

ALGEBRAIC TOPOLOGY OF PDES

A THESIS SUBMITTED TO THE UNIVERSITY OF MANCHESTER
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE FACULTY OF ENGINEERING AND PHYSICAL SCIENCES

2011

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(Word count: 12034)

The University of Manchester

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Doctor of Philosophy

Algebraic Topology of PDES

December 15, 2011

We consider a compact, oriented, smooth Riemannian manifold M (with or without boundary) and we suppose G is a torus acting by isometries on M . Given X in the Lie algebra of G and corresponding vector field X_M on M , one defines Witten's inhomogeneous coboundary operator $d_{X_M} = d + \iota_{X_M} : \Omega_G^\pm \rightarrow \Omega_G^\mp$ (even/odd invariant forms on M) and its adjoint δ_{X_M} .

First, Witten [35] showed that the resulting cohomology classes have X_M -harmonic representatives (forms in the null space of $\Delta_{X_M} = (d_{X_M} + \delta_{X_M})^2$), and the cohomology groups are isomorphic to the ordinary de Rham cohomology groups of the set $N(X_M)$ of zeros of X_M . The first principal purpose is to extend Witten's results to manifolds with boundary. In particular, we define relative (to the boundary) and absolute versions of the X_M -cohomology and show the classes have representative X_M -harmonic fields with appropriate boundary conditions. To do this we present the relevant version of the Hodge-Morrey-Friedrichs decomposition theorem for invariant forms in terms of the operators d_{X_M} and δ_{X_M} ; the proof involves showing that certain boundary value problems are elliptic. We also elucidate the connection between the X_M -cohomology groups and the relative and absolute equivariant cohomology, following work of Atiyah and Bott. This connection is then exploited to show that every harmonic field with appropriate boundary conditions on $N(X_M)$ has a unique corresponding an X_M -harmonic field on M to it, with corresponding boundary conditions. Finally, we define the interior and boundary portion of X_M -cohomology and then we define the X_M -Poincaré duality angles between the interior subspaces of X_M -harmonic fields on M with appropriate boundary conditions.

Second, In 2008, Belishev and Sharafutdinov [9] showed that the Dirichlet-to-Neumann (DN) operator Λ inscribes into the list of objects of algebraic topology by proving that the de Rham cohomology groups are determined by Λ .

In the second part of this thesis, we investigate *to what extent is the equivariant topology of a manifold determined by a variant of the DN map?*. Based on the results in the first part above, we define an operator Λ_{X_M} on invariant forms on the boundary ∂M which we call the X_M -DN map and using this we recover the long exact X_M -cohomology sequence of the topological pair $(M, \partial M)$ from an isomorphism with the long exact sequence formed from the generalized boundary data. Consequently, This shows that for a Zariski-open subset of the Lie algebra, Λ_{X_M} determines the free part of the relative and absolute equivariant cohomology groups of M . In addition, we partially determine the mixed cup product of X_M -cohomology groups from Λ_{X_M} . This shows that Λ_{X_M} encodes more information about the equivariant algebraic topology of M than does the operator Λ on ∂M . Finally, we elucidate the connection between Belishev-Sharafutdinov's boundary data on $\partial N(X_M)$ and ours on ∂M .

Third, based on the first part above, we present the (even/odd) X_M -harmonic cohomology which is the cohomology of certain subcomplex of the complex (Ω_G^*, d_{X_M}) and we prove that it is isomorphic to the total absolute and relative X_M -cohomology groups.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Publications

The last four chapters of this thesis which contain the new results are based on the following publications:

- Chapter 3 is based on the paper “Witten-Hodge theory for manifolds with boundary and equivariant cohomology.” (with J. Montaldi) [2], *Differential Geometry and its Applications* (2011), doi:10.1016/j.difgeo.2011.11.002.
- Chapter 4 is based on [2] and the paper “Generalized Dirichlet to Neumann operator on invariant differential forms and equivariant cohomology.” (with J. Montaldi) [3], *Topology and its Applications*, **159**, 823–832, 2012, doi:10.1016/j.topol.2011.11.052.
- Chapter 5 is based on the paper [3].
- Chapter 6 is based on the Preprint “ X_M -Harmonic Cohomology and Equivariant Cohomology on Riemannian Manifolds With Boundary” [4].

It was a great honour for me to present the results of [2] and the main goal of [3] in the 25th British Topology Meeting which took place in Merton Collage at the University of Oxford, 6th – 8th September 2010 (see, <http://www.maths.ox.ac.uk/groups/topology/btm2010>).

Acknowledgements

First and foremost, my great thanks go to the almighty *Allah* who made this work possible.

I am heartily thankful to my supervisor, Professor James Montaldi, whose encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of this huge subject. He provided enthusiasm, inspiration, sound advice, perfect teaching, excellent supervision, and lots of great ideas. He always seemed to know what I needed and I was shown through him how a mathematician should be. I offer him my deepest thanks.

I highly appreciate the valuable comments given by the examiners on the whole thesis which improved the exposition of the thesis. Thank you.

I wish to express my gratitude to Professor Bill Lionheart from University of Manchester for his suggestion of the references [11] and [33]. I am also thankful to Dr Clayton Shonkwiler from University of Georgia in the USA for his valuable discussions with me about the cohomology ring structures. In addition, I would like to thank Dr Marta Mazzocco from Loughborough University in the UK who introduced me to the field of isomonodromic deformation systems in 2008.

I humbly thank all my old and new friends and all the exceptional people involved in my life who wish good things on me and in this occasion I would like to mention my wife's parents who always keep supporting me.

A special thanks should go to my wife for her patience in the past four years and for her continuous support to achieve my best. I should not forget to give warm hug with sweet thanks to my beloved Shadan (my daughter) and Muhammad (my son) who have meant a lot to my life.

I offer my sincerest thanks to my brother who has sacrificed a lot in the past years to push me to be better and encourage me towards my best. Thank you for looking after our parents during my absence. In addition, I would like to thank my sister for her constant support to me and I wish her a bright and promising future in Medical College.

Lastly, and most importantly, I wish to express my deep and sincere gratitude to my mother and father. They bore me, raised me, supported me, taught me, and loved me. Thank you for reading to me as a kid. I graciously thank them for everything they have done.

Chapter 1

Introduction

Hodge theory which is named after W. V. D. Hodge, is one aspect of the study of the algebraic topology of a smooth manifold M without boundary. In the 1950's, much effort had been made by Morrey [30] and Friedrichs [18] to extend Hodge theory to a manifold M with boundary ∂M , leading to the *Hodge-Morrey-Friedrichs* decompositions theory [31]. These theorems work out the consequences for the cohomology groups of M with real coefficients. More concretely, Hodge theory is a fundamental theory which shows how the de Rham cohomology groups of a manifold M (with or without boundary) can be realized from the analysis (harmonic forms (or fields)) point of view. This thesis can be thought of as a continuation of this trend but in the setting of equivariant topology, showing the analysis can be used as powerful tools to encode more information about the equivariant algebraic topology of the manifold in question, leading to Witten-Hodge theory and consequently to the generalized boundary data on the boundary of the manifold.

We briefly outline the structure of the thesis. Chapter 2 is devoted to background material while the final four to new results. All of the new results found within this thesis can also be found in the papers [2, 3, 4].

Chapter 2 covers more background material, namely the classical Hodge theory with some of Witten's results [35] and Hodge-Morrey-Friedrichs decompositions theory with the recent modification [15] to this theory. In addition, we briefly outline the relation between algebraic topology and the Dirichlet-to-Neumann (DN) map Λ [11] and [33]. However, there is some different notation which is explained there and it is not familiar in the literature.

In chapter 3, the new material begins. When $\partial M = \emptyset$, Witten, in his well-known paper [35] which is regarded as the seed to the subject of Topological Quantum Field Theory (TQFT) [7], deforms the de Rham coboundary operator and shows that the resulting cohomology classes have K -harmonic representatives and the cohomology groups of M are isomorphic to the ordinary de Rham cohomology groups of the set of zeros of a killing

vector field K on M . In more general context, in this thesis we suppose G is a torus (unless otherwise indicated) acting by isometries on a compact, oriented, smooth Riemannian manifold M of dimension n (with or without boundary) and in this setting, we reconsider Witten's inhomogeneous coboundary operator $d_{X_M} = d + \iota_{X_M} : \Omega_G^\pm \rightarrow \Omega_G^\mp$ (even/odd invariant forms on M) and its adjoint δ_{X_M} where X_M is the corresponding vector field on M to a vector X which is in the Lie algebra of G . Since $d_{X_M}^2 = 0$, there are corresponding cohomology groups which we call X_M -cohomology groups. The main new results are these:

- 1- When $\partial M \neq \emptyset$, we present the relevant version of the Hodge-Morrey-Friedrichs decomposition for the square integrable invariant differential forms $L^2\Omega_G^\pm(M)$ in terms of d_{X_M} and δ_{X_M} which we call it within this thesis *X_M -Hodge-Morrey-Friedrichs decomposition*. The proof is based on the ellipticity of a certain BVP. This gives a new decomposition to the space of $L^2\Omega_G^\pm(M)$ rather than to the space of smooth invariant differential form $\Omega_G^\pm(M)$.
- 2- Using the setting above, we extend the Localization Theorem (to the fixed point set) of Atiyah-Bott of [8] to manifolds with boundary which leads to relate the relative and absolute X_M -cohomology groups with the relative and absolute equivariant cohomology groups of M .
- 3- No.(1-2) above gives insight to extend some of Witten's original results of [35] to manifolds with boundary as follows:
 - (i) Based on *X_M -Hodge-Morrey-Friedrichs decomposition*, we show the classes of the relative and absolute X_M -cohomology groups have representative X_M -harmonic fields (invariant forms in $\ker d_{X_M} \cap \ker \delta_{X_M}$) with appropriate boundary conditions. Thus, these spaces are a concrete realization of the relative and absolute X_M -cohomology groups inside $\Omega_G^\pm(M)$.
 - (ii) Based on No.(2), we prove that the relative and absolute X_M -cohomology of M are isomorphic to the ordinary relative and absolute de Rham cohomology groups of the set $N(X_M)$ of zeros of X_M respectively and consequently to the relative and absolute singular homology groups of $N(X_M)$. This reduction of cohomology on M to cohomology on $N(X_M)$ is crucial to make computation possible in Quantum Fields Theory when $\partial M \neq \emptyset$ (see, No. (3) in section 3.5).

In addition, all the results above and the other within this chapter show that the Witten-Hodge theory gives additional equivariant topological insight.

In chapter 4, we extend the recent work of DeTurck and Gluck [15] which is used to define the interior and boundary portion of the ordinary de Rham cohomology groups to the context of X_M -cohomology and we give here a list of the main new results:

- 1- We first prove that the concrete realization of the relative and absolute X_M -cohomology groups meet only at the origin in $\Omega_G^\pm(M)$. We use a different argument in the proof, based on Hadamard's lemma and the *boundary normal coordinates* because the technique which is used to prove the classical case does not appear to extend to the present setting and in fact this new argument is also valid in the classical case.
- 2- The consideration of X_M allows to define the long exact sequence in X_M -cohomology of the topological pair $(M, \partial M)$ derived from the inclusion $i : \partial M \hookrightarrow M$. This is used to define the interior and boundary portion of the absolute and relative X_M -cohomology respectively.
- 3- We decompose the concrete realization of the relative and absolute X_M -cohomology groups to the direct sum of interior and boundary subspaces with appropriate boundary conditions. We give a direct proof involving only the cohomology theory while the proof by DeTurck and Gluck of the analogous result uses the duality between de Rham cohomology and singular homology and we do not have such a result for X_M -cohomology. Moreover, the same argument can be used to prove DeTurck and Gluck's original results [15]. This gives the following results:
 - (i) We refine the results of chapter 3 to the interior and boundary subspaces and this gives more concrete understanding to the extension results of chapter 3. In addition, we refine the generalized Localization Theorem (i.e. $\partial M \neq \emptyset$) of Atiyah-Bott to the style of interior and boundary portions and it is used to give an alternative argument to prove the results in No. (3) above.
 - (ii) The results of this chapter are used to define the X_M -Poincaré duality angles between the interior subspaces of X_M -harmonic fields with appropriate boundary conditions and we prove that they are all acute angles.

Moreover, the results above show that the Witten-Hodge theory gives additional equivariant geometric insight rather than the topological insight. Finally, at the very end of this chapter we state a geometric question which follows from the above results.

In Chapter 5, we consider the following open problem which is of great theoretical and applied interest [11]: “*To what extent are the topology and geometry of M determined by the DN map*”? Recently, Belishev and Sharafutdinov [11] give an answer to the topological aspect of this question when they prove that the real additive de Rham cohomology of a smooth Riemannian manifold M with boundary is completely determined by its boundary data $(\partial M, \Lambda)$ where $\Lambda : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$ is the DN map.

In general the de Rham cohomology of M is isomorphic to the de Rham cohomology of invariant forms when G (or any compact connected Lie group) acts on M . It means in this case that we can “forget” the whole de Rham cochain complex of differential forms

in M and regard only those that are invariant under the group in question. In this light and if the group action on M is by isometries then Belishev-Sharafutdinov's boundary data does not give further information about the equivariant topology (e.g. equivariant cohomology) of M . Therefore, in this chapter we consider this motivation which leads us to be interested in the equivariant topology of M analogue of the above interesting open problem. More precisely, the X_M -Hodge-Morrey-Friedrichs decomposition of smooth invariant differential forms are used to create boundary data which is a generalization of Belishev-Sharafutdinov's boundary data on $\Omega_G^\pm(\partial M)$. The investigations give the following new results:

- 1- We define the X_M -DN operator $\Lambda_{X_M} : \Omega_G^\pm(\partial M) \longrightarrow \Omega_G^{n-\mp}(\partial M)$ which is a generalization to the DN map on $\Omega_G^\pm(\partial M)$. The definition of Λ_{X_M} is based on the solvability of certain BVP.
- 2- Based on Λ_{X_M} , we recover the X_M -cohomology groups and we partially determine the ring structure of X_M -cohomology groups from the generalized boundary data $(\partial M, \Lambda_{X_M})$. In addition, it follows that Λ_{X_M} determines the free part of the relative and absolute equivariant cohomology groups of M when the set $N(X_M)$ of zeros of X_M is equal to the fixed point set F for the G -action.
- 3- Under certain condition, we prove the \pm relative and absolute de Rham cohomology groups of $N(X_M)$ are also determined by the generalized boundary data $(\partial M, \Lambda_{X_M})$. This means that the Belishev-Sharafutdinov's boundary data $(\partial N(X_M), \Lambda)$ can be determined from the generalized boundary data $(\partial M, \Lambda_{X_M})$ and vice versa.

Hence, these results contribute to explain to what extent the equivariant topology of the manifold in question is determined by the X_M -DN map Λ_{X_M} . Moreover, following Witten but for the case when $\partial M \neq \emptyset$, these results suggest a possible relation between Λ_{X_M} and Quantum Field Theory and possibly to other mathematical and physical interpretations (see, No. (3) in Section 5.6). Finally, this imposes a topological open problem which asks about the possibility to determine the torsion part of the absolute and relative equivariant cohomology groups as well from Λ_{X_M} .

Finally, in chapter 6 we prove the (even/odd) cohomology of the subcomplex $(\ker \Delta_{X_M}, d_{X_M})$ of the complex $(\Omega_G^*(M), d_{X_M})$ is enough to determine the total absolute and relative X_M -cohomology groups with few other conclusions where we call the operator $\Delta_{X_M} = d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M}$ within this thesis the *Witten-Hodge-Laplacian* operator, extending a result of S. Cappell et al. [13].

Remark on typesetting: Since the letter H plays three roles in this thesis, we use three different typefaces: a script \mathcal{H} for harmonic fields, a sans-serif H for Sobolev spaces and a normal (italic) H for cohomology. We hope that will prevent any confusion.

Chapter 2

Preliminaries

2.1 Introduction

This chapter covers the background material of this thesis. Much of this material is standard and can be found in the literature, though, some remarks are different and are specified here. Section 2.2 discusses basic but necessary concepts of the left group actions on smooth manifolds. Section 2.3 introduces an overview of the Hodge theory and some of Witten's results [35] on a smooth manifold M without boundary and also we review the Hodge-Morrey-Friedrichs theorem for manifolds with boundary. In addition, we review the recent modification of this theorem by DeTurck and Gluck [15]. Section 2.4 gives the necessary background on the Dirichlet-to-Neumann map Λ for differential forms and states the recent results of [11] and some of the results of [33] which relate Λ to algebraic topology.

2.2 Group actions on smooth manifolds

We start by looking at the definition of group actions on manifolds and some other basic notions because we will need this in the equivariant algebraic topology of manifolds.

Definition 2.2.1 [14, 20] Let G be a group (it could be a Lie group) with identity element e and M a smooth manifold. We say that G acts on M if there exists a smooth map $F : G \times M \longrightarrow M$ (where $F(g, x)$ is denoted by $F(g, x) = g.x$) such that

$$(i) \quad g.(h.x) = (gh).x \quad \text{for } g, h \in G, x \in M$$

$$(ii) \quad e.x = x \quad \text{for } x \in M$$

Definition 2.2.2 [16, 19] Let G be a Lie group acting on a manifold M , and let $x \in M$. The isotropy (stabilizer) subgroup of a point $x \in M$ is the Lie subgroup $G_x = \{g \in G \mid g.x = x\}$. It has Lie algebra $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$, where \mathfrak{g} is the Lie algebra of G and X_M is the

vector field on M corresponding to X . Moreover, G_x is a proper isotropy subgroup of G if $G_x \neq G$.

Remark 2.2.3 (1) We can look at $F(g, x)$ as defining a mapping $g \mapsto F_g$ by $F_g(x) = F(g, x)$. Thus, if F_g is the mapping $F_g : M \rightarrow M$ associated with the action of g on M , it is seen that the left action satisfies the homomorphism property $F_g \circ F_h = F_{gh}$. Since $F_{g^{-1}} = (F_g)^{-1}$, F_g is a diffeomorphism of M [16].

(2) Now suppose that M is a smooth manifold with boundary ∂M and that G acts on M . Number (1) above asserts that for each $x \in \partial M$, F_g is the mapping $F_g : \partial M \rightarrow \partial M$ associated with the action of g on ∂M because F_g is a diffeomorphism of M .

(3) A differential k -form ω in M is said to be *invariant* if $F_g^* \omega = g^* \omega = \omega$ for every $g \in G$, where $F_g^* = g^*$ denotes the pullback induced by $F_g = g$. We write $\Omega_G^k(M)$ for the space of G -invariant differential k -forms.

2.2.1 Averaging with respect to a compact Lie group action

Let M be a smooth manifold of dimension n . For each $0 \leq k \leq n$ denote by $\Omega^k = \Omega^k(M)$ the space of smooth differential k -forms on M . The exterior differential $d : \Omega^k \rightarrow \Omega^{k+1}$ is the de Rham coboundary operator (i.e. $d^2 = 0$) and $(\Omega^k(M), d)$ is de Rham cochain complex where a differential form $\omega \in \Omega^k$ is closed (cocycle) if $d\omega = 0$ and is exact (coboundary) if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}$. The de Rham cohomology of M is defined to be $H^k(M) = \ker d_k / \text{im } d_{k-1}$, where d_k is the restriction of the exterior differential d to Ω^k [12, 14].

We now consider actions of a compact Lie group G on a manifold M . The Haar measure allows to employ *averaging arguments* with respect to compact Lie group actions [19]. From each k -form $\omega \in \Omega^k(M)$ we can construct an invariant form in $\Omega_G^k(M)$ by taking on its “translations”. Following this idea one defines a projection map $J : \Omega^k(M) \rightarrow \Omega_G^k(M)$ by

$$J(\omega)(X_1, \dots, X_k) := \int_G (g^* \omega)(X_1, \dots, X_k) dg$$

where X_1, \dots, X_k are vector fields of M . More precisely, we have the following theorem which follows from Corollary B.13 (The complex of invariant forms) in [19].

Theorem 2.2.4 *Let a compact Lie group G act smoothly on a manifold M . For any differential k -form ω , its average $J(\omega)(X_1, \dots, X_k)$ is a G -invariant differential k -form in $\Omega_G^k(M)$, which is in the same de Rham cohomology class as ω if G is connected.*

$(\Omega_G(M), d)$ forms a subcomplex of the de Rham cochain complex because of $F_g^* d = d F_g^*$ for all $g \in G$. Let $H^k(\Omega_G(M))$ be the cohomology of this subcomplex, the following remark which will be used later, proves that the de Rham cohomology groups are just the cohomology groups $H^k(\Omega_G(M))$ if G is connected.

Remark 2.2.5 If G is compact, connected Lie Group acts on M then the action F_g induces a trivial action on $H^k(M)$ because in this case F_g and the identity map I_M are homotopic. This proves that the inclusion map $I_G : \Omega_G^k(M) \hookrightarrow \Omega^k(M)$ induces an isomorphism $H^k(\Omega_G(M)) \cong H^k(M)$. Thus, any k -differential form and its average are in the same de Rham cohomology class if G is connected.

2.3 Hodge theory

2.3.1 Hodge theorem for manifolds without boundary

Let M be a compact oriented smooth Riemannian manifold of dimension n without boundary, let $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$ be the algebra of all differential forms on M .

Based on the Riemannian structure, there is a natural L^2 -inner product on each Ω^k defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star \beta), \quad (2.1)$$

where $\star : \Omega^k \rightarrow \Omega^{n-k}$ is the Hodge star operator [1, 31]. One defines $\delta : \Omega^k \rightarrow \Omega^{k-1}$ by

$$\delta \omega = (-1)^{n(k+1)+1} (\star d \star) \omega. \quad (2.2)$$

This is seen to be the formal adjoint of d relative to the inner product (2.1): $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$. The Hodge Laplacian is defined by $\Delta = (d + \delta)^2 = d\delta + \delta d$, and a form ω is said to be *harmonic* if $\Delta\omega = 0$.

In the 1930s, Hodge [21] proved the fundamental result that each de Rham cohomology class contains a unique harmonic form, i.e. $H^k(M) \cong \ker(\Delta|_{\Omega^k})$ where $\Delta|_{\Omega^k}$ is the restriction of the Hodge Laplacian Δ to Ω^k . A more precise statement is that the decomposition of the space of differential form, for each k ,

$$\Omega^k(M) = \ker(\Delta|_{\Omega^k}) \oplus d\Omega^{k-1} \oplus \delta\Omega^{k+1}. \quad (2.3)$$

The direct sums are orthogonal with respect to the inner product (2.1), and the direct sum of the first two subspaces is equal to the subspace of all closed k -forms (that is, $\ker d_k$). A complete proof is given in [34].

Furthermore, on a manifold without boundary, any harmonic k -form $\omega \in \ker(\Delta|_{\Omega^k})$ is both closed ($d\omega = 0$) and co-closed ($\delta\omega = 0$), as

$$0 = \langle \Delta\omega, \omega \rangle = \langle d\delta\omega, \omega \rangle + \langle \delta d\omega, \omega \rangle = \langle \delta\omega, \delta\omega \rangle + \langle d\omega, d\omega \rangle = \|\delta\omega\|^2 + \|d\omega\|^2. \quad (2.4)$$

For manifolds with boundary this is no longer true, and in general we write

$$\mathcal{H}^k = \mathcal{H}^k(M) = \ker d \cap \ker \delta.$$

Thus for manifolds without boundary $\mathcal{H}^k(M) = \ker(\Delta|_{\Omega^k})$, the space of harmonic k -forms, and it follows that the Hodge star operator realizes Poincaré duality for the de Rham cohomology of M (i.e. $H^k(M) \cong H^{n-k}(M)$) [34] at the level of harmonic forms (i.e. $\mathcal{H}^k \cong \mathcal{H}^{n-k}$).

On the other hand, we conclude with the following remark which explains how the Hodge Theorem works when we have a group action on M :

Remark 2.3.1 An interesting observation which follows from the theorem of Hodge is the following. If a group G acts on M then there is an induced action on each $H^k(M)$, and if this action is trivial on $H^k(M)$, i.e. $g^*([w]) = [w]$, $\forall [w] \in H^k(M)$ (for example, if G is a compact, connected Lie group (see remark 2.2.5)) and the action is by isometries, then each harmonic form is invariant under this action because each de Rham cohomology class has a unique harmonic form.

2.3.2 Witten's deformation of Hodge theorem when $\partial M = \emptyset$

Now suppose K is a Killing vector field on M (meaning that the Lie derivative of the metric vanishes). Witten [35] defines, for each $s \in \mathbb{R}$, an operator on differential forms by

$$d_s := d + s \iota_K,$$

where ι_K is interior derivative of a form with K . This operator is no longer homogeneous in the degree of the form: if $\omega \in \Omega^k(M)$ then $d_s \omega \in \Omega^{k+1} \oplus \Omega^{k-1}$. Note then that $d_s : \Omega^\pm \rightarrow \Omega^\mp$, where Ω^\pm is the space of forms of even (+) or odd (−) degree. Let us write $\delta_s = d_s^*$ for the formal adjoint of d_s (so given by $\delta_s = \delta + s(-1)^{n(k+1)+1}(\star \iota_K \star)$ on each homogenous form of degree k). By Cartan's formula, $d_s^2 = s \mathcal{L}_K$ (the Lie derivative along sK). On the space $\Omega_s^\pm = \Omega^\pm \cap \ker \mathcal{L}_K$ of invariant forms, $d_s^2 = 0$ so one can define two cohomology groups $H_s^\pm := \ker d_s^\pm / \text{im } d_s^\mp$. Witten then defines

$$\Delta_s := (d_s + \delta_s)^2 : \Omega_s^\pm(M) \rightarrow \Omega_s^\pm(M),$$

(which he denotes H_s as it represents a Hamiltonian operator, but for us this would cause confusion), and he observes that using standard Hodge theory arguments, there is an isomorphism

$$\mathcal{H}_s^\pm := \ker \Delta_s \simeq H_s^\pm(M), \tag{2.5}$$

although no details of the proof are given nor are they to be found elsewhere in the literature (in chapter 3 we outline a proof of Witten's results using classical Hodge Theorem arguments and then we extend Witten's results to deal with the case of manifolds with boundary). Witten also shows, among other things, that for $s \neq 0$, the dimensions of \mathcal{H}_s^\pm are respectively equal to the total even and odd Betti numbers of the subset $N(K)$ of zeros of K , which in particular implies the finiteness of $\dim \mathcal{H}_s$. Atiyah and Bott [8] relate this result of Witten to their Localization Theorem in equivariant cohomology which in the next chapter, we describe and generalize to the case of manifolds with boundary.

2.3.3 Hodge-Morrey-Friedrichs theorem for manifolds with boundary

In this section, we recall the standard extension of Hodge theory to manifolds with boundary, leading to the Hodge-Morrey-Friedrichs decompositions [1, 31]. So now we let M be a compact orientable Riemannian manifold with boundary ∂M , and let $i : \partial M \hookrightarrow M$ be the inclusion. In this setting, there are two types of de Rham cohomology, the absolute cohomology $H^k(M)$ and the relative cohomology $H^k(M, \partial M)$. The first is the cohomology of the de Rham complex $(\Omega^k(M), d)$, while the second is the cohomology of the subcomplex $(\Omega_D^k(M), d)$, where $\omega \in \Omega_D^k$ if it satisfies $i^* \omega = 0$ (the D is for Dirichlet boundary condition). One also defines $\Omega_N^k(M) = \{\alpha \in \Omega^k(M) \mid i^*(\star \alpha) = 0\}$ (Neumann boundary condition). Here i^* is the pullback by the inclusion map. Clearly, the Hodge star provides an isomorphism

$$\star : \Omega_D^k \xrightarrow{\sim} \Omega_N^{n-k}.$$

Furthermore, because d and i^* commute, it follows that d preserves Dirichlet boundary conditions while δ preserves Neumann boundary conditions.

As alluded to before, because of boundary terms, the null space of Δ no longer coincides with the closed and co-closed forms. Elements of $\ker \Delta$ are called *harmonic forms*, while ω satisfying $d\omega = \delta\omega = 0$ are called *harmonic fields* (following Kodaira); it is clear that every harmonic field is a harmonic form, but the converse is false. The space of harmonic k -fields is denoted $\mathcal{H}^k(M)$ (so $\mathcal{H}^k(M) \subset \ker \Delta$). In fact, the space $\mathcal{H}^k(M)$ is infinite dimensional and so is much too big to represent the cohomology, and to recover the Hodge isomorphism one has to impose boundary conditions. One restricts $\mathcal{H}^k(M)$ into each of two finite dimensional subspaces, namely $\mathcal{H}_D^k(M)$ and $\mathcal{H}_N^k(M)$ with the obvious meanings (Dirichlet and Neumann harmonic k -fields, respectively). There are therefore two different candidates for harmonic representatives when the boundary is present.

