

PERFECT ISOMETRY GROUPS FOR BLOCKS OF FINITE GROUPS

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Contents

| | |
|--|-----------|
| Abstract | 6 |
| Declaration | 7 |
| Copyright Statement | 8 |
| Acknowledgements | 10 |
| 1 Introduction | 11 |
| 2 Preliminary | 16 |
| 2.1 p-modular system | 16 |
| 2.2 Notations and basic definitions | 18 |
| 2.3 Block theory | 19 |
| 2.3.1 Definition of blocks | 19 |
| 2.3.2 Defect groups | 22 |
| 2.3.3 First main theorem | 23 |
| 2.3.4 Blocks and normal subgroups | 23 |
| 2.4 Grothendieck groups | 25 |
| 2.5 Equivalences of blocks | 27 |
| 2.5.1 Morita equivalence and Picard group | 27 |
| 2.5.2 Derived equivalence and derived Picard group | 29 |
| 3 Perfect Isometries | 32 |
| 3.1 Definitions | 32 |

| | | |
|----------|--|------------|
| 3.2 | Properties | 37 |
| 3.3 | Examples of perfect isometries | 40 |
| 4 | Perfect isometry groups | 44 |
| 4.1 | Structure of $\text{PI}(B)$ | 45 |
| 4.2 | Standard subgroups of $\text{PI}(B_0)$ for a principal block B_0 | 52 |
| 4.3 | $R_K(B \otimes_{\mathcal{O}} B^\circ)$ as a ring | 54 |
| 4.4 | Connection to Picard groups and derived Picard groups | 57 |
| 4.5 | Factor blocks | 62 |
| 4.6 | Isometries extended from a normal subgroup | 63 |
| 4.6.1 | Direct Product | 63 |
| 4.6.2 | Isometries commuting with induction | 66 |
| 5 | p-groups | 69 |
| 5.1 | Abelian p -groups | 70 |
| 5.2 | Extra special p -groups | 73 |
| 5.2.1 | Irreducible characters of G | 73 |
| 5.2.2 | The proof | 74 |
| 5.2.3 | Picard group and derived Picard group | 79 |
| 6 | Blocks with cyclic defect groups | 81 |
| 6.1 | The group $D \rtimes E$ and its character table | 83 |
| 6.2 | Perfect isometry groups | 86 |
| 6.3 | Special case when defect group is C_p | 93 |
| 6.4 | Comparison with Picard groups and derived Picard groups | 101 |
| 7 | Blocks with TI defect groups | 106 |
| 7.1 | Some Observations | 107 |
| 7.2 | Suzuki group $Sz(q)$ | 108 |
| 7.2.1 | Principal block of G | 109 |
| 7.2.2 | Principal block of $N_G(D)$ | 115 |
| 7.2.3 | Non-existence of perfect embedding when $q > 8$ | 118 |

| | | |
|-----------|---|------------|
| 7.2.4 | $PI_s(A) \cong PI_s(B)$ | 119 |
| 7.3 | McLaughlin group McL | 121 |
| 7.3.1 | Principal block of G | 121 |
| 7.3.2 | Principal block of $N_G(D)$ | 122 |
| 7.4 | The group $PSU_3(q)$ | 123 |
| 7.4.1 | $PSU_3(3)$, $p = 3$ | 123 |
| 7.4.2 | $PSU_3(5)$, $p = 5$ | 125 |
| 8 | Sporadic groups | 128 |
| 8.1 | Convention | 129 |
| 8.2 | Mathieu group M_{11} | 130 |
| 8.3 | Mathieu group M_{12} | 131 |
| 8.4 | Mathieu group M_{22} | 132 |
| 8.5 | Mathieu group M_{23} | 133 |
| 8.6 | Mathieu group M_{24} | 134 |
| 8.7 | Janko group J_1 | 135 |
| 8.8 | Janko group J_2 | 135 |
| 8.9 | Janko group J_3 | 136 |
| 8.10 | Higman-Sims group HS | 137 |
| 8.11 | McLaughlin group McL | 139 |
| 8.12 | Held group He | 140 |
| 8.13 | Rudvalis group Ru | 142 |
| 9 | Other blocks | 145 |
| 9.1 | Blocks with defect group $C_2 \times C_2$ | 145 |
| 9.2 | Blocks with defect group $C_3 \times C_3$ and cyclic inertial quotients of order 4146 | |
| 10 | Isotypies | 148 |
| 10.1 | Cyclic p -groups | 153 |
| 10.2 | Suzuki group $Sz(q)$ | 154 |
| | Bibliography | 156 |

| | |
|---------------------------------|------------|
| A Computer-aided results | 162 |
| A.1 Codes | 163 |

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Perfect Isometry Groups for Blocks of Finite Groups

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Our aim is to investigate perfect isometry groups, which are invariants for blocks of finite groups. There are two subgoals. First is to study some properties of perfect isometry groups in general. We found that every perfect isometry has essentially a unique sign. This allowed us to show that, in many cases, a perfect isometry group contains a direct factor generated by $-id$.

The second subgoal is to calculate perfect isometry groups for various blocks. Notable results include the perfect isometry groups for blocks with defect 1, abelian p -groups, extra special p -groups, and the principal 2-block of the Suzuki group $Sz(q)$. In the case of blocks with defect 1, we also showed that every perfect isometry can be induced by a derived equivalence. With the help of a computer, we also calculated perfect isometry groups for some blocks of sporadic simple groups.

Apart from perfect isometries, we also investigated self-isotypies in the special case where $C_G(x)$ is a p -group whenever x is a p -element. We applied our result to calculate isotypies in cyclic p -groups and the principal 2-blocks of the Suzuki group $Sz(q)$.

Declaration

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Chapter 1

Introduction

Let p be a prime number. Let G be a finite group whose order is divisible by p . Let \mathcal{O} be a complete discrete valuation ring whose residue field has characteristic p . In order to study the modular representation theory of the group algebra $\mathcal{O}G$, it is convenient to decompose $\mathcal{O}G$ into a direct sum of smaller objects called *blocks*, and study the representation theory of each block individually.

Many equivalences between blocks have been defined and studied. For example, let B be a block of $\mathcal{O}G$ and A a block of $\mathcal{O}H$, where H is another finite group, a *Morita equivalence* between B and A is a relation of the two blocks via an invertible (B, A) -bimodule (if it exists). A *derived equivalence* (as triangulated categories) is a relation via an invertible complex of perfect (B, A) -bimodules, called a *tilting complex*, (if it exists). Another concept of equivalence of blocks, which is the main focus of this thesis, was defined by Broué [8] in 1989, relating two blocks via a *perfect isometry* at the level of characters. These equivalences can be related by the following chain of implications:

$$\text{Morita equivalence} \implies \text{Derived equivalence} \implies \text{Perfect isometry}.$$

When two blocks are equivalent, many invariants associated to blocks are also carried over. For example, the set of invertible (B, B) -bimodules are shown to form a group $\text{Pic}(B)$ called the *Picard group*. If two blocks A and B are Morita equivalent, then $\text{Pic}(A) \cong \text{Pic}(B)$. Similarly, the Derived Picard groups $\text{DPic}(B)$ of invertible

tilting complexes of projective (B, B) -bimodules are invariant under derived equivalences. Since this thesis focuses on perfect isometries, we shall study invariants under perfect isometries. Broué showed in Theorem 1.5 in [8] that if A and B are two perfectly isometric blocks, then, amongst other things, $k(A) = k(B)$ and $l(A) = l(B)$. Here $k(B)$ denotes the number of irreducible characters in B , and $l(B)$ the number of irreducible Brauer characters in B .

Similar to Picard groups and Derived Picard groups, we wish to study structured invariants under perfect isometries other than the numerical invariants $k(B), l(B)$. Therefore, we will introduce a *perfect isometry group*, denoted by $\text{PI}(B)$, of all perfect isometries in the block B . We will prove some structural properties of $\text{PI}(B)$ and calculate and compare it for many blocks. It is not difficult to see that perfectly isometric blocks have isomorphic perfect isometry groups. The converse, however, does not hold in general as we found the following counter-example: let B be the principal 3-block of $\mathcal{O}G$ where G is the Ree group $G = {}^2G_2(3)$ and let A be the principal 3-block of $\mathcal{O}N_G(P)$ where P is a Sylow 3-subgroup of G , then $\text{PI}(A) \cong \text{PI}(B) \cong D_{12} \times C_2$ but there is no perfect isometry between A and B . In this case, $P \cong 3_-^{1+2}$ and $N_G(P) = P \rtimes C_2$.

Let A be a block of $\mathcal{O}N_G(P)$ with a defect group P . Let B be the Brauer correspondent block of $\mathcal{O}G$ with defect group P . Some information about B can often be derived from that about A . We state two conjectures relating the two blocks.

Conjecture 1.0.1 (Alperin-McKay). *Let A be the unique block of $\mathcal{O}N_G(P)$ with Brauer correspondent block B of $\mathcal{O}G$. Then the number of irreducible characters of height zero in A and B are equal.*

Conjecture 1.0.2 (Broué). *Let A be the unique block of $\mathcal{O}N_G(P)$ with Brauer correspondent block B of $\mathcal{O}G$. Suppose P is abelian. Then the two blocks A, B are derived equivalent (as triangulated categories).*

Since a derived equivalence implies the existence of a perfect isometry, a weaker conjecture (which is also still unproved) can also be stated:

Conjecture 1.0.3 (Broué). *Let A be the unique block of $\mathcal{O}N_G(P)$ with Brauer correspondent block B of $\mathcal{O}G$. Suppose P is abelian. Then the two blocks A, B are perfectly isometric.*

Conjecture 1.0.3 has been verified in many cases including when P is cyclic [30, 43, 44], when B is a principal 2-block by Fong and Harris [18], when B is the principal block of $GL_2(p^n)$ with $p \neq 2$ [8], when B is a block of a symmetric group by Rouquier [42], when B is the principal blocks of sporadic simple groups by Rouquier [42] and when the inertial quotient E of B is small by Puig and Usami [36], [37], [50], [51], [52].

In order to find the perfect isometry group of a block B , it is sometimes easier to work with a block A that is perfectly isometric to B but has a simpler structure. Linckelmann [30] showed that if B is a block of $\mathcal{O}G$ whose defect group P is cyclic and if E is the inertial quotient of B , then B and $\mathcal{O}(P \rtimes E)$ are derived equivalent, and consequently perfectly isometric. Our task of finding the perfect isometry group of B then reduces to finding the perfect isometry group of $P \rtimes E$ whose structure is well understood.

Apart from proving existence of a perfect isometry, showing the non-existence of a perfect isometry is also interesting. Cliff [10] showed that the centers of the principal 2-block B of the Suzuki group and the principal 2-block A of its Sylow 2-subgroup P are isomorphic as algebras over k but not isomorphic as algebras over \mathcal{O} , where \mathcal{O} is a discrete valuation ring of characteristic 0 whose residue field is k . As we shall see later that a perfect isometry induces an algebra isomorphism $Z(\mathcal{O}A) \rightarrow Z(\mathcal{O}B)$, this implies that there is no perfect isometry between A and B . At the same time, Robinson [40] showed that, under certain conditions (when $N_G(P)/O_{p'}(N_G(P))$ is a Frobenius group), the existence of a perfect isometry requires that there are at least $l(B)$ irreducible characters of B which take constant (non-zero) values on p -singular elements. This condition immediately rules out the case of the Suzuki group above. We will show in this thesis that the two blocks A and B have visibly distinct perfect isometry groups, and therefore, there cannot be a perfect isometry between them.

The knowledge of invariants under perfect isometries, especially perfect isometry groups, may therefore be a useful tool when showing the non-existence of a perfect isometry.

This thesis can be summarized as follows.

In Chapter 2 we introduce the basic definitions and standard results from modular representation theory, character theory and block theory as far as required in this thesis. Some notations introduced in this chapter will be fixed and used throughout the thesis.

In Chapter 3 we give a definition of a perfect isometry, as defined by Broué [8], between blocks A, B of finite groups. Some properties and numerical invariants under perfect isometries will be given, most of which are results due to Broué [8]. We will also show that every perfect isometry has essentially a unique sign. Examples of perfect isometries that occur naturally will be given.

In Chapter 4 we restrict our attention to the case where $A = B$ and define the perfect isometry group $\text{PI}(B)$. The structure of $\text{PI}(B)$ will be studied. In particular we will show that, in many cases we have $\text{PI}(B) = H \times \langle -id \rangle$ where $\langle -id \rangle$ is the subgroup generated by $-id$ and H is isomorphic to a subgroup of the symmetric group $S_{k(B)}$. In the case where B is the principal block, we will show that $\text{PI}(B)$ contains subgroups that can be related to the structure of $\text{Aut}(G)$ and $\text{Hom}(G, \mathcal{O}^\times)$. A relationship between perfect isometry groups and Picard groups and derived Picard groups will be explained here.

In Chapter 5-6 we calculate the perfect isometry groups for abelian p -groups, extra special p -groups and blocks with cyclic defect groups. In the case of blocks with cyclic defect groups, we obtain a complete result for perfect isometry groups when the defect is 1. This allows us to compare them with the corresponding Picard groups and derived Picard groups. In particular, we show that the canonical homomorphism $\text{DPic}(B) \rightarrow \text{PI}(B)$ is surjective if B is a block with defect 1. When B is a block with cyclic defect group and the defect is greater than 1, we obtain a partial result for the perfect isometry group. If G is an extra special group p^{1+2} , we show that the

homomorphism $\text{DPic}(\mathcal{O}G) \rightarrow \text{PI}(\mathcal{O}G)$ is not surjective.

In Chapter 7 we study some blocks with TI defect groups. One of the properties of blocks with TI defect groups is that we have $k(B) = k(A)$ where B is a block of $\mathcal{O}G$ with TI defect group D , and A is a block of $\mathcal{O}N_G(D)$ such that $A^G = B$. This guarantees that there is at least an isometry between B and A . We will calculate both $\text{PI}(B)$, $\text{PI}(A)$ and compare them. When B is the principal 2-block of $Sz(q)$, we will show that $\text{PI}(B)$ and $\text{PI}(A)$ are visibly distinct. This gives yet another proof that there is no perfect isometry between B and A in this case.

In Chapter 9 we calculate perfect isometry groups for some blocks of sporadic groups. And in chapter 10 we calculate perfect isometry groups for some blocks that have been classified or shown to be perfectly isometric to blocks of groups with simpler structures.

Finally, in Chapter 10 we define isotypies as in [8] and calculate the subgroup of $\text{PI}(B)$ consisting of isotypies I for cyclic p -groups and Suzuki group $Sz(q)$.

Chapter 2

Preliminary

2.1 p -modular system

Let p be a prime number. Let \mathcal{O} be a complete discrete valuation ring of characteristic 0. Let \mathfrak{p} be the unique maximal ideal of \mathcal{O} . Let $k = \mathcal{O}/\mathfrak{p}$ be the residue field of characteristic p and $*$: $\mathcal{O} \rightarrow k$ be the natural projection map. Let K be a field of fraction of \mathcal{O} . The triple (K, \mathcal{O}, k) is called a p -modular system. Let G be a finite group. We say that a p -modular system (K, \mathcal{O}, k) is *sufficiently large* for G if K, \mathcal{O} and k contain a primitive m -th root of unity where m is the exponent of G .

One example of a p -modular system for G to have in mind is as follows. Let $\zeta = e^{2\pi i/|G|}$ be a primitive $|G|$ -th root of unity. Let \mathbb{Q}_p be the field of p -adic numbers. It is the completion of \mathbb{Q} with respect to the usual p -adic norm $\|\cdot\|_p$. Let $K := \mathbb{Q}_p(\zeta)$ be a finite extension of \mathbb{Q}_p by adjoining the element ζ . The norm $\|\cdot\|_p$ can be extended uniquely to a norm on K , still denoted $\|\cdot\|_p$, by

$$\|x\|_p = (\|\text{Norm}_{K/\mathbb{Q}_p}(x)\|_p)^{1/[K:\mathbb{Q}_p]}, \quad \text{for } x \in K \quad (2.1)$$

where

$$\text{Norm}_{K/\mathbb{Q}_p}(x) = (-1)^{[K:\mathbb{Q}_p]} F(0)^{[K:\mathbb{Q}_p]/d}$$

where F is the minimal polynomial of x over \mathbb{Q}_p and $d = \deg(F)$. Similarly, the usual

p -adic valuation v_p on \mathbb{Q}_p can be extended to K via

$$v_p(x) = \frac{1}{[K : \mathbb{Q}_p]} v_p(\text{Norm}_{K/\mathbb{Q}_p}(x)), \quad (2.2)$$

for $x \in K^\times$ and define $v_p(0) = +\infty$. The (extended) p -adic norm and the (extended) p -adic valuation are related by

$$\|x\|_p = p^{-v_p(x)},$$

for $x \in K$. The reader is referred to [48] or [20, Section 5] for a more detailed discussion on extensions of p -adic numbers.

With the above setup, the valuation ring is then

$$\mathcal{O} = \{x \in K : \|x\|_p \leq 1\} = \{x \in K : v_p(x) \geq 0\}. \quad (2.3)$$

The maximal ideal is

$$\mathfrak{p} = \{x \in K : \|x\|_p < 1\} = \{x \in K : v_p(x) > 0\},$$

and the invertible elements in \mathcal{O} is

$$\mathcal{O}^\times = \{x \in K : \|x\|_p = 1\} = \{x \in K : v_p(x) = 0\}.$$

An important consequence due to the uniqueness of norm extension [20, Corollary 5.3.2] is the following. Suppose we have field extensions $\mathbb{Q}_p \subset K \subset L$ and $|\cdot|_K, |\cdot|_L$ are the norms on K, L extending the norm $\|\cdot\|_p$ on \mathbb{Q}_p . If $x \in K$ then we must have $|x|_K = |x|_L$. In other words, the extended p -adic norm of x does not depend on the field where x lives in. The following lemma will be useful when calculating perfect isometry groups (to be defined later).

Lemma 2.1.1. *Let $\mathbb{Q}_p \subset K \subset L$ be field extensions of \mathbb{Q}_p and let $\mathcal{O}_K, \mathcal{O}_L$ be the valuation rings of K and L with respect to the extended norm $\|\cdot\|_p$. Then*

$$\mathcal{O}_L \cap K = \mathcal{O}_K.$$

Proof. Let $x \in \mathcal{O}_K$. Then $x \in K \subset L$ and $\|x\|_p \leq 1$. Since $\|x\|_p$ does not depend on the context, we still have $\|x\|_p \leq 1$ in L and therefore $x \in \mathcal{O}_L$. On the other hand, if $x \in \mathcal{O}_L \cap K$ then $x \in K$ and $\|x\|_p \leq 1$ in L (and so in K). Hence $x \in \mathcal{O}_K$. \square

Assumption 2.1.2. Throughout this thesis we will assume the following. When working with a finite group G , we will work over a p -modular system (K, \mathcal{O}, k) where we assume that p divides $|G|$, the order of the group G . We will also assume that the p -modular system is sufficiently large for G . If we are working with two finite groups G and H at the same time then we will also assume that (K, \mathcal{O}, k) is sufficiently large for both groups.

Remark 2.1.3. The symbols $K, \mathcal{O}, \mathfrak{p}, p$ will be reserved for their respective meanings as defined above throughout the thesis. The symbol k may clash with some well-known notations (for example $k(B)$). In some proofs, we might use k for a dummy variable (for example, as summation indexes). Since in this thesis we rarely work over k , this should be clear from the context and no confusion should arise.

2.2 Notations and basic definitions

Let G be a finite group and p a prime number. Denote by

- $|G|$ the order of the group,
- $\text{cl}(G)$ the set of conjugacy classes of G ,
- $\text{ccl}_G(x)$ the conjugacy class of an element $x \in G$,
- $C_G(x)$ the centralizer in G of an element $x \in G$,
- $Z(G)$ the center of G ,
- $\text{Aut}(G)$ the automorphism group of G .

An element $g \in G$ is called a *p -singular element* if p divides the order of g . If p does not divide the order of g , it is called a *p -regular element*. If the order of g is a power of p , then g is called a *p -element*. Denote by $G_{p'}$ the set of p -regular elements of G . If $g \in G$, we can write $g = g_p g_{p'}$ where g_p is a p -singular element and $g_{p'}$ is a p -regular element.

Let n be a positive integer. Write $n = p^a m$ where $p \nmid m$. Then p^a is called the p -part of n , denoted by $n_p = p^a$, and m is the p' -part of n , denoted by $n_{p'} = m$.

All modules are assumed to be left modules by default.

An $\mathcal{O}G$ -lattice is an $\mathcal{O}G$ -module having a finite \mathcal{O} -basis. Let V be a finitely generated KG -module. A *full $\mathcal{O}G$ -lattice* M in V is an $\mathcal{O}G$ -lattice M contained in V , such that $V = KM \cong K \otimes_{\mathcal{O}} M$. It follows from [12, (16.15)] that every finitely generated KG -module contains full $\mathcal{O}G$ -lattices.

Let M be an $\mathcal{O}G$ -lattice. The K -character afforded by M is the character of the KG -module $K \otimes_{\mathcal{O}} M$. Denote by $\text{Irr}(G)$ the set of irreducible K -characters of G . The trivial character will be denoted by $\mathbf{1}$. A *generalized character* of G is a \mathbb{Z} -linear combination of irreducible characters in $\text{Irr}(G)$. The set of generalized characters of G is denoted by $\text{ch}(G)$. Denote by $\text{CF}(G; K)$ the space of K -valued class functions on G . Then $\text{Irr}(G)$ is a K -basis of $\text{CF}(G; K)$. Denote by $\text{CF}_{p'}(G; K)$ the subspace of $\text{CF}(G; K)$ consisting of class functions vanishing on p -singular elements. The set of irreducible Brauer characters of G is denoted by $\text{IBr}(G)$.

2.3 Block theory

Since in most of the thesis we will work on blocks of a finite group G . We will give some background materials on blocks theory in this section. The main references for this section are [34], [12], [13].

2.3.1 Definition of blocks

Let G be a finite group. We can decompose the group algebra $\mathcal{O}G$ as

$$\mathcal{O}G = B_1 \oplus \dots \oplus B_n$$

where B_i are primitive two-sided ideals. This corresponds to a decomposition of $1 \in Z(\mathcal{O}G)$ into primitive idempotents of $Z(\mathcal{O}G)$: $1 = e_1 + \dots + e_n$ where $e_i \in B_i$ and $\mathcal{O}Ge_i = B_i$. Each B_i is called a *block* of $\mathcal{O}G$ and e_i a *block idempotent*.

Similarly, we can define blocks for KG and kG . By our settings of the p -modular system, there is a one-to-one correspondence between blocks of $\mathcal{O}G$ and blocks of kG . For a block B of $\mathcal{O}G$ we set $KB := K \otimes_{\mathcal{O}} B$ and $kB := k \otimes_{\mathcal{O}} B$. In this thesis, a block will always mean a block over \mathcal{O} . The set of all blocks of $\mathcal{O}G$ is denoted by $\text{Bl}(G)$.

If M is an indecomposable $\mathcal{O}G$ -module, then $M = B_i M$ for some unique block B_i , and we say that M belongs to B_i . If V is an irreducible KG -module affording a character $\chi \in \text{Irr}(G)$, then there is an indecomposable full $\mathcal{O}G$ -lattice M in V such that $V = KM$. We say that χ belongs to a block to which M belongs. This is independent of the choice of M .

Let S be a simple kG -module affording a Brauer character φ . If S belongs to a block corresponding to a block B of $\mathcal{O}G$, then we also say that φ belongs to B .

For any block $B \in \text{Bl}(G)$ we denote by $\text{Irr}(B)$ the set of irreducible characters $\chi \in \text{Irr}(G)$ belonging to B , and by $\text{IBr}(B)$ the set of irreducible Brauer characters $\varphi \in \text{IBr}(G)$ belonging to B . The unique block containing $\mathbf{1}$ is called the *principal block*, and will be denoted by B_0 . As in standard notations, we use $k(B)$ for $|\text{Irr}(B)|$ and $l(B)$ for $|\text{IBr}(B)|$.

There is an equivalent way of deciding when a character $\chi \in \text{Irr}(G)$ belongs to which block: we say that χ belongs to B_i if $\chi(e_i) = \chi(1)$ and $\chi(e_j) = 0$ for all $j \neq i$.

Another way to determine whether two irreducible characters are in the same block is via central characters. If $\chi \in \text{Irr}(G)$, then χ uniquely determines an algebra homomorphism $\omega_\chi : Z(KG) \rightarrow K$ defined by

$$\omega_\chi(\hat{\mathcal{C}}) = \frac{|\mathcal{C}|\chi(x_{\mathcal{C}})}{\chi(1)},$$

where $\mathcal{C} \in \text{cl}(G)$, $x_{\mathcal{C}} \in \mathcal{C}$ and $\hat{\mathcal{C}} = \sum_{x \in \mathcal{C}} x$.

It is well known that the value $\omega_\chi(\hat{\mathcal{C}})$ is in \mathcal{O} , so we can define a homomorphism $\lambda_\chi : Z(kG) \rightarrow k$ by setting

$$\lambda_\chi(\hat{\mathcal{C}}) = \omega_\chi(\hat{\mathcal{C}})^*.$$

Definition 2.3.1. [34, Definition 3.1] Let $B \in \text{Bl}(G)$. Then $\text{Irr}(B)$ is an equivalence

class in $\text{Irr}(G)$ under the relation $\chi \sim \varphi$ if $\lambda_\chi = \lambda_\varphi$ for $\chi, \varphi \in \text{Irr}(G)$. That is, if

$$\left(\frac{|\mathcal{C}|\chi(x_{\mathcal{C}})}{\chi(1)} \right)^* = \left(\frac{|\mathcal{C}|\varphi(x_{\mathcal{C}})}{\varphi(1)} \right)^*$$

for all $\mathcal{C} \in \text{cl}(G)$. We write $\lambda_B = \lambda_\chi$ for some $\chi \in \text{Irr}(B)$.

Recall that the primitive idempotents of $Z(KG)$ are $\{e_\chi \mid \chi \in \text{Irr}(G)\}$ where

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g.$$

If $B \in \text{Bl}(G)$ has a block idempotent $e_B \in Z(\mathcal{O}G)$, then

$$e_B = \sum_{\chi \in \text{Irr}(B)} e_\chi.$$

If $\chi, \varphi \in \text{Irr}(G)$, then we have $\omega_\chi(e_\varphi) = \delta_{\chi\varphi}$ and so $\omega_\chi(e_B) = 1$ if and only if $\chi \in \text{Irr}(B)$, and $\omega_\chi(e_B) = 0$ if $\chi \notin \text{Irr}(B)$.

Let $\theta = \sum_{\chi \in \text{Irr}(G)} \langle \theta, \chi \rangle \chi$ be a class function where $\langle \cdot, \cdot \rangle$ is the usual inner product of class functions. For $B \in \text{Bl}(G)$, we let

$$\theta_B = \sum_{\chi \in \text{Irr}(B)} \langle \theta, \chi \rangle \chi.$$

We say that θ_B is the B -part of θ .

Lemma 2.3.2. [34, Lemma 3.31] *Let $\theta \in \text{CF}(G; K)$ be a class function on G . Let $B \in \text{Bl}(G)$ with a block idempotent e . Then for all $x \in KG$,*

$$\theta(xe) = \theta_B(x).$$

For a block idempotent $e \in Z(\mathcal{O}G)$, let

$$\text{Irr}(G, e) = \{\chi \in \text{Irr}(G) : \chi(g) = \chi(ge) \forall g \in G\}.$$

Let $\chi \in \text{Irr}(G)$, then by the previous lemma $\chi \in \text{Irr}(G, e)$ if and only if $\chi(g) = \chi_B(g) \forall g \in G$, that is, $\text{Irr}(G, e) = \text{Irr}(B)$.

2.3.2 Defect groups

Let G be a finite group. Let $B \in \text{Bl}(G)$ with block idempotent $e_B \in Z(\mathcal{O}G)$. Write

$$(e_B)^* = \sum_{\mathcal{C} \in \text{cl}(G)} a_B(\hat{\mathcal{C}})\hat{\mathcal{C}}.$$

Then

$$1 = \lambda_B((e_B)^*) = \sum_{\mathcal{C} \in \text{cl}(G)} a_B(\hat{\mathcal{C}})\lambda_B(\hat{\mathcal{C}}).$$

This means that there is at least one conjugacy class $\mathcal{C} \in \text{cl}(G)$ such that both $a_B(\hat{\mathcal{C}})$ and $\lambda_B(\hat{\mathcal{C}})$ are not 0. We call such a class a *defect class* for B . A *defect group* of B is then defined to be a Sylow p -subgroup of $C_G(x)$ where x is in a defect class of B . This is well-defined and all defect groups of B are G -conjugate.

If D is a defect group of B , define the *defect* of B to be the non-negative integer $d(B)$ such that $|D| = p^{d(B)}$. In fact, $d(B)$ can also be determined by the irreducible characters contained in B as follows.

For $\chi \in \text{Irr}(G)$, define the *defect* of χ to be the integer $d(\chi)$ such that

$$p^{d(\chi)}\chi(1)_p = |G|_p.$$

Then $d(B)$ can be defined as

$$d(B) = \max\{d(\chi) : \chi \in \text{Irr}(B)\}.$$

The *height* of $\chi \in \text{Irr}(B)$ is defined to be the integer $h(\chi)$ such that

$$h(\chi) = d(B) - d(\chi).$$

An irreducible character $\chi \in \text{Irr}(G)$ is said to have *p -defect zero* if $\chi(1)_p = |G|_p$.

A defect group of the principal block of G is a Sylow p -subgroup of G . So the principal block has the maximal defect whereas a block of defect zero has trivial defect group and contains only one irreducible character (cf. [34, Theorem 3.18]).

We denote by $\text{Bl}(G|D)$ for the set of blocks of $\mathcal{O}G$ with defect group D and denote the set of defect groups of a block B by $\delta(B)$.

2.3.3 First main theorem

Suppose that D is a p -subgroup of G and let $C_G(D) \leq H \leq N_G(D)$. We can define a homomorphism $\text{Br}_D : Z(kG) \longrightarrow Z(kH)$ by

$$\text{Br}_D(\hat{\mathcal{C}}) = \sum_{x \in \mathcal{C} \cap C_G(D)} x,$$

for $\mathcal{C} \in \text{cl}(G)$ and extend linearly. If $\mathcal{C} \cap C_G(D) = \emptyset$ then we set $\text{Br}_D(\hat{\mathcal{C}}) = 0$.

If H is a subgroup of G and $A \in \text{Bl}(H)$, we can extend $\lambda_A : Z(kH) \longrightarrow k$ to a linear map $\lambda_A^G : Z(kG) \longrightarrow k$ by

$$\lambda_A^G(\hat{\mathcal{C}}) = \lambda_A \left(\sum_{x \in \mathcal{C} \cap H} x \right),$$

for $\mathcal{C} \in \text{cl}(G)$ and extend linearly. If $\mathcal{C} \cap H = \emptyset$ then we set $\lambda_A^G(\hat{\mathcal{C}}) = 0$. If λ_A^G is an algebra homomorphism then there is a unique block $A^G \in \text{Bl}(G)$ such that

$$\lambda_A^G = \lambda_{A^G}.$$

In this case we say that A^G is defined and call A^G the *induced block* of A .

Theorem 2.3.3. *Suppose that D is a p -subgroup of G and let H be a subgroup of G satisfying $DC_G(D) \leq H \leq N_G(D)$. If $A \in \text{Bl}(H)$, then A^G is defined. Moreover, if $B \in \text{Bl}(G)$, then $B = A^G$ for some block $A \in \text{Bl}(H)$ if and only if D is contained in some defect group of B .*

Proof. See [34, Theorem 4.14]. □

Theorem 2.3.4 (Brauer's First Main Theorem). *The map $A \mapsto A^G$ defines a bijection from $\text{Bl}(N_G(D)|D)$ to $\text{Bl}(G|D)$. Furthermore, if e_A, e_{A^G} are block idempotents of A, A^G respectively, then $\text{Br}_D(e_{A^G}) = e_A$.*

Proof. See [34, Theorem 4.17]. □

2.3.4 Blocks and normal subgroups

Let G be a finite group. Let $\sigma \in \text{Aut}(G)$ and let $H \leq G$ be a subgroup such that $H^\sigma = H$. Then σ acts on $\text{CF}(H; K)$ via $\varphi^\sigma(h) = \varphi(h^{\sigma^{-1}})$, where $\varphi \in \text{CF}(H; K)$ and

$h \in H$. The next lemma shows that the action by σ preserves the block structure of H .

Lemma 2.3.5. *Let $\sigma \in \text{Aut}(G)$ and $H \leq G$ be such that $H^\sigma = H$. If $\chi \in \text{Irr}(H)$ then $\chi^\sigma \in \text{Irr}(H)$. Furthermore, if $\chi, \varphi \in \text{Irr}(H)$ are in the same block then $\chi^\sigma, \varphi^\sigma$ are also in the same block.*

Proof. First, if $\chi \in \text{Irr}(H)$, then

$$\langle \chi^\sigma, \chi^\sigma \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h^{\sigma^{-1}}) \overline{\chi(h^{\sigma^{-1}})} = \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\chi(h)} = \langle \chi, \chi \rangle_H = 1.$$

So $\chi^\sigma \in \text{Irr}(H)$. Suppose $\chi, \varphi \in \text{Irr}(H)$ are in the same block. Then we have

$$|\mathcal{C}| \left(\frac{\chi(h_{\mathcal{C}})}{\chi(1)} - \frac{\varphi(h_{\mathcal{C}})}{\varphi(1)} \right) \equiv 0 \pmod{\mathfrak{p}},$$

for all $\mathcal{C} \in \text{cl}(H)$, where $h_{\mathcal{C}} \in \mathcal{C}$. If $\mathcal{C} \in \text{cl}(H)$, denote by $\mathcal{C}^\sigma \in \text{cl}(H)$ the class of $(h_{\mathcal{C}})^\sigma$ for some $h_{\mathcal{C}} \in \mathcal{C}$. Since $|\mathcal{C}^{\sigma^{-1}}| = |\mathcal{C}|$, we have

$$|\mathcal{C}| \left(\frac{\chi^\sigma(h_{\mathcal{C}})}{\chi^\sigma(1)} - \frac{\varphi^\sigma(h_{\mathcal{C}})}{\varphi^\sigma(1)} \right) = |\mathcal{C}^{\sigma^{-1}}| \left(\frac{\chi((h_{\mathcal{C}})^{\sigma^{-1}})}{\chi(1)} - \frac{\varphi((h_{\mathcal{C}})^{\sigma^{-1}})}{\varphi(1)} \right) \equiv 0 \pmod{\mathfrak{p}},$$

for all $\mathcal{C} \in \text{cl}(G)$. Hence $\chi^\sigma, \varphi^\sigma$ are in the same block. \square

Let A be a block of $\mathcal{O}H$. We denote by A^σ another block of H with $\text{Irr}(A^\sigma) = \{\chi^\sigma : \chi \in \text{Irr}(A)\}$. If D is a defect group of A then D^σ is a defect group of A^σ ([34, Problem 4.4]).

Suppose now that $N \triangleleft G$. Then G acts on $\text{Irr}(N)$ by conjugation. We say that $A_1, A_2 \in \text{Bl}(N)$ are G -conjugate if $A_2 = A_1^g$ for some $g \in G$. If $A \in \text{Bl}(N)$ we denote the stabilizer of A in G by $I_G(A) = \{g \in G : A^g = A\}$.

Definition 2.3.6. We say that a block $B \in \text{Bl}(G)$ covers a block $A \in \text{Bl}(N)$ if any of the following equivalent conditions hold:

- (a) If $\chi \in \text{Irr}(B)$, then every irreducible constituent of χ_N lies in a G -conjugate of A .
- (b) There is a $\chi \in \text{Irr}(B)$ such that χ_N has an irreducible constituent in A .

A proof of the above equivalence can be found in [34, Theorem 9.2].

Theorem 2.3.7 (Extended First Main Theorem). *If $B \in \text{Bl}(G)$ has defect group D , then there exists a unique $N_G(D)$ -orbit of blocks $A_D \in \text{Bl}(DC_G(D))$ such that $(A_D)^G = B$. All these blocks have defect group D and $((A_D)^{N_G(D)})^G = B$.*

Proof. See [34, Theorem 9.7]. □

The block $A_D \in \text{Bl}(DC_G(D))$ from the last theorem is called a *root* of B .

Theorem 2.3.8 (Brauer's Third Main Theorem). *Let $H \leq G$ and let A be a block of $\mathcal{O}H$. Suppose that A^G is defined. Then, A is the principal block of $\mathcal{O}H$ if and only if A^G is the principal block of $\mathcal{O}G$.*

Proof. See [34, Theorem 6.7]. □

Definition 2.3.9. Let $B \in \text{Bl}(G)$ be a block with defect group D and $A_D \in \text{Bl}(DC_G(D))$ be a root of B . We define the *inertial quotient* of B to be the quotient group $E = I_{N_G(D)}(A_D)/DC_G(D)$. The order of E is called the *inertial index* of B .

Since all the roots of B are conjugate in $N_G(D)$ by the Extended First Main Theorem, the inertial index is well defined. We will use inertial quotient and inertial index extensively when we study blocks with cyclic defect groups.

Theorem 2.3.10. *Let $B \in \text{Bl}(G)$. Then the inertial index of B is not divisible by p .*

Proof. See [13, (61.15)]. □

2.4 Grothendieck groups

In what follows, all modules will be finitely generated left modules, unless explicitly stated otherwise.

Definition 2.4.1. Let A be a ring. The Grothendieck group $R(A)$ is the abelian group generated by the symbols $[M]$ corresponding to isomorphism classes (M) of

A -modules M with relations

$$[M] = [M'] + [M'']$$

for each short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of A -modules.

Let G be a finite group. The structures of $R(KG)$ and $R(kG)$ can be seen by the following proposition.

Proposition 2.4.2. *Let $A = KG$ or kG . Let $\{S_1, \dots, S_n\}$ be the complete set of simple A -modules. Then*

$$R(A) = \bigoplus_{i=1}^n \mathbb{Z}[S_i].$$

Proof. See [12, (16.6)]. □

If $B \in \text{Bl}(G)$ we write $R_K(B)$ for $R(KB)$, where $KB = K \otimes_{\mathcal{O}} B$.

Now let M, N be two KG -modules. We can define another KG -module by $M \otimes_K N$. It turns out that we can also define multiplication in $R(KG)$ by $[M][N] := [M \otimes_K N]$. If M and N afford characters χ and φ respectively, then $M \otimes_K N$ affords the character $\chi\varphi$ defined by $(\chi\varphi)(g) = \chi(g)\varphi(g)$ for $g \in G$. The next proposition shows that we can identify $R(KG)$ with the ring of generalized characters $\text{ch}(G)$.

Proposition 2.4.3. *Let G be a finite group. There is a ring isomorphism $R(KG) \cong \text{ch}(KG)$ defined by $[M] \mapsto \mu$ where μ is the character of G afforded by M .*

Proof. See [12, (16.10)]. □

Suppose that V is a KG -module. There is an $\mathcal{O}G$ -lattice M such that $V = KM \cong K \otimes_{\mathcal{O}} M$. We then have a kG -module $k \otimes_{\mathcal{O}} M \cong M/\mathfrak{p}M$.

Proposition 2.4.4. *The map sending $[V] \in R(KG)$ to $[k \otimes_{\mathcal{O}} M] \in R(kG)$ where $V = KM$ gives a homomorphism of abelian groups $d : R(KG) \rightarrow R(kG)$.*

Proof. See [12, (16.17)]. □

The above homomorphism is called the *decomposition map* associated with the p -modular system (K, \mathcal{O}, k) .

2.5 Equivalences of blocks

Let G, H be two finite groups. Let $B \in \text{Bl}(G)$ and $A \in \text{Bl}(H)$. In this section we will discuss briefly some types of equivalences of blocks that can be related to perfect isometries later on. Good references for this section can be found, for example, in [9], [23], [28], [5].

To each of these equivalences is associated an invariant group which can be obtained when we let $A = B$. In some cases where these groups can be determined, we shall compare them to our invariant group of perfect isometries, which will be defined and studied in more details later.

Unless stated otherwise, all modules in this section are assumed to be finitely generated left modules. Recall that an (A, B) -bimodule is an abelian group M which is simultaneously a left A -module and a right B -module such that $(am)b = a(mb)$ and $rm = mr$ for all $a \in A, b \in B, m \in M$ and $r \in \mathcal{O}$. Denote by A° the opposite algebra of A . An A° -module can be regarded as a right A -module and a $(B \otimes_{\mathcal{O}} A^\circ)$ -module can be regarded as a (B, A) -bimodule. Denote by $\mathbf{Mod}(B)$ the category of finitely generated B -modules. So $\mathbf{Mod}(B \otimes_{\mathcal{O}} A^\circ)$ is the category of finitely generated (B, A) -bimodules.

2.5.1 Morita equivalence and Picard group

We begin with the Morita equivalence. The following well-known theorem is extracted from [9].

Theorem 2.5.1 (Morita Theorem). *The following are equivalent*

1. *The categories $\mathbf{Mod}(A)$ and $\mathbf{Mod}(B)$ are equivalent.*
2. *There exist*
 - *a (B, A) -bimodule M , which is projective as a left B -module and as a right A -module,*
 - *a (A, B) -bimodule N , which is projective as a left A -module and as a right B -module,*

- an (B, A) -compatible \mathcal{O} -duality between M and N such that $M \otimes_A N \cong B$ in $\mathbf{Mod}(B \otimes_{\mathcal{O}} B^{\circ})$ and $N \otimes_B M \cong A$ in $\mathbf{Mod}(A \otimes_{\mathcal{O}} A^{\circ})$.

Definition 2.5.2. We say that A and B are *Morita equivalent* if $\mathbf{Mod}(A)$ and $\mathbf{Mod}(B)$ are equivalent.

If A and B are Morita equivalent then, with the notation of Theorem 2.5.1, the functors

$$M \otimes_A - : \mathbf{Mod}(A) \longrightarrow \mathbf{Mod}(B)$$

$$N \otimes_B - : \mathbf{Mod}(B) \longrightarrow \mathbf{Mod}(A)$$

provide inverse equivalences of module categories. Moreover, they send simple modules to simple modules ([23, Remark 1.5]).

By tensoring everything with K , Morita equivalence of A and B also implies Morita equivalence of $K \otimes_{\mathcal{O}} A$ and $K \otimes_{\mathcal{O}} B$. Similarly, tensoring with k gives Morita equivalence of $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$. We summarize some of the well-known invariants preserved by a Morita equivalence [9].

Proposition 2.5.3. *Suppose that A and B are Morita equivalent. Then*

1. $Z(A) \cong Z(B)$.
2. $k(A) = k(B)$.
3. $l(A) = l(B)$.
4. A and B have the same decomposition matrix.
5. A and B have the same Cartan matrix.
6. A and B have the same defect.

Definition 2.5.4. A (B, A) -bimodule M is called *invertible* if it gives a Morita equivalence between A and B .

Now suppose $A = B$ and consider all invertible (B, B) -bimodules.

Definition 2.5.5. Let $B \in \text{Bl}(G)$. Denote by (M) the bimodule isomorphism class of a (B, B) -bimodule M . The *Picard group* of B relative to \mathcal{O} , denoted by $\text{Pic}_{\mathcal{O}}(B)$, is the multiplicative group consisting of all classes (M) of invertible (B, B) -bimodules. Multiplication in $\text{Pic}_{\mathcal{O}}(B)$ is defined by

$$(M)(M') = (M \otimes_B M').$$

The identity element is (B) , and the inverse of (M) is $(M)^{-1} = (M^{-1})$.

In this thesis, we shall write $\text{Pic}(B)$ for $\text{Pic}_{\mathcal{O}}(B)$.

Lemma 2.5.6. *Let G, H be finite groups. Let $B \in \text{Bl}(G), A \in \text{Bl}(H)$. If B and A are Morita equivalent, then $\text{Pic}(B) \cong \text{Pic}(A)$.*

Proof. Let M be a (B, A) -bimodule and N be an (A, B) -bimodule such that M, N give a Morita equivalent between B and A . It is elementary to check that the map $P \mapsto N \otimes_B P \otimes_B M$ is a required isomorphism $\text{Pic}(B) \rightarrow \text{Pic}(A)$. \square

2.5.2 Derived equivalence and derived Picard group

Morita equivalence is a strong type of equivalences between blocks. It does not happen often, for example, in non-abelian simple groups. A somewhat weaker type of equivalence, and a generalization of a Morita equivalence, is called a derived equivalence.

Let $\mathcal{D}^b(B)$ be the derived category of bounded complexes of B -modules. A complex of B -modules is called *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective B -modules. The following theorem, due to Rickard [38] [39], is essential.

