outputs which orbit of $\mathcal{G}$ the base Ω_x belongs to. This function works in the obvious way, firstly calculating the transposition profile for Ω_x, and then using the `IsConjugate` command on all the orbit representative of $\mathcal{G}$ with the same profile as Ω_x to determine exactly which orbit Ω_x belongs to.

We can now calculate all the neighbor data for our graph $\mathcal{G}$. That is, we can compute an array named $NeighbourData$, whose entries are themselves 1518 element arrays. Now say we calculate all 1518 neighbours for the ith orbit representative of $\mathcal{G}$ (where we order all 120 orbits of $\mathcal{G}$ first by which disc they are in, and then by stabilizer size), and suppose the jth neighbour was in orbit $\Delta_m^n(a)$, then the entry $NeighbourData[i][j] = [m, n]$. This array was calculated as expected, by running through all G_a orbit representatives x, calculating all 1518 neighbours of x and then using `WhereAmI` on each of them. As expected this was a considerable amount of work, in fact it took in excess of 28 days, running on 10 different machines (each a 3.2GHz machine running MAGMA V2.11-15 with 8GB of memory), giving us a total computational time of 280 days. At this point I would like to apologize to anybody who was trying to run calculations in the Mathematics Department of Manchester University over christmas 2008.

From this neighbour data, working out the collapsed adjacency matrix is very easy, we just needed to run through each of the 120 G_a orbit representatives and count up the number of neighbours from each G_a orbit. As all the hard work is already done this takes a matter of seconds. We give the full collapsed adjacency matrix in Section 2.6.

2.5 The Computer Files

In this section we will give descriptions for all the files associated with the investigation of $\mathcal{G}$. These files will be included both online at

`www.maths.manchester.ac.uk/~bwright/Fi24.zip`

and on CD. We first remark that the easiest way to load all the relevant files is to call the file `Fi24load.m` in MAGMA.

Fi24perms.m

In this file we have included the following:

- Generators $a_1, \ldots, a_{12}$ of Fi_{24} stored as permutations in $Sym(306936)$.
- Commands to define $G = Fi_{24}$ and $F = Fi_{24}'$.
- Generators $g_1, \ldots, g_9$, again stored as permutations in $Sym(306936)$ which generate G_a, a subgroup of shape $2^{12}.M_{24}$. This is the stabilizer of some base Ω_a of G, which corresponds to our fixed vertex a of $\mathcal{G}$.

- The base Ω_a, calculated as the smallest of the orbits as G_a acts on $\{1, \ldots, 306936\}$ and stored as the set `a`, as well as $\mathcal{O}_a$, the octadic transpositions for a, stored as `OctTran`, and the base Ω_b, stored as `b`, a neighbour of a.

- Words in the generators $g_1, \ldots, g_9$ which generate G_{al}, a subgroup of shape $2^{12}.2^4.Alt(8)$ which is the stabilizer in G_a of a line l, corresponding to the octad O of a, which is the intersection of a and b.

- An array named `Neighboursa` giving all 1518 neighbours of our fixed vertex a in $\mathcal{G}$. For a base Ω_x, such that $\Omega_x = \Omega_a^g$ for some $g \in G$, then the neighbours of x in $\mathcal{G}$ are given by `Neighboursa^g`.

- A word in the generators of G for the element `twiddle`. This is the element which takes us from x_1 to x_2, where a, x_1 and x_2 are the three points incident with the line l corresponding to the octad $O = \Omega_a \cap \Omega_b$.

reps_for_all_discs.m

- Contains words in the generators for G for group elements which take us from a to each of the 120 G_a orbits contained in the five discs of $\mathcal{G}$. These words are stored as arrays named `DisciOrbitj` corresponding to a representative in $\Delta_i^j(a)$. Use the function `MultiplyRandomWord` to convert this array into a usable group element.

- Contains arrays named `Disci`, containing the words for all representatives in $\Delta_i(a)$.

- Contains the array `Orbits`, containing all representatives.

MultiplyRandomWord.m

Contains the function `MultiplyRandomWord` used to convert a word in the generators of G into a usable permutation. To use type

```
MultiplyRandomWord(~z,Disc4Orbit23,G)
```

to convert, for example, the representative of $\Delta_4^{23}(a)$ into a usable group element, stored in Magma as the element `z`.

Tran.m

Contains an array named `Tran`, which contains a transversal for G_{al} in G_a, stored as words in the generators for G. Use the function `MultiplyRandomWord` to convert these into usable group elements. We remark that since we wanted these elements stored as words instead of actual permutations we couldn't simply use the `Transversal` command in Magma. This saved a considerable amount of memory - instead of needing 1.5GB to store the transversal, we only need 70KB. Storing the transversal in this way also guaranteed that we got the same coset representative every time, making our results reproducible. This transversal was produced using the following procedure.

1. Recall that a base Ω_a of G is a certain 24 element subset of Ω, where $\Omega = \{1 \dots 306936\}$. Therefore we calculate the action of each of the generators g_i of G_a on Ω_a. These permutations (in $Sym(24)$) $\overline{g_i}$ will generate a subgroup $\overline{G_a}$ of $Sym(24)$ isomorphic to M_{24}.

2. We now take the image of the generators for G_{al} under this mapping, to get elements in $\overline{G_a}$ which generate a subgroup $\overline{G_{al}}$ isomorphic to $2^4.Alt(8)$.

3. By generating random words in $\overline{G_a}$, in the generators $\overline{g_i}$, we can produce a representative for each of the 759 cosets of $\overline{G_{al}}$ in $\overline{G_a}$.

4. Finally we convert these words in the generators $\overline{g_i}$ to exactly the same words in the generators g_i (by simply removing the bar) to get a transversal for G_{al} in G_a as required.

Note that this procedure would have been impossible if we had stayed within the group G_a in the $Sym(306936)$ setting, as generating enough random elements to produce representatives for each of the 759 cosets would have taken too long.

TransProfile.m

Contains a function `Transprofile(x)`, which gives the transposition profile for the base Ω_x. Note that we have not stored the duadic transpositions for a, however the transposition profile for x can be calculated as $l_1|l_2|(24 - l_1 - l_2)$ where $l_1 = \Omega_x \cap \Omega_a$ and $l_2 = \Omega_x \cap \mathcal{O}_a$.

Octadsa.m

Gives all 759 octads for the base Ω_a, stored in the array `Octadsa`. To calculate the octads for the base Ω_x such that $\Omega_x = \Omega_a^g$ for $g \in G$, calculate `Octadsa^g`.

IsDistance3.m

Contains a function `IsDistance3(g)`, which quickly determines whether the base $\Omega_x = \Omega_a^g$ is contained within the first three discs of $\mathcal{G}$, and if so which orbit it is in. It will output an array $[i, j]$ corresponding to the orbit $\Delta_i^j(a)$, and will output $[0, 0]$ if Ω_x is not contained in the first three discs. This function is much faster than the `WhereAmI` command below, as it utilizes the fact that transposition profiles in the first three discs are (mostly) unique.

WhereAmI.m

Contains a function `WhereAmI(g)`, that determines which orbit of $\mathcal{G}$ the base $\Omega_x = \Omega_a^g$ belongs to. Outputs an ordered pair $[i, j]$ corresponding to the orbit $\Delta_i^j(a)$.

CollapsedAdjacencyMatrix.m

- Contains the collapsed adjacency matrix for $\mathcal{G}$, stored as an array (of arrays) called `CollapsedAdjacencyMatrix`. To calculate the number of points in the jth orbit connected to a single point in the ith orbit type

```
CollapsedAdjacencyMatrix[i][j]
```

- Contains two functions `NumberToName` and `NameToNumber`. The first converts an orbit number into its name (given as an array $[i, j]$ corresponding to $\Delta_i^j(a)$ and the other converts a orbit name to its number. Thus to calculate the number of elements in $\Delta_5^{30}(a)$ connected to a single point in $\Delta_4^{40}(a)$ type

```
CollapsedAdjacencyMatrix[NameToNumber([4,40])][NameToNumber([5,30])]
```

and you should get 18.

NeighbourData.m

Contains an array `NeighbourData` which gives information on the 1518 neighbours for each of the 120 G_a orbit representatives for $\mathcal{G}$. For the kth orbit (use `NameToNumber` to determine what k is for a particular orbit), `NeighbourData[k]` is an array of length 1518 listing the location of each neighbour, as an ordered pair $[i, j]$ corresponding to the orbit $\Delta_i^j(a)$.

Qa.m

Gives generators as words in the generators of G, for Q_a, the elementary abelian subgroup of G_a of order 2^{12}.

2.6 The Collapsed Adjacency Matrix for $\mathcal{G}$

In this section we will give the collapsed adjacency matrix for $\mathcal{G}$. As this matrix is rather large it is spread over a multiple number of pages, therefore to make it more usable we have included a map at the start to make finding a particular entry of interest easier. We have of course omitted any page completely filled with zeros, and this is indicated on the map. The entry, say d in the row indexed by Δ_i^j and column indexed by Δ_m^n gives the number of points in the orbit $\Delta_m^n(a)$ connected to a single point in $\Delta_i^j(a)$. For example the top row of our matrix tells us that the 1518 neighbours of the single point a in $\Delta_0^1(a)$ are in $\Delta_1^1(a)$ as expected and looking elsewhere in the matrix we can see that a vertex in $\Delta_4^{36}(a)$ is connected to 36 vertices in $\Delta_5^{28}(a)$.

	Δ_0^1 to Δ_3^7		Δ_3^8 to Δ_4^9		Δ_4^{10} to Δ_4^{21}		Δ_4^{22} to Δ_4^{33}		Δ_4^{34} to Δ_4^{45}
Δ_0^1 to Δ_4^{25}	Page 47	Δ_0^1 to Δ_4^{25}	Page 48	Δ_0^1 to Δ_4^{25}	Page 49	Δ_0^1 to Δ_4^{25}	Page 50	Δ_0^1 to Δ_4^{25}	Page 51
	Δ_0^1 to Δ_3^7		Δ_3^8 to Δ_4^9		Δ_4^{10} to Δ_4^{21}		Δ_4^{22} to Δ_4^{33}		Δ_4^{34} to Δ_4^{45}
Δ_4^{26} to Δ_5^{19}	Page 57	Δ_4^{26} to Δ_5^{19}	Page 58	Δ_4^{26} to Δ_5^{19}	Page 59	Δ_4^{26} to Δ_5^{19}	Page 60	Δ_4^{26} to Δ_5^{19}	Page 61
	Δ_0^1 to Δ_3^7		Δ_3^8 to Δ_4^9		Δ_4^{10} to Δ_4^{21}		Δ_4^{22} to Δ_4^{33}		Δ_4^{34} to Δ_4^{45}
Δ_5^{20} to Δ_5^{59}	All Zero	Δ_5^{20} to Δ_5^{59}	Page 67	Δ_5^{20} to Δ_5^{59}	Page 68	Δ_5^{20} to Δ_5^{59}	Page 69	Δ_5^{20} to Δ_5^{59}	Page 70
	Δ_4^{46} to Δ_5^{11}		Δ_5^{12} to Δ_5^{23}		Δ_5^{24} to Δ_5^{35}		Δ_5^{36} to Δ_5^{47}		Δ_5^{48} to Δ_5^{59}
Δ_0^1 to Δ_4^{25}	Page 52	Δ_0^1 to Δ_4^{25}	Page 53	Δ_0^1 to Δ_4^{25}	Page 54	Δ_0^1 to Δ_4^{25}	Page 55	Δ_0^1 to Δ_4^{25}	Page 56
	Δ_4^{46} to Δ_5^{11}		Δ_5^{12} to Δ_5^{23}		Δ_5^{24} to Δ_5^{35}		Δ_5^{36} to Δ_5^{47}		Δ_5^{48} to Δ_5^{59}
Δ_4^{26} to Δ_5^{19}	Page 62	Δ_4^{26} to Δ_5^{19}	Page 63	Δ_4^{26} to Δ_5^{19}	Page 64	Δ_4^{26} to Δ_5^{19}	Page 65	Δ_4^{26} to Δ_5^{19}	Page 66
	Δ_4^{46} to Δ_5^{11}		Δ_5^{12} to Δ_5^{23}		Δ_5^{24} to Δ_5^{35}		Δ_5^{36} to Δ_5^{47}		Δ_5^{48} to Δ_5^{59}
Δ_5^{20} to Δ_5^{59}	Page 71	Δ_5^{20} to Δ_5^{59}	Page 72	Δ_5^{20} to Δ_5^{59}	Page 73	Δ_5^{20} to Δ_5^{59}	Page 74	Δ_5^{20} to Δ_5^{59}	Page 75

	Δ_0^1	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7
Δ_0^1	0	1518	0	0	0	0	0	0	0	0	0	0
Δ_1^1	1	1	60	560	896	0	0	0	0	0	0	0
Δ_2^1	0	3	3	0	0	0	168	0	0	0	0	1344
Δ_2^2	0	5	0	5	0	0	20	128	480	0	240	0
Δ_2^3	0	1	0	0	1	16	0	0	120	16	60	120
Δ_3^1	0	0	0	0	77	0	0	0	0	77	0	0
Δ_3^2	0	0	9	6	0	0	15	0	0	0	0	0
Δ_3^3	0	0	0	21	0	0	0	21	0	336	0	0
Δ_3^4	0	0	0	7	14	0	0	0	21	112	0	0
Δ_3^5	0	0	0	0	1	1	0	16	60	76	0	0
Δ_3^6	0	0	0	1	2	0	0	0	0	0	3	0
Δ_3^7	0	0	1	0	4	0	0	0	0	0	0	5
Δ_3^8	0	0	0	1	4	0	0	0	0	0	0	0
Δ_3^9	0	0	0	0	1	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	1	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	7	0	8	0	42	14
Δ_4^2	0	0	0	0	0	0	7	0	8	0	42	14
Δ_4^3	0	0	0	0	0	0	0	0	15	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	15
Δ_4^5	0	0	0	0	0	1	0	0	0	0	14	7
Δ_4^6	0	0	0	0	0	1	0	0	0	0	14	7
Δ_4^7	0	0	0	0	0	1	0	0	0	0	0	0
Δ_4^8	0	0	0	0	0	0	0	1	0	0	9	0
Δ_4^9	0	0	0	0	0	0	0	1	0	0	9	0
Δ_4^{10}	0	0	0	0	0	0	1	0	0	0	0	12
Δ_4^{11}	0	0	0	0	0	0	1	0	0	0	0	12
Δ_4^{12}	0	0	0	0	0	0	1	0	4	0	8	18
Δ_4^{13}	0	0	0	0	0	0	0	0	0	0	0	3
Δ_4^{14}	0	0	0	0	0	0	0	0	0	1	0	1
Δ_4^{15}	0	0	0	0	0	0	0	0	0	1	0	1
Δ_4^{16}	0	0	0	0	0	0	0	1	7	0	0	0
Δ_4^{17}	0	0	0	0	0	0	0	0	0	11	0	0
Δ_4^{18}	0	0	0	0	0	0	0	0	3	0	0	0
Δ_4^{19}	0	0	0	0	0	0	0	0	0	1	1	0
Δ_4^{20}	0	0	0	0	0	0	0	0	0	1	1	0
Δ_4^{21}	0	0	0	0	0	0	0	0	0	5	0	0
Δ_4^{22}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_4^{23}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_4^{24}	0	0	0	0	0	0	0	0	0	0	0	5
Δ_4^{25}	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_3^8	Δ_3^9	Δ_3^{10}	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	640	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	320	384	480	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	330	330	704	0	0
Δ_3^2	0	0	0	144	144	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	210	210
Δ_3^4	0	0	0	8	8	32	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	12	12	0	0	32	32	0	48	48
Δ_3^7	0	0	0	4	4	0	24	16	16	0	0	0
Δ_3^8	5	0	0	0	0	16	0	0	0	64	10	10
Δ_3^9	0	97	0	0	0	0	0	0	0	56	0	0
Δ_3^{10}	0	0	1	0	0	0	0	15	15	0	0	0
Δ_4^1	0	0	0	7	22	0	0	0	0	0	0	0
Δ_4^2	0	0	0	22	7	0	0	0	0	0	0	0
Δ_4^3	70	0	0	0	0	15	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^5	0	0	105	0	0	0	0	0	1	0	0	0
Δ_4^6	0	0	105	0	0	0	0	1	0	0	0	0
Δ_4^7	35	147	0	0	0	0	0	0	0	1	0	0
Δ_4^8	5	0	0	0	0	0	0	0	0	0	6	9
Δ_4^9	5	0	0	0	0	0	0	0	0	0	9	6
Δ_4^{10}	8	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{11}	8	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{12}	0	0	32	2	2	0	0	0	0	0	0	0
Δ_4^{13}	0	0	0	0	0	0	3	0	0	0	0	0
Δ_4^{14}	0	0	1	6	0	0	0	2	0	0	0	24
Δ_4^{15}	0	0	1	0	6	0	0	0	2	0	24	0
Δ_4^{16}	35	42	0	0	0	0	0	0	0	0	0	0
Δ_4^{17}	0	11	0	0	0	11	0	0	0	11	0	0
Δ_4^{18}	14	84	0	0	0	2	0	0	0	32	0	0
Δ_4^{19}	0	0	3	0	1	0	0	1	3	0	4	0
Δ_4^{20}	0	0	3	1	0	0	0	3	1	0	0	4
Δ_4^{21}	5	51	0	0	0	0	0	0	0	20	0	0
Δ_4^{22}	4	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{23}	2	0	16	0	0	0	0	0	0	0	0	0
Δ_4^{24}	4	0	12	0	0	0	0	0	0	0	0	0
Δ_4^{25}	0	0	8	0	0	0	4	0	0	0	0	0

	Δ_4^{10}	Δ_4^{11}	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	384	384	432	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	720	0	0	0	0	0
Δ_3^4	0	0	84	0	0	0	448	0	224	0	0	0
Δ_3^5	0	0	0	0	30	30	0	384	0	60	60	320
Δ_3^6	0	0	48	0	0	0	0	0	0	32	32	0
Δ_3^7	64	64	108	24	16	16	0	0	0	0	0	0
Δ_3^8	16	16	0	0	0	0	240	0	112	0	0	64
Δ_3^9	0	0	0	0	0	0	60	16	140	0	0	136
Δ_3^{10}	0	0	12	0	1	1	0	0	0	6	6	0
Δ_4^1	0	0	42	0	336	0	0	0	0	0	112	0
Δ_4^2	0	0	42	0	0	336	0	0	0	112	0	0
Δ_4^3	0	0	0	0	0	0	0	336	70	0	0	0
Δ_4^4	0	0	0	15	0	0	0	0	0	0	0	0
Δ_4^5	0	0	0	0	14	0	0	0	0	14	42	0
Δ_4^6	0	0	0	0	0	14	0	0	0	42	14	0
Δ_4^7	0	0	0	0	0	0	0	42	140	0	0	140
Δ_4^8	0	0	0	0	0	72	0	0	0	24	0	0
Δ_4^9	0	0	0	0	72	0	0	0	0	0	24	0
Δ_4^{10}	13	8	0	0	24	0	0	0	0	72	0	0
Δ_4^{11}	8	13	0	0	0	24	0	0	0	0	72	0
Δ_4^{12}	0	0	27	16	16	16	0	0	0	80	80	0
Δ_4^{13}	0	0	12	12	0	0	0	0	0	0	0	0
Δ_4^{14}	8	0	6	0	14	27	8	0	0	12	0	0
Δ_4^{15}	0	8	6	0	27	14	8	0	0	0	12	0
Δ_4^{16}	0	0	0	0	7	7	43	168	42	7	7	168
Δ_4^{17}	0	0	0	0	0	0	165	132	55	0	0	110
Δ_4^{18}	0	0	0	0	0	0	36	48	95	0	0	192
Δ_4^{19}	12	0	15	0	6	0	4	0	0	25	20	0
Δ_4^{20}	0	12	15	0	0	6	4	0	0	20	25	0
Δ_4^{21}	0	0	0	0	0	0	90	60	120	0	0	155
Δ_4^{22}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{23}	0	0	0	0	0	0	0	0	0	12	12	0
Δ_4^{24}	0	0	0	0	0	0	0	0	0	16	16	0
Δ_4^{25}	1	1	1	4	0	0	0	0	0	4	4	0

	Δ_4^{22}	Δ_4^{23}	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_4^{31}	Δ_4^{32}	Δ_4^{33}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	224	0	0	336	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	240	240	0	0	0	0
Δ_3^6	64	0	0	0	384	0	0	0	0	0	0	0
Δ_3^7	0	0	320	0	192	128	0	0	0	0	0	0
Δ_3^8	96	48	96	0	144	0	0	0	0	0	0	0
Δ_3^9	0	0	0	0	0	20	0	0	0	0	0	0
Δ_3^{10}	0	64	48	48	84	0	0	0	8	8	96	96
Δ_4^1	0	0	0	0	0	0	0	0	448	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	448	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	0	0	240	0	0	0	0	0	0	0	0
Δ_4^5	0	0	0	0	0	0	0	0	0	56	0	0
Δ_4^6	0	0	0	0	0	0	0	0	56	0	0	0
Δ_4^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^8	0	0	0	0	0	0	72	168	0	0	0	0
Δ_4^9	0	0	0	0	0	0	168	72	0	0	0	0
Δ_4^{10}	0	0	0	18	0	0	0	0	216	120	0	0
Δ_4^{11}	0	0	0	18	0	0	0	0	120	216	0	0
Δ_4^{12}	0	0	0	16	32	0	64	64	128	128	0	0
Δ_4^{13}	0	0	0	48	0	0	0	0	0	0	0	0
Δ_4^{14}	0	0	0	0	72	0	0	24	0	48	0	0
Δ_4^{15}	0	0	0	0	72	0	24	0	48	0	0	0
Δ_4^{16}	0	0	0	0	0	0	28	28	42	42	0	0
Δ_4^{17}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{19}	0	24	32	12	30	0	24	4	32	12	0	0
Δ_4^{20}	0	24	32	12	30	0	4	24	12	32	0	0
Δ_4^{21}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{22}	5	0	0	24	0	0	4	4	12	12	0	0
Δ_4^{23}	0	19	0	0	0	0	52	52	0	0	0	0
Δ_4^{24}	0	0	9	6	12	0	0	0	52	52	0	0
Δ_4^{25}	16	0	4	4	0	0	0	0	0	0	0	0

	Δ_4^{34}	Δ_4^{35}	Δ_4^{36}	Δ_4^{37}	Δ_4^{38}	Δ_4^{39}	Δ_4^{40}	Δ_4^{41}	Δ_4^{42}	Δ_4^{43}	Δ_4^{44}	Δ_4^{45}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	0	384	384	0	0	0	0	0	0
Δ_3^7	256	256	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	96	96	0	0	384	0	0	0	0	0
Δ_3^9	40	40	20	20	60	60	80	96	96	0	0	240
Δ_3^{10}	32	32	16	16	72	72	192	0	0	96	96	192
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^9	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{11}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{12}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{13}	0	0	0	0	96	96	0	0	0	0	0	0
Δ_4^{14}	0	0	0	96	48	24	0	0	0	0	0	0
Δ_4^{15}	0	0	96	0	24	48	0	0	0	0	0	0
Δ_4^{16}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{17}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{19}	0	0	0	0	36	24	0	0	0	144	0	96
Δ_4^{20}	0	0	0	0	24	36	0	0	0	0	144	0
Δ_4^{21}	0	0	0	0	0	0	0	0	0	90	90	0
Δ_4^{22}	0	0	0	0	0	0	0	0	0	96	96	0
Δ_4^{23}	0	0	0	0	0	0	0	0	0	48	48	0
Δ_4^{24}	0	0	0	0	0	0	0	0	0	24	24	0
Δ_4^{25}	32	32	0	0	20	20	0	64	64	48	48	0

	Δ_4^{46}	Δ_5^1	Δ_5^2	Δ_5^3	Δ_5^4	Δ_5^5	Δ_5^6	Δ_5^7	Δ_5^8	Δ_5^9	Δ_5^{10}	Δ_5^{11}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^9	240	0	0	0	0	0	0	0	0	0	0	0
Δ_3^{10}	192	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	32	0	0	224	0	0	0	0	224	0
Δ_4^2	0	32	0	0	0	0	224	0	0	224	0	0
Δ_4^3	0	16	16	0	0	0	0	0	0	210	210	0
Δ_4^4	0	0	0	0	0	0	0	120	120	0	0	0
Δ_4^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^6	0	0	0	0	0	0	0	0	0	0	0	32
Δ_4^7	0	0	0	0	0	0	0	0	0	0	0	30
Δ_4^8	0	0	0	0	0	0	48	0	48	0	0	0
Δ_4^9	0	0	0	0	0	48	0	48	0	0	0	0
Δ_4^{10}	0	8	0	0	16	0	8	12	12	24	0	0
Δ_4^{11}	0	0	8	16	0	8	0	12	12	0	24	96
Δ_4^{12}	0	0	0	0	0	0	0	32	32	0	0	0
Δ_4^{13}	0	0	0	0	0	0	0	24	24	0	0	0
Δ_4^{14}	0	0	0	16	0	0	0	48	0	0	0	0
Δ_4^{15}	0	0	0	0	16	0	0	0	48	0	0	0
Δ_4^{16}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{17}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	0	0	4	4	12	12	6	6	0
Δ_4^{19}	0	0	0	0	0	0	0	0	0	0	24	0
Δ_4^{20}	96	0	0	0	0	0	0	0	0	24	0	0
Δ_4^{21}	0	0	0	1	1	0	0	0	0	0	0	0
Δ_4^{22}	0	0	0	0	0	0	0	32	32	12	12	0
Δ_4^{23}	0	2	2	0	0	2	2	0	0	8	8	0
Δ_4^{24}	0	0	0	0	0	4	4	20	20	0	0	0
Δ_4^{25}	0	0	0	0	0	0	0	2	2	0	0	0

	Δ_5^{12}	Δ_5^{13}	Δ_5^{14}	Δ_5^{15}	Δ_5^{16}	Δ_5^{17}	Δ_5^{18}	Δ_5^{19}	Δ_5^{20}	Δ_5^{21}	Δ_5^{22}	Δ_5^{23}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^9	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	0	0	240	0	0	0	0	0	0	0	0
Δ_4^5	32	0	112	168	0	0	0	0	0	0	0	0
Δ_4^6	0	112	0	168	0	0	0	0	0	0	0	0
Δ_4^7	30	35	35	0	0	0	0	0	0	0	0	0
Δ_4^8	0	0	64	0	0	64	0	0	0	0	0	0
Δ_4^9	0	64	0	0	64	0	0	0	0	0	0	0
Δ_4^{10}	96	16	0	18	0	0	0	0	0	0	0	48
Δ_4^{11}	0	0	16	18	0	0	0	0	0	0	0	0
Δ_4^{12}	0	0	0	16	0	0	0	0	0	0	0	0
Δ_4^{13}	0	0	0	48	0	0	0	0	0	0	0	0
Δ_4^{14}	0	0	0	0	0	64	0	192	0	0	0	128
Δ_4^{15}	0	0	0	0	64	0	192	0	0	0	0	16
Δ_4^{16}	0	0	0	0	0	0	0	0	32	0	0	0
Δ_4^{17}	0	0	0	0	0	0	55	55	0	66	66	55
Δ_4^{18}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{19}	0	0	0	0	0	32	32	0	0	96	0	0
Δ_4^{20}	0	0	0	0	32	0	0	32	0	0	96	0
Δ_4^{21}	0	0	0	0	10	10	0	0	0	0	0	15
Δ_4^{22}	0	0	0	0	16	16	0	0	0	0	0	0
Δ_4^{23}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{24}	0	0	0	6	16	16	0	0	64	0	0	0
Δ_4^{25}	0	16	16	7	0	0	0	0	0	0	0	0

	Δ_5^{24}	Δ_5^{25}	Δ_5^{26}	Δ_5^{27}	Δ_5^{28}	Δ_5^{29}	Δ_5^{30}	Δ_5^{31}	Δ_5^{32}	Δ_5^{33}	Δ_5^{34}	Δ_5^{35}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^9	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	560	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^5	0	224	112	0	0	0	0	0	168	0	0	0
Δ_4^6	0	112	224	0	0	0	0	168	0	0	0	0
Δ_4^7	0	105	105	0	0	0	0	0	0	0	0	0
Δ_4^8	0	192	0	0	128	128	0	0	0	0	0	0
Δ_4^9	0	0	192	128	0	128	0	0	0	0	0	0
Δ_4^{10}	0	48	0	192	0	0	144	0	0	96	0	0
Δ_4^{11}	48	0	48	0	192	0	144	0	0	0	96	0
Δ_4^{12}	0	0	0	0	0	0	64	64	64	0	0	0
Δ_4^{13}	0	0	0	0	0	0	0	96	96	0	0	0
Δ_4^{14}	16	0	0	0	0	0	0	0	24	32	0	0
Δ_4^{15}	128	0	0	0	0	0	0	24	0	0	32	0
Δ_4^{16}	0	0	0	56	56	112	84	0	0	0	0	0
Δ_4^{17}	55	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	16	16	0	0	36	36	0	0	0
Δ_4^{19}	0	0	0	0	0	0	0	36	12	64	0	0
Δ_4^{20}	0	0	0	0	0	0	0	12	36	0	64	0
Δ_4^{21}	15	0	0	0	0	0	0	0	0	30	30	0
Δ_4^{22}	0	0	0	80	80	112	36	12	12	32	32	0
Δ_4^{23}	0	48	48	0	0	48	48	0	0	32	32	0
Δ_4^{24}	0	24	24	0	0	32	96	0	0	0	0	48
Δ_4^{25}	0	0	0	0	0	0	0	28	28	0	0	128

	Δ_5^{36}	Δ_5^{37}	Δ_5^{38}	Δ_5^{39}	Δ_5^{40}	Δ_5^{41}	Δ_5^{42}	Δ_5^{43}	Δ_5^{44}	Δ_5^{45}	Δ_5^{46}	Δ_5^{47}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^9	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	768
Δ_4^5	0	0	0	0	0	0	0	0	0	448	0	0
Δ_4^6	0	0	0	0	0	0	0	0	0	0	448	0
Δ_4^7	0	0	0	0	0	0	0	0	0	210	210	252
Δ_4^8	0	0	192	0	0	0	0	0	0	0	0	0
Δ_4^9	0	192	0	0	0	0	0	0	0	0	0	0
Δ_4^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{11}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{12}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{13}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{14}	0	0	0	0	192	0	0	0	0	0	0	0
Δ_4^{15}	0	0	0	192	0	0	0	0	0	0	0	0
Δ_4^{16}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{17}	0	55	55	110	110	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	0	0	0	0	48	48	0	0	0
Δ_4^{19}	0	128	64	32	0	192	96	96	0	0	0	0
Δ_4^{20}	0	64	128	0	32	96	192	0	96	0	0	0
Δ_4^{21}	0	60	60	30	30	90	90	0	0	0	0	0
Δ_4^{22}	0	0	0	0	0	0	0	0	0	48	48	192
Δ_4^{23}	0	0	0	48	48	0	0	96	96	80	80	0
Δ_4^{24}	48	0	0	48	48	0	0	16	16	64	64	0
Δ_4^{25}	128	0	0	0	0	64	64	96	96	32	32	0