The *Hodge-Morrey decomposition* [30] states that

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1}.$$

This decomposition is again orthogonal with respect to the inner product given above. Friedrichs [18] subsequently showed that

$$\mathcal{H}^k = \mathcal{H}_D^k \oplus \mathcal{H}_{\text{co}}^k; \quad \mathcal{H}^k = \mathcal{H}_N^k \oplus \mathcal{H}_{\text{ex}}^k$$

where $\mathcal{H}_{\text{ex}}^k$ are the exact harmonic fields and $\mathcal{H}_{\text{co}}^k = \mathcal{H}^k \cap \delta\Omega^k$ the co-exact ones. These give the orthogonal *Hodge-Morrey-Friedrichs* decompositions [31],

$$\begin{aligned} \Omega^k(M) &= d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathcal{H}_D^k \oplus \mathcal{H}_{\text{co}}^k \\ &= d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathcal{H}_N^k \oplus \mathcal{H}_{\text{ex}}^k. \end{aligned}$$

The two decompositions are related by the Hodge star operator. The consequence for cohomology is that each class in $H^k(M)$ is represented by a unique harmonic field in $\mathcal{H}_N^k(M)$ (i.e. $H^k(M) \cong \mathcal{H}_N^k(M)$), and each relative class in $H^k(M, \partial M)$ is represented by a unique harmonic field in $\mathcal{H}_D^k(M)$ (i.e. $H^k(M, \partial M) \cong \mathcal{H}_D^k(M)$). Again, the Hodge star operator acts as Poincaré-Lefschetz duality for the de Rham cohomology of M with boundary (i.e. $H^k(M) \cong H^{n-k}(M, \partial M)$) [31, 14] on the harmonic fields, sending Dirichlet fields to Neumann fields (i.e. $\mathcal{H}_N^k(M) \cong \mathcal{H}_D^{n-k}(M)$). By expanding remark 2.3.1 into the manifold with boundary case, we can again see how the Hodge-Morrey-Friedrichs Theorem works when there is a group action. Hence, if a group G acts by isometries on $(M, \partial M)$ in a manner that is trivial on the cohomology, then the harmonic fields are invariant.

Example 2.3.2 Consider $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i^2 \leq 1\}$ (the solid unit ball in \mathbb{R}^3) and $\partial M = S^2$ (the unit 2-sphere in \mathbb{R}^3). The absolute and relative de Rham cohomology of M (by using Poincaré-Lefschetz duality) are

$$H^k(M) \simeq H^{3-k}(M, \partial M) = \begin{cases} \mathbb{R} & k = 0. \\ 0 & k = 1, 2, 3, \dots \end{cases}$$

Moreover, the constructions above prove $H^0(M) \simeq \mathcal{H}_N^0(M)$ and $H^3(M, \partial M) \simeq \mathcal{H}_D^3(M)$. In fact one can see easily that

$$\mathcal{H}_N^0(M) = \{\text{constant functions}\} \quad \text{and} \quad \mathcal{H}_D^3(M) = \{c \, dx_1 \wedge dx_2 \wedge dx_3 \mid c \in \mathbb{R}\}.$$

Clearly, Hodge star \star provides the isomorphism $\mathcal{H}_N^0(M) \simeq \mathcal{H}_D^3(M)$.

2.3.4 Modified Hodge-Morrey-Friedrichs theorem

It is proven in [31] that $\mathcal{H}_D^k(M) \cap \mathcal{H}_N^k(M) = \{0\}$, so the sum $\mathcal{H}_D^k(M) + \mathcal{H}_N^k(M)$ is a direct sum but unfortunately $\mathcal{H}_D^k(M)$ and $\mathcal{H}_N^k(M)$ are not orthogonal in general and hence

cannot both appear in the same orthogonal decomposition of Hodge-Morrey-Friedrichs of $\Omega^k(M)$. Therefore, DeTurck and Gluck in [15] (cf. [32], Theorem 2.1.1 for details) modify this decomposition by observing that the best that can be done is the following five-term decomposition eq. (2.6) which is implied immediately by the Hodge-Morrey-Friedrichs decomposition of $\Omega^k(M)$.

$$\Omega^k(M) = d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus (\mathcal{H}_D^k(M) + \mathcal{H}_N^k(M)) \oplus \mathcal{H}_{\text{ex,co}}^k, \quad (2.6)$$

where $\mathcal{H}_{\text{ex,co}}^k = \mathcal{H}_{\text{ex}}^k \cap \mathcal{H}_{\text{co}}^k$ and the symbol $+$ indicates to a direct sum whereas \oplus indicates an orthogonal direct sum.

In addition, there is a long exact sequence in de Rham cohomology associated to the pair $(M, \partial M)$, [14, 20]

$$\dots \xrightarrow{i^*} H^{k-1}(\partial M) \xrightarrow{\partial^*} H^k(M, \partial M) \xrightarrow{\rho^*} H^k(M) \xrightarrow{i^*} H^k(\partial M) \xrightarrow{\partial^*} H^{k+1}(M, \partial M) \longrightarrow \dots$$

and one can use this to define two subspaces of $H^k(M)$ and $H^k(M, \partial M)$ as follows:

- the *interior* subspace $IH^k(M)$ of $H^k(M)$ is the kernel of $i^* : H^k(M) \rightarrow H^k(\partial M)$
- the *boundary* subspace $BH^k(M, \partial M)$ of $H^k(M, \partial M)$ is the image of $\partial^* : H^{k-1}(\partial M) \rightarrow H^k(M, \partial M)$, where ∂^* is derived from d .

At the level of cohomology there is no ‘natural’ definition for the boundary part of the absolute cohomology nor the interior part of the relative cohomology. However, DeTurck and Gluck [15] use the metric and harmonic representatives to provide these. Firstly the subspaces defined above are realized as

$$\begin{aligned} \mathcal{IH}_N^k &= \{\omega \in \mathcal{H}_N^k(M) \mid i^* \omega = d\theta, \text{ for some } \theta \in \Omega^{k-1}(\partial M)\} \\ \mathcal{BH}_D^k &= \mathcal{H}_D^k(M) \cap \mathcal{H}_{\text{ex}}^k \end{aligned}$$

respectively (these are denoted $\mathcal{E}_\partial \mathcal{H}_N^k(M)$ and $\mathcal{E} \mathcal{H}_D^k(M)$ respectively in [15, 32]). They then use the Hodge star operator to define the other spaces:

$$\begin{aligned} \mathcal{IH}_D^k &= \{\omega \in \mathcal{H}_D^k(M) : i^* \star \omega = d\kappa, \text{ for some } \kappa \in \Omega^{n-k-1}(\partial M)\} \\ \mathcal{BH}_N^k &= \mathcal{H}_N^k(M) \cap \mathcal{H}_{\text{co}}^k \end{aligned}$$

(denoted $c\mathcal{E}_\partial \mathcal{H}_D^k(M)$ and $c\mathcal{E} \mathcal{H}_N^k(M)$ in [15, 32]).

The main theorems of DeTurck and Gluck on this subject are

Theorem 2.3.3 (DeTurck and Gluck [15]) (i) *The boundary subspace $\mathcal{BH}_N^\pm(M)$ of $\mathcal{H}_N^\pm(M)$ is orthogonal to all of $\mathcal{H}_D^\pm(M)$ and the boundary subspace $\mathcal{BH}_D^\pm(M)$ of $\mathcal{H}_D^\pm(M)$ is orthogonal to all of $\mathcal{H}_N^\pm(M)$.*

(ii) No larger subspace of $\mathcal{H}_N^\pm(M)$ is orthogonal to all of $\mathcal{H}_D^\pm(M)$ and no larger subspace of $\mathcal{H}_D^\pm(M)$ is orthogonal to all of $\mathcal{H}_N^\pm(M)$.

Theorem 2.3.4 (DeTurck and Gluck [15]) Both \mathcal{H}_D^k and \mathcal{H}_N^k have orthogonal decompositions,

$$\begin{aligned}\mathcal{H}_N^k(M) &= \mathcal{I}\mathcal{H}_N^k \oplus \mathcal{B}\mathcal{H}_N^k \\ \mathcal{H}_D^k(M) &= \mathcal{B}\mathcal{H}_D^k \oplus \mathcal{I}\mathcal{H}_D^k.\end{aligned}$$

Furthermore, the two boundary subspaces are mutually L^2 -orthogonal inside Ω^k .

However the interior subspaces are not orthogonal, and they prove

Theorem 2.3.5 (DeTurck-Gluck [15]) The principal angles between the interior subspaces $\mathcal{I}\mathcal{H}_N^k$ and $\mathcal{I}\mathcal{H}_D^k$ are all acute.

Part of the motivation for considering these principal angles, called *Poincaré duality angles*, is that they should measure in some sense how far the Riemannian manifold M is from being closed.

In his thesis [32], Shonkwiler measures these Poincaré duality angles in interesting examples of manifolds with boundary derived from complex projective spaces and Grassmannians and shows that in these examples the angles do indeed tend to zero as the boundary shrinks to zero.

2.4 The Dirichlet-to-Neumann (DN) operator for differential forms

The classical Dirichlet-to-Neumann (DN) operator $\Lambda_{cl} : C^\infty(\partial M) \longrightarrow C^\infty(\partial M)$ is defined by $\Lambda_{cl}\theta = \frac{\partial\omega}{\partial\nu}$, where ω is the solution to the Dirichlet problem

$$\Delta\omega = 0, \quad \omega|_{\partial M} = \theta \tag{2.7}$$

and ν is the unit outer normal to the boundary. The classical DN operator arises in connection with the problem of Electrical Impedance Tomography which is also of interest in medical imaging application [22].

In the scope of inverse problems of reconstructing a manifold from the boundary measurements, the following question is of great theoretical and applied interest [11]:

To what extent are the topology and geometry of M determined by the Dirichlet-to-Neumann map?

The geometry aspect of the above question has been studied in [25] and [27]. Much effort has been made to answer the topology aspect of this question. For instance, in the case of a two-dimensional manifold M with a connected boundary, an explicit formula is obtained which expresses the Euler characteristic of M in terms of Λ_{cl} where the Euler characteristic completely determines the topology of M in this case [10]. In the three-dimensional case [9], some formulas are obtained which express the Betti numbers $\beta_1(M)$ and $\beta_2(M)$ in terms of Λ_{cl} and the vector DN map $\vec{\Lambda} : C^\infty(T(\partial M)) \longrightarrow C^\infty(T(\partial M))$ is defined on the space of vector fields in [9].

For more topological aspects, Belishev and Sharafutdinov [11] prove that the real additive de Rham cohomology of a compact, connected, oriented smooth Riemannian manifold M of dimension n with boundary is completely determined by its boundary data $(\partial M, \Lambda)$ where $\Lambda : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$ is a generalization of the classical Dirichlet-to-Neumann operator Λ_{cl} to the space of differential forms. More precisely, they define the DN- operator Λ as follows [11]: given $\theta \in \Omega^k(\partial M)$, the boundary value problem

$$\Delta \omega = 0, \quad i^* \omega = \theta, \quad i^*(\delta \omega) = 0 \quad (2.8)$$

is solvable and the operator Λ is given by the formula

$$\Lambda \theta = i^*(\star d\omega).$$

In the case of $k = 0$, Λ is equivalent to Λ_{cl} . Indeed, suppose $f \in \Omega^0(M)$ is a harmonic function which restricts to θ on the boundary. Since $f \in \Omega^0(M)$ satisfies $\delta f = 0$ then the BVP (2.8) coincides with the BVP (2.7) and the definition of Λ gives

$$\Lambda \theta = i^*(\star df) = \frac{\partial f}{\partial \mathbf{v}} \mu_\partial = \Lambda_{cl}(\theta) \mu_\partial,$$

where $\mu_\partial \in \Omega^{n-1}(\partial M)$ is the boundary volume form. So, the operator Λ differs from the classical operator Λ_{cl} by the presence of the factor μ_∂ .

Their main results are these.

Theorem 2.4.1 (Belishev-Sharafutdinov [11]) *For any $0 \leq k \leq n - 1$, the range of the operator*

$$\Lambda + (-1)^{nk+k+n} d\Lambda^{-1}d : \Omega^k(\partial M) \longrightarrow \Omega^{n-k-1}(\partial M)$$

is $i^ \mathcal{H}_N^{n-k-1}(M)$.*

But, a Neumann harmonic field λ_N is uniquely determined by its trace $i^* \lambda_N$. Hence,

$$(\Lambda + (-1)^{nk+k+1} d\Lambda^{-1}d) \Omega^{n-k-1}(\partial M) \cong H^k(M) \cong \mathcal{H}_N^k(M).$$

Using, Poincaré-Lefschetz duality, $H^k(M) \cong H^{n-k}(M, \partial M)$. So the above theorem immediately implies that the data $(\partial M, \Lambda)$ determines the absolute and relative de Rham cohomology groups.

Moreover, in section 5 of [11], they present one of the equivalent definitions of the classical *Hilbert transform* T on the unit circle S^1 which is as follows. Let $f = \varepsilon + i\omega$ be a holomorphic function on the disc $\{re^{i\theta} | 0 \leq \theta \leq 1\}$ so that ω and ε are conjugate by Cauchy-Riemann: $d\omega = \star d\varepsilon$. If $\varphi = \omega|_{S^1}$ and $\psi = \varepsilon|_{S^1}$ are the boundary trace, then $T \frac{d\varphi}{d\theta} = \frac{d\psi}{d\theta}$.

In addition, they define the generalized *Hilbert transform* T as $T = d\Lambda^{-1} : i^* \mathcal{H}^k(M) \longrightarrow i^* \mathcal{H}^{n-k}(M)$. In particular, T is defined on exact boundary forms $\mathcal{E}^k(\partial M)$ as well.

Let $\omega \in \Omega^k(M)$ and $\varepsilon \in \Omega^{n-k-2}(M)$ ($0 \leq k \leq n-2$) be two co-closed forms (i.e. $\delta\omega = \delta\varepsilon = 0$). The form ε is named the conjugate of ω if $d\omega = \star d\varepsilon$ (for details, see section 5 of [11]). Their main results about *Hilbert transform* T is the following theorem.

Theorem 2.4.2 (Belishev-Sharafutdinov [11]) *A form $\omega \in \Omega^k(M)$ satisfying $\Delta\omega = 0$ and $\delta\omega = 0$ has conjugate form if and only if the trace $\theta = i^*\omega$ satisfies*

$$(\Lambda + (-1)^{nk+k+n} d\Lambda^{-1} d)\theta = 0.$$

In the case, if ε is the conjugate form of ω and $\psi = i^\varepsilon$, then $Td\theta = d\psi$.*

In addition, they present the following theorem which gives the lower bound for the Betti numbers $\beta_k(M)$ (i.e. $\dim \mathcal{H}_D^k(M) = \dim \mathcal{H}_N^{n-k}(M) = \beta_k(M)$) of the manifold M through the DN-operator Λ .

Theorem 2.4.3 (Belishev-Sharafutdinov [11]) *The kernel $\ker \Lambda^k$ contains the space $\mathcal{E}^k(\partial M)$ of exact forms and*

$$\dim[\ker \Lambda^k / \mathcal{E}^k(\partial M)] \leq \min\{\beta_k(\partial M), \beta_k(M)\}$$

where $\beta_k(\partial M)$ and $\beta_k(M)$ are the Betti numbers, and Λ^k is the restriction of Λ to $\Omega^k(\partial M)$.

2.4.1 DN-operator Λ and cohomology ring structure

At the end of their paper [11], Belishev and Sharafutdinov posed the following topological open problem:

Can the multiplicative structure of cohomologies be recovered from our data $(\partial M, \Lambda)$?

In 2009, Shonkwiler in [33] gave a partial answer to the above question. He presents a well-defined map which is

$$(\phi, \psi) \longmapsto \Lambda((-1)^k \phi \wedge \Lambda^{-1} \psi), \quad \forall (\phi, \psi) \in i^* \mathcal{H}_N^k(M) \times i^* \star \mathcal{H}_D^l(M) \quad (2.9)$$

and then uses it to give a partial answer to that question. More precisely, by using the classical wedge product between the differential forms, he considers the mixed cup product between the absolute cohomology $H^k(M)$ and the relative cohomology $H^l(M, \partial M)$, i.e.

$$\cup : H^k(M) \times H^l(M, \partial M) \longrightarrow H^{k+l}(M, \partial M)$$

and then he restricts $H^l(M, \partial M)$ to come from the boundary subspace described above and then he presents the following theorem as a partial answer to Belishev and Sharafutdinov's question:

Theorem 2.4.4 (Shonkwiler [33]) *The boundary data $(\partial M, \Lambda)$ completely determines the mixed cup product in terms of the map (2.9) when the relative cohomology class is restricted to come from the boundary subspace.*

Chapter 3

Witten-Hodge theory and equivariant cohomology

3.1 Introduction

The immediate purpose of this chapter is to extend Witten's results which are given in chapter 2 to manifolds with boundary. In order to do this, in section 3.2 we outline a proof of Witten's results (but in terms of the setting below) using classical Hodge theory arguments and also we add more topological properties to X_M -cohomology, which in section 3.3 we extend to deal with the case of manifolds with boundary. In Section 3.4 we describe Atiyah and Bott's localization and its conclusions in the case of manifolds with boundary, and its relation to X_M -cohomology. Finally, Section 3.5 provides a few conclusions.

Henceforth we have the following setting: Recalling Witten's results and as it is well-known that the group of isometries of a Riemannian manifold (with or without boundary) is compact, so that a Killing vector field K generates an action of a torus. In this light, and because of Remark 2.3.1 (and its extension to Witten's setting), Witten's analysis can be cast in a slightly more general context.

Let G be a torus acting by isometries on M , with Lie algebra \mathfrak{g} , and denote by $\Omega_G = \Omega_G(M)$ the space of smooth G -invariant forms on M : $\omega \in \Omega_G$ if $g^*\omega = \omega$ for all $g \in G$. Note that because the action preserves the metric and the orientation it follows that, for each $g \in G$, $\star(g^*\omega) = g^*(\star\omega)$, so if $\omega \in \Omega_G$ then $\star\omega \in \Omega_G$.

Given any $X \in \mathfrak{g}$ we denote the corresponding vector field on M by X_M . Note that, if M has a boundary then the G -action necessarily restricts to an action on the boundary and X_M must therefore be tangent to the boundary. Following Witten we define $d_{X_M} = d + \iota_{X_M}$. Then d_{X_M} defines an operator $d_{X_M} : \Omega_G^\pm \rightarrow \Omega_G^\mp$, with the Lie derivative $\mathcal{L}_{X_M}\omega = d_{X_M}^2\omega = 0$. For each $X \in \mathfrak{g}$ there are therefore two corresponding cohomology groups

$H_{X_M}^\pm = \ker d_{X_M}^\pm / \text{im } d_{X_M}^\mp$, which we call X_M -cohomology groups, and a corresponding operator we call the *Witten-Hodge-Laplacian*

$$\Delta_{X_M} = (d_{X_M} + \delta_{X_M})^2 : \Omega_G^\pm \rightarrow \Omega_G^\pm.$$

According to Witten (see, subsection 2.3.4) there is an isomorphism $\mathcal{H}_{X_M}^\pm \cong H_{X_M}^\pm(M)$, where $\mathcal{H}_{X_M}^\pm$ is the space of X_M -harmonic forms, that is those forms annihilated by Δ_{X_M} . Of course, Witten's presentation is no less general than this, and is obtained by putting $X_M = sK$; the only difference is we are thinking of X as a variable element of \mathfrak{g} , while for Witten varying s only gives a 1-dimensional subspace of \mathfrak{g} (although one may change K as well). The results of this chapter are given in [2].

3.2 Witten-Hodge theory for manifolds without boundary

In this section we prove some of the results of Witten [35], providing details we will need in the next section for manifolds with boundary. We will use the notation from the introduction.

We have an oriented boundaryless compact Riemannian manifold M with an action of a torus G which acts by isometries on M , and we fix an element $X \in \mathfrak{g}$. The associated vector field on M is X_M , and using this one defines Witten's inhomogeneous coboundary operator $d_{X_M} : \Omega_G^\pm \rightarrow \Omega_G^\mp$, $d_{X_M} \omega = d\omega + \iota_{X_M} \omega$, and the corresponding operator (cf. eq. (2.2))

$$\delta_{X_M} = (-1)^{n(k+1)+1} \star d_{X_M} \star = \delta + (-1)^{n(k+1)+1} \star \iota_{X_M} \star$$

(which is the operator adjoint to d_{X_M} by Proposition 3.2.2 below). The resulting *Witten-Hodge-Laplacian* is $\Delta_{X_M} : \Omega_G^\pm \rightarrow \Omega_G^\pm$ defined by $\Delta_{X_M} = (d_{X_M} + \delta_{X_M})^2 = d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M}$. We write the space of X_M -harmonic fields

$$\mathcal{H}_{X_M} = \ker d_{X_M} \cap \ker \delta_{X_M},$$

which (for manifolds without boundary) satisfies $\mathcal{H}_{X_M} = \ker \Delta_{X_M}$. The last equality follows for the same reason as for ordinary Hodge theory, namely the argument in (2.4), with Δ replaced by Δ_{X_M} etc.

The Sobolev space $W^{s,p} \Omega(M)$ is the vector space of differential forms equipped with a norm that is a sum of L^p -norms of the differential forms itself as well as its derivatives up to a given order $s \in \mathbb{N}$. The space $W^{0,p} \Omega(M)$ and $W^{s,2} \Omega(M)$ are also denoted by $L^p \Omega(M)$ and $H^s \Omega(M)$, respectively. In fact, the L^2 -norm is already given in chapter 2 (eq. (2.1)).

Moreover, Schwarz in [31] (Proposition 2.1.1) proves that Stokes' theorem is still true for all differential forms of Sobolev class $W^{1,1} \Omega(M)$. His argument uses the fact that

for any $\omega \in W^{1,1}\Omega(M)$ there exists an approximation sequence of smooth forms which is convergent to ω in the $W^{1,1}$ -norm and then he uses the classical Stokes' theorem for these smooth forms.

In this light, we can recast Stokes' theorem in terms of the operators d_{X_M} by defining $\int_M \omega = 0$ if $\omega \in W^{1,1}\Omega^k(M)$ with $k \neq n$. For any form $\omega \in W^{1,1}\Omega(M)$ one has $\int_M \iota_{X_M} \omega = 0$ as $\iota_{X_M} \omega$ has no term of degree n , and the following version of Stokes' theorem follows from the ordinary Stokes' theorem of Sobolev class $W^{1,1}\Omega(M)$. For future use, we allow M to have a boundary.

Theorem 3.2.1 (Stokes' theorem for d_{X_M}) *Let M be a compact manifold with boundary ∂M (possibly empty) for all differential forms $\omega \in W^{1,1}\Omega_G(M)$ then*

$$\int_M d_{X_M} \omega = \int_{\partial M} i^* \omega,$$

where $i : \partial M \hookrightarrow M$ is the inclusion, and where the right-hand-side is taken to be zero if M has no boundary.

Using this, we can present the following Green's formula in terms of the operators d_{X_M} and δ_{X_M} .

Proposition 3.2.2 (Green's formula for d_{X_M} and δ_{X_M}) *Let $\alpha, \beta \in H^1\Omega_G$ be invariant differential forms on the compact manifold M with boundary ∂M (possibly empty), then*

$$\langle d_{X_M} \alpha, \beta \rangle = \langle \alpha, \delta_{X_M} \beta \rangle + \int_{\partial M} i^* (\alpha \wedge \star \beta), \quad (3.1)$$

where as always $i : \partial M \hookrightarrow M$ is the inclusion.

PROOF: For technical reasons we write α and β as:

$$\alpha = \alpha^+ + \alpha^-, \quad \beta = \beta^+ + \beta^- \in H^1\Omega_G$$

then

$$\begin{aligned} d_{X_M}(\alpha \wedge (\star \beta)) &= d_{X_M}(\alpha^+ + \alpha^-) \wedge \star(\beta^+ + \beta^-) + \\ &\quad \alpha^+ \wedge d_{X_M}(\star(\beta^+ + \beta^-)) - \alpha^- \wedge d_{X_M}(\star(\beta^+ + \beta^-)). \end{aligned}$$

Since, $\alpha, \beta \in H^1\Omega_G$ then the term $\alpha \wedge \star \beta$ belongs in Sobolev class $W^{1,1}\Omega_G(M)$, [31]. Moreover, all the terms of right hand side above belong in $L^1\Omega_G(M)$. Hence, we can apply theorem 3.2.1. Now, integrating both sides over M and using $\star \delta_{X_M} = \pm d_{X_M} \star$ on $H^1\Omega_G^\pm(M)$ and then by using the linearity and orthogonality of $H^1\Omega_G(M) = H^1\Omega_G^+(M) \oplus H^1\Omega_G^-(M)$ we obtain eq. (3.1). \square

Returning now to the case of a manifold without boundary, we obtain the following.

Theorem 3.2.3 *The Witten-Hodge-Laplacian Δ_{X_M} is a self-adjoint elliptic operator.*

PROOF: The self-adjoint property follows from the same argument as for the classical Hodge Laplacian, namely that δ_{X_M} is the adjoint of d_{X_M} . For the ellipticity, we can expand Δ_{X_M} from its definition as,

$$\Delta_{X_M} = \Delta + (-1)^{n(k+1)+1} (d \star \iota_{X_M} \star + \iota_{X_M} \star \iota_{X_M} \star) + (-1)^{nk+1} (\star \iota_{X_M} \star d + \star \iota_{X_M} \star \iota_{X_M} \star) + \iota_{X_M} \delta + \delta \iota_{X_M}. \quad (3.2)$$

It follows that Δ_{X_M} and Δ have the same principal symbol (indeed $\Delta_{X_M} - \Delta$ is a first order differential operator). Since Δ is elliptic [23, 31], it follows that so too is Δ_{X_M} . \square

Every elliptic operator on a compact manifold is Fredholm [23], in the sense that for each $s \in \mathbb{N}$,

$$\Delta_{X_M} : H^s \Omega_G^\pm \rightarrow H^{s-2} \Omega_G^\pm$$

is a Fredholm operator, so has finite dimensional kernel and cokernel, and closed range.

The regularity and Fredholm properties of elliptic operators [23, 31] imply the following.

Corollary 3.2.4 *The set of X_M -harmonic (even/odd) forms $\mathcal{H}_{X_M}^\pm$ is finite dimensional and consists of smooth C^∞ forms.*

The following result is the analogue of the Hodge decomposition theorem, and is a standard consequence of the fact that Δ_{X_M} is self-adjoint.

Theorem 3.2.5 *The following is an orthogonal decomposition*

$$\Omega_G^\pm = \mathcal{H}_{X_M}^\pm \oplus d_{X_M} \Omega_G^\mp \oplus \delta_{X_M} \Omega_G^\mp,$$

and in terms of Sobolev spaces ($\forall s \in \mathbb{N}$)

$$H^s \Omega_G^\pm = \mathcal{H}_{X_M}^\pm \oplus d_{X_M} H^{s+1} \Omega_G^\mp \oplus \delta_{X_M} H^{s+1} \Omega_G^\mp.$$

The orthogonality is with respect to the L^2 -inner product, given in (2.1).

PROOF: Since, Δ_{X_M} is elliptic and self adjoint operator then the decomposition above follows immediately from Elliptic Splitting Theorem (cf. Theorem 7.5.6 [1]). \square

As consequences for our decomposition above to the invariant differential forms Ω_G^\pm , we have the following topological properties for X_M -cohomology.

Proposition 3.2.6 *Every X_M -cohomology class has a unique X_M -harmonic form (=field) representative (i.e $H_{X_M}^\pm(M) \cong \mathcal{H}_{X_M}^\pm$).*

PROOF: We define a map $P : \mathcal{H}_{X_M}^\pm \longrightarrow H_{X_M}^\pm(M)$ by $P(\omega) = [\omega]_{X_M}$ for $\omega \in \mathcal{H}_{X_M}^\pm$, where we denote $[\]_{X_M}$ for X_M -cohomology classes. Clearly, P is well-defined and $[\omega]_{X_M} \in H_{X_M}^\pm(M)$ for all $\omega \in \mathcal{H}_{X_M}^\pm$.

Now, we need first to prove P is injective. Suppose $\omega \in \ker P$ then $P(\omega) = [\omega]_{X_M} = 0$. But $[\omega]_{X_M} = 0$ means that ω is an X_M -exact form; $\omega = d_{X_M} \alpha$. But $\delta_{X_M} \omega = 0$ and ω is orthogonal to $d_{X_M} \alpha$. Hence, the orthogonality of theorem 3.2.5 asserts that ω is orthogonal to itself, so $\omega = 0$. Thus, $\ker P = \{0\}$ which proves that P is injective.

Next, let $[\omega]_{X_M} \in H_{X_M}^\pm(M)$, then Theorem 3.2.5 shows that ω can be decomposed as $\omega = \lambda + d_{X_M} \alpha + \delta_{X_M} \beta$ where $d_{X_M} \omega = d_{X_M} \delta_{X_M} \beta = 0$. So, $0 = \langle \beta, d_{X_M} \delta_{X_M} \beta \rangle = \langle \delta_{X_M} \beta, \delta_{X_M} \beta \rangle$, so $\delta_{X_M} \beta = 0$. Thus $[\omega]_{X_M} = [\lambda]_{X_M}$ where $\lambda \in \mathcal{H}_{X_M}^\pm$. So, P is surjective. Hence, P is bijection.

Now, suppose we have two X_M -harmonic forms λ_1 and λ_2 differ by an X_M -exact form $d_{X_M} \mu$ then we get

$$0 = (\lambda_1 - \lambda_2) + d_{X_M} \mu.$$

Again the orthogonality of theorem 3.2.5 and the injectivity of P prove that $d_{X_M} \mu = 0$ and thus $\lambda_1 = \lambda_2$. Hence, there is a unique X_M -harmonic form in each X_M -cohomology class. \square

Corollary 3.2.7 *The X_M -cohomology groups $H_{X_M}^\pm(M)$ for a compact, oriented differentiable manifold M with an action of a torus G are all finite dimensional.*

PROOF: Any differentiable manifold can be equipped with a Riemannian metric and by averaging, there exists a G -invariant Riemannian metric [19]. The corollary then follows immediately from proposition 3.2.6 and corollary 3.2.4. \square

We infer the following form of Poincaré duality but in terms of X_M -cohomology. Here and elsewhere we write $n - \pm$ for the parity (modulo 2) resulting from subtracting an even/odd number from n .