Theorem 2.5.7. *The following are equivalent*

1. *The categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.*
2. *There are a complex $T \in \mathcal{D}^b(B \otimes_{\mathcal{O}} A^{\circ})$ whose restriction to B and A° are perfect and a complex $S \in \mathcal{D}^b(A \otimes_{\mathcal{O}} B^{\circ})$ whose restriction to B and A° are perfect such*

that

$$T \otimes_A S \cong B \text{ in } \mathcal{D}^b(B \otimes_{\mathcal{O}} B^\circ) \text{ and } S \otimes_B T \cong A \text{ in } \mathcal{D}^b(A \otimes_{\mathcal{O}} A^\circ).$$

Definition 2.5.8. We say that A and B are *derived equivalent* if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

The complexes T and S as in Theorem 2.5.7 are called (*two-sided*) *tilting complexes*, inverse to each other. The functors

$$\begin{aligned} T \otimes_A^L - &: \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B) \\ S \otimes_B^L - &: \mathcal{D}^b(B) \longrightarrow \mathcal{D}^b(A) \end{aligned}$$

then give inverse equivalences between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$.

There is a full embedding of $\mathbf{Mod}(B)$ (respectively $\mathbf{Mod}(B \otimes_{\mathcal{O}} A^\circ)$) into $\mathcal{D}^b(B)$ (respectively $\mathcal{D}^b(B \otimes_{\mathcal{O}} A^\circ)$) by sending M to the complex with component M concentrated in degree 0. We will identify M with this complex. Using this, it is clear that if $M \in \mathbf{Mod}(B \otimes_{\mathcal{O}} A^\circ)$ is an invertible bimodule, then it is also a tilting complex when considered as a complex in $\mathcal{D}^b(B \otimes_{\mathcal{O}} A^\circ)$. So a Morita equivalence induces a derived equivalence.

Suppose A and B are derived equivalent. Similar to Morita equivalence, tensoring everything with K and k gives derived equivalences between $K \otimes_{\mathcal{O}} A$ and $K \otimes_{\mathcal{O}} B$ and between $k \otimes_{\mathcal{O}} A$ and $k \otimes_{\mathcal{O}} B$ respectively.

We will consider now the case $A = B$.

Definition 2.5.9. Let $B \in \text{Bl}(G)$. Denote by (T) the isomorphism class of $T \in \mathcal{D}^b(B \otimes_{\mathcal{O}} B^\circ)$. The *derived Picard group* of B relative to \mathcal{O} , denoted by $\text{DPic}_{\mathcal{O}}(B)$, is the multiplicative group consisting of all classes (T) of tilting complexes in $\mathcal{D}^b(B \otimes_{\mathcal{O}} B^\circ)$. Multiplication is defined by

$$(T)(S) = (T \otimes_B^L S).$$

The identity element is (B) . That this is a group follows from [45, Proposition 2.4].

In this thesis, we shall write $\mathrm{DPic}(B)$ for $\mathrm{DPic}_{\mathcal{O}}(B)$. Note that the usual embedding of $\mathbf{Mod}(B \otimes_{\mathcal{O}} B^{\circ})$ into $\mathcal{D}^b(B \otimes_{\mathcal{O}} B^{\circ})$ gives the canonical injection

$$\mathrm{Pic}(B) \longrightarrow \mathrm{DPic}(B).$$

Let $\mathrm{Sh}(B)$ be a subgroup of $\mathrm{DPic}(B)$ generated by $B[1]$. Then $\mathrm{Sh}(B)$ is an infinite cyclic group and lies in the center of $\mathrm{DPic}(B)$. The direct product $\mathrm{Pic}(B) \times \mathrm{Sh}(B)$ is a subgroup of $\mathrm{DPic}(B)$.

Lemma 2.5.10. *Let G, H be finite groups. Let $B \in \mathrm{Bl}(G), A \in \mathrm{Bl}(H)$. If B and A are derived equivalent, then $\mathrm{DPic}(B) \cong \mathrm{DPic}(A)$.*

Proof. Let $T \in \mathcal{D}^b(B \otimes_{\mathcal{O}} A^{\circ})$ and $S \in \mathcal{D}^b(A \otimes_{\mathcal{O}} B^{\circ})$ be tilting complexes giving inverse derived equivalences between $\mathcal{D}^b(B)$ and $\mathcal{D}^b(A)$. It is elementary to check that the map $X \mapsto S \otimes_B^L X \otimes_B^L T$ is a required isomorphism $\mathrm{DPic}(B) \longrightarrow \mathrm{DPic}(A)$. \square

Chapter 3

Perfect Isometries

Let G be a finite group. Let B be a block of $\mathcal{O}G$. We identify $R_K(B)$ with the abelian group generated by $\text{Irr}(B)$, the irreducible characters of simple KB -modules. Let H be a second finite group and A be a block of $\mathcal{O}H$. We will identify $R_K(B \otimes_{\mathcal{O}} A^\circ)$ with the abelian group on the irreducible characters of $B \otimes_{\mathcal{O}} A^\circ$ -modules.

3.1 Definitions

Let $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$. We can regard μ as a generalized character of $G \times H$ by $\mu(g, h) = \mu(g \otimes (h^{-1})^\circ)$.

Definition 3.1.1 ([8]). We say that a generalized character μ of $G \times H$ is *perfect* if the following two conditions hold:

(i) [Integrality] For all $(g, h) \in G \times H$ we have

$$\frac{\mu(g, h)}{|C_G(g)|} \in \mathcal{O} \quad \text{and} \quad \frac{\mu(g, h)}{|C_H(h)|} \in \mathcal{O}.$$

(ii) [Separation] For all $(g, h) \in G \times H$, if $\mu(g, h) \neq 0$ then, g is p -regular if and only if h is p -regular.

Perfect characters can arise from bimodules that are projective on both sides, as is shown by the following proposition.

Proposition 3.1.2. [8, Proposition 1.2] *Let G, H be two finite groups. Let M be an $\mathcal{O}(G, H)$ -bimodule which is projective as left $\mathcal{O}G$ -module and as right $\mathcal{O}H$ -module. Then the character afforded by $K \otimes_{\mathcal{O}} M$ is perfect.*

Following Broué [8], each $\mu \in R_K(B \otimes_{\mathcal{O}} A^{\circ})$ determines two linear maps

$$I_{\mu} : R_K(A) \longrightarrow R_K(B) \quad , \quad R_{\mu} : R_K(B) \longrightarrow R_K(A)$$

defined by

$$I_{\mu}(\beta)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1})\beta(h) = \langle \mu(g, \cdot), \beta \rangle_H \quad (3.1)$$

and

$$R_{\mu}(\alpha)(h) = \frac{1}{|G|} \sum_{g \in G} \mu(g^{-1}, h)\alpha(g) = \langle \mu(\cdot, h), \alpha \rangle_G \quad (3.2)$$

for $\alpha \in R_K(B)$ and $\beta \in R_K(A)$. Here, $\langle \cdot, \cdot \rangle$ is the standard inner product of class functions.

Lemma 3.1.3. *Given $\mu \in R_K(B \otimes_{\mathcal{O}} A^{\circ})$. The linear maps I_{μ}, R_{μ} defined by (3.1) and (3.2) are adjoint to each other with respect to $\langle \cdot, \cdot \rangle$. That is,*

$$\langle I_{\mu}(\beta), \alpha \rangle_G = \langle \beta, R_{\mu}(\alpha) \rangle_H$$

for all $\alpha \in R_K(B), \beta \in R_K(A)$.

Proof. It is sufficient to prove for the irreducible characters. Let $\alpha \in \text{Irr}(B), \beta \in \text{Irr}(A)$. Then

$$\begin{aligned} \langle I_{\mu}(\beta), \alpha \rangle_G &= |G|^{-1} \sum_{g \in G} I_{\mu}(\beta)(g)\alpha(g^{-1}) \\ &= |G|^{-1} \sum_{g \in G} |H|^{-1} \sum_{h \in H} \mu(g, h^{-1})\beta(h)\alpha(g^{-1}) \\ &= |H|^{-1} \sum_{h \in H} \left(|G|^{-1} \sum_{g \in G} \mu(g^{-1}, h^{-1})\alpha(g) \right) \beta(h) \\ &= |H|^{-1} \sum_{h \in H} R_{\mu}(\alpha)(h^{-1})\beta(h) \\ &= \langle \beta, R_{\mu}(\alpha) \rangle_H. \end{aligned}$$

□

If X is a KB -module and Y is a KA -module. Then they determine a $K(B \otimes_{\mathcal{O}} A^{\circ})$ -module $X \otimes_{\mathcal{O}} Y^{\vee}$ where $Y^{\vee} := \text{Hom}_K(Y, K)$ is a right KA -module. If χ is the image of X in $R_K(B)$ and φ is the image of Y in $R_K(A)$, then we denote by $\chi \cdot \varphi^{\vee}$ the image of $X \otimes_{\mathcal{O}} Y^{\vee}$ in $R_K(B \otimes_{\mathcal{O}} A^{\circ})$. Now let $I : R_K(A) \rightarrow R_K(B)$ be a linear map, we can regard I as an element in $R_K(B \otimes_{\mathcal{O}} A^{\circ})$, denoted μ_I , by $\mu_I = \sum_{\varphi \in \text{Irr}(A)} I(\varphi) \cdot \varphi^{\vee}$. The following lemma shows that the map induced by μ_I is I .

Lemma 3.1.4. *Let $I : R_K(A) \rightarrow R_K(B)$ be a linear map. Define*

$$\mu = \sum_{\varphi \in \text{Irr}(A)} I(\varphi) \cdot \varphi^{\vee}.$$

Then $\mu \in R_K(B \otimes_{\mathcal{O}} A^{\circ})$ and $I = I_{\mu}$. When viewed as a generalized character of $G \times H$ we have

$$\mu(g, h) = \sum_{\varphi \in \text{Irr}(A)} I(\varphi)(g)\varphi(h). \quad (3.3)$$

for $g \in G$ and $h \in H$. Furthermore, μ is unique.

Proof. It is clear by definition of I that $\mu \in R_K(B \otimes_{\mathcal{O}} A^{\circ})$. Since $(\chi \cdot \varphi^{\vee})(g \otimes h^{\circ}) = \chi(g)\varphi^{\vee}(h) = \chi(g)\varphi(h^{-1})$, we have

$$\mu(g, h) = \mu(g \otimes (h^{-1})^{\circ}) = \sum_{\varphi \in \text{Irr}(A)} I(\varphi)(g)\varphi^{\vee}(h^{-1}) = \sum_{\varphi \in \text{Irr}(A)} I(\varphi)(g)\varphi(h)$$

for $g \in G$ and $h \in H$. Let $\beta \in R_K(A)$ and $g \in G$, then

$$\begin{aligned} I_{\mu}(\beta)(g) &= \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1})\beta(h) \\ &= \frac{1}{|H|} \sum_{h \in H} \left(\sum_{\varphi \in \text{Irr}(A)} I(\varphi)(g)\varphi(h^{-1}) \right) \beta(h) \\ &= \sum_{\varphi \in \text{Irr}(A)} \frac{1}{|H|} \left(\sum_{h \in H} \varphi(h^{-1})\beta(h) \right) I(\varphi)(g) \\ &= \sum_{\varphi \in \text{Irr}(A)} \langle \beta, \varphi \rangle I(\varphi)(g) \\ &= I \left(\sum_{\varphi \in \text{Irr}(A)} \langle \beta, \varphi \rangle \varphi \right) (g) \\ &= I(\beta)(g). \end{aligned}$$

To show uniqueness of μ , suppose there is $\tau \in R_K(B \otimes_{\mathcal{O}} A^\circ)$ such that $I = I_\tau$. Let

$$\tau = \sum_{\chi \in \text{Irr}(B)} \sum_{\varphi \in \text{Irr}(A)} (C_{\chi\varphi})\chi \cdot \varphi^\vee, \quad C_{\chi\varphi} \in \mathbb{Z}.$$

The same arguments as above applied to $\beta \in \text{Irr}(A)$ shows that

$$\sum_{\chi \in \text{Irr}(B)} (C_{\chi\beta})\chi = I(\beta).$$

That is,

$$\tau = \sum_{\varphi \in \text{Irr}(A)} \left(\sum_{\chi \in \text{Irr}(B)} (C_{\chi\varphi})\chi \right) \cdot \varphi^\vee = \sum_{\varphi \in \text{Irr}(A)} I(\varphi) \cdot \varphi^\vee = \mu.$$

□

The following result of Masao Kiyota [47] is useful and convenient when we want to check an integrality condition of a generalized character μ of $G \times H$ once we know that μ satisfies the separation condition. The author would like to thank Dr. Charles Eaton for showing him this paper.

Theorem 3.1.5 (Kiyota). *Let G, H be finite groups. Let $B \in \text{Bl}(G)$ and $A \in \text{Bl}(H)$. Let $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$ be a generalized character of $G \times H$ satisfying the separation condition. Then, in order to check the integrality condition, it is sufficient to check only for p -singular elements $g \in G$ and $h \in H$.*

Let $\text{CF}(G, B; K)$ be the subspace of $\text{CF}(G; K)$ of class functions generated by $\text{Irr}(B)$. Let $\text{CF}(G, B; \mathcal{O})$ be the subspace of $\text{CF}(G, B; K)$ of \mathcal{O} -valued class functions. Let $\text{CF}_{p'}(G, B; K)$ be the subspace of class functions $\alpha \in \text{CF}(G, B; K)$ vanishing outside p -regular elements.

If $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$, the linear maps I_μ, R_μ defined in (3.1),(3.2) can be extended in the usual way to the linear maps

$$I_\mu : \text{CF}(H, A; K) \longrightarrow \text{CF}(G, B; K) \quad , \quad R_\mu : \text{CF}(G, A; K) \longrightarrow \text{CF}(H, B; K).$$

Proposition 3.1.6. [8, Proposition 4.1] *A character $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$ is perfect if and only if*

(i*) I_μ gives a map from $\text{CF}(H, A; \mathcal{O})$ to $\text{CF}(G, B; \mathcal{O})$ and R_μ gives a map from $\text{CF}(G, B; \mathcal{O})$ to $\text{CF}(H, A; \mathcal{O})$.

(ii*) I_μ gives a map from $\text{CF}_{p'}(H, A; K)$ to $\text{CF}_{p'}(G, B; K)$ and R_μ gives a map from $\text{CF}_{p'}(G, B; K)$ to $\text{CF}_{p'}(H, A; K)$.

Corollary 3.1.7. *Let $I_\mu : R_K(A) \longrightarrow R_K(B)$ be a perfect isometry. Let $\alpha \in \text{CF}(H, A; K)$ be a class function vanishing on p -regular elements. Then $I_\mu(\alpha)$ also vanishes on p -regular elements.*

Proof. Let $g \in G$ be a p -regular element. Consider

$$I_\mu(\alpha)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1}) \alpha(h).$$

If h is p -regular then $\alpha(h) = 0$ while if h is p -singular then $\mu(g, h^{-1}) = 0$ as μ is perfect. So $I_\mu(\alpha)(g) = 0$. \square

We already know that a character $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$ defines I_μ and R_μ which are adjoint to each other. If, in addition, I_μ is an isometry, then we can say more about R_μ .

Lemma 3.1.8. *Let $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$. Suppose that I_μ is an isometry. Then R_μ is also an isometry and $(I_\mu)^{-1} = R_\mu$.*

Proof. Let $\beta \in \text{Irr}(A)$. Since $I_\mu(\beta)$ is a \mathbb{Z} -linear combination of characters in $\text{Irr}(B)$ (by identification of $R_K(B)$), we have $I_\mu(\beta) \in \pm \text{Irr}(B)$. Let $\alpha \in \text{Irr}(B)$. Since I_μ and R_μ are adjoint,

$$\langle I_\mu(\beta), \alpha \rangle_G = \langle \beta, R_\mu(\alpha) \rangle_H.$$

As the left hand side takes values in $\{0, \pm 1\}$, this implies that $R_\mu(\alpha) \in \pm \text{Irr}(A)$. By adjointness again,

$$\langle I_\mu(R_\mu(\alpha)), \alpha \rangle_G = \langle R_\mu(\alpha), R_\mu(\alpha) \rangle_H = 1.$$

Since $I_\mu(R_\mu(\alpha)) \in \pm \text{Irr}(B)$, this forces $I_\mu(R_\mu(\alpha)) = \alpha$. Similarly,

$$1 = \langle I_\mu(\beta), I_\mu(\beta) \rangle_G = \langle \beta, R_\mu(I_\mu(\beta)) \rangle_H$$

implies that $R_\mu(I_\mu(\beta)) = \beta$. Hence $R_\mu = (I_\mu)^{-1}$ and R_μ is an isometry. \square

Definition 3.1.9. We say that $I : R_K(A) \rightarrow R_K(B)$ is a *perfect isometry* if I is an isometry and $I = I_\mu$ where $\mu \in R_K(B \otimes_{\mathcal{O}} A^\circ)$ is perfect.

Definition 3.1.10. Two blocks A, B are said to be *perfectly isometric* if there is a perfect isometry $I : R_K(A) \rightarrow R_K(B)$.

If $I : R_K(A) \rightarrow R_K(B)$ is a (perfect) isometry, then $I(\chi) \in \pm \text{Irr}(B)$ for all $\chi \in \text{Irr}(A)$. So I defines a bijection $I^+ : \text{Irr}(A) \rightarrow \text{Irr}(B)$ and a sign function $\varepsilon_I : \text{Irr}(A) \rightarrow \{\pm 1\}$ such that $I(\chi) = \varepsilon_I(\chi)I^+(\chi)$ for $\chi \in \text{Irr}(A)$. This gives a bijection with signs between $\text{Irr}(A)$ and $\text{Irr}(B)$.

We denote the sign function ε_I such that $\varepsilon_I(\chi) = 1, \forall \chi \in \text{Irr}(A)$ by 1 and the sign function ε_I such that $\varepsilon_I(\chi) = -1, \forall \chi \in \text{Irr}(A)$ by -1 .

3.2 Properties

Let B, A be blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. It is easy to see that if $I : R_K(A) \rightarrow R_K(B)$ is a perfect isometry then so is $-I$. It turns out that a perfect isometry has essentially a unique sign. This means that, if J is another perfect isometry with $J^+ = I^+$, then $\varepsilon_J = \varepsilon_I$ or $\varepsilon_J = -\varepsilon_I$, in other words, $J = \pm I$. We need the following theorem, due to Osima, which is a converse of block orthogonality.

Theorem 3.2.1. [35, Theorem 3] *Let S be a set of ordinary characters of G such that*

$$\sum_{\chi \in S} \chi(g)\chi(h) = 0$$

for any p -regular element g and any p -singular element h . Then S is a union of $\text{Irr}(B)$ for some blocks $B \in \text{Bl}(G)$. In other words, if $\chi \in S$ and $\chi \in \text{Irr}(B)$ for some block B , then $\text{Irr}(B) \subseteq S$.

Proposition 3.2.2. *Let $I : R_K(A) \rightarrow R_K(B)$ be a perfect isometry such that I^+ is the identity map. Then $\varepsilon_I = 1$ or -1 .*

Proof. Suppose $I = I_\mu$. Then, for $g, h \in G$, $\mu(g, h) = \sum_{\chi \in \text{Irr}(B)} \varepsilon_I(\chi)\chi(g)\chi(h)$. Set

$$B^+ = \{\chi \in \text{Irr}(B) : \varepsilon_I(\chi) = 1\}$$

and

$$B^- = \{\chi \in \text{Irr}(B) : \varepsilon_I(\chi) = -1\}.$$

Then $\mu(g, h) = \sum_{\chi \in B^+} \chi(g)\chi(h) - \sum_{\chi \in B^-} \chi(g)\chi(h)$. Hence

$$\sum_{\chi \in B^+} \chi(g)\chi(h) - \sum_{\chi \in B^-} \chi(g)\chi(h) = 0 \quad \forall g \in G_{p'}, \forall h \in G - G_{p'}.$$

Since $id : R_K(B) \rightarrow R_K(B)$ is also a perfect isometry,

$$\sum_{\chi \in B^+} \chi(g)\chi(h) + \sum_{\chi \in B^-} \chi(g)\chi(h) = 0 \quad \forall g \in G_{p'}, \forall h \in G - G_{p'}.$$

Hence, $2 \sum_{\chi \in B^+} \chi(g)\chi(h) = 0$ and so $\sum_{\chi \in B^+} \chi(g)\chi(h) = 0$ for any p -regular element g and p -singular element h . By Theorem 3.2.1, $B^+ = B$ or ϕ . If $B^+ = B$, then $\varepsilon_I = 1$. If $B^+ = \phi$, then $\varepsilon_I = -1$. \square

Lemma 3.2.3 (Uniqueness of signs). *Suppose $I, J : R_K(A) \rightarrow R_K(B)$ are perfect isometries such that $I^+ = J^+$. Then $J = I$ or $J = -I$.*

Proof. Let $\delta = \varepsilon_I$ and $\tau = \varepsilon_J$. Then $I \circ J^{-1} : R_K(B) \rightarrow R_K(B)$ is a perfect isometry with $(I \circ J^{-1})^+ = id$ and $\varepsilon = \varepsilon_{I \circ J^{-1}}$ is given by $\varepsilon(\chi) = \delta((J^+)^{-1}(\chi)) \cdot \tau((J^+)^{-1}(\chi))$. By the previous proposition, $\varepsilon = 1$ or -1 and hence $\tau = \delta$ or $\tau = -\delta$. \square

Suppose μ induces a perfect isometry $I_\mu : R_K(A) \rightarrow R_K(B)$. Then I_μ defines a bijection between primitive idempotents of $Z(KA)$ and $Z(KB)$ by $\chi \mapsto e_{I(\chi)}$. Note that by definition we have $e_{-\chi} = e_\chi$. We next follow the result of Broué [8, Theorem 1.5] and define two linear maps

$$I_\mu^\circ : Z(KA) \rightarrow Z(KB) \quad , \quad R_\mu^\circ : Z(KB) \rightarrow Z(KA)$$

by

$$I_\mu^\circ(y) = \sum_{g \in G} \left(\frac{1}{|H|} \sum_{h \in H} \mu(g^{-1}, h)y(h) \right) g \quad (3.4)$$

where $y = \sum_{h \in H} y(h)h \in Z(KA)$, and

$$R_\mu^\circ(x) = \sum_{h \in H} \left(\frac{1}{|G|} \sum_{g \in G} \mu(g, h^{-1})x(g) \right) h \quad (3.5)$$

where $x = \sum_{g \in G} x(g)g \in Z(KB)$.

Since μ is perfect, the maps I_μ° and R_μ° also defined \mathcal{O} -linear maps on $Z(KA)$ and $Z(KB)$ respectively. Let $\chi \in \text{Irr}(A)$. Then

$$I_\mu^\circ(e_\chi) = \frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)} e_{I_\mu(\chi)}.$$

It is easy to check that $I_\mu^\circ(e_\chi R_\mu^\circ(e)) = e_{I_\mu(\chi)}$, where e is the block idempotent of B . So we have an algebra isomorphism $Z(KA) \longrightarrow Z(KB)$ by defining

$$y \mapsto I_\mu^\circ(y R_\mu^\circ(e)) \quad (3.6)$$

with the inverse $Z(KB) \longrightarrow Z(KA)$ defined by

$$x \mapsto R_\mu^\circ(I_\mu^\circ(f)x) \quad (3.7)$$

where f is the block idempotent of A . These also give algebra isomorphisms between $Z(A)$ and $Z(B)$.

Theorem 3.2.4. [8, Theorem 1.5] *Suppose μ induces a perfect isometry $I_\mu : R_K(A) \longrightarrow R_K(B)$. Then I_μ induces an algebra isomorphism from $Z(A)$ to $Z(B)$. Furthermore, I_μ preserves the defect of the blocks, the number of irreducible ordinary and Brauer characters and the Cartan invariants.*

Lemma 3.2.5. [8, Lemma 1.6] *Suppose $I_\mu : R_K(A) \longrightarrow R_K(B)$ is a perfect isometry. Let $\chi \in \text{Irr}(A)$. Let*

$$r_\chi = \frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)}.$$

Then r_χ is an invertible element in \mathcal{O} and $(r_\chi)^$ is constant for all $\chi \in \text{Irr}(A)$.*

Proof. Let $\rho_\mu : x \mapsto R_\mu^\circ(I_\mu^\circ(f)x)$ be the algebra isomorphism from $Z(B)$ onto $Z(A)$ as above. Let $x = \rho_\mu^{-1}(R_\mu^\circ(e)) \in Z(B)$. Then

$$I_\mu^\circ(f)x = R_\mu^{\circ-1}(\rho_\mu(x)) = e.$$

This shows that $I_\mu^\circ(f)$ is invertible in $Z(B)$. But

$$I_\mu^\circ(f) = \sum_{\chi \in \text{Irr}(A)} \frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)} e_{I_\mu(\chi)}.$$

Thus, $\frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)}$ are invertible elements in \mathcal{O} and $\left(\frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)}\right)^*$ exist for all $\chi \in \text{Irr}(A)$.

So

$$\lambda_\chi(I_\mu^\circ(f)) = (\omega_\chi(I_\mu^\circ(f)))^* = \left(\frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)}\right)^*.$$

But $\lambda_\chi = \lambda_\phi$ for all $\chi, \phi \in \text{Irr}(A)$. Hence $\left(\frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)}\right)^*$ is constant for all $\chi \in \text{Irr}(A)$.

Since $\frac{|G|/I_\mu(\chi)(1)}{|H|/\chi(1)} \in \mathbb{Q}$, it is constant modulo p . \square

Since r_χ is invertible, the previous lemma also shows that a perfect isometry preserves defects of irreducible characters.

Other numerical invariants preserved by a perfect isometry is given by the following theorem.

Theorem 3.2.6 (Theorem 4.11). *[9] Let $A \in \text{Bl}(H)$ and $B \in \text{Bl}(G)$ be perfectly isometric blocks. Then,*

- $k(A) = k(B)$.
- $l(A) = l(B)$.
- $d(A) = d(B)$.

3.3 Examples of perfect isometries

Let G be a finite group. In this section we will describe some typical examples of perfect isometries that arise “naturally”.

Galois Action

Let $\zeta = \exp(2\pi i/|G|)$. Denote by $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ the Galois group of the field extension. For an integer n coprime to $|G|$, let $\sigma_n \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ where $\zeta^{\sigma_n} = \zeta^n$. Since every irreducible character $\chi \in \text{Irr}(G)$ has values in $\mathbb{Q}(\zeta)$, the group $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ acts on $\text{Irr}(G)$ by $(\chi^{\sigma_n})(g) = \chi(g)^{\sigma_n}, \forall \chi \in \text{Irr}(G), \forall g \in G$. It is clear that $\chi(g)^{\sigma_n} = \chi(g^n)$.

The next lemma shows that the action of $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ preserves the block structure on $\text{Irr}(G)$.

Lemma 3.3.1. *Let $\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. Suppose $\chi, \varphi \in \text{Irr}(G)$ are in the same block then $\chi^\sigma, \varphi^\sigma \in \text{Irr}(G)$ are also in the same block.*

Proof. Since χ, φ are in the same block we have

$$|\mathcal{C}| \left(\frac{\chi(g_{\mathcal{C}})}{\chi(1)} - \frac{\varphi(g_{\mathcal{C}})}{\varphi(1)} \right) \equiv 0 \pmod{\mathfrak{p}},$$

for all $\mathcal{C} \in \text{cl}(G)$, where $g_{\mathcal{C}} \in \mathcal{C}$. Then

$$|\mathcal{C}| \left(\frac{\chi^\sigma(g_{\mathcal{C}})}{\chi^\sigma(1)} - \frac{\varphi^\sigma(g_{\mathcal{C}})}{\varphi^\sigma(1)} \right) = \left[|\mathcal{C}| \left(\frac{\chi(g_{\mathcal{C}})}{\chi(1)} - \frac{\varphi(g_{\mathcal{C}})}{\varphi(1)} \right) \right]^\sigma \equiv 0 \pmod{\mathfrak{p}},$$

for all $\mathcal{C} \in \text{cl}(G)$. Hence $\chi^\sigma, \varphi^\sigma$ are in the same block. \square

Let $\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and $B \in \text{Bl}(G)$. Denote by B^σ the block of G with $\text{Irr}(B^\sigma) = \{\chi^\sigma : \chi \in \text{Irr}(B)\}$. We can define an isometry $I_\sigma : R_K(B) \rightarrow R_K(B^\sigma)$ by $I_\sigma(\chi) = \chi^\sigma$ for $\chi \in \text{Irr}(G)$.

Lemma 3.3.2. *Let $\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and $B \in \text{Bl}(G)$. The isometry $I_\sigma : R_K(B) \rightarrow R_K(B^\sigma)$ is perfect.*

Proof. For short, write $\mu_\sigma = \mu_{I_\sigma}$. Let $g, h \in G$. Suppose that $g^\sigma = g^n$ where n is coprime to $|G|$. Then

$$\begin{aligned} \mu_\sigma(g, h) &= \sum_{\chi \in \text{Irr}(B)} I_\sigma(\chi)(g)\chi(h) = \sum_{\chi \in \text{Irr}(B)} \chi(g)^\sigma \chi(h) \\ &= \sum_{\chi \in \text{Irr}(B)} \chi(g^n)\chi(h) = \mu_{id}(g^n, h) \end{aligned}$$

where $id : R_K(B) \rightarrow R_K(B)$ is the identity map.

We claim that $C_G(g) = C_G(g^n)$. To see this, first it is clear that $C_G(g) \subseteq C_G(g^n)$. Let $x \in C_G(g^n)$. There is an integer m such that $g = (g^n)^m$. Then $xgx^{-1} = xg^{nm}x^{-1} = (xg^n x^{-1})^m = (g^n)^m = g$. Thus $x \in C_G(g)$. Now, since n is coprime to $|G|$, we have that g is p -regular if and only if g^n is p -regular. Since the identity map is a perfect isometry,

$$\begin{aligned} \frac{\mu_\sigma(g, h)}{|C_G(g)|} &= \frac{\mu_{id}(g^n, h)}{|C_G(g)|} = \frac{\mu_{id}(g^n, h)}{|C_G(g^n)|} \in \mathcal{O} \\ \frac{\mu_\sigma(g, h)}{|C_H(h)|} &= \frac{\mu_{id}(g^n, h)}{|C_H(h)|} \in \mathcal{O}. \end{aligned}$$

To show the separation condition, notice that if exactly one of g, h is p -singular, then exactly one of g^n, h is p -singular. Thus $\mu_\sigma(g, h) = \mu_{id}(g^n, h) = 0$. Hence I_σ is a perfect isometry as claimed. \square

In fact, Kessar [26] showed that the map I_σ is an “isotypy” (to be defined later).

Automorphism Action

Let $\sigma \in \text{Aut}(G)$ and $H \leq G$ such that $H^\sigma = H$. Then σ acts on $\text{CF}(H; K)$ via $\theta^\sigma(h) = \theta(h^{\sigma^{-1}})$. By Lemma 2.3.5, the action by σ permutes elements in the set $\text{Irr}(H)$ and also preserves the block structure of H . Let $A \in \text{Bl}(H)$. Define $I_\sigma : R_K(A) \rightarrow R_K(A^\sigma)$ by $I_\sigma(\chi) = \chi^\sigma$ for $\chi \in \text{Irr}(A)$.

Lemma 3.3.3. *Let $\sigma \in \text{Aut}(G)$ and $H \leq G$ be invariant under σ . Let $A \in \text{Bl}(H)$. Then $I_\sigma : R_K(A) \rightarrow R_K(A^\sigma)$ is a perfect isometry.*

Proof. The isometry I_σ is also defined on $\text{CF}(H, A; K)$ in the obvious way. Since σ is an automorphism, an element $h \in H$ is p -singular if and only if $h^{\sigma^{-1}}$ is p -singular. Thus, if $\theta \in \text{CF}(H, A; K)$ vanishes on p -singular elements, then so does $I_\sigma(\theta)$. Also, if $\theta \in \text{CF}(H, A; K)$ has values in \mathcal{O} for all $h \in H$ then so does $I_\sigma(\theta)$. Hence I_σ is a perfect isometry by Proposition 3.1.6. \square

As the principal block $A_0 \in \text{Bl}(H)$ contains the trivial character which is invariant under σ , we have $(A_0)^\sigma = A_0$ and so I_σ is a perfect isometry in A_0 .

An important special case is when H is a normal subgroup of G . Then H is invariant under any inner automorphism of G . So, conjugation by any element of G gives a perfect isometry in the principal block of H .

Multiplication by a linear character

Let $\chi, \lambda \in \text{Irr}(G)$ where λ is a linear character. It is easy to see that $\lambda\chi \in \text{Irr}(G)$ where $\lambda\chi(g) = \lambda(g)\chi(g), \forall g \in G$. The map $I : \chi \mapsto \lambda\chi$ then defines a bijection on $\text{Irr}(G)$.

Lemma 3.3.4. *Let $\lambda \in \text{Irr}(G)$ be a linear character. Suppose $\chi, \varphi \in \text{Irr}(G)$ are in the same block then $\lambda\chi, \lambda\varphi$ are also in the same block.*

Proof. Let $\mathcal{C} \in \text{cl}(G)$ and $g_{\mathcal{C}} \in \mathcal{C}$. Then

$$\begin{aligned} \left(\frac{|\mathcal{C}|\lambda\chi(g_{\mathcal{C}})}{\lambda\chi(1)} \right)^* - \left(\frac{|\mathcal{C}|\lambda\varphi(g_{\mathcal{C}})}{\lambda\varphi(1)} \right)^* &= \lambda(g_{\mathcal{C}})^* \left[\left(\frac{|\mathcal{C}|\chi(g_{\mathcal{C}})}{\chi(1)} \right)^* - \left(\frac{|\mathcal{C}|\varphi(g_{\mathcal{C}})}{\varphi(1)} \right)^* \right] \\ &= 0. \end{aligned}$$

Hence $\lambda\chi, \lambda\varphi$ are in the same block. □

Let $\lambda \in \text{Irr}(G)$ be a linear character. For a block $B \in \text{Bl}(G)$, denote by λB the block of $\lambda\chi$ for some $\chi \in \text{Irr}(B)$. This is well-defined by the previous lemma.

Lemma 3.3.5. *Let B be a block of $\mathcal{O}G$. Let $\lambda \in \text{Irr}(G)$ be a linear character. Then the map $I : \text{Irr}(B) \rightarrow \text{Irr}(\lambda B)$ defined by $I(\chi) = \lambda\chi$ is a perfect isometry.*

Proof. Let $\theta \in \text{CF}(G, B; K)$ be a class function. Then $I(\theta) = \lambda\theta$. Thus, if θ vanishes on p -singular elements, then so does $I(\theta)$. And if θ has values in \mathcal{O} , then so does $I(\theta)$. Hence I is a perfect isometry by Proposition 3.1.6. □

Chapter 4

Perfect isometry groups

Let G be a finite group. Let B be a block of $\mathcal{O}G$. In this chapter we will consider the perfect isometries from $R_K(B)$ to itself. We denote by $\Gamma(B)$ the group of all isometries in $R_K(B)$ under compositions. Using Proposition 3.1.6, it is clear that the set of all perfect isometries in $R_K(B)$ forms a subgroup of $\Gamma(B)$.

Definition 4.0.6. The group of all perfect isometries in $R_K(B)$ is denoted by $\text{PI}(B)$ and called the *perfect isometry group* of B .

One of the reasons to study the perfect isometry groups is because they are invariant for perfectly isometric blocks.

Proposition 4.0.7. *Let G, H be finite groups and let B, A be blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Suppose there exists a perfect isometry*

$$I : R_K(A) \longrightarrow R_K(B).$$

Then $\text{PI}(A)$ and $\text{PI}(B)$ are isomorphic. An isomorphism

$$\theta : \text{PI}(A) \longrightarrow \text{PI}(B)$$

is given by $\theta(\alpha) = I\alpha I^{-1}$ for $\alpha \in \text{PI}(A)$.

Proof. We first need to check that θ is well-defined, that is, it maps a perfect isometry in $\text{PI}(A)$ to a perfect isometry in $\text{PI}(B)$. Let $\alpha \in \text{PI}(A)$. Then $\theta(\alpha)$ is an isometry from $R_K(B)$ to $R_K(B)$, since it is a composition of isometries

$$R_K(B) \xrightarrow{I^{-1}} R_K(A) \xrightarrow{\alpha} R_K(A) \xrightarrow{I} R_K(B).$$

Since $I, I^{-1}, \alpha, \alpha^{-1}$ are all perfect isometries, $\theta(\alpha)$ gives a map

$$\mathrm{CF}(G, B; \mathcal{O}) \xrightarrow{I^{-1}} \mathrm{CF}(H, A; \mathcal{O}) \xrightarrow{\alpha} \mathrm{CF}(H, A; \mathcal{O}) \xrightarrow{I} \mathrm{CF}(G, B; \mathcal{O})$$

and

$$\mathrm{CF}_{p'}(G, B; K) \xrightarrow{I^{-1}} \mathrm{CF}_{p'}(H, A; K) \xrightarrow{\alpha} \mathrm{CF}_{p'}(H, A; K) \xrightarrow{I} \mathrm{CF}_{p'}(G, B; K),$$

whereas $\theta(\alpha)^{-1} = I\alpha^{-1}I^{-1}$ gives a map

$$\mathrm{CF}(G, B; \mathcal{O}) \xrightarrow{I^{-1}} \mathrm{CF}(H, A; \mathcal{O}) \xrightarrow{\alpha^{-1}} \mathrm{CF}(H, A; \mathcal{O}) \xrightarrow{I} \mathrm{CF}(G, B; \mathcal{O})$$

and

$$\mathrm{CF}_{p'}(G, B; K) \xrightarrow{I^{-1}} \mathrm{CF}_{p'}(H, A; K) \xrightarrow{\alpha^{-1}} \mathrm{CF}_{p'}(H, A; K) \xrightarrow{I} \mathrm{CF}_{p'}(G, B; K).$$

So, by Proposition 3.1.6, $\theta(\alpha)$ is a perfect isometry and thus lies in $\mathrm{PI}(B)$. The map θ is clearly an isomorphism. \square

The converse, however, is not true in general.

Example 4.0.8. Let D be a Sylow 2-subgroup of the group $G = L_2(17)$. Let B_0, A_0 be principal 2-blocks of G and $N_G(D)$ respectively. Calculations in GAP [19] show that $\mathrm{PI}(B_0) \cong \mathrm{PI}(A_0) \cong D_8 \times C_2 \times C_2$, but there does not exist a perfect isometry between $R_K(B)$ and $R_K(A)$.

We have seen in the previous chapter that the Galois action, automorphism action and multiplication by linear characters all give rise to perfect isometries from a block B to its corresponding block. Since the principal block B_0 are invariant under these actions, we always have elements in $\mathrm{PI}(B_0)$ arising from these actions. We will see later that, in some cases, the subgroups generated by perfect isometries from some of these actions contribute a significant part in the structure of $\mathrm{PI}(B_0)$.

4.1 Structure of $\mathrm{PI}(B)$

Let $I \in \Gamma(B)$. Then there is a bijection $I^+ : \mathrm{Irr}(B) \longrightarrow \mathrm{Irr}(B)$ and a sign function $\varepsilon_I : \mathrm{Irr}(B) \longrightarrow \{\pm 1\}$ such that $I(\chi) = \varepsilon_I(\chi)I^+(\chi)$ for all $\chi \in \mathrm{Irr}(B)$. For the

purpose of investigating the structure of $\text{PI}(B)$, we will label the irreducible characters in $\text{Irr}(B)$ as $\{\chi_1, \chi_2, \dots, \chi_n\}$. Then we can regard I^+ as a permutation $\sigma \in S_n$. We will treat elements in S_n as functions, so in a product of elements in S_n , the rightmost permutation is applied first. We will also regard ε_I as an element $\varepsilon = (\varepsilon_I(\chi_1), \dots, \varepsilon_I(\chi_n)) \in (C_2)^n$. Denote by $\varepsilon(i)$ the i -th component of ε . Then

$$I(\chi_i) = \varepsilon(i)\chi_{\sigma(i)}, \quad i \in \{1, \dots, n\}.$$

Multiplications in $(C_2)^n$ are pointwise.

Every isometry $I \in \Gamma(B)$ can then be identified with (σ, ε) where $\sigma \in S_n$ and $\varepsilon \in (C_2)^n$. For convenience, we will denote $(1, 1, \dots, 1) \in (C_2)^n$ by 1 and $(-1, -1, \dots, -1)$ by -1 .

Definition 4.1.1. An isometry $I \in \Gamma(B)$ is said to have a *homogenous sign* if $I = (\sigma, \varepsilon)$ where $\varepsilon = \pm 1$. It has the *all-positive sign* if $\varepsilon = 1$. In other words, an isometry $I : R_K(B) \rightarrow R_K(B)$ has all-positive sign if I sends $\text{Irr}(B)$ to $\text{Irr}(B)$.

The group S_n acts on the group $(C_2)^n$ by $\varepsilon^\sigma(i) = \varepsilon(\sigma(i))$, where $\sigma \in S_n$ and $\varepsilon \in (C_2)^n$. Let $\phi : S_n \rightarrow \text{Aut}((C_2)^n)$ be the corresponding action. The next lemma gives a structure of the group $\Gamma(B)$ via this action.

Lemma 4.1.2. *Let $I \in \Gamma(B)$ be an isometry with $I^+ = \sigma \in S_n$ and $\varepsilon_I = \varepsilon \in (C_2)^n$. Then the map $I \mapsto (\sigma, \varepsilon)$ is a group isomorphism from $\Gamma(B)$ onto $S_n \rtimes_\phi (C_2)^n$. Multiplications in $S_n \rtimes_\phi (C_2)^n$ are as follows. If $(\sigma_1, \varepsilon_1), (\sigma_2, \varepsilon_2) \in S_n \rtimes_\phi (C_2)^n$, then*

$$(\sigma_1, \varepsilon_1)(\sigma_2, \varepsilon_2) = (\sigma_1\sigma_2, (\varepsilon_1^{\sigma_2})(\varepsilon_2)).$$

The inverse of (σ, ε) is $(\sigma^{-1}, \varepsilon^{\sigma^{-1}})$.

Proof. Define $f : \Gamma(B) \rightarrow S_n \rtimes_\phi (C_2)^n$ by $f(I) = (\sigma, \varepsilon)$, where $I(\chi_i) = \varepsilon(i)\chi_{\sigma(i)}, \forall \chi_i \in \text{Irr}(B)$ for some $\varepsilon \in (C_2)^n$ and $\sigma \in S_n$. We will show that this is an isomorphism. Let $I_1, I_2 \in \Gamma(B)$. Suppose $I_1(\chi_i) = \varepsilon_1(i)\chi_{\sigma_1(i)}$ and $I_2(\chi_i) = \varepsilon_2(i)\chi_{\sigma_2(i)}$ for $i = 1, \dots, n$.

Then

$$\begin{aligned}
I_1(I_2(\chi_i)) &= I_1(\varepsilon_2(i)\chi_{\sigma_2(i)}) = \varepsilon_2(i)I_1(\chi_{\sigma_2(i)}) \\
&= \varepsilon_2(i)\varepsilon_1(\sigma_2(i))\chi_{\sigma_1(\sigma_2(i))} \\
&= \varepsilon_2(i)\varepsilon_1^{\sigma_2}(i)\chi_{\sigma_1(\sigma_2(i))} \\
&= (\varepsilon_2 \cdot \varepsilon_1^{\sigma_2})(i)\chi_{\sigma_1(\sigma_2(i))}.
\end{aligned}$$

Therefore, $f(I_1 \circ I_2) = (\sigma_1\sigma_2, (\varepsilon_1)^{\sigma_2} \cdot \varepsilon_2)$. On the other hand,

$$f(I_1)f(I_2) = (\sigma_1, \varepsilon_1)(\sigma_2, \varepsilon_2) = (\sigma_1\sigma_2, (\varepsilon_1)^{\sigma_2} \cdot \varepsilon_2).$$

So f is a homomorphism. Note that $f(I) = (1, 1)$ if and only if $I(\chi_i) = 1(i)\chi_{1(i)} = \chi_i \forall i = 1, \dots, n$ if and only if $I = id$. Hence f is injective. Finally, given $\sigma \in S_n$ and $\varepsilon \in (C_2)^n$, the map I defined by $I(\chi_i) = \varepsilon(i)\chi_{\sigma(i)}$, $i \in \{1, \dots, n\}$ is clearly an isometry. \square

Since $\text{PI}(B)$ is a subgroup of $\Gamma(B)$, it therefore inherits the structure of $\Gamma(B)$ and we shall identify a perfect isometry $I \in \text{PI}(B)$ with $(\sigma, \varepsilon) \in S_n \times (C_2)^n$ and write $I = (\sigma, \varepsilon)$ when $I(\chi_i) = \varepsilon(i)\chi_{\sigma(i)} \forall i \in \{1, \dots, n\}$. We will use I and (σ, ε) interchangeably to denote a perfect isometry. The following facts can be easily proved:

- If $I = (\sigma, \varepsilon)$ then $-I = (\sigma, -\varepsilon)$.
- $\varepsilon^{-1} = \varepsilon$.
- $(-\varepsilon)^\sigma = -(\varepsilon^\sigma)$.