	Δ_5^{48}	Δ_5^{49}	Δ_5^{50}	Δ_5^{51}	Δ_5^{52}	Δ_5^{53}	Δ_5^{54}	Δ_5^{55}	Δ_5^{56}	Δ_5^{57}	Δ_5^{58}	Δ_5^{59}
Δ_0^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_1^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_2^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^8	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^9	0	0	0	0	0	0	0	0	0	0	0	0
Δ_3^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^1	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^2	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^5	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^6	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^7	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^8	0	0	0	0	288	0	0	0	0	0	0	0
Δ_4^9	0	0	0	288	0	0	0	0	0	0	0	0
Δ_4^{10}	0	0	0	0	0	288	0	0	0	0	0	0
Δ_4^{11}	0	0	0	0	0	0	288	0	0	0	0	0
Δ_4^{12}	0	0	0	0	0	256	256	0	0	0	0	0
Δ_4^{13}	192	0	0	0	0	0	0	0	0	0	0	768
Δ_4^{14}	0	288	0	0	96	0	0	0	0	0	0	0
Δ_4^{15}	0	0	288	96	0	0	0	0	0	0	0	0
Δ_4^{16}	0	0	0	84	84	0	0	0	0	0	336	0
Δ_4^{17}	0	165	165	0	0	0	0	0	0	0	0	0
Δ_4^{18}	0	0	0	72	72	144	144	0	96	96	144	0
Δ_4^{19}	0	48	0	0	0	0	0	0	0	0	0	0
Δ_4^{20}	0	0	48	0	0	0	0	0	0	0	0	0
Δ_4^{21}	0	90	90	0	0	0	0	0	0	0	0	180
Δ_4^{22}	24	0	0	0	0	0	0	0	96	96	240	0
Δ_4^{23}	0	0	0	0	0	144	144	0	0	0	240	0
Δ_4^{24}	0	0	0	24	24	120	120	0	64	64	192	0
Δ_4^{25}	112	0	0	32	32	16	16	64	64	64	0	0

	Δ_0^1	Δ_1^1	Δ_2^1	Δ_2^2	Δ_2^3	Δ_3^1	Δ_3^2	Δ_3^3	Δ_3^4	Δ_3^5	Δ_3^6	Δ_3^7
Δ_4^{26}	0	0	0	0	0	0	0	0	1	0	4	2
Δ_4^{27}	0	0	0	0	0	0	0	0	0	0	0	1
Δ_4^{28}	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{29}	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{30}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{31}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{32}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{33}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{34}	0	0	0	0	0	0	0	0	0	0	0	1
Δ_4^{35}	0	0	0	0	0	0	0	0	0	0	0	1
Δ_4^{36}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{37}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{38}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_4^{39}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_4^{40}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{41}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{42}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{43}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{44}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{45}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{46}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{1}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{4}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{5}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{6}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{7}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{8}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{9}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{10}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{11}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{12}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{13}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{14}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{15}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{16}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{17}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{18}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{19}	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_3^8	Δ_3^9	Δ_3^{10}	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9
Δ_4^{26}	4	0	14	0	0	0	0	0	0	0	0	0
Δ_4^{27}	0	2	0	0	0	0	0	0	0	0	0	0
Δ_4^{28}	0	0	0	0	0	0	0	0	0	0	3	7
Δ_4^{29}	0	0	0	0	0	0	0	0	0	0	7	3
Δ_4^{30}	0	0	1	1	0	0	0	0	1	0	0	0
Δ_4^{31}	0	0	1	0	1	0	0	1	0	0	0	0
Δ_4^{32}	0	0	9	0	0	0	0	0	0	0	0	0
Δ_4^{33}	0	0	9	0	0	0	0	0	0	0	0	0
Δ_4^{34}	0	2	2	0	0	0	0	0	0	0	0	0
Δ_4^{35}	0	2	2	0	0	0	0	0	0	0	0	0
Δ_4^{36}	1	1	1	0	0	0	0	0	0	0	0	0
Δ_4^{37}	1	1	1	0	0	0	0	0	0	0	0	0
Δ_4^{38}	0	2	3	0	0	0	0	0	0	0	0	0
Δ_4^{39}	0	2	3	0	0	0	0	0	0	0	0	0
Δ_4^{40}	1	1	3	0	0	0	0	0	0	0	0	0
Δ_4^{41}	0	1	0	0	0	0	0	0	0	0	0	0
Δ_4^{42}	0	1	0	0	0	0	0	0	0	0	0	0
Δ_4^{43}	0	0	1	0	0	0	0	0	0	0	0	0
Δ_4^{44}	0	0	1	0	0	0	0	0	0	0	0	0
Δ_4^{45}	0	1	1	0	0	0	0	0	0	0	0	0
Δ_4^{46}	0	1	1	0	0	0	0	0	0	0	0	0
Δ_5^1	0	0	0	0	15	16	0	0	0	0	0	0
Δ_5^2	0	0	0	15	0	16	0	0	0	0	0	0
Δ_5^3	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^4	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^5	0	0	0	3	0	0	0	0	0	0	0	12
Δ_5^6	0	0	0	0	3	0	0	0	0	0	12	0
Δ_5^7	0	0	0	0	0	0	3	0	0	0	0	4
Δ_5^8	0	0	0	0	0	0	3	0	0	0	4	0
Δ_5^9	0	0	0	0	1	2	0	0	0	0	0	0
Δ_5^{10}	0	0	0	1	0	2	0	0	0	0	0	0
Δ_5^{11}	0	0	0	0	0	0	0	0	1	2	0	0
Δ_5^{12}	0	0	0	0	0	0	0	1	0	2	0	0
Δ_5^{13}	0	0	0	0	0	0	0	0	3	2	0	4
Δ_5^{14}	0	0	0	0	0	0	0	3	0	2	4	0
Δ_5^{15}	0	0	0	0	0	0	4	4	4	0	0	0
Δ_5^{16}	0	0	0	0	0	0	0	0	0	0	0	3
Δ_5^{17}	0	0	0	0	0	0	0	0	0	0	3	0
Δ_5^{18}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{19}	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_4^{10}	Δ_4^{11}	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}
Δ_4^{26}	0	0	2	0	12	12	0	0	0	10	10	0
Δ_4^{27}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{28}	0	0	3	0	0	3	4	0	0	6	1	0
Δ_4^{29}	0	0	3	0	3	0	4	0	0	1	6	0
Δ_4^{30}	9	5	6	0	0	6	6	0	0	8	3	0
Δ_4^{31}	5	9	6	0	6	0	6	0	0	3	8	0
Δ_4^{32}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{33}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{34}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{35}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{36}	0	0	0	0	0	6	0	0	0	0	0	0
Δ_4^{37}	0	0	0	0	6	0	0	0	0	0	0	0
Δ_4^{38}	0	0	0	2	2	1	0	0	0	3	2	0
Δ_4^{39}	0	0	0	2	1	2	0	0	0	2	3	0
Δ_4^{40}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{41}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{42}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{43}	0	0	0	0	0	0	0	0	0	3	0	2
Δ_4^{44}	0	0	0	0	0	0	0	0	0	0	3	2
Δ_4^{45}	0	0	0	0	0	0	0	0	0	1	0	0
Δ_4^{46}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_5^{1}	70	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	70	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	0	15	0	0	45	0	0	0	0	0	0	6
Δ_5^{4}	15	0	0	0	0	45	0	0	0	0	0	6
Δ_5^{5}	0	2	0	0	0	0	0	0	4	0	0	0
Δ_5^{6}	2	0	0	0	0	0	0	0	4	0	0	0
Δ_5^{7}	1	1	3	3	12	0	0	0	4	0	0	0
Δ_5^{8}	1	1	3	3	0	12	0	0	4	0	0	0
Δ_5^{9}	2	0	0	0	0	0	0	0	2	0	12	0
Δ_5^{10}	0	2	0	0	0	0	0	0	2	12	0	0
Δ_5^{11}	0	7	0	0	0	0	0	0	0	0	0	0
Δ_5^{12}	7	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{13}	1	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{14}	0	1	0	0	0	0	0	0	0	0	0	0
Δ_5^{15}	1	1	1	4	0	0	0	0	0	0	0	0
Δ_5^{16}	0	0	0	0	0	9	0	0	0	0	9	3
Δ_5^{17}	0	0	0	0	9	0	0	0	0	9	0	3
Δ_5^{18}	0	0	0	0	0	27	0	9	0	9	0	0
Δ_5^{19}	0	0	0	0	27	0	0	9	0	0	9	0

	Δ_4^{22}	Δ_4^{23}	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_4^{31}	Δ_4^{32}	Δ_4^{33}
Δ_4^{26}	0	0	8	0	15	0	44	44	44	44	0	0
Δ_4^{27}	0	0	0	0	0	3	6	6	6	6	0	0
Δ_4^{28}	2	26	0	0	33	6	33	30	0	0	8	0
Δ_4^{29}	2	26	0	0	33	6	30	33	0	0	0	8
Δ_4^{30}	6	0	26	0	33	6	0	0	33	36	0	4
Δ_4^{31}	6	0	26	0	33	6	0	0	36	33	4	0
Δ_4^{32}	0	0	0	0	0	0	6	0	0	3	0	0
Δ_4^{33}	0	0	0	0	0	0	0	6	3	0	0	0
Δ_4^{34}	0	0	0	12	0	0	0	12	6	6	0	0
Δ_4^{35}	0	0	0	12	0	0	12	0	6	6	0	0
Δ_4^{36}	0	0	0	0	0	0	0	6	0	6	0	0
Δ_4^{37}	0	0	0	0	0	0	6	0	6	0	0	0
Δ_4^{38}	0	0	0	5	1	0	0	18	2	2	0	0
Δ_4^{39}	0	0	0	5	1	0	18	0	2	2	0	0
Δ_4^{40}	0	0	0	0	0	12	6	6	3	3	0	0
Δ_4^{41}	0	0	0	5	0	0	0	0	5	0	0	10
Δ_4^{42}	0	0	0	5	0	0	0	0	0	5	10	0
Δ_4^{43}	4	2	1	3	1	0	4	2	1	5	5	0
Δ_4^{44}	4	2	1	3	1	0	2	4	5	1	0	5
Δ_4^{45}	0	0	0	0	0	6	2	1	0	2	6	4
Δ_4^{46}	0	0	0	0	0	6	1	2	2	0	4	6
Δ_5^{1}	0	210	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	210	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{4}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{5}	0	6	12	0	0	36	0	36	0	36	0	0
Δ_5^{6}	0	6	12	0	0	36	36	0	36	0	0	0
Δ_5^{7}	32	0	20	3	12	0	0	0	0	0	0	0
Δ_5^{8}	32	0	20	3	12	0	0	0	0	0	0	0
Δ_5^{9}	12	8	0	0	12	0	0	0	0	0	0	0
Δ_5^{10}	12	8	0	0	12	0	0	0	0	0	0	0
Δ_5^{11}	0	0	0	0	0	42	0	0	0	0	0	0
Δ_5^{12}	0	0	0	0	0	42	0	0	0	0	0	0
Δ_5^{13}	0	0	0	18	18	0	0	0	0	6	24	0
Δ_5^{14}	0	0	0	18	18	0	0	0	6	0	0	24
Δ_5^{15}	0	0	4	7	8	0	0	0	0	0	48	48
Δ_5^{16}	9	0	9	0	0	0	0	27	0	9	18	0
Δ_5^{17}	9	0	9	0	0	0	27	0	9	0	0	18
Δ_5^{18}	0	0	0	0	0	0	45	27	0	36	0	0
Δ_5^{19}	0	0	0	0	0	0	27	45	36	0	0	0

	Δ_4^{34}	Δ_4^{35}	Δ_4^{36}	Δ_4^{37}	Δ_4^{38}	Δ_4^{39}	Δ_4^{40}	Δ_4^{41}	Δ_4^{42}	Δ_4^{43}	Δ_4^{44}	Δ_4^{45}
Δ_4^{26}	0	0	0	0	4	4	0	0	0	16	16	0
Δ_4^{27}	0	0	0	0	0	0	96	0	0	0	0	144
Δ_4^{28}	0	24	0	12	0	54	48	0	0	48	24	48
Δ_4^{29}	24	0	12	0	54	0	48	0	0	24	48	24
Δ_4^{30}	12	12	0	12	6	6	24	48	0	12	60	0
Δ_4^{31}	12	12	12	0	6	6	24	0	48	60	12	48
Δ_4^{32}	0	0	0	0	0	0	0	0	72	45	0	108
Δ_4^{33}	0	0	0	0	0	0	0	72	0	0	45	72
Δ_4^{34}	3	26	0	24	0	30	0	48	0	6	72	24
Δ_4^{35}	26	3	24	0	30	0	0	0	48	72	6	72
Δ_4^{36}	0	24	24	3	0	24	0	72	24	24	108	0
Δ_4^{37}	24	0	3	24	24	0	0	24	72	108	24	156
Δ_4^{38}	0	20	0	16	3	20	16	48	16	16	76	48
Δ_4^{39}	20	0	16	0	20	3	16	16	48	76	16	96
Δ_4^{40}	0	0	0	0	6	6	5	48	48	33	33	42
Δ_4^{41}	10	0	15	5	15	5	40	26	26	40	40	125
Δ_4^{42}	0	10	5	15	5	15	40	26	26	40	40	105
Δ_4^{43}	1	12	4	18	4	19	22	32	32	57	40	70
Δ_4^{44}	12	1	18	4	19	4	22	32	32	40	57	56
Δ_4^{45}	2	6	0	13	6	12	14	50	42	35	28	83
Δ_4^{46}	6	2	13	0	12	6	14	42	50	28	35	91
Δ_5^{1}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	0	45	0	90	0	0	0	0	216	0	0	0
Δ_5^{4}	45	0	90	0	0	0	0	216	0	0	0	0
Δ_5^{5}	0	48	0	72	36	144	0	0	0	0	0	0
Δ_5^{6}	48	0	72	0	144	36	0	0	0	0	0	0
Δ_5^{7}	48	24	24	0	36	0	32	0	0	24	0	96
Δ_5^{8}	24	48	0	24	0	36	32	0	0	0	24	0
Δ_5^{9}	0	16	80	0	84	0	96	0	0	0	0	48
Δ_5^{10}	16	0	0	80	0	84	96	0	0	0	0	0
Δ_5^{11}	42	35	7	21	0	0	0	84	84	0	84	0
Δ_5^{12}	35	42	21	7	0	0	0	84	84	84	0	126
Δ_5^{13}	3	0	0	90	0	72	84	72	0	36	36	0
Δ_5^{14}	0	3	90	0	72	0	84	0	72	36	36	108
Δ_5^{15}	16	16	32	32	28	28	0	64	64	64	64	96
Δ_5^{16}	0	27	0	0	0	27	18	0	0	27	81	54
Δ_5^{17}	27	0	0	0	27	0	18	0	0	81	27	54
Δ_5^{18}	0	0	0	9	0	27	0	0	0	27	27	0
Δ_5^{19}	0	0	9	0	27	0	0	0	0	27	27	27

	Δ_4^{46}	Δ_5^{1}	Δ_5^{2}	Δ_5^{3}	Δ_5^{4}	Δ_5^{5}	Δ_5^{6}	Δ_5^{7}	Δ_5^{8}	Δ_5^{9}	Δ_5^{10}	Δ_5^{11}
Δ_4^{26}	0	0	0	0	0	0	0	8	8	8	8	0
Δ_4^{27}	144	0	0	0	0	6	6	0	0	0	0	24
Δ_4^{28}	24	0	0	0	0	0	6	0	0	0	0	0
Δ_4^{29}	48	0	0	0	0	6	0	0	0	0	0	0
Δ_4^{30}	48	0	0	0	0	0	6	0	0	0	0	0
Δ_4^{31}	0	0	0	0	0	6	0	0	0	0	0	0
Δ_4^{32}	72	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{33}	108	0	0	0	0	0	0	0	0	0	0	0
Δ_4^{34}	72	0	0	0	1	0	4	12	6	0	4	12
Δ_4^{35}	24	0	0	1	0	4	0	6	12	4	0	10
Δ_4^{36}	156	0	0	0	2	0	6	6	0	20	0	2
Δ_4^{37}	0	0	0	2	0	6	0	0	6	0	20	6
Δ_4^{38}	96	0	0	0	0	2	8	6	0	14	0	0
Δ_4^{39}	48	0	0	0	0	8	2	0	6	0	14	0
Δ_4^{40}	42	0	0	0	0	0	0	2	2	6	6	0
Δ_4^{41}	105	0	0	0	1	0	0	0	0	0	0	5
Δ_4^{42}	125	0	0	1	0	0	0	0	0	0	0	5
Δ_4^{43}	56	0	0	0	0	0	0	1	0	0	0	0
Δ_4^{44}	70	0	0	0	0	0	0	0	1	0	0	4
Δ_4^{45}	91	0	0	0	0	0	0	2	0	1	0	0
Δ_4^{46}	83	0	0	0	0	0	0	0	2	0	1	3
Δ_5^{1}	0	15	16	0	0	0	70	0	0	0	210	0
Δ_5^{2}	0	16	15	0	0	70	0	0	0	210	0	0
Δ_5^{3}	0	0	0	0	6	0	0	0	0	0	0	0
Δ_5^{4}	0	0	0	6	0	0	0	0	0	0	0	90
Δ_5^{5}	0	0	2	0	0	3	16	12	0	54	0	0
Δ_5^{6}	0	2	0	0	0	16	3	0	12	0	54	0
Δ_5^{7}	0	0	0	0	0	4	0	19	10	0	0	0
Δ_5^{8}	96	0	0	0	0	0	4	10	19	0	0	0
Δ_5^{9}	0	0	2	0	0	18	0	0	0	3	8	32
Δ_5^{10}	48	2	0	0	0	0	18	0	0	8	3	0
Δ_5^{11}	126	0	0	0	7	0	0	0	0	28	0	1
Δ_5^{12}	0	0	0	7	0	0	0	0	0	0	28	2
Δ_5^{13}	108	0	0	0	4	0	8	24	0	0	0	0
Δ_5^{14}	0	0	0	4	0	8	0	0	24	0	0	0
Δ_5^{15}	96	0	0	0	0	0	0	2	2	0	0	0
Δ_5^{16}	54	0	0	0	0	0	12	0	0	0	0	0
Δ_5^{17}	54	0	0	0	0	12	0	0	0	0	0	0
Δ_5^{18}	27	3	0	0	0	0	3	0	0	9	0	0
Δ_5^{19}	0	0	3	0	0	3	0	0	0	0	9	0

	Δ_5^{12}	Δ_5^{13}	Δ_5^{14}	Δ_5^{15}	Δ_5^{16}	Δ_5^{17}	Δ_5^{18}	Δ_5^{19}	Δ_5^{20}	Δ_5^{21}	Δ_5^{22}	Δ_5^{23}
Δ_4^{26}	0	16	16	8	0	0	0	0	0	0	0	0
Δ_4^{27}	24	0	0	0	0	0	0	0	0	8	8	8
Δ_4^{28}	0	0	0	0	0	24	40	24	0	16	16	72
Δ_4^{29}	0	0	0	0	24	0	24	40	0	16	16	0
Δ_4^{30}	0	0	4	0	0	8	0	32	0	24	48	60
Δ_4^{31}	0	4	0	0	8	0	32	0	0	48	24	0
Δ_4^{32}	0	12	0	27	12	0	0	0	0	0	0	0
Δ_4^{33}	0	0	12	27	0	12	0	0	0	0	0	0
Δ_4^{34}	10	1	0	6	0	12	0	0	0	0	0	27
Δ_4^{35}	12	0	1	6	12	0	0	0	0	0	0	0
Δ_4^{36}	6	0	30	12	0	0	0	4	0	0	8	0
Δ_4^{37}	2	30	0	12	0	0	4	0	0	8	0	2
Δ_4^{38}	0	0	16	7	0	8	0	8	0	0	0	0
Δ_4^{39}	0	16	0	7	8	0	8	0	0	0	0	4
Δ_4^{40}	0	7	7	0	2	2	0	0	12	0	0	2
Δ_4^{41}	5	5	0	5	0	0	0	0	0	1	1	0
Δ_4^{42}	5	0	5	5	0	0	0	0	0	1	1	5
Δ_4^{43}	4	2	2	4	2	6	2	2	0	4	2	17
Δ_4^{44}	0	2	2	4	6	2	2	2	0	2	4	4
Δ_4^{45}	3	0	3	3	2	2	0	1	8	2	1	4
Δ_4^{46}	0	3	0	3	2	2	1	0	8	1	2	2
Δ_5^{1}	0	0	0	0	0	0	560	0	0	336	0	0
Δ_5^{2}	0	0	0	0	0	0	0	560	0	0	336	0
Δ_5^{3}	90	0	60	0	0	0	0	0	0	0	0	0
Δ_5^{4}	0	60	0	0	0	0	0	0	0	0	0	225
Δ_5^{5}	0	0	32	0	0	64	0	16	0	0	0	0
Δ_5^{6}	0	32	0	0	64	0	16	0	0	0	0	0
Δ_5^{7}	0	32	0	3	0	0	0	0	0	32	32	24
Δ_5^{8}	0	0	32	3	0	0	0	0	0	32	32	0
Δ_5^{9}	0	0	0	0	0	0	16	0	0	0	0	0
Δ_5^{10}	32	0	0	0	0	0	0	16	0	0	0	96
Δ_5^{11}	2	0	0	0	0	0	0	0	0	0	14	7
Δ_5^{12}	1	0	0	0	0	0	0	0	0	14	0	14
Δ_5^{13}	0	0	2	0	4	24	0	0	0	0	0	42
Δ_5^{14}	0	2	0	0	24	4	0	0	0	0	0	0
Δ_5^{15}	0	0	0	4	0	0	0	0	0	0	0	0
Δ_5^{16}	0	3	18	0	9	30	15	0	0	0	0	0
Δ_5^{17}	0	18	3	0	30	9	0	15	0	0	0	0
Δ_5^{18}	0	0	0	0	15	0	48	66	0	0	45	81
Δ_5^{19}	0	0	0	0	0	15	66	48	0	45	0	0

	Δ_5^{24}	Δ_5^{25}	Δ_5^{26}	Δ_5^{27}	Δ_5^{28}	Δ_5^{29}	Δ_5^{30}	Δ_5^{31}	Δ_5^{32}	Δ_5^{33}	Δ_5^{34}	Δ_5^{35}
Δ_4^{26}	0	16	16	32	32	128	0	24	24	0	0	0
Δ_4^{27}	8	8	8	0	0	0	0	30	30	16	16	0
Δ_4^{28}	0	0	0	0	0	0	6	6	0	0	48	0
Δ_4^{29}	72	0	0	0	0	0	6	0	6	48	0	0
Δ_4^{30}	0	0	0	0	24	0	6	6	6	24	16	0
Δ_4^{31}	60	0	0	24	0	0	6	6	6	16	24	0
Δ_4^{32}	0	9	18	0	0	0	0	0	45	0	0	36
Δ_4^{33}	0	18	9	0	0	0	0	45	0	0	0	36
Δ_4^{34}	0	33	0	12	24	4	36	30	12	2	28	12
Δ_4^{35}	27	0	33	24	12	4	36	12	30	28	2	12
Δ_4^{36}	2	26	0	12	36	12	0	6	24	2	30	0
Δ_4^{37}	0	0	26	36	12	12	0	24	6	30	2	0
Δ_4^{38}	4	40	24	0	0	8	4	20	2	0	32	0
Δ_4^{39}	0	24	40	0	0	8	4	2	20	32	0	0
Δ_4^{40}	2	15	15	13	13	26	18	0	0	18	18	12
Δ_4^{41}	5	0	5	20	5	5	5	20	20	10	5	26
Δ_4^{42}	0	5	0	5	20	5	5	20	20	5	10	26
Δ_4^{43}	4	11	0	14	12	10	8	19	20	4	12	16
Δ_4^{44}	17	0	11	12	14	10	8	20	19	12	4	16
Δ_4^{45}	2	13	8	8	0	4	9	11	5	8	12	18
Δ_4^{46}	4	8	13	0	8	4	9	5	11	12	8	18
Δ_5^{1}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	225	0	0	180	0	0	0	270	0	0	0	0
Δ_5^{4}	0	0	0	0	180	0	0	0	270	0	0	0
Δ_5^{5}	0	0	0	0	32	0	0	0	0	0	144	0
Δ_5^{6}	0	0	0	32	0	0	0	0	0	144	0	0
Δ_5^{7}	0	0	0	32	0	0	0	0	36	0	0	0
Δ_5^{8}	24	0	0	0	32	0	0	36	0	0	0	0
Δ_5^{9}	96	0	0	0	0	0	12	36	0	0	0	0
Δ_5^{10}	0	0	0	0	0	0	12	0	36	0	0	0
Δ_5^{11}	14	0	0	28	0	0	0	0	0	14	42	0
Δ_5^{12}	7	0	0	0	28	0	0	0	0	42	14	0
Δ_5^{13}	0	3	0	8	24	0	0	0	0	0	78	0
Δ_5^{14}	42	0	3	24	8	0	0	0	0	78	0	0
Δ_5^{15}	0	0	0	0	0	0	16	20	20	0	0	64
Δ_5^{16}	0	27	0	3	27	0	0	27	27	0	27	0
Δ_5^{17}	0	0	27	27	3	0	0	27	27	27	0	0
Δ_5^{18}	0	27	9	12	54	138	27	0	27	0	0	0
Δ_5^{19}	81	9	27	54	12	138	27	27	0	0	0	0

	Δ_5^{36}	Δ_5^{37}	Δ_5^{38}	Δ_5^{39}	Δ_5^{40}	Δ_5^{41}	Δ_5^{42}	Δ_5^{43}	Δ_5^{44}	Δ_5^{45}	Δ_5^{46}	Δ_5^{47}
Δ_4^{26}	0	0	0	0	0	32	32	32	32	32	32	64
Δ_4^{27}	0	24	24	72	72	0	0	0	0	48	48	48
Δ_4^{28}	0	48	40	0	104	72	24	0	24	0	8	0
Δ_4^{29}	0	40	48	104	0	24	72	24	0	8	0	0
Δ_4^{30}	0	48	96	24	48	72	56	24	0	0	0	0
Δ_4^{31}	0	96	48	48	24	56	72	0	24	0	0	0
Δ_4^{32}	72	0	0	36	0	30	36	36	18	54	78	36
Δ_4^{33}	72	0	0	0	36	36	30	18	36	78	54	36
Δ_4^{34}	36	0	8	0	48	24	8	0	28	36	24	72
Δ_4^{35}	36	8	0	48	0	8	24	28	0	24	36	72
Δ_4^{36}	24	0	48	0	36	28	24	0	36	24	36	48
Δ_4^{37}	24	48	0	36	0	24	28	36	0	36	24	48
Δ_4^{38}	16	8	48	16	32	24	16	24	56	40	40	32
Δ_4^{39}	16	48	8	32	16	16	24	56	24	40	40	32
Δ_4^{40}	12	12	12	15	15	15	15	31	31	42	42	42
Δ_4^{41}	30	40	15	25	10	15	20	15	15	20	30	66
Δ_4^{42}	30	15	40	10	25	20	15	15	15	30	20	66
Δ_4^{43}	24	20	18	10	50	36	20	26	18	26	24	28
Δ_4^{44}	24	18	20	50	10	20	36	18	26	24	26	28
Δ_4^{45}	28	15	17	12	36	16	12	31	29	45	36	28
Δ_4^{46}	28	17	15	36	12	12	16	29	31	36	45	28
Δ_5^{1}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{4}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{5}	0	0	48	0	144	48	48	0	96	0	96	0
Δ_5^{6}	0	48	0	144	0	48	48	96	0	96	0	0
Δ_5^{7}	96	128	0	0	0	128	96	0	32	0	32	0
Δ_5^{8}	96	0	128	0	0	96	128	32	0	32	0	0
Δ_5^{9}	0	48	0	208	48	0	0	0	32	0	96	0
Δ_5^{10}	0	0	48	48	208	0	0	32	0	96	0	0
Δ_5^{11}	0	28	42	0	0	0	28	0	112	0	0	0
Δ_5^{12}	0	42	28	0	0	28	0	112	0	0	0	0
Δ_5^{13}	0	0	36	0	72	0	12	0	48	0	0	72
Δ_5^{14}	0	36	0	72	0	12	0	48	0	0	0	72
Δ_5^{15}	0	0	0	0	0	32	32	64	64	0	0	0
Δ_5^{16}	54	0	63	0	36	27	27	27	63	0	45	0
Δ_5^{17}	54	63	0	36	0	27	27	63	27	45	0	0
Δ_5^{18}	0	0	54	45	54	45	0	0	0	27	0	0
Δ_5^{19}	0	54	0	54	45	0	45	0	0	0	27	0

	Δ_5^{48}	Δ_5^{49}	Δ_5^{50}	Δ_5^{51}	Δ_5^{52}	Δ_5^{53}	Δ_5^{54}	Δ_5^{55}	Δ_5^{56}	Δ_5^{57}	Δ_5^{58}	Δ_5^{59}
Δ_4^{26}	0	16	16	96	96	32	32	0	0	0	256	64
Δ_4^{27}	0	48	48	12	12	72	72	0	32	32	0	240
Δ_4^{28}	0	96	120	60	0	72	0	0	0	0	48	96
Δ_4^{29}	0	120	96	0	60	0	72	0	0	0	48	96
Δ_4^{30}	24	168	60	0	12	12	12	0	48	0	24	96
Δ_4^{31}	24	60	168	12	0	12	12	0	0	48	24	96
Δ_4^{32}	36	9	0	36	54	45	0	144	0	72	36	216
Δ_4^{33}	36	0	9	54	36	0	45	144	72	0	36	216
Δ_4^{34}	36	24	24	24	24	96	60	48	96	24	72	72
Δ_4^{35}	36	24	24	24	24	60	96	48	24	96	72	72
Δ_4^{36}	24	24	0	42	90	24	0	48	56	0	84	96
Δ_4^{37}	24	0	24	90	42	0	24	48	0	56	84	96
Δ_4^{38}	20	24	8	48	104	72	20	32	32	0	72	112
Δ_4^{39}	20	8	24	104	48	20	72	32	0	32	72	112
Δ_4^{40}	12	21	21	66	66	51	51	72	42	42	168	126
Δ_4^{41}	55	20	40	30	10	15	40	50	65	60	50	160
Δ_4^{42}	55	40	20	10	30	40	15	50	60	65	50	160
Δ_4^{43}	32	35	36	44	20	72	32	48	76	44	76	124
Δ_4^{44}	32	36	35	20	44	32	72	48	44	76	76	124
Δ_4^{45}	34	26	17	47	40	51	33	68	76	50	71	164
Δ_4^{46}	34	17	26	40	47	33	51	68	50	76	71	164
Δ_5^{1}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{2}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{3}	0	0	270	0	0	0	0	0	0	0	0	0
Δ_5^{4}	0	270	0	0	0	0	0	0	0	0	0	0
Δ_5^{5}	0	0	0	0	72	144	0	0	0	0	0	0
Δ_5^{6}	0	0	0	72	0	0	144	0	0	0	0	0
Δ_5^{7}	0	0	144	96	24	24	48	0	32	32	0	0
Δ_5^{8}	0	144	0	24	96	48	24	0	32	32	0	0
Δ_5^{9}	24	0	96	48	0	0	144	0	32	96	48	0
Δ_5^{10}	24	96	0	0	48	144	0	0	96	32	48	0
Δ_5^{11}	84	0	84	84	0	84	0	0	84	112	84	0
Δ_5^{12}	84	84	0	0	84	0	84	0	112	84	84	0
Δ_5^{13}	72	54	0	0	0	90	0	0	168	24	0	72
Δ_5^{14}	72	0	54	0	0	0	90	0	24	168	0	72
Δ_5^{15}	240	16	16	0	0	32	32	0	64	64	0	0
Δ_5^{16}	0	81	27	27	135	54	0	0	0	54	81	108
Δ_5^{17}	0	27	81	135	27	0	54	0	54	0	81	108
Δ_5^{18}	0	135	54	54	27	0	27	0	0	0	162	0
Δ_5^{19}	0	54	135	27	54	27	0	0	0	0	162	0