Theorem 3.2.8 (Poincaré duality for $H_{X_M}^\pm$) *Let M be a compact, oriented smooth Riemannian manifold of dimension n and with an action of a torus G . The bilinear function*

$$(\ , \) : H_{X_M}^\pm \times H_{X_M}^{n-\pm} \longrightarrow \mathbb{R}$$

defined by setting

$$([\alpha]_{X_M}, [\beta]_{X_M}) = \int_M \alpha \wedge \beta \quad (3.3)$$

is a well-defined, non-singular pairing and consequently gives isomorphisms of $H_{X_M}^{n-\pm}$ with the dual space of $H_{X_M}^{\pm}$. i.e.

$$H_{X_M}^{n-\pm} \cong (H_{X_M}^{\pm})^*.$$

PROOF: It is easy to prove that the bilinear map (3.3) is well-defined while the non-singularity follows from Proposition 3.2.6 as follows: given a non-zero X_M -cohomology class $[\omega]_{X_M} \in H_{X_M}^{\pm}$, we must find a non-zero X_M -cohomology class $[\xi]_{X_M} \in H_{X_M}^{n-\pm}$ such that $([\omega]_{X_M}, [\xi]_{X_M}) \neq 0$. According to Proposition 3.2.6, that ω is the X_M -harmonic form representative of the non zero X_M -cohomology class $[\omega]_{X_M}$, it follows that ω is not identically zero. Applying the fact that $\star\Delta_{X_M} = \Delta_{X_M}\star$, it gives that $\star\omega$ is also X_M -harmonic form and represents an X_M -cohomology class $[\star\omega]_{X_M} \in H_{X_M}^{n-\pm}$. Thus the pairing (3.3)

$$([\omega]_{X_M}, [\star\omega]_{X_M}) = \int_M \omega \wedge \star\omega = \|\omega\|^2 \neq 0$$

is non-singular while the isomorphisms $H_{X_M}^{n-\pm} \cong (H_{X_M}^{\pm})^*$ follow from the finite dimensionality of X_M -cohomology (cf. Corollary 3.2.4 and Proposition 3.2.6) and the non-singularity above. \square

Remark 3.2.9 Theorem 3.2.8 shows that the Hodge star operator provides the isomorphism $H_{X_M}^{n-\pm} \cong (H_{X_M}^{\pm})^*$. In addition, a finite dimensional vector space has the same dimension as its dual space. Thus, $H_{X_M}^{n-\pm} \cong H_{X_M}^{\pm}$.

Let $N(X_M)$ be the set of zeros of X_M , and $j : N(X_M) \hookrightarrow M$ the inclusion. As observed by Witten, on $N(X_M)$ one has $X_M = 0$, so that $j^*d_{X_M}\omega = d(j^*\omega)$, and in particular if ω is X_M -closed then its pullback to $N(X_M)$ is closed in the usual (de Rham) sense. And X_M -exact forms pull back to exact forms. Consequently, pullback defines a natural map $H_{X_M}^{\pm}(M) \rightarrow H^{\pm}(N(X_M))$, where $H^{\pm}(N(X_M))$ is the direct sum of the even/odd de Rham cohomology groups of $N(X_M)$.

Theorem 3.2.10 (Witten [35]) *The pullback to $N(X_M)$ induces an isomorphism between the X_M -cohomology groups $H_{X_M}^{\pm}(M)$ and the cohomology groups $H^{\pm}(N(X_M))$.*

Witten gave a fairly explicit proof of this theorem by extending closed forms on $N(X_M)$ to X_M -closed forms on M . Atiyah and Bott [8] give a proof using their localization theorem in equivariant cohomology which we discuss, and adapt to the case of manifolds with boundary, in Section 3.4.

Example 3.2.11 Consider $M = S^2$ (the unit 2-sphere in \mathbb{R}^3), and use cylindrical polar coordinates $z \in [-1, 1]$ and $\phi \in [0, 2\pi]$. Let the group $G = S^1$ act on S^2 by rotations about the z -axis, with infinitesimal generator $\partial/\partial\phi$. Let $X \in \mathfrak{g}$, so $X_M = s\partial/\partial\phi$, for some $s \in \mathbb{R}$. Invariant even and odd forms are of the form

$$\omega_+ = f_0(z) + f_2(z) d\phi \wedge dz \in \Omega_G^+, \quad \omega_- = f_1(z) dz + g_1(z) d\phi \in \Omega_G^-.$$

In order that ω_- is smooth, g_1 must vanish at the poles $z = \pm 1$. The invariant volume form is $d\phi \wedge dz$, with total volume 4π , and the metric is $(1 - z^2)^{-1} dz^2 + (1 - z^2) d\phi^2$. Consequently, $\star(dz) = -(1 - z^2) d\phi$ and $\star(d\phi) = (1 - z^2)^{-1} dz$, so

$$d_{X_M} \omega_+ = (f'_0(z) + s f_2(z)) dz, \quad \delta_{X_M} \omega_+ = -(1 - z^2)^2 (f'_2(z) + s f_0(z)) d\phi.$$

One finds ω_+ is X_M -harmonic if and only if

$$\omega_+ = A e^{sz} (1 - d\phi \wedge dz) + B e^{-sz} (1 + d\phi \wedge dz), \quad (3.4)$$

for $A, B \in \mathbb{R}$, and one finds that there are no non-zero odd X_M -harmonic forms. Furthermore, the pullback of ω_+ to $N(X_M)$ (which here is the two poles at $z = \pm 1$) is $A(e^s, e^{-s}) + B(e^{-s}, e^s)$ which for $s \neq 0$ are linearly independent, as predicted by Theorem 3.2.10.

Remark 3.2.12 Extending remark 2.3.1, suppose X generates the torus $G(X)$, and G is a larger torus containing $G(X)$ and acting on M by isometries. Then the action of G preserves X_M because G is an abelian Lie group. It follows that G acts trivially on the de Rham cohomology of $N(X_M)$, and hence on the X_M -cohomology of M , and consequently on the space of X_M -harmonic forms. Now, replacing d by d_{X_M} and $\Omega^k(M)$ by $\Omega_{G(X)}^\pm(M)$ in remark 2.2.5, this proves that $H_{X_M}^\pm(M) \cong H_{X_M, G(X)}^\pm(M)$ and more concretely, Proposition 3.2.6 implies that $\mathcal{H}_{X_M}^\pm = \mathcal{H}_{X_M, G(X)}^\pm$ where $H_{X_M, G(X)}^\pm(M)$ and $\mathcal{H}_{X_M, G(X)}^\pm$ are defined using $G(X)$ -invariant forms. There is therefore no loss in considering just forms invariant under the action of the larger torus in that the X_M -cohomology, or the space of X_M -harmonic forms, is independent of the choice of torus, provided it contains $G(X)$.

3.3 Witten-Hodge theory for manifolds with boundary

In this section we adapt the results and methods of Hodge theory for manifolds with boundary to study the X_M -cohomology and the space of X_M -harmonic forms and fields for manifolds with boundary. As for ordinary (singular) cohomology, there are both absolute and relative X_M -cohomology groups. So from now on our manifold will be with boundary and with torus action which acts by isometry on this manifold unless otherwise indicated, and

as before $i : \partial M \hookrightarrow M$ denotes the inclusion of the boundary.

3.3.1 The difficulties if the boundary is present

Firstly, d_{X_M} and δ_{X_M} are no longer adjoint because the boundary terms arise when we integrate by parts and then Δ_{X_M} will not be self-adjoint. In addition, the space of all harmonic fields is infinite dimensional and there is no reason to expect the X_M -harmonic fields $\mathcal{H}_{X_M}(M)$ to be any different. To overcome these difficulties, at the beginning we follow the method which is used to solve this problem in the classical case, i.e. with d and δ [1, 31], and impose certain boundary conditions on the invariant forms $\Omega_G(M)$. Hence we make the following definitions.

Definition 3.3.1 (1) We define the following two sets of smooth invariant forms on the manifold M with boundary and with torus action

$$\Omega_{G,D} = \Omega_G \cap \Omega_D = \{\omega \in \Omega_G \mid i^* \omega = 0\} \quad (3.5)$$

$$\Omega_{G,N} = \Omega_G \cap \Omega_N = \{\omega \in \Omega_G \mid i^*(\star \omega) = 0\} \quad (3.6)$$

and the spaces $H^s \Omega_{G,D}$ and $H^s \Omega_{G,N}$ are the corresponding closures with respect to suitable Sobolev norms, for $s > \frac{1}{2}$. This can be refined to take into account the parity of the forms, so defining $\Omega_{G,D}^\pm$ etc. Since $\omega \in \Omega^k$ implies $\star \omega \in \Omega^{n-k}$ we write that for $\omega \in \Omega_G^\pm$ we have $\star \omega \in \Omega_G^{n-\pm}$.

(2) We define the two subspaces of $\mathcal{H}_{X_M}(M)$

$$\mathcal{H}_{X_M,D}(M) = \{\omega \in H^1 \Omega_{G,D} \mid d_{X_M} \omega = 0, \delta_{X_M} \omega = 0\} \quad (3.7)$$

$$\mathcal{H}_{X_M,N}(M) = \{\omega \in H^1 \Omega_{G,N} \mid d_{X_M} \omega = 0, \delta_{X_M} \omega = 0\} \quad (3.8)$$

which we call Dirichlet and Neumann X_M -harmonic fields, respectively. We will show below that these forms are smooth. Clearly, the Hodge star operator \star defines an isomorphism $\mathcal{H}_{X_M,D}(M) \cong \mathcal{H}_{X_M,N}(M)$. Again, these can be refined to take the parity into account, defining $\mathcal{H}_{X_M,D}^\pm(M)$ etc.

As for ordinary Hodge theory, on a manifold with boundary one has to distinguish between X_M -harmonic *forms* (i.e. $\ker \Delta_{X_M}$) and X_M -harmonic *fields* (i.e. $\mathcal{H}_{X_M}(M)$) because they are not equal: one has $\mathcal{H}_{X_M}(M) \subseteq \ker \Delta_{X_M}$ but not conversely. The following proposition shows the conditions on ω to be fulfilled in order to ensure $\omega \in \ker \Delta_{X_M} \implies \omega \in \mathcal{H}_{X_M}(M)$ when $\partial M \neq \emptyset$.

Proposition 3.3.2 *If $\omega \in \Omega_G(M)$ is an X_M -harmonic form (i.e. $\Delta_{X_M} \omega = 0$) and in addition*

any one of the following four pairs of boundary conditions is satisfied then $\omega \in \mathcal{H}_{X_M}(M)$.

- (1) $i^*\omega = 0, i^*(\star\omega) = 0;$ (2) $i^*\omega = 0, i^*(\delta_{X_M}\omega) = 0;$
 (3) $i^*(\star\omega) = 0, i^*(\star d_{X_M}\omega) = 0;$ (4) $i^*(\delta_{X_M}\omega) = 0, i^*(\star d_{X_M}\omega) = 0.$

PROOF: Because $\Delta_{X_M}\omega = 0$, one has $\langle \Delta_{X_M}\omega, \omega \rangle = 0$. Now applying Proposition 3.2.2 to this, so we get that:

$$0 = \langle \Delta_{X_M}\omega, \omega \rangle = \langle d_{X_M}\omega, d_{X_M}\omega \rangle + \langle \delta_{X_M}\omega, \delta_{X_M}\omega \rangle - \int_{\partial M} i^*\omega \wedge i^*\star d_{X_M}\omega + \int_{\partial M} i^*\delta_{X_M}\omega \wedge i^*\star\omega.$$

Using any of these conditions (1)–(4) ensures that the integrals above are zero and then ω is an X_M -harmonic field. \square

Remark 3.3.3 Using Theorem 2.2.4, an averaging argument shows that $H^1\Omega_{G,D}$ and $H^1\Omega_{G,N}$ are dense in $L^2\Omega_G$, because the corresponding statements hold for the spaces of all (not only invariant) forms [31].

3.3.2 Elliptic boundary value problem

The essential ingredients that Schwarz [31] needs to prove the classical *Hodge-Morrey-Friedrichs* decomposition are his Theorem 2.1.5 and Gaffney’s inequality. However, these results do not appear to extend to the context of d_{X_M} and δ_{X_M} . Therefore, we use a different approach to overcome this problem, based on the ellipticity of a certain boundary value problem (BVP), namely (3.9) below. This theorem represents the keystone to extending the Hodge-Morrey and Friedrichs decomposition theorems to the present setting and then to extending Witten’s results to manifolds with boundary.

Consider the BVP

$$\begin{cases} \Delta_{X_M}\omega = \eta & \text{on } M \\ i^*\omega = 0 & \text{on } \partial M \\ i^*(\delta_{X_M}\omega) = 0 & \text{on } \partial M. \end{cases} \quad (3.9)$$

where $\eta \in \Omega_G(M)$.

Remark 3.3.4 It is well-known that the ellipticity of the BVP on compact manifolds is often defined by the Lopatinskiĭ-Šapiro condition. Moreover, the most characteristic properties of an elliptic operator on compact manifolds are the regularity of the solutions of the corresponding equations and the Fredholm property of elliptic operators. In this thesis, we do not give description to Lopatinskiĭ-Šapiro condition because we are not interested in its own right and we therefore refer to the well established literature on the theory of elliptic operators, in particular to the book of Hörmander [23] for those who are interested

in this condition. So, in the proof of theorem 3.3.5, we will use the ellipticity in the sense of Lopatinskiĭ-Šapiro condition as a tool to obtain information about the regularity and Fredholm property of the certain BVP (3.9).

Theorem 3.3.5

1. The BVP (3.9) is elliptic in the sense of Lopatinskiĭ-Šapiro, where $\Delta_{X_M} : \Omega_G(M) \longrightarrow \Omega_G(M)$.
2. The BVP (3.9) is Fredholm of index 0.
3. All $\omega \in \mathcal{H}_{X_M,D} \cup \mathcal{H}_{X_M,N}$ are smooth.

PROOF:

(1) Firstly, as in the proof of Theorem 3.2.3, we can see that Δ and Δ_{X_M} have the same principal symbol. Similarly, expanding the second boundary condition gives

$$\delta_{X_M} = \delta + (-1)^{n(k+1)+1} \star \iota_{X_M} \star$$

so δ_{X_M} and δ have the same first-order part. Hence our BVP (3.9) has the same principal symbol as the following BVP

$$\left\{ \begin{array}{ll} \Delta \varepsilon = \xi & \text{on } M \\ i^* \varepsilon = 0 & \text{on } \partial M \\ i^*(\delta \varepsilon) = 0 & \text{on } \partial M \end{array} \right. \quad (3.10)$$

for $\varepsilon, \xi \in \Omega(M)$, because the principal symbol does not change when terms of lower order are added to the operator. However the BVP (3.10) is elliptic in the sense of Lopatinskiĭ-Šapiro conditions [23, 31], and thus so is (3.9).

(2) From part (1), since the BVP (3.9) is elliptic, by using Theorem 1.6.2 in [31] or Theorem 20.1.2 in [23] we conclude that the BVP (3.9) is a Fredholm operator and the regularity theorem holds. In addition, we observe that the only differences between BVP (3.10) and the BVP (3.9) are all lower order operators and it is proved in [31] that the index of BVP (3.10) is zero but Theorem 20.1.8 in [23] asserts generally that if the difference between two BVP's are just lower order operators then they must have the same index. Hence, the index of the BVP (3.9) must be zero.

(3) Let $\omega \in \mathcal{H}_{X_M,D} \cup \mathcal{H}_{X_M,N}$. If $\omega \in \mathcal{H}_{X_M,D}$ then it satisfies the BVP (3.9) with $\eta = 0$, so by the regularity properties of elliptic BVPs, the smoothness of ω follows. If on the other hand $\omega \in \mathcal{H}_{X_M,N}$ then $\star \omega \in \mathcal{H}_{X_M,D}$ which is therefore smooth and consequently $\omega = \pm \star(\star \omega)$ is smooth as well. \square

We consider the resulting operator obtained by restricting Δ_{X_M} to the subspace of smooth invariant forms satisfying the boundary conditions

$$\overline{\Omega}_G(M) = \{\omega \in \Omega_G(M) \mid i^* \omega = 0, i^*(\delta_{X_M} \omega) = 0\} \quad (3.11)$$

Since the trace map i^* is well-defined on $H^s \Omega_G$ for $s > 1/2$ it follows that it makes sense to consider $H^2 \overline{\Omega}_G(M)$, which is a closed subspace of $H^2 \Omega_G(M)$ and hence a Hilbert space. For simplicity, we rewrite the BVP (3.9) as follows: consider the restriction/extension of Δ_{X_M} to this space:

$$A = \Delta_{X_M}|_{H^2 \overline{\Omega}_G(M)} : H^2 \overline{\Omega}_G(M) \longrightarrow L^2 \Omega_G(M).$$

and consider the BVP,

$$A\omega = \eta \quad (3.12)$$

for $\omega \in H^2 \overline{\Omega}_G(M)$ and $\eta \in L^2 \Omega_G(M)$ instead of BVP (3.9) which are in fact compatible. In addition, from Theorem 3.3.5 we deduce that A is an elliptic and Fredholm operator and

$$\text{index}(A) = \dim(\ker A) - \dim(\ker A^*) = 0 \quad (3.13)$$

where A^* is the adjoint operator of A .

From Green's formula (Proposition 3.2.2) we deduce the following property.

Lemma 3.3.6 *A is L^2 -self-adjoint on $H^2 \overline{\Omega}_G(M)$, meaning that for all $\alpha, \beta \in H^2 \overline{\Omega}_G(M)$ we have*

$$\langle A\alpha, \beta \rangle = \langle \alpha, A\beta \rangle,$$

where $\langle -, - \rangle$ is the L^2 -pairing.

PROOF: Clearly, because for all $\alpha, \beta \in H^2 \overline{\Omega}_G(M)$ we have that α and β satisfy No.(2) of proposition 3.3.2. Now, using this fact together with proposition 3.2.2, we can prove that $\langle A\alpha, \beta \rangle = \langle \alpha, A\beta \rangle$. \square

Theorem 3.3.7 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action (by isometries) of a torus G . The space $\mathcal{H}_{X_M, D}(M)$ is finite dimensional and*

$$L^2 \Omega_G(M) = \mathcal{H}_{X_M, D}(M) \oplus \mathcal{H}_{X_M, D}(M)^\perp. \quad (3.14)$$

PROOF: We begin by showing that $\ker A = \mathcal{H}_{X_M, D}(M)$. It is clear that $\mathcal{H}_{X_M, D}(M) \subseteq \ker A$, so we need only prove that $\ker A \subseteq \mathcal{H}_{X_M, D}(M)$.

Let $\omega \in \ker A$. Then ω satisfies the BVP (3.9). Therefore, by condition (2) of Proposition 3.3.2, it follows that $\omega \in \mathcal{H}_{X_M, D}(M)$, as required.

Now, $\ker A = \mathcal{H}_{X_M, D}(M)$ but $\dim \ker A$ is finite, it follows that so too is $\dim \mathcal{H}_{X_M, D}(M)$. This implies that $\mathcal{H}_{X_M, D}(M)$ is a closed subspace of the Hilbert space $L^2\Omega_G(M)$, hence eq. (3.14) holds. \square

Theorem 3.3.8

$$\text{Range}(A) = \mathcal{H}_{X_M, D}(M)^\perp \quad (3.15)$$

where \perp denotes the orthogonal complement in $L^2\Omega_G(M)$.

PROOF: Firstly, we should observe that eq. (3.13) asserts that $\ker A \cong \ker A^*$ but Theorem 3.3.7 shows that $\ker A = \mathcal{H}_{X_M, D}(M)$, thus

$$\ker A^* \cong \mathcal{H}_{X_M, D}(M) \quad (3.16)$$

Since $\text{Range}(A)$ is closed in $L^2\Omega_G(M)$ because A is Fredholm operator, it follows from the closed range theorem in Hilbert spaces that

$$\text{Range}(A) = (\ker A^*)^\perp \iff \text{Range}(A)^\perp = \ker A^*. \quad (3.17)$$

Hence, we just need to prove that $\ker A^* = \mathcal{H}_{X_M, D}(M)$, and to show that we need first to prove

$$\text{Range}(A) \subseteq \mathcal{H}_{X_M, D}(M)^\perp. \quad (3.18)$$

So, if $\alpha \in \overline{H^2\Omega_G(M)}$ and $\beta \in \mathcal{H}_{X_M, D}(M)$ then applying Lemma 3.3.6 gives

$$\langle A\alpha, \beta \rangle = 0$$

hence, eq. (3.18) holds. Moreover, equations (3.17) and (3.18) and the closedness of $\mathcal{H}_{X_M, D}(M)$ imply

$$\mathcal{H}_{X_M, D}(M) \subseteq \ker A^* \quad (3.19)$$

but eq. (3.16) and eq. (3.19) force $\ker A^* = \mathcal{H}_{X_M, D}(M)$. Hence, $\text{Range}(A) = \mathcal{H}_{X_M, D}(M)^\perp$. \square

Following [31], we denote the L^2 -orthogonal complement of $\mathcal{H}_{X_M, D}(M)$ in the space $H^2\Omega_{G, D}$ by

$$\mathcal{H}_{X_M, D}(M)^\ominus = H^2\Omega_{G, D} \cap \mathcal{H}_{X_M, D}(M)^\perp \quad (3.20)$$

(although in [31] it denotes H^1 -forms rather than H^2).

Proposition 3.3.9 *For each $\eta \in \mathcal{H}_{X_M, D}(M)^\perp$ there is a unique differential form $\omega \in \mathcal{H}_{X_M, D}(M)^\oplus$ satisfying the BVP (3.9).*

PROOF: Let $\eta \in \mathcal{H}_{X_M, D}(M)^\perp$. Because of Theorem (3.3.8) there is a differential form $\gamma \in H^2\overline{\Omega}_G(M)$ such that γ satisfies the BVP (3.9). Since $\gamma \in H^2\overline{\Omega}_G(M) \subseteq L^2\Omega_G(M)$ then there are unique differential forms $\alpha \in \mathcal{H}_{X_M, D}(M)$ and $\omega \in \mathcal{H}_{X_M, D}(M)^\perp$ such that $\gamma = \alpha + \omega$ because of eq. (3.14).

Since γ satisfies the BVP (3.9) it follows that ω satisfies the BVP (3.9) as well because $\alpha \in \mathcal{H}_{X_M, D}(M) = \ker(\Delta_{X_M}|_{H^2\overline{\Omega}_G(M)})$. Since $\omega = \gamma - \alpha$, it follows that $\omega \in H^2\Omega_{G, D}$, hence $\omega \in \mathcal{H}_{X_M, D}(M)^\oplus$ and it is unique \square

Remark 3.3.10

(1) ω satisfying the BVP (3.9) in Proposition 3.3.9 can be recast to the condition

$$\langle d_{X_M}\omega, d_{X_M}\xi \rangle + \langle \delta_{X_M}\omega, \delta_{X_M}\xi \rangle = \langle \eta, \xi \rangle, \quad \forall \xi \in H^1\Omega_{G, D} \quad (3.21)$$

(2) All the results above can be recovered but in terms of $\mathcal{H}_{X_M, N}(M)$ because the Hodge star operator \star defines an isomorphism $L^2\Omega_G \cong L^2\Omega_G$ which restricts to $\mathcal{H}_{X_M, D}(M) \cong \mathcal{H}_{X_M, N}(M)$. In addition, \star takes orthogonal direct sum to orthogonal direct sum because it is an L^2 -isometry of Ω_G .

3.3.3 Decomposition theorems

The results above provide the basic ingredients needed to extend the Hodge-Morrey and Freidrichs decompositions arising for Hodge theory on manifolds with boundary, to the present setting with d_{X_M} and δ_{X_M} . Depending on these results, the proofs in this subsection rely heavily on the analogues of the corresponding statements for the usual Laplacian Δ on a manifold with boundary, as described in the book of Schwarz [31].

Definition 3.3.11 Define the following two sets of X_M -exact and X_M -coexact forms on the manifold M with boundary and with an action of the torus G :

$$\mathcal{E}_{X_M}(M) = \{d_{X_M}\alpha \mid \alpha \in H^1\Omega_{G, D}\} \subseteq L^2\Omega_G(M), \quad (3.22)$$

$$\mathcal{C}_{X_M}(M) = \{\delta_{X_M}\beta \mid \beta \in H^1\Omega_{G, N}\} \subseteq L^2\Omega_G(M). \quad (3.23)$$

Clearly, $\mathcal{E}_{X_M}(M) \perp \mathcal{C}_{X_M}(M)$ because of Proposition 3.2.2. We denote by $L^2\mathcal{H}_{X_M}(M) = \overline{\mathcal{H}_{X_M}(M)}$ the L^2 -closure of the space $\mathcal{H}_{X_M}(M)$.

Proposition 3.3.12 (Algebraic decomposition and L^2 -closedness)

(a) Each $\omega \in L^2\Omega_G(M)$ can be split uniquely into

$$\omega = d_{X_M}\alpha_\omega + \delta_{X_M}\beta_\omega + \kappa_\omega$$

where $d_{X_M}\alpha_\omega \in \mathcal{E}_{X_M}(M)$, $\delta_{X_M}\beta_\omega \in \mathcal{C}_{X_M}(M)$ and $\kappa_\omega \in (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$.

(b) The spaces $\mathcal{E}_{X_M}(M)$ and $\mathcal{C}_{X_M}(M)$ are closed subspaces of $L^2\Omega_G(M)$.

(a) and (b) mean that there is the following orthogonal decomposition

$$L^2\Omega_G(M) = \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp \quad (3.24)$$

PROOF: (a) We have shown that

$$L^2\Omega_G(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,D}(M)^\perp = \mathcal{H}_{X_M,N}(M) \oplus \mathcal{H}_{X_M,N}(M)^\perp.$$

Let $\omega \in L^2\Omega_G(M)$ then corresponding to these decompositions we can split it uniquely into

$$\omega = \lambda_D + (\omega - \lambda_D), \quad \omega = \lambda_N + (\omega - \lambda_N)$$

where $(\omega - \lambda_D) \in \mathcal{H}_{X_M,D}(M)^\perp$ and $(\omega - \lambda_N) \in \mathcal{H}_{X_M,N}(M)^\perp$. By Proposition 3.3.9 there are unique elements $\theta_D \in \mathcal{H}_{X_M,D}(M)^\oplus$ and $\theta_N \in \mathcal{H}_{X_M,N}(M)^\oplus$ satisfying the BVP (3.9) with η replaced by $(\omega - \lambda_D)$ and $(\omega - \lambda_N)$ respectively.

From Proposition (3.3.9) we infer that θ_D and θ_N are of Sobolev class H^2 , so define

$$\alpha_\omega = \delta_{X_M}\theta_D \in H^1\Omega_{G,D} \quad \text{and} \quad \beta_\omega = d_{X_M}\theta_N \in H^1\Omega_{G,N} \quad (3.25)$$

Now let

$$\kappa_\omega = \omega - d_{X_M}\alpha_\omega - \delta_{X_M}\beta_\omega \in L^2\Omega_G(M)$$

The next step is to show that κ_ω is orthogonal to $\mathcal{E}_{X_M}(M)$ but from proposition 3.2.2 we can prove that $\lambda_D, \delta_{X_M}\beta \in \mathcal{E}_{X_M}(M)^\perp$, in addition, $(\omega - \lambda_D) = \Delta_{X_M}\theta_D$ then

$$\langle \kappa_\omega, d_{X_M}\alpha \rangle = \langle \Delta_{X_M}\theta_D, d_{X_M}\alpha \rangle - \langle d_{X_M}\delta_{X_M}\theta_D + \delta_{X_M}d_{X_M}\theta_D, d_{X_M}\alpha \rangle = 0, \quad \forall d_{X_M}\alpha \in \mathcal{E}_{X_M}(M)$$

Analogously we can show that $\langle \kappa_\omega, \delta_{X_M}\beta \rangle = 0, \forall \delta_{X_M}\beta \in \mathcal{C}_{X_M}(M)$. Therefore $\kappa_\omega \in (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$.

(b) Let $\{d_{X_M}\alpha_j\}_{j \in \mathbb{N}}$ be an L^2 -Cauchy sequence in $\mathcal{E}_{X_M}(M)$ then $d_{X_M}\alpha_j \rightarrow \gamma \in L^2\Omega_G(M)$. Hence we get from part (a) above that

$$\gamma = d_{X_M}\alpha_\gamma + \delta_{X_M}\beta_\gamma + \kappa_\gamma$$

where $d_{X_M}\alpha_\gamma \in \mathcal{E}_{X_M}(M)$, $\delta_{X_M}\beta_\gamma \in \mathcal{C}_{X_M}(M)$ and $\kappa_\gamma \in (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$. Because $\mathcal{E}_{X_M}(M) \perp \mathcal{C}_{X_M}(M) \perp (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$ and $\langle \gamma - d_{X_M}\alpha_j, \gamma - d_{X_M}\alpha_j \rangle \rightarrow 0$ it follows that $\delta_{X_M}\beta_\gamma = 0$ and $\kappa_\gamma = 0$, thus $\gamma = d_{X_M}\alpha_\gamma \in \mathcal{E}_{X_M}(M)$. Hence $\mathcal{E}_{X_M}(M)$ is closed. The corresponding argument applies to $\mathcal{C}_{X_M}(M)$. \square

Now we can present the main theorems for this section.