We will now define an important group $\text{PI}^+(B)$ which, in many cases, characterizes the group $\text{PI}(B)$. First, it is clear that id and $-id$ are perfect isometries, where id is the identity isometry and $-id(\chi) = -\chi, \forall \chi \in \text{Irr}(B)$. We will denote by $\langle -id \rangle$ the subgroup generated by $-id$.

Lemma 4.1.3. *There exists a subgroup $H \leq S_n$ such that $\text{PI}(B)/\langle -id \rangle \cong H$.*

Proof. Embed the sign group $(C_2)^n$ into $\Gamma(B) = S_n \rtimes_\phi (C_2)^n$. By Proposition 3.2.2, $\text{PI}(B) \cap (C_2)^n = \langle -\text{id} \rangle$. By the Second Isomorphism Theorem,

$$\frac{\text{PI}(B)}{\langle -\text{id} \rangle} = \frac{\text{PI}(B)}{\text{PI}(B) \cap (C_2)^n} \cong \frac{\text{PI}(B)(C_2)^n}{(C_2)^n} \leq \frac{S_n \rtimes_\phi (C_2)^n}{(C_2)^n} \cong S_n.$$

□

Definition 4.1.4. Let B be a block of $\mathcal{O}G$ and let $\text{Irr}(B) = \{\chi_1, \dots, \chi_n\}$. Let $\sigma \in S_n$. We say that σ is a *perfect permutation* if there is a sign function ε such that $I = (\sigma, \varepsilon)$ is a perfect isometry. Denote by $\text{PI}^+(B)$ the group of perfect permutations of the block B . If $H \leq \text{PI}(B)$, we say that H *contains* a perfect permutation $\sigma \in \text{PI}^+(B)$ if $(\sigma, \varepsilon) \in H$ for some sign ε .

The group $\text{PI}^+(B)$ therefore represents how $\text{PI}(B)$ acts on $\text{Irr}(B)$ via permutation (ignoring the signs). Note that $\text{PI}^+(B)$ is the projection of $\text{PI}(B) \leq S_n \rtimes_\phi (C_2)^n$ onto S_n . From the previous lemma it is clear that we have a short exact sequence

$$1 \longrightarrow \langle -\text{id} \rangle \longrightarrow \text{PI}(B) \longrightarrow \text{PI}^+(B) \longrightarrow 1.$$

We will now show that, in many cases, the sequence splits. First we need the following lemma.

Lemma 4.1.5. *Suppose there exists a (normal) subgroup H of $\text{PI}(B)$ of index 2 such that $-\text{id} \notin H$. Then $\text{PI}(B) = H \times \langle -\text{id} \rangle$ and $H \cong \text{PI}^+(B)$. Conversely, if $\text{PI}(B) = H \times \langle -\text{id} \rangle$ for some subgroup $H \triangleleft \text{PI}(B)$ of index 2, then H contains all perfect permutations and $H \cong \text{PI}^+(B)$.*

Proof. Suppose there is a subgroup $H \leq \text{PI}(B)$ of index 2 and $-\text{id} \notin H$. Let $I \in H$. Then $-I \notin H$, otherwise we would have $(-I) \circ I^{-1} = -\text{id} \in H$. Since $|\text{PI}(B)| = 2|\text{PI}^+(B)|$, and H has index 2, we have that H contains all perfect permutations with one copy each. Any perfect isometry in $\text{PI}(B)$ can therefore be written as I or $-I$ where $I \in H$. Since $-\text{id}$ commutes with every element in H , we have $\text{PI}(B) = H \times \langle -\text{id} \rangle$ and the projection $(\sigma, \varepsilon) \mapsto \sigma$ gives a desired isomorphism $H \cong \text{PI}^+(B)$.

Assume conversely that $\text{PI}(B) = H \times \langle -\text{id} \rangle$. If $(\sigma, \varepsilon) \in H$ then $(\sigma, -\varepsilon) \notin H$ or we would have $(\sigma, \varepsilon)^{-1}(\sigma, -\varepsilon) = (1, -1) = -\text{id} \in H$, a contradiction, since $-\text{id} \notin H$. So

H contains at most one copy of each perfect permutation. Thus $|H| \leq |\text{PI}^+(B)| = |\text{PI}(B)|/2$. Since H has index 2, we have the equality and so H contains all perfect permutations with one copy each. An isomorphism $H \cong \text{PI}^+(B)$ is given by the projection $(\sigma, \varepsilon) \mapsto \sigma$. \square

Lemma 4.1.6. *Suppose that every perfect isometry $I \in \text{PI}(B)$ has a homogenous sign. Then*

$$\text{PI}(B) \cong \text{PI}(B)^+ \times \langle -id \rangle.$$

Proof. If $I = (\sigma, \varepsilon) \in \text{PI}(B)$, then, by assumption, we have $\varepsilon = \pm 1$. Let

$$H = \{(\sigma, 1) : \sigma \in \text{PI}^+(B)\}.$$

Then it is clear that H is a subgroup of $\text{PI}(B)$ of index 2, and the result follows from Lemma 4.1.5. \square

Proposition 4.1.7. *Suppose $|\text{Irr}(B)| = n$ is an odd integer. Then*

$$\text{PI}(B) \cong \text{PI}^+(B) \times \langle -id \rangle.$$

Proof. For $\varepsilon \in (C_2)^n$, define the weight function

$$wt(\varepsilon) = \text{number of } -1 \text{ in } \varepsilon.$$

It can easily be checked that

- $wt(-\varepsilon) = n - wt(\varepsilon)$
- $wt(\varepsilon^\sigma) = wt(\varepsilon) \quad \forall \sigma \in S_n$

and if we denote by $[\varepsilon, \tau]$ the number of positions where ε and τ both have -1, then

$$wt(\varepsilon \cdot \tau) = wt(\varepsilon) + wt(\tau) - 2[\varepsilon, \tau].$$

For each $\sigma \in \text{PI}^+(B)$, let ε_σ be such that $(\sigma, \varepsilon_\sigma) \in \text{PI}(B)$ and $(-1)^{wt(\varepsilon_\sigma)} = 1$. Since n is odd, $(-1)^{wt(-\varepsilon_\sigma)} = (-1)^n (-1)^{wt(\varepsilon_\sigma)} = -1$, so the choice of ε_σ is unique.

Now define

$$H = \{(\sigma, \varepsilon_\sigma) : \sigma \in \text{PI}^+(B)\}.$$

We claim that $H \leq \text{PI}(B)$ is a subgroup of index 2 and does not contain $-id$. The result will then follow from Lemma 4.1.5.

To see this, first it is clear that $id \in H$ and $-id \notin H$ since n is odd. If $(\sigma, \varepsilon_\sigma) \in H$, then $(\sigma, \varepsilon_\sigma)^{-1} = (\sigma^{-1}, (\varepsilon_\sigma)^{\sigma^{-1}})$. But $wt((\varepsilon_\sigma)^{\sigma^{-1}}) = wt(\varepsilon_\sigma)$. So $(\sigma, \varepsilon_\sigma)^{-1} \in H$. If $(\sigma, \varepsilon_\sigma), (\tau, \varepsilon_\tau) \in H$, then $(\sigma, \varepsilon_\sigma)(\tau, \varepsilon_\tau) = (\sigma\tau, (\varepsilon_\sigma)^\tau \cdot \varepsilon_\tau)$. But

$$\begin{aligned} wt((\varepsilon_\sigma)^\tau \cdot \varepsilon_\tau) &= wt((\varepsilon_\sigma)^\tau) + wt(\varepsilon_\tau) - 2[(\varepsilon_\sigma)^\tau, \varepsilon_\tau] \\ &= wt(\varepsilon_\sigma) + wt(\varepsilon_\tau) - 2[(\varepsilon_\sigma)^\tau, \varepsilon_\tau] \end{aligned}$$

So $(-1)^{wt((\varepsilon_\sigma)^\tau \cdot \varepsilon_\tau)} = (-1)^{wt(\varepsilon_\sigma)}(-1)^{wt(\varepsilon_\tau)} = 1$ and $(\sigma, \varepsilon_\sigma)(\tau, \varepsilon_\tau) \in H$.

Finally since H contains only one copy of each perfect permutation, we have $|\text{PI}(B)/H| = 2$. \square

Let $I \in \text{PI}(B)$. Define the ratio $r_I = \left(\frac{I(\chi)(1)}{\chi(1)}\right)^*$ for some $\chi \in \text{Irr}(B)$. By Lemma 3.2.5, this is well-defined and a non-zero element in k . Since $\left(\frac{I(-\chi)(1)}{-\chi(1)}\right)^* = \left(\frac{I(\chi)(1)}{\chi(1)}\right)^*$, the ratio $\left(\frac{I(\chi)(1)}{\chi(1)}\right)^*$ is also constant for all $\chi \in \pm \text{Irr}(B)$. It is easy to see that $r_{-I} = -r_I$ for all $I \in \text{PI}(B)$.

Lemma 4.1.8. *The map $I \mapsto r_I$ is a group homomorphism from $\text{PI}(B)$ to k^\times , the group of non-zero elements of k .*

Proof. First, it is clear that $r_{id} = 1$. By Lemma 3.2.5, we have $r_I \neq 0$ for all $I \in \text{PI}(B)$. Let $I, J \in \text{PI}(B)$. For $\chi \in \text{Irr}(B)$ let $\varphi = I^{-1}(\chi) \in \pm \text{Irr}(B)$. Then

$$r_{I^{-1}} = \left(\frac{I^{-1}(\chi)(1)}{\chi(1)}\right)^* = \left(\frac{\varphi(1)}{I(\varphi)(1)}\right)^* = \frac{1}{r_I} = (r_I)^{-1}.$$

Let $\theta = J(\chi) \in \pm \text{Irr}(B)$, then

$$\begin{aligned} r_{I \circ J} &= \left(\frac{I(J(\chi))(1)}{\chi(1)}\right)^* \\ &= \left(\frac{I(\theta)(1)}{J^{-1}(\theta)(1)}\right)^* \\ &= \left(\frac{I(\theta)(1)}{\theta(1)}\right)^* \left(\frac{\theta(1)}{J^{-1}(\theta)(1)}\right)^* \\ &= (r_I)(1/r_{J^{-1}}) = (r_I)(r_J). \end{aligned}$$

This shows that $I \mapsto r_I$ is a group homomorphism. \square

We make the following observation based on our computer-generated examples.

Conjecture 4.1.9. *Let B be a block of $\mathcal{O}G$. Let $I \in \text{PI}(B)$. Then $r_I = \pm 1$.*

Remark 4.1.10. If B is the principal block of $\mathcal{O}G$ with defect group D , and A is the principal block of $\mathcal{O}N_G(D)$, then by Sylow Theorem we have $|G : N_G(D)| \equiv 1 \pmod{p}$. Let $I : R_K(A) \rightarrow R_K(B)$ be a perfect isometry. By Lemma 3.2.5, we can still define a well-defined ratio

$$r_I = \left(\frac{|N_G(D)|/\chi(1)}{|G|/I(\chi)(1)} \right)^* = \left(\frac{I(\chi)(1)}{\chi(1)} \right)^*,$$

for some $\chi \in \text{Irr}(A)$. Broué [8] noted in his remark after Lemma 1.6 that $I(\chi)(1) \equiv \chi(1) \pmod{p}, \forall \chi \in \text{Irr}(A)$. Since I is not specified, this seems to suggest that $r_I = 1$ for any perfect isometry I in this case. (Although in fact, it should be $r_I = \pm 1$ as $r_{-I} = -r_I$.)

Proposition 4.1.11. *Let G be a finite group. Let p be an odd prime. Let B be a block of $\mathcal{O}G$. Suppose that Conjecture 4.1.9 holds for B . Then*

$$\text{PI}(B) \cong \text{PI}^+(B) \times \langle -id \rangle.$$

Proof. Let

$$H = \{I \in \text{PI}(B) : r_I = 1\}.$$

Since p is odd, $r_{-id} = -1 \neq 1$ and so $-id \notin H$. From the proof of Lemma 4.1.8, it is clear that H is a subgroup of $\text{PI}(B)$. Finally, since $r_I = \pm 1$ for all $I \in \text{PI}(B)$ we have that either $I \in H$ or $-I \in H$. So H has index 2 in $\text{PI}(B)$. The result now follows from Lemma 4.1.5. \square

Apart from Conjecture 4.1.9, we also suspect that we always have $\text{PI}(B) \cong \text{PI}^+(B) \times \langle -id \rangle$ for any blocks of a finite group G . We are not able to prove this for all blocks so far but we will list the cases where we have proved it.

Proposition 4.1.12. *Let G be a finite group. Let B be a block of $\mathcal{O}G$. Then we have $\text{PI}(B) \cong \text{PI}^+(B) \times \langle -id \rangle$ in the following cases:*

- if every perfect isometry $I \in \text{PI}(B)$ has a homogenous sign,

- if $|\text{Irr}(B)|$ is an odd integer,
- if p is odd and $r_I = \pm 1$ for all $I \in \text{PI}(B)$.

Conjecture 4.1.13. *Let G be a finite group. Let $B \in \text{Bl}(G)$. Then the short exact sequence*

$$1 \longrightarrow \langle -id \rangle \longrightarrow \text{PI}(B) \longrightarrow \text{PI}^+(B) \longrightarrow 1$$

splits.

4.2 Standard subgroups of $\text{PI}(B_0)$ for a principal block B_0

Suppose now that B_0 is the principal block of $\mathcal{O}G$. We have seen that for each $\sigma \in \text{Aut}(G)$ we have a perfect isometry $I_\sigma : \chi \mapsto \chi^\sigma$. Let $\text{Aut}_c(G)$ be the group of class-preserving automorphisms of G , that is, the group of automorphisms σ such that, for each $g \in G$, $\sigma(g)$ and g are conjugate in G . We then have a sequence of normal subgroups (cf. [46]):

$$\text{Inn}(G) \leq \text{Aut}_c(G) \leq \text{Aut}(G).$$

Lemma 4.2.1. *The map $\sigma \mapsto I_{\sigma^{-1}}$ is a group homomorphism from $\text{Aut}(G)$ to $\text{PI}(B_0)$ with $\text{Aut}_c(G)$ in the kernel.*

Proof. Let $\sigma, \tau \in \text{Aut}(G)$. For each $\chi \in \text{Irr}(B_0)$ we have

$$I_{(\sigma\tau)^{-1}}(\chi) = \chi^{(\sigma\tau)^{-1}} = \chi^{\tau^{-1}\sigma^{-1}} = I_{\sigma^{-1}}(\chi^{\tau^{-1}}) = I_{\sigma^{-1}}(I_{\tau^{-1}}(\chi)) = (I_{\sigma^{-1}} \circ I_{\tau^{-1}})(\chi).$$

This shows that the map $\sigma \mapsto I_{\sigma^{-1}}$ is a group homomorphism. If $\sigma \in \text{Aut}_c(G)$, then $I_{\sigma^{-1}}(\chi)(g) = \chi^{\sigma^{-1}}(g) = \chi(g^\sigma) = \chi(g)$ for all $\chi \in \text{Irr}(B_0)$, all $g \in G$. Thus, I_σ is the identity map. \square

Denote by $\tilde{\mathcal{A}}(B_0)$ the image of the map $\text{Aut}(G) \longrightarrow \text{PI}(B_0)$. If M is the kernel then $\text{Aut}_c(G) \leq M \leq \text{Aut}(G)$ by the previous lemma.

Remark 4.2.2. It should be noted that M depends on the block B_0 and, in fact, both bounds can be achieved. If B_0 contains only $\mathbf{1}$ then clearly $M = \text{Aut}(G)$ since we must have $\mathbf{1}^\sigma = \mathbf{1}$ for all $\sigma \in \text{Aut}(G)$. On the other hand, if $\text{Irr}(B_0) = \text{Irr}(G)$ then for any $\sigma \in M$ we have $I_{\sigma^{-1}}(f) = f$ for any class function f on G (since $\text{Irr}(B_0)$ is a basis for class functions on G). Let $x \in G$ and δ_x be the indicator class function of x (that is, $\delta_x(g) = 1$ if g and x are conjugate, and zero otherwise). If $\sigma \in M$, then

$$1 = \delta_x(x) = I_{\sigma^{-1}}(\delta_x)(x) = (\delta_x)^{\sigma^{-1}}(x) = \delta_x(x^\sigma).$$

This shows that x and x^σ are conjugate. As x is arbitrary, σ is a class-preserving automorphism. Hence $M = \text{Aut}_c(G)$ in this case.

Let $\mathcal{L}(B_0)$ be a multiplicative group of linear characters in $\text{Irr}(B_0)$. We have seen that for each linear character $\lambda \in \mathcal{L}(B_0)$, there is a perfect isometry $I_\lambda \in \text{PI}(B_0)$ defined by $I_\lambda(\chi) = \lambda\chi$ for $\chi \in \text{Irr}(B_0)$.

Lemma 4.2.3. *The map $\lambda \mapsto I_\lambda$ is a group monomorphism from $\mathcal{L}(B_0)$ into $\text{PI}(B_0)$.*

Proof. Let $\lambda, \lambda' \in \mathcal{L}(B_0)$. For each $\chi \in \text{Irr}(B_0)$ we have

$$I_{\lambda\lambda'}(\chi) = \lambda\lambda'\chi = \lambda I_{\lambda'}(\chi) = I_\lambda(I_{\lambda'}(\chi)) = (I_\lambda \circ I_{\lambda'}) (\chi).$$

This shows that the map $\lambda \mapsto I_\lambda$ is a group homomorphism. Let λ be in the kernel. Then $I_\lambda(\mathbf{1}) = \mathbf{1} = \lambda \cdot \mathbf{1}$. So $\lambda = \mathbf{1}$. \square

Let $\tilde{\mathcal{L}}(B_0) \cong \mathcal{L}(B_0)$ be the image of the map $\lambda \mapsto I_\lambda$. Then $\tilde{\mathcal{L}}(B_0)$ and $\tilde{\mathcal{A}}(B_0)$ are subgroups of $\text{PI}(B_0)$. The next lemma describes their structure inside $\text{PI}(B_0)$.

Lemma 4.2.4. *The group $\text{PI}(B_0)$ contains the subgroup $\tilde{\mathcal{L}}(B_0) \rtimes \tilde{\mathcal{A}}(B_0)$ where all perfect isometries in $\tilde{\mathcal{L}}(B_0) \rtimes \tilde{\mathcal{A}}(B_0)$ have all-positive signs.*

Proof. First we will show that $\tilde{\mathcal{L}}(B_0)$ intersects $\tilde{\mathcal{A}}(B_0)$ trivially. Suppose $I_\lambda = I_\sigma$ for some $\lambda \in \mathcal{L}(B_0)$ and some $\sigma \in \text{Aut}(G)$. Then $\lambda\chi = \chi^\sigma$ for all $\chi \in \text{Irr}(B_0)$. Taking $\chi = \mathbf{1}$ we get $\lambda = \mathbf{1}$. So $\tilde{\mathcal{L}}(B_0) \cap \tilde{\mathcal{A}}(B_0) = \{\text{id}\}$. The group $\tilde{\mathcal{A}}(B_0)$ acts on $\tilde{\mathcal{L}}(B_0)$ by

$(I_\lambda)^{I_\sigma} = I_{I_\sigma(\lambda)}$. Let $S \leq \text{PI}(B_0)$ be generated by elements in $\tilde{\mathcal{L}}(B_0)$ and $\tilde{\mathcal{A}}(B_0)$. If $I_\lambda, I_{\lambda'} \in \tilde{\mathcal{L}}(B_0)$ and $I_\sigma, I_\tau \in \tilde{\mathcal{A}}(B_0)$, then for $\chi \in \text{Irr}(B_0)$,

$$\begin{aligned} (I_\lambda, I_\sigma) \circ (I_{\lambda'}, I_\tau)(\chi) &= (I_\lambda, I_\sigma)(\lambda' \cdot \chi^\tau) \\ &= \lambda \cdot (\lambda')^\sigma \cdot \chi^{\tau\sigma} \\ &= \lambda \cdot I_\sigma(\lambda') \cdot I_\sigma(I_\tau(\chi)) \\ &= (I_\lambda \circ I_{I_\sigma(\lambda')} \circ I_\sigma \circ I_\tau)(\chi) \\ &= (I_\lambda \circ (I_{\lambda'})^{I_\sigma}, I_\sigma \circ I_\tau)(\chi). \end{aligned}$$

So $S = \tilde{\mathcal{L}}(B_0) \rtimes \tilde{\mathcal{A}}(B_0)$. Finally, since $\lambda\chi$ and χ^σ are irreducible characters for any linear character λ , any irreducible character χ and any automorphism σ of G . We see that the perfect isometries I_λ and I_σ have all-positive signs. \square

4.3 $R_K(B \otimes_{\mathcal{O}} B^\circ)$ as a ring

Let G be a finite group and B be a block of $\mathcal{O}G$. We can regard $R_K(B \otimes_{\mathcal{O}} B^\circ)$ as a ring where multiplication is defined as follows. Define a map $-\otimes_B - : R_K(B \otimes_{\mathcal{O}} B^\circ) \times R_K(B \otimes_{\mathcal{O}} B^\circ) \rightarrow R_K(B \otimes_{\mathcal{O}} B^\circ)$ by

$$(\mu \otimes_B \lambda)(g, h) = \frac{1}{|G|} \sum_{x \in G} \mu(g, x^{-1}) \lambda(x, h) \quad (4.1)$$

for $\mu, \lambda \in R_K(B \otimes_{\mathcal{O}} B^\circ)$ and $g, h \in G$. It can be easily seen that \otimes_B is a bilinear map. Note also that the formula is induced by the tensor product $-\otimes_{K \otimes_{\mathcal{O}} B} -$ of bimodules.

The following lemma shows that perfectness are closed under \otimes_B

Lemma 4.3.1. *Let $\mu, \lambda \in R_K(B \otimes_{\mathcal{O}} B^\circ)$. If μ, λ are perfect, then $\mu \otimes_B \lambda$ is also perfect.*

Proof. Let $g, h \in G$. Then

$$\begin{aligned} \frac{(\mu \otimes_B \lambda)(g, h)}{|C_G(g)|} &= \frac{1}{|G|} \sum_{x \in G} \frac{\mu(g, x^{-1})}{|C_G(g)|} \lambda(x, h) \\ &= \sum_{x_i \in G} \frac{\mu(g, x_i^{-1})}{|C_G(g)|} \frac{\lambda(x_i, h)}{|C_G(x_i)|} \end{aligned}$$

where x_i runs over all conjugacy class representatives of G . Since μ, λ are perfect, $\mu(g, x_i^{-1})/|C_G(g)|$ and $\lambda(x_i, h)/|C_G(x_i)|$ are in \mathcal{O} for all x_i . So $(\mu \otimes_B \lambda)(g, h)/|C_G(g)| \in \mathcal{O}$. Similarly $(\mu \otimes_B \lambda)(g, h)/|C_G(h)| \in \mathcal{O}$. So $\mu \otimes_B \lambda$ satisfies the integrality condition. If g is p -singular and h is p -regular, then $\mu(g, x^{-1}) = 0$ for all p -regular elements x while $\lambda(x, h) = 0$ for all p -singular element x . So $(\mu \otimes_B \lambda)(g, h) = 0$. Similarly, we can show $(\mu \otimes_B \lambda)(g, h) = 0$ for p -regular g and p -singular h . Hence $\mu \otimes_B \lambda$ is perfect. \square

The motivation for our definition of the product (4.1) is the following.

Lemma 4.3.2. *Let $\mu, \lambda \in R_K(B \otimes_{\mathcal{O}} B^{\circ})$ and let I_{μ}, I_{λ} be the corresponding isometries. Then $I_{\mu} \circ I_{\lambda} = I_{\mu \otimes_B \lambda}$.*

Proof. Let $\chi \in R_K(B)$ and $g \in G$. Then

$$\begin{aligned}
 (I_{\mu} \circ I_{\lambda})(\chi)(g) &= I_{\mu}(I_{\lambda}(\chi))(g) \\
 &= \frac{1}{|G|} \sum_{x \in G} \mu(g, x^{-1}) I_{\lambda}(\chi)(x) \\
 &= \frac{1}{|G|} \sum_{x \in G} \mu(g, x^{-1}) \frac{1}{|G|} \sum_{y \in G} \lambda(x, y^{-1}) \chi(y) \\
 &= \frac{1}{|G|} \sum_{y \in G} \left(\frac{1}{|G|} \sum_{x \in G} \mu(g, x^{-1}) \lambda(x, y^{-1}) \right) \chi(y) \\
 &= \frac{1}{|G|} \sum_{y \in G} (\mu \otimes_B \lambda)(g, y^{-1}) \chi(y) \\
 &= I_{\mu \otimes_B \lambda}(\chi)(g).
 \end{aligned}$$

\square

Corollary 4.3.3. *With the multiplication \otimes_B , $R_K(B \otimes_{\mathcal{O}} B^{\circ})$ is a ring with μ_{id} as the identity element, where*

$$\mu_{id}(g, h) = \sum_{\chi \in \text{Irr}(B)} \chi(g) \chi(h),$$

for $g, h \in G$.

Proof. Let $\mu, \lambda \in R_K(B \otimes_{\mathcal{O}} B^{\circ})$. Since $I_{\mu} \circ I_{\lambda} : R_K(B) \rightarrow R_K(B)$ is a linear map, by Lemma 3.1.4, $I_{\mu} \circ I_{\lambda} = I_{\tau}$ for some $\tau \in R_K(B \otimes_{\mathcal{O}} B^{\circ})$. By Lemma 4.3.2, $\tau = \mu \otimes_B \lambda$,

so $\mu \otimes_B \lambda \in R_K(B \otimes_{\mathcal{O}} B^\circ)$. Since μ_{id} induces the identity map, using Lemma 4.3.2 again, it is clear that μ_{id} is the identity of the ring $R_K(B \otimes_{\mathcal{O}} B^\circ)$. \square

If $\mu \in R_K(B \otimes_{\mathcal{O}} B^\circ)$ is a perfect character, then by Theorem 3.2.4 the perfect isometry I_μ induces linear maps $I_\mu^\circ, R_\mu^\circ : Z(KB) \rightarrow Z(KB)$ defined by

$$I_\mu^\circ(y) = \sum_{g \in G} \left(\frac{1}{|G|} \sum_{h \in G} \mu(g^{-1}, h) y(h) \right) g,$$

and

$$R_\mu^\circ(x) = \sum_{h \in G} \left(\frac{1}{|G|} \sum_{g \in G} \mu(g, h^{-1}) x(g) \right) h,$$

where $y = \sum_{h \in G} y(h)h \in Z(KB)$ and $x = \sum_{g \in G} x(g)g \in Z(KB)$.

If e is the block idempotent of B , then the map ρ_μ , defined by $\rho_\mu(y) = I_\mu^\circ(yR_\mu^\circ(e))$ gives an automorphism on $Z(KB)$ which is also an automorphism on $Z(B)$. Note that if $\chi \in \text{Irr}(B)$, then $\rho_\mu(e_\chi) = e_{I_\mu(\chi)}$ where $e_{-\chi} = e_\chi$.

Lemma 4.3.4. *The map $\Psi : I_\mu \mapsto \rho_\mu$ is a group homomorphism from $\text{PI}(B)$ to $\text{Aut}_{\mathcal{O}}(Z(B))$ with kernel $\langle -id \rangle$.*

Proof. Define $\tilde{\Psi} : \text{PI}(B) \rightarrow \text{Aut}_K(Z(KB))$ by $\tilde{\Psi}(I_\mu) = \Psi(I_\mu)$. First we show that $\tilde{\Psi}$ is a homomorphism. Let $\chi \in \text{Irr}(B)$ and let $I_\mu, I_\lambda \in \text{PI}(B)$. Then

$$\begin{aligned} \tilde{\Psi}(I_\mu \circ I_\lambda)(e_\chi) &= \rho_{\mu \otimes_B \lambda}(e_\chi) = e_{I_{\mu \otimes_B \lambda}(\chi)} \\ &= e_{(I_\mu \circ I_\lambda)(\chi)} \\ &= \rho_\mu(e_{I_\lambda(\chi)}) = \rho_\mu(\rho_\lambda(e_\chi)) \\ &= (\tilde{\Psi}(I_\mu) \circ \tilde{\Psi}(I_\lambda))(e_\chi). \end{aligned}$$

By linearity, this gives a group homomorphism. Since the image of $\tilde{\Psi}$ preserves $Z(B)$, Ψ is also a group homomorphism.

Let $I_\mu \in \text{Ker}(\Psi)$. For any $x \in Z(KB)$ there is $m \in \mathcal{O}$ such that $mx \in Z(B)$.

Then

$$\Psi(I_\mu)(mx) = mx$$

$$\tilde{\Psi}(I_\mu)(mx) = mx$$

$$m\tilde{\Psi}(I_\mu)(x) = mx$$

$$\tilde{\Psi}(I_\mu)(x) = x.$$

Thus $I_\mu \in \text{Ker}(\tilde{\Psi})$, and so

$$e_\chi = \tilde{\Psi}(e_\chi) = \rho_\mu(e_\chi) = e_{I_\mu(\chi)},$$

for all $\chi \in \text{Irr}(B)$. This implies that $I_\mu(\chi) = \pm\chi$ for all $\chi \in \text{Irr}(B)$. By Proposition 3.2.2, we have $I_\mu = \pm id$. \square

We shall see later in the case of abelian p -groups that the map Ψ , as defined in the previous lemma, becomes an isomorphism. Nevertheless, if we know the structure of $\text{Aut}_{\mathcal{O}}(Z(B))$, this gives a good upper bound for the size of the perfect isometry group.

Corollary 4.3.5. *Let B be a block of $\mathcal{O}G$. Then*

$$|\text{PI}(B)| \leq 2|\text{Aut}_{\mathcal{O}}(Z(B))|.$$

4.4 Connection to Picard groups and derived Picard groups

We have seen that the perfect isometry group is an invariant for perfectly isometric blocks. In the same way, Picard group and Derived Picard group are invariants for Morita equivalent and derived equivalent blocks respectively. In this section, we will describe how these equivalences and their associated invariant groups can be related.

Let A, B be blocks of $\mathcal{O}H, \mathcal{O}G$ for two finite groups H, G respectively. As we saw in Section 2.5, if A and B are Morita equivalent then they are derived equivalent. This in turn implies that they are perfectly isometric as well.

Theorem 4.4.1. [8, Theorem 3.1] *Suppose that A and B are derived equivalent. Then they are perfectly isometric.*

The converse does not hold in general.

Example 4.4.2. [29] Let $p = 2$. The blocks $\mathcal{O}D_8$ and $\mathcal{O}Q_8$ are perfectly isometric, but they are not derived equivalent.

Let U be a bounded complex of A -lattices. Set

$$[K \otimes_{\mathcal{O}} U] = \sum_{i \in \mathbb{Z}} (-1)^i [K \otimes_{\mathcal{O}} U_i] \in R_K(A),$$

where U_i is the component of U in degree i . Following the discussion after Lemma 3.4 in [27], if U, U' are two quasi-isomorphic complexes as above then $[K \otimes_{\mathcal{O}} U] = [K \otimes_{\mathcal{O}} U']$. Now suppose that A, B are derived equivalent, induced by a tilting complex $X \in \mathcal{D}^b(B \otimes_{\mathcal{O}} A^{\circ})$. Then the functor

$$X \otimes_A^L - : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B),$$

induces an isometry $\Phi_X : R_K(A) \longrightarrow R_K(B)$ given by

$$\Phi_X([K \otimes_{\mathcal{O}} U]) = [K \otimes_{\mathcal{O}} (X \otimes_A^L U)]. \quad (4.2)$$

The map (4.2) can be related to a perfect isometry in the following way. Let X be a bounded complex of (B, A) -bimodules with components X_i in degree i . Suppose each X_i is projective as a left B -module and a right A -module. Let μ_i be the character of $K \otimes_{\mathcal{O}} X_i$. Define the generalized character μ of the complex $K \otimes_{\mathcal{O}} X$ by

$$\mu = \sum_{i \in \mathbb{Z}} (-1)^i \mu_i \in R_K(B \otimes_{\mathcal{O}} A^{\circ}) \quad (4.3)$$

(where, as usual, we identify $R_K(B \otimes_{\mathcal{O}} A^{\circ})$ with the group of generalized characters of (B, A) -bimodules).

By Proposition 3.1.2, each μ_i is perfect. Since any \mathcal{O} -linear combination of perfect characters is perfect, the generalized character μ is perfect. The map Φ_X is then equal to the map I_{μ} in (3.1) on the character level. If, furthermore, $X \in \mathcal{D}^b(B \otimes_{\mathcal{O}} A^{\circ})$

is a tilting complex, then [8, Theorem 3.1] implies that Φ_X is a perfect isometry.

Suppose now that $A = B$. If $X \in \text{DPic}(B)$, then by the above remarks we have $\Phi_X \in \text{PI}(B)$.

Lemma 4.4.3. *The map*

$$\begin{aligned} \text{DPic}(B) &\longrightarrow \text{PI}(B) \\ (X) &\mapsto \Phi_X. \end{aligned}$$

is a group homomorphism. Hence, the restriction to $\text{Pic}(B)$, considered as a subgroup of $\text{DPic}(B)$ via the canonical injection $\text{Pic}(B) \longrightarrow \text{DPic}(B)$, also gives a group homomorphism

$$\begin{aligned} \text{Pic}(B) &\longrightarrow \text{PI}(B) \\ (M) &\mapsto \Phi_M. \end{aligned}$$

Proof. We only need to prove the first homomorphism. Let $X, Y \in \text{DPic}(B)$ and let U be a bounded complex of A -lattices. Then

$$\begin{aligned} \Phi_X \circ \Phi_Y([K \otimes_{\mathcal{O}} U]) &= \Phi_X([K \otimes_{\mathcal{O}} (Y \otimes_B^L U)]) \\ &= [K \otimes_{\mathcal{O}} (X \otimes_B^L (Y \otimes_B^L U))] \\ &= [K \otimes_{\mathcal{O}} ((X \otimes_B^L Y) \otimes_B^L U)] \\ &= \Phi_{X \otimes_B^L Y}([K \otimes_{\mathcal{O}} U]). \end{aligned}$$

So $(X) \mapsto \Phi_X$ is a group homomorphism. Since Φ_X only depends on $[K \otimes_{\mathcal{O}} X]$, if $X, X' \in \mathcal{D}^b(B \otimes_{\mathcal{O}} B^{\circ})$ are isomorphic, then $\Phi_X = \Phi_{X'}$. As each component of X, Y is projective on the right hand side, Φ_X, Φ_Y preserve exact sequences. So the above map is well defined. \square

We will refer to the homomorphisms in Lemma 4.4.3 as the canonical homomorphisms $\text{DPic}(B), \text{Pic}(B) \longrightarrow \text{PI}(B)$. Actually, we can define the map $\text{Pic}(B) \longrightarrow \text{PI}(B)$ directly, since all the complexes involved are concentrated in degree 0, by

$$(M) \mapsto \Phi_M : [K \otimes_{\mathcal{O}} U] \mapsto [K \otimes_{\mathcal{O}} (M \otimes_B U)]$$

for any B -lattices U . This gives a map which is compatible with the character product defined in (4.1). More specifically, if $M, N \in \text{Pic}(B)$ afford the K -characters μ, λ respectively, then on the character level, $\Phi_{M \otimes_B N} = \Phi_M \circ \Phi_N = I_\mu \circ I_\lambda = I_{\mu \otimes_B \lambda}$. If $X \in \text{DPic}(B)$, then it can be easily checked that

$$\Phi_X = \sum_{i \in \mathbb{Z}} (-1)^i \Phi_{X_i}$$

where X_i is the component in degree i of X . So if each X_i affords a K -character μ_i and X affords the generalized K -character $\mu = \sum_{i \in \mathbb{Z}} (-1)^i \mu_i$ then, on the character level,

$$I_\mu = \sum_{i \in \mathbb{Z}} (-1)^i I_{\mu_i}.$$

If $M \in \text{Pic}(B)$, then the map Φ_M is a perfect isometry that sends irreducible characters to irreducible characters. The image of $\text{Pic}(B)$ in $\text{PI}(B)$ are therefore perfect isometries with all-positive signs. This raises the following question: Given a perfect isometry $I \in \text{PI}(B)$ with all-positive sign, is it always induced by a Morita equivalence? We will see later, when we study blocks with cyclic defect groups, that the answer is no.

The images in $\text{PI}(B)$ of some elements of $\text{Pic}(B)$ can be described as follows. Let M be any (B, B) -bimodule and let $\sigma, \pi \in \text{Aut}_{\mathcal{O}}(B)$. We define the bimodule ${}_{\sigma}M_{\pi}$ to have the same elements as M but with actions of G defined by σ and π :

$$g \cdot m \cdot h := (g^{\sigma})m(h^{\pi}),$$

for all $g, h \in G$, $m \in M$. The following proposition is easily verified.

Proposition 4.4.4. *[13, (55.10)] Let B be an \mathcal{O} -algebra viewed as (B, B) -bimodule. For each $\sigma, \sigma'\pi, \pi', \rho \in \text{Aut}_{\mathcal{O}} B$, there are bimodule isomorphisms*

$${}_{\sigma}B_{\pi} \cong {}_{\rho\sigma}B_{\rho\pi} \cong {}_1B_{\sigma^{-1}\pi} \cong {}_{\pi^{-1}\sigma}B_1,$$

$$\sigma'({}_{\sigma}B_{\pi})_{\pi'} \cong {}_{\sigma'\sigma}B_{\pi\pi'},$$

$${}_1B_{\pi} \otimes_B {}_1B_{\pi'} \cong {}_1B_{\pi\pi'}.$$

Lemma 4.4.5. *The Picard group $\text{Pic}(B)$ contains a subgroup isomorphic to $\text{Out}_{\mathcal{O}}(B)$.*

Proof. Define a map $\text{Aut}_{\mathcal{O}}(B) \longrightarrow \text{Pic}(B)$ by $\sigma \mapsto (B_{\sigma})$. By [13, (55.11)], this map is a group homomorphism with kernel $\text{Inn}_{\mathcal{O}}(B)$. \square

The next lemma further explains Lemma 4.2.4.

Lemma 4.4.6. *Let B be the principal block of $\mathcal{O}G$. Denote by $\widetilde{\text{Out}}_{\mathcal{O}}(B)$ the image of $\text{Out}_{\mathcal{O}}(B)$ (considered as a subgroup of $\text{Pic}(B)$) in $\text{PI}(B)$. Then, with the notations in Lemma 4.2.4,*

$$\widetilde{\mathcal{L}}(B) \rtimes \widetilde{\mathcal{A}}(B) \leq \widetilde{\text{Out}}_{\mathcal{O}}(B).$$

Proof. Let $\Phi : \text{Pic}(B) \longrightarrow \text{PI}(B)$ be the canonical homomorphism. Let $\sigma \in \text{Aut}_{\mathcal{O}}(B)$ and U be a B -lattice. Then

$$\Phi_{B_{\sigma}}([K \otimes_{\mathcal{O}} U]) = [K \otimes_{\mathcal{O}} (B_{\sigma} \otimes_B U)] = [K \otimes_{\mathcal{O}} (\sigma^{-1}B \otimes_B U)] = [K \otimes_{\mathcal{O}} \sigma^{-1}U].$$

So if U affords a K -character χ then $\sigma^{-1}U$ affords the K -character χ^{σ} (where $(\chi^{\sigma}(g) = \chi(g^{\sigma^{-1}}), g \in G)$). So the perfect isometry induced by B_{σ} is $\chi \mapsto \chi^{\sigma}$.

Since B is the principal block, we have $\text{Aut}(G) \leq \text{Aut}_{\mathcal{O}}(B)$. So $\widetilde{\mathcal{A}}(B)$ is the collection of perfect isometries of the form $I_{\sigma} : \chi \mapsto \chi^{\sigma}$ where $\sigma \in \text{Aut}(G) \leq \text{Aut}_{\mathcal{O}}(B)$. But $\sigma \in \text{Aut}(G)$ is in $\text{Inn}_{\mathcal{O}}(B_0)$ if and only if $\sigma \in \text{Inn}(G)$. Thus $\widetilde{\mathcal{A}}(B) \leq \widetilde{\text{Out}}_{\mathcal{O}}(B)$.

Let λ be a linear character in $\text{Irr}(B)$. Define an element $\sigma \in \text{Aut}_{\mathcal{O}}(B)$ by $g^{\sigma} = \lambda(g^{-1})g, \forall g \in G$ and extend linearly. Clearly, $\sigma \in \text{Inn}_{\mathcal{O}}(B)$ if and only if λ is the trivial character. Then B_{σ} induces the map $\chi \mapsto \chi^{\sigma}$ where $\chi^{\sigma}(g) = \chi(g^{\sigma^{-1}}) = \chi(\lambda(g)g) = \lambda(g)\chi(g)$. This is the same as the map $I_{\lambda} : \chi \mapsto \lambda\chi$ (cf. Lemma 4.2.3). So $\widetilde{\mathcal{L}}(B) \leq \widetilde{\text{Out}}_{\mathcal{O}}(B)$. \square

We can also consider the canonical homomorphism $\text{DPic}(B) \longrightarrow \text{PI}(B)$. It is obvious that the homomorphism is never injective as B and $B[2]$ give the identity isometry. Whether or not it is surjective is, however, a more interesting question. A perfect isometry induced by a tilting complex does not have to have a homogenous sign. Therefore, one could be tempted to try to construct a tilting complex inducing a given perfect isometry. We will show that it is always possible to do this in the case

of blocks with defect group C_p . However, when $B = \mathcal{O}G$ where $G = p^{1+2}$ is an extra special group of order p^3 , we will see (Example 5.2.3) that there is a perfect isometry $I : R_K(B) \rightarrow R_K(B)$ that cannot be given by any derived equivalence.

4.5 Factor blocks

This section contains results that will be used later. It allows us, under certain conditions, to work with the quotient group G/N instead of G . Let N be a normal subgroup of a finite group G whose order is coprime to p . Let $\bar{G} = G/N$. Let $\pi : G \rightarrow \bar{G}$ be the quotient map and let $\pi_{\mathcal{O}} : \mathcal{O}G \rightarrow \mathcal{O}\bar{G}$ be the algebra homomorphism induced by π .

Let B be a block of $\mathcal{O}G$. The kernel of B is defined by

$$\text{Ker } B = \{g \in G : \chi(g) = \chi(1), \forall \chi \in \text{Irr}(B)\}.$$

If $N \leq \text{Ker } B$ then by [17, Chapter V, Lemma 4.3], $\bar{B} = \pi_{\mathcal{O}}(B)$ is also a block of $\mathcal{O}\bar{G}$. The map $B \mapsto \bar{B}$ is a bijection between the set of blocks B of $\mathcal{O}G$ with $N \leq \text{Ker } B$ and the set of blocks \bar{B} of $\mathcal{O}\bar{G}$. The characters χ in $\text{Irr}(B)$ are inflation of the characters $\bar{\chi}$ in $\text{Irr}(\bar{B})$ under π , and the map $\chi \mapsto \bar{\chi}$ is a bijection $\text{Irr}(B) \rightarrow \text{Irr}(\bar{B})$.