	Δ_3^8	Δ_3^9	Δ_3^{10}	Δ_4^1	Δ_4^2	Δ_4^3	Δ_4^4	Δ_4^5	Δ_4^6	Δ_4^7	Δ_4^8	Δ_4^9
Δ_5^{20}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{21}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{22}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{23}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{24}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{25}	0	0	0	0	0	0	0	2	1	2	4	0
Δ_5^{26}	0	0	0	0	0	0	0	1	2	2	0	4
Δ_5^{27}	0	0	0	0	0	0	0	0	0	0	0	2
Δ_5^{28}	0	0	0	0	0	0	0	0	0	0	2	0
Δ_5^{29}	0	0	0	0	0	1	0	0	0	0	2	2
Δ_5^{30}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{31}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_5^{32}	0	0	0	0	0	0	0	1	0	0	0	0
Δ_5^{33}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{34}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{35}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{36}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{37}	0	0	0	0	0	0	0	0	0	0	0	1
Δ_5^{38}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_5^{39}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{40}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{41}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{42}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{43}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{44}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{45}	0	0	0	0	0	0	0	1	0	1	0	0
Δ_5^{46}	0	0	0	0	0	0	0	0	1	1	0	0
Δ_5^{47}	0	0	0	0	0	0	1	0	0	1	0	0
Δ_5^{48}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{49}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{50}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{51}	0	0	0	0	0	0	0	0	0	0	0	1
Δ_5^{52}	0	0	0	0	0	0	0	0	0	0	1	0
Δ_5^{53}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{54}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{55}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{56}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{57}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{58}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{59}	0	0	0	0	0	0	0	0	0	0	0	0

	Δ_4^{10}	Δ_4^{11}	Δ_4^{12}	Δ_4^{13}	Δ_4^{14}	Δ_4^{15}	Δ_4^{16}	Δ_4^{17}	Δ_4^{18}	Δ_4^{19}	Δ_4^{20}	Δ_4^{21}
Δ_5^{20}	0	0	0	0	0	0	3	0	0	0	0	0
Δ_5^{21}	0	0	0	0	0	0	0	6	0	15	0	0
Δ_5^{22}	0	0	0	0	0	0	0	6	0	0	15	0
Δ_5^{23}	1	0	0	0	8	1	0	4	0	0	0	2
Δ_5^{24}	0	1	0	0	1	8	0	4	0	0	0	2
Δ_5^{25}	1	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{26}	0	1	0	0	0	0	0	0	0	0	0	0
Δ_5^{27}	3	0	0	0	0	0	3	0	1	0	0	0
Δ_5^{28}	0	3	0	0	0	0	3	0	1	0	0	0
Δ_5^{29}	0	0	0	0	0	0	6	0	0	0	0	0
Δ_5^{30}	2	2	1	0	0	0	4	0	0	0	0	0
Δ_5^{31}	0	0	1	2	0	1	0	0	2	3	1	0
Δ_5^{32}	0	0	1	2	1	0	0	0	2	1	3	0
Δ_5^{33}	1	0	0	0	1	0	0	0	0	4	0	2
Δ_5^{34}	0	1	0	0	0	1	0	0	0	0	4	2
Δ_5^{35}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{36}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{37}	0	0	0	0	0	0	0	1	0	4	2	2
Δ_5^{38}	0	0	0	0	0	0	0	1	0	2	4	2
Δ_5^{39}	0	0	0	0	0	3	0	2	0	1	0	1
Δ_5^{40}	0	0	0	0	3	0	0	2	0	0	1	1
Δ_5^{41}	0	0	0	0	0	0	0	0	0	6	3	3
Δ_5^{42}	0	0	0	0	0	0	0	0	0	3	6	3
Δ_5^{43}	0	0	0	0	0	0	0	0	1	3	0	0
Δ_5^{44}	0	0	0	0	0	0	0	0	1	0	3	0
Δ_5^{45}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{46}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{47}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{48}	0	0	0	1	0	0	0	0	0	0	0	0
Δ_5^{49}	0	0	0	0	3	0	0	2	0	1	0	2
Δ_5^{50}	0	0	0	0	0	3	0	2	0	0	1	2
Δ_5^{51}	0	0	0	0	0	1	1	0	1	0	0	0
Δ_5^{52}	0	0	0	0	1	0	1	0	1	0	0	0
Δ_5^{53}	1	0	1	0	0	0	0	0	2	0	0	0
Δ_5^{54}	0	1	1	0	0	0	0	0	2	0	0	0
Δ_5^{55}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{56}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_5^{57}	0	0	0	0	0	0	0	0	1	0	0	0
Δ_5^{58}	0	0	0	0	0	0	2	0	1	0	0	0
Δ_5^{59}	0	0	0	1	0	0	0	0	0	0	0	1

	Δ_4^{22}	Δ_4^{23}	Δ_4^{24}	Δ_4^{25}	Δ_4^{26}	Δ_4^{27}	Δ_4^{28}	Δ_4^{29}	Δ_4^{30}	Δ_4^{31}	Δ_4^{32}	Δ_4^{33}
Δ_5^{20}	0	0	21	0	0	0	0	0	0	0	0	0
Δ_5^{21}	0	0	0	0	0	5	10	10	15	30	0	0
Δ_5^{22}	0	0	0	0	0	5	10	10	30	15	0	0
Δ_5^{23}	0	0	0	0	0	4	36	0	30	0	0	0
Δ_5^{24}	0	0	0	0	0	4	0	36	0	30	0	0
Δ_5^{25}	0	12	6	0	6	4	0	0	0	0	6	12
Δ_5^{26}	0	12	6	0	6	4	0	0	0	0	12	6
Δ_5^{27}	15	0	0	0	9	0	0	0	0	9	0	0
Δ_5^{28}	15	0	0	0	9	0	0	0	9	0	0	0
Δ_5^{29}	21	9	6	0	36	0	0	0	0	0	0	0
Δ_5^{30}	6	8	16	0	0	0	2	2	2	2	0	0
Δ_5^{31}	2	0	0	7	6	10	2	0	2	2	0	20
Δ_5^{32}	2	0	0	7	6	10	0	2	2	2	20	0
Δ_5^{33}	4	4	0	0	0	4	0	12	6	4	0	0
Δ_5^{34}	4	4	0	0	0	4	12	0	4	6	0	0
Δ_5^{35}	0	0	5	20	0	0	0	0	0	0	10	10
Δ_5^{36}	0	0	3	12	0	0	0	0	0	0	12	12
Δ_5^{37}	0	0	0	0	0	3	6	5	6	12	0	0
Δ_5^{38}	0	0	0	0	0	3	5	6	12	6	0	0
Δ_5^{39}	0	3	3	0	0	9	0	13	3	6	6	0
Δ_5^{40}	0	3	3	0	0	9	13	0	6	3	0	6
Δ_5^{41}	0	0	0	6	3	0	9	3	9	7	5	6
Δ_5^{42}	0	0	0	6	3	0	3	9	7	9	6	5
Δ_5^{43}	0	6	1	9	3	0	0	3	3	0	6	3
Δ_5^{44}	0	6	1	9	3	0	3	0	0	3	3	6
Δ_5^{45}	3	5	4	3	3	6	0	1	0	0	9	13
Δ_5^{46}	3	5	4	3	3	6	1	0	0	0	13	9
Δ_5^{47}	10	0	0	0	5	5	0	0	0	0	5	5
Δ_5^{48}	1	0	0	7	0	0	0	0	2	2	4	4
Δ_5^{49}	0	0	0	0	1	4	8	10	14	5	1	0
Δ_5^{50}	0	0	0	0	1	4	10	8	5	14	0	1
Δ_5^{51}	0	0	1	2	6	1	5	0	0	1	4	6
Δ_5^{52}	0	0	1	2	6	1	0	5	1	0	6	4
Δ_5^{53}	0	6	5	1	2	6	6	0	1	1	5	0
Δ_5^{54}	0	6	5	1	2	6	0	6	1	1	0	5
Δ_5^{55}	0	0	0	3	0	0	0	0	0	0	12	12
Δ_5^{56}	3	0	2	3	0	2	0	0	3	0	0	6
Δ_5^{57}	3	0	2	3	0	2	0	0	0	3	6	0
Δ_5^{58}	5	5	4	0	8	0	2	2	1	1	2	2
Δ_5^{59}	0	0	0	0	1	5	2	2	2	2	6	6

	Δ_4^{34}	Δ_4^{35}	Δ_4^{36}	Δ_4^{37}	Δ_4^{38}	Δ_4^{39}	Δ_4^{40}	Δ_4^{41}	Δ_4^{42}	Δ_4^{43}	Δ_4^{44}	Δ_4^{45}
Δ_5^{20}	0	0	0	0	0	0	63	0	0	0	0	126
Δ_5^{21}	0	0	0	10	0	0	0	6	6	30	15	30
Δ_5^{22}	0	0	10	0	0	0	0	6	6	15	30	15
Δ_5^{23}	27	0	0	2	0	6	8	0	24	102	24	48
Δ_5^{24}	0	27	2	0	6	0	8	24	0	24	102	24
Δ_5^{25}	33	0	26	0	60	36	60	0	24	66	0	156
Δ_5^{26}	0	33	0	26	36	60	60	24	0	0	66	96
Δ_5^{27}	9	18	9	27	0	0	39	72	18	63	54	72
Δ_5^{28}	18	9	27	9	0	0	39	18	72	54	63	0
Δ_5^{29}	3	3	9	9	9	9	78	18	18	45	45	36
Δ_5^{30}	24	24	0	0	4	4	48	16	16	32	32	72
Δ_5^{31}	20	8	4	16	20	2	0	64	64	76	80	88
Δ_5^{32}	8	20	16	4	2	20	0	64	64	80	76	40
Δ_5^{33}	1	14	1	15	0	24	36	24	12	12	36	48
Δ_5^{34}	14	1	15	1	24	0	36	12	24	36	12	72
Δ_5^{35}	5	5	0	0	0	0	20	52	52	40	40	90
Δ_5^{36}	9	9	6	6	6	6	12	36	36	36	36	84
Δ_5^{37}	0	2	0	12	3	18	12	48	18	30	27	45
Δ_5^{38}	2	0	12	0	18	3	12	18	48	27	30	51
Δ_5^{39}	0	12	0	9	6	12	15	30	12	15	75	36
Δ_5^{40}	12	0	9	0	12	6	15	12	30	75	15	108
Δ_5^{41}	6	2	7	6	9	6	15	18	24	54	30	48
Δ_5^{42}	2	6	6	7	6	9	15	24	18	30	54	36
Δ_5^{43}	0	7	0	9	9	21	31	18	18	39	27	93
Δ_5^{44}	7	0	9	0	21	9	31	18	18	27	39	87
Δ_5^{45}	9	6	6	9	15	15	42	24	36	39	36	135
Δ_5^{46}	6	9	9	6	15	15	42	36	24	36	39	108
Δ_5^{47}	15	15	10	10	10	10	35	66	66	35	35	70
Δ_5^{48}	6	6	4	4	5	5	8	44	44	32	32	68
Δ_5^{49}	4	4	4	0	6	2	14	16	32	35	36	52
Δ_5^{50}	4	4	0	4	2	6	14	32	16	36	35	34
Δ_5^{51}	4	4	7	15	12	26	44	24	8	44	20	94
Δ_5^{52}	4	4	15	7	26	12	44	8	24	20	44	80
Δ_5^{53}	16	10	4	0	18	5	34	12	32	72	32	102
Δ_5^{54}	10	16	0	4	5	18	34	32	12	32	72	66
Δ_5^{55}	6	6	6	6	6	6	36	30	30	36	36	102
Δ_5^{56}	12	3	7	0	6	0	21	39	36	57	33	114
Δ_5^{57}	3	12	0	7	0	6	21	36	39	33	57	75
Δ_5^{58}	6	6	7	7	9	9	56	20	20	38	38	71
Δ_5^{59}	3	3	4	4	7	7	21	32	32	31	31	82

	Δ_4^{46}	Δ_5^1	Δ_5^2	Δ_5^3	Δ_5^4	Δ_5^5	Δ_5^6	Δ_5^7	Δ_5^8	Δ_5^9	Δ_5^{10}	Δ_5^{11}
Δ_5^{20}	126	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{21}	15	1	0	0	0	0	0	10	10	0	0	0
Δ_5^{22}	30	0	1	0	0	0	0	10	10	0	0	5
Δ_5^{23}	24	0	0	0	5	0	0	6	0	0	24	2
Δ_5^{24}	48	0	0	5	0	0	0	0	6	24	0	4
Δ_5^{25}	96	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{26}	156	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{27}	0	0	0	3	0	0	2	6	0	0	0	6
Δ_5^{28}	72	0	0	0	3	2	0	0	6	0	0	0
Δ_5^{29}	36	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{30}	72	0	0	0	0	0	0	0	0	2	2	0
Δ_5^{31}	40	0	0	4	0	0	0	0	6	6	0	0
Δ_5^{32}	88	0	0	0	4	0	0	6	0	0	6	0
Δ_5^{33}	72	0	0	0	0	0	6	0	0	0	0	2
Δ_5^{34}	48	0	0	0	0	6	0	0	0	0	0	6
Δ_5^{35}	90	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{36}	84	0	0	0	0	0	0	6	6	0	0	0
Δ_5^{37}	51	0	0	0	0	0	1	8	0	3	0	2
Δ_5^{38}	45	0	0	0	0	1	0	0	8	0	3	3
Δ_5^{39}	108	0	0	0	0	0	3	0	0	13	3	0
Δ_5^{40}	36	0	0	0	0	3	0	0	0	3	13	0
Δ_5^{41}	36	0	0	0	0	1	1	8	6	0	0	0
Δ_5^{42}	48	0	0	0	0	1	1	6	8	0	0	2
Δ_5^{43}	87	0	0	0	0	0	2	0	2	0	2	0
Δ_5^{44}	93	0	0	0	0	2	0	2	0	2	0	8
Δ_5^{45}	108	0	0	0	0	0	2	0	2	0	6	0
Δ_5^{46}	135	0	0	0	0	2	0	2	0	6	0	0
Δ_5^{47}	70	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{48}	68	0	0	0	0	0	0	0	0	1	1	4
Δ_5^{49}	34	0	0	0	1	0	0	0	6	0	4	0
Δ_5^{50}	52	0	0	1	0	0	0	6	0	4	0	4
Δ_5^{51}	80	0	0	0	0	0	1	4	1	2	0	4
Δ_5^{52}	94	0	0	0	0	1	0	1	4	0	2	0
Δ_5^{53}	66	0	0	0	0	2	0	1	2	0	6	4
Δ_5^{54}	102	0	0	0	0	0	2	2	1	6	0	0
Δ_5^{55}	102	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{56}	75	0	0	0	0	0	0	1	1	1	3	3
Δ_5^{57}	114	0	0	0	0	0	0	1	1	3	1	4
Δ_5^{58}	71	0	0	0	0	0	0	0	0	1	1	2
Δ_5^{59}	82	0	0	0	0	0	0	0	0	0	0	0

	Δ_5^{12}	Δ_5^{13}	Δ_5^{14}	Δ_5^{15}	Δ_5^{16}	Δ_5^{17}	Δ_5^{18}	Δ_5^{19}	Δ_5^{20}	Δ_5^{21}	Δ_5^{22}	Δ_5^{23}
Δ_5^{20}	0	0	0	0	0	0	0	0	24	42	42	0
Δ_5^{21}	5	0	0	0	0	0	0	25	40	46	36	30
Δ_5^{22}	0	0	0	0	0	0	25	0	40	36	46	0
Δ_5^{23}	4	14	0	0	0	0	36	0	0	24	0	32
Δ_5^{24}	2	0	14	0	0	0	0	36	0	0	24	26
Δ_5^{25}	0	1	0	0	12	0	12	4	0	0	8	4
Δ_5^{26}	0	0	1	0	0	12	4	12	0	8	0	0
Δ_5^{27}	0	2	6	0	1	9	4	18	0	18	6	45
Δ_5^{28}	6	6	2	0	9	1	18	4	0	6	18	12
Δ_5^{29}	0	0	0	0	0	0	46	46	0	39	39	60
Δ_5^{30}	0	0	0	4	0	0	8	8	0	32	32	16
Δ_5^{31}	0	0	0	5	8	8	0	8	0	0	0	4
Δ_5^{32}	0	0	0	5	8	8	8	0	0	0	0	0
Δ_5^{33}	6	0	13	0	0	6	0	0	8	12	0	12
Δ_5^{34}	2	13	0	0	6	0	0	0	8	0	12	6
Δ_5^{35}	0	0	0	10	0	0	0	0	0	2	2	0
Δ_5^{36}	0	0	0	0	6	6	0	0	0	0	0	0
Δ_5^{37}	3	0	3	0	0	7	0	6	16	14	6	15
Δ_5^{38}	2	3	0	0	7	0	6	0	16	6	14	0
Δ_5^{39}	0	0	6	0	0	4	5	6	4	2	14	3
Δ_5^{40}	0	6	0	0	4	0	6	5	4	14	2	0
Δ_5^{41}	2	0	1	3	3	3	5	0	12	12	8	6
Δ_5^{42}	0	1	0	3	3	3	0	5	12	8	12	6
Δ_5^{43}	8	0	4	6	3	7	0	0	0	0	0	0
Δ_5^{44}	0	4	0	6	7	3	0	0	0	0	0	6
Δ_5^{45}	0	0	0	0	0	5	3	0	0	9	0	0
Δ_5^{46}	0	0	0	0	5	0	0	3	0	0	9	0
Δ_5^{47}	0	5	5	0	0	0	0	0	0	1	1	10
Δ_5^{48}	4	4	4	15	0	0	0	0	0	0	0	0
Δ_5^{49}	4	3	0	1	6	2	10	4	8	12	6	9
Δ_5^{50}	0	0	3	1	2	6	4	10	8	6	12	24
Δ_5^{51}	0	0	0	0	2	10	4	2	8	6	2	12
Δ_5^{52}	4	0	0	0	10	2	2	4	8	2	6	0
Δ_5^{53}	0	5	0	2	4	0	0	2	0	10	4	1
Δ_5^{54}	4	0	5	2	0	4	2	0	0	4	10	4
Δ_5^{55}	0	0	0	0	0	0	0	0	0	0	0	0
Δ_5^{56}	4	7	1	3	0	3	0	0	6	6	0	3
Δ_5^{57}	3	1	7	3	3	0	0	0	6	0	6	3
Δ_5^{58}	2	0	0	0	3	3	6	6	4	10	10	15
Δ_5^{59}	0	1	1	0	2	2	0	0	8	2	2	2

	Δ_5^{24}	Δ_5^{25}	Δ_5^{26}	Δ_5^{27}	Δ_5^{28}	Δ_5^{29}	Δ_5^{30}	Δ_5^{31}	Δ_5^{32}	Δ_5^{33}	Δ_5^{34}	Δ_5^{35}
Δ_5^{20}	0	0	0	0	0	0	0	0	0	21	21	0
Δ_5^{21}	0	0	10	30	10	65	60	0	0	30	0	6
Δ_5^{22}	30	10	0	10	30	65	60	0	0	0	30	6
Δ_5^{23}	26	4	0	60	16	80	24	6	0	24	12	0
Δ_5^{24}	32	0	4	16	60	80	24	0	6	12	24	0
Δ_5^{25}	0	2	2	0	0	0	0	0	54	28	2	24
Δ_5^{26}	4	2	2	0	0	0	0	54	0	2	28	24
Δ_5^{27}	12	0	0	21	25	8	9	18	18	0	0	18
Δ_5^{28}	45	0	0	25	21	8	9	18	18	0	0	18
Δ_5^{29}	60	0	0	8	8	34	0	9	9	0	0	0
Δ_5^{30}	16	0	0	8	8	0	31	18	18	16	16	16
Δ_5^{31}	0	0	36	16	16	8	18	11	20	0	16	16
Δ_5^{32}	4	36	0	16	16	8	18	20	11	16	0	16
Δ_5^{33}	6	14	1	0	0	0	12	0	12	15	22	0
Δ_5^{34}	12	1	14	0	0	0	12	12	0	22	15	0
Δ_5^{35}	0	10	10	10	10	0	10	10	10	0	0	30
Δ_5^{36}	0	0	0	6	6	0	0	0	0	18	18	33
Δ_5^{37}	0	12	7	4	6	11	33	0	9	6	24	6
Δ_5^{38}	15	7	12	6	4	11	33	9	0	24	6	6
Δ_5^{39}	0	21	3	0	27	18	15	6	3	0	33	6
Δ_5^{40}	3	3	21	27	0	18	15	3	6	33	0	6
Δ_5^{41}	6	6	6	23	12	12	27	6	3	12	15	12
Δ_5^{42}	6	6	6	12	23	12	27	3	6	15	12	12
Δ_5^{43}	6	9	2	3	6	3	3	9	0	15	21	24
Δ_5^{44}	0	2	9	6	3	3	3	0	9	21	15	24
Δ_5^{45}	0	0	0	6	4	0	0	18	0	16	12	18
Δ_5^{46}	0	0	0	4	6	0	0	0	18	12	16	18
Δ_5^{47}	10	0	0	0	0	0	0	15	15	0	0	20
Δ_5^{48}	0	8	8	4	4	0	3	9	9	12	12	24
Δ_5^{49}	24	2	6	32	18	30	16	20	5	16	2	0
Δ_5^{50}	9	6	2	18	32	30	16	5	20	2	16	0
Δ_5^{51}	0	4	12	6	2	6	8	6	11	10	26	8
Δ_5^{52}	12	12	4	2	6	6	8	11	6	26	10	8
Δ_5^{53}	4	4	8	8	2	8	6	5	12	30	8	16
Δ_5^{54}	1	8	4	2	8	8	6	12	5	8	30	16
Δ_5^{55}	0	0	0	0	0	0	3	6	6	12	12	33
Δ_5^{56}	3	1	12	16	2	3	6	9	9	18	6	18
Δ_5^{57}	3	12	1	2	16	3	6	9	9	6	18	18
Δ_5^{58}	15	2	2	3	3	7	4	10	10	13	13	8
Δ_5^{59}	2	4	4	5	5	0	6	6	6	13	13	20

	Δ_5^{36}	Δ_5^{37}	Δ_5^{38}	Δ_5^{39}	Δ_5^{40}	Δ_5^{41}	Δ_5^{42}	Δ_5^{43}	Δ_5^{44}	Δ_5^{45}	Δ_5^{46}	Δ_5^{47}
Δ_5^{20}	0	84	84	21	21	63	63	0	0	0	0	0
Δ_5^{21}	0	70	30	10	70	60	40	0	0	45	0	6
Δ_5^{22}	0	30	70	70	10	40	60	0	0	0	45	6
Δ_5^{23}	0	60	0	12	0	24	24	0	24	0	0	48
Δ_5^{24}	0	0	60	0	12	24	24	24	0	0	0	48
Δ_5^{25}	0	48	28	84	12	24	24	36	8	0	0	0
Δ_5^{26}	0	28	48	12	84	24	24	8	36	0	0	0
Δ_5^{27}	18	12	18	0	81	69	36	9	18	18	12	0
Δ_5^{28}	18	18	12	81	0	36	69	18	9	12	18	0
Δ_5^{29}	0	33	33	54	54	36	36	9	9	0	0	0
Δ_5^{30}	0	88	88	40	40	72	72	8	8	0	0	0
Δ_5^{31}	0	0	24	16	8	16	8	24	0	48	0	48
Δ_5^{32}	0	24	0	8	16	8	16	0	24	0	48	48
Δ_5^{33}	36	12	48	0	66	24	30	30	42	32	24	0
Δ_5^{34}	36	48	12	66	0	30	24	42	30	24	32	0
Δ_5^{35}	55	10	10	10	10	20	20	40	40	30	30	40
Δ_5^{36}	36	12	12	12	12	6	6	36	36	18	18	72
Δ_5^{37}	12	47	54	14	61	42	31	24	36	27	15	18
Δ_5^{38}	12	54	47	61	14	31	42	36	24	15	27	18
Δ_5^{39}	12	14	61	24	50	12	15	16	33	18	21	24
Δ_5^{40}	12	61	14	50	24	15	12	33	16	21	18	24
Δ_5^{41}	6	42	31	12	15	55	57	18	27	18	21	36
Δ_5^{42}	6	31	42	15	12	57	55	27	18	21	18	36
Δ_5^{43}	36	24	36	16	33	18	27	39	25	26	30	42
Δ_5^{44}	36	36	24	33	16	27	18	25	39	30	26	42
Δ_5^{45}	18	27	15	18	21	18	21	26	30	25	25	30
Δ_5^{46}	18	15	27	21	18	21	18	30	26	25	25	30
Δ_5^{47}	60	15	15	20	20	30	30	35	35	25	25	2
Δ_5^{48}	68	8	8	8	8	20	20	28	28	36	36	40
Δ_5^{49}	8	30	18	42	28	26	34	16	8	16	18	20
Δ_5^{50}	8	18	30	28	42	34	26	8	16	18	16	20
Δ_5^{51}	12	30	30	32	66	22	12	18	20	28	22	48
Δ_5^{52}	12	30	30	66	32	12	22	20	18	22	28	48
Δ_5^{53}	24	54	28	28	10	44	42	30	20	12	20	32
Δ_5^{54}	24	28	54	10	28	42	44	20	30	20	12	32
Δ_5^{55}	51	12	12	12	12	12	12	36	36	24	24	48
Δ_5^{56}	42	33	3	16	10	21	21	29	15	29	35	36
Δ_5^{57}	42	3	33	10	16	21	21	15	29	35	29	36
Δ_5^{58}	12	36	36	42	42	40	40	18	18	16	16	20
Δ_5^{59}	32	21	21	20	20	20	20	35	35	27	27	19

	Δ_5^{48}	Δ_5^{49}	Δ_5^{50}	Δ_5^{51}	Δ_5^{52}	Δ_5^{53}	Δ_5^{54}	Δ_5^{55}	Δ_5^{56}	Δ_5^{57}	Δ_5^{58}	Δ_5^{59}
Δ_5^{20}	0	63	63	63	63	0	0	0	63	63	63	252
Δ_5^{21}	0	90	45	45	15	75	30	0	60	0	150	60
Δ_5^{22}	0	45	90	15	45	30	75	0	0	60	150	60
Δ_5^{23}	0	54	144	72	0	6	24	0	24	24	180	48
Δ_5^{24}	0	144	54	0	72	24	6	0	24	24	180	48
Δ_5^{25}	48	12	36	24	72	24	48	0	8	96	24	96
Δ_5^{26}	48	36	12	72	24	48	24	0	96	8	24	96
Δ_5^{27}	18	144	81	27	9	36	9	0	96	12	27	90
Δ_5^{28}	18	81	144	9	27	9	36	0	12	96	27	90
Δ_5^{29}	0	135	135	27	27	36	36	0	18	18	63	0
Δ_5^{30}	12	64	64	32	32	24	24	16	32	32	32	96
Δ_5^{31}	36	80	20	24	44	20	48	32	48	48	80	96
Δ_5^{32}	36	20	80	44	24	48	20	32	48	48	80	96
Δ_5^{33}	36	48	6	30	78	90	24	48	72	24	78	156
Δ_5^{34}	36	6	48	78	30	24	90	48	24	72	78	156
Δ_5^{35}	60	0	0	20	20	40	40	110	60	60	40	200
Δ_5^{36}	102	12	12	18	18	36	36	102	84	84	36	192
Δ_5^{37}	12	45	27	45	45	81	42	24	66	6	108	126
Δ_5^{38}	12	27	45	45	45	42	81	24	6	66	108	126
Δ_5^{39}	12	63	42	48	99	42	15	24	32	20	126	120
Δ_5^{40}	12	42	63	99	48	15	42	24	20	32	126	120
Δ_5^{41}	30	39	51	33	18	66	63	24	42	42	120	120
Δ_5^{42}	30	51	39	18	33	63	66	24	42	42	120	120
Δ_5^{43}	42	24	12	27	30	45	30	72	58	30	54	210
Δ_5^{44}	42	12	24	30	27	30	45	72	30	58	54	210
Δ_5^{45}	54	24	27	42	33	18	30	48	58	70	48	162
Δ_5^{46}	54	27	24	33	42	30	18	48	70	58	48	162
Δ_5^{47}	50	25	25	60	60	40	40	80	60	60	50	95
Δ_5^{48}	65	8	8	28	28	34	34	132	68	68	48	216
Δ_5^{49}	8	76	84	32	42	41	44	24	40	44	162	120
Δ_5^{50}	8	84	76	42	32	44	41	24	44	40	162	120
Δ_5^{51}	28	32	42	55	42	40	24	32	60	40	54	160
Δ_5^{52}	28	42	32	42	55	24	40	32	40	60	54	160
Δ_5^{53}	34	41	44	40	24	23	44	44	28	44	62	152
Δ_5^{54}	34	44	41	24	40	44	23	44	44	28	62	152
Δ_5^{55}	99	18	18	24	24	33	33	87	60	60	36	228
Δ_5^{56}	51	30	33	45	30	21	33	60	52	68	51	186
Δ_5^{57}	51	33	30	30	45	33	21	60	68	52	51	186
Δ_5^{58}	24	81	81	27	27	31	31	24	34	34	78	164
Δ_5^{59}	54	30	30	40	40	38	38	76	62	62	82	198

Chapter 3

A Commuting Involution Graph for the Baby Monster

3.1 Literature Review

Suppose G is a finite group and X is a subset of G. Then the commuting graph on X, denoted $\mathcal{C}(G, X)$, is a graph whose vertex set is X, with any two points connected by an edge if and only if they commute. If the set X is a conjugacy class of involutions then we call the graph $\mathcal{C}(G, X)$ the commuting involution graph for G with respect to X. These graphs have been studied by many different authors and a brief history will be outlined here.