Theorem 3.3.13 (X_M -Hodge-Morrey decomposition theorem) *Let M be a compact, oriented, smooth Riemannian manifold of dimension n with boundary and with an action of a torus G . Then*

$$L^2\Omega_G(M) = \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus L^2\mathcal{H}_{X_M}(M) \quad (3.26)$$

PROOF: We use the decomposition (3.24) from Proposition 3.3.12 and observe that the spaces $\mathcal{E}_{X_M}(M)$, $\mathcal{C}_{X_M}(M)$ and $L^2\mathcal{H}_{X_M}(M)$ are mutually orthogonal with respect to the L^2 -inner product which is an immediate consequence of Green's formula (Proposition 3.2.2), and hence

$$L^2\mathcal{H}_{X_M}(M) \subseteq (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp.$$

So we need only to prove the converse and then using eq. (3.24) we will get the decomposition (3.26). Let $\omega \in (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$, so

$$\begin{aligned} \langle \omega, d_{X_M}\alpha \rangle &= \langle \delta_{X_M}\omega, \alpha \rangle = 0 \quad \forall \alpha \in H^1\Omega_{G,D} \\ \langle \omega, \delta_{X_M}\beta \rangle &= \langle d_{X_M}\omega, \beta \rangle = 0 \quad \forall \beta \in H^1\Omega_{G,N}. \end{aligned} \quad (3.27)$$

From Remark 3.3.3 we know that $H^1\Omega_{G,D}$ and $H^1\Omega_{G,N}$ are dense in $L^2\Omega_G(M)$, hence eq. (3.27) implies that $d_{X_M}\omega = 0$ and $\delta_{X_M}\omega = 0$ which shows that $\omega \in L^2\mathcal{H}_{X_M}(M)$. Hence $L^2\mathcal{H}_{X_M}(M) = (\mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M))^\perp$. \square

Theorem 3.3.14 (X_M -Friedrichs Decomposition Theorem) *Let M be a compact, oriented smooth Riemannian manifold with boundary of dimension n and with an action of a torus G . Then the space $\mathcal{H}_{X_M}(M) \subseteq H^1\Omega_G(M)$ of X_M -harmonic fields can respectively be decomposed into*

$$\mathcal{H}_{X_M}(M) = \mathcal{H}_{X_M,D}(M) \oplus \mathcal{H}_{X_M,\text{co}}(M) \quad (3.28)$$

$$\mathcal{H}_{X_M}(M) = \mathcal{H}_{X_M,N}(M) \oplus \mathcal{H}_{X_M,\text{ex}}(M) \quad (3.29)$$

where the right hand terms are the X_M -coexact and X_M -exact harmonic fields respectively:

$$\mathcal{H}_{X_M,\text{co}}(M) = \{\eta \in \mathcal{H}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\} \quad (3.30)$$

$$\mathcal{H}_{X_M,\text{ex}}(M) = \{\xi \in \mathcal{H}_{X_M}(M) \mid \xi = d_{X_M}\sigma\} \quad (3.31)$$

For $L^2\mathcal{H}_{X_M}(M)$ these decompositions are valid accordingly.

PROOF: We prove eq. (3.28); the argument for the dual eq. (3.29) is analogous. Proposition 3.2.2 shows the orthogonality of the decomposition (3.28), i.e.

$$\langle \delta_{X_M} \alpha, \lambda_D \rangle = 0 \quad \forall \lambda_D \in \mathcal{H}_{X_M, D}(M). \quad (3.32)$$

The space $\mathcal{H}_{X_M}(M) \subseteq L^2\Omega_G(M)$, hence equation (3.14) asserts that $\mathcal{H}_{X_M}(M)$ can be decomposed into:

$$\mathcal{H}_{X_M}(M) = \mathcal{H}_{X_M, D}(M) \oplus \mathcal{H}_{X_M, D}(M)^\perp \cap \mathcal{H}_{X_M}(M) \quad (3.33)$$

where $\mathcal{H}_{X_M, D}(M)^\perp \cap \mathcal{H}_{X_M}(M)$ is the orthogonal complement of $\mathcal{H}_{X_M, D}(M)$ inside the space $\mathcal{H}_{X_M}(M)$. So, we need only prove that

$$\mathcal{H}_{X_M, \text{co}}(M) = \mathcal{H}_{X_M, D}(M)^\perp \cap \mathcal{H}_{X_M}(M).$$

But, it is clear that $\mathcal{H}_{X_M, \text{co}}(M) \subseteq \mathcal{H}_{X_M, D}(M)^\perp \cap \mathcal{H}_{X_M}(M)$ so, we just need to prove that

$$\mathcal{H}_{X_M, D}(M)^\perp \cap \mathcal{H}_{X_M}(M) \subseteq \mathcal{H}_{X_M, \text{co}}(M).$$

Now, let $\omega \in \mathcal{H}_{X_M}(M) \cap \mathcal{H}_{X_M, D}(M)^\perp$ then Proposition 3.3.9 asserts that there is a unique element $\theta_D \in \mathcal{H}_{X_M, D}(M)^\ominus$ such that θ_D satisfies the BVP (3.9). One can infer from eq. (3.32) that also $\omega - \delta_{X_M} d_{X_M} \theta_D \in \mathcal{H}_{X_M, D}(M)^\perp$. Hence,

$$\omega - \delta_{X_M} d_{X_M} \theta_D = \Delta_{X_M} \theta_D - \delta_{X_M} d_{X_M} \theta_D = d_{X_M} \delta_{X_M} \theta_D.$$

The above equation gives that

$$i^*(\omega - \delta_{X_M} d_{X_M} \theta_D) = 0, \quad d_{X_M}(\omega - \delta_{X_M} d_{X_M} \theta_D) = 0, \quad \text{and} \quad \delta_{X_M}(\omega - \delta_{X_M} d_{X_M} \theta_D) = 0$$

which mean that $\omega - \delta_{X_M} d_{X_M} \theta_D \in \mathcal{H}_{X_M, D}(M)$. However, $\omega - \delta_{X_M} d_{X_M} \theta_D \in \mathcal{H}_{X_M, D}(M)^\perp$, so $\omega - \delta_{X_M} d_{X_M} \theta_D \in \mathcal{H}_{X_M, \text{co}}(M)$ as required. Thus, equation (3.28) holds.

For $\omega \in L^2\mathcal{H}_{X_M}(M)$ all the arguments up to $\omega - \delta_{X_M} d_{X_M} \theta_D$ apply similarly. \square

The following remark will be used later in chapter 5.

Remark 3.3.15 The definition of $\mathcal{H}_{X_M, \text{co}}^\pm(M)$ and $\mathcal{H}_{X_M, \text{ex}}^\pm(M)$ and the proof of Theorem 3.3.14 show that the differential forms α can be chosen to satisfy $d_{X_M} \alpha = \Delta_{X_M} \alpha = 0$ while σ can be chosen to satisfy $\delta_{X_M} \sigma = \Delta_{X_M} \sigma = 0$.

Now, combining Theorems 3.3.13 and 3.3.14 gives the following.

Corollary 3.3.16 (The X_M -Hodge-Morrey-Friedrichs decompositions) *The space $L^2\Omega_G(M)$ can be decomposed into L^2 -orthogonal direct sums as follows:*

$$L^2\Omega_G(M) = \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus \mathcal{H}_{X_M,D}(M) \oplus L^2\mathcal{H}_{X_M,\text{co}}(M) \quad (3.34)$$

$$L^2\Omega_G(M) = \mathcal{E}_{X_M}(M) \oplus \mathcal{C}_{X_M}(M) \oplus \mathcal{H}_{X_M,N}(M) \oplus L^2\mathcal{H}_{X_M,\text{ex}}(M) \quad (3.35)$$

Remark 3.3.17 All the results above can be recovered but in terms of \pm -spaces, for instance,

$$\mathcal{H}_{X_M,D}^\pm(M) \cong \mathcal{H}_{X_M,N}^{n-\pm}(M), \quad L^2\Omega_G^\pm(M) = \mathcal{E}_{X_M}^\pm(M) \oplus \mathcal{C}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M,D}^\pm(M) \oplus L^2\mathcal{H}_{X_M,\text{co}}^\pm(M)$$

... etc.

3.3.4 Relative and absolute X_M -cohomology

Using d_{X_M} and δ_{X_M} we can form a number of \mathbb{Z}_2 -graded complexes. A \mathbb{Z}_2 -graded complex is a pair of Abelian groups C^\pm with homomorphisms between them:

$$C^+ \begin{array}{c} \xrightarrow{d_+} \\ \xleftarrow{d_-} \end{array} C^-$$

satisfying $d_+ \circ d_- = 0 = d_- \circ d_+$. The two (co)homology groups of such a complex are defined in the obvious way: $H^\pm = \ker d_\pm / \text{im } d_\mp$.

The complexes we have in mind are,

$$\begin{array}{cc} (\Omega_G^\pm, d_{X_M}) & (\Omega_G^\pm, \delta_{X_M}) \\ (\Omega_{G,D}^\pm, d_{X_M}) & (\Omega_{G,N}^\pm, \delta_{X_M}). \end{array}$$

The two on the lower line are subcomplexes of the corresponding upper ones. These are subcomplexes because i^* commutes with d_{X_M} . By analogy with the de Rham groups, we denote

$$\begin{aligned} H_{X_M}^\pm(M) &:= H^\pm(\Omega_G, d_{X_M}), \\ H_{X_M}^\pm(M, \partial M) &:= H^\pm(\Omega_{G,D}, d_{X_M}). \end{aligned}$$

The theorem of Hodge is often quoted as saying that every (de Rham) cohomology class on a compact Riemannian manifold without boundary contains a unique harmonic form. The corresponding statement for X_M -cohomology on a manifold with boundary is,

Theorem 3.3.18 (X_M -Hodge Isomorphism) *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which*

acts by isometries on M . Let $X \in \mathfrak{g}$. We have

- (a) Each relative X_M -cohomology class contains a unique Dirichlet X_M -harmonic field, i.e. $H_{X_M}^\pm(M, \partial M) \cong \mathcal{H}_{X_M, D}^\pm(M)$.
- (b) Each absolute X_M -cohomology class contains a unique Neumann X_M -harmonic field, i.e. $H_{X_M}^\pm(M) \cong \mathcal{H}_{X_M, N}^\pm(M)$.
- (c) (X_M -Poincaré-Lefschetz duality): The Hodge star operator \star on $\Omega_G(M)$ induces an isomorphism

$$H_{X_M}^\pm(M) \cong H_{X_M}^{n-\pm}(M, \partial M).$$

PROOF: We use the various decomposition theorems to prove (a). Part (b) is proved similarly, and part (c) follows from (a), (b) and the fact that the Hodge star operator defines an isomorphism $\mathcal{H}_{X_M, D}^\pm(M) \cong \mathcal{H}_{X_M, N}^{n-\pm}(M)$.

For the first isomorphism in (a), Theorem 3.3.13 (the X_M -Hodge-Morrey decomposition theorem) implies a unique splitting of any $\gamma \in \Omega_{G, D}^\pm$ into,

$$\gamma = d_{X_M} \alpha_\gamma + \delta_{X_M} \beta_\gamma + \kappa_\gamma$$

where $d_{X_M} \alpha_\gamma \in \mathcal{E}_{X_M}^\pm(M)$, $\delta_{X_M} \beta_\gamma \in \mathcal{C}_{X_M}^\pm(M)$ and $\kappa_\gamma \in L^2 \mathcal{H}_{X_M}^\pm(M)$. If $d_{X_M} \gamma = 0$ then $\delta_{X_M} \beta_\gamma = 0$, but $i^* \gamma = 0$ implies $i^*(\kappa_\gamma) = 0$ so that $\kappa_\gamma \in \mathcal{H}_{X_M, D}^\pm(M)$. Thus,

$$\gamma \in \ker d_{X_M}|_{\Omega_{G, D}^\pm} \iff \gamma = d_{X_M} \alpha_\gamma + \kappa_\gamma.$$

This establishes the isomorphism $H_{X_M}^\pm(M, \partial M) \cong \mathcal{H}_{X_M, D}^\pm(M)$.

Now, to prove the uniqueness, suppose we have two Dirichlet X_M -harmonic field κ_γ and $\bar{\kappa}_\gamma$ belong in the same relative X_M -cohomology class $[\gamma]_{(X_M, M, \partial M)}$. This means that

$$\kappa_\gamma - \bar{\kappa}_\gamma = d_{X_M} \bar{\alpha}_\gamma$$

where $d_{X_M} \bar{\alpha}_\gamma \in \mathcal{E}_{X_M}^\pm(M)$. Proposition 3.2.2 (Green's formula for d_{X_M} and δ_{X_M}) asserts that $d_{X_M} \bar{\alpha}_\gamma = 0$ and thus $\kappa_\gamma = \bar{\kappa}_\gamma$ as desired. \square

The decomposition theorems above lead to the following result.

Corollary 3.3.19 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M . Let $X \in \mathfrak{g}$. There are the following isomorphisms of vector spaces:*

- (a) $\mathcal{H}_{X_M, D}^\pm(M) \cong H^\pm(\Omega_G^\pm, \delta_{X_M})$

$$(b) \mathcal{H}_{X_M, N}^\pm(M) \cong H^\pm(\Omega_{G, N}^\pm, \delta_{X_M})$$

PROOF: We prove part (a) while part (b) is proved similarly.

For the isomorphism in (a), the X_M -Hodge-Morrey-Friedrichs decomposition (Corollary 3.3.16) eq. (3.34) implies a unique splitting of any $\gamma \in \Omega_G^\pm(M)$ into,

$$\gamma = d_{X_M} \xi_\gamma + \delta_{X_M} \eta_\gamma + \lambda_\gamma + \delta_{X_M} \zeta_\gamma$$

where $d_{X_M} \xi_\gamma \in \mathcal{E}_{X_M}^\pm(M)$, $\delta_{X_M} \eta_\gamma \in \mathcal{C}_{X_M}^\pm(M)$, $\lambda_\gamma \in \mathcal{H}_{X_M, D}^\pm(M)$ and $\delta_{X_M} \zeta_\gamma \in L^2 \mathcal{H}_{X_M, \text{co}}^\pm(M)$.

If $\delta_{X_M} \gamma = 0$, then $d_{X_M} \xi_\gamma = 0$, and hence

$$\gamma \in \ker \delta_{X_M} \iff \gamma = \delta_{X_M}(\eta_\gamma + \zeta_\gamma) + \lambda_\gamma.$$

This establishes the isomorphism $\mathcal{H}_{X_M, D}^\pm(M) \cong H^\pm(\Omega_G^\pm, \delta_{X_M})$. □

Remark 3.3.20 Analogously to the case of $\partial M = \emptyset$ (Remark 3.2.12), if G acts on the Riemannian manifold with boundary, preserving X_M , then Theorem 3.3.18 (X_M -Hodge Isomorphism) provides that the G -invariant relative and absolute X_M -cohomology groups or the corresponding spaces of Dirichlet and Neumann X_M -harmonic fields are independent of the choice of torus, provided it contains $G(X)$.

3.4 Relation with equivariant cohomology and singular homology

3.4.1 X_M -cohomology and equivariant cohomology

When the manifold in question has no boundary, Atiyah and Bott [8] discuss the relationship between equivariant cohomology and X_M -cohomology by using their localization theorem. In this section we will relate the relative and absolute X_M -cohomology with the relative and absolute equivariant cohomology $H_G^\pm(M, \partial M)$ and $H_G^\pm(M)$; the arguments are no different to the ones in [8]. First we recall briefly the basic definitions of equivariant cohomology, and the relevant localization theorem, and then state the conclusions for the relative and absolute X_M -cohomology.

If a torus G acts on a manifold M (with or without boundary), the Cartan model for the equivariant cohomology is defined as follows. Let $\{X_1, \dots, X_\ell\}$ be a basis of \mathfrak{g} and $\{u_1, \dots, u_\ell\}$ the corresponding coordinates. The *Cartan complex* consists of polynomial¹

¹we use real valued polynomials, though complex valued ones works just as well, and all tensor products are thus over \mathbb{R} , unless stated otherwise

maps from \mathfrak{g} to the space of invariant differential forms, so is equal to $\Omega_G^*(M) \otimes R$ where $R = \mathbb{R}[u_1, \dots, u_\ell]$, with differential

$$d_{\text{eq}}(\omega) = d\omega + \sum_{j=1}^{\ell} u_j \iota_{X_j} \omega. \quad (3.36)$$

The equivariant cohomology $H_G^*(M)$ is the cohomology of this complex. The relative equivariant cohomology $H_G^*(M, \partial M)$ (if M has non-empty boundary) is formed by taking the subcomplex with forms that vanish on the boundary $i^* \omega = 0$, with the same differential.

The cohomology groups are graded by giving the u_i weight 2 and a k -form weight k , so the differential d_{eq} is of degree 1. Furthermore, as the cochain groups are R -modules, and d_{eq} is a homomorphism of R -modules, it follows that the equivariant cohomology is an R -module. The localization theorem of Atiyah and Bott [8] gives information on the module structure (there it is only stated for absolute cohomology, but it is equally true in the relative setting, with the same proof; see also Appendix C of [19]).

First we define the following subset of \mathfrak{g} ,

$$Z := \bigcup_{\widehat{K} \subsetneq G} \mathfrak{k}$$

where the union is over proper isotropy subgroups \widehat{K} (and \mathfrak{k} its Lie algebra) of the action on M . If M is compact, then Z is a finite union of proper subspaces of \mathfrak{g} . Let $F = \text{Fix}(G, M) = \{x \in M \mid G \cdot x = x\}$ be the set of fixed points in M . It follows from the local structure of group actions that F is a submanifold of M , with boundary $\partial F = F \cap \partial M$.

Theorem 3.4.1 (Atiyah-Bott [8, Theorem 3.5]) *The inclusion $j : F \hookrightarrow M$ induces homomorphisms of R -modules*

$$\begin{aligned} H_G^*(M) &\xrightarrow{j^*} H_G^*(F) \\ H_G^*(M, \partial M) &\xrightarrow{j^*} H_G^*(F, \partial F) \end{aligned}$$

whose kernel and cokernel have support in Z .

In particular, this means that if $f \in I(Z)$ (the ideal in R of polynomials vanishing on Z) then the localizations² $H_G^*(M)_f$ and $H_G^*(F)_f$ are isomorphic R_f -modules. Notice that the act of localization destroys the integer grading of the cohomology, but since the u_i have weight 2, it preserves the parity of the grading, so that the separate even and odd parts are maintained: $H_G^\pm(M)_f \cong H_G^\pm(F)_f$. The same reasoning applies to the cohomology relative to the boundary, so $H_G^\pm(M, \partial M)_f \cong H_G^\pm(F, \partial F)_f$

²The localized ring R_f consists of elements of R divided by a power of f and if K is an R -module, its localization is $K_f := K \otimes_R R_f$; they correspond to restricting to the open set where f is non-zero. See the notes by Libine [29] for a good discussion of localization in this context.

Since the action on F is trivial, it is immediate from the definition that there is an isomorphism of R -modules, $H_G^*(F) \cong H^*(F) \otimes R$ so that the localization theorem shows j^* induces an isomorphism of R_f -modules,

$$H_G^\pm(M)_f \xrightarrow{j^*} H^\pm(F) \otimes R_f. \quad (3.37)$$

It follows that $H_G^\pm(M)_f$ is a free R_f module whenever $f \in I(Z)$. Of course, analogous statements hold for the relative versions. Since localization does not alter the rank of a module (it just annihilates torsion elements), we have that

$$\text{rank } H_G^\pm(M) = \dim H^\pm(F), \quad \text{rank } H_G^\pm(M, \partial M) = \dim H^\pm(F, \partial F).$$

For $X \in \mathfrak{g}$, define $N(X_M) = \{x \in M \mid X_M(x) = 0\}$, the set of zeros of the vector field X_M . Since X generates a torus action, $N(X_M)$ is a manifold with boundary $\partial N(X_M) = N(X_M) \cap \partial M$. Clearly $N(X_M) \supset F$, and $N(X_M) = F$ if and only if $X \notin Z$.

Theorem 3.4.2 *Let $X = \sum_j s_j X_j \in \mathfrak{g}$. If the set of zeros of the corresponding vector field X_M is equal to the fixed point set for the G -action (i.e. $N(X_M) = F$) then*

$$H_{X_M}^\pm(M, \partial M) \cong H_G^\pm(M, \partial M) / \mathfrak{m}_X H_G^\pm(M, \partial M), \quad (3.38)$$

and

$$H_{X_M}^\pm(M) \cong H_G^\pm(M) / \mathfrak{m}_X H_G^\pm(M) \quad (3.39)$$

where $\mathfrak{m}_X = \langle u_1 - s_1, \dots, u_l - s_l \rangle$ is the ideal of polynomials vanishing at X .

PROOF: Our assumption $N(X_M) = F$ is equivalent to $X \in \mathfrak{g} \setminus Z$. Therefore there is a polynomial $f \in I(Z)$ such that $f(X) \neq 0$. In addition, we can use f and replace the ring R by R_f and then localize $H_G^\pm(M)$ and $H_G^\pm(M, \partial M)$ to make $H_G^\pm(M)_f$ and $H_G^\pm(M, \partial M)_f$ which are free R_f -modules.

We now apply the lemma stated below, in which the left-hand side is obtained by putting $u_i = s_i$ before taking cohomology, so results in $H_{X_M}^\pm(M)$ (or similar for the relative case), while the right-hand side is the right-hand side of (3.38) and (3.39), so proving the theorem. \square

Lemma 3.4.3 (Atiyah-Bott [8, Lemma 5.6]) *Let (C^*, d) be a cochain complex of free R -modules and assume that, for some polynomial f , $H(C^*, d)_f$ is a free module over the localized ring R_f . Then, if $s \in \mathbb{R}^l$ with $f(s) \neq 0$,*

$$H^\pm(C_s^*, d_s) \cong H^\pm(C^*, d) \text{ mod } \mathfrak{m}_s$$

where \mathfrak{m}_s is the (maximal) ideal $\langle u_1 - s_1, \dots, u_l - s_l \rangle$ at X in $\mathbb{R}[\mathfrak{g}]$.

Corollary 3.4.4 *Let $X \in \mathfrak{g}$ and $j_X : N(X_M) \hookrightarrow M$, then j_X^* induces the following isomorphisms*

$$1- H_{X_M}^\pm(M) \cong H^\pm(N(X_M)),$$

$$2- H_{X_M}^\pm(M, \partial M) \cong H^\pm(N(X_M), \partial N(X_M)).$$

Moreover, if $N(X_M) = F$ then $\dim H_{X_M}^\pm(M, \partial M) = \text{rank} H_G^\pm(M, \partial M)$ and $\dim H_{X_M}^\pm(M) = \text{rank} H_G^\pm(M)$.

PROOF: First suppose $X \notin Z$ which implies $N(X_M) = F$. Then the isomorphisms above follow by reducing equation (3.37) modulo \mathfrak{m}_X and applying Theorem 3.4.2. This proves the equality between the dimension of X_M -cohomology and the rank of equivariant cohomology.

If on the other hand, $X \in Z$, then let G' be the corresponding isotropy subgroup, so that $N(X_M) = \text{Fix}(G', M)$ (it is clear that $G' \supset G(X)$, the subgroup of G generated by X). The considerations above show that $H_{X_M, G'}^\pm(M, \partial M) \cong H^\pm(N(X_M), \partial N(X_M))$ and $H_{X_M, G'}^\pm(M) \cong H^\pm(N(X_M))$, where $H_{X_M, G'}^\pm(M)$ and $H_{X_M, G'}^\pm(M, \partial M)$ are defined using G' -invariant forms, and $\mathfrak{m}_{G', X}$ is the maximal ideal at X in the ring $\mathbb{R}[\mathfrak{g}']$. Moreover, all classes in $H_{X_M, G'}^\pm(M)$ and $H_{X_M, G'}^\pm(M, \partial M)$ have representatives which are G -invariant, not only G' -invariant (either by an averaging argument, or by using the unique X_M -harmonic representatives, see remark 3.3.20). So, this gives $H_{X_M, G}^\pm(M) \cong H_{X_M, G'}^\pm(M)$ and $H_{X_M, G}^\pm(M, \partial M) \cong H_{X_M, G'}^\pm(M, \partial M)$, $\forall X \in \mathfrak{g}' \subset \mathfrak{g}$ as desired. □

3.4.2 X_M -cohomology and singular homology

One of the fundamental result in algebraic topology of manifolds is that: If M is a compact manifold with or without boundary then the absolute $H^k(M)$ (relative $H^k(M, \partial M)$) de Rham cohomology is isomorphic with the absolute $H_k(M)$ (relative $H_k(M, \partial M)$) singular homology, all over \mathbb{R} , [34, 17]. Using this fact together with corollary 3.4.4, we get the following theorem:

Theorem 3.4.5 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and G act by isometries on M . Then for $X \in \mathfrak{g}$,*

$$H_{X_M}^\pm(M) \cong H_\pm(N(X_M)) \tag{3.40}$$

$$H_{X_M}^\pm(M, \partial M) \cong H_\pm(N(X_M), \partial N(X_M)) \tag{3.41}$$

where,

$H_+(N(X_M)) = \bigoplus_{i=0}^{\infty} H_{2i}(N(X_M))$ and $H_-(N(X_M), \partial N(X_M)) = \bigoplus_{i=0}^{\infty} H_{2i+1}(N(X_M), \partial N(X_M))$,
by using the map

$$[\omega]_{X_M}(\{c\}) = \int_c j_X^* \omega, \quad (3.42)$$

where ω is an X_M -closed \pm -form representing the absolute (or relative) X_M -cohomology class $[\omega]_{X_M}$ on M and c is a \pm -cycle representing the absolute (or relative) singular homology class $\{c\}$ on $N(X_M)$.

The real numbers determined by the integral (3.42) of $j_X^* \omega$ over \pm -cycles on $N(X_M)$ are called the *periods* of $j_X^* \omega$ over all the \pm -cycles on $N(X_M)$. In the setting above, Stokes' theorem for d_{X_M} (theorem 3.2.1) and corollary 3.4.4 assert that the periods of an X_M -exact form are all zero. Conversely, If X_M -closed form ω has all of the periods of $j_X^* \omega$ zero over the cycles of $N(X_M)$ then it is X_M -exact where this follows immediately from the injectivity of the isomorphisms in theorem 3.4.5. In this light, theorem 3.4.5 proves that:

an X_M -closed form ω is X_M -exact iff all the periods of $j_X^ \omega$ over \pm -cycles of $N(X_M)$ vanish.*

3.5 Conclusions

- (1) Theorem 3.3.18 proves that $\dim H_{X_M}^{\pm}(M, \partial M) = \dim \mathcal{H}_{X_M, D}^{\pm}(M)$ and $\dim H_{X_M}^{\pm}(M) = \dim \mathcal{H}_{X_M, N}^{\pm}(M)$. These results show the following: the definition of the Dirichlet and Neumann X_M -harmonic fields depend on the Riemannian metric and so their dimension whereas the relative and absolute X_M -cohomology are defined regardless of the Riemannian metric and hence the dimension of X_M -cohomology does not depend on the metric. This argument can be applied also when M has no boundary ∂M (proposition 3.2.6) as the Witten-Hodge-Laplacian Δ_{X_M} depends on the Riemannian metric. This fact could be useful somehow to relate X_M -cohomology with the Atiyah-Singer index theorem because like this fact (for standard de Rham cohomology) was used as a key stone in Atiyah-Singer index theorem to relate the analytic index with the topological index [16].
- (2) The relation of the X_M -cohomology with the equivariant cohomology and singular homology in section 3.4 prove that the X_M -cohomology groups depend only on the underlying topological structure of $N(X_M)$ and do not depend on the smooth or differentiable structure.
- (3) The following example is to support the above conclusions: consider $M = S^3$ (the unit 3-sphere in \mathbb{C}^2). Let a 2-torus $G = S^1 \times S^1$ acts on M by $(e^{i\theta}, e^{i\phi}) \cdot (z_1, z_2) =$

$(e^{i\theta}z_1, e^{i\phi}z_2)$. Let $X = (a, b) \in \mathfrak{g}$, so $X_M(z_1, z_2) = (aiz_1, biz_2)$. We have the following four cases:

- Case 1: If $a \neq 0$ and $b \neq 0$ then we have that $F = N(X_M) = \emptyset$. In this case, corollary 3.4.4 asserts that $H_{X_M}^\pm(M) = 0$ which represents the free part of $H_G^\pm(M)$ and this gives $\text{rank } H_G^\pm(M) = 0$.
- Case 2: If $a = 0$ and $b \neq 0$ then we have that $N(X_M) = \{(z_1, 0)\} \simeq S^1$. Hence, in this case corollary 3.4.4 asserts $H_{X_M}^\pm(M) \simeq H^\pm(S^1) \simeq \mathbb{R}$. This gives some of the torsion part of $H_G^\pm(M)$ because in this case $X = (0, b) \in Z \subset \mathfrak{g}$.
- Case 3: If $a \neq 0$ and $b = 0$ then we have that $N(X_M) = \{(0, z_2)\} \simeq S^1$. Hence, in this case corollary 3.4.4 asserts $H_{X_M}^\pm(M) \simeq H^\pm(S^1) \simeq \mathbb{R}$. This gives some of the torsion part of $H_G^\pm(M)$ because in this case $X = (a, 0) \in Z \subset \mathfrak{g}$.
- Case 4: If $a = b = 0$ then we have $X_M = 0$. In this case 0-cohomology of M reduces to the standard de Rham cohomology of M . This means that $H_0^\pm(M) = H^\pm(M) \simeq \mathbb{R}$.

The four cases above show that the dependence of X_M -cohomology on the vector field X even when M is not changed.

- (3) **Generalization of Witten's results:** In previous sections, we began with the action of a torus G ; here we state results for a given Killing vector field K on a compact Riemannian manifold M (with or without boundary), more in keeping with Witten's original work [35]. Recall that the group $\text{Isom}(M)$ of isometries of M is a compact Lie group, and the smallest closed subgroup $G(K)$ containing K in its Lie algebra is abelian, so a torus. Furthermore, the submanifold $N(K)$ of zeros of K coincides with $\text{Fix}(G(K), M)$.