Lemma 4.5.1. *Let N be a normal subgroup of G whose order is coprime to p . Let $B \in \text{Bl}(G)$. Suppose that $N \leq \text{Ker } B$. Let $I : R_K(\bar{B}) \rightarrow R_K(B)$ be defined by $\bar{\chi} \mapsto \chi$ for $\chi \in \text{Irr}(B)$ and extend linearly. Then I is a perfect isometry. Consequently,*

$$\text{PI}(\bar{B}) \cong \text{PI}(B).$$

Proof. We have $\mu_I \in R_K(B \otimes_{\mathcal{O}} (\bar{B})^{\circ})$ where

$$\begin{aligned} \mu_I(g, \bar{h}) &= \sum_{\bar{\chi} \in \text{Irr}(\bar{B})} I(\bar{\chi})(g) \bar{\chi}(\bar{h}) = \sum \chi(g) \bar{\chi}(\bar{h}) \\ &= \sum \bar{\chi}(\bar{g}) \bar{\chi}(\bar{h}) \\ &= \mu_{id}(\bar{g}, \bar{h}). \end{aligned}$$

Since N is a p' -group, gN is a p -regular element if and only if g is a p -regular element. Thus μ_I satisfies the separation condition. Since μ_{id} is perfect, we have

$\mu_{id}(\bar{g}, \bar{h})/(|C_{\bar{G}}(\bar{g})||C_N(g)|) \in \mathcal{O}$ as $1/|C_N(g)| \in \mathcal{O}$. Since $|C_G(g)|$ divides $|C_{\bar{G}}(\bar{g})||C_N(g)|$, $\mu_{id}(\bar{g}, \bar{h})/|C_G(g)| \in \mathcal{O}$. Also $\mu_{id}(\bar{g}, \bar{h})/|C_{\bar{G}}(\bar{h})| \in \mathcal{O}$. So μ_I satisfies the integrality condition. The last statement follows from Proposition 4.0.7. \square

Corollary 4.5.2. *Suppose G contains a normal p' -subgroup $N \triangleleft G$ such that G/N is a p -group. Let $B \in \text{Bl}(G)$ and suppose that $N \leq \text{Ker } B$. Then*

$$\text{PI}(B) \cong \text{PI}(\mathcal{O}(G/N)).$$

Proof. Since G/N is a p -group, it has only one block, the principal block. So the corresponding block \bar{B} of G/N is this block. The result now follows from the previous lemma. \square

4.6 Isometries extended from a normal subgroup

Let G be a finite group and N a normal subgroup of G . Let B, A be blocks of $\mathcal{O}G$ and $\mathcal{O}N$ respectively, where B covers A . In this section we will investigate isometries $R_K(B) \rightarrow R_K(B)$ that, in some sense, extend perfect isometries $R_K(A) \rightarrow R_K(A)$. Under suitable conditions, this allows us to construct perfect isometries in $\text{PI}(B)$ from known perfect isometries in $\text{PI}(A)$.

4.6.1 Direct Product

Let $G = N \times P$ where P is a p -group. Let B, A be the principal blocks of $\mathcal{O}G$ and $\mathcal{O}N$ respectively. Given perfect isometries $\bar{I} \in \text{PI}(A)$ and $J \in \text{PI}(\mathcal{O}P)$, we will show how to construct a perfect isometry $I \in \text{PI}(B)$ from \bar{I} and J .

Let $\theta \in \text{Irr}(A)$. Then θ extends to an irreducible character $E(\theta) \in \text{Irr}(B)$. Since irreducible characters of G are of the form $\varphi\chi$ where $\varphi \in \text{Irr}(N)$ and $\chi \in \text{Irr}(P)$, we can take $E(\theta) = \theta\mathbf{1}$. Then by [24, Corollary 6.17],

$$\{\chi \in \text{Irr}(B) : \chi \downarrow_N = \theta\} = \{WE(\theta) : W \in \text{Irr}(G/N)\},$$

where $WE(\theta)(g) = W(gN)E(\theta)(g)$, $g \in G$. So,

$$\text{Irr}(B) = \{WE(\theta) : W \in \text{Irr}(G/N), \theta \in \text{Irr}(A)\}.$$

Theorem 4.6.1. *Let $\bar{I} \in \text{PI}(A)$ and $J \in \text{PI}(\mathcal{O}(G/N))$. Define an isometry $I : R_K(B) \rightarrow R_K(B)$ by*

$$I : WE(\theta) \mapsto \varepsilon(\theta)J(W)E(\sigma(\theta)), \quad W \in \text{Irr}(G/N), \theta \in \text{Irr}(A), \quad (4.4)$$

where $\bar{I}(\theta) = \varepsilon(\theta)\sigma(\theta)$ for a sign function ε and a bijection σ on $\text{Irr}(A)$. Then $I \in \text{PI}(B)$. Furthermore, if $J : \text{Irr}(G/N) \rightarrow \text{Irr}(G/N)$ then $I(\theta \uparrow^G) = \bar{I}(\theta) \uparrow^G$ for all $\theta \in \text{Irr}(A)$.

Proof. Let $\mu \in R_K(B \otimes B^\circ)$ be the character induced by I . For $g, h \in G$,

$$\begin{aligned} \mu(g, h) &= \sum_{\theta} \sum_W I(WE(\theta))(g)WE(\theta)(h) \\ &= \sum_{\theta} \sum_W \varepsilon(\theta)J(W)(gN)E(\sigma(\theta))(g)W(hN)E(\theta)(h) \\ &= \left[\sum_{\theta} \varepsilon(\theta)E(\sigma(\theta))(g)E(\theta)(h) \right] \left[\sum_W J(W)(gN)W(hN) \right]. \end{aligned}$$

For convenience, we will denote

$$\left[\sum_{\theta} \varepsilon(\theta)E(\sigma(\theta))(g)E(\theta)(h) \right]$$

by $[\sum_{\theta}]$ and

$$\left[\sum_W J(W)(gN)W(hN) \right]$$

by $[\sum_W]$. Suppose $\mu(g, h) \neq 0$ then $[\sum_{\theta}] \neq 0$ and $[\sum_W] \neq 0$. We will show that g, h are both p -singular or both p -regular elements and hence satisfying the separation condition.

Since J is perfect, $[\sum_W] \neq 0$ implies that gN, hN are both p -regular elements or both p -singular elements of G/N . If gN, hN are both p -singular elements of G/N , then $g, h \notin N$ as G/N is a p -group. Since $G = N \times P$, this implies that g, h are both p -singular elements of G .

On the other hand if gN, hN are both p -regular elements of G/N , then $g, h \in N$. In this case

$$\begin{aligned} \sum_{\theta} \varepsilon(\theta)E(\sigma(\theta))(g)E(\theta)(h) &= \sum_{\theta} \varepsilon(\theta)\sigma(\theta)(g)\theta(h) \\ &= \sum_{\theta} \bar{I}(\theta)(g)\theta(h). \end{aligned}$$

Since \bar{I} is perfect and $[\sum_{\theta}] \neq 0$ this implies that g, h are both p -singular elements or both p -regular elements of G .

To prove the integrality condition, let $g = n_1a, h = n_2b$ where $n_1, n_2 \in N$ and $a, b \in P$. By our choice of E , $E(\sigma(\theta))(g) = \sigma(\theta)(n_1)$, $E(\theta)(h) = \theta(n_2)$. So

$$\begin{aligned} \mu(g, h) &= \left[\sum_{\theta} \varepsilon(\theta) \sigma(\theta)(n_1) \theta(n_2) \right] \left[\sum_W J(W)(aN)W(bN) \right] \\ &= \left[\sum_{\theta} \bar{I}(\theta)(n_1) \theta(n_2) \right] \left[\sum_W J(W)(aN)W(bN) \right]. \end{aligned}$$

For convenience, denote

$$\left[\sum_{\theta} \bar{I}(\theta)(n_1) \theta(n_2) \right]$$

by $[\sum_{\theta}]$ and

$$\left[\sum_W J(W)(aN)W(bN) \right]$$

by $[\sum_W]$. Since \bar{I} is perfect, $[\sum_{\theta}]$ is in $|C_N(n_1)|\mathcal{O}$ and $|C_N(n_2)|\mathcal{O}$. Since J is perfect, $[\sum_W]$ is in $|C_{G/N}(aN)|\mathcal{O}$ and $|C_{G/N}(bN)|\mathcal{O}$. But

$$\begin{aligned} |C_G(n_1a)| &= |C_N(n_1)| \times |C_P(a)| \\ &= |C_N(n_1)| \times |C_{G/N}(aN)|. \end{aligned}$$

Therefore $\mu(g, h) \in |C_G(g)|\mathcal{O}$. Similarly $\mu(g, h) \in |C_G(h)|\mathcal{O}$. This proves the integrality condition.

Finally, suppose that $J : \text{Irr}(G/N) \rightarrow \text{Irr}(G/N)$. Then, for $\theta \in \text{Irr}(A)$,

$$\begin{aligned} I((\theta) \uparrow^G) &= I \left(\sum_{W \in \text{Irr}(G/N)} WE(\theta) \right) \\ &= \sum_{W \in \text{Irr}(G/N)} \varepsilon(\theta) J(W) E(\sigma(\theta)) \\ &= \varepsilon(\theta) \sum_{W \in \text{Irr}(G/N)} WE(\sigma(\theta)) \\ &= \varepsilon(\theta) \sigma(\theta) \uparrow^G \\ &= \bar{I}(\theta) \uparrow^G. \end{aligned}$$

□

4.6.2 Isometries commuting with induction

Let G be a finite group and N a normal subgroup of G . Let B be a block of $\mathcal{O}G$ covering a block A of $\mathcal{O}N$. In Theorem 4.6.1 we have seen a perfect isometry $I \in \text{PI}(B)$ such that $I(\theta \uparrow^G) = \tilde{I}(\theta) \uparrow^G, \forall \theta \in \text{Irr}(A)$, where $\tilde{I} \in \text{PI}(A)$. Therefore, given a perfect isometry $\tilde{I} \in \text{PI}(A)$, one could try to construct a perfect isometry $I \in \text{PI}(B)$ making the following diagram commutative.

$$\begin{array}{ccc} R_K(B) & \xrightarrow{I} & R_K(B) \\ \text{Induction} \uparrow & & \uparrow \text{Induction} \\ R_K(A) & \xrightarrow{\tilde{I}} & R_K(A) \end{array} \quad (4.5)$$

In this section we will look at isometries $R_K(B) \rightarrow R_K(B)$ with the above property and show that, under suitable conditions, such isometries already satisfy the separation condition.

Remark 4.6.2. If we take $\tilde{I} = id \in \text{PI}(A)$, then the set of perfect isometries $I \in \text{PI}(B)$ making the diagram (4.5) commutative forms a subgroup of $\text{PI}(B)$. These are perfect isometries preserving inductions from N , that is $I(\theta \uparrow^G) = \theta \uparrow^G, \forall \theta \in \text{Irr}(A)$.

Lemma 4.6.3. *Let $N \triangleleft G$. Let B be a block of $\mathcal{O}G$ covering a block A of $\mathcal{O}N$. If $\alpha \in \text{CF}(N, A; K)$ vanishes on p -singular elements. Then $\alpha \uparrow^G \in \text{CF}(G, B; K)$ vanishes on p -singular elements.*

Proof. First note that since B covers A , $\alpha \uparrow^G \in \text{CF}(G, B; K)$. Let $g \in G$ be a p -singular element. If $g \notin N$ then $\alpha \uparrow^G(g) = 0$ by definition. If $g \in N$ then $\alpha \uparrow^G(g) = \frac{1}{|N|} \sum_{x \in G} \alpha(x^{-1}gx)$. But $x^{-1}gx$ is a p -singular element for all $x \in G$. Thus $\alpha \uparrow^G(g) = 0$, proving the lemma. \square

For an irreducible Brauer character φ of G , let Φ_φ denote the *projective indecomposable character* of G associated to φ .

Proposition 4.6.4. *Let $N \triangleleft G$. Let B be a block of $\mathcal{O}G$ covering a block A of $\mathcal{O}N$. Assume that for all $\varphi \in \text{IBr}(B)$ there exists $\tilde{\varphi} \in \text{IBr}(A)$ such that $\Phi_\varphi = (\Phi_{\tilde{\varphi}}) \uparrow^G$. Let $\tilde{I} \in \text{PI}(A)$ and $I : R_K(B) \rightarrow R_K(B)$ be an isometry making the diagram (4.5) commutative. Then I satisfies the separation condition.*

Proof. Since $\{\Phi_\varphi : \varphi \in \text{IBr}(B)\}$ is a K -basis of $\text{CF}_{p'}(G, B; K)$, it suffices to show that $I(\Phi_\varphi) \in \text{CF}_{p'}(G, B; K)$, $\forall \varphi \in \text{IBr}(B)$. Let $\varphi \in \text{IBr}(B)$ then, by the assumption, $\exists \tilde{\varphi} \in \text{IBr}(A)$ such that $\Phi_\varphi = (\Phi_{\tilde{\varphi}}) \uparrow^G$. If $\{d_{\tilde{\chi}\tilde{\varphi}} : \tilde{\chi} \in \text{Irr}(A), \tilde{\varphi} \in \text{IBr}(A)\}$ are decomposition numbers, then

$$\begin{aligned} I(\Phi_\varphi) &= I((\Phi_{\tilde{\varphi}}) \uparrow^G) = I\left(\sum_{\tilde{\chi} \in \text{Irr}(A)} d_{\tilde{\chi}\tilde{\varphi}} \tilde{\chi} \uparrow^G\right) \\ &= \sum_{\tilde{\chi}} d_{\tilde{\chi}\tilde{\varphi}} I(\tilde{\chi} \uparrow^G) = \sum_{\tilde{\chi}} d_{\tilde{\chi}\tilde{\varphi}} \tilde{I}(\tilde{\chi}) \uparrow^G \\ &= \tilde{I}\left(\sum_{\tilde{\chi}} d_{\tilde{\chi}\tilde{\varphi}} \tilde{\chi}\right) \uparrow^G \\ &= \tilde{I}(\Phi_{\tilde{\varphi}}) \uparrow^G. \end{aligned}$$

Since \tilde{I} is perfect and $\Phi_{\tilde{\varphi}} \in \text{CF}_{p'}(N, A; K)$, we have $\tilde{I}(\Phi_{\tilde{\varphi}}) \in \text{CF}_{p'}(N, A; K)$. Thus by the previous lemma, $\tilde{I}(\Phi_{\tilde{\varphi}}) \uparrow^G \in \text{CF}_{p'}(G, B; K)$. Hence I satisfies the separation condition. \square

Example 4.6.5. Let $G = C_{p^2}$ and $N = C_p \triangleleft G$. In this case, $\text{IBr}(A) = \{\tilde{\mathbf{1}}\}$ and $\text{IBr}(B) = \{\mathbf{1}\}$ and $\Phi_{\mathbf{1}} = (\Phi_{\tilde{\mathbf{1}}}) \uparrow^G$. Suppose $G = \langle g \rangle$ and $N = \langle g^p \rangle$. Let $\lambda \in \text{Irr}(G)$ where $\lambda(g) = e^{2\pi i/p}$. Then $\lambda(h) = 1, \forall h \in N$. So the isometry $I : \chi \mapsto \lambda\chi, \forall \chi \in \text{Irr}(G)$ satisfies $I(\theta \uparrow^G) = \theta \uparrow^G, \forall \theta \in \text{Irr}(N)$. So, by Proposition 4.6.4, I satisfies the separation condition. (In fact we already know from Lemma 3.3.5 that I is a perfect isometry.)

Another situation where we can apply Proposition 4.6.4 is the following [22].

Example 4.6.6. Let N be a normal subgroup of a finite group G such that G/N is a p -group. Let A be a G -invariant block of $\mathcal{O}N$. Let B be a block of $\mathcal{O}G$ covering A . By [22, Lemma 2.1], B is unique.

- Suppose that the map $\text{IBr}(B) \rightarrow \text{IBr}(A)$ defined by $\varphi \mapsto \varphi \downarrow_N$ is a bijection.

Then

$$\Phi_\varphi = (\Phi_{\tilde{\varphi}}) \uparrow^G,$$

where $\tilde{\varphi} = \varphi \downarrow_N$ [22, Lemma 2.4].

- Suppose that $|\mathrm{IBr}(A)| < p$. Then the map $\mathrm{IBr}(B) \longrightarrow \mathrm{IBr}(A)$ defined by $\varphi \mapsto \varphi \downarrow_N$ is a bijection [22, Lemma 2.2].

Chapter 5

p -groups

In this chapter we will study perfect isometries in blocks of p -groups. An interesting feature of perfect isometries in these group is that they always have homogenous signs. It therefore follows from Lemma 4.1.6 that we can decompose the perfect isometry group as the product of subgroup of perfect isometries with all-positive signs and the subgroup generated by $-id$.

Lemma 5.0.7. *Let G be a p -group and $B = \mathcal{O}G$. Let $I \in \text{PI}(B)$ be a perfect isometry. Then*

(i) *I has homogenous sign.*

(ii) *Either $I(\chi)(1) = \chi(1) \forall \chi \in \text{Irr}(G)$ or $I(\chi)(1) = -\chi(1) \forall \chi \in \text{Irr}(G)$.*

Proof. Suppose $|G| = p^n$. If $\chi \in \text{Irr}(B)$, we know that $\frac{I(\chi)(1)}{\chi(1)}$ is an invertible element in \mathcal{O} , by Lemma 3.2.5. Since both $I(\chi)(1)$ and $\chi(1)$ are powers of p , we must have $I(\chi)(1) = \pm\chi(1)$. Let $\mu_I \in R_K(B \otimes_{\mathcal{O}} B^{\circ})$ be the generalized character induced by I . Consider

$$\frac{\mu_I(1, 1)}{|G|} = \frac{\sum_{\chi \in \text{Irr}(G)} I(\chi)(1)\chi(1)}{p^n} = \frac{\sum_{\chi \in \text{Irr}(G)} (\pm\chi(1)^2)}{p^n}.$$

Since $\mu_I(1, 1)/|G| \in \mathcal{O}$ and $|\sum_{\chi \in \text{Irr}(G)} (\pm\chi(1)^2)| \leq p^n$ this means that either

- $\sum_{\chi \in \text{Irr}(G)} (\pm\chi(1)^2) = p^n$ in which case all the signs are positive, or
- $\sum_{\chi \in \text{Irr}(G)} (\pm\chi(1)^2) = -p^n$ in which case all the signs are negative.

This proves (i), and (ii) follows from $I(\chi)(1) = \pm\chi(1)$ by above. \square

5.1 Abelian p -groups

Since the most basic p -groups are the abelian ones, it is natural to start with these groups. In this section, we will find perfect isometry groups for abelian p -groups. Then we will deduce a result for the general abelian groups, using Corollary 4.5.2.

Let G be an abelian p -group. We know from Lemma 4.3.4 that every perfect isometry $I \in \text{PI}(\mathcal{O}G)$ gives rise to an, essentially unique, automorphism in $\text{Aut}_{\mathcal{O}}(Z\mathcal{O}G)$. We shall see that the converse also holds in the abelian case. This enables us to prove the following main result.

Theorem 5.1.1. *Let G be an abelian p -group. Then every perfect isometry has a homogenous sign and*

$$\text{PI}(\mathcal{O}G) \cong (G \rtimes \text{Aut}(G)) \times \langle -id \rangle.$$

Furthermore, if $\text{Irr}(G) = \{\chi_1, \dots, \chi_{|G|}\}$, then $G \leq \text{PI}(\mathcal{O}G)$ is the subgroup of perfect isometries $\chi_i \mapsto \lambda\chi_i, \forall i$ for some $\lambda \in \text{Irr}(G)$ and $\text{Aut}(G) \leq \text{PI}(\mathcal{O}G)$ is the subgroup of perfect isometries $\chi_i \mapsto \chi_i^\sigma, \forall i$ for some $\sigma \in \text{Aut}(G)$.

Lemma 5.1.2. *Let G be a p -group. Then*

$$\text{Out}_{\mathcal{O}} \mathcal{O}G \cong \text{Hom}(G, \mathcal{O}^\times) \rtimes \text{Out}(G).$$

Proof. We follow the remarks before Corollary 4 in [41]. There is an injective homomorphism $\text{Hom}(G, \mathcal{O}^\times) \longrightarrow \text{Aut}_{\mathcal{O}}(\mathcal{O}G)$ given by $\lambda \mapsto (\tilde{\lambda} : g \mapsto \lambda(g)g)$. An image is inner if and only if it is trivial. Let $n\text{Out}_{\mathcal{O}}(\mathcal{O}G)$ be the image in $\text{Out}_{\mathcal{O}}(\mathcal{O}G)$ of the normalized (augmentation-preserving) automorphisms of $\mathcal{O}G$. Then we can write

$$\text{Out}_{\mathcal{O}}(\mathcal{O}G) \cong \text{Hom}(G, \mathcal{O}^\times) \cdot n\text{Out}_{\mathcal{O}}(\mathcal{O}G).$$

Since G is a p -group, no prime divisor of $|G|$ is invertible in \mathcal{O} . Since G is a p -group, it is well-known that G is nilpotent. So, [41, Corollary 3] gives

$$n\text{Out}_{\mathcal{O}}(\mathcal{O}G) = (\text{Outcent}(\mathcal{O}G))(\text{Out}(G)),$$

where $\text{Outcent}(\mathcal{O}G) = \text{Out}_{Z(\mathcal{O}G)}(\mathcal{O}G)$, regarding $\mathcal{O}G$ as an algebra over its center $Z(\mathcal{O}G)$.

To find $\text{Outcent } \mathcal{O}G$, we first define $\text{Outcent}(G) = (\text{Outcent}(\mathcal{O}G)) \cap \text{Out}(G)$. Then [41, Corollary 4] implies that $\text{Outcent}(\mathcal{O}G) = \text{Outcent}(G)$. By definition of $\text{Outcent}(G)$, this means that $\text{Outcent}(\mathcal{O}G) = (\text{Outcent}(\mathcal{O}G)) \cap \text{Out}(G)$. This implies that $\text{Outcent}(\mathcal{O}G) \subseteq \text{Out}(G)$, and so $n \text{Out}_{\mathcal{O}}(\mathcal{O}G) = \text{Out}(G)$.

Finally, the group $\text{Out}(G)$ acts on $\text{Hom}(G, \mathcal{O}^\times)$ by $\lambda^\sigma(g) = \lambda(g^{\sigma^{-1}})$ for $\lambda \in \text{Hom}(G, \mathcal{O}^\times)$, $\sigma \in \text{Out}(G)$, $g \in G$, and so $(\widetilde{\lambda})^\sigma = \widetilde{(\lambda)^\sigma}$. With this action, we have $\text{Out}_{\mathcal{O}} \mathcal{O}G \cong \text{Hom}(G, \mathcal{O}^\times) \rtimes (\text{Out}(G))$ as claimed. \square

Corollary 5.1.3. *Let G be an abelian p -group. Then*

$$\text{Aut}_{\mathcal{O}}(Z\mathcal{O}G) \cong G \rtimes \text{Aut}(G).$$

Proof. When G is abelian, we have the following:

- $\text{Aut}_{\mathcal{O}}(Z\mathcal{O}G) = \text{Aut}_{\mathcal{O}}(\mathcal{O}G)$,
- $\text{Aut}_{\mathcal{O}}(\mathcal{O}G) = \text{Out}_{\mathcal{O}}(\mathcal{O}G)$,
- $\text{Hom}(G, \mathcal{O}^\times) \cong G$,
- $\text{Out}(G) = \text{Aut}(G)$.

Applying the previous lemma gives the result. \square

Proof of Theorem 5.1.1. Because G is a p -group, all perfect isometries have homogeneous signs. So we can write

$$\text{PI}(\mathcal{O}G) = S \times \langle -\text{id} \rangle,$$

where S is a subgroup consisting of all perfect isometries with all-positive signs. Let \mathcal{L} be the multiplicative group of $\text{Irr}(G)$. Let $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{A}}$ be the images of the maps $\mathcal{L} \rightarrow \text{PI}(\mathcal{O}G)$ and $\text{Aut}(G) \rightarrow \text{PI}(\mathcal{O}G)$ respectively (see Section 4.2). Then, by Lemma 4.2.4, S contains $\widetilde{\mathcal{L}} \rtimes \widetilde{\mathcal{A}}$. It is clear that we have $\widetilde{\mathcal{L}} \cong G$. Since G is an abelian p -group, by Remark 4.2.2, we have $\widetilde{\mathcal{A}} \cong \text{Aut}(G)$. Hence, by Lemma 4.2.4,

$$\text{PI}(\mathcal{O}G) \geq (G \rtimes \text{Aut}(G)) \times \langle -\text{id} \rangle.$$

On the other hand, Corollary 4.3.5 together with Corollary 5.1.3 imply that

$$|\text{PI}(\mathcal{O}G)| \leq 2|G| |\text{Aut}(G)|.$$

Hence we have $\text{PI}(\mathcal{O}G) \cong (G \rtimes \text{Aut}(G)) \times \langle -\text{id} \rangle$. The subgroups of $\text{PI}(\mathcal{O}G)$ isomorphic to G and $\text{Aut}(G)$ are already described in Section 4.2. \square

The perfect isometry group for a general abelian group can be easily deduced.

Theorem 5.1.4. *Let G be a finite abelian group. Let $B \in \text{Bl}(G)$. Then*

$$\text{PI}(B) \cong (O_p(G) \rtimes \text{Aut}(O_p(G))) \times \langle -\text{id} \rangle.$$

Proof. It suffices to assume that B is the principal block. Since G is abelian, by the fundamental theorem of finite abelian groups we can write

$$G = O_p(G) \times H,$$

where H is an abelian group whose order is coprime to p . By Corollary 4.5.2, we have $\text{PI}(B) \cong \text{PI}(\mathcal{O}(G/H))$ and so, by Theorem 5.1.1,

$$\text{PI}(B) \cong (G/H \rtimes \text{Aut}(G/H)) \times \langle -\text{id} \rangle.$$

\square

We will finish this section with the following useful lemma.

Lemma 5.1.5. *Let G be an abelian p -group. Let $I : R_K(\mathcal{O}G) \rightarrow R_K(\mathcal{O}G)$ be an isometry with a homogenous sign. If $I = I_\mu$ then μ satisfies the separation condition. Hence, in order to check if μ is perfect, we only need to check $\mu(g, h)$ where g, h are non-identity.*

Proof. Let $1 \neq g \in G$. Since $\chi(1) = 1$ for all $\chi \in \text{Irr}(G)$,

$$\begin{aligned} \mu_I(1, g) &= \sum_{\chi \in \text{Irr}(G)} I(\chi)(1)\chi(g) = (\pm 1) \sum_{\chi \in \text{Irr}(G)} \chi(g) = 0 \\ \mu_I(g, 1) &= \sum_{\chi \in \text{Irr}(G)} I(\chi)(g)\chi(1) = (\pm 1) \sum_{\chi \in \text{Irr}(G)} \chi(g) = 0. \end{aligned}$$

Since 1 is the only p -regular element of G , μ satisfies the separation condition. The last statement follows from Theorem 3.1.5. \square

5.2 Extra special p -groups

In this section, we will find perfect isometry groups of extra special p -groups. Recall that a p -group G is called *extra special* if the center $Z(G)$ is cyclic of order p and the quotient G/Z is a non-trivial elementary abelian p -group. The main result for this section is the following theorem.

Theorem 5.2.1. *Let G be an extra special p -group of order p^{1+2n} . Let $Z = Z(G)$ be the center of G . Then we can write*

$$\text{Irr}(G) = \{\lambda_1, \dots, \lambda_{p^{2n}}\} \cup \{\chi_1, \dots, \chi_{p-1}\}$$

where $\{\lambda_1, \dots, \lambda_{p^{2n}}\}$ can be identified with $\text{Irr}(G/Z)$ and $\{\chi_1, \dots, \chi_{p-1}\}$ can be identified with $\text{Irr}(Z) - \{\mathbf{1}\}$. With these identifications, every perfect isometry $\text{Irr}(G) \rightarrow \text{Irr}(G)$ is of the form $\bar{I} \circ \bar{J}$ where $\bar{I} \in \text{PI}(\mathcal{O}(G/Z))$ permutes only the characters $\{\lambda_1, \dots, \lambda_{p^{2n}}\}$ in $\text{Irr}(G)$ and $\bar{J} \in \text{PI}(\mathcal{O}Z)$ fixes $\mathbf{1} \in \text{Irr}(Z)$ and permutes only the characters $\{\chi_1, \dots, \chi_{p-1}\}$ in $\text{Irr}(G)$. In particular,

$$\text{PI}(\mathcal{O}G) \cong (G/Z \rtimes \text{Aut}(G/Z)) \times C_{p-1} \times \langle -id \rangle.$$

5.2.1 Irreducible characters of G

First we summarize results about the irreducible characters of G . The details of the proofs can be found in [21, Lemma 3]. Let $Z = Z(G)$, $\bar{G} = G/Z$ and $g \mapsto \bar{g} = gZ$ be the canonical projection map. Let $\bar{\lambda}_1, \dots, \bar{\lambda}_{p^{2n}}$ be the irreducible characters of \bar{G} . Then there are irreducible characters $\lambda_1, \dots, \lambda_{p^{2n}}$ of G such that, for each i ,

$$\lambda_i(g) = \bar{\lambda}_i(\bar{g}),$$

for all $g \in G$. Let $\hat{\chi}_1, \dots, \hat{\chi}_{p-1}$ be the non-trivial irreducible characters of Z . For each i , we extend $\hat{\chi}_i$ to G by setting $\hat{\chi}_i(g) = 0$ if $g \notin Z$. Then there are irreducible characters $\chi_1, \dots, \chi_{p-1}$ of G such that, for each i ,

$$\chi_i(g) = p^n \hat{\chi}_i(g),$$

for all $g \in G$. We then have

$$\text{Irr}(G) = \{\lambda_1, \dots, \lambda_{p^{2n}}\} \cup \{\chi_1, \dots, \chi_{p-1}\},$$

where $\{\lambda_1, \dots, \lambda_{p^{2n}}\}$ can be identified with $\text{Irr}(G/Z)$ and $\{\chi_1, \dots, \chi_{p-1}\}$ can be identified with $\text{Irr}(Z) - \{1\}$.

Table 5.1: Character table of G

| G | $\{1\}$ | $g \in Z - \{1\}$ | $g \in G - Z$ |
|--------------------|----------|-----------------------|----------------------------|
| λ_1 | 1 | 1 | 1 |
| \vdots | \vdots | \vdots | |
| λ_i | 1 | 1 | $\bar{\lambda}_i(\bar{g})$ |
| \vdots | \vdots | \vdots | |
| $\lambda_{p^{2n}}$ | 1 | 1 | |
| χ_1 | p^n | | 0 |
| \vdots | \vdots | | \vdots |
| χ_j | p^n | $p^n \hat{\chi}_j(g)$ | 0 |
| \vdots | \vdots | | \vdots |
| χ_{p-1} | p^n | | 0 |

5.2.2 The proof

We will now prove Theorem 5.2.1. Since G is a p -group, all perfect isometries have homogenous signs. It is sufficient to prove that the subgroup of $\text{PI}(\mathcal{O}G)$ generated by perfect isometries with all-positive signs is isomorphic to $(G/Z \rtimes \text{Aut}(G/Z)) \times C_{p-1}$. We denote generic characters in $\{\lambda_1, \dots, \lambda_{p^{2n}}\}$ and $\{\chi_1, \dots, \chi_{p-1}\}$ by λ and χ respectively and $\bar{\lambda}, \hat{\chi}$ the corresponding characters in G/Z and Z .

Let $\bar{I} \in \text{PI}(\mathcal{O}(G/Z))$ be a perfect isometry with all-positive sign and $\hat{J} \in \text{PI}(\mathcal{O}Z)$ be a perfect isometry with all-positive sign such that \hat{J} fixes the trivial character of Z . Then we can define an isometry $I : \text{Irr}(G) \rightarrow \text{Irr}(G)$ as follows.

$$I(\lambda)(g) = \bar{I}(\bar{\lambda})(\bar{g}) \tag{5.1a}$$

$$I(\chi)(g) = p^n \hat{J}(\hat{\chi})(g) \tag{5.1b}$$

for all $g \in G$.

On the other hand, since $\lambda(1)_p = 1$ and $\chi(1)_p = p^n$, every perfect isometry in $\text{PI}(\mathcal{O}G)$ must preserve the sets $\{\lambda_1, \dots, \lambda_{p^{2n}}\}$ and $\{\chi_1, \dots, \chi_{p-1}\}$ up to signs. Let $I \in \text{PI}(\mathcal{O}G)$ be a perfect isometry with all-positive sign ($I : \text{Irr}(G) \rightarrow \text{Irr}(G)$), then it can be written as in (5.1) for some isometries $\bar{I} : \text{Irr}(G/Z) \rightarrow \text{Irr}(G/Z)$ and $\hat{J} : \text{Irr}(Z) \rightarrow \text{Irr}(Z)$ where \hat{J} fixes the trivial character.

Since G/Z is an abelian p -group and $Z \cong C_p$, we know from Theorem 5.1.1 that the group of all-positive-sign perfect isometries $\bar{I} \in \text{PI}(\mathcal{O}(G/Z))$ is $G/Z \rtimes \text{Aut}(G/Z)$ and that the subgroup of perfect isometries $\hat{J} \in \text{PI}(\mathcal{O}Z)$ fixing the trivial character is $\text{Aut}(Z) \cong C_{p-1}$. It remains to show that I is a perfect isometry if and only if \bar{I} and \hat{J} are perfect isometries.

In order to simplify the proof we first make some observations and pre-calculations. Let $g \in G$. The values of $|C_G(g)|$, $|C_{\bar{G}}(\bar{g})|$ and $|C_Z(g)|$ can be summarized in the following table.

| $g \in G$ | $ C_G(g) $ | $ C_{\bar{G}}(\bar{g}) $ | $ C_Z(g) $ |
|-------------------|------------|--------------------------|------------|
| $g = 1$ | p^{2n+1} | p^{2n} | p |
| $g \in Z - \{1\}$ | p^{2n+1} | p^{2n} | p |
| $g \notin Z$ | p^{2n} | p^{2n} | p |

Note that if $g \notin Z$ then, by the second orthogonality of characters,

$$\begin{aligned}
 |C_G(g)| &= \sum_{\phi \in \text{Irr}(G)} |\phi(g)|^2 \\
 &= \sum_{\lambda} |\lambda(g)|^2 + \sum_{\chi} |\chi(g)|^2 \\
 &= \sum_{\lambda} |\lambda(g)|^2 = \sum_{\bar{\lambda}} |\bar{\lambda}(\bar{g})|^2 \\
 &= |C_{\bar{G}}(\bar{g})| = p^{2n}.
 \end{aligned}$$

Let I be an isometry as in (5.1) and $\mu_I \in R_K(\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G^{\circ})$ be the corresponding

character. If $g, h \in Z$, then

$$\begin{aligned}\mu_I(g, h) &= \sum_{\lambda} I(\lambda)(g)\lambda(h) + \sum_{\chi} I(\chi)(g)\chi(h) \\ &= \sum_{\bar{\lambda}} \bar{I}(\bar{\lambda})(\bar{g})\bar{\lambda}(\bar{h}) + p^{2n} \sum_{\hat{\chi} \neq \mathbf{1}} \hat{J}(\hat{\chi})(g)\hat{\chi}(h) \\ &= \mu_{\bar{I}}(\bar{g}, \bar{h}) + p^{2n}(\mu_{\hat{J}}(g, h) - 1),\end{aligned}$$

whereas if $g \notin Z$ or $h \notin Z$ or both then

$$\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}).$$

If $g = h = 1$ then

$$\mu_{\hat{J}}(g, h) = \sum_{\hat{\chi}} \hat{J}(\hat{\chi})(1)\hat{\chi}(1) = p.$$

If $g = 1, h \in Z - \{1\}$ or $g \in Z - \{1\}, h = 1$ then

$$\mu_{\hat{J}}(g, h) = \sum_{\hat{\chi}} \hat{J}(\hat{\chi})(g)\hat{\chi}(g) = 0.$$

If $\bar{g} = \bar{h} = 1$ in \bar{G} then

$$\mu_{\bar{I}}(\bar{g}, \bar{h}) = \sum_{\bar{\lambda}} \bar{I}(\bar{\lambda})(1)\bar{\lambda}(1) = p^{2n}.$$

If $\bar{g} = 1, \bar{h} \neq 1$ in \bar{G} or $\bar{g} \neq 1, \bar{h} = 1$ in \bar{G} , then

$$\mu_{\bar{I}}(\bar{g}, \bar{h}) = \sum_{\hat{\chi}} \hat{J}(\hat{\chi})(\bar{g})\hat{\chi}(\bar{h}) = 0.$$

The following table summarizes computable values of $\mu_{\bar{I}}(\bar{g}, \bar{h})$, $\mu_{\hat{J}}(g, h)$ and $\mu_I(g, h)$. These values do not depend on the isometries \bar{I} , \hat{J} and I (as long as they take the form as in (5.1)).

| $g \in G$ | $h \in G$ | $\mu_{\bar{I}}(\bar{g}, \bar{h})$ | $\mu_j(g, h)$ | $\mu_I(g, h)$ | $\mu_I =$ |
|-------------------|-------------------|-----------------------------------|---------------|---------------|-------------------------------------|
| $g = 1$ | $h = 1$ | p^{2n} | p | p^{2n+1} | $\mu_{\bar{I}} + p^{2n}(\mu_j - 1)$ |
| $g = 1$ | $h \in Z - \{1\}$ | p^{2n} | 0 | 0 | $\mu_{\bar{I}} + p^{2n}(\mu_j - 1)$ |
| $g = 1$ | $h \notin Z$ | 0 | | 0 | $\mu_{\bar{I}}$ |
| $g \in Z - \{1\}$ | $h = 1$ | p^{2n} | 0 | 0 | $\mu_{\bar{I}} + p^{2n}(\mu_j - 1)$ |
| $g \in Z - \{1\}$ | $h \in Z - \{1\}$ | p^{2n} | | | $\mu_{\bar{I}} + p^{2n}(\mu_j - 1)$ |
| $g \in Z - \{1\}$ | $h \notin Z$ | 0 | | 0 | $\mu_{\bar{I}}$ |
| $g \notin Z$ | $h = 1$ | 0 | | 0 | $\mu_{\bar{I}}$ |
| $g \notin Z$ | $h \in Z - \{1\}$ | 0 | | 0 | $\mu_{\bar{I}}$ |
| $g \notin Z$ | $h \notin Z$ | | | | $\mu_{\bar{I}}$ |

Proof that if I is perfect then \bar{I} is perfect

Since \bar{G} is an abelian p -group and \bar{I} has the all-positive sign, by Lemma 5.1.5, we only need to consider $\mu_{\bar{I}}$ for p -singular elements of \bar{G} .

Let \bar{g}, \bar{h} be p -singular elements of \bar{G} . Then $g, h \notin Z$ and so g, h are p -singular elements in G . Thus $\mu_{\bar{I}}(\bar{g}, \bar{h}) = \mu_I(g, h) \in p^{2n}\mathcal{O}$. Since $|C_{\bar{G}}(\bar{g})| = |C_{\bar{G}}(\bar{h})| = p^{2n}$, this satisfies the integrality condition.

Hence, we conclude that \bar{I} is a perfect isometry.

Proof that if I is perfect then \hat{J} is perfect

Since Z is an abelian p -group and \hat{J} has the all-positive sign, by Lemma 5.1.5, we only need to consider $\mu_{\hat{J}}$ for p -singular elements of Z .

Let g, h be p -singular elements of Z . Then $g, h \neq 1$. But $\bar{g}, \bar{h} = 1$ in \bar{G} . So $\mu_{\bar{I}}(\bar{g}, \bar{h}) = p^{2n}$, and hence $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) + p^{2n}(\mu_j(g, h) - 1) = p^{2n}\mu_j(g, h)$. But $\mu_I(g, h) \in p^{2n+1}\mathcal{O}$, so $\mu_j(g, h) \in p\mathcal{O}$. Since $|C_Z(g)| = |C_Z(h)| = p$, this satisfies the integrality condition.

Hence, we conclude that \hat{J} is a perfect isometry.

Proof that if \bar{I}, \hat{J} are perfect then I is perfect

Let g, h be p -regular elements of G . Then $g = h = 1$ and $\bar{g} = \bar{h} = 1$ in \bar{G} . So $\mu_{\bar{I}}(\bar{g}, \bar{h}) = p^{2n}$ and $\mu_{\hat{J}}(g, h) = p$. Hence $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) + p^{2n}(\mu_{\hat{J}}(g, h) - 1) = p^{2n} + p^{2n}(p - 1) = p^{2n+1}$. Since $|C_G(g)| = |C_G(h)| = p^{2n+1}$, this satisfies the integrality condition.

Let g be a p -regular element, h be a p -singular element of G . There are two cases to consider.

1. Suppose $g = 1, h \in Z - \{1\}$. Then $\mu_{\hat{J}}(g, h) = 0$. Also $\bar{g} = \bar{h} = 1$ in \bar{G} . Thus $\mu_{\bar{I}}(\bar{g}, \bar{h}) = p^{2n}$. Hence $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) + p^{2n}(\mu_{\hat{J}}(g, h) - 1) = p^{2n} + p^{2n}(0 - 1) = 0$. This satisfies the separation condition.
2. Suppose $g = 1, h \notin Z$. Then $\bar{g} = 1, \bar{h} \neq 1$ in \bar{G} and so $\mu_{\bar{I}}(\bar{g}, \bar{h}) = 0$. Hence $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) = 0$. This satisfies the separation condition.

Let g be a p -singular element, h be a p -regular element of G . Then the same argument as in the previous paragraph shows that $\mu_I(g, h) = 0$, which satisfies the separation condition.

Let g, h be p -singular elements of G . Then $g, h \neq 1$. There are four cases to consider.

1. Suppose $g, h \in Z - \{1\}$ then $\bar{g} = \bar{h} = 1$ in \bar{G} . So $\mu_{\bar{I}}(\bar{g}, \bar{h}) = p^{2n}$. Since \hat{J} is perfect, we also have $\mu_{\hat{J}}(g, h) \in p\mathcal{O}$ and so $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) + p^{2n}(\mu_{\hat{J}}(g, h) - 1) = p^{2n}\mu_{\hat{J}}(g, h) \in p^{2n+1}\mathcal{O}$. This satisfies the integrality condition.
2. Suppose $g \in Z - \{1\}, h \notin Z$ then $\bar{g} = 1, \bar{h} \neq 1$ in \bar{G} . So $\mu_{\bar{I}}(\bar{g}, \bar{h}) = 0$. Hence $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) = 0$. This satisfies the integrality condition.
3. Suppose $g \notin Z, h \in Z - \{1\}$ then the similar argument as in case 2 shows that $\mu_I(g, h) = 0$, satisfying the integrality condition.
4. Suppose $g, h \notin Z$. Then $\bar{g}, \bar{h} \neq 1$ in \bar{G} . Since \bar{I} is perfect, $\mu_{\bar{I}}(\bar{g}, \bar{h}) \in p^{2n}\mathcal{O}$. Hence $\mu_I(g, h) = \mu_{\bar{I}}(\bar{g}, \bar{h}) \in p^{2n}\mathcal{O}$. Since $|C_G(g)| = |C_G(h)| = p^{2n}$, this satisfies the integrality condition.

We have now considered all cases, thus I is a perfect isometry as claimed. This proves Theorem 5.2.1.

5.2.3 Picard group and derived Picard group

We will now determine $\text{Pic}(\mathcal{O}G)$ and $\text{DPic}(\mathcal{O}G)$ and show that the homomorphism $\text{DPic}(\mathcal{O}G) \rightarrow \text{PI}(\mathcal{O}G)$ is not surjective when $G = p^{1+2}$.

The following theorem gives us the structures of $\text{Pic}(B)$ and $\text{DPic}(B)$ in the case where B is local. Note that it applies over \mathcal{O} also by [53, Remark 1.12].

Theorem 5.2.2. *Let G be a p -group. Then*

$$\text{Pic}(\mathcal{O}G) \cong \text{Out}_{\mathcal{O}}(\mathcal{O}G)$$

$$\text{DPic}(\mathcal{O}G) \cong \text{Pic}(\mathcal{O}G) \times \text{Sh}(\mathcal{O}G),$$

where $\text{Sh}(\mathcal{O}G)$ is the subgroup generated by $\mathcal{O}G[1]$.

Proof. This follows from [53, Proposition 3.4] since the block $\mathcal{O}G$ is local by [12, (5.25)]. \square

Recall that, by Lemma 5.1.2, we have $\text{Out}_{\mathcal{O}}(\mathcal{O}G) \cong \text{Hom}(G, \mathcal{O}^{\times}) \rtimes \text{Out}(G)$. Let $\mathcal{L}(G)$ be the group of linear characters in $\text{Irr}(G)$. Then it is easy to see that $\text{Hom}(G, \mathcal{O}^{\times}) \cong \mathcal{L}(G) \cong G/Z$. Thus, by Theorem 5.2.2,

$$\text{Pic}(\mathcal{O}G) \cong G/Z \rtimes \text{Out}(G) \tag{5.2}$$

$$\text{DPic}(\mathcal{O}G) \cong (G/Z \rtimes \text{Out}(G)) \times \text{Sh}(\mathcal{O}G). \tag{5.3}$$

Example 5.2.3. Let $G = p^{1+2}$ be an extra special group of order p^3 . Let $B = \mathcal{O}G$. Since $\text{Aut}(C_p \times C_p) \cong \text{GL}_2(p)$, by Theorem 5.2.1, we get

$$\text{PI}(B) \cong ((C_p \times C_p) \rtimes \text{GL}_2(p)) \times C_{p-1} \times \langle -\text{id} \rangle.$$

By (5.3) we get

$$\text{DPic}(B) \cong ((C_p \times C_p) \rtimes \text{Out}(G)) \times \text{Sh}(B).$$

We know that $(C_p \times C_p) \leq \text{DPic}(B)$ induces perfect isometries given by multiplication by linear characters. So $(C_p \times C_p) \leq \text{DPic}(B)$ maps bijectively onto $(C_p \times C_p) \leq$

$\text{PI}(B)$. Furthermore, $\text{Sh}(B) \leq \text{DPic}(B)$ maps onto $\langle -\text{id} \rangle \leq \text{PI}(B)$. So the image of $\text{DPic}(B)$ in $\text{PI}(B)$ is

$$((C_p \times C_p) \rtimes \widetilde{\text{Out}(G)}) \times \langle -\text{id} \rangle,$$

where $\widetilde{\text{Out}(G)}$ is the image of $\text{Out}(G) \leq \text{DPic}(B)$ in $\text{PI}(B)$. But if G has exponent p then $\text{Out}(G) \cong \text{GL}_2(p)$ while if G has exponent p^2 then $\text{Out}(G) \cong C_p \rtimes C_{p-1}$ [15, §A 20.8]. So,

$$|\widetilde{\text{Out}(G)}| \leq |\text{Out}(G)| < |\text{GL}_2(p) \times C_{p-1}|.$$

Comparing the order of $((C_p \times C_p) \rtimes \widetilde{\text{Out}(G)}) \times \langle -\text{id} \rangle$ with $|\text{PI}(B)|$ shows that the canonical homomorphism $\text{DPic}(B) \rightarrow \text{PI}(B)$ is not surjective.