3.1.1 The Work of Brauer and Fowler

In Brauer and Fowler's famous 1955 paper On Groups of Even Order, [5], the case was studied where G was a group of even order and X the set of non identity elements. One result states that if G has more than one conjugacy class of involutions then the distance between any two involutions is at most 3. The proof is included here as it is elementary, fairly short and elegant.

Lemma 3.1.1 (R. Brauer and K. Fowler)**.** *Let G be a finite Group of even order* *with more than one class of involutions. If x and y are two non-conjugate involutions then there exists an involution w which commutes with both x and y.*

Proof. Consider the subgroup $D = \langle x, y \rangle$ of G. It is a well known fact that D is a dihedral group of order $2m$ where m is the order of xy. Furthermore if m is even then $(xy)^{\frac{m}{2}}$ is an involution contained in the centre of D and in particular commutes with both x and y. Therefore if we can prove that the order of xy is even then we are done.

So suppose that m is odd. Then if S_1 and S_2 are Sylow 2-subgroups of D containing x and y respectively, then $|S_1| = |S_2| = 2$. However by Sylow's Theorems, S_1 is conjugate to S_2 implying that x is conjugate to y, a contradiction. Hence m must be even and we are done. □

Theorem 3.1.2 (R. Brauer and K. Fowler)**.** *If a group G of even order contains more than one class of involutions then for any two involutions $x, y \in G$, we have $d(x, y) \leq 3$.*

Proof. If x and y are not conjugate in G, then by Lemma 3.1.1, $d(x, y) \leq 2$. Thus suppose that x, y are contained in the same involution conjugacy class C. Now let z be an involution not in C. Then again by Lemma 3.1.1, there exists an involution $w \in G$ such that w commutes with both y and z. First suppose that $w \notin C$, then by Lemma 3.1.1, $d(x, w) \leq 2$ and since w commutes with y, $d(x, y) \leq 3$. So assume that $w \in C$, then there exists a $g \in G$ such that $x = w^g$. Then as w commutes with z, $x = w^g$ commutes with z^g. However as $z \notin C$ we have $z^g \notin C$, and thus by Lemma 3.1.1, $d(z^g, y) \leq 2$. Hence $d(x, y) \leq 3$ as required. □

This result also gives us two easy corollaries.

Corollary 3.1.3 (R. Brauer and K. Fowler)**.** *If G has even order and contains more than one class of involutions then any two elements g_1 and g_2 such that $|C_G(g_1)|$ and $|C_G(g_2)|$ are even have distance at most 5.*

Proof. Since $C_G(g_1)$ and $C_G(g_2)$ have even order they both contain involutions x_1 and x_2. Thus by Theorem 3.1.2, $d(x_1, x_2) \leq 3$, and we have $d(g_1, x_1) = 1$ and $d(g_2, x_2) = 1$, hence our result follows. □

A similar argument gives us the second corollary

Corollary 3.1.4 (R. Brauer and K. Fowler)**.** *Let G be a group of even order which contains a real element g such that $C_G(h)$ has odd order for every non-identity h in $C_G(g)$. Then G contains involutions which have distance greater than 2.*

3.1.2 The Work of Fischer

Commuting graphs came up in Fischer's work on 3-transposition groups. A group G is said to be a 3-transposition group if it is generated by a set D of involutions of G such that D is a union of conjugacy classes of G and for all $a, b \in D$, the product ab has order 1, 2 or 3. A good example of a 3-transposition group is the symmetric group S_n, where the set D is the conjugacy class of transpositions.

The study of the commuting graph $\mathcal{C}(G, D)$ where D is a conjugacy class of 3-transpositions in part led to the proof of Fischer's Theorem, a classification of all almost simple 3-transposition groups and led to the discovery of three new sporadic simple groups.

Theorem 3.1.5 (Fischer's Theorem, B. Fischer)**.** *Let D be a conjugacy class of 3-transpostions of the finite group G. Assume the centre of G is trivial and the derived subgroup of G is simple. Then one of the following holds:*

1. *$G \cong S_n$ and D is the set of transpositions of G.*

2. *$G \cong Sp_n(2)$ and D is the set of transvections of G.*

3. *$G \cong U_n(2)$ and D is the set of transvections of G.*

4. *$G \cong O_n^{\epsilon}(2)$ and D is the set of transvections of G.*

5. *$G \cong PO_n^{\mu,\pi}(3)$ is the subgroup of an n-dimensional projective orthogonal group over the field of order 3 generated by a conjugacy class D of reflections.*

6. *G is one of the three Fischer sporadic groups Fi_{22}, Fi_{23} or Fi_{24}, where D is a uniquely determined class of involutions.*

A proof of this theorem is given in [3].

3.1.3 The Work of Segev

In 2001, Segev published the following result in [38].

Theorem 3.1.6 (Y. Segev)**.** *Let G be a minimal non-soluble group, that is G is not soluble but every proper quotient of G is soluble, and suppose X consists of all non-identity elements of G. Then the commuting graph for G with vertex set X has diameter at least 3.*

This theorem was part of the solution of the Finite Soluble Quotients Conjecture, that is that finite quotients of the multiplicative group of finite dimensional division algebras are soluble. In an early paper by Rapinchuk and Segev [23], they proved the following result

Theorem 3.1.7 (Non-Existence Theorem at Diameter ≥ 4, Y. Segev)**.** *Let $\mathcal{G}$ be a class of finite groups. Then a member $G \in \mathcal{G}$ is called minimal if no proper quotient of G belongs to $\mathcal{G}$. If we assume that*

1. *The members of $\mathcal{G}$ are non-soluble.*

2. *If $G \in \mathcal{G}$ and $N \trianglelefteq G$ with G/N soluble, then $N \in \mathcal{G}$.*

3. *If $G \in \mathcal{G}$ and $N \trianglelefteq G$ is a soluble normal subgroup of G then $G/N \in \mathcal{G}$.*

4. *The commuting graph of minimal members of $\mathcal{G}$ has diameter ≥ 4.*

Then no member of $\mathcal{G}$ is a quotient of the multiplicative group of a finite-dimensional division algebra.

Now if we could replace the bound in condition 4 above with ≥ 3, and if we take $\mathcal{G}$ to be the class of non-soluble finite groups then the Finite Soluble Quotients Conjecture will follow by using Theorem 3.1.6.

Important in the proof of Theorem 3.1.6 is the following idea. Let

$$\mathcal{C}_2(L) = \{(a, b) \in L \times L \mid C_{Aut(L)}(a) \cap C_{Aut(L)}(b) = 1\}$$

where L is a finite group. Now $Aut(L)$ acts naturally on $\mathcal{C}_2(L)$ in the following way

$$(a, b) \mapsto (a^{\alpha}, b^{\alpha})$$

for $\alpha \in Aut(L)$. Consider the following property:

$$Aut(L) \text{ has at least 5 orbits on } \mathcal{C}_2(L).$$

Now suppose that G is a finite group, $K \neq 1$ is a normal subgroup of G and $L \leq K$ is a subgroup such that

$$K = L^{g_1} \times L^{g_2} \times \ldots \times L^{g_n}$$

for $g_i \in G$. We assume that G acts transitively on

$$X = \{L^{g_1}, L^{g_2}, \ldots, L^{g_n}\}$$

by conjugation and suppose $\Sigma \leq Syn(n)$ is the permutation group induced from this action. We assume Σ is soluble and that $C_G(K) = 1$. Now it is true that G having the structure as above, with L being non-abelian simple, is the same as saying that G is minimal non-soluble. Now if we assume further that L has the property mentioned above then we have the following lemma

Lemma 3.1.8 (Y. Segev)**.** *If G is as above, then the commuting graph for G on the* *set of all non-identity elements, has diameter at least 3.*

Then using this lemma with the observation above we come to the proof of Theorem 3.1.6. It must be noted that the proof of Lemma 3.1.8 relies on the following theorem, originally proved in [15], which uses the Classification of Finite Simple Groups in its proof.

Theorem 3.1.9 (G. Malle, J. Saxl and T. Weigel)**.** *Every Finite simple group except* $U_3(3)$ *can be generated by two elements, one strongly real and the other an involution.*

The proof of the Finite Soluble Quotients Conjecture was completed by A. Rapinchuk, G. Seitz and Y. Segev in [2].

3.1.4 The Work of Bundy, Bates, Rowley and Perkins

Peter Rowley has been the driving force behind the recent surge of results concerning commuting involution graphs, where the set X is a conjugacy class of involutions of a group G. It is the overall aim to calculate these graphs, to some extent, for all the involution conjugacy classes for all the finite simple groups and their automorphism groups as well as a few other interesting examples. Over the last 10 years Rowley and three of his former PhD students, D. Bundy, C. Bates and S. Perkins (now S. Hart) have written four papers [9],[11],[12] and [10] on the subject which cover many of the simple groups, as well as the finite Coxeter groups. More recently A. Everett and P. Taylor, two more of Rowley's students, have completed work on some of the remaining cases.

Commuting Involution Graphs for Symmetric Groups

In [10], Bundy, Bates, Rowley and Perkins carried out an extensive amount of work on the commuting involution graphs for $G \cong S_n$, see [10].

Now let G be the symmetric group on n objects, and let X be a conjugacy class of involutions. A typical element of X will be the product of disjoint transpositions, hence we may assume that any element of X has cycle type $1^r 2^m$ for a fixed m. Two results were proved,

Theorem 3.1.10 (Bundy, Bates, Rowley and Perkins)**.** *For $G \cong S_n$, $\mathcal{C}(G, X)$ is disconnected if and only if $n = 2m + 1$ or $n = 4$ and $m = 1$.*

Theorem 3.1.11 (Bundy, Bates, Rowley and Perkins)**.** *If we suppose that $\mathcal{C}(G, X)$ is connected then one of the following holds:*

1. *The diameter of $\mathcal{C}(G, X)$ is at most 3.*

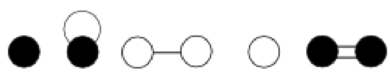
2. *$2m + 2 = n \in \{6, 8, 10\}$ and the diameter of $\mathcal{C}(G, X)$ is at most 4.*

Important in the proofs of these two theorems is the idea of an x-graph. We now pick a fixed $a \in X$ and without loss of generality suppose $a = (1, 2)(3, 4) \dots (2m - 1, 2m)$, so in particular a has cycle type $1^{(n-2m)} 2^m$. We now let G act on $\Omega = \{1 \dots n\}$ in the usual manner and let

$$\mathcal{V} = \{\{1, 2\}, \{3, 4\} \dots \{2m - 1, 2m\}, \{2m + 1\}, \{2m + 2\}, \dots \{n\}\}$$

so $\mathcal{V}$ is the set of orbits as a acts on Ω. Now for $x \in X$ we will define a graph, denoted $\mathcal{G}_x$, whose vertex set is $\mathcal{V}$ with $v_1, v_2 \in \mathcal{V}$ connected by an edge if and only if there exist $\alpha \in v_1$ and $\beta \in v_2$ with $\alpha \neq \beta$ such that x interchanges α and β. Furthermore the vertices corresponding to the 2-cycles of a will be coloured black, and the points fixed by a coloured white. The x-graph gives us valuable information on $\mathcal{C}(G, X)$. The following lemma gives a good example.

Lemma 3.1.12 (Bundy, Bates, Rowley and Perkins)**.** *Let $x \in X$. Then $x \in \Delta_1(a) \cup \{a\}$ if and only if each connected component of $\mathcal{G}_x$ is one of the following:*



The structure of the x-graph also gives us the sizes of each G_a orbit, and two involutions $x, y \in X$ are in the same G_a orbit if and only if their corresponding x-graphs are isomorphic. These two facts alone give us a wealth of knowledge about

$\mathcal{C}(G, X)$. This essentially means that all information about $\mathcal{C}(G, X)$ can be worked out via these x-graphs and as they are purely combinatorial in nature, and fairly easy to deal with, this simplifies the problem greatly.

Commuting Involution Graphs in Coxeter Groups

In [9], Bates, Bundy, Perkins and Rowley studied the commuting involution graphs for the finite irreducible Coxeter groups. We recall there are three infinite families of finite Coxeter groups, that is A_n, the symmetric group on $n+1$ points, B_n and D_n as well as the 7 exceptional finite coxeter groups E_6, E_7, E_8, F_4, H_3, H_4 and I_n.

We recall that we can think of B_n as the group of signed permutations on n objects. That is, we define the sign change to be the element which sends i to $-i$ and fixes all other j. Then take this element and combine it with S_n to get B_n. More precisely we write a permutation in S_n (including 1-cycles) and add a plus or minus sign above each i. For example if

$$w = (\overset{+}{1}, \overset{-}{2})(\overset{-}{3}, \overset{+}{4}) \in B_4$$

then $w(1) = 2$, $w(2) = -1$, $w(3) = -4$ and $w(4) = 3$. The Coxeter Group D_n is the subgroup of index 2 in B_n consisting of all elements which involve an even number of sign changes. Now if we express an element w as a product of disjoint cycles, then we say a cycle $(i_1, \ldots i_n)$ is positive if it contains an even number of negative signs, and negative if it contains an odd number. We can now define an obvious extension of cycle type in the symmetric group, the signed cycle type, that is the usual cycle type, but with a $+$ or $-$ sign above each cycle, where we again include cycles of length 1. As expected, it is true that elements in B_n are conjugate if and only if they have the same signed cycle type, and conjugacy classes in D_n are parameterized by signed cycle type, with one class for each cycle type except in the cases where the signed cycle type contains only even length positive cycles, in which there are two. In [9] the authors proved two main theorems, which we are now in a position to state.

Theorem 3.1.13 (Bundy, Bates, Rowley and Perkins)**.** *Suppose that $G \cong B_n$ or D_n and let*

$$a = (\overset{+}{1}, \overset{+}{2}) \ldots (\overset{+}{2m-1}, \overset{+}{2m})(\overset{+}{2m+1}) \ldots (\overset{+}{2m+k_1})(\overset{-}{2m+k_1+1}) \ldots (\overset{-}{2m+k_1+k_2}).$$

Let $X = a^G$ and $k = \max\{k_1, k_2\}$. Then we have the following:

(i) If $m = 0$ then $\mathcal{C}(G, X)$ is the complete graph.

(ii) If $k = 0$, then the diameter of $\mathcal{C}(G, X)$ is at most 2.

(iii) If $k = 1$ and $m > 0$ then $\mathcal{C}(G, X)$ is disconnected.

(iv) If $k \geq 2$ and $n > 5$ then the diameter of $\mathcal{C}(G, X)$ is at most 4.

(v) If $n = 5$, $m = 1$ and $k = 2$ then the diameter of $\mathcal{C}(G, X)$ is 5.

(vi) If $n = 5$, $m = 1$ and $k = 3$ then the diameter of $\mathcal{C}(G, X)$ is 2.

(vii) If $n = 4$, $m = 1$ and $k = 2$ then $\mathcal{C}(G, X)$ is disconnected.

For the exceptional Coxeter groups we have the following result

Theorem 3.1.14 (Bundy, Bates, Rowley and Perkins)**.** *Suppose that G is an exceptional finite Coxeter group, X a conjugacy class of G and $a \in X$.*

(i) If $G \cong I_n$ then $\mathcal{C}(G, X)$ is either disconnected or consists of a single vertex.

(ii) If $G \cong E_6$ then the diameter of $\mathcal{C}(G, X)$ is at most 5.

(ii) If $G \cong E_7$ or E_8 then the diameter of $\mathcal{C}(G, X)$ is at most 4.

(iv) If $G \cong F_4$ and $|X| > 1$ then either $\mathcal{C}(G, X)$ is disconnected or has diameter 2.

(v) If $G \cong H_3$ or H_4 and $|X| > 1$ either $\mathcal{C}(G, X)$ is disconnected or has diameter 2.

Note that the commuting involution graph for the family A_n has already been calculated in [10] as $A_n \cong Sym(n+1)$.

Theorem 3.1.14 is proved by using MAGMA and calculating the commuting involution graph directly. As these groups are relatively small this problem is computationally fairly easy, and just consists of some easy number crunching. For Theorem 3.1.13 as they had a more concrete understanding of the elements of B_n and D_n, the commuting involution graphs can be constructed without use of a machine. As in [10], central to the calculation is the idea of an x-graph. Indeed, for $x \in X$ we define a graph $\mathcal{G}_x$ as follows. Without loss of generality we fix $a \in X$,

$$a = (\overset{+}{1}, \overset{+}{2}) \ldots (2\overset{+}{m} - 1, 2\overset{+}{m})(2m+1) \ldots (n)$$

and define

$$\mathcal{V} = \{\{1,2\}, \ldots, \{2m-1, 2m\}, \{2m+1\} \ldots \{n\}\}.$$

Then $\mathcal{G}_x$ has vertex set $\mathcal{V}$, with $v_1, v_2 \in \mathcal{V}$ connected by an edge if and only if there exists a $\alpha \in v_1$ and $\beta \in v_2$ with $\alpha \neq \beta$ such that x interchanges $\pm\alpha$ and $\pm\beta$. Within $\mathcal{G}_x$ we will colour the vertices corresponding to the 2 cycles black and the others white.

As in [10], information on these x-graphs can be pulled across to $\mathcal{C}(G, X)$, however whereas in the case of the symmetric groups two elements $x, y \in X$ are in the same G_a orbit if and only if $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic, for Coxeter groups this in general is not true, however these graphs are still a great deal of use.

Commuting Involution Graphs in Special Linear Groups

In [11], the authors Bates, Bundy, Perkins and Rowley gave bounds on the commuting involution graph for special linear groups over fields of characteristic 2, and gave the exact disc sizes for 2 and 3 dimensional special linear groups over any finite field. They proved the following theorems.

Theorem 3.1.15 (Bundy, Bates, Rowley and Perkins). *Suppose $G \cong L_2(q)$, the 2 dimensional projective special linear group over the field of q elements, then*

(i) If q is even then $\mathcal{C}(G,X)$ consists of $q+1$ cliques of size $q-1$, that is $\mathcal{C}(G,X)$ consists of $q+1$ copies of the complete graph on $q-1$ vertices.

(ii) If $q \equiv 3 \mod 4$ with $q > 3$ then $\mathcal{C}(G,X)$ is connected with diameter 3. Furthermore

$$\begin{aligned}|\Delta_1(t)| &= (q+1)/2\\|\Delta_2(t)| &= (q+1)(q-3)/4\\|\Delta_3(t)| &= (q+1)(q-3)/4\end{aligned}$$

(iii) If $q \equiv 1 \mod 4$ with $q > 13$ then $\mathcal{C}(G,X)$ is connected with diameter 3. Furthermore

$$\begin{aligned}|\Delta_1(t)| &= (q+1)/2\\|\Delta_2(t)| &= (q+1)(q-5)/4\\|\Delta_3(t)| &= (q+1)(q+7)/4\end{aligned}$$

Note that this theorem misses out the cases where $q = 3, 5, 9$ and 13. However in three of the cases we have a isomorphism into the class of alternating groups, which have already been studied, that is $L_2(3) \cong Alt(4)$,$L_2(9) \cong Alt(6)$ and $L_2(5) \cong Alt(5)$, and hence these graphs are given in [10]. Finally the graph for $L_2(13)$ is calculated separately. We remark that the graphs for $L_2(9)$ and $L_2(13)$ are both connected and have diameter 4 and that the graph for $L_2(3)$ is in fact the complete graph on 3 vertices.

Theorem 3.1.16 (Bundy, Bates, Rowley and Perkins). *Suppose that $G \cong SL_3(q)$. Then $\mathcal{C}(G,X)$ is connected and has diameter 3. Furthermore we have* *(i) If q is even then*

$$\begin{aligned} |\Delta_1(t)| &= 2q^2 - q - 2 \\ |\Delta_2(t)| &= 2q^2(q-1) \\ |\Delta_3(t)| &= q^3(q-1) \end{aligned}$$

(i) If q is odd then

$$\begin{aligned} |\Delta_1(t)| &= q(q+1) \\ |\Delta_2(t)| &= (q^2-1)(q^2+1) \\ |\Delta_3(t)| &= (q+1)(q-1)^2 \end{aligned}$$

We also have that the commuting involution graphs for $L_3(q)$ and $SL_3(q)$ are isomorphic.

Theorem 3.1.17 (Bundy, Bates, Rowley and Perkins)**.** *Let K be a possibly infinite field of characteristic 2 and suppose that $G \cong SL_n(K)$. Also suppose that V is the natural KG-module associated to G, and set $k = dim_K[V,t]$, where $[V,t] = \langle v^t + v \mid v \in V \rangle$. Then*

(i) if $n > 4k$ then the diameter of $\mathcal{C}(G,X)$ is 2;

(ii) if $3k \leq n < 4k$ then the diameter of $\mathcal{C}(G,X)$ is at most 3;

(iii) if $2k < n < 3k$ or k is even such that $n = 2k$, then the diameter of $\mathcal{C}(G,X)$ is at most 5;

(iv) if $n = 2k$ where k is odd then the diameter of $\mathcal{C}(G,X)$ is at most 6.

Central to the proof of Theorem 3.1.17, is the following lemma

Lemma 3.1.18 (Bundy, Bates, Rowley and Perkins)**.** *Suppose $x, y \in X$ then*

(i) $[V,x] \leq C_V(x)$

(ii) if $[V,x][V,y] \leq C_V(x) \cap C_V(y)$ *then* $[x,y]=1$.

Now if $[x,y]=1$ then $d(x,y)=1$ and so we can prove the following corollary

Corollary 3.1.19. *Let* $x,y \in X$ *with* $x \neq y$. *If* $C_V(x) = C_V(y)$ *then* $d(x,y)=1$.

By using this corollary we can determine which vertices should be joined by an edge by studying their fixed spaces. This converts our problem to simply studying linear algebra.

Commuting Involution Graphs for Sporadic Groups

In [12], Bundy, Bates, Rowley and Perkins studied commuting involution graphs for the 26 sporadic simple groups and their automorphism groups. All cases were covered in this paper apart from J_4 with the class $2B$, Fi_{24}' with the classes $2B$ and $2D$, the Baby Monster, BM, with the classes $2C$ and $2D$ and the Monster $\mathbb{M}$ with the class $2B$. The J_4 and Fi_{24}' cases have recently been calculated by Rowley and P. Taylor and will be published in the near future.

The idea of the calculations was to pick a fixed vertex t and split the involution class X into smaller chunks, that is into the sets $X_C = \{x \in X \mid tx \in C\}$ where C is any conjugacy class of the group G in question. They then determined which disc of $\mathcal{C}(G,X)$ each X_C belonged to. In all cases they found that the diameter of $\mathcal{C}(G,X)$ was at most 4, only being 4 in a limited number of cases.

For many of the sporadic simple groups, the commuting involution graph for the class $2A$ was calculated as part of the primary investigation into the group. For example in Fi_{24} the commuting involution graph for the class $2A$, the class of 3-transpositions which generate the group, was calculated during Fischer's investigation into 3-transposition groups. Similarly for the class $2A$ of the Baby Monster, similar graphs were studied by Fischer and by Ivanov and data from these commuting involution graphs can easily be extracted from these papers.

For the other cases a mixture of brain and machine was used. For the smaller sporadic groups they used the following computational method, using the smallest degree non-trivial faithful permutation representation given in the online Atlas.

- Calculate $C = C_G(t)$ and $S \in Syl_2(C)$.
- Compute $T = S \cap X$. This can be done easily by using the dimension of the fixspace as a conjugacy class invariant, that is the subspace of the natural G-module which is fixed by an element of the conjugacy class.
- Calculate $\Delta_1(t)$, the first disc of $\mathcal{C}(G, X)$, which is the union of the conjugacy classes of C in $T \setminus \{t\}$. Let R_1 be a full set of representatives for these conjugacy classes.

For $i \geq 2$ carry out the following steps

- Compute representatives R_i of the $C_G(t)$ orbits of $\Delta_i(t)$. This is done as follows
  1. For each $r \in R_{i-1}$ find $g \in G$ such that $r = t^g$.
  2. Calculate $\Delta_1(r)$ as $\Delta_1(t)^g$.
  3. Run through $\Delta_1(r)$ and discard element in orbits that have already been found.
- Calculate $|\Delta_i(t)| = \sum_{r \in R_i} \frac{|C_G(t)|}{|C_G(\langle t,r \rangle)|}$.
- Stop when $\sum_i |\Delta_i(t)| = |X|$.

This method works well in Magma for the smaller sporadic groups, however fails in larger ones as we often have to store many elements in a large matrix representation, and we run out of memory. For the larger sporadic groups they changed tactics by instead of considering the element t and varying the product $z = tx$, they fixed an element $z \in C$, for a conjugacy class C, and considered all the possible elements t which could arise. Using this method we can now consider the maximal p-local subgroup M which contains z. In most cases a smaller permutation representation for M is given in the online Atlas, which makes calculations possible. In this paper, the authors also extensively used Bray's algorithm [6], a very efficient method for calculating the centralizer of an involution. As this algorithm is fairly restrictive, as it is only applicable to involutions, the authors used a slight modification, given in [4], that can be applied to real elements, that is elements which are conjugate to their inverse.

3.2 Basic Definitions and Results

From now on we will assume that X is a conjugacy class of involutions and $\mathcal{C}(G, X)$ is the commuting involution graph of G with respect to X.

Now the following simple lemma shows that our graph is invariant under action by G.

Lemma 3.2.1. *The map $\varphi_g : X \mapsto X$ given by $x^{\varphi_g} = x^g$ is a graph automorphism.*

Proof. Clearly φ_g is a bijection as X is a conjugacy class, therefore we just need to show φ_g is compatable with the graph structure of $\mathcal{C}$, that is x and y are joined by an edge if and only if x^{φ_g} and y^{φ_g} are joined by an edge. So suppose that $xy = yx$ then

$$\begin{aligned} x^{\varphi_g} y^{\varphi_g} &= g^{-1}xgg^{-1}yg \\ &= g^{-1}xyg \\ &= g^{-1}yxg \\ &= g^{-1}ygg^{-1}xg \\ &= y^{\varphi_g} x^{\varphi_g} \end{aligned}$$

Clearly the opposite direction is also true, and hence φ_g is a graph automorphism. □

Hence the distance between any two vertices x and y is the same as the distance between x^g and y^g for any $g \in G$. Therefore the sizes and structures of the discs $\Delta_i(t)$

are independent on our choice of t. We will frequently make use of this by choosing a particular t which makes our life as easy as possible.

We have the following elementary result, proved in [12], which will be a very powerful tool in the study of these graphs.

Lemma 3.2.2 (Bundy, Bates, Rowley and Perkins)**.** *Let $x \in X$ and let $z = tx$. Suppose z has order m, then the following are true.*

(i) $x \in \Delta_1(t)$ if and only if $m = 2$.

(ii) If m is even, greater or equal to 4 and $z^{m/2} \in X$, then $x \in \Delta_2(t)$.

(iii) If $C_{C_G(z)}(x) \cap X = \emptyset$ then $d(t, x) \geq 3$. In particular if $C_{C_G(z)}(x)$ has odd order, then $d(t, x) \geq 3$.

(iv) Suppose m is odd and assume that there doesn't exist any elements $g \in G$ of order $2m$ such that $g^2 = z$ and $g^m \in X$. Then $d(t, x) \geq 3$.

Proof. We follow the proof given in [12]. We first note that z being an involution is equivalent to x and t commuting (as t and x are involutions). Hence $m = 2$ if and only if $x \in \Delta_1(t)$. Part (ii) follows from the properties of dihedral groups. Indeed firstly note that t and x generate a dihedral group of order $2m$ and as m is even, $z^{m/2} \in Z(\langle t, x\rangle)$. Hence $z^{m/2}$ commutes with both t and x, and thus $d(t, x) \leq 2$. On the other hand, as $m > 2$, $d(t, x) \geq 2$. Thus $d(t, x) = 2$ as required.

Now note that $C_G(t) \cap C_G(x) \cap X = C_{C_G(z)}(x) \cap X$ which we will assume to be empty. Therefore there are no elements in X which commute with both t and x and thus $d(t, x) \geq 3$. So in particular if $C_{C_G(z)}(x)$ has odd order, then it cannot contain any involutions and thus its intersection with X must be empty. Hence (iii) follows.

Finally for (iv) note that if m is odd then $d(t, x) \geq 2$ by (i). Now suppose that $d(t, x) = 2$, then there exists $y \in C_G(t) \cap C_G(x) \cap X$. Now z has odd order, so there exists an integer i such that $(z^i)^2 = z$. Let $w = yz^i$, then

$$\begin{aligned} w^2 &= yz^iyz^i \\ &= y^2(z^i)^2 \text{ as } y \text{ commutes with both } t \text{ and } x \\ &= z. \end{aligned}$$

We also have

$$\begin{aligned} w^m &= (yz^i)^m \\ &= y^m z^{im} \\ &= y \text{ as } y \text{ is an involution, and } z \text{ as order } m. \end{aligned}$$

However by hypothesis G has no such element, and hence $d(t, x) \geq 3$ as required. □

The following elementary Lemma about centralizers of involutions will be useful.

Lemma 3.2.3. *Let t and x be involutions in G and suppose $z = tx$. Then*

$$C_G(t) \cap C_G(x) = C_{C_G(t)}(x) = C_{C_G(z)}(t).$$

Proof. It is clear that $C_G(t) \cap C_G(x) = C_{C_G(t)}(x)$. Now suppose that $g \in C_{C_G(z)}(t)$, then g commutes with both z and t. Now $x = tz$ and hence $gx = gtz = tzg = xg$ and hence g commutes with both t and x, so $C_{C_G(z)}(t) \subseteq C_G(t) \cap C_G(x)$. The other inclusion is similar. □

Crucial to the study of $\mathcal{C}(G, X)$ is the following idea.

Definition 3.2.4. For two conjugacy classes X and C of G, with t a fixed element of X we define

$$X_C = \{x \in X \mid tx \in C\}.$$

We first note that as both X and C are conjugacy classes, the sets X_C will be independent on the choice of t. Indeed we have the following easy lemma.

Lemma 3.2.5. *For t a fixed involution in X, C a conjugacy class of G and $g \in G$ we have*

$$\{x \in X \mid tx \in C\} = \{x \in X \mid t^g x \in C\}$$

Proof. The condition that $t^g x \in C$ is equivalent to $tx^{g^{-1}} \in C$ as C is a conjugacy class. Hence

$$\begin{aligned} \{x \in X \mid t^g x \in C\} &= \{x \in X \mid tx^{g^{-1}} \in C\} \\ &= \{x \in X \mid tx \in C\} \text{ as } X \text{ is a conjugacy class.} \end{aligned}$$

□

The following lemma is an important observation about X_C, and will play an important role when we study commuting involution graphs, especially in the case of sporadic simple groups.