The equivariant cohomology constructions of Section 3.4 give us the proof of the following result, which extends the theorem of Witten (our Theorem 3.2.10) to manifolds with boundary.

Theorem 3.5.1 *Let K be a Killing vector field on the compact Riemannian manifold M (with or without boundary), and let $N(K)$ be the submanifold of zeros of K . Then pullback to $N(K)$ induces isomorphisms*

$$H_K^\pm(M) \cong H^\pm(N(K)), \quad \text{and} \quad H_K^\pm(M, \partial M) \cong H^\pm(N(K), \partial N(K)).$$

PROOF: Apply Corollary 3.4.4 to the equivariant cohomology for the action of the torus $G(K)$. □

Furthermore, using the Hodge star operator, the Poincaré-Lefschetz duality of Theorem 3.3.18(c) corresponds under the isomorphisms in the theorem above, to Poincaré-Lefschetz duality on the fixed point space.

Translating this theorem into the language of harmonic fields, shows

$$\mathcal{H}_{K,N}^{\pm}(M) \cong \mathcal{H}_N^{\pm}(N(K)) \quad \text{and} \quad \mathcal{H}_{K,D}^{\pm}(M) \cong \mathcal{H}_D^{\pm}(N(K)). \quad (3.43)$$

where $\mathcal{H}_N^{\pm}(N(K))$ and $\mathcal{H}_D^{\pm}(N(K))$ are the ordinary Neumann and Dirichlet harmonic fields on $N(K)$ respectively.

Corollary 3.5.2 *Given any harmonic field on $N(K)$ with either Dirichlet or Neumann boundary conditions, there is a unique K -harmonic field on M with the corresponding boundary conditions whose pullback to $N(K)$ is cohomologous to the given harmonic field.*

Note that if $\partial N(K) = \emptyset$ then the boundary condition on $N(K)$ is non-existent, and so every harmonic form (= field) on $N(K)$ has corresponding to it both a unique Dirichlet and a unique Neumann K -harmonic field on M .

As an application, we have the fact that theorem 3.5.1 and corollary 3.5.2 can be used to extend the other results of Witten in [35]. In addition, Witten mentions in [35] that *in Quantum Fields Theory, M will be infinite dimensional and $N(K)$ finite dimensional. The reduction of a problem on M to a problem on $N(K)$ is crucial to make computation possible*, so following this we hope that this extension (i.e. theorem 3.5.1 and corollary 3.5.2) will be useful in Quantum Field Theory and other mathematical and physical applications when $\partial M \neq \emptyset$.

- (4) **Euler characteristics:** As is well known, given a complex of $\mathbb{R}[s]$ (or $\mathbb{C}[s]$) modules whose cohomology is finitely generated, the Euler characteristic of the complex is independent of s . This remains true for a \mathbb{Z}_2 -graded complex, for the same reasons (briefly, the cohomology is the direct sum of a torsion module and a free module, and the torsion cancels in the Euler characteristic).

Applying this to the complexes for X_M -cohomology, with $X_M = sK$, it follows that $\chi(M) = \chi(N)$ and $\chi(M, \partial M) = \chi(N, \partial N)$ (where $N = N(K)$), and furthermore applying the same arguments to the manifold ∂M , one has $\chi(\partial M) = \chi(\partial N)$, i.e.

$$\chi(M) = \chi(\partial M) + \chi(M, \partial M) = \chi(\partial N) + \chi(N, \partial N) = \chi(N).$$

- (5) **Other Applications:** We have shown that the Witten-Hodge theory can shed light to

give additional topological insight. In addition, the fact that we can use the new decompositions of $L^2\Omega_G^\pm(M)$ given in theorem 3.3.13 and corollary 3.3.16 and also the relation between the X_M -cohomology and X_M -harmonic fields (theorem 3.3.18 (X_M -Hodge Isomorphism)) as powerful tools (under topological aspects) in the theory of differential equations on $L^2\Omega_G^\pm(M)$ to obtain the solvability of various BVPs. In particular, we can extend most of the results of chapter three of [31] on $L^2\Omega_G^\pm(M)$ to the context of the operators d_{X_M} , δ_{X_M} and Δ_{X_M} . Moreover, the classical Hodge theory plays a fundamental role in incompressible hydrodynamics and it has applications to many other area of mathematical physics and engineering [1]. So, following these, we hope that the Witten-Hodge theory will be useful as tools in these applications as well.

Chapter 4

Interior and boundary portions of X_M -cohomology

4.1 Introduction

It seems reasonable to think that we can extend the results of chapter 3 further in the style of DeTurck and Gluck [15], and break down the Neumann and Dirichlet X_M -harmonic fields into interior and boundary subspaces. If so, we ask the following question, *does the natural extension of corollary 3.4.4 hold?* The affirmative answer of this question which is given in corollary 4.3.5 gives more concrete understanding to improve the main results of chapter 3.

However, the results of chapter 3 do not extend immediately to the style of DeTurck and Gluck [15] because the proofs of DeTurck and Gluck's results in [15] do not appear to extend to the present setting. But, in section 4.2 we do the first step by using a new argument, providing that the concrete realization of the relative and absolute X_M -cohomology groups meet only at the origin in $\Omega_G^\pm(M)$. In section 4.3, we define the interior and boundary portions of X_M -cohomology and we present a new different argument to break down the Neumann and Dirichlet X_M -harmonic fields into interior and boundary subspaces. Finally, a few conclusions are given in section 4.4 with a geometric open problem. The results of this chapter are given in [2] while part of them (in particular theorem 4.2.1) are given in [3].

4.2 The intersection of $\mathcal{H}_{X_M, N}^\pm(M)$ and $\mathcal{H}_{X_M, D}^\pm(M)$

An important classical result is that any harmonic field satisfying both Neumann and Dirichlet boundary conditions (so one vanishing on the boundary) is necessarily zero: see theorem 3.4.4 in [31] or lemma 2 in [13]. So, we present this theorem.

Theorem 4.2.1 *Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M . If an X_M -harmonic field $\lambda \in \mathcal{H}_{X_M}^\pm(M)$ vanishes on the boundary ∂M , then $\lambda \equiv 0$, i.e.*

$$\mathcal{H}_{X_M, N}^\pm(M) \cap \mathcal{H}_{X_M, D}^\pm(M) = \{0\} \quad (4.1)$$

In general, the proof consists in showing that a form which is both Neumann and Dirichlet harmonic fields has a zero of infinite order at every boundary point and then applying the Strong Unique Continuation Theorem below. However, the proof that there are zeros of infinite order in [31, 13] does not appear to extend to our present setting, so we give a different argument, based on Hadamard's lemma and the *boundary normal coordinates*. Moreover, this basic technique is also valid in the classical case which gives another proof of theorem 3.4.4 in [31].

First, we state the Strong Unique Continuation Theorem, due to Aronszajn [5], Aronszajn, Krzywicki and Szarski [6]. In [26], Kazdan writes this theorem in terms of Laplacian operator Δ but he mentions that it is still valid for any operator having the diagonal form $P = \Delta I +$ lower-order terms, where I is the identity matrix. Hence, one can state this theorem in terms of diagonal form operator by the following form:

Theorem 4.2.2 (Strong Unique Continuation Theorem [26]) *Let \bar{M} be a Riemannian manifold with Lipschitz continuous metric, and let ω be a differential form having first derivatives in L^2 that satisfies $P(\omega) = 0$ where P is a diagonal form operator. If ω has a zero of infinite order at some point in \bar{M} , then ω is identically zero on \bar{M} .*

In addition, the following remark will be used during the proof.

Remark 4.2.3 Along the boundary of M , any smooth differential form ω has a natural decomposition into tangential ($t\omega$) and normal($n\omega$) components. i.e.

$$\omega|_{\partial M} = t\omega + n\omega$$

and the tangential component $t\omega$ is uniquely determined by the pull-back $i^*\omega$ and it has been denoted in a slight abuse of notation by $i^*\omega = i^*t\omega = t\omega$. The normal and tangential components of ω are Hodge adjoint to each other [31], i.e.

$$\star(n\omega) = t(\star\omega) = i^*\star\omega.$$

Proof of Theorem 4.2.1. Suppose $\lambda \in \mathcal{H}_{X_M, N}^\pm(M) \cap \mathcal{H}_{X_M, D}^\pm(M)$, then λ is smooth by theorem 3.3.5(c). Since $i^*\lambda = i^*\star\lambda = 0$ then remark 4.2.3 asserts that $t\lambda = n\lambda = 0$. Hence $\lambda|_{\partial M} \equiv 0$ and we get that $(\iota_{X_M}\lambda)|_{\partial M} = 0$ as well.

The proof is local so we can consider M to be the upper half space in \mathbb{R}^n with $\partial M = \mathbb{R}^{n-1}$. Since the metric, the differential form λ and the vector field X_M are given in the upper half space, we can extend them from there to all of \mathbb{R}^n by making them invariant under the reflection in $\partial M = \mathbb{R}^{n-1}$. The resulting objects are: the extended metric, which will be Lipschitz continuous [13]; the extended form λ and extended vector field X_M which will be a Lipschitz continuous vector field. But the original λ satisfies $\lambda|_{\partial M} \equiv 0$ and $d_{X_M}\lambda = \delta_{X_M}\lambda = 0$ on the upper half space, hence the extended one will be of class C^1 and satisfy $d_{X_M}\lambda = \delta_{X_M}\lambda = 0$ on \mathbb{R}^n , i.e. the extended λ satisfies $P(\lambda) = \Delta_{X_M}\lambda = 0$ on all of \mathbb{R}^n where the operator Δ_{X_M} has diagonal form, i.e. $P = \Delta_{X_M} = \Delta I +$ lower-order terms. So far, we satisfy the first condition of theorem 4.2.2.

Now, we need to satisfy the remaining hypotheses of theorem 4.2.2. Let $x = (x', x_n)$ be a coordinates chart where $x' = (x_1, x_2, \dots, x_{n-1})$ is a chart on the boundary ∂M and x_n is the distance to the boundary. In these coordinates $x_n > 0$ in M and ∂M is locally characterized by $x_n = 0$. These coordinates are called boundary normal coordinates and the Riemannian metric in these coordinates has the form $\sum_{m,r=1}^{n-1} h_{m,r}(x) dx^m \otimes dx^r + dx^n \otimes dx^n$, [25] and [28].

Now, we consider a neighborhood of $p \in \partial M$ where our boundary normal coordinates are well defined. We can write $\lambda = \alpha + \beta \wedge dx_n$ where $\alpha = \sum f_{I_1}(x) dx^{I_1}$, $\beta = \sum g_{I_2}(x) dx^{I_2}$ and $I_1, I_2 \subset \{1, 2, \dots, n-1\}$. Our goal is to prove that all the partial derivatives of the coefficients of λ (i.e. $f_{I_1}(x)$ and $g_{I_2}(x)$) vanish at $p \in \partial M$. Now, $\lambda|_{\partial M} \equiv 0$ which implies that $f_{I_1}(x', 0) = g_{I_2}(x', 0) = 0$. Hence, we can apply Hadamard's lemma [24] to $f_{I_1}(x)$ and $g_{I_2}(x)$ and deduce that $f_{I_1}(x) = x_n \bar{f}_{I_1}(x)$ and $g_{I_2}(x) = x_n \bar{g}_{I_2}(x)$ for some smooth functions $\bar{f}_{I_1}(x)$ and $\bar{g}_{I_2}(x)$. Moreover, these representations for $f_{I_1}(x)$ and $g_{I_2}(x)$ help us to conclude that all the higher partial derivatives of $f_{I_1}(x)$ and $g_{I_2}(x)$ with respect to the coordinates of x' (i.e. except the normal direction coordinate x_n) at the point p are all zero. i.e.

$$\frac{\partial^{|s|} f_{I_1}(x', 0)}{\partial x_1^{s_1} \dots \partial x_{n-1}^{s_{n-1}}} = \frac{\partial^{|s|} g_{I_2}(x', 0)}{\partial x_1^{s_1} \dots \partial x_{n-1}^{s_{n-1}}} = 0, \quad \forall s_1, s_2, \dots, s_{n-1} = 0, 1, 2, \dots$$

Therefore, we only need to prove that all the higher partial derivatives of $f_{I_1}(x)$ and $g_{I_2}(x)$ in the normal direction are zero to deduce that the Taylor series of $f_{I_1}(x)$ and $g_{I_2}(x)$ around $x_n = 0$ are zero.

For contradiction, suppose the Taylor series of $f_{I_1}(x)$ and $g_{I_2}(x)$ around $x_n = 0$ are not zero at $p \in \partial M$ which means that there exist the largest positive integer numbers k and j such that $f_{I_1}(x) = x_n^k \widehat{f}_{J_1}(x)$ and $g_{I_2}(x) = x_n^j \widehat{g}_{J_2}(x)$ where $\widehat{f}_{J_1}(x', 0) \neq 0$ and $\widehat{g}_{J_2}(x', 0) \neq 0$ for some J_1, J_2 . Thus, we can always write λ in the following form $\lambda = x_n^k \tau + x_n^j \rho \wedge dx_n$ where the differential forms τ and ρ do not contain dx_n . Applying $d_{X_M}\lambda = 0$, we get

$$0 = d_{X_M}\lambda = kx_n^{k-1} dx_n \wedge \tau + x_n^k d\tau + x_n^j d\rho \wedge dx_n + x_n^k \iota_{X_M} \tau + x_n^j \iota_{X_M} (\rho \wedge dx_n).$$

Now, reducing this equation modulo x_n^k we conclude that the term $x_n^j(d\rho \wedge dx_n + \iota_{X_M}(\rho \wedge dx_n)) \not\equiv 0$ modulo x_n^k because the term $kx_n^{k-1}dx_n \wedge \tau \not\equiv 0$ modulo x_n^k and as a consequence, we infer that $k > j$.

Similarly, we can calculate $\delta_{X_M}\lambda = -(\mp)^n(\star d\star\lambda + \star\iota_{X_M}\star\lambda) = 0$ (using the Riemannian metric above). For simplicity, it is enough to calculate $d\star\lambda + \iota_{X_M}\star\lambda = 0$ where $\star\lambda = x_n^k\xi \wedge dx_n + x_n^j\zeta$ such that the differential forms ξ and ζ do not contain dx_n and both of them should contain many of the coefficients $h_{m,r}(x)$. Hence, we get

$$0 = d\star\lambda + \iota_{X_M}\star\lambda = x_n^k d\xi \wedge dx_n + jx_n^{j-1}dx_n \wedge \zeta + x_n^j d\zeta + x_n^k \iota_{X_M}(\xi \wedge dx_n) + x_n^j \iota_{X_M}\zeta.$$

Reducing this equation modulo x_n^j and for the same reason above but replacing k by j , then we can infer that $k < j$, but this is a contradiction, then there are not such largest positive integer numbers k and j . Hence, the Taylor series for the coefficients $f_{I_1}(x)$ and $g_{I_2}(x)$ around $x_n = 0$ must be zero at $p \in \partial M$, i.e.

$$\frac{\partial^r f_{I_1}(x', 0)}{\partial x_n^r} = \frac{\partial^r g_{I_2}(x', 0)}{\partial x_n^r} = 0, \quad \forall r = 0, 1, 2, \dots$$

It means that all the higher partial derivatives of $f_{I_1}(x)$ and $g_{I_2}(x)$ we have already considered vanish at all points of the boundary ∂M . Thus, this facts are enough to show the mixed partial derivatives including x_n also vanish at the boundary. Hence, λ has a zero of infinite order at $p \in \partial M$.

The remaining possibility of one of the coefficients $f_{I_1}(x)$ and $g_{I_2}(x)$ having finite order and the other infinite order in x_n follows from the same argument as above.

Thus, λ satisfies all the hypotheses of the strong Unique Continuation Theorem 4.2.2 then λ must be zero on all of \mathbb{R}^n . Since M is assumed to be connected, λ must be identically zero on all of M , i.e. $\lambda \equiv 0$. \square

In addition, theorem 4.2.1 has not only an interest of its own, but is also crucial for creating the generalized boundary data which will be given in chapter 5 and possibly for solving various BVPS. However, it will be used in section 4.3 in order to refine the X_M -Hodge-Morrey-Friedrichs decompositions and to prove that the X_M -Poincaré duality angles are all acute.

4.3 X_M -cohomology in the style of DeTurck-Gluck

In this section, we define the interior and boundary portions of X_M -cohomology and then we force the main results of chapter 3 to be written in the style of DeTurck and Gluck by using a new different argument in the proofs of the main results in this section. In addition,

this new argument can also be used to prove the main original results of [15].

4.3.1 Refinement of the X_M -Hodge-Morrey-Friedrichs decomposition

Theorem 4.2.1 proves that the sum $\mathcal{H}_{X_M,N}^\pm(M) + \mathcal{H}_{X_M,D}^\pm(M)$ is a direct sum, and by using Green's formula (3.1), one finds that the orthogonal complement of $\mathcal{H}_{X_M,N}^\pm(M) + \mathcal{H}_{X_M,D}^\pm(M)$ inside $\mathcal{H}_{X_M}^\pm(M)$ is $\mathcal{H}_{X_M,\text{ex,co}}^\pm(M) = \mathcal{H}_{X_M,\text{ex}}^\pm(M) \cap \mathcal{H}_{X_M,\text{co}}^\pm(M)$. Therefore, we can refine the X_M -Friedrichs decomposition (theorem 3.3.14) into

$$\mathcal{H}_{X_M}^\pm(M) = (\mathcal{H}_{X_M,N}^\pm(M) + \mathcal{H}_{X_M,D}^\pm(M)) \oplus \mathcal{H}_{X_M,\text{ex,co}}^\pm(M).$$

Consequently, following DeTurck and Gluck's decomposition (2.6), we can refine the X_M -Hodge-Morrey-Friedrichs decompositions (Corollary 3.3.16) into the following five terms decomposition:

$$\Omega_G^\pm(M) = \mathcal{E}_{X_M}^\pm(M) \oplus \mathcal{C}_{X_M}^\pm(M) \oplus (\mathcal{H}_{X_M,N}^\pm(M) + \mathcal{H}_{X_M,D}^\pm(M)) \oplus \mathcal{H}_{X_M,\text{ex,co}}^\pm(M). \quad (4.2)$$

Here as usual, \oplus is an orthogonal direct sum, while $+$ is just a direct sum.

4.3.2 Interior and boundary portions and decomposition theorems

The non-orthogonality of $\mathcal{H}_{X_M,N}^\pm(M)$ and $\mathcal{H}_{X_M,D}^\pm(M)$ has to do that some of the X_M -cohomology of M comes from the "interior" of M and some comes from the boundary.

Since the vector field X_M which we are considering is always tangent to the boundary ∂M then we can still define X_M -cohomology on ∂M , i.e. $H_{X_M}^\pm(\partial M)$ and also we have that d_{X_M} and i^* commute. So, we can define the following long exact sequence in X_M -cohomology of the topological pair $(M, \partial M)$ derived from the inclusion $i : \partial M \hookrightarrow M$,

$$\dots \xrightarrow{i^*} H_{X_M}^\mp(\partial M) \xrightarrow{\partial^*} H_{X_M}^\pm(M, \partial M) \xrightarrow{\rho^*} H_{X_M}^\pm(M) \xrightarrow{i^*} H_{X_M}^\pm(\partial M) \xrightarrow{\partial^*} H_{X_M}^\mp(M, \partial M) \longrightarrow \dots \quad (4.3)$$

where ∂^* is derived from d_{X_M} as follows: given an X_M -closed form λ on ∂M , let $\tilde{\lambda}$ be an extension form on M . Then $[d_{X_M}\tilde{\lambda}]_{(X_M,M,\partial M)}$ defines a well-defined element of $H_{X_M}^\pm(M, \partial M)$. The operator ρ^* is induced by the embedding of pairs $\rho : (M, \emptyset) \subset (M, \partial M)$. Thus, ρ^* is well-defined.

Hence, in this light we can define interior and boundary portions of the absolute and relative X_M -cohomology respectively,

$$\begin{aligned} IH_{X_M}^\pm(M) &= \ker[i^* : H_{X_M}^\pm(M) \rightarrow H_{X_M}^\pm(\partial M)] \\ BH_{X_M}^\pm(M, \partial M) &= \text{im}[\partial^* : H_{X_M}^\mp(\partial M) \rightarrow H_{X_M}^\pm(M, \partial M)]. \end{aligned} \quad (4.4)$$

These spaces are realized as X_M -harmonic fields through Theorem 3.3.18 (X_M -Hodge Isomorphism) as

$$\begin{aligned}\mathcal{IH}_{X_M,N}^\pm &= \{\omega \in \mathcal{H}_{X_M,N}^\pm(M) \mid i^* \omega = d_{X_M} \theta, \text{ for some } \theta \in \Omega_G^\mp(\partial M)\} \\ \mathcal{BH}_{X_M,D}^\pm &= \mathcal{H}_{X_M,D}^\pm(M) \cap \mathcal{H}_{X_M,\text{ex}}^\pm\end{aligned}$$

respectively. Now use the Hodge star operator to define two other spaces:

$$\begin{aligned}\mathcal{IH}_{X_M,D}^\pm &= \{\omega \in \mathcal{H}_{X_M,D}^\pm(M) : i^* \star \omega = d_{X_M} \kappa, \text{ for some } \kappa \in \Omega_G^{n-\mp}(\partial M)\} \\ \mathcal{BH}_{X_M,N}^\pm &= \mathcal{H}_{X_M,N}^\pm(M) \cap \mathcal{H}_{X_M,\text{co}}^\pm.\end{aligned}$$

Note that Hodge star maps boundary to boundary and interior to interior; it follows that, for example $\mathcal{BH}_{X_M,N}^\pm \cong \mathcal{BH}_{X_M,D}^{n-\pm}$.

The following theorem gives more information for the boundary subspaces, analogous to theorem 2.3.3.

Theorem 4.3.1 *The boundary subspace $\mathcal{BH}_{X_M,N}^\pm(M)$ is the largest subspace of $\mathcal{H}_{X_M,N}^\pm(M)$ orthogonal to all of $\mathcal{H}_{X_M,D}^\pm(M)$ while the boundary subspace $\mathcal{BH}_{X_M,D}^\pm(M)$ is the largest subspace of $\mathcal{H}_{X_M,D}^\pm(M)$ orthogonal to all of $\mathcal{H}_{X_M,N}^\pm(M)$.*

PROOF: The proof of the orthogonality follows immediately from the definition of these subspaces and from proposition 3.2.2 (Green's formula for d_{X_M} and δ_{X_M}).

Next, in theorem 3.3.14, we prove that

$$\mathcal{H}_{X_M,\text{co}}^\pm(M) = \mathcal{H}_{X_M,D}^\pm(M)^\perp \cap \mathcal{H}_{X_M}^\pm(M), \quad \mathcal{H}_{X_M,\text{ex}}^\pm(M) = \mathcal{H}_{X_M,N}^\pm(M)^\perp \cap \mathcal{H}_{X_M}^\pm(M)$$

and these relations are also true for the class of smooth invariant forms.

So, if a form $\omega \in \mathcal{H}_{X_M,N}^\pm(M) \subset \mathcal{H}_{X_M}^\pm(M)$ is orthogonal to all of $\mathcal{H}_{X_M,D}^\pm(M)$, i.e. $\omega \in \mathcal{H}_{X_M,D}^\pm(M)^\perp \cap \mathcal{H}_{X_M,N}^\pm(M) \subset \mathcal{H}_{X_M,\text{co}}^\pm(M)$, then $\omega \in \mathcal{H}_{X_M,\text{co}}^\pm(M) \cap \mathcal{H}_{X_M,N}^\pm(M)$ and therefore, $\omega \in \mathcal{BH}_{X_M,N}^\pm(M)$.

Likewise, if $\alpha \in \mathcal{H}_{X_M,D}^\pm(M) \subset \mathcal{H}_{X_M}^\pm(M)$ is orthogonal to all of $\mathcal{H}_{X_M,N}^\pm(M)$, then $\alpha \in \mathcal{H}_{X_M,\text{ex}}^\pm(M) \cap \mathcal{H}_{X_M,D}^\pm(M)$ and therefore, $\alpha \in \mathcal{BH}_{X_M,D}^\pm(M)$. \square

The main goal of this subsection is to prove the following theorem and to answer the question which is given in section 4.1.

Theorem 4.3.2 *Analogous to theorem 2.3.4, we have the orthogonal decompositions*

$$\begin{aligned}\mathcal{H}_{X_M,N}^\pm(M) &= \mathcal{IH}_{X_M,N}^\pm(M) \oplus \mathcal{BH}_{X_M,N}^\pm(M) \\ \mathcal{H}_{X_M,D}^k(M) &= \mathcal{BH}_{X_M,D}^\pm(M) \oplus \mathcal{IH}_{X_M,D}^\pm(M).\end{aligned}$$

Remark 4.3.3 The proof by DeTurck and Gluck of the analogous result uses the duality between de Rham cohomology and singular homology. However, we do not have such a result on M (see, subsection 3.4.2), so we give a direct proof involving only the cohomology—the same argument can be used to prove Deturck and Gluck’s original theorem (replacing d_{X_M} by d and \pm by k everywhere). In addition, an alternative argument will be given later using the localization to the fixed point set (corollary 3.4.4).

The direct proof of theorem 4.3.2 is as follows.

PROOF: The orthogonality of the right hand sides follows from Green’s formula (3.1). It follows that

$$\mathcal{IH}_{X_M, N}^{\pm} \oplus \mathcal{BH}_{X_M, N}^{\pm} \subset \mathcal{H}_{X_M, N}^{\pm}(M) \quad \text{and} \quad \mathcal{BH}_{X_M, D}^{\pm} \oplus \mathcal{IH}_{X_M, D}^{\pm} \subset \mathcal{H}_{X_M, D}^{\pm}(M). \quad (4.5)$$

Now reconsider the long exact sequence (4.3) in X_M -cohomology derived from the inclusion $i : \partial M \hookrightarrow M$

$$\dots \xrightarrow{i^*} H_{X_M}^{\mp}(\partial M) \xrightarrow{\partial^*} H_{X_M}^{\pm}(M, \partial M) \xrightarrow{\rho^*} H_{X_M}^{\pm}(M) \xrightarrow{i^*} H_{X_M}^{\pm}(\partial M) \xrightarrow{\partial^*} H_{X_M}^{\mp}(M, \partial M) \longrightarrow \dots$$

It follows from the exactness that

$$IH_{X_M}^{\pm}(M) = \text{im } \rho^*, \quad \text{and} \quad BH_{X_M}^{\pm}(M, \partial M) = \ker \rho^*.$$

Thus, $H_{X_M}^{\pm}(M, \partial M) \cong BH_{X_M}^{\pm}(M, \partial M) + IH_{X_M}^{\pm}(M)$, (direct sum) or equivalently

$$\mathcal{H}_{X_M, D}^{\pm} \cong \mathcal{BH}_{X_M, D}^{\pm} + \mathcal{IH}_{X_M, N}^{\pm}. \quad (4.6)$$

It follows from equations (4.5) and (4.6) that $\dim(\mathcal{IH}_{X_M, D}^{\pm}) \leq \dim(\mathcal{IH}_{X_M, N}^{\pm})$. However, the Hodge star operator identifies $\mathcal{IH}_{X_M, N}^{\pm}$ with $\mathcal{IH}_{X_M, D}^{n \mp}$ which implies that the inequality in dimensions is in fact an equality: for even n this is immediate, while for odd n one has the sequence,

$$\dim \mathcal{IH}_{X_M, D}^{+} \leq \dim \mathcal{IH}_{X_M, N}^{+} = \dim \mathcal{IH}_{X_M, D}^{-} \leq \dim \mathcal{IH}_{X_M, N}^{-} = \dim \mathcal{IH}_{X_M, D}^{+}$$

and the result follows. \square

4.3.3 Interior and boundary portions of equivariant cohomology

The first purpose of this subsection is to refine theorem 3.4.1 (Atiyah-Bott [8, Theorem 3.5]) while the second purpose is to use an alternative argument to give another proof

to theorem 4.3.2 (remark 4.3.3), based on the localization to the fixed point set (corollary 3.4.4).

To do the first aim, we first define the interior and boundary portions for absolute and relative equivariant cohomology respectively. For clarification, we denote in this subsection the following inclusion maps as follows: $i_{\partial M} : \partial M \hookrightarrow M$, $i_{\partial F} : \partial F \hookrightarrow F$, $j_{\partial F} : \partial F \hookrightarrow \partial M$ and $j_F : F \hookrightarrow M$. Similarly to sequence (4.3), there is a long exact sequence in equivariant cohomology of the topological pair $(M, \partial M)$ derived from the inclusion $i_{\partial M}$ but replacing the absolute (relative) X_M -cohomology groups by absolute (relative) equivariant cohomology groups and d_{X_M} by Cartan coboundary operator d_{eq} (see, equation (3.36)).