Chapter 6

Blocks with cyclic defect groups

One of the most studied families of blocks are blocks with cyclic defect groups. Brauer [6, 7] first studied blocks of defect 1 and applied his results to study groups whose order contains a prime number to the first power. Later on, Dade [14] extended Brauer results and gave a structure of any block with a cyclic defect group. The equivalences for blocks with cyclic defect groups are also well-understood. Broué's Abelian Defect Group Conjecture (Conjecture 1.0.3) is shown to be true in this case [30, 43, 44]. In particular, Linckelmann [30] showed that the derived category of any block with cyclic defect group is equivalent to the derived category of the semidirect product of its defect group and its inertial quotient. Since the latter block is easier to work with, this gives us a very powerful reduction tool when studying objects invariant under a derived equivalence, for example, perfect isometry groups.

The aim of this chapter is to give a description of the perfect isometry groups for blocks with cyclic defect groups. When the defect is 1, the perfect isometry groups are completely determined. When the defect is greater than 1 we give a partial result based on conditions of the inertial indexes and a conjecture for the remaining cases. As a consequence of these results, we will also show that the homomorphism from the derived Picard group to the perfect isometry group is surjective in the cases where we can completely determine the perfect isometry groups.

Let G be a finite group and B be a block of $\mathcal{O}G$ with a cyclic defect group D .

Let E be the inertial quotient of B and $e = |E|$ be the inertial index. We can form a group $D \rtimes E$ with the evident action of E on D . In order to find $\text{PI}(B)$ we use the following reduction.

Theorem 6.0.4 (Linckelmann [30]). *The derived categories of B and $\mathcal{O}(D \rtimes E)$ are equivalent.*

As a consequence, B and $\mathcal{O}(D \rtimes E)$ are perfectly isometric and so, by Proposition 4.0.7, $\text{PI}(B) \cong \text{PI}(\mathcal{O}(D \rtimes E))$. This allows us to work with the group $D \rtimes E$ which has a simpler structure than G . (Note: that $\mathcal{O}(D \rtimes E)$ is a block follows from [25, Remark 49.8].) The main result is the following.

Theorem 6.0.5. *Let G be a finite group. Let B be a block of $\mathcal{O}G$ with a cyclic defect group D of order p^n . Let e be the inertial index of B and let $t = (p^n - 1)/e$. There are $\{\chi_1, \dots, \chi_e\} \subset \text{Irr}(B)$ and $\{\theta_1, \dots, \theta_t\} \subset \text{Irr}(B)$ such that $\text{Irr}(B) = \{\chi_1, \dots, \chi_e\} \cup \{\theta_1, \dots, \theta_t\}$.*

(i) *If $e = 1$ then*

$$\text{PI}(B) \cong \text{PI}(\mathcal{O}D) \cong (D \rtimes \text{Aut}(D)) \times \langle -id \rangle .$$

(ii) *If $e > 1$ and $t = 1$ then $|\text{Irr}(B)| = p$. Every permutation on $\text{Irr}(B)$ gives a perfect isometry (with a choice of sign), and*

$$\text{PI}(B) \cong S_p \times \langle -id \rangle .$$

(iii) *If $e > 1$ and $t > 1$ then every perfect isometry permutes (with signs) the sets $\{\chi_1, \dots, \chi_e\}$ and $\{\theta_1, \dots, \theta_t\}$ separately. We have,*

$$\text{PI}(B) \cong S_e \times X \times \langle -id \rangle$$

where S_e is the symmetric group on $\{\chi_1, \dots, \chi_e\}$ and X is a permutation group on $\{\theta_1, \dots, \theta_t\}$. Furthermore, X contains a subgroup isomorphic to $\text{Aut}(D)/E$. If $D \cong C_p$, then $X \cong \text{Aut}(D)/E \cong C_{(p-1)/e}$.

Unfortunately, in the case $e > 1$ and $t > 1$ we were unable to give the complete description of $\text{PI}(B)$ when B has defect greater than 1. However, based on computer-generated results of many examples we can speculate that the subgroup X is isomorphic to $\text{Aut}(D)/E \cong C_{p^{n-1}(p-1)/e}$.

Conjecture 6.0.6. *Let B be a block of \mathcal{OG} with a cyclic defect group D of order p^n . Let e be the inertial index of B and let $t = (p^n - 1)/e$. Suppose $e > 1$ and $t > 1$. Then*

$$\text{PI}(B) \cong S_e \times C_{p^{n-1}(p-1)/e} \times \langle -id \rangle.$$

The case $D \cong C_p$ ($n = 1$) in Conjecture 6.0.6 is included in Theorem 6.0.5(iii). We will prove this in Theorem 6.3.1.

As discussed before, it suffices to work with the group $D \rtimes E$. Note that a derived equivalence between B and $\mathcal{O}(D \rtimes E)$ implies that there is a bijection with signs between $\text{Irr}(\mathcal{O}(D \rtimes E))$ and $\text{Irr}(B)$. This means that characteristics of perfect isometry groups such as every permutation gives a perfect isometry (with some sign), every perfect isometry permutes two subsets of irreducible characters in the block separately, will be preserved. It does not, however, preserve properties related to signs, such as having homogeneous signs. So if $e = 1$ then $\text{PI}(B)$ may not have homogenous signs even though $\text{PI}(D)$ has (by Theorem 5.1.1).

6.1 The group $D \rtimes E$ and its character table

Let $H = D \rtimes E$. Let $|D| = p^n$. We first need some remarks on the group H .

Lemma 6.1.1. *The following hold.*

- (i) e is not divisible by p .
- (ii) e divides $p - 1$
- (iii) The group H is uniquely determined by e .

(iv) The group E acts frobeniusly on D , that is, if $g^h = g$ for $g \in D, h \in E$ then $g = 1$ or $h = 1$.

Proof. (i) is Theorem 2.3.10.

Let b_D be a root of B . Since D is abelian, $DC_G(D) = C_G(D)$. So the inertial quotient $E = I_{N_G(D)}(b_D)/C_G(D)$ acts faithfully on D and is therefore a subgroup of $\text{Aut}(D)$. If $|D| = p^n$ then we have $e|p^{n-1}(p-1)$. Since e does not divide p by (i), e must divide $p-1$, proving (ii).

To prove (iii), notice that if $e = 1$ then $H = D$. If $e > 1$ then by (ii), p must be odd, and so $\text{Aut}(D)$ and E are cyclic. Since two automorphisms of D of the same order are powers of each other and $E \leq \text{Aut}(D)$, the isomorphism class of H is uniquely determined by the order of E .

Finally, (iv) follows from [14, Section 1]. \square

Since E acts on D , it also acts on $\text{Irr}(D)$. Let $\lambda \in \text{Irr}(D)$ be a non-trivial character. Since the action of E on D is faithful, the stabilizer in E of λ is 1. By the orbit-stabilizer theorem, the size of orbit of λ is e . So the $p^n - 1$ non-trivial characters of $\text{Irr}(D)$ break up into orbits of equal size e . Let $\{\phi_1, \dots, \phi_t\} \subseteq \text{Irr}(D)$ be non-trivial E -orbit representatives where $t = (p^n - 1)/e$. By [24, Theorem 6.34], each ϕ_i induces to an irreducible character of H , denoted by Φ_i . We call Φ_1, \dots, Φ_t the *exceptional characters* of H .

On the other hand, for each irreducible character $\bar{\chi}_i$ of E there is a corresponding irreducible character χ_i of H such that $\chi_i(g) = \bar{\chi}_i(\bar{g})$ where $g \mapsto \bar{g}$ is the canonical map from H to $E = H/D$. The irreducible characters χ_1, \dots, χ_e are called *non-exceptional characters* of H . We label in such a way that χ_1 is the trivial character. We have now described all irreducible characters of H :

$$\text{Irr}(H) = \{\chi_1, \dots, \chi_e\} \cup \{\Phi_1, \dots, \Phi_t\}.$$

Let a, b be generators of D and E respectively. Since E acts frobeniusly on D , $D - \{1\}$ breaks up into orbits of equal size e . Let $\{a_1, \dots, a_t\}$ be non-trivial E -orbit representatives of D . The conjugacy classes and the character table of H are as

follow:

$$\begin{aligned} & \{1\}, \\ & (a_i)^G, \quad (0 \leq i \leq t-1), \\ & (b^y)^G = \{gb^y : g \in D\} \quad (1 \leq y \leq e-1). \end{aligned}$$

Table 6.1: Character table of H

| H | $\{1\}$ | ... | $\{a_i\}$ | ... | ... | $\{b^y\}$ | ... |
|----------|---------|-----|---------------------------|-----|-----|---------------------|-----|
| χ_1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| ... | 1 | 1 | 1 | 1 | | | |
| χ_j | 1 | 1 | 1 | 1 | | $\bar{\chi}_j(b^y)$ | |
| ... | 1 | 1 | 1 | 1 | | | |
| ... | e | | | | 0 | 0 | 0 |
| Φ_j | e | | $\phi_j \uparrow^G (a_i)$ | | 0 | 0 | 0 |
| ... | e | | | | 0 | 0 | 0 |

Lemma 6.1.2. *Let $g \in H$. Then g is p -singular if and only if $g \in D - \{1\}$.*

Proof. If $g \in D - \{1\}$, then it is clear that g is a p -singular element. Suppose that $g \notin D$. We will show that g is p -regular. First write $g = xy$ where $x \in D, y \in E$ and $y \neq 1$. If $x = 1$, then $g = y$ is clearly a p -regular element. So suppose $x \neq 1$. Since $D \triangleleft H$, $yx = x^k y$ for some $k \not\equiv 0 \pmod{p^n}$. Then $y^m x = x^{k^m} y^m$ and so $k^{o(y)} \equiv 1 \pmod{o(x)}$, where $o(x), o(y)$ are the orders of x, y respectively. Note that $g^r = x^{1+k+k^2+\dots+k^{r-1}} y^r = x^{(k^r-1)/(k-1)} y^r$. Suppose $g^r = 1$, then $o(y)|r$. But if $r = o(y)$, then $k^r \equiv 1 \pmod{o(x)}$ and so $g^{o(y)} = x^0 y^{o(y)} = 1$. Hence, g has order $o(y)$ which is not divisible by p . So g is p -regular. \square

The next lemma summarizes the values of $|C_H(g)|$ for all $g \in H$.

Lemma 6.1.3. (i) *If $g = 1$ then $|C_H(g)| = |H|$.*

(ii) *If $g \neq 1$ and g is p -regular then $|C_H(g)| = |E|$*

(iii) *If $g \neq 1$ and g is p -singular then $|C_H(g)| = |D|$*

Proof. (i) is obvious. For (ii), the previous lemma shows that $g \notin D$ so $\Phi_j(g) = 0, \forall j$. By column orthogonality of irreducible characters,

$$\begin{aligned} |C_H(g)| &= \sum_{\theta \in \text{Irr}(H)} |\theta(g)|^2 \\ &= \sum_{i=1}^e |\chi_i(g)|^2 + \sum_{j=1}^t |\Phi_j(g)|^2 \\ &= \sum_{i=1}^e |\bar{\chi}_i(\bar{g})|^2 \\ &= |C_E(\bar{g})| = |E|. \end{aligned}$$

For (iii), by the previous lemma, $g \in D - \{1\}$. Since D is cyclic, it is clear that $D \leq C_H(g)$. By Lemma 6.1.1(iv), there is no non-trivial $h \in E$ with $g^h = g$. Thus, $|C_H(g)| = |D|$. \square

6.2 Perfect isometry groups

We will now prove Theorem 6.0.5 for the group $H = D \rtimes E$. Let $A = \mathcal{O}H$. We divide the proof into 3 cases based on the values of e and t .

Case: $e = 1$

In this case $H = D$, and therefore the result on abelian p -groups (Theorem 5.1.1) applies.

Lemma 6.2.1. *Suppose $e = 1$. Then*

$$\text{PI}(A) \cong (D \rtimes \text{Aut}(D)) \times \langle -id \rangle.$$

Case: $e > 1$ and $t = 1$

Since $e \mid p - 1$, we can deduce that p is odd. Since $t = 1$, we have $e = |D| - 1$, implying that $D \cong C_p$ and $e = p - 1$. Therefore, $H = C_p \rtimes C_{p-1}$. Let $\chi_p = \Phi_1$ so that $\text{Irr}(A) = \{\chi_1, \dots, \chi_p\}$. There is only one p -singular class: $\{a\}$. There are e p -regular classes: $\{1\}, \{b^y\}$ for $y = 1, \dots, e - 1$.

Table 6.2: Character Table of $C_p \rtimes C_{p-1}$

| G | 1 | $\{a\}$ | $\{b^y\}$ |
|----------|-------|---------|---------------------|
| χ_1 | 1 | 1 | 1 |
| \dots | 1 | 1 | |
| χ_j | 1 | 1 | $\bar{\chi}_j(b^y)$ |
| \dots | 1 | 1 | |
| χ_p | $p-1$ | -1 | 0 |

Proposition 6.2.2. *Suppose that $e = p - 1$ and $t = 1$. Then*

$$\text{PI}(A) = S_p \times \langle -id \rangle.$$

Proof. Since $|\text{Irr}(A)| = p$ is odd, by Proposition 4.1.7, $\text{PI}(A) \cong \text{PI}^+(A) \times \langle -id \rangle$. It is therefore enough to show that, for any $\sigma \in S_p$ there is a sign function ε such that $I = (\sigma, \varepsilon)$ is a perfect isometry. Let $\delta_i = 1$ for $i = 1, \dots, p - 1$ and $\delta_p = -1$. Define an isometry I by

$$I(\chi_i) = \delta_i \delta_{\sigma(i)} \chi_{\sigma(i)}, \quad i = 1, \dots, p.$$

We will show that I is perfect.

Let μ_I be the character corresponding to I . For $g, h \in H$,

$$\begin{aligned} \mu_I(g, h) &= \sum_{i=1}^p \delta_i \delta_{\sigma(i)} \chi_{\sigma(i)}(g) \chi_i(h) \\ &= \sum_{i \neq p} \delta_i \delta_{\sigma(i)} \chi_{\sigma(i)}(g) \chi_i(h) - \delta_{\sigma(p)} \chi_{\sigma(p)}(g) \chi_p(h). \end{aligned}$$

Suppose $g = h = 1$. Then

$$\mu_I(1, 1) = \sum_{i \neq p} \delta_{\sigma(i)} \chi_{\sigma(i)}(1) - \delta_{\sigma(p)} \chi_{\sigma(p)}(1)(p - 1).$$

If $\sigma(p) = p$, then $\mu_I(1, 1) = (p - 1) - (-1)(p - 1)^2 = (p - 1)p$. Since $|C_H(1)| = p(p - 1)$ and $1/(p - 1) \in \mathcal{O}$, this satisfies the integrality condition. If $\sigma(p) \neq p$ then $\mu_I(1, 1) = (p - 2) + (-1)(p - 1) - (p - 1) = -p$. Since $|C_H(1)| = p(p - 1)$, this satisfies the integrality condition.

Suppose $g = 1$, $h \neq 1$, h is p -regular. If $\sigma(p) = p$, then $\mu_I(g, h) = \sum_{i \neq p} \chi_i(h) = 0$. If $\sigma(p) \neq p$, then $\sigma(j) = p$ for some $j \in \{1, \dots, p - 1\}$. So $\mu_I(g, h) = (-1)(p - 1)\chi_j(h) + \sum_{i \neq p, i \neq j} \chi_i(h) = -p\chi_j(h)$. In both cases, $\mu_I(g, h) \in p\mathcal{O}$. This satisfies the integrality condition.

Suppose $h = 1$, $g \neq 1$, g is p -regular. If $\sigma(p) = p$, then $\mu_I(g, h) = \sum_{i \neq p} \chi_{\sigma(i)}(g) = 0$. If $\sigma(p) \neq p$, then $\sigma(j) = p$ for some $j \in \{1, \dots, p-1\}$. So $\mu_I(g, h) = 0 + \sum_{i \neq p, i \neq j} \chi_{\sigma(i)}(g) - (1)\chi_{\sigma(p)}(g)(p-1) = -p\chi_{\sigma(p)}(g)$. In both cases, $\mu_I(g, h) \in p\mathcal{O}$. This satisfies the integrality condition.

Suppose $g = 1$, $h \neq 1$, h is p -singular. If $\sigma(p) = p$, then $\mu_I(g, h) = \sum_{i \neq p} (1)(1)(1) - (-1)(p-1)(-1) = 0$. If $\sigma(p) \neq p$, then $\sigma(j) = p$ for some $j \in \{1, \dots, p-1\}$. So, $\mu_I(g, h) = (-1)(p-1)(1) + \sum_{i \neq p, i \neq j} (1)(1)(1) - (1)(1)(-1) = 0$. Both cases satisfy the separation condition.

Suppose $h = 1$, $g \neq 1$, g is p -singular. If $\sigma(p) = p$, then $\mu_I(g, h) = \sum_{i \neq p} (1)(1)(1) - (-1)(-1)(p-1) = 0$. If $\sigma(p) \neq p$, then $\sigma(j) = p$ for some $j \in \{1, \dots, p-1\}$. So, $\mu_I(g, h) = (-1)(-1)(1) + \sum_{i \neq p, i \neq j} (1)(1)(1) - (1)(1)(p-1) = 0$. Both cases satisfy the separation condition.

Suppose $g \neq 1$, $h \neq 1$ and g, h are both p -regular. Then, $|C_H(g)| = |C_H(h)| = e = p-1$. But $\mu_I(g, h) \in (p-1)\mathcal{O}$. This satisfies the integrality condition.

Suppose $g \neq 1$, $h \neq 1$ and g, h are both p -singular. If $\sigma(p) = p$, then $\mu_I(g, h) = \sum_{i \neq p} (1)(1)(1) - (-1)(-1)(-1) = p$. If $\sigma(p) \neq p$, then $\sigma(j) = p$ for some $j \in \{1, \dots, p-1\}$. So, $\mu_I(g, h) = (-1)(-1)(1) + \sum_{i \neq p, i \neq j} (1)(1)(1) - (1)(1)(-1) = p$. Both cases satisfy the integrality condition.

Therefore I is perfect as claimed. \square

Case: $e > 1$ and $t > 1$

In this case we have $p > 2$ (since e divides $p-1$). First we show that every perfect isometry preserves $\pm\{\chi_1, \dots, \chi_e\}$ and $\pm\{\Phi_1, \dots, \Phi_t\}$.

Lemma 6.2.3. *Let $I : R_K(H) \rightarrow R_K(H)$ be a perfect isometry. Then I sends $\{\chi_1, \dots, \chi_e\}$ to $\pm\{\chi_1, \dots, \chi_e\}$ and sends $\{\Phi_1, \dots, \Phi_t\}$ to $\pm\{\Phi_1, \dots, \Phi_t\}$.*

Proof. Suppose that $I(\Phi_x) = \pm\chi_r$ for some x and r . Without loss of generality, we can take $I(\Phi_x) = \chi_r$. Since, for any i, j , $\Phi_i - \Phi_j = 0$ on p -regular elements, so $I(\Phi_i) - I(\Phi_j) = 0$ on p -regular elements as I is perfect. Since b is a p -regular element and $t > 1$, there is $y \neq x$ such that $I(\Phi_y)(b) = I(\Phi_x)(b) = \chi_r(b)$.

Since $\Phi_i(b) = 0$ for all i and $\chi_r(b) \neq 0$, we must have $I(\Phi_y) \in \pm\{\chi_1, \dots, \chi_e\}$, say $I(\Phi_y) = \pm\chi_s$ where $s \neq r$. Then $\pm\chi_s(b) = \chi_r(b)$. But $\{\chi_1(b), \dots, \chi_e(b)\}$ are distinct e -th roots of unity, we cannot have $\chi_s(b) = \chi_r(b)$. So we must have $I(\Phi_y) = -\chi_s$ and $-\chi_s(b) = \chi_r(b)$.

But we also have $I(\Phi_x)(1) = I(\Phi_y)(1)$, since 1 is a p -regular element. This gives $\chi_r(1) = -\chi_s(1)$, that is, $1 = -1$, a contradiction. Hence I sends $\{\Phi_1, \dots, \Phi_t\}$ to $\pm\{\Phi_1, \dots, \Phi_t\}$, and consequently, I sends $\{\chi_1, \dots, \chi_e\}$ to $\pm\{\chi_1, \dots, \chi_e\}$. \square

The next lemma shows that every perfect isometry has a homogenous sign.

Lemma 6.2.4. *Let $I : R_K(H) \rightarrow R_K(H)$ be a perfect isometry. Then I has a homogenous sign.*

Proof. By Lemma 6.1.2, the p -singular elements of H are in $D - \{1\}$. So $\chi_i - \chi_j = 0$ on p -singular elements for all i, j . Hence, $I(\chi_i) - I(\chi_j) = 0$ on p -singular elements for all i, j . In particular, $I(\chi_i)(a)$ is constant for all i . Since $\chi_i(a) = 1$ for all i , this shows that I has a homogenous sign on $\{\chi_1, \dots, \chi_e\}$. We can assume without loss of generality that I has positive signs on $\{\chi_1, \dots, \chi_e\}$.

The same reasoning also applies to $\Phi_i - \Phi_j$, vanishing on p -regular elements for all i, j . So $I(\Phi_i)(1)$ is constant for all i . This shows that I also has a homogenous sign on $\{\Phi_1, \dots, \Phi_t\}$. Let $r = |\{i \in \{1, \dots, t\} : I(\Phi_i) \in \{\Phi_1, \dots, \Phi_t\}\}|$, that is, r is the number of $\Phi_i \in \{\Phi_1, \dots, \Phi_t\}$ whose image has positive sign. By above, $r = 0$ or t . We will show that $r = t$.

Let μ_I be the corresponding character of I . Then

$$\begin{aligned} \mu_I(1, 1) &= \sum_{i=1}^e I(\chi_i)(1)\chi_i(1) + \sum_{i=1}^t I(\Phi_i)(1)\Phi_i(1) \\ &= e + re^2 - (t - r)e^2 = e + 2re^2 - te^2 \\ &= e + 2re^2 - \left(\frac{p^n - 1}{e}\right)e^2 = e + 2re^2 - ep^n + e \\ &= 2e(1 + re) - ep^n. \end{aligned}$$

Since I is perfect, we have $\mu_I(1, 1) \equiv 0 \pmod{p^n}$. So $2e(1 + re) \equiv 0 \pmod{p^n}$. Since p is odd and $p \nmid e$, this yields $1 + re \equiv 0 \pmod{p^n}$ and so $r \neq 0$. Thus $r = t$ and I has

a homogenous sign on $\text{Irr}(H)$ as claimed. \square

Proposition 6.2.5. *Let $\sigma \in \text{Sym}\{\chi_1, \dots, \chi_e\}$. Define $I : R_K(H) \rightarrow R_K(H)$ by*

$$I(\chi) = \sigma(\chi), \quad \forall \chi \in \{\chi_1, \dots, \chi_e\}$$

$$I(\Phi) = \Phi, \quad \forall \Phi \in \{\Phi_1, \dots, \Phi_t\}.$$

Then I is a perfect isometry.

Proof. Let $\mu = \mu_I$ be the character corresponding to I and let μ_{id} be the character corresponding to the identity isometry. Let $g, h \in H$.

$$\mu(g, h) = \sum_{\chi} \sigma(\chi)(g)\chi(h) + \sum_{\Phi} \Phi(g)\Phi(h).$$

If $g = 1$, then

$$\mu(1, h) = \sum_{\chi} \sigma(\chi)(1)\chi(h) + \sum_{\Phi} \Phi(1)\Phi(h) = \mu_{id}(1, h).$$

So $\mu(1, h)$ satisfies the conditions of perfect characters.

If $h = 1$, then

$$\sum_{\chi} \sigma(\chi)(g)\chi(1) = \sum_{\chi} \sigma(\chi)(g) = \sum_{\chi} \chi(g) = \sum_{\chi} \chi(g)\chi(1).$$

So

$$\mu(g, 1) = \sum_{\chi} \sigma(\chi)(g)\chi(1) + \sum_{\Phi} \Phi(g)\Phi(1) = \mu_{id}(g, 1).$$

So $\mu(g, 1)$ satisfies the conditions of perfect characters.

Assume from now on that $g \neq 1$ and $h \neq 1$. If g is p -regular and h is p -regular, then by Lemma 6.1.3(ii), $|C_H(g)| = |C_H(h)| = e$. Since $p \nmid e$, $\mu(g, h)/e \in \mathcal{O}$. So $\mu(g, h)$ satisfies the integrality condition.

If g is p -regular and h is p -singular, then $g \notin D$ and $\bar{g} \neq 1$, $\bar{h} = 1$. So

$$\mu(g, h) = \sum_{\chi} \overline{\sigma(\chi)}(\bar{g})\bar{\chi}(1) + 0 = \sum_{\chi} \overline{\sigma(\chi)}(\bar{g}) = 0.$$

So $\mu(g, h)$ satisfies the separation condition.

If g is p -singular and h is p -regular, then $h \notin D$ and $\bar{g} = 1$, $\bar{h} \neq 1$. So

$$\mu(g, h) = \sum_{\chi} \overline{\sigma(\chi)}(1)\bar{\chi}(\bar{h}) + 0 = \sum_{\chi} \bar{\chi}(\bar{h}) = 0.$$

So $\mu(g, h)$ satisfies the separation condition.

Finally, if g, h are p -singular, then $\bar{g} = \bar{h} = 1$ and, by Lemma 6.1.3(iii), $|C_H(g)| = |C_H(h)| = p^n$. So

$$\begin{aligned} \mu(g, h) &= \sum_{\chi} \overline{\sigma(\chi)}(1)\bar{\chi}(1) + \sum_{\Phi} \Phi(g)\Phi(h) = \sum_{\chi} \bar{\chi}(1)\bar{\chi}(1) + \sum_{\Phi} \Phi(g)\Phi(h) \\ &= \mu_{id}(g, h). \end{aligned}$$

So $\mu(g, h)$ satisfies the integrality condition. Hence I is perfect. \square

Corollary 6.2.6. *Suppose $e > 1$ and $t > 1$. Then*

$$\text{PI}(A) \cong S_e \times X \times \langle -id \rangle$$

where $X \leq S_t$ is a permutation group on the exceptional characters.

Proof. Since all perfect isometries have homogenous signs, by Lemma 4.1.6 we have $\text{PI}(A) \cong \text{PI}(A)^+ \times \langle -id \rangle$ where $\text{PI}^+(A)$ is a permutation group on $\text{Irr}(A)$. By Lemma 6.2.3, $\text{PI}^+(A)$ permutes the exceptional and non-exceptional characters separately. By Proposition 6.2.5, perfect isometries permuting the non-exceptional characters only is the full symmetric group S_e . We deduce that $\text{PI}(A) \cong S_e \times X \times \langle -id \rangle$ where X is a permutation group on the exceptional characters. \square

Now it remains to find the group X , that is the subgroup of perfect isometries permuting only the exceptional characters. We are going to show that X contains $\text{Aut}(D)/E$, the cyclic group of order $p^{n-1}(p-1)/e$. Recall from Section 4.2 that there is a homomorphism $\text{Aut}(H) \rightarrow \text{PI}(A)$. Let $\tilde{\mathcal{A}}(A)$ be the image of this map. Since $\text{Irr}(A) = \text{Irr}(H)$, by Remark 4.2.2, $\tilde{\mathcal{A}}(A) \cong \text{Aut}(H)/\text{Aut}_c(H)$.

Proposition 6.2.7. *The group X contains $\tilde{\mathcal{A}}(A)$ and $\tilde{\mathcal{A}}(A) \cong \text{Aut}(D)/E$, a cyclic group of order $p^{n-1}(p-1)/e$.*

Proof. First we will show that $\text{Aut}(H) \cong D \rtimes \text{Aut}(D)$. As before, let a, b be generators of D and E respectively. The group $H = D \rtimes E$ can be written as

$$H = \langle a, b : a^{p^n} = b^e = 1, a^b = a^u \rangle,$$

where u has order e in $\mathbb{Z}_{p^n}^\times$. Let $\theta \in \text{Aut}(H)$. Since a is a p -singular element, $\theta(a)$ is also p -singular and so $\theta(a) = a^r$ for some positive integer r with $(r, p^n) = 1$. Let $\theta(b) = a^x b^y$ for some $x \in \{0, \dots, p^n - 1\}, y \in \{0, \dots, e\}$. Also

$$\begin{aligned} a^{ru} &= \theta(a^u) = \theta(a^b) = \theta(a)^{\theta(b)} \\ &= (a^r)^{a^x b^y} = (a^{a^x b^y})^r = (a^{b^y})^r \\ &= a^{u^y r}. \end{aligned}$$

This implies $a^{u^y r - ru} = 1$, and so

$$\begin{aligned} (u^y - u)r &\equiv 0 \pmod{p^n} \\ u^y - u &\equiv 0 \pmod{p^n} \quad \text{since } (r, p^n) = 1 \\ u^{y-1} - 1 &\equiv 0 \pmod{p^n} \quad \text{since } (u, p^n) = 1 \\ \therefore y &\equiv 1 \pmod{e} \quad \text{since } u \text{ has order } e \\ \therefore b^y &= b. \end{aligned}$$

So, for each $\theta \in \text{Aut}(H)$ we must have,

$$\begin{aligned} \theta(a) &= a^r, \quad \text{for some } r \text{ with } (r, p^n) = 1 \\ \theta(b) &= a^x b, \quad \text{for some } x \text{ with } x \in \{0, \dots, p^n - 1\}. \end{aligned}$$

For each $\sigma \in \text{Aut}(D), g \in D$, define θ_σ, θ_g by

$$\begin{aligned} \theta_\sigma(a) &= \sigma(a), & \theta_\sigma(b) &= b \\ \theta_g(a) &= a^g = a, & \theta_g(b) &= b^g. \end{aligned}$$

Then, we see that $\text{Aut}(H)$ is generated by elements of the forms θ_σ, θ_g for $\sigma \in \text{Aut}(D), g \in D$. Moreover, we have $\theta_\sigma \theta_g (\theta_\sigma)^{-1} = \theta_{\sigma(g)}$. Thus

$$\text{Aut}(H) \cong D \rtimes \text{Aut}(D), \tag{6.1}$$

where multiplications are defined by $(\theta_g, \theta_\sigma)(\theta_h, \theta_\pi) = (\theta_g \theta_{\sigma(h)}, \theta_\sigma \theta_\pi)$ for $g, h \in D$ and $\sigma, \pi \in \text{Aut}(D)$.

Next we show that $\text{Aut}_c(H) \cong D \rtimes E$. For convenience, we write $g \sim h$ for “ g is conjugate in H to h ”. Let $g \in D$ be such that $\theta_g \in \text{Aut}_c(H)$. Then, $\theta_g(b) = b^g \sim b$. But the conjugacy class of b is $\{xb, x \in D\}$. Thus,

$$\{g \in D : \theta_g \in \text{Aut}_c(H)\} = D.$$

Let $\sigma \in \text{Aut}(D)$ be such that $\theta_\sigma \in \text{Aut}_c(H)$. Then, $\theta_\sigma(a) = \sigma(a) \sim a$. But $\sigma(a) \sim a$ if and only if $\sigma(a) = a^y$ for some $y \in E$. Thus,

$$\{\sigma \in \text{Aut}(D) : \theta_\sigma \in \text{Aut}_c(H)\} \cong E.$$

Hence,

$$\text{Aut}_c(H) \cong D \rtimes E. \tag{6.2}$$

Therefore, from (6.1) and (6.2), the image of $\text{Aut}(H)$ in $\text{PI}(A)$ is

$$\tilde{\mathcal{A}}(A) \cong \frac{\text{Aut}(H)}{\text{Aut}_c(H)} \cong \frac{D \rtimes \text{Aut}(D)}{D \rtimes E} \cong \frac{\text{Aut}(D)}{E}.$$

Since p is odd, $\text{Aut}(D)$ is cyclic of order $p^{n-1}(p-1)$ and hence $\text{Aut}(D)/E$ is cyclic of order $p^{n-1}(p-1)/e$.

Finally, we show that $\tilde{\mathcal{A}}(A) \leq X$. Let $\sigma \in \text{Aut}(H)$. Let $g \in D, h \in E$. Then, as in the beginning of the proof, $\sigma^{-1}(gh) = g'h$, where $g' \in D$. For $\chi \in \{\chi_1, \dots, \chi_e\}$,

$$\chi^\sigma(gh) = \chi(\sigma^{-1}(gh)) = \chi(g'h) = \chi(gh),$$

since $gh \sim g'h$. So the perfect isometry $I_\sigma : \chi \mapsto \chi^\sigma$ fixes every character $\chi \in \{\chi_1, \dots, \chi_e\}$. Hence I_σ is a permutation only on the exceptional characters and thus, $\tilde{\mathcal{A}}(A) \leq X$ as claimed. \square

6.3 Special case when defect group is C_p

In this section, we will prove the last statement in Theorem 6.0.5(iii) in the case $e > 1$ and $t > 1$. Although we cannot completely determine the subgroup $X \leq \text{PI}(B)$ when $n > 1$. The case when $n = 1$ (that is, $D \cong C_p$) can be done. Our proof is direct and elementary.

Theorem 6.3.1. *Let B be a block of $\mathcal{O}G$ with a defect group $D \cong C_p$. Let e be the inertial index of B , and let $t = (p - 1)/e$. Suppose that $e > 1$ and $t > 1$. Then*

$$\text{PI}(B) \cong S_e \times C_t \times \langle -id \rangle.$$

As before we will work with the group $H = D \rtimes E$. Let $A = \mathcal{O}H$. Let a, b be generators of D, E respectively. In order to calculate the perfect isometry group, we need to describe the group structure and character values in more details than in the previous section. We will also slightly change the indexes of our irreducible characters from the previous section, so that the formulas for character values are easy to manipulate.

Write

$$H = \langle a, b : a^p = b^e = 1, a^b = a^u \rangle$$

where u has order e in \mathbb{Z}_p^\times . Let $S = \langle u \rangle \triangleleft \mathbb{Z}_p^\times$ and let $\{1, v, v^2, \dots, v^{t-1}\}$ be coset representatives of S in \mathbb{Z}_p^\times , so that $\mathbb{Z}_p^\times = \bigsqcup_{i=0}^{t-1} v^i S$.

The conjugacy classes of G are:

$$\{1\},$$

$$(a^{v^i})^G = \{a^{v^i s} : s \in S\} \quad (0 \leq i \leq t - 1),$$

$$(b^y)^G = \{a^x b^y : 0 \leq x \leq p - 1\} \quad (1 \leq y \leq e - 1).$$

Let $\zeta = e^{2\pi i/e}$ and $\omega = e^{2\pi i/p}$ be primitive roots of unity. Then the irreducible characters of G are

$$\{\chi_0, \chi_1, \dots, \chi_{e-1}\} \cup \{\Phi_0, \Phi_1, \dots, \Phi_{t-1}\},$$

where

$$\chi_i(a^x b^y) = \zeta^{iy}$$

$$\Phi_j(a^x b^y) = 0 \quad \text{if } 1 \leq y \leq e - 1$$

$$\Phi_j(a^x) = \sum_{s \in S} \omega^{v^j s x}.$$

Here, the non-exceptional characters are $\{\chi_0, \chi_1, \dots, \chi_{e-1}\}$ and the exceptional characters are $\{\Phi_0, \Phi_1, \dots, \Phi_{t-1}\}$.

Table 6.3: Character table of H

| G | $\{1\}$ | \dots | $\{a^x\}$ | \dots | \dots | $\{b^y\}$ | \dots |
|----------|---------|-----------------------------------|-----------|---------|---------|--------------|---------|
| χ_0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \dots | 1 | 1 | 1 | 1 | | | |
| χ_j | 1 | 1 | 1 | 1 | | ζ^{jy} | |
| \dots | 1 | 1 | 1 | 1 | | | |
| \dots | e | | | | 0 | 0 | 0 |
| Φ_j | e | $\sum_{s \in S} \omega^{v^j s x}$ | | | 0 | 0 | 0 |
| \dots | e | | | | 0 | 0 | 0 |

Recall from the previous section that any perfect isometries $I \in \text{PI}(A)$ has a homogenous sign. If I has all-positive sign, then I sends exceptional characters to exceptional characters and sends non-exceptional characters to non-exceptional characters. Furthermore, we can write

$$\text{PI}(A) \cong S_e \times X \times \langle -\text{id} \rangle$$

where X is the subgroup of $\text{PI}(A)$ permuting only the exceptional characters.

Let $T = \{0, 1, \dots, t-1\}$. For each $\pi \in \text{Sym}(T)$, define $I_\pi : R_K(A) \rightarrow R_K(A)$ by

$$I_\pi(\chi_i) = \chi_i, \quad i = 0, \dots, e-1$$

$$I_\pi(\Phi_j) = \Phi_{\pi(j)}, \quad j = 0, \dots, t-1.$$

Then X consists of perfect isometries of the form I_π for some $\pi \in \text{Sym}(T)$. Let $\mu_\pi \in R_K(A \otimes_{\mathcal{O}} A^\circ)$ be the character corresponding to I_π . Theorem 6.3.1 will be proved once we show that μ_π is perfect if and only if π is of the form $\pi(j) = j + \alpha \pmod t, \forall j \in T$ where $\alpha \in T$. This will then imply that $X \cong \mathbb{Z}_t \cong C_t$.

It is easy to see from the character table that μ_π satisfies the separation condition (for any $\pi \in \text{Sym}(T)$). Since $|C_H(a^x)| = p, \forall x \in \mathbb{Z}_p^\times$, we have that μ_π is perfect if and only if $\mu_\pi(a^x, a^y)/p \in \mathcal{O}, \forall x, y \in \mathbb{Z}_p^\times$, that is, if and only if

$$\|\mu_\pi(a^x, a^y)\|_p \leq \frac{1}{p} \quad \forall x, y \in \mathbb{Z}_p^\times.$$

Let $x, y \in \mathbb{Z}_p^\times$. We have

$$\begin{aligned}
 \mu_\pi(a^x, a^y) &= \sum_i \chi_i(a^x) \chi_i(a^y) + \sum_j \Phi_{\pi(j)}(a^x) \Phi_j(a^y) \\
 &= e + \sum_j \left(\sum_{r \in S} \omega^{v^{\pi(j)} r x} \right) \left(\sum_{s \in S} \omega^{v^j s y} \right) \\
 &= e + \sum_{j \in T} \sum_{r \in S} \sum_{s \in S} \omega^{v^{\pi(j)} r x + v^j s y}. \tag{6.3}
 \end{aligned}$$

Recall that K is sufficiently large for H , so K is an extension of $\mathbb{Q}_p(\omega)$. It is evident from (6.3) that $\mu_\pi(a^x, a^y)/p \in \mathbb{Q}_p(\omega)$. If $\|\mu_\pi(a^x, a^y)/p\|_p \leq 1$, then $\mu_\pi(a^x, a^y)/p \in \mathbb{Q}_p(\omega) \cap \mathcal{O}$. So, by Lemma 2.1.1, $\mu_\pi(a^x, a^y)/p$ is in the valuation ring of $\mathbb{Q}_p(\omega)$. This ring can be described by the following lemma.

Lemma 6.3.2. *The valuation ring of $\mathbb{Q}_p(\omega)$ with respect to $\|\cdot\|_p$ is $\mathbb{Z}_p(\omega)$.*

Proof. See the discussion in Section 5.6 in [20]. □

Lemma 6.3.3. *Let $\alpha = C_0 + C_1\omega + \cdots + C_{p-1}\omega^{p-1}$ where $C_i \in \mathbb{Z}$ for all i . Suppose that $\|\alpha\|_p \leq 1/p$. Then $C_i \equiv C_j \pmod{p}$ for all i, j .*

Proof. Since $\alpha/p \in \mathbb{Q}_p(\omega)$ and $\|\alpha/p\|_p = p\|\alpha\|_p \leq 1$, we have that α/p is in the valuation ring of $\mathbb{Q}_p(\omega)$. By the previous lemma, we can therefore write $\alpha = pa_0 + pa_1\omega + \cdots + pa_{p-2}\omega^{p-2}$ where $a_i \in \mathbb{Z}_p \forall i$, as $\{1, \omega, \dots, \omega^{p-2}\}$ is a basis of $\mathbb{Z}_p(\omega)$ over \mathbb{Z}_p . Write $\omega^{p-1} = -1 - \omega - \cdots - \omega^{p-2}$. Then

$$\begin{aligned}
 \alpha &= (C_0 - C_{p-1}) + (C_1 - C_{p-1})\omega + \cdots + (C_{p-2} - C_{p-1})\omega^{p-2} \\
 &= pa_0 + pa_1\omega + \cdots + pa_{p-2}\omega^{p-2}.
 \end{aligned}$$

Comparing the coefficients, we have $C_i - C_{p-1} \in p\mathbb{Z}_p$ for $i = 1, \dots, p-2$. But $C_i - C_{p-1}$ are integers, we have $C_i - C_{p-1} \in p\mathbb{N}$ and so $C_i \equiv C_j \pmod{p}$ for all i, j as claimed. □

We are going to write $\mu_\pi(a^x, a^y)$ in a nicer form so we can apply Lemma 6.3.3. For $k \in \{0, 1, \dots, p-1\}$, define

$$F_k^\pi = \{(j, r, s) \in T \times S \times S : v^{\pi(j)} r x + v^j s y \equiv k \pmod{p}\}.$$

Then, from (6.3), we see that

$$\mu_\pi(a^x, a^y) = C_0 + C_1\omega + \dots + C_{p-1}\omega^{p-1},$$

where $C_0 = e + |F_0^\pi|$ and $C_i = |F_i^\pi|$ for $i \in \{1, \dots, p-1\}$ and

$$\begin{aligned} C_0 + \dots + C_{p-1} &= e + tee = e + (p-1)e \\ &= pe. \end{aligned}$$

We can derive some properties of the coefficients $\{C_0, \dots, C_{p-1}\}$. For $k \in T$, define

$$\beta_k = |\{j \in T : \pi(j) - j \equiv k \pmod{t}\}|.$$

Lemma 6.3.4. *Let $k \in T$ be a unique integer such that $x^{-1}(-y) \in v^k S$. Then $|F_0^\pi| = \beta_k e$ and so $C_0 = (\beta_k + 1)e$.*

Proof. Consider

$$\begin{aligned} v^{\pi(j)}rx + v^j sy &\equiv 0 \pmod{p} \\ v^{\pi(j)}rx &\equiv v^j s(-y) \pmod{p} \\ v^{\pi(j)-j}rs^{-1} &\equiv x^{-1}(-y) \pmod{p}. \end{aligned}$$

Since $x^{-1}(-y)$ belongs to a unique S -coset, write $x^{-1}(-y) = v^k z$ for a unique k and $z \in S$. Then

$$v^{\pi(j)-j}rs^{-1} \equiv v^k z \pmod{p}.$$

But $v^{\pi(j)-j}rs^{-1}$ belongs to the coset $v^{\pi(j)-j}S$. This is possible if and only if $\pi(j) - j \equiv k \pmod{t}$ (v has order t) and $rs^{-1} \equiv z \pmod{p}$. Suppose there exists j with $\pi(j) - j \equiv k \pmod{t}$, then the number of pairs (r, s) such that $rs^{-1} \equiv z \pmod{p}$ is equal to e ($= |S|$). Hence

$$|F_0^\pi| = |\{(j, r, s) \in T \times S \times S : v^{\pi(j)}rx + v^j sy \equiv 0 \pmod{p}\}| = \beta_k e.$$

□

Lemma 6.3.5. *If $m, n \in \mathbb{Z}_p^\times$ are in the same S -coset, then $C_m = C_n$.*

Proof. Let $m = v^k g, n = v^k h$ where $g, h \in S$. If $(j, r, s) \in F_{v^k g}^\pi$, then

$$\begin{aligned} v^{\pi(j)}x(r) + v^j y(s) &\equiv v^k g \pmod{p} \\ v^{\pi(j)}x(rg^{-1}h) + v^j y(sg^{-1}h) &\equiv v^k h \pmod{p}. \end{aligned}$$

So $(j, rg^{-1}h, sg^{-1}h) \in F_{v^k h}^\pi$. Conversely, if $(j, r, s) \in F_{v^k h}^\pi$, it is easy to verify that $(j, rh^{-1}g, sh^{-1}g) \in F_{v^k g}^\pi$. So we have a 1-1 correspondence $F_{v^k g}^\pi \leftrightarrow F_{v^k h}^\pi$, given by $(j, r, s) \leftrightarrow (j, rg^{-1}h, sg^{-1}h)$. Hence,

$$C_m = |F_m^\pi| = |F_{v^k g}^\pi| = |F_{v^k h}^\pi| = |F_n^\pi| = C_n.$$

□

So far, the results about the coefficients $\{C_0, \dots, C_{p-1}\}$ are valid for any $\pi \in \text{Sym}(T)$. However, if π is of the form $\pi(j) = j + \alpha \pmod{t}, \forall j$ for some $\alpha \in T$, then the coefficients $\{C_0, \dots, C_{p-1}\}$ have a particularly nice property.