Lemma 3.2.6. *For C a conjugacy class of G, the set X_C is a union of $C_G(t)$ orbits, as $C_G(t)$ acts on X_C by conjugation.*

Proof. We must show that for $g \in C_G(t)$ and $x \in X_C$, $x^g \in X_C$, then our result will follow. That is, we must show that $tx^g \in C$.

$$\begin{aligned} tx^g &= tg^{-1}xg \\ &= g^{-1}txg \text{ as } g \text{ commutes with } t \\ &= g^{-1}cg \text{ where } c \in C \end{aligned}$$

Hence tx^g is an element of C as required. □

Lemma 3.2.7. *For $x \in X$ and $g \in C_G(t)$, we have $d(t, x) = d(t, x^g)$.*

Proof. Suppose $d(t, x) = n$, then there exists a chain of elements $t = x_0, x_1, x_2, \ldots, x_n = x$ such that $x_i \in X$ and $x_i x_{i+1} = x_{i+1} x_i$, and no shorter chain exists. If we conjugate each element of the chain by g, then each pair of adjacent elements still commute, so we get the following chain,

$$t = t^g = x_0^g, x_1^g, x_2^g, \ldots, x_n^g = x^g$$

In this case, no shorter chain can exist between t and x^g, as if there were, we could conjugate back to t and x, producing a shorter chain between them. Hence $d(t, x) = n = d(t, x^g)$ as required. □

Now Lemma 3.2.7 shows us that the discs of $\mathcal{C}(G, X)$ consist of unions of $C_G(t)$ orbits of X. Therefore our general tactic will be to pick a particular $x \in X$, calculate which disc it belongs to and then note that the entire orbit $x^{C_G(t)}$ belongs to this disc. Now from Lemma 3.2.6, we see that the sets X_C are also unions of $C_G(t)$ orbits, for C a conjugacy class of G. So we will break down the sets X_C into their constituent orbits and determine in which disc each orbit belongs to. It is usually the case that every orbit contained in a particular X_C will belong to the same disc of $\mathcal{C}(G, X)$.

For example for the sporadic simple group J_2 and the conjugacy class of involutions $X = 2A$ the set X_C such that $C = 2A$ make up the first disc, the set X_C such that $C = 4A$ makes up the second disc, the set X_C such that $C = 3B$ makes up the third disc and finally the sets X_C such that $C = 5A, 5B*$ make up the fourth disc. All the other sets X_C are empty.

If we have a set X_C splitting between two discs we will simply write the size of the intersection of X_C and that disc in brackets after the Atlas name for C. For example in the sporadic simple group McL, with the class $X = 2B$ the set X_{4A} splits between the second and third discs, so we will write $4A(1980)$ in the second disc, and $4A(990)$ in the third.

Now due to some ingenious character theory by Burnside we can easily calculate the sizes of the sets X_C from the character table of G.

Definition 3.2.8. Let G be a finite group and C_i, C_j and C_k be three conjugacy classes of G. Let a_{ijk} be the number of pairs (a, b) with $a \in C_i$ and $b \in C_j$, such that

$ab = g$ where g is a fixed element in C_k. Then the integers a_{ijk} for all possible i, j and k are called the class structure constants for G.

Using some character theory we can easily calculate the values of the class structure constants.

Lemma 3.2.9. *Let $C_1, \ldots C_n$ denote the conjugacy classes of G, and suppose that $g_i \in C_i$. Then for all i, j and k*

$$a_{ijk} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{\chi} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)}$$

where the sum is over all irreducible characters of G.

Proof. See [16], page 128, Lemma 2.12. □

Lemma 3.2.10.

$$|X_C| = \frac{|G|}{|C_G(g)||C_G(t)|} \sum_{\chi} \frac{\chi(g)\chi(t)^2}{\chi(1)}$$

again where the sum is over all ireducible characters of G, and g is a representative of C.

Proof. We must first show that

$$|X_C| = |\{(g, h) \in C \times X \mid gh = t\}|$$

and then using Lemma 3.2.9 our result will follow. Indeed,

$$\begin{aligned} |\{(g, h) \in C \times X \mid gh = t\}| &= |\{(g, h) \in C \times X \mid g = th\}| \\ &= |\{h \in X \mid th \in C\}| \\ &= |X_C|. \end{aligned}$$

□

Another concept that will be important to us will be the extended centralizer of an element g in G. The extended centralizer, $C^*_G(g)$ for $g \in G$ is defined as follows

$$C^*_G(g) = \{x \in G \mid g^x = g \text{ or } g^x = g^{-1}\}.$$

Note that $C^*_G(g) = N_G(\{g, g^{-1}\})$, so in particular $C^*_G(g)$ is a subgroup of G. The size of the extended centralizer of an element with respect to the size of the centralizer is closely related to that element being real or not.

Definition 3.2.11. An element $g \in G$ is said to be real (in G) if there exists an $x \in G$ such that $g^x = g^{-1}$. Furthermore, g is said to be strongly real if there is a conjugating element which is an involution.

Lemma 3.2.12. *Let $g \in G$, then*

(i) If g is an involution then $C^*_G(g) = C_G(g)$.

(ii) If g is not real then $C^*_G(g) = C_G(g)$.

(iii) if g is real and not an involution, then $|C^*_G(g)| = 2|C_G(g)|$

Proof. Parts (i) and (ii) follow easily from the definition. For part (iii), let $C^{-1} = \{x \in G \mid g^x = g^{-1}\}$. Clearly $C^*_G(g)$ is the disjoint union of $C_G(g)$ and C^{-1} as any element which centralizes g cannot invert it. Therefore if we show there exists a bijection between $C_G(g)$ and C^{-1} then we are done. Indeed, consider the following map

$$\begin{aligned} \varphi : C_G(x) &\mapsto C^{-1} \text{ such that} \\ g^{\varphi} &= hg \end{aligned}$$

for a fixed $h \in C^{-1}$. Firstly, this map is well defined. Indeed, consider a $g \in C_G(x)$, then $x^{hg} = (x^{-1})^g = x^{-1}$, hence $g^{\varphi} = hg \in C^{-1}$. Clearly φ is injective, therefore we just need to show it is surjective. So take $z \in C^{-1}$ and let $y = h^{-1}z$, then clearly $y^{\varphi} = z$, thus we need to show is that $y \in C_G(x)$. Indeed,

$$\begin{aligned} xy &= xh^{-1}z \\ &= h^{-1}hxh^{-1}z \\ &= h^{-1}x^{-1}z \\ &= h^{-1}zxz^{-1}z \\ &= h^{-1}zx \\ &= yx. \end{aligned}$$

Hence $y \in C_G(x)$and our map is indeed a bijection. Therefore our lemma holds.

□

Lemma 3.2.13. *Let t, x be non-commuting involutions from a finite group G and let $z = tx$. Then*

(i) z is strongly real in G

(ii) $|C_G^*(z)| = 2|C_G(z)|$

(iii) $C_G^*(z) = \langle C_G(z), t\rangle$

Proof. The element z is clearly strongly real as $z^t = (tx)^t = xt = z^{-1}$. Part (ii) of the lemma follows straight from Lemma 3.2.12. For part (iii), it is clear that $\langle C_G(z), t\rangle \subseteq C_G^*(z)$ as both $C_G(z)$ and t are contained in $C_G^*(z)$. Now suppose that $w \in C_G^*(z)$ and thus either $z^w = z$ or $z^w = z^{-1}$. Now if $z^w = z$ then $w \in C_G(z)$ and we are done, so suppose that $z^w = z^{-1}$. Then

$$\begin{aligned} z^{wt} &= (z^{-1})^t \\ &= txtt \\ &= tx \\ &= z. \end{aligned}$$

Hence $wt \in C_G(t)$ and therefore as $w = wtt$, we have $w \in \langle C_G(t), t\rangle$ implying that $C^*_G(z) \subseteq \langle C_G(t), t\rangle$, and we are done. □

We will finally give two more useful tools, both of which we will use extensively when studying $\mathcal{C}(G, X)$.

3.2.1 The Fix Space

Let $\rho : G \mapsto GL_n(\mathbb{F})$ be a representation of a finite group G, where $\mathbb{F}$ is some field of positive characteristic. Let V be the associated G-module, a copy of the n-dimensional vector space over $\mathbb{F}$ with the obvious action. For an element $g \in G$, we define the fixspace of g as follows

$$Fix_g = \{v \in V \mid v^g = v\}.$$

Note that Fix_g is the eigenspace of the matrix $\rho(g)$ corresponding to the eigenvalue 1, where 1 is to the multiplicative identity in $\mathbb{F}$. Clearly, if 1 is not an eigenvalue of $\rho(g)$, then Fix_g is trivial.

Lemma 3.2.14. *For $g \in G$, Fix_g is a subspace of V.*

Proof. Let $v, w \in Fix_g$ and let $\lambda_1, \lambda_2 \in \mathbb{F}$. Consider the following

$$\begin{aligned} (\lambda_1 v + \lambda_2 w)^g &= \lambda_1 v^g + \lambda_2 w^g \\ &= \lambda_1 v + \lambda_2 w \end{aligned}$$

Hence $\lambda_1 v + \lambda_2 w \in Fix_g$, and our lemma follows. □

The following Lemma will give us an important tool when studying $\mathcal{C}(G, X)$.

Lemma 3.2.15. *Let g, h be two conjugate elements in G. Then*

$$Fix_g \cong Fix_h.$$

In particular the dimensions of the two fix spaces are equal.

Proof. As g and h are conjugate in G, there exists an $a \in G$ such that $g^a = h$. Consider the following map,

$$\begin{aligned} \theta : Fix_g &\mapsto Fix_h \\ \theta(v) &= v^a \end{aligned}$$

Firstly suppose $v \in Fix_g$, then

$$\begin{aligned} \theta(v)^h &= (v^a)^h \\ &= v^{ah} \\ &= v^{ga} \\ &= v^a \\ &= \theta(v) \end{aligned}$$

Hence $\theta(v) \in Fix_h$, and this map is well defined. Now by its definition, θ is clearly linear, so all there is left to prove is that it is a bijection. Now θ is obviously injective, so suppose $w \in Fix_h$, and consider $w^{a^{-1}}$. Hence $(w^{a^{-1}})^g = w^{a^{-1}g} = w^{ha^{-1}} = w^{a^{-1}}$, and thus $w^{a^{-1}} \in Fix_g$. Clearly as $\theta(w^{a^{-1}}) = w$, our lemma easily follows.

□

Lemma 3.2.15 shows that the dimension of the fix space is a conjugacy class invariant and gives us an easy way to see if two elements are in different conjugacy classes. Assuming you are working inside a linear representation, the fixspace can be easily computed in Magma as the eigenspace of 1. When we are dealing with groups with very large dimension linear representations we can more often than not tell exactly which conjugacy class an element is in by simply using the dimension of the fixspace. For the Baby Monster, Rob Wilson [43] gave the dimension of the fixed space for representatives for all conjugacy classes of elements of even order in the 4370 dimensional representation over the field of 2 elements. During our investigation into the commuting involution graph for BM we will make extensive use of this.

3.2.2 Bray's Algorithm and Generalizations

In this section we will give details of an algorithm which computes elements which commute with a given involution. We follow the details which are given in [6].

The following elementary observation is the main justification for the algorithm.

Lemma 3.2.16 (J. Bray)**.** *For $g, h \in G$ with g an involution we have*

$$g[g,h]^{-n} = [g,h]^n g$$

for all $n \in \mathbb{N}$.

Proof. Consider the following,

$$\begin{aligned} g[g,h]^{-n} &= g\,\underbrace{(h^{-1}ghg)\dots(h^{-1}ghg)}_{n \text{ times}} \\ &= \underbrace{(gh^{-1}gh)\dots(gh^{-1}gh)}_{n \text{ times}}\,g \\ &= [g,h]^n g. \end{aligned}$$

□

Therefore if $[g,h]$ has even order, say $2m$, then $g[g,h]^m = g[g,h]^{-m} = [g,h]^m g$ and hence $[g,h]^m$ commutes with g. On the other hand, if $[g,h]$ has odd order, say $2m+1$

then $gh[g,h]^m = hg[g,h]^{m+1} = hg[g,h]^{-m} = h[g,h]^m g$, and thus $h[g,h]^m$ commutes with g . Therefore in both cases we have produced an element which commutes with g. We also note that $[g,h^{-1}] = ([g,h]^{h^{-1}})^{-1}$ and thus $[g,h^{-1}]$ has the same order as $[g,h]$ and therefore in the even case the $[g,h]$ above can be replaced by $[g,h^{-1}]$ producing two elements instead of one (in the odd case these two elements are equal). So we propose the following algorithm to produce a set S of elements which commute with g,

1. Initialise S to be $\{g\}$.

2. Choose a random element h, which isn't an involution.

3. If $[g,h]$ has even order, $2m$, then add $[g,h]^m$ and $[g,h^{-1}]^m$ to S.

4. If $[g,h]$ has odd order, $2m+1$ then add $h[g,h]^m$ to S.

5. Make another random element h.

6. Go to Step 3 unless you have enough elements.

Obviously if we have enough elements then $C_G(g) = \langle S \rangle$, however in the case of large groups in which calculating $|\langle S \rangle|$ is difficult we may not know when to stop. However in our case we often do not require all of $C_G(t)$, just part of it, so this algorithm will be sufficient. We will refer to this algorithm as Bray's Algorithm.

At this point we make an important remark on how we make random elements . We obviously want our results to be reproducible and therefore any random elements created will need to be stored. Say our group has a large degree matrix representation in which we work, for example in the Baby Monster. Then we will not want to store these elements as matrices, as this will take up far too much memory. Instead, suppose that our group G is generated by a number (normally two) of known elements, say x and y. Then to produce a random element we produce a random string of xs and ys and store this in an array, which will hopefully only take a few bytes of memory. Then to produce the element we just write a procedure that goes through the array multiplying the required elements together - this is the approach we will usually take. A full code listing for producing random elements and the algorithms given in this section can be found in Appendix 4.

In [4], Rowley and Bates made the following improvement to Bray's Algorithm so that it will work on strongly real elements. The following elementary facts underpin the method,

Lemma 3.2.17 (C. Bates and P. Rowley)**.** *Suppose that we have $t \in G$, z a real elements of G which is inverted by t, and let $h \in C_G(t)$. Then for any $i \in \mathbb{N}$,*

$$z[z,h]^{-i} = ([z,h]^i)^t z.$$

Proof. Since $z^t = z^{-1}$, we have

$$z[z,h]^{-1} = zh^{-1}z^{-1}hz = zh^{-1}z^t hz.$$

Now since $h \in C_G(t)$ and $zt^{-1} = t^{-1}z^{-1}$, we have

$$\begin{aligned} zh^{-1}z^t hz &= zh^{-1}t^{-1}zthz \\ &= zt^{-1}h^{-1}zhtz \\ &= t^{-1}z^{-1}h^{-1}zhtz \\ &= [z,h]^t z \end{aligned}$$

and thus, $z[z,h]^{-1} = [z,h]^t z$. To complete the proof of the lemma, a simple induction argument suffices. □

Lemma 3.2.18 (C. Bates and P. Rowley)**.** *Suppose that $t \in G$, z is a real element of G inverted by t, and let $h \in C_G(t)$. If we let $\mathcal{R}(t)$ denote the set of real elements of G inverted by t, then*

$$\langle [z,h] \rangle \cap \mathcal{R}(t) \subseteq C_G(z).$$

Proof. Suppose $[z,h]^i \in \mathcal{R}(t)$ then $([z,h]^i)^t = [z,h]^{-i}$, and thus by Lemma 3.2.17 we have $[z,h]^i \in C_G(z)$. Therefore $\langle [z,h] \rangle \cap \mathcal{R}(t) \subseteq C_G(z)$ as required. □

Now suppose t and x are involutions in G. Now as $z = tx$ is a real element in G inverted by t, Lemma 3.2.18 leads us to the following algorithm to compute elements in $C_G(z)$.

1. Use Bray's algorithm to produce an element h in $C_G(t)$.
2. Calculate $w = [z,h]$ and n, the order of w.
3. Test whether w^i is inverted by t, where $1 \leq i \leq n$.
4. If so output w^i and go to Step 1.

As for the previous algorithm if we produce enough elements in $C_G(z)$ we may hope to generate the entire centralizer, however knowing when to terminate is a difficult question. In practice this algorithm isn't nearly as efficient as Bray's algorithm - it will often only compute elements in $\langle z \rangle$.

3.3 The Baby Monster

The Baby Monster, BM is the second largest of the sporadic simple groups, having an order of

$$4,154,781,481,226,426,191,177,580,544,000,000$$

with a factorisation of

$$2^{41} \times 3^{13} \times 5^6 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 31 \times 47.$$

It is a so called $\{3,4\}$-transposition group as it is generated by the class $2A$ of $\{3,4\}$-transpositions, elements which product to an element of order 1,2,3 or 4. During Fischer's investigations on $\{3,4\}$-transposition groups he calculated $\mathcal{C}(BM, 2A)$ before the baby monster was even constructed. During this work, Fischer was led to predict a simple group of this order, but could not construct it. Eventually, after extensive computation, Leon and Sims [19] gave a computational construction of a group of the correct order, and proved it had the properties Fischer predicted and showed it was unique. Later Griess gave a non computational construction of the baby monster, related to the 196,884 dimensional Griess Algebra also used to construct the monster. The baby monster has 184 conjugacy classes, with four involution conjugacy classes and the maximum element order is 70. The smallest faithful linear representation of the baby monster is 4370 dimensional over the field of two elements, meaning that calculations inside BM are rather difficult and ingenious workarounds need to be found for even simple calculations. This representation was originally constructed by Rob Wilson [42] and can be found in the online Atlas [22]. We will use standard Atlas notation for all conjugacy classes.

As has already been noted, the commuting involution graph for $2A$ was known even before the construction of the baby monster. The class $2B$ was calculated by Bundy, Bates, Rowley and Perkins in [12], using the point line collinearity graph for the maximal 2-local geometry for the baby monster, computed by Rowley and Walker in [26] and [27] . The commuting involution graphs for the classes $2C$ and $2D$ are still open, with the class $2C$ being investigated in this thesis.

From now on in this chapter G will be the Baby Monster, X the class $2C$ and t will be a fixed element in X. We will denote the commuting involution graph of G with respect to X by $\mathcal{C}(G,X)$. As in [12], we wish to calculate the diameter of $\mathcal{C}(G,X)$, calculate the sizes of each of the discs and give the conjugacy classes of products tx for x running through each of the discs. Not all of this has been possible, however all classes X_C have been located within the disc structure of $\mathcal{C}(G,X)$, except for those C of elements of 2-power order, and the classes $7A$ and $14D$. The results will be summarized in the following theorem.

Theorem 3.3.1 (B. Wright)**.** *The following table gives the locations of the sets* X_C *in the graph* $\mathcal{C}(G,X)$*, where* G *is the Baby Monster and* X *is the conjugacy class* $2C$*, for various conjugacy classes* C*.*

Table 3.1: Location of X_C in $\mathcal{C}(G,X)$ for various classes C

$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$
$2B, 2C, 2D$	$3A, 5A, 6C, 6G, 6H, 6I, 6K, 9B, 10B, 10C, 12B, 12D, 12F, 12L, 12O, 12R, 13A, 15A, 17A, 20D, 20F, 21A, 24A, 24C, 24H, 26A, 40D$	$3B, 5B, 10D, 10F, 11A, 12G, 12J, 12M, 19A, 20G, 22B, 24G, 33A, 35A, 48A$

Furthermore the sets X_{18C} and X_{30D} split over two discs of $\mathcal{C}(G,X)$, with 3311126603366400 elements from X_{18C} contained in $\Delta_2(t)$ and the other 1103708867788800 elements in $\Delta_3(t)$ and 3311126603366400 elements from X_{30D} contained in $\Delta_2(t)$ and the other 3311126603366400 elements in $\Delta_3(t)$.

The rest of this chapter will be devoted to the details of the calculation of this graph. We first give a table of the sizes of the sets X_C where C runs over all conjugacy classes of G. These were computed in GAP, using the `ClassMultiplicationCoefficient(tbl,i,j,k)` command, where `tbl` is the character table for BM stored in the Character Table Library of GAP, `i` and `k` are equal to 4, as $2C$ is the fourth conjugacy class of BM, and `j` runs from 1 to 184 corresponding to all conjugacy classes of BM. The classes C for which X_C is zero are obviously omitted to conserve space. There are 77 non zero class structure constants.

Table 3.2: Class Structure Constants For $2C$.

C	Structure Constant $\|X_C\|$	factors
1A	1	1
2A	4524975	$3^2 \times 5^2 \times 7 \times 13^2 \times 17$
2C	184246272	$2^{13} \times 3^3 \times 7^2 \times 17$
2D	350859600	$2^4 \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17$
3A	4004675584	$2^{15} \times 7 \times 13 \times 17 \times 79$
3B	141937868800	$2^{19} \times 5^2 \times 7^2 \times 13 \times 17$
4B	6629575680	$2^{10} \times 3^3 \times 5 \times 7 \times 13 \times 17 \times 31$
4C	185253868800	$2^8 \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13 \times 17$
4E	224550144000	$2^{11} \times 3^4 \times 5^3 \times 7^2 \times 13 \times 17$
4F	235777651200	$2^9 \times 3^5 \times 5^2 \times 7^3 \times 13 \times 17$
4G	1005984645120	$2^{15} \times 3^4 \times 5 \times 7^3 \times 13 \times 17$
4H	1482030950400	$2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13 \times 17$
4J	3233522073600	$2^{14} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
5A	4598786949120	$2^{20} \times 3^4 \times 5 \times 7^2 \times 13 \times 17$
5B	11037088677888	$2^{22} \times 3^5 \times 7^2 \times 13 \times 17$
6C	6882212413440	$2^{15} \times 3^2 \times 5 \times 7^2 \times 13 \times 17 \times 431$
6G	22993934745600	$2^{20} \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17$
6H	14371209216000	$2^{17} \times 3^4 \times 5^3 \times 7^2 \times 13 \times 17$
6I	11496967372800	$2^{19} \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17$
6K	30658579660800	$2^{22} \times 3^3 \times 5^2 \times 7^2 \times 13 \times 17$
7A	110370886778880	$2^{23} \times 3^5 \times 5 \times 7^2 \times 13 \times 17$
8B	17245451059200	$2^{18} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
8D	30179539353600	$2^{16} \times 3^5 \times 5^2 \times 7^3 \times 13 \times 17$
8E	17245451059200	$2^{18} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
8F	25868176588800	$2^{17} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
8G	51736353177600	$2^{18} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$

8H	103472706355200	$2^{19} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
8I	51736353177600	$2^{18} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
8J	155209059532800	$2^{18} \times 3^7 \times 5^2 \times 7^2 \times 13 \times 17$
8K	137963608473600	$2^{21} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
8L	248334495252480	$2^{21} \times 3^7 \times 5 \times 7^2 \times 13 \times 17$
8N	275927216947200	$2^{22} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
9B	572293487001600	$2^{25} \times 3^2 \times 5^2 \times 7^3 \times 13 \times 17$
10B	344909021184000	$2^{20} \times 3^5 \times 5^3 \times 7^2 \times 13 \times 17$
10C	331112660336640	$2^{23} \times 3^6 \times 5 \times 7^2 \times 13 \times 17$
10D	331112660336640	$2^{23} \times 3^6 \times 5 \times 7^2 \times 13 \times 17$
10F	275927216947200	$2^{22} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
11A	2207417735577600	$2^{25} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
12B	45987869491200	$2^{21} \times 3^4 \times 5^2 \times 7^2 \times 13 \times 17$
12D	99640383897600	$2^{20} \times 3^3 \times 5^2 \times 7^2 \times 13^2 \times 17$
12F	229939347456000	$2^{21} \times 3^4 \times 5^3 \times 7^2 \times 13 \times 17$
12G	310418119065600	$2^{19} \times 3^7 \times 5^2 \times 7^2 \times 13 \times 17$
12J	413890825420800	$2^{21} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
12L	551854433894400	$2^{23} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
12M	551854433894400	$2^{23} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
12O	413890825420800	$2^{21} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
12R	827781650841600	$2^{22} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
13A	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
14D	1655563301683200	$2^{23} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
15A	1765934188462080	$2^{27} \times 3^5 \times 5 \times 7^2 \times 13 \times 17$
16A	827781650841600	$2^{22} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
16C	1655563301683200	$2^{23} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
16D	827781650841600	$2^{22} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
16E	827781650841600	$2^{22} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$

16F	1655563301683200	$2^{23} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
17A	6622253206732800	$2^{25} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
18C	4414835471155200	$2^{26} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
19A	13244506413465600	$2^{26} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
20D	827781650841600	$2^{22} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
20F	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
20G	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
21A	2207417735577600	$2^{25} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
22B	6622253206732800	$2^{25} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
24A	1103708867788800	$2^{24} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
24C	1103708867788800	$2^{24} \times 3^5 \times 5^2 \times 7^2 \times 13 \times 17$
24G	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
24H	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
26A	6622253206732800	$2^{25} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
28A	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
28B	3311126603366400	$2^{24} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
30D	6622253206732800	$2^{25} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
32A	6622253206732800	$2^{25} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
32B	6622253206732800	$2^{25} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
33A	13244506413465600	$2^{26} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
35A	13244506413465600	$2^{26} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
40D	13244506413465600	$2^{26} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$
48A	13244506413465600	$2^{26} \times 3^6 \times 5^2 \times 7^2 \times 13 \times 17$

At this point we make a comment on how we differentiate between conjugacy classes in BM. Obviously, as the order of BM is large and it has a large matrix representation dimension, using the `IsConjugate` command in MAGMA is impossible. Instead we use the co-dimension of the fixspace of an element to distinguish between classes. In [43], Rob Wilson gave the co-dimensions for all the classes of even order elements in BM. In most cases this will tell us exactly which class a particular element is in. If we have a number of classes with the same co-dimension of fixspace, we can load the element into the 4371 dimensional representation for BM over $\mathbb{F}_3$ and check the trace of the elements in question.

For elements of odd order it is fairly straight forward to calculate the dimension of the fixspace from the character table, however in most cases (apart from elements of order 3 and 5) the order of an element uniquely defines which class it belongs to.

From the ATLAS, we glean the following information about centralizers of involutions

Class	Shape of Centralizer	Size of Centralizer
$2A$	$2.^2E_6(2):2$	$2^{38}\times 3^9\times 5^2\times 7^2\times 11\times 13\times 17\times 19$
$2B$	$2^{1+22}.Co_2$	$2^{41}\times 3^6\times 5^3\times 7\times 11\times 23$
$2C$	$(2^2\times F_4(2)):2$	$2^{27}\times 3^6\times 5^2\times 7^2\times 13\times 17$
$2D$	$2^9.2^{16}.O_8^+(2):2$	$2^{38}\times 3^5\times 5^2\times 7$

Using Lemma 3.2.2 part (i), we can quickly calculate the first disc of $\mathcal{C}(G,X)$. Indeed $\Delta_1(t)=X_{2B}\cup X_{2C}\cup X_{2D}$, and thus $|\Delta_1(t)|=539,630,847$. We can also use Lemma 3.2.2 to gather some information about the other discs of $\mathcal{C}(G,X)$. The ATLAS[18] gives us information about which conjugacy class different powers of elements of G are contained in - for example we know that the cube of a $6A$ element in G is contained in $2A$, and the fourth power of a $20B$ element lives in $5B$. We can use this information as well as part (ii) from the lemma to determine whether certain sets X_C are contained in the second disc.

Indeed, consider the class $C=6K$. From the ATLAS we know the cube of an element from $6K$ is contained in $2C$. Hence by Lemma 3.2.2 (ii), $X_{6K}\subseteq\Delta_2(t)$. The same argument can be used to show X_{10C} and X_{26A} are both contained in the second disc.

Now part (iv) of Lemma 3.2.2 gives us a final bit of easy information about $\mathcal{C}(G, X)$. Indeed, consider $x \in X$ such that $z = tx \in 19A$. Now suppose there existed a $g \in G$ such that $g^2 = z$. From the ATLAS, we know that $g \in 38A$. The 19th power of such an element lives in $2A$, and hence there does not exist a $g \in G$ such that $g^2 = z$ and $g^{19} \in X$. Thus $d(t, x) \geq 3$. Again the same argument can be used to show for $x \in X_C$ with $C \in \{5B, 11A, 33A, 35A\}$ that $d(t, x) \geq 3$.

For the sets X_{19A}, X_{33A} and X_{35A} we can prove something further, that they are all in fact orbits of X as $C_G(t)$ acts by conjugation. Indeed, consider $x \in X_{19A}$, that is $z = tx \in 19A$. We wish to prove that $X_{19A} = x^{C_G(t)}$. Now by the Orbit Stabilizer Theorem and Lemma 3.2.3

$$|x^{C_G(t)}| = \frac{|C_G(t)|}{|C_{C_G(t)}(x)|} = \frac{|C_G(t)|}{|C_{C_G(z)}(t)|}.$$

We now note that

$$\frac{|C_G(t)|}{|X_{19A}|} = 2.$$

Now as $x^{C_G(t)} \subseteq X_{19A}$, if we can prove $|C_{C_G(z)}(t)| = 2$ then we must have $X_C = x^{C_G(t)}$ and thus X_C is a $C_G(t)$ orbit. Consulting the ATLAS we see that $|C_G(z)| = 38 = 2 \times 19$, so the possible orders of $C_{C_G(z)}(t)$ are 1, 2, 19 and 38. Now $C_{C_G(z)}(t)$ cannot have order 38 as $z \in C_G(z)$ and t inverts z, so definitely doesn't commute with it. So if we prove $C_{C_G(z)}(t)$ contains an involution we are done.

On the other hand $t \in C_G^*(z)$, and since z is real, Lemma 3.2.12 tells us that $[C_G^*(z) : C_G(z)] = 2$, so in particular $C_G(z) \trianglelefteq C_G^*(z)$. Hence we must have $C_G(z)^t = C_G(z)$. Now by Sylow's Theorems, any Sylow 2-subgroup of $C_G(z)$ will have order 2, that is they are just the identity and an involution, and there will be an odd number of such subgroups. Thus, as t is an involution, there must exist a Sylow 2-subgroup P such that $P^t = P$. Therefore the single involution in P must commute with t and thus we are done.

This same method can be also used to show that X_{33A} and X_{35A} are both $C_G(t)$ orbits. Now we can easily prove these elements are in the third disc of $\mathcal{C}(G,X)$ by finding an element in the second disc which commutes with our x in question, proving that $d(t,x) \leq 3$, and thus must be equal to 3. For $x \in X_{19A}$, using the 4370 dimensional representation for BM, and Bray's Algorithm, we can find a $\tau \in X_{40D}$ such that x commutes with τ. Since $X_{40D} \subseteq \Delta_2(t)$ and $d(t,x) \geq 3$, we have $d(t,x) = 3$. Now as X_{19A} is a $C_G(t)$ orbit, we have $X_{19A} \subseteq \Delta_3(t)$. Similarly we can find a $\tau \in X_{40D}$ and $\xi \in X_{17A}$ such that τ commutes with an $x \in X_{33A}$ and ξ commutes with a $y \in X_{35A}$. Hence $X_{33A}, X_{35A} \subseteq \Delta_3(t)$. Details of these calculations will be given in Appendix 4.