Hence, in this light we can define interior and boundary portions of the absolute and relative equivariant cohomology respectively,

$$\begin{aligned} IH_G^\pm(M) &= \ker[i_{\partial M}^* : H_G^\pm(M) \rightarrow H_G^\pm(\partial M)] \\ BH_G^\pm(M, \partial M) &= \text{im}[\partial^* : H_G^\mp(\partial M) \rightarrow H_G^\pm(M, \partial M)]. \end{aligned} \quad (4.7)$$

Here, ∂^* is derived from the d_{eq} as described above. Hence, we refine the localization theorem of Atiyah-Bott into the following theorem

Theorem 4.3.4 *The inclusion $j_F : F \hookrightarrow M$ induces isomorphisms of R_f -modules*

$$\begin{aligned} IH_G^\pm(M)_f &\xrightarrow{j_F^*} IH_G^\pm(F)_f \\ BH_G^\pm(M, \partial M)_f &\xrightarrow{j_F^*} BH_G^\pm(F, \partial F)_f \end{aligned}$$

where $f \in I(Z)$. Moreover, $IH_G^\pm(F)_f \cong IH^\pm(F) \otimes R_f$ and $BH_G^\pm(F, \partial F)_f \cong BH^\pm(F) \otimes R_f$.

PROOF: Since j_F^* is bijection (theorem 3.4.1), then the definition of the interior and boundary portions of the absolute and relative equivariant cohomology (eq. (4.7)) imply the isomorphisms above as desired. \square

Since localization does not alter the rank of a module, we have that

$$\text{rank } IH_G^\pm(M) = \dim IH^\pm(F), \quad \text{rank } BH_G^\pm(M, \partial M) = \dim BH^\pm(F, \partial F).$$

Now, to satisfy the second purpose of this subsection, we first refine corollary 3.4.4 into the following corollary 4.3.5 which makes the following natural extension of corollary 3.4.4 possible.

Corollary 4.3.5 *Let $F' = N(X_M)$, then $j_{F'}$ induces the isomorphisms,*

$$1- IH_{X_M}^\pm(M) \cong IH^\pm(F'), \quad H_{X_M}^\pm(M)/IH_{X_M}^\pm(M) \cong H^\pm(F')/IH^\pm(F'),$$

2- $BH_{X_M}^\pm(M, \partial M) \cong BH^\pm(F', \partial F')$, $H_{X_M}^\pm(M, \partial M)/BH_{X_M}^\pm(M, \partial M) \cong H^\pm(F', \partial F')/BH^\pm(F', \partial F')$.

Moreover, if $N(X_M) = F$ then $\dim IH_{X_M}^\pm(M) = \text{rank } IH_G^\pm(M)$ and $\dim BH_{X_M}^\pm(M, \partial M) = \text{rank } BH_G^\pm(M, \partial M)$

PROOF: The following square is commutative,

$$\begin{array}{ccc} H_{X_M}^\pm(M) & \xrightarrow{j_{F'}^*} & H^\pm(F') \\ \downarrow i_{\partial M}^* & & \downarrow i_{\partial F'}^* \\ H_{X_M}^\pm(\partial M) & \xrightarrow{j_{\partial F'}^*} & H^\pm(\partial F') \end{array} \quad (4.8)$$

because of $j_{F'} \circ i_{\partial F'} = i_{\partial M} \circ j_{\partial F'}$. But, $j_{F'}^*$ and $j_{\partial F'}^*$ are bijections (corollary 3.4.4). Hence, the commutativity of diagram 4.8 and corollary 3.4.4 prove that $j_{F'}^*$ moves the interior portion of absolute X_M -cohomology of M onto the interior portion of the absolute de Rham cohomology of F' and the boundary portion of the relative X_M -cohomology of M onto the boundary portion of the relative de Rham cohomology of F' , and the result follows as desired. \square

In this light, using corollary 4.3.5, we can neatly break down the isomorphisms $\mathcal{H}_{X_M, N}^\pm(M) \cong \mathcal{H}_N^\pm(F')$ and $\mathcal{H}_{X_M, D}^\pm(M) \cong \mathcal{H}_D^\pm(F')$ (these isomorphisms follow from Theorem 3.3.18 (X_M -Hodge Isomorphism) and corollary 3.4.4) into the following more precise ones:

Theorem 4.3.6 *Let $F' = N(X_M)$. We have isomorphisms,*

$$\begin{aligned} \mathcal{IH}_{X_M, N}^\pm(M) &\cong \mathcal{IH}_N^\pm(F'), & \mathcal{BH}_{X_M, D}^\pm(M) &\cong \mathcal{BH}_D^\pm(F'), \\ \mathcal{IH}_{X_M, D}^\pm(M) &\cong \mathcal{IH}_D^\pm(F'), & \mathcal{BH}_{X_M, N}^\pm(M) &\cong \mathcal{BH}_N^\pm(F'). \end{aligned}$$

PROOF: We prove the first two; the other two follow by applying the Hodge star operator (on M and on F'). $j_{F'} : F' \hookrightarrow M$ induces a chain map between the long exact sequences of X_M -cohomology on M and de Rham cohomology on F' , which by corollary 3.4.4 is an isomorphism.

But, corollary 4.3.5 proves that $j_{F'}$ induces isomorphisms

$$IH_{X_M}^\pm(M) \cong IH^\pm(F'), \quad \text{and} \quad BH_{X_M}^\pm(M, \partial M) \cong BH^\pm(F', \partial F').$$

It then follows from the Theorem 3.3.18 (X_M -Hodge Isomorphism) and corollary 4.3.5 that there are isomorphisms $\mathcal{IH}_{X_M, N}^\pm(M) \cong \mathcal{IH}_N^\pm(F')$ and $\mathcal{BH}_{X_M, D}^\pm(M) \cong \mathcal{BH}_D^\pm(F')$. \square

Now, based on the results above, we can give another proof of theorem 4.3.2. We just prove $\mathcal{H}_{X_M, N}^\pm(M) = \mathcal{IH}_{X_M, N}^\pm(M) \oplus \mathcal{BH}_{X_M, N}^\pm(M)$ while the other one follows by applying

the Hodge star operator.

Green's formula (3.1) and theorem 4.3.6 give the following orthogonal direct sum

$$\mathcal{IH}_{X_M, N}^{\pm}(M) \oplus \mathcal{BH}_{X_M, N}^{\pm}(M) \cong \mathcal{IH}_N^{\pm}(F') \oplus \mathcal{BH}_N^{\pm}(F')$$

but, theorem 2.3.4 asserts that $\mathcal{H}_N^{\pm}(F') = \mathcal{IH}_N^{\pm}(F') \oplus \mathcal{BH}_N^{\pm}(F')$ while Theorem 3.3.18 (X_M -Hodge Isomorphism) and corollary 3.4.4 imply $\mathcal{H}_{X_M, N}^{\pm}(M) \cong \mathcal{H}_N^{\pm}(F')$. Hence, we prove the isomorphism

$$\mathcal{H}_{X_M, N}^{\pm}(M) \cong \mathcal{IH}_{X_M, N}^{\pm}(M) \oplus \mathcal{BH}_{X_M, N}^{\pm}(M).$$

Thus, this isomorphism together with the fact that $\mathcal{IH}_{X_M, N}^{\pm}(M) \oplus \mathcal{BH}_{X_M, N}^{\pm}(M) \subset \mathcal{H}_{X_M, N}^{\pm}(M)$ imply the equality of $\mathcal{H}_{X_M, N}^{\pm}(M) = \mathcal{IH}_{X_M, N}^{\pm}(M) \oplus \mathcal{BH}_{X_M, N}^{\pm}(M)$, as desired.

The analogue of Gluck and DeTurck's theorem for the Poincaré duality angles (theorem 2.3.5) also holds. The X_M -Poincaré duality angles are defined in the obvious way, as the principal angles between $\mathcal{IH}_{X_M, D}^{\pm}$ and $\mathcal{IH}_{X_M, N}^{\pm}$.

Proposition 4.3.7 *The X_M -Poincaré duality angles are all acute.*

PROOF: These angles can be neither 0 nor $\pi/2$, firstly because $\mathcal{H}_{X_M, N}^{\pm}(M) \cap \mathcal{H}_{X_M, D}^{\pm}(M) = \{0\}$ (Theorem 4.2.1), and secondly because of theorem 4.3.1. Hence they must all be acute. \square

4.4 Conclusions and geometric open problem

- 1- Recalling, the generalization of Witten's results (section 3.5, No.(3)), we have that theorem 3.5.1 and eq. (3.43) can be refined to the style of interior and boundary portions (corollary 4.3.5 and theorem 4.3.6 respectively) and this gives a more precise meaning for these isomorphisms.

Moreover, if $\partial N(K) = \emptyset$ then there is no boundary part of the cohomology of $N(K)$. In other words, it means that all the de Rham cohomology of $N(K)$ must come only from the interior portion, i.e. $H^{\pm}(N(K)) = H^{\pm}(N(K), \partial N(K))$, which shows that every interior de Rham cohomology class on $N(K)$ has corresponding to it both a unique relative and a unique absolute K -cohomology class on M .

- 2- The X_M -Poincaré duality angles are invariants of the Riemannian structure on M and they do not depend on the group action on M (see, remark 3.3.20). In addition, this provides that the Witten-Hodge theory gives additional equivariant geometric insight rather than the topological insight.

Geometric question: Finally, we proved that $\mathcal{IH}_{X_M, N}^\pm(M) \cong \mathcal{IH}_N^\pm(N(X_M))$ and $\mathcal{IH}_{X_M, D}^\pm(M) \cong \mathcal{IH}_D^\pm(N(X_M))$ and that the principal angles between the corresponding interior subspaces are all acute. Hence, it would be interesting to answer the following:

How do the X_M -Poincaré duality angles between the interior subspaces $\mathcal{IH}_{X_M, N}^\pm(M)$ and $\mathcal{IH}_{X_M, D}^\pm(M)$ depend on X , and how do they compare to the Poincaré duality angles between the interior subspaces $\mathcal{IH}_N^\pm(N(X_M))$ and $\mathcal{IH}_D^\pm(N(X_M))$?

Chapter 5

Generalized DN-operator on invariant differential forms

5.1 Introduction

The construction of the X_M -Hodge-Morrey-Friedrichs decompositions (corollary 3.3.16) of smooth invariant differential forms gives us insight to create boundary data which is a generalization of Belishev-Sharafutdinov's boundary data on $\Omega_G^\pm(\partial M)$.

In this chapter, we give an answer to the topological open problem which states,

“To what extent is the equivariant topology of a manifold determined by the DN or a variant map”?

So, we take a more topological approach, looking to determine the absolute and relative X_M -cohomology groups and consequently the free part of the absolute and relative equivariant cohomology groups from the generalized boundary data which we set up in this chapter. To this end, we first in section 5.2 prove fundamental results which follow from Theorem 4.2.1 while in section 5.3 we define the X_M -DN operator Λ_{X_M} on $\Omega_G^\pm(\partial M)$, the definition involves showing that certain boundary value problems are solvable. In fact, the definition of Λ_{X_M} represents a generalization of Belishev-Sharafutdinov's DN-operator Λ on $\Omega_G^\pm(\partial M)$ in the sense that when $X = 0$, we have $\Lambda_0 = \Lambda$. Finally, in the remaining sections, we explain to what extent the equivariant topology of the manifold in question is determined by Λ_{X_M} . The results of this chapter are given as well in [3].

5.2 Preparing to the generalized boundary data

Theorem 4.2.1 plays a fundamental role to obtain the following results.

Corollary 5.2.1

$$\mathcal{H}_{X_M}^\pm(M) = \mathcal{H}_{X_M, \text{ex}}^\pm(M) + \mathcal{H}_{X_M, \text{co}}^\pm(M) \quad (5.1)$$

where “+” is not direct sum.

PROOF: The X_M -Friedrichs Decomposition Theorem 3.3.14 shows the following intersections: $(\mathcal{H}_{X_M,D}^\pm(M))^\perp \cap \mathcal{H}_{X_M}^\pm(M) = \mathcal{H}_{X_M,\text{co}}^\pm(M)$ and $(\mathcal{H}_{X_M,N}^\pm(M))^\perp \cap \mathcal{H}_{X_M}^\pm(M) = \mathcal{H}_{X_M,\text{ex}}^\pm(M)$. Hence, using these facts together with Theorem 4.2.1, we conclude eq.(5.1). \square

Corollary 5.2.2 *The trace map $i^* : \mathcal{H}_{X_M,N}^\pm(M) \longrightarrow i^*\mathcal{H}_{X_M,N}^\pm(M)$ defines an isomorphism.*

PROOF: It is clear that i^* is surjective and Theorem 4.2.1 implies the kernel of the linear map i^* is zero (i.e. $\ker i^* = \{0\}$) which means that i^* is injective. Thus, i^* is a bijection. \square

Corollary 5.2.3 1- *The map $f : i^*\mathcal{H}_{X_M,N}^\pm(M) \longrightarrow H_{X_M}^\pm(M)$ defined by $f(i^*\lambda_N) = [\lambda_N]_{(X_M,M)}$ for $\lambda_N \in \mathcal{H}_{X_M,N}^\pm(M)$ is an isomorphism.*

2- *The map $h : i^*\mathcal{H}_{X_M,N}^{n-\pm}(M) \longrightarrow H_{X_M}^\pm(M, \partial M)$ defined by $h(i^*\lambda_N) = [\star\lambda_N]_{(X_M,M,\partial M)}$ for $\lambda_N \in \mathcal{H}_{X_M,N}^{n-\pm}(M)$ is an isomorphism.*

PROOF:

1- f is a well-defined map because $\ker i^* = \{0\}$ (corollary 5.2.2). Furthermore, f is a bijection because there exists a unique Neumann X_M -harmonic field in any absolute X_M -cohomology class (Theorem 3.3.18) hence part (1) holds.

2- It follows from part (1) by using X_M -Poincaré-Lefschetz duality (Theorem 3.3.18(c)). \square

Corollary 5.2.4 $\dim(\mathcal{H}_{X_M,N}^\pm(M)) = \dim(i^*\mathcal{H}_{X_M,N}^\pm(M)) = \dim(H_{X_M}^\pm(M)) = \dim(H_{X_M}^{n-\pm}(M, \partial M))$.

As we state in section 3.5 No. (5) that Witten-Hodge theory can be used to obtain the solvability of various BVPS. Therefore, in this chapter we will need the following theorem which can be proved by using the X_M -Hodge-Morrey-Friedrichs decompositions.

Theorem 5.2.5 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M . Given $\chi, \rho \in \Omega_G^\mp(M)$ and $\psi \in \Omega_G^\pm(\partial M)$, the boundary value problem*

$$\begin{cases} d_{X_M} \omega = \chi & \text{on } M \\ \delta_{X_M} \omega = \rho & \text{on } M \\ i^* \omega = \psi & \text{on } \partial M \end{cases} \quad (5.2)$$

is solvable for $\omega \in \Omega_G^\pm(M)$ if and only if the data obey the integrability conditions

$$\delta_{X_M} \rho = 0, \quad \langle \rho, \kappa \rangle = 0, \quad \forall \kappa \in \mathcal{H}_{X_M, D}^\mp(M) \quad (5.3)$$

and

$$d_{X_M} \chi = 0, \quad i^* \chi = d_{X_M} \psi, \quad \langle \chi, \kappa \rangle = \int_{\partial M} \psi \wedge i^* \star \kappa, \quad \forall \kappa \in \mathcal{H}_{X_M, D}^\mp(M). \quad (5.4)$$

The solution of eq.(5.2) is unique up to arbitrary Dirichlet X_M -harmonic fields $\kappa \in \mathcal{H}_{X_M, D}^\pm(M)$.

PROOF: The proof is identical to the proof of theorem 3.2.5 of [31] but replacing d by d_{X_M} and δ by δ_{X_M} . \square

Now, let us prove the following lemma which will be used later.

Lemma 5.2.6

$$i^* \mathcal{H}_{X_M}^\pm(M) = \mathcal{E}_{X_M}^\pm(\partial M) + i^* \mathcal{H}_{X_M, N}^\pm(M) \quad (5.5)$$

where $\mathcal{E}_{X_M}^\pm(\partial M) = \{d_{X_M} \alpha \mid \alpha \in \Omega_G^\mp(\partial M)\}$.

PROOF: We first prove that, $i^* \mathcal{H}_{X_M}^\pm(M) \subseteq \mathcal{E}_{X_M}^\pm(\partial M) + i^* \mathcal{H}_{X_M, N}^\pm(M)$.

Suppose $\lambda \in \mathcal{H}_{X_M}^\pm(M)$ then the X_M -Friedrichs Decomposition (3.29) implies that

$$\lambda = d_{X_M} \alpha + \lambda_N \in \mathcal{H}_{X_M, \text{ex}}^\pm(M) \oplus \mathcal{H}_{X_M, N}^\pm(M).$$

Hence,

$$i^* \lambda = d_{X_M} i^* \alpha + i^* \lambda_N.$$

Conversely, it is clear that $i^* \mathcal{H}_{X_M, N}^\pm(M) \subseteq i^* \mathcal{H}_{X_M}^\pm(M)$. So, we only need to prove that $\mathcal{E}_{X_M}^\pm(\partial M) \subseteq i^* \mathcal{H}_{X_M}^\pm(M)$. Suppose, $\eta = d_{X_M} \alpha \in \mathcal{E}_{X_M}^\pm(\partial M)$ then η satisfies

$$d_{X_M} \eta = 0, \quad \int_{\partial M} d_{X_M} \alpha \wedge i^* \star \kappa = 0, \quad \forall \kappa \in \mathcal{H}_{X_M, D}^\mp(M). \quad (5.6)$$

Clearly, theorem 5.2.5 asserts that the condition (5.6) is a necessary and sufficient condition for the existence of $\lambda \in \mathcal{H}_{X_M}^\pm(M)$ such that $\eta = i^* \lambda$. \square

5.3 X_M -DN operator

Before defining this operator, we first need to prove the solvability of a certain boundary value problem (5.7). The proof depends on the main results in chapter 3 and there is not any corresponding statement of it in [31]. When $X = 0$, this gives an independent proof of the

solvability of Belishev-Sharafutdinov's BVP (2.8). Theorem 5.3.1 represents the keystone to defining the X_M -DN operator and then to exploiting a connection between this X_M -DN operator and X_M -cohomology via the Neumann X_M -trace space $i^*\mathcal{H}_{X_M,N}^\pm(M)$.

Theorem 5.3.1 *Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M . Given $\theta \in \Omega_G^\pm(\partial M)$ and $\eta \in \Omega_G^\pm(M)$, then the BVP*

$$\begin{cases} \Delta_{X_M} \omega = \eta & \text{on } M \\ i^* \omega = \theta & \text{on } \partial M \\ i^*(\delta_{X_M} \omega) = 0 & \text{on } \partial M. \end{cases} \quad (5.7)$$

is solvable for $\omega \in \Omega_G^\pm(M)$ if and only if

$$\langle \eta, \kappa_D \rangle = 0, \quad \forall \kappa_D \in \mathcal{H}_{X_M,D}^\pm(M) \quad (5.8)$$

The solution of BVP (5.7) is unique up to an arbitrary Dirichlet X_M -harmonic field $\mathcal{H}_{X_M,D}^\pm(M)$.

PROOF: Suppose eq.(5.7) has a solution then one can easily show that the condition (5.8) holds by using Green's formula (3.1).

Now, suppose the condition $\langle \eta, \kappa_D \rangle = 0, \quad \forall \kappa_D \in \mathcal{H}_{X_M,D}^\pm(M)$ is given (i.e. $\eta \in \mathcal{H}_{X_M,D}^\pm(M)^\perp$). Since $\theta \in \Omega_G^\pm(\partial M)$, we can construct an extension $\omega_1 \in \Omega_G^\pm(M)$ of the differential form $\theta \in \Omega_G^\pm(\partial M)$ such that

$$i^* \omega_1 = \theta, \quad \omega_1 = \delta_{X_M} \beta_{\omega_1} + \lambda_{\omega_1} \in \mathcal{C}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M}^\pm(M).$$

But $\Delta_{X_M} \omega_1 = \delta_{X_M} d_{X_M} \delta_{X_M} \beta_{\omega_1}$, then (3.1) implies that $\Delta_{X_M} \omega_1 \in \mathcal{H}_{X_M,D}^\pm(M)^\perp$ as well. Hence, $\eta - \Delta_{X_M} \omega_1 \in \mathcal{H}_{X_M,D}^\pm(M)^\perp$. We now apply proposition 3.3.9 which for smooth invariant forms states that for each $\bar{\eta} \in \mathcal{H}_{X_M,D}^\pm(M)^\perp$ there is a unique smooth differential form $\bar{\omega} \in \Omega_{G,D}^\pm \cap \mathcal{H}_{X_M,D}^\pm(M)^\perp$ satisfying the BVP (5.7) but with $\eta = \bar{\eta}$ and $\theta = 0$. Since $\eta - \Delta_{X_M} \omega_1 \in \mathcal{H}_{X_M,D}^\pm(M)^\perp$ is smooth, it follows from this there is a unique smooth differential form $\omega_2 \in \Omega_{G,D}^\pm \cap \mathcal{H}_{X_M,D}^\pm(M)^\perp$ which satisfies the BVP

$$\begin{cases} \Delta_{X_M} \omega_2 = \eta - \Delta_{X_M} \omega_1 & \text{on } M \\ i^* \omega_2 = 0 & \text{on } \partial M \\ i^*(\delta_{X_M} \omega_2) = 0 & \text{on } \partial M. \end{cases} \quad (5.9)$$

Now, let $\omega_2 = \omega - \omega_1$, then the BVP (5.9) turns into the BVP (5.7). Hence, there exists a solution to the BVP (5.7) which is $\omega = \omega_1 + \omega_2$, where the uniqueness of ω is up to an arbitrary Dirichlet X_M -harmonic fields. \square

Definition 5.3.2 (X_M -DN operator Λ_{X_M}) Let M be the manifold in question. We consider the BVP (5.7) with $\eta = 0$, i.e.

$$\left\{ \begin{array}{l} \Delta_{X_M} \omega = 0 \quad \text{on } M \\ i^* \omega = \theta \quad \text{on } \partial M \\ i^*(\delta_{X_M} \omega) = 0 \quad \text{on } \partial M \end{array} \right. \quad (5.10)$$

then the BVP (5.10) is solvable and the solution is unique up to an arbitrary Dirichlet X_M -harmonic field $\kappa_D \in \mathcal{H}_{X_M, D}^\pm(M)$ (Theorem 5.3.1). We can therefore define the X_M -DN operator $\Lambda_{X_M} : \Omega_G^\pm(\partial M) \longrightarrow \Omega_G^{n-(\mp)}(\partial M)$ by

$$\Lambda_{X_M} \theta = i^*(\star d_{X_M} \omega).$$

Note that taking $d_{X_M} \omega$ eliminates the ambiguity in the choice of the solution ω which means $\Lambda_{X_M} \theta$ is well defined.

In the case of $X = 0$, definition (5.3.2) reduces to the definition of the DN-operator Λ given by Belishev and Sharafutdinov [11].

The results above and those in chapter 3 provide the basic ingredients needed to extend (by analogy) the results in [11] and some of the results in [32] (on the ring structure) to the context of X_M -cohomology and the X_M -DN map. However, some results in sections 5.4 and 5.6 are different and are specified here.

Lemma 5.3.3 Let $\omega \in \Omega_G^\pm(M)$ be a solution to the BVP (5.10) where $\theta \in \Omega_G^\pm(\partial M)$ is given. Then $d_{X_M} \omega \in \mathcal{H}_{X_M}^\mp(M)$ and $\delta_{X_M} \omega = 0$.

PROOF: Since d_{X_M} commutes with i^* and Δ_{X_M} then the BVP (5.10) and the fact that $\Lambda_{X_M} \theta = i^*(\star d_{X_M} \omega)$ show that $d_{X_M} \omega$ solves the BVP

$$\Delta_{X_M} d_{X_M} \omega = 0, \quad i^*(\star d_{X_M}^2 \omega) = 0, \quad i^*(\delta_{X_M} d_{X_M} \omega) = 0.$$

But proposition 3.3.2(4) implies that $d_{X_M} \omega \in \mathcal{H}_{X_M}^\mp(M)$.

Since $d_{X_M} \omega \in \mathcal{H}_{X_M}^\mp(M)$, one can easily verify that $d_{X_M} \delta_{X_M} \omega = -\delta_{X_M} d_{X_M} \omega = 0$ and $\delta_{X_M}^2 \omega = 0$ which means that $\delta_{X_M} \omega \in \mathcal{H}_{X_M, \text{co}}^\pm(M)$ but the second condition (i.e. $i^*(\delta_{X_M} \omega) = 0$) of the BVP (5.10) gives that $\delta_{X_M} \omega \in \mathcal{H}_{X_M, D}^\pm(M)$. Using (3.28), this then implies that $\delta_{X_M} \omega \in \mathcal{H}_{X_M, D}^\pm(M) \cap \mathcal{H}_{X_M, \text{co}}^\pm(M) = \{0\}$, i.e. $\delta_{X_M} \omega = 0$. \square

Lemma 5.3.4 *The operator Λ_{X_M} is nonnegative in the sense that the integral*

$$\int_{\partial M} \theta \wedge \Lambda_{X_M} \theta$$

is nonnegative for any $\theta \in \Omega_G^\pm(\partial M)$.

PROOF: For given θ , let $\omega \in \Omega_G^\pm(M)$ be a solution to the BVP (5.10). Then it follows from (3.1) that

$$0 = \langle \Lambda_{X_M} \omega, \omega \rangle = \langle d_{X_M} \omega, d_{X_M} \omega \rangle + \langle \delta_{X_M} \omega, \delta_{X_M} \omega \rangle - \int_{\partial M} i^* \omega \wedge i^* (\star d_{X_M} \omega)$$

whence

$$\int_{\partial M} \theta \wedge \Lambda_{X_M} \theta = \|d_{X_M} \omega\|^2 + \|\delta_{X_M} \omega\|^2 \geq 0. \quad (5.11)$$

□

Lemma 5.3.5

$$\ker \Lambda_{X_M} = \text{Ran } \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$$

where $\mathcal{H}_{X_M} = \mathcal{H}_{X_M}^+ \oplus \mathcal{H}_{X_M}^-$.

PROOF: We first prove that $\ker \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$. If $\theta = i^* \lambda \in i^* \mathcal{H}_{X_M}(M)$ for $\lambda \in \mathcal{H}_{X_M}(M)$, then λ is a solution to the BVP (5.10). But $d_{X_M} \lambda = \delta_{X_M} \lambda = 0$, therefore $\Lambda_{X_M} \theta = i^* (\star d_{X_M} \lambda) = 0$. Conversely, if $\theta \in \ker \Lambda_{X_M}$ and λ is a solution to the BVP (5.10) then $\theta = i^* \lambda$ and equation (5.11) implies that $d_{X_M} \lambda = \delta_{X_M} \lambda = 0$. i.e. $\theta = i^* \lambda \in i^* \mathcal{H}_{X_M}(M)$. Hence, $\ker \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$.

Now, to prove $\text{Ran } \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$, suppose $\phi \in \text{Ran } \Lambda_{X_M}$ then $\phi = \Lambda_{X_M} \theta$ where $\theta = i^* \lambda$ such that λ is a solution of the BVP (5.10). But, $d_{X_M} \lambda \in \mathcal{H}_{X_M}(M)$ (Lemma 5.3.3) then $\star d_{X_M} \lambda \in \mathcal{H}_{X_M}(M)$ too. Hence, $\phi = \Lambda_{X_M} \theta = i^* (\star d_{X_M} \lambda) \in i^* \mathcal{H}_{X_M}(M)$. Conversely, let $\phi = i^* \lambda \in i^* \mathcal{H}_{X_M}(M)$, i.e. $\lambda \in \mathcal{H}_{X_M}(M)$. Applying, the X_M -Friedrichs Decomposition (3.29), we can decompose $\star \lambda$ as

$$\star \lambda = d_{X_M} \omega + \lambda_N \in \mathcal{H}_{X_M, \text{ex}}(M) \oplus \mathcal{H}_{X_M, N}(M). \quad (5.12)$$

Remark 3.3.15 asserts that ω can be chosen such that

$$\Delta_{X_M} \omega = 0, \quad \delta_{X_M} \omega = 0$$

which implies that

$$\Lambda_{X_M} i^* \omega = i^* (\star d_{X_M} \omega).$$

We can obtain from eq. (5.12) that

$$i^*(\star d_{X_M} \omega) = \pm i^* \lambda.$$

Comparing the last two equation with $\phi = i^* \lambda$, we obtain $\phi = \Lambda_{X_M}(\pm i^* \omega) \in \text{Ran } \Lambda_{X_M}$. \square

Corollary 5.3.6 *The operator Λ_{X_M} satisfies the following relations:*

$$\Lambda_{X_M} d_{X_M} = 0, \quad d_{X_M} \Lambda_{X_M} = 0, \quad \Lambda_{X_M}^2 = 0. \quad (5.13)$$

PROOF: The first relation of (5.13) means that any form in the space $\mathcal{E}_{X_M}(\partial M)$ is the trace of an X_M -harmonic field which is true by $\mathcal{E}_{X_M}(\partial M) \subseteq i^* \mathcal{H}_{X_M}(M) = \ker \Lambda_{X_M}$ (lemma 5.2.6 and lemma 5.3.5) while the second and third of equalities (5.13) follow from Lemma 5.3.5. \square

In this corollary, we introduce the X_M -Hilbert transform T_{X_M} which is of course the analogue of the usual Hilbert transform (see section 5 in [11]) and it will be used in section 5.5.