Lemma 6.3.6. *Suppose π is of the form $\pi(j) = j + \alpha \pmod{t}, \forall j$ for some $\alpha \in T$. Then $C_m = C_n$ for all $m, n \in \mathbb{Z}_p^\times$.*

Proof. Let $m, n \in \mathbb{Z}_p^\times$. By Lemma 6.3.5, it suffices to assume $m = v^k, n = v^l$. Suppose $(j, r, s) \in F_{v^k}^\pi$. Then

$$\begin{aligned} v^{\pi(j)}x(r) + v^j y(s) &\equiv v^k \pmod{p} \\ v^{j+\alpha}x(r) + v^j y(s) &\equiv v^k \pmod{p} \\ v^{j+\alpha+l-k}x(r) + v^{j+l-k}y(s) &\equiv v^l \pmod{p} \\ v^{\pi(j+l-k)}x(r) + v^{j+l-k}y(s) &\equiv v^l \pmod{p}. \end{aligned}$$

So, $((j+l-k) \pmod{t}, r, s) \in F_{v^l}^\pi$. Similarly, if $(j, r, s) \in F_{v^l}^\pi$, then $((j+k-l) \pmod{t}, r, s) \in F_{v^k}^\pi$. So we have a 1-1 correspondence $F_{v^k}^\pi \leftrightarrow F_{v^l}^\pi$. Hence,

$$C_m = |F_{v^k}^\pi| = |F_{v^l}^\pi| = C_n.$$

□

Lemma 6.3.7. *Suppose π is of the form $\pi(j) = j + \alpha \pmod{t}, \forall j$ for some $\alpha \in T$. Then μ_π is perfect.*

Proof. Let $x, y \in \mathbb{Z}_p^\times$. It suffices to show that $\|\mu_\pi(a^x, a^y)\|_p \leq 1/p$. Using the notation discussed above, we can write

$$\mu(a^x, a^y) = C_0 + C_1\omega + \dots + C_{p-1}\omega^{p-1}, \quad (6.4)$$

where $C_0 = e + |F_0^\pi|$ and $C_m = |F_m^\pi|$ for $m \in \{1, \dots, p-1\}$ and

$$C_0 + C_1 + \dots + C_{p-1} = pe.$$

Let $k \in T$ be such that $x^{-1}(-y) \in v^k S$. Then, by Lemma 6.3.4,

$$C_0 = (\beta_k + 1)e.$$

By Lemma 6.3.4 and Lemma 6.3.6, and using the fact that $te = p-1$, we have,

$$\begin{aligned} (\beta_k + 1)e + (p-1)C_1 &= pe \\ (\beta_k + 1)e + teC_1 &= pe \\ C_1 &= \frac{p-1-\beta_k}{t} = \frac{te-\beta_k}{t} \\ &= e - \frac{\beta_k}{t}. \end{aligned}$$

So, $C_m = e - \beta_k/t$ for all $m \in \mathbb{Z}_p^\times$. Substituting in (6.4) yields

$$\begin{aligned} \mu(a^x, a^y) &= (\beta_k + 1)e + \left(e - \frac{\beta_k}{t}\right)(\omega + \omega^2 + \dots + \omega^{p-1}) \\ &= (\beta_k + 1)e - \left(e - \frac{\beta_k}{t}\right) \\ &= \beta_k e + \frac{\beta_k}{t}. \end{aligned}$$

But,

$$\begin{aligned} \beta_k &= |\{j \in T : \pi(j) - j \equiv k \pmod{t}\}| \\ &= |\{j \in T : j + \alpha - j \equiv k \pmod{t}\}| \\ &= |\{j \in T : \alpha \equiv k \pmod{t}\}|. \end{aligned}$$

This implies that $\beta_k = t\delta_{k\alpha}$ (where δ is the Kronecker delta function). Thus,

$$\begin{aligned} \mu(a^x, a^y) &= te\delta_{k\alpha} + \frac{t\delta_{k\alpha}}{t} \\ &= (te + 1)\delta_{k\alpha} \\ &= p\delta_{k\alpha}. \end{aligned}$$

Therefore, $\|\mu_\pi(a^x, a^y)\|_p \leq 1/p$ and μ_π is perfect as claimed. \square

Lemma 6.3.8. *Suppose that μ_π is perfect. Then π is of the form $\pi(j) = j + \alpha \pmod t, \forall j$ for some $\alpha \in T$.*

Proof. Let $\alpha = \pi(0)$. Choose $x, y \in \mathbb{Z}_p^\times$ such that $x^{-1}(-y) \in v^\alpha S$. This is possible, for example, choose $x^{-1} = v^\alpha$ and $y = p - 1$. Then

$$\begin{aligned} \beta_\alpha &= |\{j \in T : \pi(j) - j \equiv \alpha \pmod t\}| \\ &\geq 1 \quad (\because \text{the set contains } 0). \end{aligned}$$

We claim that $\beta_\alpha = t$. This will imply that $\pi(j) = j + \alpha \pmod t$ for all $j \in T$, and the lemma will be proved. To see this, as before we can write

$$\mu(a^x, a^y) = C_0 + C_1\omega + \dots + C_{p-1}\omega^{p-1},$$

where $C_0 = e + |F_0^\pi|$ and $C_m = |F_m^\pi|$ for $m \in \{1, \dots, p-1\}$ and

$$C_0 + C_1 + \dots + C_{p-1} = pe. \quad (6.5)$$

By Lemma 6.3.4, we have $C_0 = (\beta_\alpha + 1)e$ and by Lemma 6.3.5, we have $C_m = C_n$ whenever $m, n \in \mathbb{Z}_p^\times$ are in the same S -coset. Thus, (6.5) becomes

$$\begin{aligned} (\beta_\alpha + 1)e + eC_1 + eC_v + eC_{v^2} + \dots + eC_{v^{t-1}} &= pe \\ (\beta_\alpha + 1) + C_1 + C_v + C_{v^2} + \dots + C_{v^{t-1}} &= p. \end{aligned} \quad (6.6)$$

Since $\beta_\alpha \geq 1$ and $C_m \geq 0 \forall m \in \mathbb{Z}_p^\times$, this implies $0 \leq C_{v^i} \leq p - 2 \forall i$, and hence, by Lemma 6.3.5,

$$0 \leq C_m \leq p - 2, \quad \forall m \in \mathbb{Z}_p^\times.$$

Since μ_π is perfect, $\|\mu_\pi(a^x, a^y)\|_p \leq 1/p$. So $C_m \equiv C_n \pmod p$ for all $m, n \in \{0, 1, \dots, p-1\}$, by Lemma 6.3.3. This means

$$C_m = C_n \quad \forall m, n \in \mathbb{Z}_p^\times.$$

Therefore, (6.6) becomes

$$\begin{aligned} (\beta_\alpha + 1) + tC_1 &= p \\ C_1 &= \frac{p - \beta_\alpha - 1}{t} = \frac{(te + 1) - \beta_\alpha - 1}{t} \\ &= e - \frac{\beta_\alpha}{t}. \end{aligned}$$

But C_1 is an integer and $1 \leq \beta_\alpha \leq t$, this forces $\beta_\alpha = t$ as required. \square

To summarize, we have shown that μ_π is perfect if and only if π is of the form $\pi(j) = j + \alpha \pmod t, \forall j$ for some $\alpha \in T$. So the subgroup $X \leq \text{PI}(A)$ consists of perfect isometries I_π where $\pi \in \text{Sym}(T)$ is of the above form. Hence $X \cong C_t$, and so $\text{PI}(A) \cong S_e \times C_t \times \langle -\text{id} \rangle$. Since $\text{PI}(B) \cong \text{PI}(A)$, this proves Theorem 6.3.1.

6.4 Comparison with Picard groups and derived Picard groups

Let B be a block of $\mathcal{O}G$ with a cyclic defect group D of order p^n and inertial quotient group E of order e . As we already know, a derived equivalence gives rise to a perfect isometry. It is interesting to know, in this case, if every perfect isometry in $\text{PI}(B)$ can also be given by a derived equivalence, in other words, if the homomorphism $\text{DPic}(B) \longrightarrow \text{PI}(B)$ is surjective.

Kunugi [29] showed that if $e > 1$ then, for any perfect isometry $I \in \text{PI}(\mathcal{O}(D \rtimes E))$, there exists a tilting complex $X \in \text{DPic}(\mathcal{O}(D \rtimes E))$ such that the induced perfect isometry agrees with I on the non-exceptional characters. However, in the case $D \cong C_p$ and $e = p - 1$, we also have perfect isometries sending a non-exceptional character to the (single) exceptional character (Theorem 6.0.5(ii)). These perfect isometries are not covered by tilting complexes in Kunugi's result. In this section we will show that there are also tilting complexes that induce these perfect isometries. In fact, we will prove the following theorem.

Theorem 6.4.1. *If $e = 1$ or ($e > 1$ and $t = 1$), then the homomorphism $\text{DPic}(B) \longrightarrow \text{PI}(B)$ is surjective. If $e > 1$ and $t > 1$ and Conjecture 6.0.6 holds (for example when $D \cong C_p$), then $\text{DPic}(B) \longrightarrow \text{PI}(B)$ is surjective.*

Corollary 6.4.2. *If B is a block with defect group C_p , then every perfect isometry in $\text{PI}(B)$ can be given by a tilting complex in $\text{DPic}(B)$.*

Since B is derived equivalent to $\mathcal{O}(D \rtimes E)$, they are also perfectly isometric. Then $\text{DPic}(B) \cong \text{DPic}(\mathcal{O}(D \rtimes E))$, and $\text{PI}(B) \cong \text{PI}(\mathcal{O}(D \rtimes E))$ and we have:

$$\text{DPic}(B) \cong \text{DPic}(\mathcal{O}(D \rtimes E)) \longrightarrow \text{PI}(\mathcal{O}(D \rtimes E)) \cong \text{PI}(B).$$

So it suffices to show that $\text{DPic}(\mathcal{O}(D \rtimes E)) \longrightarrow \text{PI}(\mathcal{O}(D \rtimes E))$ is surjective.

Let $H = D \rtimes E$ and $A = \mathcal{O}H$. We will study $\text{Pic}(A)$ and $\text{DPic}(A)$ (in some cases) and their images in $\text{PI}(A)$. It is necessary to study A in three separate cases, depending on e and $t = (p^n - 1)/e$.

Case: $e = 1$

In this case we have $H = D \cong C_{p^n}$.

Proposition 6.4.3. *The homomorphism $\text{DPic}(\mathcal{O}C_{p^n}) \longrightarrow \text{PI}(\mathcal{O}C_{p^n})$ is surjective.*

Proof. By Theorem 6.0.5(i), we have

$$\text{PI}(A) \cong (D \rtimes \text{Aut}(D)) \times \langle -id \rangle.$$

By Theorem 5.2.2 and Corollary 5.1.3, we have

$$\text{Pic}(A) \cong (D \rtimes \text{Aut}(D)).$$

So, by Lemma 4.4.6, we can see that $\text{PI}(A) \cong \text{Pic}(A) \times \langle -id \rangle$.

The subgroup $\text{Sh}(A) \leq \text{DPic}(A)$ contains $A[m]$ for $m \in \mathbb{Z}$. If m is even, $A[m]$ corresponds to $id \in \text{PI}(A)$ while if m is odd, $A[m]$ corresponds to $-id$. Since, by Theorem 5.2.2,

$$\text{DPic}(A) \cong (D \rtimes \text{Aut}(D)) \times \text{Sh}(A),$$

and $\text{Sh}(A) \leq \text{DPic}(A)$ is mapped to $\langle -id \rangle \leq \text{PI}(A)$, the proposition is proved. \square

Case: $e > 1$ and $t = 1$

In this case we must have $D \cong C_p$ and so $H = C_p \rtimes C_{p-1}$.

The Picard groups for blocks with non-trivial inertial quotient are given by Linckelmann in the following theorem.

Theorem 6.4.4 ([32], Theorem 11.4.11). *Let B be a block of a finite group G having a cyclic defect group D with non-trivial inertial quotient E acting on D . Let F be*

the automorphism group of the Brauer tree of B . Then F is cyclic of order dividing $|E|$ and we have

$$\text{Pic}(B) \cong F \times \text{Aut}(D)/E.$$

In this case we have $E \cong \text{Aut}(D)$. Since a block whose cyclic defect group is normal has Brauer tree a star with exceptional vertex in the center [2, Theorem 19.1], the automorphism group of the Brauer tree of A is C_{p-1} . By Theorem 6.4.4, therefore,

$$\text{Pic}(A) \cong C_{p-1}.$$

The Picard group is generated by B_σ , where $\sigma \in \text{Aut}_{\mathcal{O}}(A)$ is defined by $g^\sigma = \lambda(g^{-1})g, \forall g \in H$, for some non-trivial linear character $\lambda \in \text{Irr}(A)$. Note that the exceptional character is fixed by the images of $\text{Pic}(A)$ in $\text{PI}(A)$.

We next need the following theorem, due to Rouquier, which we will state in a form that is useful to us.

Theorem 6.4.5. [43, Theorem 7] *Let B be a block of $\mathcal{O}G$ for a finite group G . Suppose there exists a direct summand P of $\bigoplus_i P_i \otimes_{\mathcal{O}} P_i^\vee$, where P_i runs over all the projective indecomposable B -modules, such that $0 \rightarrow P \xrightarrow{0} B \rightarrow 0$ induces an isometry from $R_K(B)$ to itself. Then there is a complex $C = 0 \rightarrow P \rightarrow B \rightarrow 0$ inducing a derived equivalence between B and itself.*

Proposition 6.4.6. *The homomorphism $\text{DPic}(\mathcal{O}(C_p \rtimes C_{p-1})) \rightarrow \text{PI}(\mathcal{O}(C_p \rtimes C_{p-1}))$ is surjective.*

Proof. Let $A = \mathcal{O}(C_p \rtimes C_{p-1})$. By Theorem 6.0.5(ii), we have

$$\text{PI}(A) \cong S_p \times \langle -\text{id} \rangle.$$

Suppose now that $\text{Irr}(A) = \{\chi_1, \dots, \chi_{e-1}, \theta\}$, where θ is the exceptional character. Since the Brauer tree is a star, we label in such a way that the projective indecomposable P_i has the K -character $\chi_i + \theta$.

Let $C_i = 0 \rightarrow P_i \otimes_{\mathcal{O}} P_i^\vee \rightarrow A \rightarrow 0$. The K -character of $P_i \otimes_{\mathcal{O}} P_i^\vee$ is

$$(\chi_i + \theta)(\chi_i + \theta) = (\chi_i)(\chi_i) + (\theta)(\theta) + (\chi_i)(\theta) + (\theta)(\chi_i).$$

The K -character of A as (A, A) -bimodule is

$$\sum_i (\chi_i)(\chi_i) + (\theta)(\theta).$$

Thus the K -character of C_i is

$$\begin{aligned} & - [(\chi_i)(\chi_i) + (\theta)(\theta) + (\chi_i)(\theta) + (\theta)(\chi_i)] + \left[\sum_i (\chi_i)(\chi_i) + (\theta)(\theta) \right] \\ & = \sum_{j \neq i} (\chi_j)(\chi_j) - (\theta)(\chi_i) - (\chi_i)(\theta). \end{aligned}$$

This generalized character of C_i induces a perfect isometry sending $\chi_i \mapsto -\theta$, $\theta \mapsto -\chi_i$ and fixes everything else. By Theorem 6.4.5, C_i is a tilting complex, and so, $C_i \in \text{DPic}(A)$, $\forall i$. The images of $\{C_i : i = 1, \dots, e-1\}$ in $\text{PI}(A)$ generates S_p , whereas $A[-1]$ gives $-id$. Thus $\text{DPic}(A) \rightarrow \text{PI}(A)$ is surjective as claimed. \square

Case: $e > 1$ and $t > 1$

By Theorem 6.0.5(iii) and Proposition 6.2.7, we have,

$$\text{PI}(A) \cong S_e \times X \times \langle -id \rangle,$$

where X contains $\text{Aut}(D)/E$. If $D \cong C_p$, then $X \cong \text{Aut}(D)/E \cong C_t$ by Theorem 6.3.1.

The Picard group is given by Theorem 6.4.4.

$$\text{Pic}(A) \cong C_e \times \text{Aut}(D)/E.$$

The (isomorphic) subgroup C_e of $\text{Pic}(A)$ comes from the automorphisms $\sigma \in \text{Aut}_{\mathcal{O}}(A)$ of the form $g^\sigma = \lambda(g^{-1})g, \forall g \in H$, for some linear character $\lambda \in \text{Irr}(A)$. This corresponds to the perfect isometry $\chi \mapsto \lambda\chi$ for $\chi \in \text{Irr}(A)$.

Let σ be a generator of the cyclic group $\text{Aut}(D)/E$. Then σ can be regarded as an element in $\text{Aut}(D \rtimes E)$ that acts trivially on E . The $\text{Aut}(D)/E$ part in $\text{Pic}(A)$ is then generated by B_σ whose induced perfect isometry $\chi \mapsto \chi^\sigma$ generates the subgroup $\text{Aut}(D)/E \leq X$ in $\text{PI}(A)$. The map $\chi \mapsto \chi^\sigma$ fixes every non-exceptional characters in $\text{Irr}(A)$.

Lemma 6.4.7. *Suppose $1 < e < |D| - 1$. If Conjecture 6.0.6 holds (for example $D \cong C_p$), then $\text{DPic}(\mathcal{O}(D \rtimes E)) \longrightarrow \text{PI}(\mathcal{O}(D \rtimes E))$ is surjective.*

Proof. Let $A = \mathcal{O}(D \rtimes E)$. By Conjecture 6.0.6 and Proposition 6.2.7, we have

$$\text{PI}(A) \cong S_e \times \text{Aut}(D)/E \times \langle -\text{id} \rangle,$$

where S_e is the symmetric group on the non-exceptional characters, and $\text{Aut}(D)/E$ is a permutation group on the exceptional characters (we know that every perfect isometry permutes the non-exceptional and exceptional characters separately by Lemma 6.2.3).

Kunugi [29, Proposition 3.2] showed that, for every permutation of the non-exceptional characters, there is a tilting complex inducing the same permutation on the non-exceptional characters. The perfect isometries from Kunugi's complexes cover the S_e part of $\text{PI}(A)$, but may incur non-identity permutations on the exceptional characters. However, since $\text{Pic}(A)$ is mapped surjectively onto $\text{Aut}(D)/E \leq \text{PI}(A)$, we can cancel out these effect by composing with complexes from $\text{Pic}(A)$ (embedded in $\text{DPic}(A)$), which fixes non-exceptional characters. So, there are tilting complexes which permute only the non-exceptional characters. Hence, $\text{DPic}(A)$ is mapped onto $S_e \times \text{Aut}(D)/E$. Finally, $A[1]$ gives $-\text{id} \in \text{PI}(A)$. Hence, $\text{DPic}(A) \longrightarrow \text{PI}(A)$ is surjective. \square

Chapter 7

Blocks with TI defect groups

Let G be a finite group. Let B be a block of $\mathcal{O}G$ with defect group D . Let $H = N_G(D)$ and A be a block of $\mathcal{O}H$ with Brauer correspondent B . If D is abelian then we can expect, by Broué's conjecture, that A and B are perfectly isometric. If D is not abelian, however, then one cannot expect such perfect isometries, especially if $k(A) \neq k(B)$. We want to study a case where we do have $k(A) = k(B)$. This happens, for example, when D is trivial intersection.

A subgroup $D \leq G$ is called *trivial intersection* (TI) if $D^g \cap D = 1$ for every $g \in G - N_G(D)$. An and Eaton [4] showed that if D is trivial intersection then, amongst other things, we do have $k(A) = k(B), l(A) = l(B)$. This means that, at least, there are isometries between $R_K(A)$ and $R_K(B)$. Such isometries may not necessarily be perfect. Cliff [10] and Robinson [40], for example, showed that there is no perfect isometry when G is a Suzuki group $Sz(q)$. On the other hands, Narasaki [33] defined a generalization of a perfect isometry and showed that such isometries always exist in TI defect group cases.

In this chapter we will investigate the relationship between perfect isometry groups $\text{PI}(A)$ and $\text{PI}(B)$ when a defect group D is TI. Blocks with TI defect groups are classified by An and Eaton in [3]. We will pick only some examples, most notably the family of the Suzuki groups $Sz(q)$.

7.1 Some Observations

In this section, we will describe some phenomena that we observed in TI cases. We were not able to prove these observations for all TI cases, even computationally, since many of TI blocks have a large number of irreducible characters. This made our GAP programs take a very long time to compute perfect isometry groups (see the Index at the end of this thesis).

Perfect embedding

Let G, H be finite groups. Let $B \in \text{Bl}(G), A \in \text{Bl}(H)$. Assume that $k(B) = k(A)$. We already know that if A and B are perfectly isometric, then $\text{PI}(A) \cong \text{PI}(B)$. If A and B are not perfectly isometric, it may happen that we can embed one perfect isometry group inside the other.

Definition 7.1.1. Let $I : R_K(A) \rightarrow R_K(B)$ be an isometry. We say that I is a *perfect embedding* (from A to B) if for every $\alpha \in \text{PI}(A)$, we have $I\alpha I^{-1} \in \text{PI}(B)$.

Suppose $I : R_K(A) \rightarrow R_K(B)$ is a perfect embedding. It is clear that the map $\alpha \mapsto I\alpha I^{-1}$ gives a monomorphism $\text{PI}(A) \rightarrow \text{PI}(B)$, hence an embedding of $\text{PI}(A)$ into $\text{PI}(B)$. Every perfect isometry is obviously a perfect embedding. An interesting case is when a perfect embedding is not a perfect isometry. In TI case, we know that there is an isometry $I : R_K(A) \rightarrow R_K(B)$. If A and B are not perfectly isometric, it may happen that a perfect embedding still exists. However, in the case of the Suzuki groups $Sz(q)$, we will show that there is neither a perfect isometry nor a perfect embedding between A and B if $q > 8$ (there is a perfect embedding when $q = 8$).

Some subgroups of perfect isometry groups

Recall that $\text{CF}_{p'}(G, B; K)$ is the subspace of $\text{CF}(G, B; K)$ consisting of class functions vanishing on p -singular elements. If $I \in \text{PI}(B)$, then I preserves $\text{CF}_{p'}(G, B; K)$ (Proposition 3.1.6). So, we can define the following subgroup.

Definition 7.1.2. Define $\text{PI}_s(B)$ by

$$\text{PI}_s(B) = \{I \in \text{PI}(B) : I(\alpha) = \alpha, \forall \alpha \in \text{CF}_{p'}(G, B; K)\}$$

Lemma 7.1.3. $\text{PI}_s(B)$ is a normal subgroup of $\text{PI}(B)$.

Proof. That $\text{PI}_s(B)$ is a subgroup is clear from the definition. We will show that $\text{PI}_s(B)$ is normal in $\text{PI}(B)$.

Let $I \in \text{PI}_s(B)$ and $J \in \text{PI}(B)$. For any class function $\alpha \in \text{CF}_{p'}(G, B; K)$,

$$\begin{aligned} JIJ^{-1}(\alpha) &= J(I(J^{-1}(\alpha))) \\ &= J(J^{-1}(\alpha)) \quad \text{since } J^{-1}(\alpha) \in \text{CF}_{p'}(G, B; K) \\ &= \alpha. \end{aligned}$$

Thus $JIJ^{-1} \in \text{PI}_s(B)$, and so $\text{PI}_s(B) \triangleleft \text{PI}(B)$ as desired. \square

If there exists a perfect isometry $I : R_K(A) \longrightarrow R_K(B)$, then the map $J \mapsto IJI^{-1}$ gives isomorphisms $\text{PI}_s(A) \cong \text{PI}_s(B)$. In TI cases, we observed that $\text{PI}_s(A) \cong \text{PI}_s(B)$ even though A and B are not perfectly isometric.

7.2 Suzuki group $Sz(q)$

In this section, we will study the principal 2-block of the Suzuki group $Sz(q)$ and the principal 2-block of its Sylow normalizer. The perfect isometry groups for both blocks will be calculated and we shall see that they are visibly different. This implies that there is no perfect isometry between the two blocks.

For the rest of the section, we fix the following notations. Let $p = 2, q = 2^{2n+1}, r = 2^{n+1}$ where n is a positive integer. Let $G = Sz(q)$ be the Suzuki group of order $q^2(q-1)(q^2+1)$. Let B be the principal block of $\mathcal{O}G$ with defect group D . Let A be the principal block of $\mathcal{O}N_G(D)$.

7.2.1 Principal block of G

The main result for this section is the following theorem.

Theorem 7.2.1. *Let $p = 2$. Let $G = Sz(q)$ where $q > 2$. Let B be the principal block of $\mathcal{O}G$. Let $r = \sqrt{2q}$. Then, every perfect isometry in $\text{PI}(B)$ has a homogenous sign. Furthermore, the irreducible characters $\text{Irr}(B)$ can be partitioned into*

$$\begin{aligned} \mathbb{X} &= \{\mathbf{1}\} \cup \{X_i : i \in \mathcal{I}\}, & |\mathbb{X}| &= q/2 \\ \mathbb{Y} &= \{Y_j : j \in \mathcal{J}\}, & |\mathbb{Y}| &= (q+r)/4 \\ \mathbb{Z} &= \{Z_k : k \in \mathcal{K}\}, & |\mathbb{Z}| &= (q-r)/4 \\ \mathbb{W} &= \{W_l : l \in \mathcal{L}\}, & |\mathbb{W}| &= 2, \end{aligned}$$

for some index sets $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}$, such that

$$\begin{aligned} \text{PI}(B) &= \text{Sym}(\mathbb{X}) \times \text{Sym}(\mathbb{Y}) \times \text{Sym}(\mathbb{Z}) \times \text{Sym}(\mathbb{W}) \times \langle -id \rangle \\ &\cong S_{q/2} \times S_{(q+r)/4} \times S_{(q-r)/4} \times S_2 \times \langle -id \rangle. \end{aligned}$$

Character table

The information about conjugacy classes and irreducible characters in $\text{Irr}(B)$ can be found in the work of Suzuki [49].

Let $\mathcal{I} = \{1, \dots, q/2-1\}$, $\mathcal{J} = \{1, \dots, (q+r)/4\}$, $\mathcal{K} = \{1, \dots, (q-r)/4\}$, $\mathcal{L} = \{1, 2\}$. Following Suzuki's notations [49, Page 141], the irreducible characters in $\text{Irr}(B)$ are denoted by $X_i (i \in \mathcal{I})$, $Y_j (j \in \mathcal{J})$, $Z_k (k \in \mathcal{K})$, $W_l (l \in \mathcal{L})$ and the trivial character is denoted by $\mathbf{1}$.

Table 7.1: Characters, their degrees and the number of characters in each type.

| Characters | Degrees | Number of characters |
|--------------|----------------|----------------------|
| $\mathbf{1}$ | 1 | 1 |
| W_l | $r(q-1)/2$ | 2 |
| Y_j | $(q-r+1)(q-1)$ | $(q+r)/4$ |
| X_i | q^2+1 | $q/2-1$ |
| Z_k | $(q+r+1)(q-1)$ | $(q-r)/4$ |

The group G contains three cyclic subgroups A_0, A_1, A_2 of order $q-1, q+r+1, q-r+1$ respectively. Let π_i denote representatives of conjugacy classes of non-identity

elements of A_i . There is an involution σ and a unique conjugacy class of involutions. There are two classes of elements of order 4; represented by ρ and ρ^{-1} . In total, there are 7 conjugacy classes of G , represented by $1, \sigma, \rho, \rho^{-1}, \pi_0, \pi_1, \pi_2$. The sizes of classes are [10, Page 80] $1, (q^2 + 1)(q - 1), \frac{q(q^2+1)(q-1)}{2}, \frac{q(q^2+1)(q-1)}{2}, q^2(q^2 + 1), q^2(q - 1)(q - r + 1), q^2(q - 1)(q + r + 1)$ respectively.

Let ω_0 be a primitive $(q - 1)$ -st root of unity. Denote by ξ_0 a generator of A_0 . Define a character λ_0^i of A_0 by

$$\lambda_0^i(\xi_0^j) = \omega_0^{ij} + \omega_0^{-ij}.$$

Let ω_1 and ω_2 be primitive $(q+r+1)$ -th and $(q-r+1)$ -th roots of unity respectively. If ξ_j is a generator of $A_j, j = 1, 2$, define a character λ_j^i of A_i by

$$\lambda_j^i(\xi_j^k) = \omega_j^{ik} + \omega_j^{ikq} + \omega_j^{-ik} + \omega_j^{-ikq}.$$

The character table (for $\text{Irr}(B)$) of $Sz(q)$ is the following ([49, Theorem 13]).

Table 7.2: Table of characters in the principal 2-block of $Sz(q)$.

| $ C_G(g) $ | $ G $ | q^2 | $2q$ | $2q$ | $\frac{q^2+1}{q+r+1}$ | $q - 1$ | $\frac{q^2+1}{q-r+1}$ |
|--------------|----------------------|----------------|-----------------|-----------------|-----------------------|----------------------|-----------------------|
| g | 1 | σ | ρ | ρ^{-1} | π_2 | π_0 | π_1 |
| \mathbb{I} | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| W_1 | $r(q - 1)/2$ | $-\frac{r}{2}$ | $\frac{ri}{2}$ | $-\frac{ri}{2}$ | -1 | 0 | 1 |
| W_2 | $r(q - 1)/2$ | $-\frac{r}{2}$ | $-\frac{ri}{2}$ | $\frac{ri}{2}$ | -1 | 0 | 1 |
| Y_j | $(q - r + 1)(q - 1)$ | $r - 1$ | -1 | -1 | 0 | 0 | $-\lambda_1^j(\pi_1)$ |
| $j \in J$ | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| X_i | $q^2 + 1$ | 1 | 1 | 1 | 0 | $\lambda_0^i(\pi_0)$ | 0 |
| $i \in I$ | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| Z_k | $(q + r + 1)(q - 1)$ | $-r - 1$ | -1 | -1 | $-\lambda_2^k(\pi_2)$ | 0 | 0 |
| $k \in K$ | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |

Proof of Theorem 7.2.1

Let $I_\mu : R_K(B) \longrightarrow R_K(B)$ be an isometry, where μ is a character as in (3.3). We will identify $I_\mu = (\beta_\mu, \varepsilon_\mu)$, where β_μ is a bijection and ε_μ is a sign function:

$$\beta_\mu : \text{Irr}(B) \mapsto \text{Irr}(B), \quad \varepsilon_\mu : \text{Irr}(B) \mapsto \{\pm 1\}$$

associated to I_μ such that $I_\mu(\chi) = \varepsilon_\mu(\chi)\beta_\mu(\chi)$ for all $\chi \in \text{Irr}(B)$.

We will partition the characters in $\text{Irr}(B)$ into the following four groups:

$$\mathbb{X} = \{\mathbf{1}\} \cup \{X_i : i \in I\}$$

$$\mathbb{Y} = \{Y_j : j \in J\}$$

$$\mathbb{Z} = \{Z_k : k \in K\}$$

$$\mathbb{W} = \{W_l : l \in L\}.$$

Our first result is that, any perfect isometry in $\text{PI}(B)$ has a homogenous sign and preserves the above partition.

Lemma 7.2.2. *Let $I_\mu = (\beta_\mu, \varepsilon_\mu)$ be a perfect isometry. Then β_μ preserves the partition $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ and $\varepsilon_\mu = \pm 1$ (homogenous sign).*

Proof. For convenience, let us define $X_0 = \mathbf{1}$ and $\tilde{\mathcal{I}} = \mathcal{I} \cup \{0\}$, so that $\mathbb{X} = \{X_i : i \in \tilde{\mathcal{I}}\}$. To show that the sign is homogenous, we assume that $\varepsilon_\mu(X_0) = 1$ and show that $\varepsilon_\mu = 1$.

For any $i, j \in \mathcal{I}$, consider the class function

$$f = X_i - X_j.$$

Since the p -singular elements of G are $\text{ccl}_G(\sigma) \cup \text{ccl}_G(\rho) \cup \text{ccl}_G(\rho^{-1})$, we see from the character table that f vanishes on all p -singular elements of G . Since I_μ is perfect, $I_\mu(f)$ must also vanish on all p -singular elements. In particular, $I_\mu(X_i)(\sigma) = I_\mu(X_j)(\sigma)$. Since i, j are arbitrary, we have that

$$I_\mu(X_i)(\sigma) = \varepsilon_\mu(X_i)\beta_\mu(X_i)(\sigma) = \text{constant}, = c \text{ say}, \quad \forall i \in \tilde{\mathcal{I}}.$$

This means that there must be $q/2$ copies of c in the sequence $[I_\mu(X_i)(\sigma), i \in \tilde{\mathcal{I}}]$. The possible values of $\beta_\mu(\chi)(\sigma)$ for $\chi \in \text{Irr}(B)$ are $1, -r/2, r-1, -r-1$ with the number of copies $q/2, 2, (q+r)/4, (q-r)/4$ respectively. Since $\varepsilon_\mu(X_0) = +1$, we must have $I_\mu(X_0)(\sigma) \neq -1$. Suppose that $I_\mu(X_0)(\sigma) \neq 1$. Then $c \neq \pm 1$. So, ε_μ must not be constant (not enough copies of c), and some numbers in $\{1, -r/2, r-1, -r-1\}$ must be equal to negatives of other numbers in the same set. But the numbers

$\pm 1, \pm(-r/2), \pm(r-1), \pm(-r-1)$ are all distinct, this is impossible. Thus, we must have $c = 1$. Hence,

$$\beta_\mu(X_i) \in \mathbb{X}, \forall i \in \tilde{\mathcal{I}} \quad \text{and} \quad \varepsilon_\mu(X_i) = +1, \forall i \in \tilde{\mathcal{I}}.$$

Next, consider the class function

$$f = Y_i - Y_j.$$

Since f vanishes on p -singular elements of G , so does $I_\mu(f)$. In particular,

$$I_\mu(Y_i)(\sigma) = \varepsilon_\mu(Y_i)\beta_\mu(Y_i)(\sigma) = \text{constant}, \quad \forall i \in \tilde{\mathcal{J}}.$$

The possible values of $\beta_\mu(\chi)(\sigma)$ for $\chi \in \text{Irr}(B) - \mathbb{X}$ are $-r/2, r-1, -r-1$ with the number of copies $2, (q+r)/4, (q-r)/4$ respectively. Since $(q+r)/4 = \max\{2, (q+r)/4, (q-r)/4\}$, the same reasoning as before shows that

$$\beta_\mu(Y_i) \in \mathbb{Y}, \forall i \in \mathcal{J} \quad \text{and} \quad \varepsilon_\mu(Y_i) = \text{constant}, \forall i \in \mathcal{J}.$$

Next, consider the class function

$$f = Z_i - Z_j.$$

Since f vanishes on p -singular elements of G , so does $I_\mu(f)$. In particular,

$$I_\mu(Z_i)(\sigma) = \varepsilon_\mu(Z_i)\beta_\mu(Z_i)(\sigma) = \text{constant}, \quad \forall i \in \tilde{\mathcal{K}}.$$

The possible values of $\beta_\mu(\chi)(\sigma)$ for $\chi \in \text{Irr}(B) - \{\mathbb{X} \cup \mathbb{Y}\}$ are $-r/2, -r-1$ with the number of copies $2, (q-r)/4$ respectively. Since $(q-r)/4 > 2$, the same reasoning as before shows that

$$\beta_\mu(Z_i) \in \mathbb{Z}, \forall i \in \mathcal{K} \quad \text{and} \quad \varepsilon_\mu(Z_i) = \text{constant}, \forall i \in \mathcal{K}.$$

Finally we must have

$$\beta_\mu(W) \in \mathbb{W}, \quad \forall W \in \mathbb{W}.$$

This proves that I_μ preserves the partition $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ (with signs). It now remains to show that $\varepsilon_\mu(\chi) = 1$ for all $\chi \in \mathbb{Y} \cup \mathbb{Z} \cup \mathbb{W}$.

Let ξ_1^k be a non-identity element in A_1 . By the column orthogonality relation:

$$\begin{aligned} \sum_{\chi \in \text{Irr}(G)} \chi(\xi_1^k) \bar{\chi}(\sigma) &= 0 \\ 1 - r/2 - r/2 + (r-1) \sum_{j \in \mathcal{J}} [-\lambda_1^j(\xi_1^k)] &= 0 \\ \therefore \sum_{j \in \mathcal{J}} [-\lambda_1^j(\xi_1^k)] &= 1. \end{aligned}$$

Consider,

$$\begin{aligned} \mu(\sigma, \xi_1^k) &= \sum_{\chi \in \text{Irr } B} I_\mu(\chi)(\sigma) \chi(\xi_1^k) \\ &= 1 + \varepsilon_\mu(W_1)(-r/2) + \varepsilon_\mu(W_2)(-r/2) + (r-1) \varepsilon_\mu(Y_1) \sum_{j \in \mathcal{J}} [-\lambda_1^j(\xi_1^k)] \\ &= 1 + \varepsilon_\mu(W_1)(-r/2) + \varepsilon_\mu(W_2)(-r/2) + (r-1) \varepsilon_\mu(Y_1) \end{aligned}$$

Since μ is perfect, $\mu(\sigma, \xi_1^k) = 0$. This implies that

$$\varepsilon_\mu(W) = +1 \quad \forall W \in \mathbb{W},$$

and $\varepsilon_\mu(Y_1) = 1$, that is,

$$\varepsilon_\mu(Y) = +1 \quad \forall Y \in \mathbb{Y}.$$

Let ξ_2^j be a non-identity element in A_2 . By orthogonality relation applied to the classes of σ and ξ_2^j , we see that,

$$\sum_{k \in \mathcal{K}} (-\lambda_2^k(\xi_2^j)) = 1.$$

Consider,

$$\begin{aligned} \mu(\sigma, \xi_2^k) &= \sum_{\chi \in \text{Irr } B} I_\mu(\chi)(\sigma) \chi(\xi_2^k) \\ &= 1 + (-1)(-r/2) + (-1)(-r/2) + (-r-1) \varepsilon_\mu(Z_1) \sum_{k \in \mathcal{K}} [-\lambda_2^k(\xi_2^j)] \\ &= 1 + r + (-r-1) \varepsilon_\mu(Z_1). \end{aligned}$$

Since μ is perfect, $\mu(\sigma, \xi_2^k) = 0$. Therefore,

$$\varepsilon_\mu(Z) = +1 \quad \forall Z \in \mathbb{Z}.$$

So, $\varepsilon_\mu(\chi) = +1, \forall \chi \in \text{Irr}(B)$, given $\varepsilon_\mu(X_0) = +1$. Hence ε_μ is homogenous. \square

Next, we will show the converse of the previous lemma.

Lemma 7.2.3. *Any isometry $I_\mu = (\beta_\mu, \varepsilon_\mu)$ such that β_μ preserves the partition $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ and $\varepsilon_\mu = \pm 1$ is perfect.*

Proof. Without loss of generality, assume that $\varepsilon_\mu = 1$. Let $g, h \in G$. Suppose exactly one of g, h is p -singular. Let $\mathbb{U} \in \{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$. We see from the character table (Table 7.2) that, either $\chi(g)$ is constant for all $\chi \in \mathbb{U}$ or $\chi(h)$ is constant for all $\chi \in \mathbb{U}$. In any case, we have

$$\sum_{\chi \in \mathbb{U}} \varepsilon_\mu(\chi) \beta_\mu(\chi)(g) \chi(h) = \sum_{\chi \in \mathbb{U}} \beta_\mu(\chi)(g) \chi(h) = \sum_{\chi \in \mathbb{U}} \chi(g) \chi(h).$$

Consequently,

$$\mu(g, h) = \sum_{\chi \in \text{Irr}(B)} \varepsilon_\mu(\chi) \beta_\mu(\chi)(g) \chi(h) = \sum_{\chi \in \text{Irr}(B)} \chi(g) \chi(h) = \mu_{id}(g, h) = 0,$$

since id is a perfect isometry. This proves the separation condition.

For the integrality condition, since p divides $|C_G(g)|$ if and only if g is a p -singular element, it remains to consider $\mu(g, h)$ when g or h is p -singular. The case where exactly one of g, h is p -singular has been covered in the separation condition. So assume now that g, h are both p -singular. Then, up to conjugation, $g, h \in \{\sigma, \rho, \rho^{-1}\}$. We see from Table 7.2 that

$$\sum_{X \in \mathbb{X}} \beta_\mu(X)(g) X(h) = q/2 \tag{7.1}$$

$$\sum_{Y \in \mathbb{Y}} \beta_\mu(Y)(g) Y(h) = \left(\frac{q+r}{4}\right) Y_1(g) Y_1(h) \tag{7.2}$$

$$\sum_{Z \in \mathbb{Z}} \beta_\mu(Z)(g) Z(h) = \left(\frac{q-r}{4}\right) Z_1(g) Z_1(h). \tag{7.3}$$

Suppose that $g, h \neq \sigma$. Then $|C_G(g)| = |C_G(h)| = 2q$. The contribution from \mathbb{Y} is $(q+r)/4$, the contribution from \mathbb{Z} is $(q-r)/4$, and the contribution from \mathbb{W} is $\pm q$. So

$$\mu(g, h) = q/2 + (q+r)/4 + (q-r)/4 \pm q = 0 \text{ or } 2q.$$

Thus $\mu(g, h)$ is divisible in \mathcal{O} by $|C_G(g)|$ and $|C_G(h)|$.

Suppose $g = \sigma$ and $h \in \{\rho, \rho^{-1}\}$. The contribution from \mathbb{Y} is $(q+r)(1-r)/4$, the contribution from \mathbb{Z} is $(q-r)(1+r)/4$, and the contribution from \mathbb{W} is 0. So

$$\mu(g, h) = q/2 + \frac{(q+r)(1-r)}{4} + \frac{(q-r)(1+r)}{4} + 0 = 0.$$

Suppose $g \in \{\rho, \rho^{-1}\}$ and $h = \sigma$. Then, contributions from $\mathbb{Y}, \mathbb{Z}, \mathbb{W}$ are the same as in the case $g = \sigma$ and $h \in \{\rho, \rho^{-1}\}$. So $\mu(g, h) = 0$.

Suppose $g = h = \sigma$. Then $|C_G(g)| = |C_G(h)| = q^2$. The contribution from \mathbb{Y} is $(q+r)(r-1)^2/4$, the contribution from \mathbb{Z} is $(q-r)(r+1)^2/4$, and the contribution from \mathbb{W} is q . So

$$\mu(g, h) = q/2 + \frac{(q+r)(r-1)^2}{4} + \frac{(q-r)(r+1)^2}{4} + q = q^2.$$

Thus $\mu(g, h)$ is divisible in \mathcal{O} by $|C_G(g)|$ and $|C_G(h)|$.

Therefore I_μ is a perfect isometry as claimed. □

With Lemma 7.2.2 and Lemma 7.2.3, we have proved Theorem 7.2.1.

7.2.2 Principal block of $N_G(D)$

We will now determine $\text{PI}(A)$.

Theorem 7.2.4. *Let $p = 2$. Let $G = Sz(q)$ where $q > 2$ and $D \in \text{Syl}_2(G)$. Let A be the principal block of $\mathcal{O}N_G(D)$. Then, every perfect isometry in $\text{PI}(A)$ has a homogenous sign. Furthermore, the irreducible characters in $\text{Irr}(A)$ can be labeled $\{\varphi_1, \dots, \varphi_{q-1}, \chi_1, \chi_2, \chi_3\}$ such that,*

$$\begin{aligned} \text{PI}(A) &= \text{Sym}\{\varphi_1, \dots, \varphi_{q-1}\} \times \text{Sym}\{\chi_2, \chi_3\} \times \langle -id \rangle \\ &\cong S_{q-1} \times S_2 \times \langle -id \rangle. \end{aligned}$$

Character table

References for the groups $N_G(D)$ and its irreducible characters can be found in [49], [10] and [33].