We now change tactic slightly, and instead of fixing an element t of X and varying $x \in X_C$ for a certain conjugacy class C, we will fix a $z \in C$ and vary t and then $x = tz$. In this case assuming z has order at least 3, we want to vary t over all $2C$ elements which invert z. Hence we want to vary t over

$$Y = (C_G^*(z) \setminus C_G(z)) \cap 2C.$$

Now as t_1 runs over Y, then for each t_1 as $t, t_1 \in 2C$ where t is our fixed element, there exists a $g \in BM$ such that $t_1^g = t$, and hence we can also spin around z, so that $d(t,x) = d(t, t_1 z)$ where x will run over X_C.

The original tactic was to let $C_G(t)$ act on X_C and take a representative x_i from each orbit, and calculate $d(t, x_i)$. Letting $C_G(z)$ act on Y will do exactly the same job for us. First note that $C_G(z)$ can act on Y, that is for $t \in Y$ and $g \in C_G(z)$, that t^g inverts z. Indeed

$$\begin{aligned} z^{t^g} &= g^{-1}tgzg^{-1}tg \\ &= g^{-1}tztg \\ &= g^{-1}z^{-1}g \\ &= z^{-1} \end{aligned}$$

Now since t^g is clearly a $2C$ element, $t^g \in Y$ and $C_G(z)$ does act on Y. Now suppose $t_1, t_2 \in Y$ are in the same $C_G(z)$ orbit. Then there exists $g \in C_G(z)$ such that $t_1^g = t_2$. Now let $x_1 = t_1 z$ and $x_2 = t_2 z$. Now as $g \in C_G(z)$, $x_1^g = x_2$ and $d(t_1, x_1) = d(t_2, x_2)$. So for $z \in C$ with $|z| \geq 3$ we carry out the following routine

1. Calculate $Y = (C_G^*(z) \setminus C_G(z)) \cap 2C$. If X_C was empty for a class C then clearly so will Y, so we will ignore it.

2. Let $C_G(z)$ act on Y and split Y into Orbits $Y_1, \ldots Y_n$.

3. For a representative $t_i \in Y_i$ Calculate $d(t_i, t_i z)$, which will correspond to $d(t, x)$ for different $C_G(t)$ orbit representatives x as $C_G(t)$ acts on X_C.

Step 3 above can be carried out using the following method. Calculate $C_i = C_{C_G(z)}(t_i)$ and see if $C_i \cap 2C \neq \emptyset$. If so then $d(t_i, t_i z) = 2$. If $C_i \cap 2C = \emptyset$ then try and find a path of length 3 or 4 between t_i and $t_i z$.

In practice this routine won't always work as calculating $C_G(z)$ and $C_G^*(z)$ inside BM is very difficult. So the general idea will be to go down to a maximal subgroup M, or part of M which contains $C_G^*(z)$. By having a stand alone version of M with a reasonably sized permutation representation and understanding the fusion between classes of M and classes of BM we hope to be able to carry out this routine.

3.3.1 The Class 17A

Let $z = tx \in 17A$. From the ATLAS it is easy to see that $C_G(z) \cong 17 \times 2^2$ and $C_G^*(z) \cong (17 : 2) \times 2^2$. So suppose $C_G^*(z) = L_1 \times L_2$ where $L_1 \cong 17 : 2$ and $L_2 \cong 2^2$. Now as $t \in C_G(z)$, $t = t_1 t_2$ where t_1 is an involution in L_1 and $t_2 \in L_2$ and either $t_2 = 1$ or an involution. Now as $C_G(z) = \langle z \rangle \times L_2$ and t inverts z we can deduce that

$$C_{C_G(z)}(t) = L_2$$

and in particular $|C_{C_G(z)}(t)| = 4$. Thus

$$|x^{C_G(t)}| = \frac{|C_G(t)|}{4} = |X_{17A}|$$

and hence X_{17A} is a $C_G(t)$ orbit.

Now by applying Bray's Algorithm to an $x \in X_{2D}$ we can find a $w \in X_{17A}$ which commutes with x. Hence $d(t, w) \leq 2$, and since tw is not an involution we deduce that $X_{17A} \subseteq \Delta_2(t)$. See Appendix 2 for calculation details.

3.3.2 The Class 3A

In this subsection we will be swopping between conjugacy classes of G and conjugacy classes of $Fi_{22} : 2$, so to make things clear we will write C_{BM} for conjugacy class C in the Baby Monster and $Y_{Fi_{22}:2}$ for class Y in $Fi_{22} : 2$.

Now suppose $z = tx \in 3A_{BM}$, and thus $x \in X_{3A}$. From the ATLAS we see that $C_G(z) = 3 \times Fi_{22} : 2 = \langle z \rangle \times Fi_{22} : 2$ and $C^*_G(z) = S_3 \times F_{22} : 2$. For compactness we will write $C^*_G(z) = S \times L$ where $S \cong S_3$ and $L \cong Fi_{22} : 2$.

We claim that $2C_{BM} \cap L = 2F_{Fi_{22}:2}$. Indeed, suppose that $u \in 2C_{BM} \cap L$, then zu is an element of order 6. Now $(zu)^2 = z^2 = z^{-1} = z^t$ and therefore zu cubes to a $2C_{BM}$ element. Similarly, zu must square to a $3A_{BM}$ element. Now from the ATLAS, G has eleven classes of elements of order 6, however only the class $6F_{BM}$ squares to a $3A_{BM}$ and cubes to a $2C_{BM}$. Hence $zu \in 6F_{BM}$. Therefore

$$|C_G(zu)| = 2^{11} \times 3^5 \times 5 \times 7.$$

Now as z and u commute and $C_G(z) = \langle z \rangle \times L$

$$|C_L(u)| = 2^{11} \times 3^4 \times 5 \times 7.$$

Consulting the ATLAS, we see that $Fi_{22} : 2$ has 6 classes of involutions, with the following centralizer sizes:

Class	Centralizer Size
$2A_{Fi_{22}:2}$	$2^{17} \times 3^6 \times 5 \times 7 \times 11$
$2B_{Fi_{22}:2}$	$2^{18} \times 3^4 \times 5$
$2C_{Fi_{22}:2}$	$2^{17} \times 3^3$
$2D_{Fi_{22}:2}$	$2^{14} \times 3^6 \times 5^2 \times 7$
$2E_{Fi_{22}:2}$	$2^{14} \times 3^4 \times 5$
$2F_{Fi_{22}:2}$	$2^{11} \times 3^4 \times 5 \times 7$

Therefore we must have that $u \in 2F_{Fi_{22}:2}$. The argument in the other direction is similar, showing that indeed, $2C_{BM} \cap L = 2F_{Fi_{22}:2}$.

Now using MAGMA and the 3510 degree permutation representation for $Fi_{22} : 2$ we found a $v \in 2F_{Fi_{22}:2}$. This was done by randomly searching for a involution and checking whether $C_{Fi_{22}:2}(v)$ had the correct size. We then found a $P \in Syl_2(C_{Fi_{22}:2}(v))$ and checked whether P contained a representative for each class of involutions in $Fi_{22} : 2$ (again done by checking whether the centralizer of each representative had the desired size).

On the other hand, $t, x \in C_G^*(z)$, and hence

$$t = t_1 u_1 \text{ and } x = t_2 u_2$$

where $t_1, t_2 \in S$ and $u_1, u_2 \in L$. Now

$$\begin{aligned} z &= tx \\ &= t_1 u_1 t_2 u_2 \\ &= t_1 t_2 u_1 u_2 \text{ as we have a direct product.} \end{aligned}$$

However $z \in S$ hence $u_1 u_2 = 1$ and $z = t_1 t_2$. Now $1 = t^2 = {t_1}^2 {u_1}^2$, and as we have a direct product we thus have both t_1 and u_1 involutions or the identity, similarly for u_2 and t_2. Note that on the other hand, z must have order 3, hence neither t_1 or t_2 can be the identity and they cannot be equal. Also note that as $u_1 u_2 = 1$ we must have $u := u_1 = u_2$. Therefore

$$t = t_1 u \text{ and } x = t_2 u$$

where t_1, t_2 are distinct involutions in S and u is either the identity, or an involution in L.

Now whichever class of involutions of $Fi_{22} : 2$ the element u belongs to, we know that a conjugate of it (in $Fi_{22} : 2$) commutes with our element $v \in 2F_{Fi22:2}$. Hence u must commute with a conjugate of v, say w, again a $2F_{Fi22:2}$ element. Therefore $w \in 2C_{BM} \cap L$. Again as we have a direct product in $C^*_G(z)$ and $w \in L$, w must also commute with both t_1 and t_2 and hence with both x and t. Now as tx is not an involution this shows that $d(t, x) = 2$ and thus $X_{3A} \subseteq \Delta_2(t)$.

3.3.3 The Class 5A

The case where $z = tx \in 5A$ can be handled in a similar manner to $3A$. From the Atlas, we have $C^*_G(z) \cong 5 : 2 \times HS : 2$ and $C_G(z) \cong 5 \times HS : 2$. Therefore if we let $C^*_G = S \times L$ where $S \cong 5 : 2$ and $L \cong HS : 2$ then $C_G(z) = \langle z \rangle \times L$.

Now we claim $2C_{BM} \cap L = 2B_{HS:2}$. Indeed, consider the element of order 10, zu. Now $(zu)^5 = u$ and hence is an element of $2C_{BM}$. The only class of elements of order 10 in G that does this is $10C_{BM}$. So in particular $|C_G(zu)| = 2^7 \times 3^2 \times 5^2$, and hence $|C_L(u)| = 2^7 \times 3^2 \times 5$. Now $HS : 2$ has 4 classes of involutions, with the following centralizer sizes

Class	Centralizer Size
$2A_{HS:2}$	$2^{10} \times 3 \times 5$
$2B_{HS:2}$	$2^7 \times 3^2 \times 5$
$2C_{HS:2}$	$2^8 \times 3^2 \times 5 \times 7$
$2D_{HS:2}$	$2^8 \times 3 \times 5$

Hence $u \in 2B_{HS:2}$ and $2C_{BM} \cap L = 2B_{HS:2}$. Now using Magma and the degree 100 permutation representation of $HS : 2$ we can find a $v \in 2B_{HS:2}$ and confirm that

$C_{HS:2}(v)$ contains a representative for each of the 4 classes of involutions. The same argument as in the $3A$ case shows that $X_{5A} \subseteq \Delta_2(t)$.

3.3.4 The Class 10B

Let $z = tx \in 10B$ and hence from the ATLAS we see that $z^2 \in 5A$. The ATLAS also tells us that

$$C_G^*(z^2) = S \times L$$

where $S \cong Dih(10)$ and $L \cong HS : 2$. Note that $t, x \in C_G^*(z^2)$, and hence $t = t_S t_L$, $x = x_L x_S$ and $z = z_L z_S$, with $t_L, x_L, z_L \in L$ and $t_S, x_S, z_S \in S$. Both t and x are involutions hence t_S, x_S must also be involutions and t_L, x_L are either involutions or the identity. Also note that z_S must have order 5 and z_L must be an involution. Now $z_S = t_S x_S$ hence $t_S \neq x_S$ and $z_L = t_L x_L$ therefore t_L and x_L must commute.

Now from the ATLAS we see that $z_L \in 2B_{BM}$, however we wish to know which class of $HS : 2$ z_L belongs to. Indeed consider the element $z_L z^2 = z^7$, a $10B_{BM}$ element. Hence

$$|C_G(z_L z^2)| = 2^{10} \times 3 \times 5^2$$

and thus

$$|C_L(z_L)| = 2^{10} \times 3 \times 5.$$

Looking this up in the ATLAS, we see that $z_L \in 2A_{HS:2}$ and more generally, $2B_{BM} \cap L = 2A_{HS:2}$.

Since $C_G(s) \geq L$ where $s = t_S$ or x_S we can easily work out which class of involutions (in the Baby Monster) t_S and x_S live in. Indeed, as $L \cong HS : 2$ and 11 divides $|HS : 2|$ but doesn't divide $|F_4(2)|$ we can deduce that $s \notin 2C_{BM}$, as the centralizer of a $2C$ element in BM has shape $(2^2 \times F_4(2)) : 2$. Similarly, 5^3 divides $|HS : 2|$, but not $|^2E_6(2)|$ or $|O_8^+(2)|$, hence $s \notin 2A_{BM}$ and $s \notin 2D_{BM}$. Therefore, we must have $s \in 2B_{BM}$ for $s = t_S$ or x_S.

We also wish to know which class of G, and thus $HS : 2$, t_L and x_L live in. Note that $t = t_S t_L$, and that $t \in 2C_{BM}$ and $t_S \in 2B_{BM}$. We see from the table in Appendix 1 that the only way a $2B_{BM}$ element and another involution can product together to get a $2C_{BM}$ element is for it to be a $2C_{BM}$ element. Hence we must have that $t_L, x_L \in 2C_{BM}$, and hence in $2B_{HS:2}$.

We wish to know pull everything across to $HS:2$ and use MAGMA to finish off the job - working in the degree 100 permutation representation of $HS:2$. We will change tack, and instead of fixing t and looking at possible zs we will fix z and look at the possible ts and thus xs. We will use the following algorithm, which we have already mentioned.

1. Pick a $z_L \in 2A_{HS:2}$.
2. Calculate $Y = (C^*_{HS:2}(z_L) \setminus (C_{HS:2}(z_L)) \cap 2B_{HS:2}$, this will give us a possible list of ts.
3. Let $C = C_{HS:2}(z_L)$ act on Y and spit into orbits U_i with representatives u_i.
4. For each representative calculate $C_i = C_C(u_i)$, this will be equal to $C_{C_{BM}(z)}(t)$ for appropriate choices of z_S and t_S.
5. For each C_i, check whether it contains a $2B_{HS:2}$ and thus a $2C_{BM}$

We note that if we find an orbit representative u_i such that the element $u_i z_L \notin 2B_{HS:2}$ then we can ignore it as $x_i = u_i z_L$ must also be a $2B_{HS:2}$ element. If we find that all C_i contain a $2B_{HS:2}$ for all relevent u_i then we may deduce that $X_{10B} \subset \Delta_2(t)$.

In this case we find that $|Y| = 200$ which splits into two orbits U_1 and U_2 under action by $C_L(z_L)$ of sizes 120 and 80. Let $u_1 \in U_1$ and $u_2 \in U_2$, then $|C_{C_L(z_L)}(u_1)| = 128$ and $|C_{C_L(z_L)}(u_1)| = 192$, with both of these centralizers containing a $2B_{HS:2}$ element. Hence $X_{10B} \subseteq \Delta_2(t)$. We also note that

$$|X_{10B}| = \frac{|C_G(t)|}{192} + \frac{|C_G(t)|}{128}$$

and thus X_{10B} must be the union of two $C_G(t)$ orbits.

The Classes $15A$, $20D$, $20F$, $30D$ and $40D$ can all be dispatched in a similar way. In these cases let $z = tx$ be a member of the required class, in all these cases z taken to an appropriate power is a $5A$ element, and hence $t, x \in S \times L$ where $S \cong Dih(10)$ and $L \cong HS : 2$. Hence let $z = z_S z_L$, $t = t_S t_L$ and $x = x_S x_L$. Note that we must still have $t_S, x_S \in 2B_{BM}$ and $t_L, x_L \in 2C_{BM}$, and hence $t_L, x_L \in 2B_{HS:2}$. In these cases we will work in the degree 100 permutation representation of $HS : 2$ and calculate $Y = (C^*_{HS:2}(z_L) \setminus (C_{HS:2}(z_L)) \cap 2B_{HS:2}$ where z_L is the element in question.

3.3.5 The Class 15A

In this case we must have $z_L \in 3A_{BM}$ and as $HS : 2$ only has one class of elements of order 3, we must have $z_L \in 3A_{HS:2}$. We now have that $|Y| = 48$ which splits into two orbits U_1, U_2 with representatives u_1, u_2 under action by $C_L(z_L)$. Now in both cases $C_{C_L(z_L)}(u_i)$ contains a $2B_{HS:2}$ element, with these centralizers having sizes 16 and 240. Consulting our table of class structure constants we see that

$$|X_{15A}| = \frac{|C_G(t)|}{16} + \frac{|C_G(t)|}{240}$$

and thus $X_{15A} \subseteq \Delta_2(t)$ and splits into two orbits under the action by $C_G(t)$.

3.3.6 The Class 20D

In this case $z_L \in 4B_{BM}$. Now we have $z_L z^4 = z^9$ is a $20D_{BM}$ element and hence $|C_G(z_L z^4)| = 2^9 \times 5$, thus $|C_L(z_L| = 2^9$. Looking this up in the ATLAS we see that $z_L \in 4B_{HS:2}$. Calculating as before we see that $|Y| = 32$ which splits into two orbits both of size 16. However we may instantly ignore one of these as for a representative u_2, $u_2 z_L \in 2A_{HS:2}$. For a representative u_1 of the other orbit we see that $C_{C_L(z_L)}(u_1)$

contains a $2B_{HS:2}$ element and has size 32. Now as we expect

$$|X_{20D}| = \frac{|C_G(t)|}{32}$$

and thus X_{20D} is a single $C_G(t)$ orbit in $\Delta_2(t)$.

3.3.7 The Class 20F

In this case we have $z_L \in 4G_{BM}$ and again $z_L z^4 = z^9$ is a $20F_{BM}$ element. Thus $|C_G(z_L z^4)| = 2^7 \times 5$ and hence $|C_L(z_L)| = 2^7$. Using our trusty companion, the Atlas, we see that $z_L \in 4C_{HS:2}$. Now $|Y| = 20$ and splits into three orbits of sizes 8,8 and 4 with representatives u_1, u_2 and u_3. Instantly we see that we can ignore u_3 as $u_3 z_L \in 2A_{HS:2}$. For the other two, $C_{C_L(z_L)}(u_i)$ contains a $2B_{HS:2}$ element in both cases, and these centralizers both have size 16. We note that

$$|X_{20F}| = \frac{|C_G(t)|}{16} + \frac{|C_G(t)|}{16}$$

and thus X_{20F} splits into two orbits under action by $C_G(t)$ and $X_{20F} \subseteq \Delta_2(t)$.

3.3.8 The Class 30D

In this case $z_L \in 6B_{BM}$. We quickly see that $z_L z^6$ is a $30D$ element, and hence $|C_G(z_L z^6)| = 2^4 \times 3 \times 5$. Therefore $|C_L(z_L)| = 2^4 \times 3$ and hence $z_L \in 6B_{HS:2}$ or $6E_{HS:2}$. Without loss of generality we pick our $z_L \in 6B_{HS:2}$ and calculate as usual. In this case we have $|Y| = 12$ which splits into 3 orbits of sizes 6,3 and 3 with representatives u_1, u_2 and u_3. Now let $C_i = C_{C_L(z_L)}(u_i)$, then $|C_1| = 8$, $|C_2| = 16$ and $|C_3| = 16$ with C_2, C_3 containing a $2B_{HS:2}$ element, and C_1 not. We now note that

$$|X_{30D}| = \frac{|C_G(t)|}{8} + \frac{|C_G(t)|}{16} + \frac{|C_G(t)|}{16}$$

and thus we must have that exactly half of X_{30D} is in $\Delta_2(t)$ and the other half has distance at least 3 from t. On the other hand, in all cases the commuting involution graph for $HS:2$ has diameter 3, hence we must have the other half of X_{30D} in $\Delta_3(t)$.

3.3.9 The Class 40D

For $z \in 40D$ we must have $z_L \in 8L_{BM}$ and $|C_G(z^8 z_L)| = 2^4 \times 5$. Hence $|C_L(z_L)| = 2^4$ and therefore $z_L \in 8B_{HS:2}$. So we again choose a $z_L \in 8B_{HS:2}$ and calculate as usual. In this case $|Y| = 12$ which splits into three orbits of size 4. We may instantly dismiss one of these as $u_1 z_L \in 2A_{HS:2}$ for a representative u_1. For the other two orbits, $|C_{C_L(z_L)}(u_i)| = 4$ for representatives u_i, with both of these centralizers containing a $2B_{HS:2}$ element. Hence $X_{40D} \subseteq \Delta_2(t)$, and by considering $|X_{40D}|$, we see that X_{40D} splits into two orbits under the action by $C_G(t)$.

3.3.10 The Class 13A

From the ATLAS we see that for $z = tx \in 13A$, $C_G^*(z) = L \times S$ where $L \cong 13:2$ and $S \cong Sym(4)$. Now S has two conjugacy classes of involutions, $2A_{Sym(4)}$ represented by $(1,2)$ and $2B_{Sym(4)}$, represented by $(1,2)(3,4)$. Clearly $|C_{Sym(4)}(2A_{Sym(4)})| = 2^2$ and $|C_{Sym(4)}(2B_{Sym(4)})| = 2^3$. Now let $v \in 2C_{BM} \cap S$, then vz is an element of order 26 which to the 13th power is in $2C_{BM}$. So by consulting the ATLAS we see that $vz \in 26A_{BM}$, and thus $|C_G(vz)| = 2^3 \times 13$. Hence $|C_S(v)| = 2^3$ and therefore, $v \in 2B_{Sym(4)}$, giving $2C_{BM} \cap S = 2B_{Sym(4)}$. Now let $t = t_L t_S$ and $x = x_L x_S$ where $t_L, x_L \in L$ and $t_S, x_S \in S$. As before it is easy to see that t_L and x_L are distinct involutions and $t_S = x_S = u$ is either the identity or an involution.

So now consider the element $v = (1,2)(3,4) \in 2B_{Sym(4)}$, an easy calculation shows that $C_{Sym(4)}(v) = \langle (1,3)(2,4), (3,4) \rangle$. In particular it is clear that $C_{Sym(4)}(v)$ contains a representative for each of the two involution conjugacy classes in $Sym(4)$. An argument identical to that in $3A$ and $5A$ shows that $X_{13A} \subseteq \Delta_2(t)$.

3.3.11 The Class 6C

Let $z = tx \in 6C$. As $z^2 \in 3A$ we can determine which disc X_{6C} is in by calculating inside $Fi_{22} : 2$ using the same method as in $10C$. Indeed note that $z, t, x \in C^*_G(z^2) = S \times L$ where $S \cong Sym(3)$ and $L \cong Fi_{22} : 2$. From the $3A$ calculation recall that $2C_{BM} \cap L = 2F_{Fi_{22}:2}$. Let $z = z_S z_L$, $t = t_S t_L$ and $x = x_S x_L$. The calculation proceeds just as in $10C$, and we see that $z_S \in 3A_{BM}$ and $z_L \in 2B_{BM}$. So consider the element $z^2 z_L$, clearly a $6C$ element, and thus $|C_G(z^2 z_L)| = 2^{18} \times 3^5 \times 5$ implying that $|C_L(z_L)| = 2^{18} \times 3^4 \times 5$. Hence $z_L \in 2B_{Fi_{22}:2}$ and more generally, $2B_{BM} \cap L = 2B_{Fi_{22}:2}$. Now as before, $C_G(t_S) \geq L$ and hence we must have $|L|$ dividing $|C_G(t_S)|$. Now note that 3^9 divides $|Fi_{22} : 2|$ but not the sizes of the centralizers of $2B_{BM}$, $2C_{BM}$ or $2D_{BM}$ elements. Thus we must have $t_S, x_S \in 2A_{BM}$. Hence we have the $2C_{BM}$ elements t and x being the products of a $2A_{BM}$ element and another involution. Looking at the Class Structure Constants given in Appendix 1 we see that $t_L, x_L \in 2A_{BM} \cup 2D_{BM}$.

Now we wish to know which classes of $Fi_{22} : 2$, $2A_{BM}$ and $2D_{BM}$ correspond to. Indeed suppose $v \in 2A_{BM} \cap L$ then $z^2 v$ is an element of order 6 which squares to a $3A_{BM}$ and cubes to a $2A_{BM}$. Hence $z^2 v \in 6A_{BM} \cup 6B_{BM}$. First suppose $z^2 v \in 6A_{BM}$ then $|C_G(z^2 v)| = 2^{17} \times 3^7 \times 5 \times 7 \times 11$ and thus $|C_L(v)| = 2^{17} \times 3^6 \times 5 \times 7 \times 11$ implying that $v \in 2A_{Fi_{22}:2}$. On the other hand if $v \in 6B_{BM}$, then a similar argument shows that $v \in 2D_{Fi_{22}:2}$. Hence $2A_{BM} \cap L = 2A_{Fi_{22}:2} \cup 2D_{Fi_{22}:2}$. Similarly, $2D_{BM} \cap L = 2C_{Fi_{22}:2} \cup 2E_{Fi_{22}:2}$. We are now in a position to write down the total fusion for the involution classes of $Fi_{22} : 2$ into involution classes of BM.

Involution Class in $Fi_{22} : 2$	Centralizer size (in $Fi_{22} : 2$)	Class in BM
2A	36,787,322,880	2A
2B	106,168,320	2B
2C	3,538,944	2D
2D	2,090,188,800	2A
2E	6,635,520	2D
2F	5,806,080	2C

By using Magma we can say more about t_L and x_L. By loading the 4370 dimensional representation of BM and feeding in the generators for $M \cong S_3 \times Fi_{22} : 2$ given in the ATLAS we can determine exactly which classes t_L and x_L belong to. Firstly we produce elements in M which have orders not among the orders of elements from $Fi_{22}:2$. We produced two elements u_1, u_2 of orders 60 and 33. As neither Fi_{22} or $Sym(3)$ have elements of these orders we know that $u_1^{20}, u_2^{11} \in S$ and $u_1^3, u_2^3 \in L$. In fact we can quickly see that u_1^{20}, u_2^{11} generate S, and by checking element orders, we can see that u_1^3, u_2^3 generate L. Now we can quickly produce the three involutions in S and by producing elements of even order and powering down, and checking the class structure constants given in [12] and the power maps given in the ATLAS, we can produce a representative for each of the 6 classes of involutions in L. Now we just need to check whether an involution from S times the representative from each class of involutions in L is a $2C$ involution in BM, which we can easily check using the dimension of its fixed space. We see that only involutions from the classes $2D_{Fi_{22}:2}$ and $2E_{Fi_{22}:2}$ when multiplied by an involution from S are in $2C$ in BM. Hence $t_L, x_L \in 2D_{Fi_{22}:2} \cup 2E_{Fi_{22}:2}$.

We will now proceed in MAGMA using the 3510 degree permutation representation of $Fi_{22}:2$. Now without loss of generality we may pick a $z_L \in 2B_{Fi_{22}:2}$ and follow the procedure in the $10C$ case, however this time we have two separate cases, corresponding to the two different possible classes for t_L. So we calculate $Y_C = C_G^*(z_L) \cap C$ where C is either $2D$ or $2E$ in $Fi_{22}:2$.

In Case 1, where $Y = C_L(z_L) \cap 2D_{Fi_{22}:2}$ we find that $|Y| = 656$ and there are 2 orbits of sizes 576 and 80. In both cases $C_{C_L(z_L)}(y_i)$, where y_i is a representative of each orbit, contain $2F_{Fi_{22}:2}$ elements.

In Case 2, where $Y = C_L(z_L) \cap 2E_{Fi_{22}:2}$ we find that $|Y| = 26928$ and there are 4 orbits of sizes 8640, 17280, 576 and 432. In all cases $C_{C_L(z_L)}(y_i)$, where y_i is a representative of each orbit, contain $2F_{Fi_{22}:2}$ elements.

So we deduce that $X_{6C} \subseteq \Delta_2(t)$.

We can use similar arguments to deal with the classes $6H$, $12D$, $12G$, $12J$, $12L$, $21A$, $24A$, $24C$, $24G$, $30D$ and $48A$. In each case let z be in the class mentioned in the section heading, z_S be the $3A$ element in S corresponding to some appropriate power of z and let z_L be the 3rd power of z living inside $L \cong Fi_{22} : 2$. In all cases we will use the routine used in the $6C$ case, with t_L and x_L in the classes $2D_{Fi_{22}:2}$ and $2E_{Fi_{22}:2}$.

3.3.12 The Class 6H

Elements in $6H$ square to $3A$, so we calculate in exactly the same way as in $6C$. In this case $z_L \in 2D_{BM}$, so $z^2 z_L = z^5 = z^{-1} = z^t$ is a $6H$ element and hence $|C_G(v^2 z_L)| = 10616832$. This implies that $|C_L(z_L)| = 3538944$ and thus $z_L \in 2C_{Fi_{22}:2}$. As in $6C$ we pick a $z_L \in 2C_{Fi_{22}:2}$ and split our calculation into two cases.

In Case 1, we let $Y = C_L(z_L) \cap 2D_{Fi_{22}:2}$ and find that $|Y| = 288$. Under the action by $C_L(z)$, Y splits into four orbits and $C_{C_L(z_L)}(y_i)$ contains a $2F_{Fi_{22}:2}$ element for each representative y_i.

In Case 2, we let $Y = C_L(z_L) \cap 2E_{Fi_{22}:2}$ and $Y = 4704$. In this case Y splits into 4 orbits and again $C_{C_L(z_L)}(y_i)$ contains a $2F_{Fi_{22}:2}$ element for each representative y_i. Hence $X_{6H} \subseteq \Delta_2(t)$.

3.3.13 The Class 12D

Elements in $12D$ to the fourth power are in $3A$, so we calculate in the usual way. In this case $z^3 \in 4B$ and clearly $z^4 z_L$ is again a $12D$ element. Hence $C_G(z^4 z_L) = 663,552$ and thus $C_L(z_L) = 221,184$ implying that $z_L \in 4A_{Fi_{22}:2}$. So using the usual routine and splitting our calculation into two cases we get the following results.

In Case 1, $Y = (C_L^*(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2}$ and $|Y| = 40$. Y splits into two orbits of sizes 36 and 4 and in both cases $C_{C_L(z_L)}(y_i)$ contains a $2F_{Fi_{22}:2}$ element for the two orbit representatives y_i.

In Case 2, $Y = (C_L^*(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2}$ and $|Y| = 1368$. In this case, Y splits into five orbits Y_1, Y_2, Y_3, Y_4 and Y_5, with sizes 576, 576, 108, 36 and 72 and representatives y_1, y_2, y_3, y_4 and y_5 respectively. We quickly see that $y_1 z_L$ isn't in one of the required classes so will be dismissed. For $i = 2, 3, 4, 5$, $C_{C_L(z_L)}(y_i)$ contains a $2F_{Fi_{22}:2}$ element, however we see that $C_{C_L(z_L)}(y_1)$ doesn't, so it was important for us to dismiss it. Hence $X_{12D} \subseteq \Delta_2(t)$.