Corollary 5.3.7 *The operator $T_{X_M} := d_{X_M} \Lambda_{X_M}^{-1} : i^* \mathcal{H}_{X_M}(M) \longrightarrow i^* \mathcal{H}_{X_M}(M)$ is well-defined ; i.e. the equation $\phi = \Lambda_{X_M} \theta$ has a solution θ for any $\phi \in i^* \mathcal{H}_{X_M}(M)$, and $d_{X_M} \theta$ is uniquely determined by $\phi = \Lambda_{X_M} \theta$. In particular, the operator $d_{X_M} \Lambda_{X_M}^{-1} d_{X_M} : \Omega_G(\partial M) \longrightarrow \Omega_G(\partial M)$ is well-defined.*

PROOF: Lemma 5.3.5 proves that $\text{Ran } \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$. Hence, if $\phi \in i^* \mathcal{H}_{X_M}(M)$ then the equation $\phi = \Lambda_{X_M} \theta$ is solvable. If $\Lambda_{X_M} \theta_1 = \Lambda_{X_M} \theta_2$ then $\theta_1 - \theta_2 \in \ker \Lambda_{X_M}$ is X_M -closed (i.e. $d_{X_M}(\theta_1 - \theta_2) = 0$) because $\ker \Lambda_{X_M} = i^* \mathcal{H}_{X_M}(M)$. Thus, $d_{X_M} \theta_1 = d_{X_M} \theta_2$ which means that $d_{X_M} \theta$ is uniquely determined by $\phi = \Lambda_{X_M} \theta$. Clearly, the operator $d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}$ is well-defined because we have shown in lemma 5.2.6 that $\mathcal{E}_{X_M}(\partial M) \subseteq i^* \mathcal{H}_{X_M}(M)$. \square

Remark 5.3.8 In the case of $X = 0$, the definition of X_M -Hilbert transform T_{X_M} reduces to the definition of the generalized Hilbert transform which is given in chapter 2 (section 2.4). The above construction enables one to extend theorem 2.4.2 to the context of Λ_{X_M} but we leave this for future work.

The above constructions provide the essential ingredients needed to extend theorem 4.2 of [11] (our theorem 2.4.1) to the present context:

Theorem 5.3.9 *The Neumann X_M -trace space $i^*\mathcal{H}_{X_M,N}^{n-(\mp)}(M)$ can be completely determined from our boundary data $(\partial M, \Lambda_{X_M})$ in particular,*

$$(\Lambda_{X_M} - (\mp 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \Omega_G^\pm(\partial M) = i^* \mathcal{H}_{X_M,N}^{n-(\mp)}(M) \quad (5.14)$$

PROOF: We need first to prove that

$$(\Lambda_{X_M} - (\mp 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \Omega_G^\pm(\partial M) \subseteq i^* \mathcal{H}_{X_M,N}^{n-(\mp)}(M).$$

Suppose $\theta \in \Omega_G^\pm(\partial M)$, let $\omega \in \Omega_G^\pm(M)$ be a solution to the BVP (5.10). Lemma (5.3.3) proves that $d_{X_M} \omega \in \mathcal{H}_{X_M}^\mp(M)$. Applying the X_M -Friedrichs decomposition to $d_{X_M} \omega$, we get

$$d_{X_M} \omega = \delta_{X_M} \alpha + \lambda_D \in \mathcal{H}_{X_M,co}^\mp(M) \oplus \mathcal{H}_{X_M,D}^\mp(M) \quad (5.15)$$

where $\alpha \in \Omega_G^\pm(M)$ and by remark 3.3.15, α can be chosen such that

$$d_{X_M} \alpha = 0, \quad \Delta_{X_M} \alpha = 0. \quad (5.16)$$

We set $\beta = \star \alpha \in \Omega_G^{n-\pm}(M)$. Hence, eq.(5.16) implies

$$\delta_{X_M} \beta = 0, \quad \Delta_{X_M} \beta = 0. \quad (5.17)$$

Substituting $\alpha = (\pm 1)^{n+1} \star \beta$ into eq.(5.15), we have

$$d_{X_M} \omega = (\pm 1)^{n+1} \delta_{X_M} \star \beta + \lambda_D \quad (5.18)$$

which implies

$$i^*(d_{X_M} \omega) = (\pm 1)^{n+1} i^*(\delta_{X_M} \star \beta). \quad (5.19)$$

But, $i^*(d_{X_M} \omega) = d_{X_M}(i^* \omega) = d_{X_M} \theta$ and $\delta_{X_M} \star \beta = \mp(-1)^n \star d_{X_M} \beta$, thus, eq.(5.19) turns into

$$d_{X_M} \theta = -(\mp 1)^n i^*(\star d_{X_M} \beta) \quad (5.20)$$

Formulas (5.17) and (5.20) mean that

$$d_{X_M} \theta = -(\mp 1)^n \Lambda_{X_M} i^* \beta. \quad (5.21)$$

Now, applying, $(i^* \star)$ to eq.(5.18) with the fact that $\Lambda_{X_M} \theta = i^*(\star d_{X_M} \omega)$, we get

$$\Lambda_{X_M} \theta = (\pm 1)^{n+1} i^*(\star \delta_{X_M} \star \beta) + i^*(\star \lambda_D). \quad (5.22)$$

Using the relation $\star\delta_{X_M}\star\beta = (\pm 1)^n d_{X_M}\beta$, then eq.(5.22) reduces to

$$\Lambda_{X_M}\theta = \pm d_{X_M}(i^*\beta) + i^*(\star\lambda_D) \quad (5.23)$$

we can obtain from eq.(5.21) that

$$d_{X_M}(i^*\beta) = -(\mp 1)^n d_{X_M}\Lambda_{X_M}^{-1}d_{X_M}\theta.$$

Putting the latter equation in eq.(5.23), we get

$$i^*(\star\lambda_D) = (\Lambda_{X_M} - (\mp 1)^{n+1}d_{X_M}\Lambda_{X_M}^{-1}d_{X_M})\theta.$$

Hence, $(\Lambda_{X_M} - (\mp 1)^{n+1}d_{X_M}\Lambda_{X_M}^{-1}d_{X_M})\theta \in i^*\mathcal{H}_{X_M,N}^{n-(\mp)}(M)$.

The next step is then to prove the converse, i.e.

$$i^*\mathcal{H}_{X_M,N}^{n-(\mp)}(M) \subseteq (\Lambda_{X_M} - (\mp 1)^{n+1}d_{X_M}\Lambda_{X_M}^{-1}d_{X_M})\Omega_G^\pm(\partial M).$$

Given $\lambda_N \in \mathcal{H}_{X_M,N}^{n-(\mp)}(M)$, then corollary 5.2.1 asserts that λ_N has the following representation

$$\lambda_N = d_{X_M}\alpha + \delta_{X_M}\beta \in \mathcal{H}_{X_M,\text{ex}}^{n-\mp}(M) + \mathcal{H}_{X_M,\text{co}}^{n-\mp}(M) \quad (5.24)$$

and also by remark 3.3.15, α and β can be chosen respectively to satisfy

$$\delta_{X_M}\alpha = 0, \quad \Delta_{X_M}\alpha = 0 \quad (5.25)$$

and

$$d_{X_M}\beta = 0, \quad \Delta_{X_M}\beta = 0. \quad (5.26)$$

We set up the transformations

$$\omega = -(\pm 1)^n \star\beta, \quad \varepsilon = -(\mp 1)^{n+1}\alpha.$$

Then eqs.(5.25)-(5.26) turn into

$$\delta_{X_M}\omega = 0, \quad \Delta_{X_M}\omega = 0 \quad (5.27)$$

$$\delta_{X_M}\varepsilon = 0, \quad \Delta_{X_M}\varepsilon = 0 \quad (5.28)$$

and eq.(5.24) implies

$$\lambda_N = \star d_{X_M}\omega - (\mp 1)^{n+1}d_{X_M}\varepsilon \quad (5.29)$$

hence,

$$\star\lambda_N = -(\mp 1)^{n+1}(\star d_{X_M}\varepsilon - d_{X_M}\omega). \quad (5.30)$$

We can define forms $\phi, \psi \in \Omega_G(\partial M)$ by setting

$$\phi = i^* \omega, \quad \psi = i^* \varepsilon. \quad (5.31)$$

Restricting eq.(5.29) to the boundary and using the fact that $i^* \star d_{X_M} \omega = \Lambda_{X_M} \phi$, we obtain

$$i^* \lambda_N = \Lambda_{X_M} \phi - (\mp 1)^{n+1} d_{X_M} \psi. \quad (5.32)$$

Restricting eq.(5.30) to the boundary

$$i^* (\star d_{X_M} \varepsilon) = d_{X_M} (i^* \omega) \quad (5.33)$$

but $i^* (\star d_{X_M} \varepsilon) = \Lambda_{X_M} \psi$ because of eq.(5.28) and the second of equality (5.31). Hence, eq.(5.33) turns to

$$\Lambda_{X_M} \psi = d_{X_M} \phi. \quad (5.34)$$

Now, we can eliminate the form ψ from eq.(5.32) and eq.(5.34) and we can obtain that

$$i^* \lambda_N = (\Lambda_{X_M} - (\mp 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \phi.$$

Hence, $i^* \lambda_N \in (\Lambda_{X_M} - (\mp 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \Omega_G^\pm(\partial M)$. \square

Now, using Theorem 5.3.9, we have this lemma.

Lemma 5.3.10 *The X_M - Hilbert transform maps $i^* \mathcal{H}_{X_M, N}^\pm(M)$ to $i^* \mathcal{H}_{X_M, N}^{n-(\pm)}(M)$.*

PROOF: Let $\phi \in i^* \mathcal{H}_{X_M, N}^\pm(M)$ then theorem 5.3.9 implies that

$$\phi = (\Lambda_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \theta$$

for some $\theta \in \Omega^{n-(\mp)}(\partial M)$. Hence, it follows that

$$\begin{aligned} T_{X_M} \phi &= d_{X_M} \Lambda_{X_M}^{-1} (\Lambda_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \theta \\ &= (d_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \theta \\ &= (\Lambda_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \Lambda_{X_M}^{-1} d_{X_M} \theta \\ &= (\Lambda_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \Lambda_{X_M}^{-1} d_{X_M} \theta \end{aligned}$$

but $\Lambda_{X_M}^{-1} d_{X_M} (\theta) \in \Omega_G^\mp(\partial M)$. Thus, by theorem (5.3.9) we find that the right hand side of the latter formula must belong to $i^* \mathcal{H}_{X_M, N}^{n-(\pm)}(M)$. \square

5.4 Λ_{X_M} operator, X_M -cohomology and equivariant cohomology

The following result is an extension of theorem 2.4.3 to X_M -cohomology. We relate the dimension of $H_{X_M}^\pm(M)$ with the kernel of Λ_{X_M} as follows:

Theorem 5.4.1 *Let $\Lambda_{X_M}^\pm$ be the restriction of X_M -DN operator to $\Omega_G^\pm(\partial M)$. Then $\mathcal{E}_{X_M}^\pm(\partial M) \subseteq \ker \Lambda_{X_M}^\pm$ and*

$$\dim[\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)] \leq \min\{\dim(H_{X_M}^\pm(\partial M)), \dim(H_{X_M}^\pm(M))\}. \quad (5.35)$$

Moreover, if every component of $F' = N(X_M)$ has a boundary then

$$\max\{\dim[\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)], \dim[\ker \Lambda^\pm / \mathcal{E}^\pm(\partial F')]\} \leq \min\{\dim(H_{X_M}^\pm(\partial M)), \dim(H_{X_M}^\pm(M))\}.$$

PROOF: We can apply the X_M -Hodge-Morrey decomposition theorem 3.3.13 (or theorem 3.2.5) for ∂M which asserts that the direct sum of the first and third subspaces is equal to the subspace of all X_M -closed invariant differential \pm -forms (that is, $\ker d_{X_M}$). Hence, this fact together with eq.(5.13) implies that

$$\mathcal{E}_{X_M}^\pm(\partial M) \subset \ker \Lambda_{X_M}^\pm \subset \mathcal{H}_{X_M}^\pm(\partial M) \oplus \mathcal{E}_{X_M}^\pm(\partial M).$$

This implies

$$\dim[\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)] \leq \dim \mathcal{H}_{X_M}^\pm(\partial M) = \dim(H_{X_M}^\pm(\partial M)).$$

By lemma 5.2.6 and lemma 5.3.5,

$$\ker \Lambda_{X_M}^\pm = \mathcal{E}_{X_M}^\pm(\partial M) + i^* \mathcal{H}_{X_M, N}^\pm(M).$$

Thus,

$$\dim[\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)] \leq \dim(i^* \mathcal{H}_{X_M, N}^\pm(M)) = \dim(H_{X_M}^\pm(M)).$$

Therefore

$$\dim[\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)] \leq \min\{\dim(H_{X_M}^\pm(\partial M)), \dim(H_{X_M}^\pm(M))\}$$

as required.

The second part follows by applying theorem 2.4.3 to F' and then using the first part of this theorem. \square

Remark 5.4.2 The second part of theorem 5.4.1 moreover refers implicitly to a possible relation between the dimensions of $\ker \Lambda_{X_M}^\pm / \mathcal{E}_{X_M}^\pm(\partial M)$ and $\ker \Lambda^\pm / \mathcal{E}^\pm(\partial F')$ which needs to be discovered. In addition, this relation will help to extend many of the results of [32] to the style of X_M -cohomology but we will leave it for future work.

To relate these inequalities to equivariant cohomology, one uses a result in chapter 3 which asserts that if $F' = F := \text{Fix}(G, M)$, then $\dim(H_{X_M}^\pm(M)) = \text{rank } H_G^\pm(M)$ (see, Corollary 3.4.4). Hence we conclude that under this assumption, the right hand side of the inequalities above can be replaced by $\min\{\text{rank } H_G^\pm(\partial M), \text{rank } H_G^\pm(M)\}$.

5.5 Recovering X_M -cohomology from the boundary data $(\partial M, \Lambda_{X_M})$

In this section, we continue extending the results of Belishev-Sharafutdinov and Shonkwiler's theorem 2.4.4 on recovering the de Rham cohomology groups and ring structure from the boundary data $(\partial M, \Lambda)$ to the context of absolute and relative X_M -cohomology and their concrete realizations $\mathcal{H}_{X_M, N}^\pm(M)$ and $\mathcal{H}_{X_M, D}^\pm(M)$ leading to the generalized boundary data $(\partial M, \Lambda_{X_M})$ on $\Omega_G(M)$.

5.5.1 Recovering the long exact X_M -cohomology sequence of $(M, \partial M)$

We show that the data $(\partial M, \Lambda_{X_M})$ determines the long exact X_M -cohomology sequence of the topological pair $(M, \partial M)$.

We reconsider the long exact sequence (4.3) in X_M -cohomology derived from the inclusion $i : \partial M \hookrightarrow M$

$$\dots \xrightarrow{i^*} H_{X_M}^\mp(\partial M) \xrightarrow{\partial^*} H_{X_M}^\pm(M, \partial M) \xrightarrow{\rho^*} H_{X_M}^\pm(M) \xrightarrow{i^*} H_{X_M}^\pm(\partial M) \xrightarrow{\partial^*} H_{X_M}^\mp(M, \partial M) \longrightarrow \dots$$

But, theorem 5.3.9 proves that we can determine the space $i^* \mathcal{H}_{X_M, N}^\pm(M)$ from the boundary data $(\partial M, \Lambda_{X_M})$ and corollary 5.2.3 gives $i^* \mathcal{H}_{X_M, N}^\pm(M) \cong H_{X_M}^\pm(M)$ and $i^* \mathcal{H}_{X_M, N}^{n-\pm}(M) \cong H_{X_M}^\pm(M, \partial M)$. This reads that the additive absolute and relative X_M -cohomology are completely determined by $(\partial M, \Lambda_{X_M})$.

So, if the boundary data $(\partial M, \Lambda_{X_M})$ is given then we can construct the sequence

$$\dots \xrightarrow{\bar{\partial}^*} i^* \mathcal{H}_{X_M, N}^{n-(\pm)}(M) \xrightarrow{\bar{\rho}^*} i^* \mathcal{H}_{X_M, N}^\pm(M) \xrightarrow{\bar{i}^*} H_{X_M}^\pm(\partial M) \xrightarrow{\bar{\partial}^*} i^* \mathcal{H}_{X_M, N}^{n-(\mp)}(M) \xrightarrow{\bar{\rho}^*} \dots \quad (5.36)$$

where we define the operators of sequence (5.36) by the following formulas:

1. $\bar{i}^* \theta = [\theta]_{(X_M, \partial M)}$, if $\theta \in i^* \mathcal{H}_{X_M, N}^\pm$ then θ is X_M -closed because i^* and d_{X_M} commute.

2. Using Lemma 5.3.10, we set, $\bar{\rho}^* \theta = -(\pm 1)^{n+1} T_{X_M} \theta$, $\forall \theta \in i^* \mathcal{H}_{X_M, N}^{n-(\pm)}$.
3. Based on theorem 5.3.9, $\Lambda_{X_M} \theta = (\Lambda_{X_M} - (\mp 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M}) \theta$, if $[\theta]_{(X_M, \partial M)} \in H_{X_M}^\pm(\partial M)$. Hence, we set

$$\bar{\partial}^* [\theta]_{(X_M, \partial M)} = (\mp 1)^{n+1} \Lambda_{X_M} \theta, \quad \forall [\theta]_{(X_M, \partial M)} \in H_{X_M}^\pm(\partial M).$$

More concretely, our goal is then to recover sequence (4.3) from sequence (5.36). It means that we need to prove the following theorem.

Theorem 5.5.1 *The following diagram (5.37) is commutative.*

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\bar{\partial}^*} & i^* \mathcal{H}_{X_M, N}^{n-(\pm)}(M) & \xrightarrow{\bar{\rho}^*} & i^* \mathcal{H}_{X_M, N}^\pm(M) & \xrightarrow{\bar{i}^*} & H_{X_M}^\pm(\partial M) & \xrightarrow{\bar{\partial}^*} & i^* \mathcal{H}_{X_M, N}^{n-(\mp)}(M) & \xrightarrow{\bar{\rho}^*} & \dots \\ & & \downarrow h & & \downarrow f & & \downarrow \iota & & \downarrow h & & \\ \dots & \xrightarrow{\partial^*} & H_{X_M}^\pm(M, \partial M) & \xrightarrow{\rho^*} & H_{X_M}^\pm(M) & \xrightarrow{i^*} & H_{X_M}^\pm(\partial M) & \xrightarrow{\partial^*} & H_{X_M}^\mp(M, \partial M) & \xrightarrow{\rho^*} & \dots \end{array} \quad (5.37)$$

where ι is the identity operator while f and h are given in corollary 5.2.3.

PROOF: We prove the commutativity of the diagram (5.37) by a method similar to that given in [11] but in terms of the operators d_{X_M} and δ_{X_M} and it is as follows:

The commutativity of the square

$$\begin{array}{ccc} i^* \mathcal{H}_{X_M, N}^\pm(M) & \xrightarrow{\bar{i}^*} & H_{X_M}^\pm(\partial M) \\ \downarrow f & & \downarrow \iota \\ H_{X_M}^\pm(M) & \xrightarrow{i^*} & H_{X_M}^\pm(\partial M) \end{array}$$

follows directly from the definitions of our operators. In fact, an invariant form $\theta = i^* \omega \in i^* \mathcal{H}_{X_M, N}^\pm(M)$ with $\omega \in \mathcal{H}_{X_M, N}^\pm(M)$ implies that

$$i^* f(\theta) = i^* [\omega]_{(X_M, M)} = [i^* \omega]_{(X_M, \partial M)} = [\theta]_{(X_M, \partial M)} = \bar{i}^* \theta.$$

Now, we check the commutativity of the second square

$$\begin{array}{ccc} i^* \mathcal{H}_{X_M, N}^{n-(\pm)}(M) & \xrightarrow{\bar{\rho}^*} & i^* \mathcal{H}_{X_M, N}^\pm(M) \\ \downarrow h & & \downarrow f \\ H_{X_M}^\pm(M, \partial M) & \xrightarrow{\rho^*} & H_{X_M}^\pm(M) \end{array}$$

Let $\theta = i^* \omega \in i^* \mathcal{H}_{X_M, N}^{n-\pm}(M)$ with $\omega \in \mathcal{H}_{X_M, N}^{n-\pm}(M)$. Then by the definitions of our operators,

we obtain

$$\rho^* h(\theta) = \rho^* [\star\omega]_{(X_M, M, \partial M)} = [\star\omega]_{(X_M, M)}. \quad (5.38)$$

And the form $\psi = \bar{\rho}^* \theta \in i^* \mathcal{H}_{X_M, N}^\pm(M)$ can be written as

$$\psi = \bar{\rho}^* \theta = i^* v, \quad v \in \mathcal{H}_{X_M, N}^\pm(M).$$

By the definition of f ,

$$f\bar{\rho}^*(\theta) = f(\psi) = [v]_{(X_M, M)}. \quad (5.39)$$

Hence, we conclude that the commutativity of the square is equivalent to prove the equality of the following X_M -cohomology classes $[\star\omega]_{(X_M, M)} = [v]_{(X_M, M)}$, which means that the X_M -Friedrichs decomposition (3.28) asserts that the invariant form $\star\omega$ must be of the form (because $\star\omega$ is harmonic field):

$$\star\omega = v + d_{X_M} \alpha \in \mathcal{H}_{X_M, N}^\pm(M) \oplus \mathcal{H}_{X_M, \text{ex}}^\pm(M). \quad (5.40)$$

But Remark 3.3.15 asserts that α can be chosen to satisfy

$$\Delta_{X_M} \alpha = 0, \quad \delta_{X_M} \alpha = 0.$$

Restricting equation (5.40) to the boundary, we have

$$\psi = i^* v = -i^* d_{X_M} \alpha = -d_{X_M} i^* \alpha. \quad (5.41)$$

Now, applying \star to (5.40), we get $(\pm)^{n+1} \omega = \star v + \star d_{X_M} \alpha$. Retracting the last equation to the boundary

$$(\pm)^{n+1} \theta = (\pm)^{n+1} i^* \omega = i^* (\star d_{X_M} \alpha) = \Lambda_{X_M} i^* \alpha.$$

We obtain from the last equation

$$i^* \alpha = (\pm)^{n+1} \Lambda_{X_M}^{-1} \theta.$$

Substitution the value of $i^* \alpha$ above into (5.41), we get

$$\psi = -(\pm)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} \theta = -(\pm)^{n+1} T_{X_M} \theta$$

i.e. $\psi = \bar{\rho}^* \theta = -(\pm)^{n+1} T_{X_M} \theta$. This is just our definition of the operator $\bar{\rho}^*$.

Finally, we check the commutativity of the last square

$$\begin{array}{ccc} H_{X_M}^\pm(\partial M) & \xrightarrow{\bar{\partial}^*} & i^* \mathcal{H}_{X_M, N}^{n-(\mp)}(M) \\ \downarrow \iota & & \downarrow h \\ H_{X_M}^\pm(\partial M) & \xrightarrow{\partial^*} & H_{X_M}^\mp(M, \partial M) \end{array}$$

Let $[\theta]_{(X_M, \partial M)} \in H_{X_M}^\pm(\partial M)$ and $\omega \in \Omega_G^\pm(M)$ be a solution to the BVP (5.10). Then

$$\partial^*[\theta]_{(X_M, \partial M)} = [d_{X_M} \omega]_{(X_M, M, \partial M)} \quad (5.42)$$

and $\Lambda_{X_M} \theta = i^*(\star d_{X_M} \omega)$. By the definition of h ,

$$h\Lambda_{X_M} \theta = [\star \star d_{X_M} \omega]_{(X_M, M, \partial M)} = (\mp)^{n+1} [d_{X_M} \omega]_{(X_M, M, \partial M)}. \quad (5.43)$$

Comparing (5.42) and (5.43), we obtain $h\Lambda_{X_M} \theta = (\mp)^{n+1} \partial^*[\theta]_{(X_M, \partial M)}$. But our definition of $\bar{\partial}^*$ asserts that the last equation proves that

$$h\bar{\partial}^*[\theta]_{(X_M, \partial M)} = \partial^*[\theta]_{(X_M, \partial M)}.$$

Thus we prove the commutativity of the diagram (5.37). \square

Actually, the above construction proves that the data $(\partial M, \Lambda_{X_M})$ recovers sequence (4.3) of the pair $(M, \partial M)$ up to an isomorphism (i.e. f and h) from the sequence (5.36).

5.5.2 Recovering the ring structure of the X_M -cohomology

We consider the following question: *Can the ring (i.e. multiplicative) structure of the real absolute and relative X_M -cohomology be completely recovered from the boundary data $(\partial M, \Lambda_{X_M})$?*

First of all, we consider the mixed cup product $\bar{\cup}$ between the absolute and relative X_M -cohomology as follows:

$$\bar{\cup} : H_{X_M}^\pm(M) \times H_{X_M}^\pm(M, \partial M) \longrightarrow H_{X_M}^\pm(M, \partial M)$$

by setting

$$[\alpha]_{(X_M, M)} \bar{\cup} [\beta]_{(X_M, M, \partial M)} = [\alpha \wedge \beta]_{(X_M, M, \partial M)}, \quad \forall ([\alpha]_{(X_M, M)}, [\beta]_{(X_M, M, \partial M)}) \in H_{X_M}^\pm(M) \times H_{X_M}^\pm(M, \partial M).$$

It is easy to check that $\bar{\cup}$ is a well-defined map. In addition, Theorem 3.3.18 asserts that any

absolute and relative X_M -cohomology classes contain a unique Neumann and Dirichlet X_M -harmonic field respectively. Hence, we can regard any absolute (relative) X_M -cohomology class as a Neumann(Dirichlet) X_M -harmonic field. But $[\alpha]_{(X_M, M)} \cup [\beta]_{(X_M, M, \partial M)} = [\alpha \wedge \beta]_{(X_M, M, \partial M)}$ is a relative X_M -cohomology class, so there exists a unique Dirichlet X_M -harmonic field $\eta \in \mathcal{H}_{X_M, D}^\pm(M)$ such that $[\alpha \wedge \beta]_{(X_M, M, \partial M)} = [\eta]_{(X_M, M, \partial M)}$, i.e.

$$\alpha \wedge \beta = \eta + d_{X_M} \xi \in \mathcal{H}_{X_M, D}^\pm(M) \oplus \mathcal{E}_{X_M}^\pm(M). \quad (5.44)$$

But, we can get from corollary 5.2.3 that

$$H_{X_M}^\pm(M, \partial M) \cong H_{X_M}^{n-(\pm)}(M) \cong i^* \mathcal{H}_{X_M, N}^{n-(\pm)}(M)$$

According to our illustrations above we know that an absolute X_M -cohomology class $[\alpha]_{(X_M, M)} \in H_{X_M}^\pm(M)$ and relative X_M -cohomology classes $[\beta]_{(X_M, M, \partial M)}, [\alpha \wedge \beta]_{(X_M, M, \partial M)} \in H_{X_M}^\pm(M, \partial M)$ are represented by the Neumann X_M -harmonic field $\alpha \in \mathcal{H}_{X_M, N}^\pm(M)$ and the Dirichlet X_M -harmonic fields $\beta, \eta \in \mathcal{H}_{X_M, D}^\pm(M)$ respectively, such that they correspond, respectively, to forms on the boundary by setting

$$\phi = i^* \alpha \in i^* \mathcal{H}_{X_M, N}^\pm(M), \quad \psi = i^* \star \beta \in i^* \star \mathcal{H}_{X_M, D}^\pm(M), \quad \vartheta = i^* \star \eta \in i^* \star \mathcal{H}_{X_M, D}^\pm(M).$$

The answer to the above question will only be partial, in the sense that we will not consider all the classes of the relative X_M -cohomology. In fact, we will just consider the boundary portion $BH_{X_M}^\pm(M, \partial M)$ of $H_{X_M}^\pm(M, \partial M)$ which we define it in subsection 4.3.2. Now, using the results of subsection 4.3.2 together with corollary 5.2.3(2), we have that $BH_{X_M}^\pm(M, \partial M) \cong i^* \star \mathcal{B}\mathcal{H}_{X_M, D}^\pm$.

Then, the above constructions help us to adapt Shonkwiler's map [32] to the context of the setting above in order to define the following map with notation as above

$$\phi \bar{\cup}_{X_M} \psi = \Lambda_{X_M}(\pm \phi \wedge \Lambda_{X_M}^{-1} \psi), \quad \forall (\phi, \psi) \in i^* \mathcal{H}_{X_M, N}^\pm(M) \times i^* \star \mathcal{H}_{X_M, D}^\pm(M). \quad (5.45)$$

Now, we use the same method as [32] but together with the definition 5.3.2 in order to prove $\bar{\cup}_{X_M}$ is well-defined and it is as follows. In general, $\Lambda_{X_M}^{-1} \psi$ is not well-defined because Λ_{X_M} has a large kernel, so for any (\mp) -invariant form $\sigma \in \ker \Lambda_{X_M}$, the form $\Lambda_{X_M}^{-1} \psi + \sigma$ is another valid choice for $\Lambda_{X_M}^{-1} \psi$. To prove that, we need only to show that, $\forall \sigma \in \ker \Lambda_{X_M}$,

$$\Lambda_{X_M}(\phi \wedge (\Lambda_{X_M}^{-1} \psi + \sigma)) = \Lambda_{X_M}(\phi \wedge \Lambda_{X_M}^{-1} \psi) + \Lambda_{X_M}(\phi \wedge \sigma) = \Lambda_{X_M}(\phi \wedge \Lambda_{X_M}^{-1} \psi). \quad (5.46)$$

It means that we need only to prove $\phi \wedge \sigma \in \ker \Lambda_{X_M}$. Lemma 5.3.5 asserts that $\sigma = i^* \tau \in$

$i^*\mathcal{H}_{X_M}^\mp(M)$. Since, α and τ are X_M -closed, it follows that so too is $\alpha \wedge \tau$. However, X_M -Hodge-Morrey decomposition theorem 3.3.13 for smooth invariant form implies that

$$\alpha \wedge \tau = \chi + d_{X_M}\varepsilon \in \mathcal{H}_{X_M}^\mp(M) \oplus \mathcal{E}_{X_M}^\mp(M),$$

so χ solves the BVP (5.10), i.e.

$$\Delta_{X_M}\chi = 0, \quad i^*\chi = \phi \wedge \sigma, \quad i^*\delta_{X_M}\chi = 0.$$

Thus, definition 5.3.2 implies that

$$\Lambda_{X_M}(\phi \wedge \sigma) = i^* \star d_{X_M}\chi = 0$$

as desired. Following [32], the map $\bar{\cup}_{X_M}$ is well-defined regardless of whether or not $\beta \in \mathcal{BH}_{X_M,D}^\pm$.