We have $|D| = q^2$ and $H = DC$ where $C = \langle a \rangle$ is a cyclic group of order $q-1$. Let $\omega = e^{2\pi i/(q-1)}$. The conjugacy classes of H are given as follows. There is a unique

class of involutions, represented by σ , $|C_H(\sigma)| = q^2$. There are two classes of elements of order 4, represented by ρ, ρ^{-1} , $|C_G(\rho)| = |C_H(\rho^{-1})| = 2q$. There are $q - 2$ classes of elements of odd orders, namely $\text{ccl}_H(a^j)$, $a^j \in C - \{1\}$, $|C_H(a^j)| = q - 1$.

The irreducible characters in $\text{Irr}(A)$ are $\{\varphi_1, \dots, \varphi_{q-1}, \chi_1, \chi_2, \chi_3\}$ and described as follows. Each irreducible character in $\text{Irr}(C)$ extends to φ_i for $i = 1, \dots, q - 1$, and we denote the trivial character by φ_1 . Each non-trivial irreducible character in $\text{Irr}(D)$ induces to χ_i for $i = 1 \dots, 3$.

Table 7.3: Table of characters in the principal 2-block of $N_G(D)$.

| $ C_H(h) $ | $ H $ | q^2 | $2q$ | $2q$ | $q - 1$ |
|-------------|--------------|----------------|-----------------|-----------------|-------------------|
| h | 1 | σ | ρ | ρ^{-1} | $\{a^j\}$ |
| φ_i | 1 | 1 | 1 | 1 | $\omega^{j(i-1)}$ |
| χ_1 | $q - 1$ | $q - 1$ | -1 | -1 | 0 |
| χ_2 | $r(q - 1)/2$ | $-\frac{r}{2}$ | $\frac{ri}{2}$ | $-\frac{ri}{2}$ | 0 |
| χ_3 | $r(q - 1)/2$ | $-\frac{r}{2}$ | $-\frac{ri}{2}$ | $\frac{ri}{2}$ | 0 |

Proof of Theorem 7.2.4

Let $\Phi = \{\varphi_1 \dots, \varphi_{q-1}\}$ and $\mathbb{X} = \{\chi_2, \chi_3\}$.

Lemma 7.2.5. *If $I \in \text{PI}(A)$, then I preserves the sets (up to signs) Φ and \mathbb{X} . In other words, $I(\Phi) \subset \pm\Phi$, $I(\chi_1) = \pm\chi_1$ and $I(\mathbb{X}) \subset \pm\mathbb{X}$.*

Proof. Since $|\varphi_i(1)|_2 = |\chi_1(1)|_2 = 1$ for $i = 1, \dots, q - 1$ and $|\chi_2(1)|_2 = |\chi_3(1)|_2 > 1$, we must have $I(\mathbb{X}) \subset \pm\mathbb{X}$. For $i, j = 1, \dots, q - 1$, consider the class function

$$f = \varphi_i - \varphi_j.$$

From the character table (Table 7.3), we see that f vanishes on p -singular elements of H . Since I is perfect, $I(f)$ also vanishes on p -singular elements of H . In particular,

$$I(\varphi_i)(\sigma) = \text{constant}, \forall i = 1, \dots, q - 1.$$

This implies that $I(\Phi) \subset \pm\Phi$, and consequently $I(\chi_1) = \pm\chi_1$. □

Let $\pi \in \text{Sym}\{1, \dots, q-1\}$ and $\tau \in \text{Sym}\{2, 3\}$. Define an isometry $I_{\pi, \tau} : R_K(A) \rightarrow R_K(A)$ by

$$\begin{aligned} I_{\pi, \tau}(\varphi_i) &= \varphi_{\pi(i)}, \quad i = 1, \dots, q-1 \\ I_{\pi, \tau}(\chi_1) &= \chi_1 \\ I_{\pi, \tau}(\chi_j) &= \chi_{\tau(j)}, \quad j = 2, 3. \end{aligned}$$

Lemma 7.2.6. *Given any $\pi \in \text{Sym}\{1, \dots, q-1\}$ and $\tau \in \text{Sym}\{2, 3\}$, the isometry $I_{\pi, \tau}$ is perfect.*

Proof. Let μ be the generalized character induced by $I_{\pi, \tau}$. Let μ_{id} be the character induced by the identity map id . We know that μ_{id} satisfies perfect conditions. Let $g, h \in H$. Then

$$\mu(g, h) = \sum_{i=1}^{q-1} \varphi_{\pi(i)}(g)\varphi(h) + \chi_1(g)\chi_1(h) + \sum_{j=2}^3 \chi_{\tau(j)}(g)\chi(h).$$

We consider the separation condition first. From the character table,

$$\begin{aligned} \mu(1, \sigma) &= \mu_{id}(1, \sigma) = 0 \\ \mu(\sigma, 1) &= \mu_{id}(\sigma, 1) = 0 \\ \mu(1, \rho^{\pm 1}) &= \mu_{id}(1, \rho^{\pm 1}) = 0 \\ \mu(\rho^{\pm 1}, 1) &= \mu_{id}(\rho^{\pm 1}, 1) = 0. \end{aligned}$$

Let j be such that $a^j \neq 1$. Since $\sum_i \omega^{j(i-1)} = 0$, we see from the character table that

$$\begin{aligned} \mu(\sigma, a^j) &= \sum_i \omega^{j(i-1)} = 0 \\ \mu(a^j, \sigma) &= \sum_i \omega^{j(i-1)} = 0 \\ \mu(\rho^{\pm 1}, a^j) &= \sum_i \omega^{j(i-1)} = 0 \\ \mu(a^j, \rho^{\pm 1}) &= \sum_i \omega^{j(i-1)} = 0. \end{aligned}$$

Thus, μ satisfies the separation condition. By Theorem 3.1.5, it suffices to consider $\mu(g, h)$ where $g, h \in H$ are p -singular elements. From the character table,

$$\begin{aligned}\mu(\sigma, \rho^{\pm 1}) &= \mu_{id}(\sigma, \rho^{\pm 1}) \in q^2\mathcal{O} \\ \mu(\rho^{\pm 1}, \sigma) &= \mu_{id}(\rho^{\pm 1}, \sigma) \in q^2\mathcal{O}.\end{aligned}$$

It now remains to consider $\mu(g, h)$ for $g, h \in \{\rho, \rho^{-1}\}$. If $g, h \in \{\rho, \rho^{-1}\}$, then

$$\begin{aligned}\mu(g, h) &= \sum_{i=1}^{q-1} \varphi_{\pi(i)}(g)\varphi(h) + \chi_1(g)\chi_1(h) + \sum_{j=2}^3 \chi_{\tau(j)}(g)\chi(h) \\ &= (q-1) + 1 \pm r^2/2 = (q-1) + 1 \pm q \\ &= 0 \text{ or } 2q \\ &\in q^2\mathcal{O}.\end{aligned}$$

So μ satisfies the integrality condition. Hence $I_{\pi, \tau}$ is perfect as claimed. \square

We have shown that any perfect isometry must preserve (up to signs) the sets Φ and \mathbb{X} , and we found a choice of sign (homogenous sign) such that any permutation of Φ and \mathbb{X} gives rise to a perfect isometry. It follows by the uniqueness of signs that every perfect isometry has a homogenous sign and, therefore, the group $\text{PI}(A)$ is of the form as described in Theorem 7.2.4.

7.2.3 Non-existence of perfect embedding when $q > 8$

Consider first when $q = 8$. By Theorem 7.2.1,

$$\begin{aligned}\text{PI}(B) &= \text{Sym}(\mathbb{X}) \times \text{Sym}(\mathbb{Y}) \times \text{Sym}(\mathbb{Z}) \times \text{Sym}(\mathbb{W}) \times \langle -\text{id} \rangle \\ &\cong S_4 \times S_3 \times S_1 \times S_2 \times \langle -\text{id} \rangle.\end{aligned}$$

By Theorem 7.2.4,

$$\begin{aligned}\text{PI}(A) &= \text{Sym}\{\varphi_1, \dots, \varphi_7\} \times \text{Sym}\{\chi_2, \chi_3\} \times \langle -\text{id} \rangle \\ &\cong S_7 \times S_2 \times \langle -\text{id} \rangle.\end{aligned}$$

We can see that there is a perfect embedding $\text{PI}(B) \longrightarrow \text{PI}(A)$ given by

$$\begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ Y_1 \\ Y_2 \\ Y_3 \\ Z_1 \\ W_1 \\ W_2 \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}.$$

Now, if $q > 8$, then $|\mathbb{X}|, |\mathbb{Y}|, |\mathbb{Z}|, |\mathbb{W}| > 1$. So, there is no irreducible character in $\text{Irr}(B)$ that is fixed by every perfect isometry (with all-positive sign) in $\text{PI}(B)$. On the other hand, $\chi_1 \in \text{Irr}(A)$ is always fixed by every perfect isometry (with all-positive sign) in $\text{PI}(A)$. Hence there is no perfect embedding $\text{PI}(B) \longrightarrow \text{PI}(A)$.

7.2.4 $\text{PI}_s(A) \cong \text{PI}_s(B)$

Lemma 7.2.7. *We have*

$$\text{PI}_s(A) \cong \text{PI}_s(B)$$

Proof. We will prove by computing both $\text{PI}_s(A), \text{PI}_s(B)$.

Let $I \in \text{PI}_s(B)$. Since $\chi_i - \chi_j \in \text{CF}_{p'}(G, B; K)$ for all $\chi_i, \chi_j \in \mathbb{X}$, we have $I(\chi_i - \chi_j) = \chi_i - \chi_j$ for all $\chi_i, \chi_j \in \mathbb{X}$. This implies that I is constant on \mathbb{X} . Similarly, I is constant on \mathbb{Y} and \mathbb{Z} .

If $\chi \in \text{Irr}(B)$, denote by χ° the restriction to p -regular elements. Let $\{d_{\chi\varphi} : \chi \in \text{Irr}(B), \varphi \in \text{IBr}(B)\}$ be the decomposition numbers. From the character table of G we have

$$W_1^\circ = \sum_{\varphi \in \text{IBr}(B)} d_{W_1\varphi}\varphi = \sum_{\varphi \in \text{IBr}(B)} d_{W_2\varphi}\varphi = W_2^\circ.$$

So

$$d_{W_1\varphi} = d_{W_2\varphi} \quad \forall \varphi \in \text{IBr}(B).$$

Since $\text{CF}_{p'}(G, B; K)$ is spanned by $\{\Phi_\varphi : \varphi \in \text{IBr}(B)\}$, we must have

$$\sum_{\chi \in \text{Irr}(B)} d_{\chi\varphi} \chi = \Phi_\varphi = I(\Phi_\varphi) = \sum_{\chi \in \text{Irr}(B)} d_{\chi\varphi} I(\chi).$$

Expanding this, and using the fact that I is constant on $\text{Irr}(B) - \mathbb{W}$ gives

$$\begin{aligned} d_{W_1\varphi} W_1 + d_{W_2\varphi} W_2 &= d_{W_1\varphi} I(W_1) + d_{W_2\varphi} I(W_2) \\ d_{W_1\varphi} (W_1 + W_2) &= d_{W_1\varphi} (I(W_1) + I(W_2)). \end{aligned}$$

It follows that

$$\text{PI}_s(B) = \text{Sym}\{W_1, W_2\} \cong S_2.$$

Next, let $I \in \text{PI}_s(A)$. Since $\varphi_i - \varphi_j \in \text{CF}_{p'}(H, A; K)$ for all $i, j = 1, \dots, q-1$, I must fix φ_i for all i .

From the character table of H , we have

$$\chi_2^\circ = \sum_{\varphi \in \text{IBr}(A)} d_{\chi_2\varphi} \varphi = \sum_{\varphi \in \text{IBr}(A)} d_{\chi_3\varphi} \varphi = \chi_3^\circ.$$

So

$$d_{\chi_2\varphi} = d_{\chi_3\varphi} \quad \forall \varphi \in \text{IBr}(A).$$

Since $\text{CF}_{p'}(H, A; K)$ is spanned by $\{\Phi_\varphi : \varphi \in \text{IBr}(A)\}$, we must have

$$\sum_{\chi \in \text{Irr}(A)} d_{\chi\varphi} \chi = \Phi_\varphi = I(\Phi_\varphi) = \sum_{\chi \in \text{Irr}(A)} d_{\chi\varphi} I(\chi).$$

Expanding this, and using the fact that I is constant on $\{\varphi_i : \forall i\}$ gives

$$\begin{aligned} d_{\chi_2\varphi} \chi_2 + d_{\chi_3\varphi} \chi_3 &= d_{\chi_2\varphi} I(\chi_2) + d_{\chi_3\varphi} I(\chi_3) \\ d_{\chi_2\varphi} (\chi_2 + \chi_3) &= d_{\chi_2\varphi} (I(\chi_2) + I(\chi_3)). \end{aligned}$$

It follows that

$$\text{PI}_s(A) = \text{Sym}\{\chi_2, \chi_3\} \cong S_2.$$

Hence $\text{PI}_s(A) \cong \text{PI}_s(B)$. □

7.3 McLaughlin group McL

From this section until the rest of this chapter, we will use the convention as in Section 8.1 to describe perfect isometry groups.

Let $G = McL$ be the McLaughlin group. Let $p = 5$. Let D be a Sylow p -subgroup of G and $H = N_G(D)$. Let B be the principal block of $\mathcal{O}G$ and A be the principal block of $\mathcal{O}H$. In this section we will calculate $PI(B)$ and $PI(A)$.

7.3.1 Principal block of G

The irreducible characters in B are as follows:

| Characters | Degrees | Characters | Degrees |
|------------------------|---------|------------------------|---------|
| χ_1 | 1 | χ_{13} | 4752 |
| χ_2 | 22 | χ_{14} | 5103 |
| χ_3 | 231 | χ_{15} | 5544 |
| χ_4 | 252 | χ_{16}, χ_{17} | 8019 |
| χ_5, χ_6 | 770 | χ_{21}, χ_{22} | 9856 |
| χ_7, χ_8 | 896 | χ_{23}, χ_{24} | 10395 |
| χ_{10}, χ_{11} | 3520 | | |

The perfect isometry group of the block B is as follows.

$$PI(B) \cong D_8 \times S_4 \times C_2 \times \langle -id \rangle,$$

where

| Subgroups | Generators |
|-----------|--|
| D_8 | $(5, 6), (\bar{2}, \bar{14})(4, 13)(5, 23, 6, 24)$ |
| S_4 | $(16, 17), (\bar{16}, 17, \bar{21}, 22)$ |
| C_2 | $(7, 8)$ |

Also,

$$PI_s(B) = \langle (5, 6) \rangle \times \langle (23, 24) \rangle \\ \cong S_2 \times S_2.$$

7.3.2 Principal block of $N_G(D)$

The irreducible characters in A are as follow:

| Characters | Degrees |
|--|---------|
| $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8$ | 1 |
| $\chi_9, \chi_{10}, \chi_{11}, \chi_{12}$ | 2 |
| $\chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}, \chi_{17}, \chi_{18}$ | 20 |
| χ_{19} | 24 |

The perfect isometry group of the block A is as follows.

$$PI(A) \cong ((S_4 \times S_4) \rtimes S_2) \times ((S_2)^4 \rtimes S_2) \times \langle -id \rangle,$$

where

| Subgroups | | |
|--------------------------------|------------|---|
| Direct factors | Components | Generators |
| $(S_4 \times S_4) \rtimes C_2$ | S_4 | $(1, 2), (1, 2, 7, 8)$ |
| | S_4 | $(3, 4), (3, 4, 5, 6)$ |
| | C_2 | $(1, 3)(2, 4)(5, 7)(6, 8)(13, 14)$ |
| $(C_2)^4 \rtimes C_2$ | $(C_2)^4$ | $(9, 10), (11, 12), (15, 16), (17, 18)$ |
| | C_2 | $(9, 11)(10, 12)(15, 17)(16, 18)$ |

Also,

$$\begin{aligned} PI_s(A) &= \langle (15, 16) \rangle \times \langle (17, 18) \rangle \\ &\cong S_2 \times S_2. \end{aligned}$$

Notice that we have $PI_s(B) \cong PI_s(A)$.

A perfect embedding $\text{PI}(B) \rightarrow \text{PI}(A)$ is given by

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 10 \\ 11 \end{pmatrix} \mapsto \begin{pmatrix} 5 \\ -9 \\ 19 \\ 10 \\ 15 \\ 16 \\ 3 \\ 4 \\ 13 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 21 \\ 22 \\ 23 \\ 24 \end{pmatrix} \mapsto \begin{pmatrix} 12 \\ 11 \\ 6 \\ -1 \\ -2 \\ 7 \\ 8 \\ 17 \\ 18 \end{pmatrix}$$

7.4 The group $PSU_3(q)$

Next we investigate a family of blocks with TI defect group. By [3, Theorem 1.1], the principal p -blocks of the group $PSU_3(p^m)$ have TI defect groups, where m is any positive integer.

7.4.1 $PSU_3(3), p = 3$

Let $G = PSU_3(3), p = 3$.

Principal block of G

Let B be the principal block of $\mathcal{O}G$. The irreducible characters in $\text{Irr}(B)$ and their degrees are as follows.

| Characters | Degrees | Characters | Degrees |
|--------------------------|---------|--------------------------|---------|
| χ_1 | 1 | χ_7, χ_8, χ_9 | 21 |
| χ_2 | 6 | χ_{11}, χ_{12} | 28 |
| χ_3, χ_4, χ_5 | 7 | χ_{13}, χ_{14} | 32 |
| χ_6 | 14 | | |

We have

$$\text{PI}(B) \cong S_3 \times C_2 \times \langle -id \rangle,$$

where

| Subgroups | Generators |
|-----------|---|
| S_3 | $(\bar{2}, \bar{8})(1, 12)(3, 5),$ $(\bar{2}, \bar{8}, 9)(1, 12, 11)(3, 5, 4)$ |
| C_2 | $(13, 14)$ |

Also,

$$\text{PI}_s(B) = \{id\}.$$

Principal block of $N_G(D)$

Let D be a Sylow p -subgroup of G . Let $H = N_G(D)$ and let A be the principal block of H .

The irreducible characters in $\text{Irr}(A)$ and their degrees are as follows.

| Characters | Degrees |
|----------------------------|---------|
| χ_1, \dots, χ_8 | 1 |
| χ_9, \dots, χ_{12} | 6 |
| χ_{13} | 8 |

We have

$$\text{PI}(A) \cong (C_2)^4 \rtimes S_4 \times \langle -id \rangle,$$

where

| Subgroups | Generators |
|-----------|---|
| $(C_2)^4$ | $(1, 2), (7, 8), (6, 3), (5, 4)$ |
| S_4 | $(9, 10)(1, 7)(2, 8),$ $(9, 10, 11, 12)(1, 7, 6, 5)(2, 8, 3, 4)$ |

Also,

$$\text{PI}_s(A) = \{id\}.$$

Again in this case we have $\text{PI}_s(B) \cong \text{PI}_s(A)$.

A perfect embedding $\text{PI}(B) \rightarrow \text{PI}(A)$ is given by the following isometry:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 11 \\ 12 \\ 13 \\ 14 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ -9 \\ 2 \\ 3 \\ 8 \\ 13 \\ 12 \\ 10 \\ 11 \\ 6 \\ 7 \\ 5 \\ 4 \end{pmatrix} .$$

7.4.2 $\text{PSU}_3(5)$, $p = 5$

Let $G = \text{PSU}_3(5)$, $p = 5$ and let B be the principal block of \mathcal{OG} .

The irreducible characters in $\text{Irr}(B)$ and their degrees are as follows.

| Characters | Degrees | Characters | Degrees |
|--------------------------|---------|-----------------------------------|---------|
| χ_1 | 1 | χ_7 | 84 |
| χ_2 | 20 | χ_8 | 105 |
| χ_3 | 21 | $\chi_{10}, \chi_{11}, \chi_{12}$ | 126 |
| χ_4, χ_5, χ_6 | 28 | χ_{13}, χ_{14} | 144 |

We have

$$\begin{aligned} \text{PI}(B) &= \text{Sym}\{4, 5, 6\} \times \langle(1, 10)\rangle \times \langle(11, 12)\rangle \times \langle(13, 14)\rangle \times \langle-\text{id}\rangle \\ &\cong S_3 \times C_2 \times C_2 \times C_2 \times \langle-\text{id}\rangle . \end{aligned}$$

Also,

$$\text{PI}_s(B) = \text{Sym}\{4, 5, 6\}.$$

Let D be a Sylow p -subgroup of G . Let $H = N_G(D)$ and let A be the principal block of $\mathcal{O}H$.

The irreducible characters in $\text{Irr}(A)$ and their degrees are as follows.

| Characters | Degrees |
|--------------------------------|---------|
| χ_1, \dots, χ_8 | 1 |
| $\chi_9, \chi_{10}, \chi_{11}$ | 8 |
| χ_{12}, χ_{13} | 20 |

We have

$$\begin{aligned} \text{PI}(A) &= (\text{Sym}\{1, 2, 7, 8\} \times \text{Sym}\{3, 4, 5, 6\}) \rtimes \langle (1, 3)(2, 4)(7, 5)(8, 6)(12, 13) \rangle \\ &\quad \times \text{Sym}\{9, 10, 11\} \times \langle -\text{id} \rangle \\ &\cong ((S_4 \times S_4) \rtimes S_2) \times S_3 \times \langle -\text{id} \rangle, \end{aligned}$$

where

| Subgroups | | |
|--------------------------------|------------|------------------------------------|
| Direct factors | Components | Generators |
| $(S_4 \times S_4) \rtimes C_2$ | S_4 | $(1, 2), (1, 2, 7, 8)$ |
| | S_4 | $(3, 4), (3, 4, 5, 6)$ |
| | C_2 | $(1, 3)(2, 4)(7, 5)(8, 6)(12, 13)$ |
| S_3 | S_3 | $(9, 10), (9, 10, 11)$ |

Also

$$\text{PI}_s(A) = \text{Sym}\{9, 10, 11\}.$$

So $\text{PI}_s(B) \cong \text{PI}_s(A)$.

A perfect embedding $\text{PI}(B) \rightarrow \text{PI}(A)$ is given by the following isometry:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 12 \\ 5 \\ 9 \\ 10 \\ 11 \\ 6 \\ 13 \\ 2 \\ 7 \\ 8 \\ 3 \\ 4 \end{pmatrix} .$$

The following table illustrates how the isometry given above embeds $\text{PI}(B)$ into $\text{PI}(A)$. We can see that every perfect isometry in $\text{PI}(B)$ “appears” in $\text{PI}(A)$.

| | | | | | | |
|-----------------|---------------------------|----------------------------|----------------------------|--------------------------|--|--|
| $\text{PI}(B)$ | $\text{Sym}\{4, 5, 6\}$ | $\langle(1, 10)\rangle$ | $\langle(11, 12)\rangle$ | $\langle(13, 14)\rangle$ | | |
| $\text{Irr}(B)$ | 4 5 6 | 1 10 11 12 | 13 14 3 7 | 2 8 | | |
| $\text{Irr}(A)$ | 9 10 11 | 1 2 7 8 | 3 4 5 6 | 12 13 | | |
| $\text{PI}(A)$ | $\text{Sym}\{9, 10, 11\}$ | $\text{Sym}\{1, 2, 7, 8\}$ | $\text{Sym}\{3, 4, 5, 6\}$ | $\langle(12, 13)\rangle$ | | |

Chapter 8

Sporadic groups

In this chapter we compute the perfect isometry groups of blocks of the first 12 Sporadic groups in ascending order¹. We do this using our programs written in GAP [19]. Since we already obtained the results for blocks with defect 1 in Chapter 6, we will only consider blocks with defect greater than 1.

Table 8.1: First 12 Sporadic groups in ascending order

| Group | Order | Factorized order |
|----------------|--------------|--|
| M_{11} | 7920 | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ |
| M_{12} | 95040 | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ |
| J_1 | 175560 | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ |
| M_{22} | 443520 | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ |
| J_2 or HJ | 604800 | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ |
| M_{23} | 10200960 | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| HS | 44352000 | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ |
| J_3 or HJM | 50232960 | $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ |
| M_{24} | 244823040 | $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| McL | 898128000 | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ |
| He | 4030387200 | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ |
| Ru | 145926144000 | $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ |

¹If a block B contains too many irreducible characters, it will take a very long time (15+ days) to compute its perfect isometry group. We found that the practical limit is $k(B) \approx 12$.

8.1 Convention

For convenience, we will introduce the following notation for (perfect) isometries. Let B be a block of G and suppose that $\text{Irr}(B) = \{\chi_i : i \in \mathcal{I}\}$ where $\mathcal{I} \subset \mathbb{N}$. Let $n = \max(\mathcal{I})$. Let $I = R_K(B) \rightarrow R_K(B)$ be an isometry. Write $I = (\sigma, \varepsilon)$, where $\sigma \in S_n$ and ε is a sign function.

Let \mathbf{c} be a cycle notation of σ . We will denote an isometry I by $\bar{\mathbf{c}}$, defined as follows. $\bar{\mathbf{c}}$ contains the same numbers in the same order as \mathbf{c} except that:

- if $i \in \mathbf{c}$ and $-\chi_i \in I(\text{Irr}(B))$ then i appears as \bar{i} in $\bar{\mathbf{c}}$,
- if $i \notin \mathbf{c}$ and $-\chi_i \in I(\text{Irr}(B))$ then add (\bar{i}) to $\bar{\mathbf{c}}$.

For example, let $\text{Irr}(B) = \{\chi_1, \dots, \chi_4\}$. An isometry $(1, \bar{2}, \bar{3})$ has the permutation $(1, 2, 3)$ and $\{-\chi_2, -\chi_3\} \subset I(\text{Irr}(B))$. So, it is the map

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} \mapsto \begin{pmatrix} -\chi_2 \\ -\chi_3 \\ \chi_1 \\ \chi_4 \end{pmatrix}.$$

An isometry $(\bar{1}, 3)(\bar{4})$ is the map

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} \mapsto \begin{pmatrix} \chi_3 \\ \chi_2 \\ -\chi_1 \\ -\chi_4 \end{pmatrix}.$$

For short, we will write $\text{Irr}(B) = \mathcal{I}$ instead of $\text{Irr}(B) = \{\chi_i : i \in \mathcal{I}\}$. We will use the subscripts of irreducible characters as in the ATLAS of finite groups [11].

Sometimes, the structure of a group S is very complicated and has a long description when using the GAP command:

```
gap> StructureDescription(S);
```

In these cases, we will list important subgroups whose union of generators generate the whole group, and, if possible, give the identity of S as in the Small Group Library (SGL) in GAP. It is obtained by the following command:

```
gap> IdGroup(S);
```

If B is a block, we denote by $\delta(B)$ a defect group of B .

8.2 Mathieu group M_{11}

$M_{11}, p = 2$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|-------------------------------|--------|
| B_0 | 4 | $QD16$ | $\{1, 2, 3, 4, 5, 8, 9, 10\}$ | 8 |

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|--------------------------------------|-----------|-----------|--|
| B_0 | $(C_2)^3 \times \langle -id \rangle$ | 16 | C_2 | $(3, 4)$ |
| | | | C_2 | $(1, 5)(9, 10)(\bar{2})(\bar{3})(\bar{4})$ |
| | | | C_2 | $(1, 9)(5, 10)(\bar{8})$ |

$M_{11}, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|----------------------------------|--------|
| B_0 | 2 | $(C_3)^2$ | $\{1, 2, 3, 4, 5, 6, 7, 8, 10\}$ | 9 |

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|--------------------------------|-----------|-----------|---|
| B_0 | $H \times \langle -id \rangle$ | 2592 | H | $(1, \bar{5}, \bar{10})(2, 4, 7, \bar{8}, \bar{3}, 6),$ |
| | | | | $(1, 4)(2, 7, \bar{8}, \bar{6})(\bar{5}, \bar{10})$ |

The group H has the identity [1296, 3490] in SGL. Moreover, H contains the following subgroups whose union of generators generate the whole group H .

| Subgroups | Generators |
|-----------|---|
| S_3 | $(1, 3), (1, 3, 4)$ |
| S_3 | $(\bar{5}, \bar{6}), (\bar{5}, \bar{6}, 7)$ |
| S_3 | $(\bar{2}, \bar{8}), (2, \bar{8}, \bar{10})$ |
| C_2 | $(2, 7)(\bar{5}, \bar{10})(\bar{6}, \bar{8})$ |
| C_6 | $(1, 2, 4, \bar{8}, \bar{3}, 10)$ |

8.3 Mathieu group M_{12}

$M_{12}, p = 2$

Blocks information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|------------------|---|--------|
| B_0 | 6 | D | $\{1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13\}$ | 11 |
| B_1 | 2 | $C_2 \times C_2$ | $\{4, 5, 14, 15\}$ | 4 |

where $D \in \text{Syl}_2(M_{12})$ and has the identity [64, 134] in SGL.

Perfect isometry groups:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---|-----------|-----------|--|
| B_0 | $C_2 \times \langle -\text{id} \rangle$ | 4 | C_2 | $(2, 3)(9, 10)$ |
| B_1 | $S_4 \times \langle -\text{id} \rangle$ | 48 | S_4 | $(5, \bar{15}, \bar{14}),$ $(4, \bar{14}, 5, \bar{15})$ |

$M_{12}, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|--------------------------------|---|--------|
| B_0 | 3 | $(C_3 \times C_3) \rtimes C_3$ | $\{1, 2, 3, 4, 5, 8, 9, 10, 11, 13, 15\}$ | 11 |

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|--|-----------|-----------|--|
| B_0 | $S_4 \times D_8 \times \langle -\text{id} \rangle$ | 384 | S_4 | (2, 3), ($\bar{2}$, 3, $\bar{4}$, 5) |
| | | | D_8 | (1, 8), (1, 9, 8, 10)(11, 13) |

8.4 Mathieu group M_{22}

$M_{22}, p = 2$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|--------------------|--------|
| B_0 | 7 | D | $\{1, \dots, 12\}$ | 12 |

where $D \in \text{Syl}_2(M_{22})$ and has the identity [128, 931] in SGL.

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---|-----------|-----------|------------|
| B_0 | $(C_2)^2 \times \langle -\text{id} \rangle$ | 8 | C_2 | (3, 4) |
| | | | C_2 | (10, 11) |

$M_{22}, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|------------------|---------------------------|--------|
| B_0 | 2 | $C_3 \times C_3$ | $\{1, 5, 7, 10, 11, 12\}$ | 6 |

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---|-----------|-----------|------------------------------|
| B_0 | $S_5 \times \langle -\text{id} \rangle$ | 240 | S_5 | (1, 5), (1, 5, 7, 10, 11) |

8.5 Mathieu group M_{23}

$M_{23}, p = 2$

Block information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|------------------------------------|--------|
| B_0 | 7 | D | $\{1, \dots, 11, 14, 15, 16, 17\}$ | 15 |

where $D \in \text{Syl}_2(M_{23})$ and has the identity [128, 931] in SGL.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---|------------------|-----------|----------------|
| B_0 | $(C_2)^3 \times \langle -\text{id} \rangle$ | 16 | C_2 | (7, 8) |
| | | | C_2 | (10, 11) |
| | | | C_2 | (3, 4)(14, 15) |

It should be noted that the defect group of B_0 is isomorphic to the defect group of the principal 2-block of M_{22} . In both blocks, the perfect isometry groups are very small compared to the number of possible permutations on the irreducible characters in the blocks. This results in a very long runtime on a computer. In the above case, it took 30 days to calculate $\text{PI}(B_0)$!

$M_{23}, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|------------------|--------------------------------------|--------|
| B_0 | 2 | $C_3 \times C_3$ | $\{1, 2, 5, 9, 10, 11, 12, 13, 17\}$ | 9 |

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---------------------------------------|------------------|-----------|--|
| B_0 | $H \times \langle -\text{id} \rangle$ | 2592 | H | $(\bar{1}, \bar{5}, 13)(\bar{2}, \bar{17}, 11, 12, \bar{9}, \bar{10}),$ $(\bar{1}, \bar{17})(\bar{2}, \bar{11}, 12, 10)(5, 13)$ |

The group H has the identity [1296, 3490] in SGL. Moreover, H contains the following subgroups whose union of generators generate the whole group H .

| Subgroups | Generators |
|-----------|---|
| S_3 | $(1, 9), (\bar{1}, 9, \bar{17})$ |
| S_3 | $(\bar{2}, \bar{12}), (\bar{2}, \bar{12}, 13)$ |
| S_3 | $(5, 10), (5, 10, 11)$ |
| C_2 | $(\bar{2}, \bar{11})(5, 13)(10, 12)$ |
| C_6 | $(\bar{1}, 2, \bar{17}, 12, \bar{9}, \bar{13})$ |

8.6 Mathieu group M_{24}

$M_{24}, p = 2$

Block information²:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|--------------------|--------|
| B_0 | 10 | D | $\{1, \dots, 26\}$ | 26 |

where $D \in \text{Syl}_2(M_{24})$.

$M_{24}, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|--|--------|
| B_0 | 3 | D | $\{1, 2, 5, 6, 8, \dots, 11, 17, 18, 19, 22, 23\}$ | 13 |

where $D \cong (C_3 \times C_3) \rtimes C_3$.

Perfect isometry group:

² B_0 contains too many irreducible characters to compute its perfect isometry group.

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|--------------------------------------|-----------|-----------|--|
| B_0 | $(C_2)^4 \times \langle -id \rangle$ | 32 | C_2 | (5, 6) |
| | | | C_2 | (10, 11) |
| | | | C_2 | (1, 8)($\bar{9}$, $\bar{23}$)($\bar{19}$, $\bar{22}$) |
| | | | C_2 | (1, 22)(2, 17)($\bar{8}$, $\bar{19}$)($\bar{9}$, $\bar{23}$) |

8.7 Janko group J_1

$$J_1, p = 2$$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|------------------------------|--------|
| B_0 | 3 | $(C_2)^3$ | {1, 6, 7, 8, 12, 13, 14, 15} | 8 |

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|--|-----------|-----------|---|
| B_0 | $((C_2)^4 \rtimes S_4) \times \langle -id \rangle$ | 768 | $(C_2)^4$ | (6, 12), (7, 13), (8, 14), (1, 15) |
| | | | S_4 | (6, 7)(12, 13), (6, 7, 8, 1)(12, 13, 14, 15) |

8.8 Janko group J_2

$$J_2, p = 2$$

Blocks information³:

³ B_0 contains too many irreducible characters to compute its perfect isometry group.

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|------------------|--|--------|
| B_0 | 7 | D | $\{1, \dots, 11, 13, 14, 15, 18, 20, 21\}$ | 17 |
| B_1 | 2 | $C_2 \times C_2$ | $\{12, 16, 17, 19\}$ | 4 |

where $D \in \text{Syl}_7(J_2)$ and has the identity [128, 934] in SGL.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---|------------------|-----------|---|
| B_1 | $S_4 \times \langle -\text{id} \rangle$ | 48 | S_4 | $(16, \bar{19}, \bar{17}),$ $(12, \bar{17}, 16, \bar{19})$ |

$J_2, p = 3$

Block information:

| Block B | Defect | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|---|--------|
| B_0 | 3 | $\{1, \dots, 5, 8, 9, 12, 13, 16, 17, 20, 21\}$ | 13 |

where $\delta(B_0) \cong (C_3 \times C_3) \rtimes C_3$.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|--|------------------|-----------|--|
| B_0 | $C_2 \times S_3 \times \langle -\text{id} \rangle$ | 24 | C_2 | $(2, 3)$ |
| | | | S_3 | $(4, 5)(8, 9)(16, 17),$ $(\bar{1}, \bar{17}, 16)(4, 21, 5)(8, 9, 12)$ |

8.9 Janko group J_3

$J_3, p = 2$

Block information⁴:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|------------------------------------|--------|
| B_0 | 7 | D | $\{1, \dots, 13, 17, 18, 20, 21\}$ | 17 |

where $D \in \text{Syl}_2(J_3)$ and has the identity [128, 934] in SGL.

⁴ B_0 contains too many irreducible characters to compute its perfect isometry group.

$J_3, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|---|--------|
| B_0 | 5 | D | $\{1, \dots, 5, 7, 8, 9, 10, 13, \dots, 19\}$ | 16 |

where $D \in \text{Syl}_3(J_3)$ and has the identity [243, 9] in SGL.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---|------------------|-----------|---|
| B_0 | $C_2 \times C_3 \times S_3 \times \langle -\text{id} \rangle$ | 72 | C_2 | (2, 3) |
| | | | C_3 | (14, 15, 16) |
| | | | S_3 | (4, 5)(7, 8)(17, 18), ($\bar{1}$, $\bar{5}$, 4)(7, $\bar{19}$, $\bar{8}$)($\bar{9}$, $\bar{18}$, 17) |

8.10 Higman-Sims group HS

$HS, p = 2$

Blocks information⁵:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|------------------|---|--------|
| B_0 | 9 | D | $\{1, \dots, 13, 16, 17, 19, \dots, 23\}$ | 20 |
| B_1 | 2 | $C_2 \times C_2$ | $\{14, 15, 18, 24\}$ | 4 |

where $D \in \text{Syl}_2(HS)$.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---|------------------|-----------|--|
| B_1 | $S_4 \times \langle -\text{id} \rangle$ | 48 | S_4 | (14, 15), ($\bar{14}$, 15, 18, $\bar{24}$) |

⁵ B_0 contains too many irreducible characters to compute its perfect isometry group.

$HS, p = 3$

Blocks information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|------------------|---------------------------------------|--------|
| B_0 | 2 | $C_3 \times C_3$ | $\{1, 2, 4, 10, 18, 19, 20, 23, 24\}$ | 9 |
| B_1 | 2 | $C_3 \times C_3$ | $\{3, 5, 6, 7, 11, 12, 14, 15, 21\}$ | 9 |

Perfect isometry groups:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---|------------------|-----------|---|
| B_0 | $H_0 \times \langle -\text{id} \rangle$ | 2592 | H_0 | $(1, 4)(2, 19)(\bar{10}, 24, 23, \bar{18}),$ $(1, 19, 2)(\bar{4}, \bar{23}, \bar{18}, \bar{20}, 10, 24)$ |
| B_1 | $H_1 \times \langle -\text{id} \rangle$ | 2592 | H_1 | $(\bar{3}, \bar{7}, \bar{21}, 11, 14, \bar{6})(\bar{5}, \bar{12}, 15),$ $(3, 11, 12)(\bar{5}, \bar{15})(6, \bar{14}, 21, \bar{7})$ |

The groups H_0, H_1 are isomorphic and have the identity [1296, 3490] in SGL. The group H_0 contains the following subgroups whose union of generators generate the whole group H_0 .

| Subgroups | Generators |
|-----------|--|
| S_3 | $(1, 4), (\bar{1}, 4, \bar{20})$ |
| S_3 | $(2, 18), (\bar{2}, 18, \bar{24})$ |
| S_3 | $(\bar{10}, \bar{19}), (10, \bar{19}, \bar{23})$ |
| C_2 | $(\bar{2}, \bar{23})(10, 24)(18, 19)$ |
| C_6 | $(\bar{1}, 2, \bar{20}, \bar{18}, 4, \bar{24})$ |

The group H_1 contains the following subgroups whose union of generators generate the whole group H_1 .

| Subgroups | Generators |
|-----------|---|
| S_3 | $(3, 11), (3, 11, 12)$ |
| S_3 | $(5, 6), (\bar{5}, 6, \bar{21})$ |
| S_3 | $(\bar{7}, \bar{14}), (\bar{7}, \bar{14}, 15)$ |
| C_2 | $(\bar{5}, \bar{15})(\bar{6}, \bar{14})(\bar{7}, \bar{21})$ |
| C_6 | $(3, \bar{5}, \bar{12}, \bar{6}, \bar{11}, 21)$ |

$HS, p = 5$

Block information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|--|--------|
| B_0 | 3 | D | $\{1, \dots, 6, 8, \dots, 12, 14, \dots, 18, 22\}$ | 17 |

where $D \cong (C_5 \times C_5) \rtimes C_5$.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|--|------------------|-----------|------------|
| B_0 | $(C_2)^3 \times \langle \text{id} \rangle$ | 16 | C_2 | $(5, 6)$ |
| | | | C_2 | $(11, 12)$ |
| | | | C_2 | $(14, 15)$ |

8.11 McLaughlin group McL

$McL, p = 2$

Block information⁶:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|---|--------|
| B_0 | 7 | D | $\{1, \dots, 6, 9, 12, \dots, 20, 23, 24\}$ | 18 |

where $D \in \text{Syl}_2(McL)$ has the identity [128, 931] in SGL.

⁶ B_0 contains too many irreducible characters to compute its perfect isometry group.

$McL, p = 3$

Block information⁷:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|---------------------------------------|--------|
| B_0 | 6 | D | $\{1, \dots, 13, 15, 18, \dots, 24\}$ | 21 |

where $D \in \text{Syl}_3(McL)$ and has the identity [729, 321] in SGL.

$McL, p = 5$

Block information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|---|--------|
| B_0 | 3 | D | $\{1, \dots, 8, 10, 11, 13, \dots, 17, 21, \dots, 24\}$ | 19 |

where $D \cong (C_5 \times C_5) \rtimes C_5$.

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|--|------------------|-----------|---|
| B_0 | $S_4 \times S_2 \times D_8 \times \langle \text{id} \rangle$ | 768 | S_4 | $(16, 17),$ $(\bar{1}6, 17, \bar{2}1, 22)$ |
| | | | S_2 | $(7, 8)$ |
| | | | D_8 | $(5, 6),$ $(\bar{2}, \bar{1}4)(4, 13)(5, 23, 6, 24)$ |

8.12 Held group He

$He, p = 2$

Blocks information⁸:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|--|--------|
| B_0 | 10 | D | $\{1, \dots, 11, 13, 17, \dots, 21, 23, \dots, 29, 32, 33\}$ | 26 |
| B_1 | 3 | D_8 | $\{12, 14, 15, 16, 22\}$ | 5 |

⁷ B_0 contains too many irreducible characters to compute its perfect isometry group.

⁸ B_0 contains too many irreducible characters to compute its perfect isometry group.

where $D \in \text{Syl}_2(He)$.

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---|-----------|-----------|---|
| B_1 | $D_8 \times \langle -\text{id} \rangle$ | 16 | D_8 | $(\bar{16}, \bar{22}),$ $(\bar{12}, 22, \bar{15}, 16)(\bar{14})$ |

$He, p = 3$

Blocks information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|------------------|--|--------|
| B_0 | 3 | D | $\{1, 6, 9, 14, 15, 22, 25, 26, 28, \dots, 31, 33\}$ | 13 |
| B_1 | 2 | $C_3 \times C_3$ | $\{2, 3, 7, 8, 10, \dots, 13, 16\}$ | 9 |

where $D \cong (C_3 \times C_3) \rtimes C_3$.

Perfect isometry groups:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---|-----------|-----------|---|
| B_0 | $(C_2)^4 \times \langle -\text{id} \rangle$ | 32 | C_2 | $(\bar{26}, \bar{33})$ |
| | | | C_2 | $(30, 31)$ |
| | | | C_2 | $(\bar{1}, \bar{22})(6, 14)(15, 29)$ |
| | | | C_2 | $(\bar{1}, \bar{29})(9, 28)(15, 22)$ |
| B_1 | $H \times \langle -\text{id} \rangle$ | 2592 | H | $(\bar{2}, \bar{8}, 16)(3, \bar{7}, \bar{10}, \bar{13}, 12, \bar{11}),$ $(2, 3)(\bar{7}, \bar{11})(8, \bar{10}, \bar{12}, 16)$ |

The group H has the identity [1296, 3490] in SGL and it contains the following subgroups whose union of generators generate the whole group H .

| Subgroups | Generators |
|-----------|--|
| S_3 | $(2, 3), (\bar{2}, 3, \bar{13})$ |
| S_3 | $(7, 8), (7, 8, 12)$ |
| S_3 | $(10, 11), (\bar{10}, 11, \bar{16})$ |
| C_2 | $(7, 16)(\bar{8}, \bar{11})(\bar{10}, \bar{12})$ |
| C_6 | $(\bar{2}, \bar{12}, 13, 8, \bar{3}, \bar{7})$ |

$He, p = 5$

Block information:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|------------------|--|--------|
| B_0 | 2 | $C_5 \times C_5$ | $\{1, \dots, 5, 7, 8, 14, \dots, 18, 28, 30, 31, 33\}$ | 16 |

Perfect isometry group:

| Block | $\text{PI}(B)$ | $ \text{PI}(B) $ | Subgroups | Generators |
|-------|---------------------------------------|------------------|-----------|---|
| B_0 | $H \times \langle -\text{id} \rangle$ | 1658880 | S_4 | $(1, 2),$ $(\bar{1}, 2, 3, \bar{3}\bar{3})$ |
| | | | S_4 | $(7, 8),$ $(7, 8, 30, 31)$ |
| | | | C_2 | $(\bar{1}, \bar{7})(\bar{2}, \bar{8})(\bar{3}, \bar{3}\bar{0})(33, 31)(\bar{1}\bar{4}, \bar{2}\bar{8})$ |
| | | | S_6 | $(4, 5),$ $(\bar{4}, 5, \bar{1}\bar{5}, \bar{1}\bar{6}, \bar{1}\bar{7}, 18)$ |

where $H \cong ((S_4 \times S_4) \rtimes C_2) \times S_6$.