3.3.14 12G

We can quickly see that elements in $12G$ to the fourth power are in $3A$ and cube down to $4C$. An easy calculation shows that $z_L \in 4C_{Fi_{22}:2}$ and hence we will pick a $z_L \in 4C_{Fi_{22}:2}$ and carry out the usual routine.

In case 1, $Y = (C^*_L(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2}$ and we see that $|Y| = 16$. We find that Y is a single orbit with representative y under action by $C_L(z_L)$. However in this case $yz_L \in 2F_{Fi_{22}:2}$ and so will be ignored.

In case 2, $Y = (C^*_L(z_L) \setminus C_L(z_L)) \cap 2E_{Fi_{22}:2}$ we see that $|Y| = 528$, which splits into four orbits of sizes 19219248 and 48. However we may instantly dismiss two of these orbits, one of size 192 and the other of size 48. So we are left with two orbits, with representatives y_1 and y_2. We can easily calculate that $|C_{C_L(z_L)}(y_1)| = 128$ and $|C_{C_L(z_L)}(y_2)| = 256$ and both of these centralizers do not contain a $2F_{Fi_{22}:2}$ element, and thus for $x \in X_{12G}$, $d(t, x) \geq 3$. As the commuting involution graph in all cases for $Fi_{22} : 2$ has diameter at most 3, we see that $d(t, x) \leq 3$. Hence $X_{12G} \subseteq \Delta_3(t)$. We also note that

$$\frac{|C_G(t)|}{128} + \frac{|C_G(t)|}{256} = |X_{12G}|$$

and so X_{12G} splits into two orbits under action by $C_G(t)$.

3.3.15 The Class 12J

Elements in $12J$ to the fourth power are in $3A$ and cube to $4E$. Hence $z_S \in 3A$ and $z_L \in 4E$ and we may again calculate inside $Fi_{22} : 2$. Firstly consider the element $z^4 z_L$, a $12J$ element and hence $|C_G(v^4 z_L)| = 2^{10} \times 3^3$, giving us $|C_L(z_L)| = 2^{10} \times 3^2$. Hence $z_L \in 4D_{Fi_{22}:2}$. Now as usual, we pick a $z_L \in 4D_{Fi_{22}:2}$ and carry out the usual routine.

Firstly we quickly note that $(C_L^*(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2} = \emptyset$. Thus in this case $t_L \notin 2D_{Fi_{22}:2}$.

For $Y = (C_L^*(z_L) \setminus C_L(z_L)) \cap 2E$ we see that $|Y| = 144$, which is itself an orbit under the action by $C_L(z_L)$ with representative y. In this case we see that $yz_L \in 2E_{Fi_{22}:2}$ and hence is a possible t_L, with $|C_{C_L(z_L)}(y)| = 64$. Now we also note that

$$\frac{|C_G(t)|}{|C_{C_L(z_L)}(y)|} = |X_{12J}|$$

Hence X_{12J} is indeed a single orbit under action by $C_G(t)$. Furthermore $C_{C_L(z_L)}(y)$ doesn't contain a $2F_{Fi_{22}:2}$ element, hence $d(t,x) \geq 3$ for $x \in X_{12J}$. However as the commuting involution graph for $Fi_{22}:2$ has diameter at most 3 in all cases we deduce that $X_{12J} \subseteq \Delta_3(t)$.

3.3.16 The Class 12L

Elements in $12L$ to the fourth power are in $3A$ and cube to $4G$. By an easy calculation we see that $z_L \in 4E_{Fi_{22}:2}$ and so we will carry out our usual routine.

In case 1, $Y = (C_L^*(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2}$ we see that $|Y| = 12$ which splits into two orbits of sizes 8 and 4. The orbit of size 4 can be instantly dismissed as $y_2 z_L \in 2F_{Fi_{22}:2}$ for a representative y_2. On the other hand, for a representative y_1 for the orbit of length 8, we see that $y_1 z_L \in 2E_{Fi_{22}:2}$ and its centralizer in $C_L(z_L)$ contains a $2F_{Fi_{22}:2}$ element.

In case 2, $Y = (C_L^*(z_L) \setminus C_L(z_L)) \cap 2E_{Fi_{22}:2}$ which has size 228. In this case, Y splits into 9 orbits under action by $C_L(z_L)$, four of which can be instantly dismissed. For representatives y_i for the other 5 orbits, $C_{C_L(z_L)}(y_i)$ contains a $2F_{Fi_{22}:2}$ element in each case. Hence $X_{12L} \subseteq \Delta_2(t)$.

3.3.17 The Class 21A

Elements in $21A$ to the seventh power are in $3A$ and cube to $7A$. Now $Fi_{22}:2$ only has one class of elements of order 7, so we must have $z_L \in 7A_{Fi_{22}:2}$. We will use the usual routine, with the following results.

In case 1, $(C^*_L(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2} = \emptyset$, and so this case will be ignored.

In case 2, $Y = (C^*_L(z_L) \setminus C_L(z_L)) \cap 2E_{Fi_{22}:2}$ with $|Y| = 7$. In this case, $C_L(z_L)$ is transitive on Y and $C_{C_L(z_L)}(y)$ contains a $2F_{Fi_{22}:2}$ element for the single representative y. Hence $X_{21A} \subseteq \Delta_2(t)$ and X_{21A} is a $C_G(t)$ orbit.

3.3.18 The Classes 24A and 24C

Let z be in one of the classes $24A$ or $24C$, then in both cases $z^8 \in 3A$ and so we may use the usual procedure with a small twist. As usual let $z = z_S z_L$, and in both cases we see that $|C_L(z_L)| = 768$. Now $Fi_{22} : 2$ has 5 classes of elements of order 8 with centralizer size 768, so we will have to check them all to cover both the $24A$ and the $24C$ cases. So we let z_L run over the classes $8A_{Fi_{22}:2}$, $8B_{Fi_{22}:2}$, $8E_{Fi_{22}:2}$, $8F_{Fi_{22}:2}$ and $8G_{Fi_{22}:2}$ and carry out the normal routine. Obviously telling exactly which classes correspond to $24A$ and which to $24C$ will be very difficult as we cannot distinguish between them easily, therefore we will produce 5 non-conjugate elements in BM of order 8 whose centralizer size in L is equal to 768, which will cover the required classes without explicitly knowing which class each z_L belongs to. We will call these five elements $z_1, \ldots, z_5$.

The results for z_1 are as follows. For $Y = (C^*_L(z_1) \setminus C_L(z_1)) \cap 2D_{Fi_{22}:2}$, $|Y| = 4$, a single orbit. In this case $C_{C_L(z_1)}(y)$ contains a $2F_{Fi_{22}:2}$ element for a representative y. For $Y = (C^*_L(z_1) \setminus C_L(z_1)) \cap 2E_{Fi_{22}:2}$, $|Y| = 68$, which splits into six orbits of sizes 12, 24, 12, 12, 4 and 4. Again $C_{C_L(z_1)}(y_i)$ contains a $2F_{Fi_{22}:2}$ element for each representative y_i.

For z_2 we get exactly the same results as in z_1.

For the z_3 case we quickly see that $(C^*_L(z_3) \setminus C_L(z_3)) \cap 2D_{Fi_{22}:2} = \emptyset$, so we only have a single case to check. For $Y = (C^*_L(z_3) \setminus C_L(z_3)) \cap 2E_{Fi_{22}:2}$ we get a single orbit of size 48, whose centralizer size (in $C_L(z_3)$) is 16. Note that $\frac{|C_G(t)|}{16} > |X_{24A}| = |X_{24C}|$ and hence we cannot have z_L in the same class as z_3 so we will ignore this case.

For z_4 we get the following results. For $Y = (C_L^*(z_4) \setminus C_L(z_4)) \cap 2D_{Fi_{22}:2}$, Y is a single orbit of size 4 with representative y. In this case $C_{C_L(z_4)}(y)$ contains a $2F_{Fi_{22}:2}$ element. For $Y = (C_L^*(z_4) \setminus C_L(z_4)) \cap 2E_{Fi_{22}:2}$, $|Y| = 52$, which splits into 5 orbits. All the centralizers in $C_L(z_4)$ for representatives of these five orbits contain a $2F_{Fi_{22}:2}$ element.

The results for z_5 are very similar to the z_4 and so will not be produced here. Thus in all cases we see that $d(t,x) = 2$ for x in either X_{24A} or X_{24C}, and therefore $X_{24A}, X_{24C} \subset \Delta_2(t)$.

3.3.19 The Class 24G

Elements in $24G$ to the sixth power are in $3A$ and cube to $8D$, so we may calculate inside $Fi_{22} : 2$. Now consider the element $z^8 z_L$, a $24G$ element, and hence $|C_G(v^8 z_L)| = 768$. Therefore $|C_L(z_L)| = 256$ implying that $z_L \in 8C_{Fi_{22}:2}$. So we do the usual job, by picking a $z_L \in 8C_{Fi_{22}:2}$ and carrying out the standard routine to get the following results.

We quickly see that $t_L \notin 2D_{Fi_{22}:2}$ as $(C_L^*(z_L) \setminus C_L(z_L)) \cap 2D_{Fi_{22}:2}$ is empty. So we only have a single case to check.

For $Y = (C_L^*(z_L) \setminus C_L(z_L)) \cap 2E_{Fi_{22}:2}$ we see that $|Y| = 48$. Y splits into 4 orbits under the action by $C_L(z_L)$, with sizes 8, 16, 16 and 8. In this case all $C_{C_L(z_L)}(y_i)$ for representatives y_i, again do not contain a $2F_{Fi_{22}:2}$ element.

Therefore we can deduce that for $x \in X_{24G}$, $d(t,x) \geq 3$. However consulting [12] we see that the diameter for the commuting involution graph for $Fi_{22} : 2$ in all cases has diameter at most 3, thus $d(t,x) \leq 3$. Hence $X_{24G} \subseteq \Delta_3(t)$.

3.3.20 The Class 48A

For $z \in 48A$ we have $z^{16} \in 3A$ so we may use the usual routine. Now $|C_G(z)| = 96$ so it is easy to see that $C_L(z_L) = 32$. Now $Fi_{22} : 2$ has two classes of elements of order 16, both of which have centralizer sizes of 32, so we will have to test both. As in the

$24A - 24C$ case, we will produce two non conjugate elements of order 16 in $Fi_{22} : 2$ and call them z_1 and z_2, without knowing precisely which class each one is in.

For z_1 we only have a single case to check as $(C^*_L(z_1) \setminus C_L(z_1)) \cap 2D_{Fi_{22}:2}$ is empty. For $Y = (C^*_L(z_1) \setminus C_L(z_1)) \cap 2E_{Fi_{22}:2}$, we see that $|Y| = 24$, which splits into three orbits under the action by $C_G(z_1)$. We can discount one of these orbits straight away as $yz_L \in 2F_{Fi_{22}:2}$ for a representative y. For the other two orbits, with representatives y_1 and y_2, we see that $|C_{C_L(z_1)}(y_i)| = 4$ and both centralizers do not contain a $2F_{Fi_{22}:2}$ element.

For z_2 we find that $(C^*_L(z_2) \setminus C_L(z_2)) \cap 2D_{Fi_{22}:2}$ is empty. In the other case we get a single $C_L(z_L)$ orbit of size 8, with centralizer of a representative y_3 in $C_L(z_2)$ containing a $2F_{Fi_{22}:2}$ elements, with this centralizer having size $6,635,520$.

We now note that

$$|X_{48A}| = \frac{|C_G(t)|}{2}$$

So the only way this is possible is for y_1 and y_2 to be in different orbits of X_{48A} and y_3 not being a possible t_L due to z_L not being in the class $16A_{Fi_{22}:2}$ or $16B_{Fi_{22}:2}$ which corresponds to z_2. Hence for $x \in X_{48A}$, $d(t, x) \geq 3$, however again as the diameter of the commuting involution graph for $Fi_{22} : 2$ in all cases is at most 3, we deduce that $X_{48A} \subseteq \Delta_3(t)$.

3.3.21 Classes Which Power to 5B

Classes $5B$, $10D$, $10F$ and $20G$ all power down to $5B$, so these classes will be treated similarly. Since $H = N_{BM}(5B) \cong 5^{1+4}_+ : 2^{1+4}_- : Alt(5).4$, calculating inside this group directly would be difficult due to its complex structure. So we wish to compute a permutation representation of H of a reasonable degree in which to carry out our calculations. Our general aim is to find τ, the central involution in the extraspecial group 2^{1+4} such that $C = C_H(\tau) \cong 5 : 2^{1+4} : Alt(5).4$. We will let the generators of H act on the cosets of C in H to produce a permutation representation of degree 5^4. Note that τ will commute with the central element of order 5 in 5^{1+4}, so this representation will not be faithful, however it will give us a faithful representation of $H/\langle w \rangle$ where w is this central 5 element, which will be sufficient for our purpose.

Our first job is to find the element τ. This is fairly straightforward, by taking a random element of order 8 and powering down to an involution we have a good chance of producing the required element, we can check by using Bray's Algorithm to produce elements in its centralizer and seeing if the element orders match those which we expected. Since we are working in a large matrix representation of BM we cannot ask directly for the coset action of H on $H/C_H(\tau)$ as simply just storing these groups would take up a huge amount of memory, so we have to be clever in our approach.

We first note that if $\mathcal{T}$ is a transversal of $C_H(\tau)$ in H, then $\mathcal{T}$ is also a transversal for 5 in 5^{1+4}, which is much easier to produce. Indeed we can easily produce the 5 linearly independent generators for 5^{1+4} in H by powering down from appropriately ordered random elements in H, with w being the central element of order 5. Now since the other 4 generators commute modulo w, a transversal for 5 in 5^{1+4} will be given by the 5^4 words in the four non-central generators in which we ignore the order of the generators.

Now $H = \langle w_1, w_2 \rangle$, with the generators w_1 and w_2 given as a straight line program in the online Atlas. We wish to calculate the action of w_1 and w_2 on $\gamma \in \mathcal{T}$. Indeed we wish to write γw_i as $\gamma' h$ where $h \in C_H(\tau)$ and $\gamma' \in \mathcal{T}$. Hence we run through all $\delta \in \mathcal{T}$ and determine whether $\delta^{-1} w_i \gamma \in C_H(\tau)$, by simply checking whether $\delta^{-1} w_i \gamma$ commutes with τ. When we find such a δ, of which there will be exactly one in $\mathcal{T}$, we will let $\gamma' = \delta$. If we then order our transversal, then if γ is the m^{th} element of of $\mathcal{T}$ and γ' is the n^{th}, then the element of our permutation representation corresponding to w_i will send m to n.

As we have 625 of these transversal elements to work through, instead of multiplying together the words in the generators of 5^{1+4} to produce a transversal, we will store it simply as a word (that is as an array containing the names of the generators in question) and act on a random vector v from the natural 4370 dimensional G-module for BM over $GF(2)$. This will at least give us a shortlist for possible elements γ', which we can go through more carefully if we get more than one possibility. The MAGMA code for this procedure is given in Appendix 3.

This procedure gives us a 625 degree permutation representation of the group $\overline{H} \cong 5^4 : 2_-^{1+4} : Alt(5).4$. Note that this group is isomorphic to $H/\langle w \rangle$, where w is the central 5 element in 5^{1+4} inside H. Inside $\overline{H}$ we want copies of $C_H(w)$ and $C_H^*(w)$ modulo $\langle w \rangle$. The first is simply given by $\overline{H}'$, the derived subgroup of $\overline{H}$, and for the second we find an involution a not in $\overline{H}'$ and calculate $\left\langle \overline{H}', a \right\rangle$. We will call these groups $\overline{C}$ and $\overline{C^*}$ respectively. These groups have the orders we expected from the ATLAS, namely 1,200,000 and 2,400,000, the sizes of the centralizer and extended centralizer in BM of a $5B$ element, divided by 5.

We now have to map the classes of involutions in $\overline{H}$ across to G. Indeed $\overline{H}$ has 4 classes of involutions. If we find words for representatives of these four classes in the generators of $\overline{H}$ and map these over to H, sitting inside G, we can easily see which class they belong to in G. This mapping works in the obvious way, if $\overline{w_1}$ and $\overline{w_2}$ are the two generators for $\overline{H}$ corresponding to the generators w_1 and w_2 of H. then we simply replace w_i with $\overline{w_i}$ in a word for a particular element. Table 3.3 gives the mapping of the involution classes of $\overline{H}$ to the involution classes in BM.

Table 3.3: Mapping between involution classes in $\overline{H}$ and BM.

Class in $\overline{H}$	Size of Centralizer in $\overline{H}$	Class in G
$2A_{\overline{H}}$	19200	$2B_{BM}$
$2B_{\overline{H}}$	9600	$2C_{BM}$
$2C_{\overline{H}}$	7680	$2D_{BM}$
$2D_{\overline{H}}$	1600	$2D_{BM}$

Now if we want to find the distance between t and x where $x \in X_{5B}$, we first note that we may pick $z \in 5B$ as our central element of order 5, w. We now want to calculate $C_{C_G(z)}(t)$ for different choices of t and see if it contains any $2C_{BM}$ elements.

We first note that as t has order 2 and $\langle w \rangle$ is a 5-group,

$$C_{C_G(z)}(t)/\langle w\rangle = C_{C_G(z)/\langle w\rangle}(\bar{t}) = C_{\overline{C}}(\bar{t})$$

where $\bar{t}$ is the image of t in $\overline{H}$.

We now calculate the possible ts in $\overline{H}$. These must be $2C_{BM}$ elements which invert, but don't centralize z. Therefore the set of possible ts is given by

$$Y = (\overline{C^*} \setminus \overline{C}) \cap 2B_{\overline{H}}.$$

We can differentiate between the different involution classes of $\overline{H}$ by simply calculating the sizes of centralizers. We find that $|Y| = 500$, and $\overline{C}$ acts transitively on Y. For y a random element from Y we see that $|C_{\overline{C}}(y)| = 2400$, which is what we expect, as that would make

$$|X_{5B}| = \frac{|C_G(t)|}{|C_{C_G(z)}(t)|}$$

agreeing with the fact that $\overline{C}$ acts transitively on Y. We also find that $C_{\overline{C}}(y)$ does not contain any $2B_{\overline{H}}$ and thus $2C_{BM}$ elements, hence $d(t,x) \geq 3$ for $x \in X_{5B}$. If we now return to our 4370 dimensional representation of BM, we can easily find an $s \in BM$ such that tt^s is a $5B$ element, and using Bray's algorithm we can find a $\tau \in X_{26A}$ such that τ commutes with t^s. Since $X_{26A} \subseteq \Delta_2(t)$ we deduce that $X_{5B} \subseteq \Delta_3(t)$. Details of this calculation are given in Appendix 4.

For $z = tx \in 20G$, we can again calculate inside $\overline{H}$ as $z^4 \in 5B$. Firstly we must work in the 4370 dimensional representation of H inside of G and find a $20G_{BM}$ element inside of H which to an appropriate power is w. Once we have it, call it z and transport it over to $\overline{H}$ to get $\overline{z}$, by taking a word of z in the generators for H, and replacing these for the generators of $\overline{H}$. We now have to find images of $C_G(z)$ and $C_G^*(z)$ inside of $\overline{H}$, however this is easy due to the following observation.

If $z \in 20G$ then $z = wf$ where $w \in 5B$ and f has order 4. Now note that

$$\begin{aligned} K = C_G(z) &= C_G(w) \cap C_G(f) \\ &= C_C(f) \end{aligned}$$

where $C = C_G(w)$. We wish to pinpoint $\overline{K}$ inside of $\overline{H}$, however

$$\overline{K} = \overline{C_C(f)} = C_{\overline{C}}(\overline{f})$$

as the order of w and f are coprime. Also note that $\overline{f}$ is equal to z once it has been transported over to $\overline{H}$.

Hence inside $\overline{H}$, let $\overline{C_z} = C_{\overline{C}}(\overline{z})$ where $\overline{C} = \overline{H'}$ as in the $5B$ case. We note that $|\overline{C_z}| = 96$, which is equal to $\frac{|C_G(z)|}{5}$ as expected. A similar technique can be used to calculate $\overline{C^*_G(z)}$, or a group close to it, however note that it is not necessary true that

$$\overline{C^*_G(z)} = C^*_{\overline{C^*}}(\overline{z})$$

however $\overline{C^*_G(z)}$ will always be contained in $C^*_{\overline{C^*}}(\overline{z})$. In fact in our case, $|C^*_{\overline{C^*}}(\overline{z})| = 384$, so if we take a $2B_{\overline{H}}$ involution, y, from $C^*_{\overline{C^*}}(\overline{z})$ such that $y\overline{z}$ is also a $2B_{\overline{H}}$ involution, then $\overline{C^*_G(z)} = \langle \overline{C_z}, y \rangle$, which we will call $\overline{C^*_z}$.

Now our list of possible ts is $Y = (\overline{C^*_z} \setminus \overline{C_z}) \cap 2B_{\overline{H}}$. By calculating this in Magma, we see that $|Y| = 12$ and $\overline{C_z}$ acts transitively on this. Again $|C_{\overline{C_z}}(y)| = 8$ for a representative $y \in Y$, which agrees with X_{20G} being a single $C_G(t)$ orbit. Now $C_{\overline{C_z}}(y)$ does not contain any $2B_{\overline{H}}$ elements, and thus $d(t, x) \geq 3$ for $x \in X_{20G}$.

Exactly the same method also works for $z \in 10D$. Indeed, in this case define $\overline{C_z}$ in exactly the same way as in $20G$, and thus we have $|\overline{C_z}| = 4800$, which is what we expected from the Atlas. Now in this case $|C^*_{\overline{C^*}}(\overline{z})| = 9600$, twice that of $|\overline{C_z}|$, and thus we must have $\overline{C^*_G(z)} = C^*_{\overline{C^*}}(\overline{z})$, and so we set $\overline{C^*_z} = C^*_{\overline{C^*}}(\overline{z})$. Again set $Y = (\overline{C^*_z} \setminus \overline{C_z}) \cap 2B_{\overline{H}}$ and let $\overline{C_z}$ act on it. Again Y splits into a single orbit with y as a representative. In this case $|C_{\overline{C_z}}(y)| = 80$, agreeing with X_{10D} being a single $C_G(t)$

orbit. Now $C_{\overline{C_z}}(y)$ again doesn't contain any $2B_{\overline{H}}$ involutions, hence $d(t,x) \geq 3$ for $x \in X_{10D}$.

Similarly for $z \in 10F$, we can define $\overline{C_z}$ and $\overline{C_z^*}$ in exactly the same way as in $10D$. In this case $|Y| = 20$ and $\overline{C_z}$ again acts transitively on Y and $|C_{\overline{C_z}}(y)| = 96$. Now $C_{\overline{C_z}}(y)$ again contains no $2B_{\overline{H}}$ involutions, hence $d(t,x) \geq 3$ for $x \in X_{10F}$.

For C one of the classes $5B$, $10D$, $10F$ and $20G$ we can easily prove that $X_C \subseteq \Delta_3(t)$. Indeed we choose a $x \in X_C$ and calculating elements in $C_G(x)$ by using Bray's algorithm. Once we have a list of elements $w \in C_G(x)$, we just check to see whether $w \in 2C$ and tw is in a known class in the second disc. If this happens (which it does in all cases) then $d(t,w) \leq 3$ and thus $X_C \subseteq \Delta_3(t)$ as X_C is made up of a single $C_G(t)$ orbit in these cases. Details of these calculations are given in Appendix 4.

3.3.22 Classes Which Power to 3B

The classes $3B, 6G, 6I, 6K, 9B, 12B, 12F, 12M, 12O, 12R, 18C$ and $24H$ all power down to a 3B element, and since H, the normalizer of a $3B$ element, has shape

$$H \cong 3_+^{1+8}.2_-^{1+6}.U_4(2).2$$

it can be treated in a similar manner to $5B$. Again we find the central involution in 2_-^{1+6}, which we will call τ, by finding a element of order 16 and taking it's eighth power. By doing this we give ourselves a good chance of finding the required involution, and then using Bray's Algorithm we check to see if elements in the centralizer of τ, which has shape $3.2_-^{1+6}.U_4(2).2$, have the required orders. As in the $5B$ case we compute a transversal for 3 in 3^{1+8}, which will also be a transversal for $C_H(\tau)$ in H. As in $5B$, H acting on these cosets will give us a faithful permutation representation for $H/\langle w \rangle$ of degree 6561 where w is the central 3 element in 3_+^{1+8}. As this degree is much larger than the representation in the $5B$ case obviously this calculation was much more time consuming. To combat this problem the program was run on ten machines each doing part of the transversal. Even so this still took seven days to calculate the coset action for the two generators for H, considerably longer than the two hours it took to calculate the representation in the $5B$ case. It is easy to see that $\overline{H} = H/\langle z\rangle$ has shape $3^8.2_-^{1+6}.U_4(2).2$.

As in the $5B$, case $\overline{C} = C_H(w)/\langle w\rangle = C_{\overline{H}}(w) \cong 3^8.2_-^{1+6}.U_4(2)$, is calculated by taking the derived subgroup of $\overline{H}$, and $\overline{C^*} = C^*_H(w)/\langle w\rangle$ is just $\overline{H}$ itself. Now by taking representatives for the 7 classes of involutions in $\overline{H}$ and transporting them over to BM, can can calculate the mapping of the involution classes of $\overline{H}$ into BM, which is given in Table 3.4.

Table 3.4: Mapping of involution classes of $\overline{H}$ into BM.

Class in $\overline{H}$	Size of Centralizer in $\overline{H}$	Class in BM
$2A_{\overline{H}}$	26,873,856	$2A_{BM}$
$2B_{\overline{H}}$	9,953,280	$2B_{BM}$
$2C_{\overline{H}}$	6,635,520	$2D_{BM}$
$2D_{\overline{H}}$	373,248	$2C_{BM}$
$2E_{\overline{H}}$	331,776	$2D_{BM}$
$2F_{\overline{H}}$	248,832	$2C_{BM}$
$2G_{\overline{H}}$	62,208	$2D_{BM}$

We now calculate in the same way as in $5B$, and as 2 is coprime to 3, we see that

$$C_{C_G(z)}(t)/\langle w\rangle = C_{C_G(z)/\langle w\rangle}(\bar{t}) = C_{\overline{C}}(\bar{t})$$

where t is an involution in H and $\bar{t}$ is its image in $\overline{H}$.

Now as $2C_{BM}$ corresponds to $2D_{\overline{H}} \cup 2F_{\overline{H}}$, we calculate

$$Y = (\overline{C^*} \setminus \overline{C}) \cap (2D_{\overline{H}} \cup 2F_{\overline{H}}).$$

However as $|C^*| = 43,535,646,720$ we have to be clever about this, as if we calculate it naively we will quickly run out of memory. Since $\overline{C^*} = \overline{H}$, we can simply calculate the two classes $2D_{\overline{H}}$ and $2F_{\overline{H}}$ on two different machines, and compute the elements not in $\overline{C}$. Once this is complete we will have two much smaller sets, which we can combine on a single machine to get Y. In fact our job is made easier as $(\overline{C^*} \setminus \overline{C}) \cap 2F_{\overline{H}}$

is empty, hence

$$Y = (\overline{C^*} \setminus \overline{C}) \cap 2D_{\overline{H}}$$

However again our job is made easy as if we pick a random $2D$ element $y_1 \in \overline{C^*} \setminus \overline{C}$ and calculate

$$\frac{|\overline{C}|}{|C_{\overline{C}}(y_1)|}$$

which is equal to the size of of the orbit Y_1 of Y containing y_1, as $\overline{C}$ acts on Y, we find that $|Y_1| = 116,640$, the size of the conjugacy class $2D$ of $\overline{H}$. Therefore we must have had that $Y = 2D_{\overline{H}}$ with $\overline{C}$ acting transitively on it. Now we can easily calculate that $|C_{\overline{C}}(y_1)| = 186,624$, and this centralizer contains either a $2D_{\overline{H}}$ or a $2F_{\overline{H}}$ involution. Since

$$|X_{3B}| = \frac{|C_G(t)|}{186624}$$

we deduce that $X_{3B} \subseteq \Delta_2(t)$ and is a single $C_G(t)$ orbit of X.

Now suppose $z \in 6G_{BM}$, such that $z^2 = w$. We then transport z over to $\overline{H}$ to get an involution, which we will call $\overline{z}$. We now calculate

$$\overline{C_z} = \overline{C_G(z)} = C_{\overline{C}}(\overline{z})$$

and note that $|C_z| = 4,976,640$ as expected. We now wish to calculate $\overline{C_z^*} = \overline{C_H^*(z)}$, however in general

$$\overline{C_H^*(z)} \neq C_{\overline{H}}^*(\overline{z})$$

though as said before, $\overline{C_H^*(z)} \subseteq C_{\overline{H}}^*(\overline{z})$. Hence we find an involution $\xi \in \overline{C^*}$ which inverts $\overline{z}$, then together with C_z, will generate C_z^*. If we carry this out, we find that C_z^* has the required size, twice that of $|\overline{C_z}|$. We follow the usual routine, by calculating $Y = (\overline{C_z^*} \setminus C_z) \cap (2D_{\overline{H}} \cup 2F_{\overline{H}})$, and we find that $|Y| = 4320$ and $\overline{C_z}$ acts transitively on this. We can also easily calculate that $|C_{C_z}(y)| = 1152$, where y is a random element from Y and $C_{C_z}(y)$ contains either a $2D_{\overline{H}}$ or a $2F_{\overline{H}}$ element. Thus as

$$|X_{6G}| = \frac{|C_G(t)|}{1152}$$

we see that $X_{6G} \subseteq \Delta_2(t)$ and consists of a single $C_G(t)$ orbit.

For $z \in 6I$, we find that $|Y| = 1440$ and again $\overline{C_z}$ acts transitively on Y. In this case $|C_{C_z}(y)| = 2304$, where y is a random element from Y and $C_{\overline{C_z}}(y)$ contains either a $2D_{\overline{H}}$ or a $2F_{\overline{H}}$ element. Thus as

$$|X_{6I}| = \frac{|C_G(t)|}{2304}$$

we see that $X_{6I} \subseteq \Delta_2(t)$ and consists of a single $C_G(t)$ orbit.

For $z \in 6K$ we have $|Y| = 576$, which splits into two orbits, Y_1, Y_2 as C_z acts on it. In this case, $|Y_1| = 432$ and $|Y_2| = 144$, however for $y_1 \in Y_1$, $y_1 z \in 2G_{\overline{H}}$, and thus is not a $2C_{BM}$ element, so can be ignored. For y_2 in the other orbit, we have $y_2 z$ a $2D_{\overline{H}}$ element. For this element, $|C_{C_z}(y_2)| = 864$, and contains either a $2D_{\overline{H}}$ or a $2F_{\overline{H}}$ element. Hence as

$$|X_{6K}| = \frac{|C_G(t)|}{864}$$

we see that $X_{6K} \subseteq \Delta_2(t)$ and consists of a single $C_G(t)$ orbit.