$He, p = 7$

Block information⁹:

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|--|--------|
| B_0 | 3 | D | $\{1, \dots, 6, 9, \dots, 14, 16, 19, \dots, 25, 27, 30, 31\}$ | 23 |

where $D \cong (C_7 \times C_7) \rtimes C_7$.

8.13 Rudvalis group Ru

$Ru, p = 2$

Blocks information¹⁰:

⁹ B_0 contains too many irreducible characters to compute its perfect isometry group.

¹⁰ B_0 contains too many irreducible characters to compute its perfect isometry group.

| Block B | Defect | Defect group | Irr(B) | $k(B)$ |
|-----------|--------|------------------|------------------------|--------|
| B_0 | 14 | D | $\{1, \dots, 31, 33\}$ | 32 |
| B_1 | 2 | $C_2 \times C_2$ | $\{32, 34, 35, 36\}$ | 4 |

where $D \in \text{Syl}_2(Ru)$.

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---|-----------|-----------|---|
| B_1 | $S_4 \times \langle -\text{id} \rangle$ | 48 | S_4 | $(\bar{3}2, \bar{3}4),$ $(32, \bar{3}4, 35, \bar{3}6)$ |

$Ru, p = 3$

Block information:

| Block B | Defect | $\delta(B)$ | Irr(B) | $k(B)$ |
|-----------|--------|-------------|--|--------|
| B_0 | 3 | D | $\{1, 4, 9, 15, 16, 20, \dots, 23, 25, 26, 32, 33, 36\}$ | 14 |

where $D \cong (C_3 \times C_3) \rtimes C_3$.

Perfect isometry group:

| Block | PI(B) | PI(B) | Subgroups | Generators |
|-------|---------------------------------------|-----------|-----------|---|
| B_0 | $H \times \langle -\text{id} \rangle$ | 144 | S_3 | $(\bar{4}, \bar{3}3)(\bar{2}1, \bar{2}3)(22, 26),$ $(\bar{1}, 4, \bar{3}3)(\bar{9}, \bar{2}6, 22)(20, \bar{2}1, \bar{2}3)$ |
| | | | S_3 | $(25, 32),$ $(\bar{2}5, 32, \bar{3}6)$ |
| | | | C_2 | $(15, 16)$ |

where $H \cong S_3 \times S_3 \times C_2$.

$Ru, p = 5$

Block information¹¹:

¹¹ B_0 contains too many irreducible characters to compute its perfect isometry group.

| Block B | Defect | $\delta(B)$ | $\text{Irr}(B)$ | $k(B)$ |
|-----------|--------|-------------|---|--------|
| B_0 | 3 | D | $\{1, \dots, 7, 10, 14, \dots, 19, 21, 22, 24, 27, 29, 30, 31, 33, \dots, 36\}$ | 25 |

where $D \cong (C_5 \times C_5) \rtimes C_5$.

Chapter 9

Other blocks

In this chapter, we will computer perfect isometry groups for some blocks that are known to be perfectly isometric to $\mathcal{O}(D \times E)$ where D is a defect group and E is the inertial quotient of the block. This allows us to consider only the group $D \times E$.

9.1 Blocks with defect group $C_2 \times C_2$

Let $p = 2$. Blocks over k with Klein four defect groups have been classified by Erdmann [16]. More specifically, let B be a block of $\mathcal{O}G$ with the Klein four group D as a defect group. Let B^* be the block of kG corresponding to B . Then, Erdmann [16, Theorem 4] showed that B^* is Morita equivalent to one of $\{kD, kA_4, C^*\}$ where C^* in the principal block of kA_5 , corresponding to the principal block C of $\mathcal{O}A_5$. Linckelmann [31], later extended Erdmann's results to \mathcal{O} , by showing that B is Morita equivalent to one of $\{\mathcal{O}D, \mathcal{O}A_4, C\}$ ([31, Corollary 1.4]) and derived equivalent to either $\mathcal{O}D$ or $\mathcal{O}A_4$ ([31, Corollary 1.5]). Since perfect isometry groups are preserved by derived equivalences, we will show that in fact the isomorphism class of $\text{PI}(B)$ is unique. Much of the results are, of course, due to the harder work of Erdmann and Linckelmann for their classifications of blocks with Klein four defect groups.

Theorem 9.1.1. *Let $p = 2$. Let B be a block of $\mathcal{O}G$ with defect group $C_2 \times C_2$. Then $|\text{Irr}(B)| = 4$, and every permutation of $\text{Irr}(B)$ gives a perfect isometry (with a choice*

of sign). Furthermore,

$$\text{PI}(B) \cong S_4 \times \langle -id \rangle.$$

Proof. By [31, Corollary 1.5], B is derived equivalent to either $\mathcal{O}D$ or $\mathcal{O}A_4$. It follows that $\text{PI}(B)$ is isomorphic to either $\text{PI}(\mathcal{O}D)$ or $\text{PI}(\mathcal{O}A_4)$, and $|\text{Irr}(B)| = |\text{Irr}(D)| = 4$.

Our calculations in GAP [19] show that

$$\text{PI}(\mathcal{O}D) \cong \text{PI}(\mathcal{O}A_4) \cong S_4 \times \langle -id \rangle.$$

Hence $\text{PI}(B) \cong S_4 \times \langle -id \rangle$ as claimed. \square

Remark 9.1.2. If E is the inertial quotient of the block B , then E is isomorphic to a subgroup of $\text{Aut}(C_2 \times C_2) \cong S_3$. Since $|E|$ is coprime to 2 (by Theorem 2.3.10), $|E| = 1$ or 3. If $|E| = 3$, then $D \rtimes E \cong A_4$. Hence B is perfectly isometric to $D \rtimes E$, a phenomenon that occurs quite often in blocks with abelian defect groups, including the cyclic defect group case (cf. Chapter 6).

9.2 Blocks with defect group $C_3 \times C_3$ and cyclic inertial quotients of order 4

Another situation where we have that a block is perfectly isometric to the block of a semidirect product of its defect group and its inertial quotient group is given by Puig and Usami. Let $p = 3$. Let B be a block of $\mathcal{O}G$ with defect group D . Let E be the inertial quotient group of B . Denote by A the block $\mathcal{O}(D \rtimes E)$. Puig and Usami [37] showed that if D is abelian and E is cyclic of order 4, then there exists a perfect isometry from $R_K(B)$ to $R_K(A)$. As a consequence, we have $\text{PI}(A) \cong \text{PI}(B)$.

Theorem 9.2.1. *Let G be a finite group. Let B be a block of $\mathcal{O}G$ with defect group $C_3 \times C_3$ and inertial quotient C_4 . Then*

$$\text{PI}(B) \cong S_4 \times C_2 \times \langle -id \rangle.$$

Proof. Let $D = C_3 \times C_3$ be a defect group and $E = C_4$ be the inertial quotient. Since E is a subgroup of $\text{Aut}(D) = \text{GL}_2(3)$, it can be checked (for example using GAP)

that if $E_1, E_2 \leq \text{Aut}(D)$ are cyclic groups of order 4 then $D \rtimes E_1 \cong D \rtimes E_2$. Hence $D \rtimes E$ is unique. Let $H = D \rtimes E$ and $A = \mathcal{O}H$. By [37], B and A are perfectly isometric and so $\text{PI}(B) \cong \text{PI}(A)$. We shall now work with the group H instead.

The irreducible characters of H are (numbers in the brackets are their degrees)

$$\text{Irr}(A) = \{\chi_1^{(1)}, \chi_2^{(1)}, \chi_3^{(1)}, \chi_4^{(1)}, \chi_5^{(4)}, \chi_6^{(4)}\}.$$

Calculations in GAP show that all perfect isometries have homogenous signs and

$$\begin{aligned} \text{PI}(A) &= \text{Sym}\{\chi_1^{(1)}, \chi_2^{(1)}, \chi_3^{(1)}, \chi_4^{(1)}\} \times \text{Sym}\{\chi_5^{(4)}, \chi_6^{(4)}\} \times \langle -\text{id} \rangle \\ &\cong S_4 \times C_2 \times \langle -\text{id} \rangle. \end{aligned}$$

□

Chapter 10

Isotypies

Let G be a finite group and B be a block of $\mathcal{O}G$. In this chapter, we will study perfect isometries $I \in \text{PI}(B)$ with an extra condition: these perfect isometries will be called isotypies. We will consider the case where $C_G(x)$ is a p -group whenever $x \in G$ is a p -element. We will then apply our result to finding isotypies in blocks of cyclic p -groups and the principal 2-blocks of the Suzuki groups $Sz(q)$.

In order to define an isotypy, we need to use *subpairs*. We will not go into the details here but instead refer the reader to [1] for more information.

Let $x \in G$ be a p -element and let P be the cyclic group generated by x . Let B_P be a block of $\mathcal{O}C_G(P)$ such that (P, B_P) is a B -subpair of G (so $(B_P)^G = B$). Let e_P be a block idempotent of B_P . We can define a linear map

$$d_G^{(x, e_P)} : \text{CF}(G, B; K) \longrightarrow \text{CF}_{p'}(C_G(P), B_P; K),$$

by

$$d_G^{(x, e_P)}(\alpha)(y) = \alpha(xy e_P),$$

for all $\alpha \in \text{CF}(G, B; K)$ and $y \in C_G(P)_{p'}$. If $y \in C_G(P)$ is a p -singular element, then we set $d_G^{(x, e_P)}(\alpha)(y) = 0$.

Remark 10.0.2. If B is a block of $\mathcal{O}G$ with block idempotent e (so $B = \mathcal{O}Ge$), we can define $\text{CF}(G, e; K)$ to be the subspace of $\text{CF}(G; K)$ of class functions α satisfying

$\alpha(ge) = \alpha(g)$ for all $g \in G$. This is the notation used in [8] where we will take the definition of isotopy from. By Lemma 2.3.2, we have $\text{CF}(G, e; K) = \text{CF}(G, B; K)$. So, for consistency with our notations used in this thesis, we will continue to use $\text{CF}(G, B; K)$ instead of $\text{CF}(G, e; K)$.

Definition 10.0.3. [8, Cf. Definition 4.6] Let B be a block of $\mathcal{O}G$ with block idempotent e and defect group D . Let (D, B_D) be a maximal B -subpair of G . Each subgroup $P \leq D$ determines a unique block B_P of $\mathcal{O}C_G(P)$ such that $(P, B_P) \leq (D, B_D)$ ([1, Theorem 3.4]). Let e_P be a block idempotent of B_P .

Let $I \in \text{PI}(B)$. We say that I is an *isotopy* if for each cyclic subgroup $P \leq D$, there is $I^P \in \text{PI}(B_P)$ (where $I^{\{1\}} = I$) such that the diagram

$$\begin{array}{ccc} \text{CF}(G, B; K) & \xrightarrow{I=I^{\{1\}}} & \text{CF}(G, B; K) \\ d_G^{(x, e_P)} \downarrow & & \downarrow d_G^{(x, e_P)} \\ \text{CF}_{P'}(C_G(P), B_P; K) & \xrightarrow{I^P} & \text{CF}_{P'}(C_G(P), B_P; K) \end{array} \quad (10.1)$$

is commutative for every generator x of P .

The family of perfect isometries $(I^P)_{P \leq D}$ with $I^{\{1\}} = I$ as above is called a *local system* for I .

Remark 10.0.4. 1. In [8, Definition 4.6], Broué considered isotopies in a more general setting. He defined isotopies between Brauer correspondent blocks B of $\mathcal{O}G$ and A of $\mathcal{O}H$ having a common defect group. Here, we will consider only in the special case $G = H$ and $B = A$.

2. Let $I \in \text{PI}(B)$ be an isotopy. A local system for I may not be unique, since for each $P \leq D$, there may be more than one choice of $I^P \in \text{PI}(B_P)$ making the diagram (10.1) commutative.

3. If $P = \{1\}$, then we always have $d_G^{(1, e)} \circ I = I \circ d_G^{(1, e)}$ for any $I \in \text{PI}(B)$, as is easily verified.

Definition 10.0.3 says that an isometry $I : R_K(B) \longrightarrow R_K(B)$ is an isotopy if I is a perfect isometry and I has a local system $(I^P)_{P \leq D}$ with $I^{\{1\}} = I$, where P runs over

all cyclic subgroups of D . Actually, the fact that I is perfect automatically follows if I has a local system $(I^P)_{\{1\} \neq P \leq D}$. This is explained in the following lemma.

Lemma 10.0.5. *Let B be a block of $\mathcal{O}G$ with defect group D . Let $I : R_K(B) \rightarrow R_K(B)$ be an isometry. Suppose there exists a family of perfect isometries*

$$(I^P : R_K(B_P) \rightarrow R_K(B_P))_{\{1\} \neq P \leq D}$$

such that, for all non-trivial cyclic subgroup $P \leq D$ and a generator x of P , we have

$$d_G^{(x, e_P)} \circ I = I^P \circ d_G^{(x, e_P)}.$$

Then I is an isotopy and $(I^P)_{P \leq D}$ with $I^{\{1\}} = I$ is a local system for I . In particular, I is a perfect isometry.

Proof. This follows from the remark after [8, Definition 4.6]. □

We will now define an isotopy subgroup of $\text{PI}(B)$.

Lemma 10.0.6. *The set of isotopies in B is a subgroup of $\text{PI}(B)$.*

Proof. For each non-trivial cyclic subgroup $P \leq D$ with a generator x , take $I^P = \text{id} \in \text{PI}(B_P)$. Then $d_G^{(x, e_P)} \circ \text{id} = d_G^{(x, e_P)} = I^P \circ d_G^{(x, e_P)}$. So $\text{id} \in \text{PI}(B)$ is an isotopy with a local system $(I^P)_{P \leq D}$ where $I^P = \text{id}, \forall P \leq D$.

Suppose $I \in \text{PI}(B)$ is an isotopy with a local system $(I^P)_{P \leq D}$. Let $\langle x \rangle = P \leq D$ be a cyclic subgroup. Let $d = d_G^{(x, e_P)}$. Let $\chi \in \text{Irr}(B)$ and suppose that $I(\chi) = \varphi \in \pm \text{Irr}(B)$. Then

$$(I^P)^{-1}(d(\varphi)) = (I^P)^{-1}(d(I(\chi))) = (I^P)^{-1}(I^P(d(\chi))) = d(\chi) = d(I^{-1}(\varphi)).$$

Since $\{I(\chi) : \chi \in \text{Irr}(B)\}$ spans $\text{CF}(G, B; K)$, it follows that the diagram (10.1) is commutative. Thus I^{-1} is an isotopy with a local system $((I^P)^{-1})_{P \leq D}$.

Finally, let $I, J \in \text{PI}(B)$ be isotopies with local systems $(I^P)_{P \leq D}$ and $(J^P)_{P \leq D}$ respectively. With similar arguments as above using associativity, it is easy to see that $I \circ J$ is also an isotopy with a local system $(I^P \circ J^P)_{P \leq D}$. This completes the proof. □

Definition 10.0.7. The subgroup of $\text{PI}(B)$ consisting of isotypies is called the *isotypy subgroup* of $\text{PI}(B)$, denoted by $\text{IPI}(B)$.

The following proposition characterizes a local system for isotypies in the case where $C_G(x)$ is a p -group for every p -element $x \in G$.

Proposition 10.0.8. *Let B be the principal block of $\mathcal{O}G$ with defect group D . Let $P \leq D$ be a cyclic subgroup with $P = \langle x \rangle$. Suppose that $C_G(x)$ is a p -group. Let $I : R_K(B) \rightarrow R_K(B)$ be an isometry. Then, the following are equivalent.*

(i) *There exists $I^P \in \text{PI}(B_P)$ such that $d_G^{(x, e_P)} \circ I = I^P \circ d_G^{(x, e_P)}$.*

(ii) *Either $I(\chi)(x) = \chi(x) \forall \chi \in \text{Irr}(B)$ or $I(\chi)(x) = -\chi(x) \forall \chi \in \text{Irr}(B)$.*

Proof. Note that since $C_G(x) = C_G(P)$ is a p -group, $B_P = \mathcal{O}C_G(P)$, $e_P = 1$ and $\text{Irr}(B_P) = \text{Irr}(C)$. The condition that B is the principal block ensures that $(B_P)^G = B$, since by Brauer's Third Main Theorem, we must have that $(B_P)^G$ is the principal block. For convenience, we will denote the group $C_G(P)$ by C and the map $d_G^{(x, e_P)}$ by d .

(i) \Rightarrow (ii). Let $\chi \in \text{Irr}(B)$. Since $e_P = 1$, by the definition, we have $d(\chi)(1) = \chi(x)$ and $d(\chi)(h) = 0$ for all p -singular elements $h \in C$. So,

$$\begin{aligned} I^P(d(\chi))(1) &= \frac{1}{|C|} \sum_{h \in C} \mu_{I^P}(1, h^{-1}) d(\chi)(h) = \frac{\mu_{I^P}(1, 1) d(\chi)(1)}{|C|} \\ &= \frac{\mu_{I^P}(1, 1)}{|C|} \chi(x). \end{aligned} \tag{10.2}$$

Let $\delta(I^P) := \mu_{I^P}(1, 1)/|C|$. By Lemma 5.0.7,

$$\delta(I^P) = \frac{\sum_{\theta \in \text{Irr}(C)} I^P(\theta)(1) \theta(1)}{|C|} = \frac{\pm \sum_{\theta \in \text{Irr}(C)} \theta(1)^2}{|C|} = \pm 1.$$

Since $\text{Irr}(B)$ is a K -basis for $\text{CF}(G, B; K)$, we have $d(I(\chi))(1) = I^P(d(\chi))(1)$ for all $\chi \in \text{Irr}(B)$. But, $d(I(\chi))(1) = I(\chi)(x)$, and $I^P(d(\chi))(1) = \delta(I^P)\chi(x)$ from (10.2). Thus,

$$I(\chi)(x) = \delta(I^P)\chi(x),$$

for all $\chi \in \text{Irr}(B)$. Substituting $\delta(I^P) = \pm 1$ yields (ii).

(ii) \Rightarrow (i). Suppose $I(\chi)(x) = \delta(I)\chi(x)$ for all $\chi \in \text{Irr}(B)$, where $\delta(I) \in \{\pm 1\}$. Take $I^P \in \{\pm id\}$ such that $\mu_{I^P}(1, 1)/|C| = \delta(I)$. Let $\chi \in \text{Irr}(B)$. Suppose $1 \neq y \in C$, then

$$d(I(\chi))(y) = 0 = I^P(d(\chi))(y),$$

by the definition of d and the fact that I^P is perfect. On the other hands, the only p -regular element in C is 1, so

$$d(I(\chi))(1) = I(\chi)(x) = \delta(I)\chi(x) = \frac{\mu_{I^P}(1, 1)}{|C|}\chi(x) = I^P(d(\chi))(1).$$

So,

$$d(I(\chi))(1) = I^P(d(\chi))(1).$$

Hence, we have $d(I(\chi)) = I^P(d(\chi))$ for all $\chi \in \text{Irr}(B)$. This proves (i). \square

Note that Proposition 10.0.8 does not require the isometry I to be perfect.

Corollary 10.0.9. *Suppose that $C_G(x)$ is a p -group for every p -element $1 \neq x \in G$. Let B be the principal block of $\mathcal{O}G$. Let $I : R_K(B) \rightarrow R_K(B)$ be an isometry. The following are equivalent.*

(i) I is an isotypy.

(ii) For each p -element $1 \neq x \in G$, either $I(\chi)(x) = \chi(x) \forall \chi \in \text{Irr}(B)$ or $I(\chi)(x) = -\chi(x) \forall \chi \in \text{Irr}(B)$.

Proof. Let $D \in \text{Syl}_p(G)$ be a defect group of B . Suppose I is an isotypy. Let $1 \neq x \in G$ be a p -element. Without loss of generality, assume that $x \in D$. Let $\langle x \rangle = P \leq D$. Then there exists $I^P \in \text{PI}(B_P)$ such that $d_G^{(x, e_P)} \circ I = I^P \circ d_G^{(x, e_P)}$. By Proposition 10.0.8, we have that either $I(\chi)(x) = \chi(x) \forall \chi \in \text{Irr}(B)$ or $I(\chi)(x) = -\chi(x) \forall \chi \in \text{Irr}(B)$. This proves (ii).

Now suppose (ii). Let $P \leq D$ be a non-trivial cyclic subgroup and let x be a generator of P . By (ii) and Proposition 10.0.8, there exists $I^P \in \text{PI}(B_P)$ such that $d_G^{(x, e_P)} \circ I = I^P \circ d_G^{(x, e_P)}$. Since this is true for any non-trivial cyclic subgroup P of D and generator x of P , by Lemma 10.0.5, it follows that I is an isotypy. \square

10.1 Cyclic p -groups

Let G be the cyclic group C_q where q is a power of a prime p and let $B = \mathcal{O}G$. Fix a generator x of G and let $\omega = e^{2\pi i/q}$. All irreducible characters of G are of the form χ^k for $k \in \{0, \dots, q-1\}$, where $\chi^k(x^a) = \omega^{ka}$. Suppose $I \in \text{PI}(B)$ is a perfect isometry. By Lemma 5.0.7, we know that I has a homogenous sign. So $I(\chi^k) = \delta(I)\chi^{\sigma_I(k)}$, $\forall k$ where $\delta(I) \in \{\pm 1\}$ and σ_I is a bijective function from the set $\{0, \dots, q-1\}$ to itself.

Theorem 10.1.1. *The isotypies in $\text{PI}(B)$ are as follows.*

- *If p is even then the isotypies are $\{\pm id, \pm J\}$ where $J(\chi^k) = \chi^{k+q/2 \pmod q}$, $\forall k$. So $\text{IPI}(B) = \langle J \rangle \times \langle -id \rangle$.*
- *If p is odd then the isotypies are $\{\pm id\}$. So $\text{IPI}(B) = \langle -id \rangle$.*

Proof. Since $I \in \text{PI}(B)$ is an isotypy if and only if $-I$ is an isotypy, it suffices to find isotypies with all-positive signs.

Suppose that $I \in \text{PI}(B)$ is an isotypy with all-positive sign, say $I(\chi^k) = \chi^{\sigma(k)}$ for some bijective function $\sigma : \{0, \dots, q-1\} \rightarrow \{0, \dots, q-1\}$. Since $C_G(g)$ is a p -group for every non-trivial p -element $g \in G$, by Corollary 10.0.9, we have $\chi^{\sigma(k)}(x) = I(\chi^k)(x) = \delta \chi^k(x)$ for all $k \in \{0, \dots, q-1\}$, where $\delta \in \{\pm 1\}$.

Suppose $\delta = 1$. Then $\chi^{\sigma(k)}(x) = \chi^k(x)$, $\forall k$. So $\omega^{\sigma(k)} = \omega^k$. Thus $\sigma(k) - k$ is divisible by q . But $\sigma(k), k \in \{0, \dots, q-1\}$, this implies that $\sigma(k) = k$. That is, I is the identity map.

Suppose $\delta = -1$. Then $\chi^{\sigma(k)}(x) = -\chi^k(x)$, $\forall k$. So $\omega^{\sigma(k)} = -\omega^k$. Thus $\sigma(k) \neq k$ and q divides $2(\sigma(k) - k)$. Suppose p is odd, then q divides $\sigma(k) - k$. This implies that $\sigma(k) - k = 0$, a contradiction. Hence, the only isotypies when p is odd are $\{\pm id\}$.

Suppose that p is even. Then $q/2$ divides $(\sigma(k) - k)$. Since $\sigma(k) \neq k$, $\forall k$, we have $0 < (\sigma(k) - k) < q$. This implies that $\sigma(k) = k + q/2$, $\forall k$, that is, $I = J$. To verify that J is an isotypy, consider

$$J(\chi^k)(x^m) = \chi^{k+q/2 \pmod q}(x^m) = \omega^{mk+mq/2} = \omega^{mq/2}\omega^{mk} = (-1)^m \chi^k(x^m).$$

This shows that, for each $g \in G$, we have $J(\theta)(g) = \delta\theta(g)$ for all $\theta \in \text{Irr}(G)$, where $\delta \in \{\pm 1\}$. By Corollary 10.0.9, this implies that J is an isotypy. Hence the isotypies when p is even are $\{\pm id, \pm J\}$. \square

10.2 Suzuki group $\text{Sz}(q)$

Let $p = 2$ and G be the Suzuki group $\text{Sz}(q)$. Let B be the principal block of $\mathcal{O}G$ with defect group D . We will use the same notations as in Section 7.2.

Theorem 10.2.1. *The isotypy subgroup of $\text{PI}(B)$ are*

$$\text{IPI}(B) \cong S_{q/2} \times S_{(q+r)/4} \times S_{(q-r)/4} \times \langle -id \rangle.$$

In other words, every perfect isometry that sends W_1 to $\{\pm W_1\}$ and sends W_2 to $\{\pm W_2\}$ is an isotypy.

Proof. We have

$$\text{Irr}(B) = \{1, X_i\} \cup \{Y_j\} \cup \{Z_k\} \cup \{W_1, W_2\}.$$

From the results in Section 7.2, every perfect isometry has a homogenous sign and

$$\text{PI}(B) = S_{q/2} \times S_{(q+r)/4} \times S_{(q-r)/4} \times S_2 \times \langle -id \rangle,$$

where $S_{q/2}, S_{(q+r)/4}, S_{(q-r)/4}, S_2$ are the permutation groups of $\{1, X_i\}, \{Y_j\}, \{Z_k\}$ and $\{W_1, W_2\}$ respectively.

The p -elements of G are, up to conjugation, $1, \sigma, \rho, \rho^{-1}$. We also have $|C_G(\sigma)| = q^2, |C_G(\rho)| = |C_G(\rho^{-1})| = 2q$. So, Corollary 10.0.9 applies. Let $I \in \text{PI}(B)$ be a perfect isometry with all-positive sign. By Corollary 10.0.9, I is an isotypy if and only if

- $I(\chi)(\sigma) = \chi(\sigma)$ for all $\chi \in \text{Irr}(B)$,
- $I(\chi)(\rho) = \chi(\rho)$ for all $\chi \in \text{Irr}(B)$ and
- $I(\chi)(\rho^{-1}) = \chi(\rho^{-1})$ for all $\chi \in \text{Irr}(B)$.

Looking at the character table of G (Table 7.2), it is clear that we must have $I(W_1) = W_1, I(W_2) = W_2$. This implies that I belongs to the subgroup $S_{q/2} \times S_{(q+r)/4} \times S_{(q-r)/4}$. But I is an isotypy if and only if $-I$ is an isotypy, we have

$$\text{IPI}(B) \cong S_{q/2} \times S_{(q+r)/4} \times S_{(q-r)/4} \times \langle -\text{id} \rangle.$$

□

Remark 10.2.2. Let $I : R_K(B) \rightarrow R_K(B)$ be an isometry with all-positive sign such that $I(\mathbb{X}) = \mathbb{X}, I(\mathbb{Y}) = \mathbb{Y}, I(\mathbb{Z}) = \mathbb{Z}, I(W_1) = W_1, I(W_2) = W_2$. Then we can see that

- $I(\chi)(\sigma) = \chi(\sigma)$ for all $\chi \in \text{Irr}(B)$,
- $I(\chi)(\rho) = \chi(\rho)$ for all $\chi \in \text{Irr}(B)$ and
- $I(\chi)(\rho^{-1}) = \chi(\rho^{-1})$ for all $\chi \in \text{Irr}(B)$.

By Corollary 10.0.9, I is an isotypy. In particular, I is perfect. This gives an alternative, quicker, and partial proof of Lemma 7.2.3.

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Appendix A

Computer-aided results

Given a group G and a character table of G . Let B is a block of $\mathcal{O}G$ and suppose we know the irreducible characters in $\text{Irr}(B)$. Then, in principal, it is always possible to calculate $\text{PI}(B)$. The basic algorithm is as follows.

1. Pick a new permutation $\sigma : \text{Irr}(B) \longrightarrow \text{Irr}(B)$. If there is no permutation left, finish.
2. Pick a new sign function $\varepsilon : \text{Irr}(B) \longrightarrow \{\pm 1\}$. If there is no sign function left, that means σ is not a perfect permutation. We must try another permutation. Repeat from 1.
3. Construct an isometry $I : R_K(B) \longrightarrow R_K(B)$ such that $I(\chi) = \varepsilon(\chi)\sigma(\chi)$.
4. Construct a character $\mu \in R_K(B \otimes_{\mathcal{O}} B^{\circ})$ such that

$$\mu(g, h) = \sum_{\chi \in \text{Irr}(B)} I(\chi)(g)\chi(h).$$

5. Check for all $g, h \in G$ if μ is perfect (Definition 3.1.1). For the integrality condition, we can use use (2.1) and (2.3) to determine if $\mu(g, h)/|C_G(g)|$ and $\mu(g, h)/|C_G(h)|$ are in \mathcal{O} .
6. If μ is perfect, record I and $-I$. Then repeat from 1. (By the uniqueness of sign, the only other sign τ such that $\chi \mapsto \tau(\chi)\sigma(\chi)$ is perfect is $\tau = -\varepsilon$.)

7. If μ is not perfect, that means the combination (σ, ε) does not work. We must try another sign. Repeat from 2.

If $|\text{Irr}(B)| = n$, then we see that we need to search through $n!2^n$ combinations of σ and ε . As n gets larger, the task rapidly becomes impractical to do by hand. So we write programs in GAP [19] to automate the searching process using the above algorithm.

There are a few tricks to improve the speed of the programs. For example, we can use Theorem 3.1.5 to check only the p -singular elements for the integrality condition. Also, if p is odd, then we can use Lemma 3.2.5 to quickly check the validity of a sign for a given permutation, before checking the perfect conditions of the resultant isometry.

Nevertheless, when n is very big (typically $n > 13$), our programs still take a very long time to compute the perfect isometry groups, due to a large number of possible permutations. This makes it impractical to deal with large blocks.

A.1 Codes

Let B be a block of $\mathcal{O}G$. Suppose $|\text{Irr}(B)| = n$. Let $\text{chind} = [i_1, \dots, i_n]$ be the subscripts of irreducible characters in $\text{Irr}(B)$. Since elements in $\text{PI}(B)$ are permutation in S_n with signs, we can embed them into S_{2n} as follows. Let $I = [i_1, \dots, i_n, -i_1, -i_2, \dots, -i_n]$. Since any isometry $J : R_K(B) \rightarrow R_K(B)$ permutes entries in I , we can identify J with a permutation on positions of entries of I .

For example, if $\text{chind} = [1, 4, 5]$, then an isometry $[1, 4, 5] \mapsto [-1, 5, -4]$ is a permutation $(1, 4)(2, 3, 5, 6)$ on positions of entries of $I = [1, 4, 5, -1, -4, -5]$. In other words, it sends the entry in position 1 of I (1) to the entry in position 4 of I (-1), sends the entry in position 2 of I (4) to the entry in position 3 of I (5), sends the entry in position 3 of I (5) to the entry in position 5 of I (-4) etc. Thus J can be represented as a list $[-1, 5, -4]$ with respect to chind or as a permutation $(1, 4)(2, 3, 5, 6) \in S_6$ with respect to I . We call $[-1, 5, -4]$ a *list* form of J (with

respect to chind), and $(1, 4)(2, 3, 5, 6)$ a *double permutation* form of J (with respect to I).

Our codes consist of 11 custom functions:

| Functions | Description |
|----------------------------------|--|
| <code>ListToDoublePerm</code> | Convert an isometry in list form to double permutation form. |
| <code>MyValuation</code> | The extended p -valuation of a given number, see (2.2). |
| <code>getTcenord</code> | Get the sizes of centralizers and orders of representatives of conjugacy classes of G |
| <code>mueval</code> | Calculate $\mu_J(g, h)$ corresponding to an isometry J . |
| <code>chkperf_integral</code> | Check if μ satisfies the integrality condition. |
| <code>chkperf_sep</code> | Check if μ satisfies the separation condition. |
| <code>chkperf</code> | Check if μ is perfect. |
| <code>getSignPerm</code> | Given a permutation σ , get the set of possible choices of signs ε . Use Lemma 3.2.5 if applicable ($p > 2$). |
| <code>chkPerfPermInternal</code> | Check (internally) if a given permutation is perfect. |
| <code>getPerfPermGroup</code> | Calculate the group $\text{PI}^+(B)$ of perfect permutations. |
| <code>getPerfect</code> | Calculate $\text{PI}(B)$ and $\text{PI}^+(B)$. |

The only functions meant to be called by the user are `getPerfPermGroup` and `getPerfect`. Other functions will be called internally, so we shall not explain their parameters.

ListToDoublePerm

```
#-----ListToPerm-----
ListToDoublePerm:=function(a,chind,I)
local i, seq, spos, smpos, tpos, tmpos;
seq:=[1..Size(I)];
for i in [1..Size(a)] do
spos := Position(I, chind[i]);
```

```

smpos := Position(I, -chind[i]);
tpos := Position(I, a[i]);
tmpos := Position(I, -a[i]);
seq[spos] := tpos;
seq[smpos] := tmpos;
od;
return PermList(seq);
end;

```

MyValuation

```

#-----MyValuation-----
MyValuation:=function(z,p)
local f,deg,const;
f:=MinimalPolynomial(Rationals, z);
deg:=Degree(f);
const:=Value(f,0);
if const=0 then return 1;
else return Valuation(const,p)/deg;
fi;
end;

```

getTcenord

```

#-----getTcenord-----
getTcenord:=function(tbl)
local Tcen, Tord;
Tcen:=SizesCentralizers(tbl);
Tord:=OrdersClassRepresentatives(tbl);
return rec(cen:=Tcen, ord:=Tord);
end;

```

mueval

```
#-----mueval-----
mueval:=function(a,i,j,irr,chind,nchars)
local k, temp,T;
  temp:=0;
  T:=irr;
  for k in [1..nchars] do
if a[k] > 0 then
  temp:= temp + T[AbsInt(a[k])] [i]*T[chind[k]] [j];
else
  temp := temp - T[AbsInt(a[k])] [i]*T[chind[k]] [j];
fi;
  od;
  return temp;
end;
```

chkperfintegral

```
#-----chkperf_integral-----
chkperf_integral:=function(a,p,irr,Tcen,Tord,chind,nchars)
local i,j,re,psingclass, valcen, uij , uji;
  re := true;
psingclass:=ListX([1..Size(Tcen)] , x->Tord[x] mod p = 0 , x->x);
  for i in psingclass do
valcen:=Valuation(Tcen[i],p);
if valcen > 0 then
  for j in psingclass do
    uij:=mueval(a,i,j,irr,chind,nchars);
  uji:=mueval(a,j,i,irr,chind,nchars);
if uij <> 0 then
```

```

if valcen > MyValuation(uij,p) then
return false;
fi;
fi;
if uji <> 0 then
if valcen > MyValuation(uji,p) then
return false;
fi;
fi;
od;
fi;
od;
return re;
end;

```

chkperfsep

```

#-----chkperf_sep-----
chkperf_sep:=function(a,p,irr,Tord,chind,nchars)
local i , j ,re;
re := true;
for i in [1..Size(Tord)] do
for j in [1..Size(Tord)] do
if mueval(a,i,j,irr,chind,nchars) <> 0 then
if (Tord[i] mod p = 0) and (Tord[j] mod p <> 0) then
return false;
fi;
if (Tord[i] mod p <> 0) and (Tord[j] mod p = 0) then
return false;
fi;
fi;

```

```

        od;
    od;
    return re;
end;

```

chkperf

```

#-----chkperf-----
chkperf:=function(a,p,irr,Tcen,Tord,chind,nchars)
local re;
    re := false;
    if chkperf_integral(a,p,irr,Tcen,Tord,chind,nchars) then
        if chkperf_sep(a,p,irr,Tord,chind,nchars) then
            re := true;
        fi;
    fi;
    return re;
end;

```

getSignPerm

```

#-----getSignPerm -----
# for each permutation to check for perfect isometry,
# find unique signs that are compatible with it
# if p=2 or #chind=2 then return a direct product of C_2
# otherwise if not perfect perm, return {[0,..,0]}
# if perfectperm, return {sgn} where sgn is a determined sign
#-----
getSignPerm:=function(g,chind,nchars,p,irr,allsigns,zerosign)
local k,sgn,s,a;
a:=OnTuples( chind, g );
    for k in [1..nchars] do

```



```

        if Valuation(irr[a[k]][1],p) <>
            Valuation(irr[chind[k]][1],p) then
return [zerosign];
# For a perfect isometry we must have I(chi)(1)/chi(1) an invertible
# element in O.
fi;
    od;
if (p = 2) or (nchars = 2) then
return allsigns;
fi;
    sgn:=[1];
    s:= irr[a[1]][1]/irr[chind[1]][1] mod p;
    for k in [2..nchars] do
        if (irr[a[k]][1]/irr[chind[k]][1] mod p) = s then
Append(sgn,[1]);
        elif (-irr[a[k]][1]/irr[chind[k]][1] mod p) = s then
Append(sgn,[-1]);
        else
return [zerosign];
        fi;
# For a PI we must have I(chi)(1)/chi(1) constant mod p
# so we add +/-1 to try to make it constant if possible.
# If not then the perm is not perfect
    od;
    return [sgn];
end;

chkPerfPermInternal

#-----chkPerfPermInternal -----
chkPerfPermInternal:=function(g,irr,chind,nchars,p,I,

```

```

                                zerosign,allsigns,Tcen,Tord,chkhomog)
local C, a, y, perm, sgn, j, found, homogsgn,chkhomog_local;
found:=false;
homogsgn:=true;
chkhomog_local:=ShallowCopy(chkhomog);
C:=getSignPerm(g,chind,nchars,p,irr,allsigns,zerosign);
for y in C do;
if Size(C) = 1 then
if y = zerosign then
break;
else
sgn:=y;
fi;
else
sgn:=[];
for j in [1..nchars] do
Append(sgn, [(-1)^((2*j)^y)]);
od;
fi;
perm:=OnTuples( chind, g );
a:=List([1..nchars] , x->perm[x]*sgn[x]); # create a with sign
    if chkperf(a,p,irr,Tcen,Tord,chind,nchars) then
if not chkhomog_local then
if not ( ForAll(a,x->x > 0) or ForAll(a,x->x < 0) ) then
homogsgn:=false;
chkhomog_local:=true;
fi;
fi;
        found:=true;
        break;

```

```

fi;
od;
    if found then
return rec(seq:=a, found:=true, homogsgn:=homogsgn );
else
return rec(seq:=[], found:=false, homogsgn:=homogsgn,);
fi;
end;

```

getPerfPermGroup

The following program computes $PI^+(B)$.

```

#-----getPerfPermGroup (using SubgroupProperty)-----
# tbl = Character table
# chind = Character indexes (subscripts)
# symgp = Permutation group on chind
# PPlist = list of already known perfect permutations.
#           Set to [] if not known
# p = prime number
# I = sequence of +- entries in chind e.g. if chind = [1,2,3] then
#     I = [1,2,3,-1,-2,-3]
#-----
getPerfPermGroup:=function(tbl,chind,symgp,PPlist,p,I)
local j, P,x,irr,zerosign, nchars, allsigns,Tcen,Tord,chkhomog,
      C2, C1;
irr:=Irr(tbl);
nchars:=Size(chind);
C2:=SymmetricGroup(2); C1:=Subgroup(C2, []);
allsigns:=C1;
for j in [2..nchars] do

```

```

allsigns:=DirectProduct(allsigns,C2);
od;
zerosign:=List(chind, x->0);
Tcen:=getTcenord(tbl).cen;
Tord:=getTcenord(tbl).ord;
chkhomog:=false;
P:=SubgroupProperty(symgp,
x -> chkPerfPermInternal(x,irr,chind,nchars,p,I,zerosign,allsigns,
    Tcen,Tord,chkhomog).found);
return P;
end;

```

getPerfect

The following program also computes $PI^+(B)$ like `getPerfPermGroup` does but may take longer to run (not using the effective `SubgroupProperty` built-in function in GAP). For quick runtime, run

```
gap>PP:=getPerfPermGroup(tbl,chind,symgp,[],p,I);
```

first to get the group $PI^+(B)$, and then

```
gap>PI:=getPerfect(tbl,chind,symgp,PP,p,I);
```

```
#-----getPerfect-----
```

```
# tbl = Character table
```

```
# chind = Character indexes (subscripts)
```

```
# symgp = Permutation group on chind
```

```
# PPlist = list of already known perfect permutations.
```

```
#           Set to [] if not known
```

```
# p = prime number
```

```
# I = sequence of +- entries in chind e.g. if chind = [1,2,3] then
```

```
#       I = [1,2,3,-1,-2,-3]
```

```

#-----
getPerfect:=function(tbl,chind,symgp,PPlist,p,I)
local x,y,i,j, perm, perminfo, Tord, Tcen, nchars, irr, GG, P, L,
      ma, PI, zerosign, chkhomog, homogsgn, D, C, found, sgn,
      allsigns, C2,C1,sizeD, sizeP, sizesym;
irr:=Irr(tbl);
nchars:=Size(chind);
Tcen:=getTcenord(tbl).cen;
Tord:=getTcenord(tbl).ord;
GG:=SymmetricGroup(2*nchars);
P:=Subgroup(symgp,PPlist);
sizeP:=Size(P);
sizesym:=Size(symgp);
L:=[]; PI:=[];
chkhomog:=false;
      homogsgn:=true;
      ma:=List([1..nchars] , x-->-chind[x]);
PI:=Subgroup(GG, [ListToDoublePerm(ma,chind,I)]);

C2:=SymmetricGroup(2); C1:=Subgroup(C2, []);
allsigns:=C1;
for j in [2..nchars] do
allsigns:=DirectProduct(allsigns,C2);
od;
zerosign:=List(chind, x->0);

if Length(PPlist) > 0 then
Print( "Initializing...\n");
j:=1;
for x in PPlist do

```

```

perminfo:=chkPerfPermInternal(x,irr,chind,nchars,p,I,
    zerosign,allsigns, Tcen,Tord,chkhomog);
if perminfo.found then
PI:=ClosureSubgroup(PI,
    ListToDoublePerm(perminfo.seq,chind,I));
if not perminfo.homogsgn and not chkhomog then
homogsgn:=false;
chkhomog:=true;
fi;
Print( "No. of perfect isometries found so far =
    ",2*j,"\n");
j:=j+1;
fi;
if Size(PI) = 2*Size(PPlist) then
Print( "No. of perfect isometries found so far =
    ",2*Size(PPlist),"\n");
break;
fi;
od;
Print( "End of initialization\n");
fi;
D:=DoubleCosetRepsAndSizes(symgp, P, P);
sizeD:=Size(D);
if D[1][1] = One(symgp) then
i:=2;
else
i:=1;
fi;

while i <= sizeD and sizeP < sizesym do

```

```

x:=D[i][1];
perminfo:=chkPerfPermInternal(x,irr,chind,nchars,p,I,
    zerosign,allsigns,Tcen,Tord,chkhomog);
if perminfo.found then
P:=ClosureSubgroup(P,x);
sizeP:=Size(P);
PI:=ClosureSubgroup(PI, ListToDoublePerm(perminfo.seq,chind,I));
if not perminfo.homogsgn and not chkhomog then
homogsgn:=false;
chkhomog:=true;
fi;
Print( "No. of perfect isometries found so far =
    ",2*sizeP,"\n");
D:=DoubleCosetRepsAndSizes(symgp, P, P);
sizeD:=Size(D);
    if D[1][1] = One(symgp) then
i:=2;
else
i:=1;
fi;
else
Print( "No. of permutations left = ",sizeD-i,"\n");
i:=i+1;
fi;
od;    #-----exit x loop -----

Print( "=====\n");
Print( "No. of perfect isometries = ",Size(PI),"\n");
Print( "=====\n");
Print( "Homogenous signs = ",homogsgn,"\n");

```

```
Print( "Returning the record with component .full and .perm\n");
# Print( "Full PI Group is ", StructureDescription(PI), "\n");;
# Print( "Perm PI Group is ", StructureDescription(P), "\n");;
return rec(full:=PI, perm:=P);
end;
```