For $z \in 12B$ we find that $|Y| = 1296$ which splits into two orbits of sizes 864 and 432. For a representative y_1 from the first orbit, we find that $y_1 z$ is a $2G_{\overline{H}}$ element, and so can be ignored. For y_2 in the other orbit, we have $y_2 z$ is a $2D_{\overline{H}}$ element and $|C_{C_z}(y_2)| = 576$, with this centralizer containing either a $2D_{\overline{H}}$ or $2F_{\overline{H}}$ element. Hence as

$$|X_{12B}| = \frac{|C_G(t)|}{576}$$

we see that $X_{12B} \subseteq \Delta_2(t)$ and consists of a single $C_G(t)$ orbit.

For $z \in 12F$, we see that $|Y| = 400$ we find that Y splits into two orbits as $\overline{C_z}$ acts upon it, of sizes 360 and 40. We can easily see that both these orbits are legitimate, with centralizer sizes, in $\overline{C_z}$, of representatives from the orbits of 128 and 1152. Both these centralizers contain either a $2D_{\overline{H}}$ or $2F_{\overline{H}}$ element, and hence $X_{12F} \subseteq \Delta_2(t)$.

We also see that

$$|X_{12F}| = \frac{|C_G(t)|}{128} + \frac{|C_G(t)|}{1152}$$

and hence X_{12F} consists of two $C_G(t)$ orbits.

For $z \in 12M$, $|Y| = 192$ which splits into 3 orbits of sizes 96, 48 and 48. The two orbits of size 48 can instantly be discounted as $y_iz \in 2G_{\overline{H}}$ for representatives y_i of the two orbits. For a representative y for the orbit of size 96 we see that $yz \in 2D_{\overline{H}}$ and that $|C_{C_z}(y)| = 48$ with this centralizer not containing either a $2D$ or $2F_{\overline{H}}$ involution. We also note that

$$|X_{12M}| = \frac{|C_G(t)|}{48}$$

so X_{12M} is a single $C_G(t)$ orbit, with $d(t, x) \geq 3$ for $x \in X_{12M}$. Now by using Bray's Algorithm on an element $x \in X_{12M}$ in BM, we can find an element $\tau \in X_{20D}$ which commutes with x. Since we know $X_{20D} \subseteq \Delta_2(t)$ and X_{12M} consists of a single $C_G(t)$ orbit, we have $d(t, x) \leq 3$ for all $x \in X_{12M}$, and thus $X_{12M} \subseteq \Delta_3(t)$. Details of this calculation are given in Appendix 4.

For $z \in 12O$, we have $|Y| = 144$ splitting into two orbits of sizes 48 and 96. The orbit of size 96 can be ignored as $y_2z \in 2G_{\overline{H}}$ for a representative y_2, however the other must be considered as $y_1z \in 2D_{\overline{H}}$. For this orbit we have $|C_{C_z}(y_1)| = 64$ with this centralizer containing either a $2D_{\overline{H}}$ or a $2F_{\overline{H}}$ involution. We note that

$$|X_{12O}| = \frac{|C_G(t)|}{64}$$

hence X_{12O} is a single $C_G(t)$ orbit contained in $\Delta_2(t)$.

For $z \in 12R$, we find that $|Y| = 32$, which splits into two orbits, both of size 16, when acted on by $\overline{C_z}$. One of these orbits can be instantly ignored, however for the other $|C_{C_z}(y)| = 32$, for a representative y. This centralizer contains either a $2D$ or a $2F$ involution, and since

$$|X_{12R}| = \frac{|C_G(t)|}{32}$$

we see that $X_{12R} \subseteq \Delta_2(t)$ and is a single $C_G(t)$ orbit.

For $z \in 24H$ we see that $|Y| = 48$, which splits into three orbits Y_1, Y_2 and Y_3 when acted upon by C_z. For y_1 a representative of Y_1 we see that $y_1 z \in 2G$, and hence Y_1 can be ignored. For the other two orbits, $y_i z \in 2D$ for representatives y_i, and in both cases $|C_{C_z}(y_i)| = 16$ and both centralizers either contain a $2D$ or a $2F$ involution. We now note that

$$|X_{24H}| = \frac{|C_G(t)|}{16} + \frac{|C_G(t)|}{16}$$

and hence X_{24H} splits into two orbits of equal size when acted upon by $C_G(t)$, both of which are contained in $\Delta_2(t)$.

The classes $9B$ and $18C$ are a little more involved as the factorization of the order of the element in question consists of multiple powers of 3. Hence it is not necessarily true that

$$C_{\overline{C}}(\overline{z}) = \overline{C_H(z)},$$

however it is true that

$$\overline{C_H(z)} \leq C_{\overline{H}}(\overline{z}).$$

Suppose $z \in 9B$ such that $z^3 = w$. Then in this case, $|C_{\overline{H}}(\overline{z})| = 11,664$, the same size as $C_G(z)$. Also $|C^*_{\overline{C^*}}(\overline{z})| = 23,328$, twice the size of $C_{\overline{H}}(\overline{z})$. We will proceed as normal with $Y = (C^*_{\overline{C^*}}(\overline{z}) \setminus C_{\overline{H}}(\overline{z}) \cap (2D_{\overline{H}} \cup 2F_{\overline{H}})$, with $C_{\overline{H}}(\overline{z})$ acting upon this. We find that $|Y| = 84$, splitting into two orbits Y_1 and Y_2, with representatives y_1 and y_2, of sizes 81 and 3. In both cases $y_i z \in 2D_{\overline{H}}$, and hence are legitimate orbits. We can also easily calculate that $|C_{C_{\overline{H}}(\overline{z})}(y_1)| = 144$ and $|C_{C_{\overline{H}}(\overline{z})}(y_2)| = 3888$, with both centralizers containing either a $2D$ of $2F$ involution in $\overline{H}$. We also note that

$$|X_{9B}| = 3 \times \left(\frac{|C_G(t)|}{114} + \frac{|C_G(t)|}{3888} \right)$$

Hence the orbits of X_{9B} with $C_G(t)$ acting upon it are 3 times larger than the orbits of Y with $C_{\overline{H}}(\overline{z})$ acting upon it. Thus X_{9B} breaks into two orbits, both of which belong to $\Delta_2(t)$.

Now suppose that $z \in 18C$ with $z^6 = w$. In this case, again $|C_{\overline{H}}(\overline{z})| = 432$, the same size as $C_G(z)$, so we need to be careful. As in the $9B$ case $|C^*_{\overline{C^*}}(\overline{z})| = 864$, twice the size of $C_{\overline{H}}(\overline{z})$. Calculating Y as usual, we find that $|Y| = 24$, which splits into three orbits when $C_{\overline{H}}(\overline{z})$ acts upon it. These orbits, with representatives y_1, y_2 and y_3 have sizes 18, 3 and 3, all of which are $2D_{\overline{H}}$ elements when multiplied by z, and hence are legitimate orbits. The centralizer sizes of these three representatives in $C_{\overline{H}}(\overline{z})$ are 24, 144 and 144 respectively, with the second two containing either a $2D_{\overline{H}}$ or a $2F_{\overline{H}}$ involution and the first not. Again we note that

$$|X_{9B}| = 3 \times \left(\frac{|C_G(t)|}{24} + \frac{|C_G(t)|}{144} + \frac{|C_G(t)|}{144} \right)$$

and thus the orbits of X_{18C} with $C_G(t)$ acting upon it are 3 times larger than the orbits of Y with $C_{\overline{H}}(\overline{z})$ acting upon it. Therefore X_{18C} breaks into 3 orbits, with 3,311,126,603,366,400 elements of distance at least 3 from t, and 1,103,708,867,788,800 in $\Delta_2(t)$.

For the elements which are a distance at least 3 from t, we can say more. Indeed, if we take the element y_1 and map it over to BM then the elements of $y_1 \langle w \rangle$ will map to y_1 in $\overline{H}$. In this coset only the element $y_1 w^2$ is an element of X_{18C}, and hence this must be a representative of the orbit contained in X_{18C} not in the second disc. Now by using Bray's Algorithm we can find an element of X_{26A}, which we know to be in the second disc, which commutes with y_1 and thus $d(t, y_1) \leq 3$. Therefore 3,311,126,603,366,400 elements of X_{18C} are contained in $\Delta_3(t)$, and 1,103,708,867,788,800 elements in $\Delta_2(t)$. Details of this calculation are given in Appendix 4.

3.3.23 Classes Which Power to $11A$

The classes $11A$ and $22B$ both power down to an $11A$ element, so will be treated similarly. After consulting the Atlas, we see that for $z \in 11A$, $C^*_G(z) \cong 11 : 2 \times Sym(5)$ and $C_G(z) \cong 11 \times Sym(5)$. Now as the 2 on the bottom inverts the 11, $11 : 2 \cong Dih(11)$, and hence $C^*_G(z) \cong Dih(11) \times Sym(5)$, which is easily produced in

Magma, using the command

```
H := DirectProduct(DihedralGroup(11),SymmetricGroup(5));.
```

Now H has 5 classes of involutions with the following centralizer sizes

Class in H	Size of Centralizer in H
$2A$	264
$2B$	240
$2C$	176
$2D$	24
$2E$	16

Now by studying the orders of products of pairs of element from each class in H we can determine that $2D_H$ corresponds to $2C_{BM}$. Now as the Baby Monster has only one class of elements of order 11, we may pick any element of order 11 to be our representative $z \in 11A_{BM}$. In Magma we can easily calculate $C = C_H(z)$, and check to see that $|C| = 1320$ which is what we expect from the Atlas, as $|C_G(z)| = 1320$ and $C_G(z) \leq H$. We also note that because of the way we have set things up, $EC = C_G^*(z) = H$.

Now let $Y = (EC \setminus C) \cap 2D_H$. This is easy to calculate using the size of the centralizer as a conjugacy class invariant. We find that $|Y| = 110$, which is a single C orbit as C acts on Y by conjugation. We also note that for a representative $y \in Y$, $yz \in 2D_H$ as expected. Now $|C_C(y)| = 12$ and $C_C(y)$ only contains $2A_H$ and $2C_H$ involutions, which do not correspond to $2C_{BM}$ involutions. Hence X_{11A} is a single $C_G(t)$ orbit and for $x \in X_{11A}$, $d(t,x) \geq 3$, which agrees with the information already gained from the power maps of G.

We also note that $|X_{11A}| = \frac{|C_G(t)|}{12}$ which agrees with the fact that X_{11A} is a single $C_G(t)$ orbit, by using the orbit stabilizer theorem. Now for $x \in X_{11A}$ there exists a $\tau \in X_{40D}$ such that τ commutes with x. Since $\tau \in \Delta_2(t)$, we see that $d(t,x) \leq 3$ and thus $d(t,x) = 3$. As X_{11A} is a single $C_G(t)$ orbit we deduce that $X_{11A} \subseteq \Delta_3(t)$. Details of this calculation will be given in Appendix 4.

Now suppose $z \in 22B_{BM}$. Now H contains ten classes of elements of order 22, five with centralizer size 120 in H, which fuse to the class $22A$ in BM and the other five of centralizer size 88 in H, which fuse to the class $22B$ in BM. Thus we pick a z in H with a centralizer size in H of 88. We now let $C = C_H(z)$ and find an involution in H which inverts z, with together with C will generate $EC = C_H^*(z)$. As per usual, we now let $Y = (EC \setminus C) \cap 2D_H$, and find that $|Y| = 22$. Now C acts transitively on Y, and $|C_C(y)| = 4$ for a representative $y \in Y$, with this centralizer again only containing either $2A_H$ or $2C_H$ involutions. Hence X_{22B} consists of a single $C_G(t)$ orbit and for $x \in X_{22B}$, $d(t,x) \geq 3$.

Again we also note that $|X_{22B}| = \frac{|C_G(t)|}{4}$ confirming that X_{22B} is a single $C_G(t)$ orbit. Now for $x \in X_{22B}$ there exists a $\tau \in X_{17A}$ such that τ commutes with x. Since $\tau \in \Delta_2(t)$, we see that $d(t,x) \leq 3$ and thus $d(t,x) = 3$. As X_{22B} is a single $C_G(t)$ orbit we deduce that $X_{22B} \subseteq \Delta_3(t)$. Details of this calculation will be given in Appendix 4.

Chapter 4

Appendices

4.1 Appendix 1

The following table gives the possible involution classes produced when you multiply two involutions together in the Baby Monster. This table was computed in Gap using the `ClassMultipicationCoeffient` command.

Class of involution u	Class of involution v	Possible involution classes of product uv
$2A_{BM}$	$2A_{BM}$	$2B_{BM}, 2C_{BM}$
$2A_{BM}$	$2B_{BM}$	$2A_{BM}, 2D_{BM}$
$2A_{BM}$	$2C_{BM}$	$2A_{BM}, 2D_{BM}$
$2A_{BM}$	$2D_{BM}$	$2B_{BM}, 2C_{BM}, 2D_{BM}$
$2B_{BM}$	$2B_{BM}$	$2B_{BM}, 2D_{BM}$
$2B_{BM}$	$2C_{BM}$	$2C_{BM}$
$2B_{BM}$	$2D_{BM}$	$2A_{BM}, 2B_{BM}, 2D_{BM}$
$2C_{BM}$	$2C_{BM}$	$2B_{BM}, 2C_{BM}, 2D_{BM}$
$2C_{BM}$	$2D_{BM}$	$2A_{BM}, 2C_{BM}, 2D_{BM}$
$2D_{BM}$	$2D_{BM}$	$2A_{BM}, 2B_{BM}, 2C_{BM}, 2D_{BM}$

4.2 Appendix 2

Details of the $17A$ calculations. Using our standard generators for BM and our standard representative t for a $2C$ element we carry out the following calculation. Note that a and b in the following calculation correspond to the two generators of

$(2^2 \times F_4(2)) : 2$ given in the online ATLAS.

```
t := (a*a*b*a*b*b*a*a*b*a*b*b*b*b*a*a*a*b*b*b*a*b*a*b*a*b)^17;
w1 := y*x*y*x*y*y*x*y*y*x*y*x*y*y*x*y*x*y*y*x*y*x;
w2 := t^w1*t*t^w1;
x2d := t^w2;
rand := y*x*y*x*y*y*x*y*x*y*y*x*y*x*y*x*y*y*x*y*x*y*y*x*y*x*y*y*x*y*
x*y*y*x*y*x*y*x*y*y*x*y*y*x*y*y*x*y*y*x*y*x*y*x*y*x*y*y*x*y*y*x*y*x*
y*y*x*y*x*y*y*x*y*y*x*y*x*y*x*y*x*y*x*y*y;
BrayLoop2(~S,rand,G,x2d);
```

First note that $x2d \in X$ such that $t{*}x2d \in 2D$, and thus $x2d \in X_{2D}$. Now S contains a single element, we'll call it s

```
s := Random(S);
```

s is an element of order 12 which powers down to a $2C$ element, which we'll create.

```
s := s^6;
```

We see that the order of ts is 17, and thus $s \in X_{17A}$, and by the way we have created it $x2d$ and s will commute.

```
Order(t*s);
```

Hence $d(t, s) = 2$.

4.3 Appendix 3

Details of calculating a permutation representation for $H \cong 5_+^{1+4} : 2_-^{1+4} : Alt(5).4$. All calculations will be carried out in the 4370 dimensional representation of BM with the generators w_1 and w_2 for H given in the online ATLAS.

The first job is to find the appropriate element τ, we find that

```
t := (w2*w2*w2*w2*w1*w2*w2*w1*w1*w1*w1)^4;
```

should do the job for us. Indeed, if we use Bray's algorithm on the element t we see that possible element orders agree with our known shape of $C_H(t)$.

The five generators for 5_+^{1+4} are given by

```
x1 := (w2*w1*w2*w1*w1*w2*w1*w2*w1*w2*w2*w1*w2*w2*w1*w2*w1*w1*w2*w1*w2*w2)^6;
x2 := (w1*w1*w1*w1*w2*w1*w1*w1*w1*w1*w1*w1*w2*w1*w1*w1*w1*w2*w1*w2*w2)^8;
x3 := x2^w1;
x4 := x2^w2;
x5 := x4^(w1*w2);
```

with $x1$ being the central generator. We can check in Magma that the group generated by these five elements is indeed an extraspecial group of the required order.

We then create the transversal for 5 in 5_+^{1+4}

```
Trans := [];
for i1 in [0 .. 4] do
        for i2 in [0 .. 4] do
                for i3 in [0 .. 4] do
                        for i4 in [0 .. 4] do
z := [];
if i4 ne 0 then
        for j in [1 .. i4] do
                z := Append(z,"x2");
        end for;
end if;
if i3 ne 0 then
        for j in [1 .. i3] do
                z := Append(z,"x3");
        end for;
end if;
if i2 ne 0 then
        for j in [1 .. i2] do
```

```
                z := Append(z,"x4");
        end for;
end if;
if i1 ne 0 then
        for j in [1 .. i1] do
                z := Append(z,"x5");
        end for;
end if;
Trans := Append(Trans,z);
end for;end for;end for;end for;
```

Note that this will only give us words for each element in the transversal, if we want to use the element we must multiply the word together first. Next we define two functions which allow us to let a word z act on a vector v in the 4370 dimensional G-module.

```
WordAct := function(z,v);
        w := v;
        for i in [1 .. #z] do
                if z[i] eq "w1" then
                        w := w^w1;
                end if;
                if z[i] eq "w2" then
                        w := w^w2;
                end if;
                if z[i] eq "x1" then
                        w := w^x1;
                end if;
                if z[i] eq "x2" then
                        w := w^x2;
                end if;
                if z[i] eq "x3" then
```

```
                w := w^x3;
        end if;
        if z[i] eq "x4" then
                w := w^x4;
        end if;
        if z[i] eq "x5" then
                w := w^x5;
        end if;
        if z[i] notin {"w1","w2","x1","x2","x3","x4","x5"} then
                print "ERROR!";
                return 0;
        end if;
    end for;
    return w;
end function;

w1inv := w1^-1;
w2inv := w2^-1;
x1inv := x1^-1;
x2inv := x2^-1;
x3inv := x3^-1;
x4inv := x4^-1;
x5inv := x5^-1;

WordActInv := function(z,v);
    w := v;
    for i in [0 .. (#z-1)] do
        if z[#z-i] eq "w1" then
                w := w^w1inv;
        end if;
        if z[#z-i] eq "w2" then
```

```
                w := w^w2inv;
            end if;
            if z[#z-i] eq "x1" then
                w := w^x1inv;
            end if;
            if z[#z-i] eq "x2" then
                w := w^x2inv;
            end if;
            if z[#z-i] eq "x3" then
                w := w^x3inv;
            end if;
            if z[#z-i] eq "x4" then
                w := w^x4inv;
            end if;
            if z[#z-i] eq "x5" then
                w := w^x5inv;
            end if;
            if z[#z-i] notin {"w1","w2","x1","x2","x3","x4","x5"} then
                print "ERROR!";
                return 0;
            end if;
        end for;
        return w;
end function;
```

WordAct produces the vector v^z and WordActInv produces the vector $v^{z^{-1}}$. We will then run the following code to create the permutation representation of $w1$.

```
V := GModule(G);
perm_w1 := [];
for i in [1 .. #Trans] do
        poss := {};
```

```
v := Random(V);
for j in [1 .. #Trans] do
        w := WordActInv(Trans[j],v);
        w := w^w1;
        w := WordAct(Trans[i],w);
        w := w^t;
        s := v^t;
        s := WordActInv(Trans[j],s);
        s := s^w1;
        s := WordAct(Trans[i],s);
        if s eq w then
                poss := poss join {j};
        end if;
end for;
if #poss ge 2 then
        poss2 := {};
        v := Random(V);
        for j in poss do
                w := WordActInv(Trans[j],v);
                w := w^w1;
                w := WordAct(Trans[i],w);
                w := w^t;
                s := v^t;
                s := WordActInv(Trans[j],s);
                s := s^w1;
                s := WordAct(Trans[i],s);
                if s eq w then
                        poss2 := poss2 join {j};
                end if;
        end for;
        poss := poss meet poss2;
```

```
        end if;
        if #poss ge 2 then
                poss2 := {};
                v := Random(V);
                for j in poss do
                        w := WordActInv(Trans[j],v);
                        w := w^w1;
                        w := WordAct(Trans[i],w);
                        w := w^t;
                        s := v^t;
                        s := WordActInv(Trans[j],s);
                        s := s^w1;
                        s := WordAct(Trans[i],s);
                        if s eq w then
                                poss2 := poss2 join {j};
                        end if;
                end for;
                poss := poss meet poss2;
        end if;
        if #poss ge 2 then
                perm_w1 eq Append(perm_w1,0);
                print "SORT OUT ENTRY ",i;
        else
                perm_w1 := Append(perm_w1,Random(poss));
        end if;
        if i mod 50 eq 0 then
                print i;
        end if;
end for;
```

Note that this code runs through the full transversal to find possible γs by acting on a random vector v, the one we want being in this set. If the set of possibilities has size one, then we know that this must be the real one however if there is more than one we run the procedure on another random vector and take the intersection of the possible γs. If there is still more than one possibility after a third attempt we put a zero in and deal with it manually. When we ran this code in all cases we never had to do this.

We then repeat this code with $w1$ replaced with $w2$ in all cases. This code was tested on smaller groups with similar maximal subgroups, in which the CosetAction command in MAGMA could be used. Exactly the same group was calculated in both cases.

The code used to produce the $3B$ representation is very similar, but involves a great deal more computational time.

4.4 Appendix 4

Details for showing $X_C \subseteq \Delta_3(t)$ for $C \in \{5B, 10D, 10F, 11A, 12M, 18C, 19A, 20G, 22B, 33A, 35A\}$. We have already proved that X_C is a single $C_G(t)$ orbit, and that $d(t,x) \geq 3$ for $x \in X_C$. So all we need to do is find a $w \in 2C$ which commutes with x and which we know to be in the second disc. Now the centralizer of a $2C$ involution in BM has shape $(2^2 \times F_4(2)) : 2$, and we can get a straight line program from the online ATLAS which gives generators a and b for a subgroup H of BM of this shape. Now by taking an involution in the central 2^2 part of H, we can find a t such that $C_{BM}(t) = H$. We will take this t to be the origin of $\mathcal{C}(G, X)$ from which we measure our discs. Thus we set t to be the following element

```
t := (a*a*b*a*b*b*a*a*b*a*b*b*b*b*a*a*a*b*b*b*a*b*a*b*a*b)^17;
```

For $5B$, if we let

```
g := t^(y*y*x*y*x*y*x*y*y*x*y)*t*t^(y*y*x*y*x*y*x*y*y*x*y);

a := y*x*y*x*y*y*x*y*x*y*y*x*y*x*y;
```

where x and y are our generators for G, then $t^g \in X_{5B}$ and if we use Bray's algorithm on t^g with a being our random element we get a $2C$ element in X_{26A}, which we know to be in the second disc.

For $10D$, if we let

```
g := y*y*x*y*x*y*x*y*x*y*y*x*y*x*y*y*x*y*y*x*y*x*y*x*y;
a := y*y*x*y*y*x*y*x*y*x*y*y;
```

then $t^g \in X_{10D}$ and again using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{26A}.

For $10F$, if we let

```
g := t^(y*x*y*y*x*y*x*y*y*x*y*x*y*x*y);
a := y*y*x*y*y*x*y*x*y*x*y*y;
```

then $t^g \in X_{10D}$, and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{17A}, which we know to be in the second disc.

For $11A$ is we let

```
g := y*y*x*y*y*x*y*x*y*x*y;
a := x*y*x*y*x*y*x*y*x*y*y*x*y*y*x*y*y;
```

then $t^g \in X_{11A}$ and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{40D}, which is in the 2nd disc.

For $12M$, if we let

```
g := y*y*x*y*y*x*y*y*x*y*x*y*x*y*x*y*y*x*y*x*y*y*x*y*x*y*y*x*y*y*x*y*y;
a := x*y*y*x*y*x*y*y*x*y*x*y*y*x*y*x*y;
```

then $t^g \in X_{12M}$, and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{20D}, which is in the 2nd disc.

For $18C$, if we let

```
y1 := (w2*w1*w2*w2*w2*w1*w2*w1*w2*w2*w2*w1*w2*w2*w2*w1*w2*w2*w2*w2*w2*w2*w2*w1*w2*w1
w := (w1*w1*w1*w2*w2*w2*w2*w1*w2*w2*w2*w2*w2*w2*w2*w2*w2*w2*w2*w2*w1*w2)^10;
a := x*y*x*y*x*y*x*y*x*y*x*y*x*y*x*y*y*x*y;
```

Then $y1 * w^2$ is a member of the orbit contained in X_{18C}, which is not in $\Delta_2(t)$. By using Bray's Algorithm on $y1 * w^2$ with a as the random element we get a $2C$ element in X_{26A} which commutes with $y1 * w^2$. As we know that X_{26A} is contained in $\Delta_2(t)$ we deduce that $d(t, y1 * w^2) \leq 3$.

For $19A$ is we let

```
g := x*y*y*x*y*y*x*y*x*y*x;
a := x*y*y*x*y*x*y*x*y*y*x*y*x*y*y;
```

Then $t^g \in X_{19A}$ and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{40D}, which is in the 2nd disc.

For $20G$, if we let

```
g := y*x*y*y*x*y*x*y*y*x*y*x*y*x*y;
a := x*y*x*y*y*x*y*x*y*y*x*y*y*x*y;
```

then $t^g \in X_{20G}$, and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{26A}, known to be in $\Delta_2(t)$.

For $22B$ is we let

```
g :=y*y*x*y*x*y*y*x*y*x*y*x*y*x*y*x*y*y*x*y*y*x*y*y*x*y*x*y*y*x*y*x*y*y*x;
a := x*y*y*x*y*y*x*y*x*y*x*y;
```

Then $t^g \in X_{22B}$ and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{17A}, which is in the 2nd disc.

For $33A$ is we let

```
g := x*y*x*y*y*x*y*x*y*y*x*y*x*y;
a := y*x*y*y*x*y*x*y*x*y*y*x*y*y*x*y;
```

Then $t^g \in X_{33A}$ and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{40D}, which is in the 2nd disc.

For $35A$ is we let

```
g := y*y*x*y*y*x*y*y*x*y*x;
a := y*y*x*y*x*y*y*x*y*y*x*y*y*x*y*x;
```

Then $t^g \in X_{35A}$ and using Bray's algorithm on t^g with a as the random element we get a $2C$ element in X_{17A}, which is in the 2nd disc.

4.5 Appendix 5

We will now give code listings for the programs used while studying the commuting involution graph for the Baby Monster.

4.5.1 BrayLoop

The is procedure carries out a single loop of Bray's algorthithm. Used as `BrayLoop(~S,h,G,g)`, where S is the set where the output is to be saved, h is the random element to be used, G is the group that is being calculated inside (simply used to get ahold of the identity) and g is the element you want to find commuting elements for.

```
BrayLoop:= procedure(~S,h,G,g)
        Z:=IntegerRing();
        S:={};
        c:=1;
```

```
        c:=c+1;
        com := (g^-1)*(h^-1)*g*h;
        order_com := Order(com);
        if (order_com mod 2) eq 0 then
                p := order_com/2;
                p := Z!p;
                w1 := com^p;
                w2 := ((g^-1)*h*g*(h^-1))^p;
                S := S join {w1,w2};
        else
                p := (order_com - 1)/2;
                p := Z!p;
                w1 := h*(com^p);
                S := S join {w1};
        end if;
end procedure;
```

4.5.2 RandomWord

RandomWord(n) is a function which produces a word of length n in the generators x and y of BM and saves it as an array. Note that as x has order 2 and y order 3, the function is carefulto make sure the word is in as compact a form as possible - that is there are no consecutive xs, and no strings of consecutive y's of length 3 or more. Note that this function simply creates a array where the entries are the names "x"and "y" , and not the actual elements, to conserve space. To convert such an array into a usable element use the function `MultiplyRandomWord`.

```
RandomWord := function(n)
if n le 2 then
        print "Don't be lazy, do it yourself!";
```

```
        return 0;
else
        i:=1;
        z:=[];
        while i le n do
                if i eq 1 then
                        a:=Random(1);
                        if a eq 0 then
                                z[1] := "x";
                                z[2] := "y";
                                i:=3;
                        else
                                z[1]:= "y";
                                b:=Random(1);
                                if b eq 0 then
                                        z[2] := "x";
                                        i:=3;
                                else
                                        z[2] := "y";
                                        i:=3;
                                end if;
                        end if;
                else
                        z1 := z[i-2];
                        z2 := z[i-1];
                        if z1 eq "x" then
                                a:=Random(1);
                                if a eq 0 then
                                        z[i] := "x";
                                        i:=i+1;
                                else
```

```
                                        z[i] := "y";
                                        i:=i+1;
                                end if;
                        else
                                if z2 eq "y" then
                                        z[i] := "x";
                                        i:=i+1;
                                else
                                        z[i] := "y";
                                        i:=i+1;
                                end if;
                        end if;
                end if;
        end while;
        return z;
end if;
end function;
```

4.5.3 MultiplyRandomWord

Used to convert an array produced using `RandomWord` into a using element. Used as `MultiplyRandomWord(~g,z,G)` where g is where you want to store the element, z is the word you want to convert, and G is a group you want to do it in.

```
MultiplyRandomWord := procedure(~a,z,G)


n:=#z;
a:=Identity(G);


for i in [1 .. n] do
        if z[i] eq "x" then
                a:=a*x;
```

```
        end if;
        if z[i] eq "y" then
                a:=a*y;
        end if;
end for;
end procedure;
```

4.6 Appendix 6

Table which gives correspondence between orbit names in Theorem 2.3.2 and orbit names in [34] and [35].

Name in [34]	Name here	Name in [34]	Name here	Name in [34]	Name here
$\Delta_1^1(a)$	$\Delta_1^1(a)$	$\Delta_3^4(a)$	$\Delta_3^1(a)$	$\Delta_4^1(a)$	$\Delta_4^{16}(a)$
$\Delta_2^1(a)$	$\Delta_2^2(a)$	$\Delta_3^5(a)$	$\Delta_3^9(a)$	$\Delta_4^2(a)$	$\Delta_4^3(a)$
$\Delta_2^2(a)$	$\Delta_2^3(a)$	$\Delta_3^6(a)$	$\Delta_3^8(a)$	$\Delta_4^3(a)$	$\Delta_4^{18}(a)$
$\Delta_2^3(a)$	$\Delta_2^1(a)$	$\Delta_3^7(a)$	$\Delta_3^6(a)$	$\Delta_4^4(a)$	$\Delta_4^{17}(a)$
$\Delta_3^1(a)$	$\Delta_3^3(a)$	$\Delta_3^8(a)$	$\Delta_3^2(a)$	$\Delta_4^5(a)$	$\Delta_4^7(a)$
$\Delta_3^2(a)$	$\Delta_3^4(a)$	$\Delta_3^9(a)$	$\Delta_3^{10}(a)$	$\Delta_4^6(a)$	$\Delta_4^{21}(a)$
$\Delta_3^3(a)$	$\Delta_3^5(a)$	$\Delta_3^{10}(a)$	$\Delta_3^7(a)$		

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