Now, we can extend Shonkwiler's theorem 2.4.4 to the style above.

Theorem 5.5.2 *The boundary data $(\partial M, \Lambda_{X_M})$ completely determines the mixed cup product structure of the X_M -cohomology when the relative X_M -cohomology classes come from the boundary subspace. That is, if $(\alpha, \beta) \in \mathcal{H}_{X_M,N}^\pm(M) \times \mathcal{BH}_{X_M,D}^\pm(M)$ such that $\alpha \wedge \beta = \eta + d_{X_M}\xi \in \mathcal{H}_{X_M,D}^\pm(M) \oplus \mathcal{E}_{X_M}^\pm(M)$ then*

$$i^* \star \eta = \Lambda_{X_M}(\pm\phi \wedge \Lambda_{X_M}^{-1}\psi)$$

where $\phi = i^*\alpha$ and $\psi = i^* \star \beta$. In fact one shows the commutativity of the following diagram,

$$\begin{array}{ccc} i^*\mathcal{H}_{X_M,N}^\pm(M) \times i^* \star \mathcal{BH}_{X_M,D}^\pm(M) & \xrightarrow{\bar{\cup}_{X_M}} & i^* \star \mathcal{BH}_{X_M,D}^\pm(M) \\ \downarrow (f,h) & & \downarrow h \\ H_{X_M}^\pm(M) \times BH_{X_M}^\pm(M, \partial M) & \xrightarrow{\bar{\cup}} & BH_{X_M}^\pm(M, \partial M), \end{array} \quad (5.47)$$

where f and h are given in corollary 5.2.3.

PROOF: Our goal is then to show that

$\forall (\phi, \psi) = (i^*\alpha, i^* \star d_{X_M}\beta_1) \in i^*\mathcal{H}_{X_M,N}^\pm(M) \times i^* \star \mathcal{BH}_{X_M,D}^\pm(M)$ we have

$$(h \circ \bar{\cup}_{X_M})(i^*\alpha, i^* \star d_{X_M}\beta_1) = (\bar{\cup} \circ (f, h))(i^*\alpha, i^* \star d_{X_M}\beta_1). \quad (5.48)$$

Then the left-hand side gives

$$h(\bar{\cup}_{X_M}(i^*\alpha, i^* \star d_{X_M}\beta_1)) = h(\Lambda_{X_M}(\pm i^*\alpha \wedge \Lambda_{X_M}^{-1} i^* \star d_{X_M}\beta_1))$$

$$= h(\Lambda_{X_M}(\pm\phi \wedge \Lambda_{X_M}^{-1}\psi)) \quad (5.49)$$

while the right-hand side together with eq.(5.44) and corollary 5.2.3 give

$$\begin{aligned} \bar{\cup}((f(i^*\alpha), h(i^*\star d_{X_M}\beta_1))) &= \bar{\cup}([\alpha]_{(X_M, M)}, [\star\star d_{X_M}\beta_1]_{(X_M, M, \partial M)}) \\ &= [\star\star(\alpha \wedge d_{X_M}\beta_1)]_{(X_M, M, \partial M)} \\ &= [\star\star\eta]_{(X_M, M, \partial M)} \\ &= h(i^*\star\eta). \end{aligned} \quad (5.50)$$

The above construction shows that we only need to prove that eq.(5.49) and eq.(5.50) are equal. This will be the case if

$$i^*\star\eta = \Lambda_{X_M}(\pm\phi \wedge \Lambda_{X_M}^{-1}\psi). \quad (5.51)$$

But, X_M -DN map together with the results of chapter 3 contribute to extend Shonkwiler's procedure (which he uses to prove theorem 2.4.4) to the style of the operators d_{X_M} , δ_{X_M} and Λ_{X_M} . Similarly as in [32], we use the extending procedure as follows to show that eq.(5.51) holds.

To this end, putting, $\beta = d_{X_M}\beta_1 \in \mathcal{BH}_{X_M, D}^\pm(M)$ and using the X_M -Hodge-Morrey decomposition theorem (3.3.13), we infer that β_1 can be chosen to solve the BVP

$$\begin{cases} \Delta_{X_M} v = 0 & \text{on } M \\ i^*v = i^*\beta_1 & \text{on } \partial M \\ i^*\delta_{X_M}v = 0 & \text{on } \partial M. \end{cases}$$

Hence,

$$\psi = i^*\star d_{X_M}\beta_1 = \Lambda_{X_M}i^*\beta_1.$$

Therefore, $\Lambda_{X_M}^{-1}\psi = i^*\beta_1$. But from eq.(5.44) we get that

$$\eta = d_{X_M}\eta' \in \mathcal{BH}_{X_M, D}^\pm(M)$$

where $\eta' = \pm\alpha \wedge \beta_1 - \xi$. Applying the X_M -Hodge-Morrey decomposition theorem (3.3.13) on η' , we infer that

$$\eta = d_{X_M}\eta' = d_{X_M}\sigma$$

such that σ solves the BVP

$$\Delta_{X_M}\varepsilon = 0, \quad i^*\varepsilon = i^*\sigma, \quad i^*\delta_{X_M}\varepsilon = 0.$$

Hence,

$$\Lambda_{X_M} i^* \sigma = i^* \star d_{X_M} \sigma = i^* \star \eta. \quad (5.52)$$

Since $\eta' = \pm \alpha \wedge \beta_1 - \xi$ implies

$$\begin{aligned} d_{X_M}(\pm \alpha \wedge \beta_1) &= d_{X_M} \eta' + d_{X_M} \xi \\ &= d_{X_M} \sigma + d_{X_M} \xi. \end{aligned} \quad (5.53)$$

Eq. (5.53) shows that the class $[\pm \alpha \wedge \beta_1 - \sigma - \xi]_{(X_M, M)} \in H_{X_M}^\mp(M)$, so the invariant form $\pm \alpha \wedge \beta_1 - \sigma - \xi$ can be decomposed as

$$\pm \alpha \wedge \beta_1 - \sigma - \xi = d_{X_M} \tau_1 + \tau_2 \in \mathcal{E}_{X_M}^\mp(M) \oplus \mathcal{H}_{X_M}^\mp(M).$$

Now, restricting the latter equation to the boundary and using Lemma 5.3.5, this implies that

$$\Lambda_{X_M} i^*(\pm \alpha \wedge \beta_1 - \sigma - \xi) = \Lambda_{X_M} i^* \tau_2 = 0.$$

Combining this with eq. (5.52) gives that

$$\begin{aligned} \Lambda_{X_M} i^*(\pm \alpha \wedge \beta_1) &= \Lambda_{X_M} i^*(\pm \alpha \wedge \beta_1 - \sigma - \xi + \sigma + \xi) \\ &= \Lambda_{X_M} i^*(\pm \alpha \wedge \beta_1 - \sigma - \xi) + \Lambda_{X_M} i^* \sigma \\ \Lambda_{X_M}(\pm \phi \wedge \Lambda_{X_M}^{-1} \psi) &= i^* \star \eta \end{aligned}$$

Hence, the diagram (5.47) is commutative as desired. \square

5.6 Conclusions and topological open problem

- 1- The key point used to recover the free part of the relative and absolute equivariant cohomology groups from the generalized boundary data $(\partial M, \Lambda_{X_M})$ is corollary 3.4.4. Now, combining corollary 3.4.4 with theorem 5.3.9, we get

Theorem 5.6.1 *Let $F' = N(X_M)$, then*

$$H_{X_M}^\pm(M, \partial M) \cong \left(\Lambda_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M} \right) \Omega_G^\mp(\partial M) \cong H^\pm(F', \partial F')$$

and

$$H_{X_M}^\pm(M) \cong \left(\Lambda_{X_M} - (\pm 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M} \right) \Omega_G^{n-\mp}(\partial M) \cong H^\pm(F').$$

Since the Neumann X_M -harmonic fields are uniquely determined by their Neumann X_M - trace spaces (corollary 5.2.3) which is in turn determined by the boundary

data $(\partial M, \Lambda_{X_M})$ (theorem 5.3.9), this means we can conclude, by using X_M -Poincaré-Lefschetz duality that we can realize the relative and absolute X_M -cohomology groups (and hence in some sense the free part of the relative and absolute equivariant cohomology groups) as particular subspaces of invariant differential forms on ∂M and they are not just determined abstractly from the generalized boundary data.

- 2- We can apply theorem 2.4.1 to the manifolds $F' = N(X_M)$ with boundary $\partial F'$. Since G acts on F' the induced action on each $H^\pm(F')$ is trivial. Now, we can use theorem 5.6.1 to exploit the connection between Belishev-Sharafutdinov's boundary data on $\partial F'$ (i.e. $(\partial F', \Lambda)$) and ours on ∂M (i.e. $(\partial M, \Lambda_{X_M})$). More concretely, we have the following.

Theorem 5.6.2 *If every component of F' has a boundary, then*

$$\left(\Lambda_{X_M} - (\mp 1)^{n+1} d_{X_M} \Lambda_{X_M}^{-1} d_{X_M} \right) \Omega_G^\pm(\partial M) \cong \left(\Lambda - (\mp 1)^{n+1} d \Lambda^{-1} d \right) \Omega^\pm(\partial F').$$

This means that the boundary data $(\partial F', \Lambda)$ can be determined from the boundary data $(\partial M, \Lambda_{X_M})$ and vice versa. In this setting, it follows that since the de Rham cohomology groups of $(F', \partial F')$ are determined by $(\partial F', \Lambda)$ (theorem 2.4.1), then the \pm de Rham cohomology groups of $(F', \partial F')$ are also determined by $(\partial M, \Lambda_{X_M})$.

- 3- When M has no boundary, Witten proves in [35] that $H_K^\pm(M) \cong H^\pm(N(K))$ where K is a Killing vector field (our X_M) on M and he shows how the K -cohomology and the isomorphism above are useful in Quantum Field Theory and other mathematical and physical applications. However, when $\partial M \neq \emptyset$, the extend isomorphism is provided by corollary 3.4.4 which gives insight that the extension for other results of Witten [35] are possible. In this light, theorem 5.6.1 suggests that Λ_{X_M} may also be relevant to Quantum Field Theory and following Witten, possibly to other mathematical and physical interpretations. This shows that Λ_{X_M} may be interesting in its own right.
- 4- The results in this chapter assert that the generalized boundary data $(\partial M, \Lambda_{X_M})$ encodes more information about the equivariant algebraic topology of M than does the boundary data $(\partial M, \Lambda)$. Hence, these results contribute to answer the topological open problem which is stated in section 5.1.
- 5- Extending the results of DeTurck and Gluck to the style of X_M -cohomology (section 4.3) and recovering the X_M -cohomology from the boundary data $(M, \partial M)$ together with the five terms decomposition eq. (4.2) will certainly provide the crude ingredients which we need to extend most of the results of Shonkwiler's Thesis [32] to the context of X_M -Poincaré duality angles but we will leave it for future work.

Topological open problem: Finally, it is worth considering the following topological problem: *Can the torsion part of the absolute and relative equivariant cohomology groups be completely recovered from the boundary data $(\partial M, \Lambda_{X_M})$?* (Here torsion is meant as a module over the ring of polynomials on \mathfrak{g} —the standard Cartan model: some torsion information is available from Corollary 3.4.4 and Theorem 5.6.1 when X is in an isotropy subalgebra, but not all.) Answering this question will indeed complete the picture of the boundary data $(\partial M, \Lambda_{X_M})$ to be added into the list of objects of equivariant cohomology of manifolds story.

Chapter 6

X_M -Harmonic cohomology on manifolds with boundary

6.1 Introduction

We denote in this chapter the space of k -harmonic forms by $\text{Harm}^k(M) = \ker(\Delta|_{\Omega^k})$. The de Rham coboundary operator d and Δ commute [16], hence $(\text{Harm}^*(M), d)$ forms a subcomplex of the de Rham complex and it is therefore natural to compute the cohomology of this subcomplex. When $\partial M = \emptyset$, the Hodge Theorem (subsection (2.3.1)) implies that $H^k(\text{Harm}^*(M), d) \cong \text{Harm}^k(M)$. However, S.Cappell, D. DeTurck et al. [13] consider the case $\partial M \neq \emptyset$ and they prove that the cohomology of this subcomplex is given by the following theorem

Theorem 6.1.1 [13]. *Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary. Then the cohomology of the complex $(\text{Harm}^*(M), d)$ of harmonic forms on M is given by the direct sum of the de Rham cohomology:*

$$H^k(\text{Harm}^*(M), d) \cong H^k(M) + H^{k-1}(M)$$

for $k = 0, 1, \dots, n$.

As consequences of the results in chapters 3, the principal idea of this chapter is to extend theorem 6.1.1 to equivariant cohomology.

More in keeping with [13], we denote in this chapter the space of X_M -harmonic forms by $\text{Harm}_{X_M}^*(M)$ where $\text{Harm}_{X_M}^*(M) = \text{Harm}_{X_M}^+(M) + \text{Harm}_{X_M}^-(M)$; it is the kernel of the Witten-Hodge-Laplacian operator Δ_{X_M} (following chapter 3), i.e.

$$\text{Harm}_{X_M}^\pm(M) = \ker \Delta_{X_M} \cap \Omega_G^\pm = \{\omega \in \Omega_G^\pm \mid \Delta_{X_M} \omega = 0\}.$$

Clearly, $\text{Harm}_{X_M}^\pm(M) \subset \Omega_G^\pm$, but Δ_{X_M} and d_{X_M} commute which means that the coboundary operator d_{X_M} preserves the X_M -harmonicity of invariant forms. i.e.

$$\text{Harm}_{X_M}^\pm(M) \xrightarrow{d_{X_M}} \text{Harm}_{X_M}^\mp(M).$$

Hence, $(\text{Harm}_{X_M}^*(M), d_{X_M})$ is a subcomplex of the \mathbb{Z}_2 -graded complex (Ω_G^*, d_{X_M}) . Therefore, we can define the cohomology of this complex which we call the X_M -harmonic cohomology and denote by $H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M})$.

In the boundaryless case, we prove that the space of X_M -harmonic fields $\mathcal{H}_{X_M}^\pm$ equal to the space of X_M -harmonic forms, i.e. $\text{Harm}_{X_M}^\pm(M) = \mathcal{H}_{X_M}^\pm$. Thus, we can conclude that all of the maps in the subcomplex $(\text{Harm}_{X_M}^*(M), d_{X_M})$ are zero which means that

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) \cong \text{Harm}_{X_M}^\pm(M).$$

But, proposition 3.2.6 asserts that $H_{X_M}^\pm(M) \cong \mathcal{H}_{X_M}^\pm$, hence,

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) \cong H_{X_M}^\pm(M). \quad (6.1)$$

On the other hand, eq. (6.1) is no longer true when the manifold in question has a boundary because the space of X_M -harmonic forms $\text{Harm}_{X_M}^\pm(M)$ no longer coincides with the space of X_M -harmonic fields $\mathcal{H}_{X_M}^\pm$ (following chapter 3). Hence, computing the cohomology $H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M})$ will not be straight away as when $\partial M = \emptyset$. Therefore, in this chapter we consider this problem and the only way to solve it is by extending theorem 6.1.1 to the style of X_M -cohomology and the resulting is theorem 6.2.3 which proves that $H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M})$ determine the whole X_M -cohomology $H_{X_M}^*(M)$. The results of this chapter are also contained in [4].

6.2 X_M -Harmonic cohomology isomorphism theorem

In this section, we use the symbol $+$ between the spaces to indicate a direct sum whereas we reserve the symbol \oplus for an orthogonal (with respect to L^2 -inner product) direct sum unless otherwise indicated.

We begin with the following remark.

Remark 6.2.1 We need to define the following subspaces:

$$E_{X_M}^\pm(M) = \{d_{X_M} \alpha \mid \alpha \in \Omega_G^\mp(M)\}$$

and

$$cE_{X_M}^\pm(M) = \{\delta_{X_M} \alpha \mid \alpha \in \Omega_G^\mp(M)\}.$$

The X_M -Hodge-Morrey decomposition (3.26) implies the following decompositions:

$$E_{X_M}^\pm = E_{X_M}^\pm(M) = \mathcal{E}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M, \text{ex}}^\pm(M)$$

and

$$cE_{X_M}^\pm = cE_{X_M}^\pm(M) = \mathcal{C}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M, \text{co}}^\pm(M).$$

The image of the *Witten-Hodge-Laplacian* operator Δ_{X_M} will be most important to obtain the main theorem 6.2.3. We therefore need first to prove the following lemma 6.2.2. The proof here is slightly different from the original proof in the classical case in [13] but it is also valid for that case .

Lemma 6.2.2 (The image of Δ_{X_M}) *The Witten-Hodge-Laplacian operator $\Delta_{X_M} = d_{X_M}\delta_{X_M} + \delta_{X_M}d_{X_M} : \Omega_G^\pm(M) \longrightarrow \Omega_G^\pm(M)$ is surjective.*

PROOF: We need to prove that $\Delta_{X_M}(\Omega_G^\pm(M)) = \Omega_G^\pm(M)$. Clearly, $\Delta_{X_M}(\Omega_G^\pm(M)) \subset \Omega_G^\pm(M)$, so we only need to prove the converse. To do so, we will first compute the image of Δ_{X_M} on each summand of the X_M -Hodge-Morrey decomposition (3.26).

It is clear that

$$\Delta_{X_M}(\mathcal{E}_{X_M}^\pm(M)) = d_{X_M}\delta_{X_M}(\mathcal{E}_{X_M}^\pm(M)) \subset E_{X_M}^\pm.$$

Now, let $\beta \in E_{X_M}^\pm$ then $\beta = d_{X_M}\alpha$ and by applying the X_M -Hodge-Morrey decomposition (3.26) on α we get $\alpha = d_{X_M}\sigma + \delta_{X_M}\rho + \lambda$, so

$$\beta = d_{X_M}\alpha = d_{X_M}\delta_{X_M}\rho$$

but also by (3.26), ρ can be written as $\rho = d_{X_M}\varepsilon + \delta_{X_M}\pi + \kappa$ which implies that

$$\beta = d_{X_M}\alpha = d_{X_M}\delta_{X_M}\rho = d_{X_M}\delta_{X_M}d_{X_M}\varepsilon \in \Delta_{X_M}(\mathcal{E}_{X_M}^\pm(M)).$$

Hence, $\Delta_{X_M}(\mathcal{E}_{X_M}^\pm(M)) = E_{X_M}^\pm$. Likewise, $\Delta_{X_M}(\mathcal{C}_{X_M}^\pm(M)) = cE_{X_M}^\pm$. Clearly, $\Delta_{X_M}(\mathcal{H}_{X_M}^\pm(M)) = 0$. Using, the above equations together with remark 6.2.1, we obtain

$$\begin{aligned} \Delta_{X_M}(\Omega_G^\pm(M)) &= E_{X_M}^\pm + cE_{X_M}^\pm \\ &= (\mathcal{E}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M, \text{ex}}^\pm(M)) + (\mathcal{C}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M, \text{co}}^\pm(M)). \end{aligned} \quad (6.2)$$

where “+” is not a direct sum.

Finally, let $\omega \in \Omega_G^\pm(M)$ then the X_M -Hodge-Morrey decomposition (3.26) together with

corollary 5.2.1 assert that ω can be decomposed as

$$\omega = d_{X_M} \alpha_\omega + \delta_{X_M} \beta_\omega + (d_{X_M} \rho_\omega + \delta_{X_M} \sigma_\omega) \in \mathcal{E}_{X_M}^\pm(M) \oplus \mathcal{C}_{X_M}^\pm(M) \oplus (\mathcal{H}_{X_M, \text{ex}}^\pm(M) + \mathcal{H}_{X_M, \text{co}}^\pm(M)) \quad (6.3)$$

Rearranging eq.(6.3), we get that eq.(6.2) shows that $\omega \in \Delta_{X_M}(\Omega_G^\pm(M))$ as desired. Thus, Δ_{X_M} is surjective. \square

Now, it is time to present the following fundamental theorem which is analogous to theorem 6.1.1.

Theorem 6.2.3 *Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M . Then the (even or odd) X_M -harmonic cohomology of the subcomplex $(\text{Harm}_{X_M}^*(M), d_{X_M})$ completely determines the total X_M -cohomology of the complex (Ω_G^*, d_{X_M}) and it is given by the direct sum:*

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) \cong H_{X_M}^\pm(M) + H_{X_M}^\mp(M) = H_{X_M}^*(M) \quad (6.4)$$

PROOF: Applying the definition of the X_M -cohomology of the subcomplex $(\text{Harm}_{X_M}^\pm(M), d_{X_M})$, we obtain that

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) = \frac{\ker d_{X_M} \upharpoonright_{\text{Harm}_{X_M}^\pm(M)}}{d_{X_M}(\text{Harm}_{X_M}^\mp(M))}$$

where $\ker d_{X_M} \upharpoonright_{\text{Harm}_{X_M}^\pm(M)} = \ker d_{X_M} \cap \text{Harm}_{X_M}^\pm(M)$. But, the X_M -Hodge-Morrey-Friedrichs decomposition (3.35) implies the following decomposition

$$\ker d_{X_M} \upharpoonright_{\text{Harm}_{X_M}^\pm(M)} = \mathcal{E}_{X_M}^\pm(M) \oplus \mathcal{H}_{X_M, N}^\pm(M) \oplus \mathcal{H}_{X_M, \text{ex}}^\pm(M) = \mathcal{H}_{X_M, N}^\pm(M) \oplus E_{X_M} \text{Harm}_{X_M}^\pm(M)$$

where $E_{X_M} \text{Harm}_{X_M}^\pm(M) = E_{X_M}^\pm(M) \cap \text{Harm}_{X_M}^\pm(M)$. But

$d_{X_M}(\text{Harm}_{X_M}^\mp(M)) \subset \ker d_{X_M} \upharpoonright_{\text{Harm}_{X_M}^\pm(M)}$, then we obtain a direct sum decomposition

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) = \frac{\ker d_{X_M} \upharpoonright_{\text{Harm}_{X_M}^\pm(M)}}{d_{X_M}(\text{Harm}_{X_M}^\mp(M))} = \mathcal{H}_{X_M, N}^\pm(M) + \frac{E_{X_M} \text{Harm}_{X_M}^\pm(M)}{d_{X_M}(\text{Harm}_{X_M}^\mp(M))}$$

However, the X_M -Hodge isomorphism theorem 3.3.18 asserts that $H_{X_M}^\pm(M) \cong \mathcal{H}_{X_M, N}^\pm(M)$.

Hence, we only need to prove that

$$\frac{E_{X_M} \text{Harm}_{X_M}^\pm(M)}{d_{X_M}(\text{Harm}_{X_M}^\mp(M))} \cong \frac{\ker d_{X_M}}{d_{X_M} \Omega_G^\pm} \cong H_{X_M}^\mp(M).$$

We define the map $\bar{\delta}_{X_M}$ as follows :

$$\bar{\delta}_{X_M}([\varphi]) = [\delta_{X_M} \varphi] \in H_{X_M}^{\mp}(M), \quad \forall [\varphi] \in \frac{E_{X_M} \text{Harm}_{X_M}^{\pm}(M)}{d_{X_M}(\text{Harm}_{X_M}^{\mp}(M))}$$

To prove $\bar{\delta}_{X_M}$ is a well-defined:

Let $\theta_1 - \theta_2 = d_{X_M} \beta$, for some $\beta \in \text{Harm}_{X_M}^{\mp}(M)$. i.e. $\Delta_{X_M} \beta = (d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M}) \beta = 0$.

Then

$$\begin{aligned} \delta_{X_M} \theta_1 - \delta_{X_M} \theta_2 &= \delta_{X_M} d_{X_M} \beta \\ &= -d_{X_M} \delta_{X_M} \beta \\ &= d_{X_M} (-\delta_{X_M} \beta) \in d_{X_M} \Omega_G^{\pm} \end{aligned} \quad (6.5)$$

Moreover, $\delta_{X_M} \beta$ is X_M -harmonic as $\Delta_{X_M}(\delta_{X_M} \beta) = \delta_{X_M} d_{X_M} \delta_{X_M} \beta = \delta_{X_M}^2(\theta_1 - \theta_2) = 0$. It means that $\delta_{X_M}(\theta_1 - \theta_2) \in d_{X_M} \text{Harm}_{X_M}^{\mp}$. Thus, $\bar{\delta}_{X_M}$ is a well-defined.

Next, we prove $\bar{\delta}_{X_M}$ is one-to-one. To this end, let $\varphi \in E_{X_M} \text{Harm}_{X_M}^{\pm}(M)$ and $\delta_{X_M} \varphi \in d_{X_M} \Omega_G^{\pm}$. We only need to prove $\varphi \in d_{X_M}(\text{Harm}_{X_M}^{\mp}(M))$. So, $\varphi = d_{X_M} \beta$, and therefore

$$\Delta_{X_M} \beta = (d_{X_M} \delta_{X_M} + \delta_{X_M} d_{X_M}) \beta = d_{X_M} \delta_{X_M} \beta + \delta_{X_M} \varphi \in d_{X_M} \Omega_G^{\pm}$$

Thus, $\Delta_{X_M} \beta = d_{X_M} \eta$ for some $\eta \in \Omega_G^{\pm}$, but Δ_{X_M} is onto by lemma (6.2.2) then we can write $\eta = \Delta_{X_M} \sigma$. Hence, $\Delta_{X_M} \beta = d_{X_M} \eta = d_{X_M} \Delta_{X_M} \sigma = \Delta_{X_M} d_{X_M} \sigma$ which implies that $\beta - d_{X_M} \sigma \in \text{Harm}_{X_M}^{\mp}(M)$. Hence, we can rewrite $\varphi = d_{X_M} \beta$ as follows, $\varphi = d_{X_M}(\beta - d_{X_M} \sigma) \in d_{X_M}(\text{Harm}_{X_M}^{\mp}(M))$.

Finally, to prove $\bar{\delta}_{X_M}$ is onto. Given $\alpha \in \ker d_{X_M}$, we should find $\varphi \in E_{X_M} \text{Harm}_{X_M}^{\pm}(M)$ such that $\delta_{X_M} \varphi - \alpha \in d_{X_M} \Omega_G^{\pm}$. Applying lemma (6.2.2) on α , then we can write $\alpha = \Delta_{X_M} \beta$ and then we take $\varphi = d_{X_M} \beta$. One should notice that $\Delta_{X_M} \varphi = \Delta_{X_M} d_{X_M} \beta = d_{X_M} \Delta_{X_M} \beta = d_{X_M} \alpha = 0$, since $\alpha \in \ker d_{X_M}$. Thus, $\varphi \in E_{X_M} \text{Harm}_{X_M}^{\pm}(M)$. Now,

$$\delta_{X_M} \varphi = \delta_{X_M} d_{X_M} \beta = \Delta_{X_M} \beta - d_{X_M} \delta_{X_M} \beta = \alpha - d_{X_M} \delta_{X_M} \beta$$

So, $\delta_{X_M} \varphi - \alpha \in d_{X_M} \Omega_G^{\pm}$, as desired. Hence $\bar{\delta}_{X_M}$ is bijection map. So, eq.(6.4) holds. \square

In addition, Δ_{X_M} and δ_{X_M} commute. Hence, the coboundary operator δ_{X_M} preserves the X_M -harmonicity of invariant forms. i.e.

$$\text{Harm}_{X_M}^{\pm}(M) \xrightarrow{\delta_{X_M}} \text{Harm}_{X_M}^{\mp}(M)$$

Thus, $(\text{Harm}_{X_M}^*(M), \delta_{X_M})$ is a subcomplex of the \mathbb{Z}_2 -graded complex $(\Omega_G^*, \delta_{X_M})$. Therefore,

we can compute the cohomology of this complex which we denote by $H^\pm(\text{Harm}_{X_M}^*(M), \delta_{X_M})$. Moreover, the Hodge star operator provides the isomorphism $H^\pm(\text{Harm}_{X_M}^*(M), \delta_{X_M}) \cong H^{n-\pm}(\text{Harm}_{X_M}^*(M), d_{X_M})$ and then applying X_M -Poincaré-Lefschetz duality on the right-hand side of eq.(6.4) to obtain the following corollary.

Corollary 6.2.4

$$H^\pm(\text{Harm}_{X_M}^*(M), \delta_{X_M}) \cong H_{X_M}^\pm(M, \partial M) + H_{X_M}^\mp(M, \partial M) = H_{X_M}^*(M, \partial M)$$

6.3 Conclusions

In chapter 3, we elucidate the connection between the X_M -cohomology groups of M and the free part of the relative and absolute equivariant cohomology groups of M . Then Theorem 6.2.3 and corollary 3.4.4 imply the following theorem.

Theorem 6.3.1 *If $N(X_M) = F$, then the (even/odd) X_M -harmonic cohomology of the sub-complexes $(\text{Harm}_{X_M}^*(M), d_{X_M})$ and $(\text{Harm}_{X_M}^*(M), \delta_{X_M})$ completely determine the free part of the absolute and relative equivariant cohomology groups, i.e.*

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) \cong H_G^*(M)/\mathfrak{m}_X H_G^*(M) \cong H^*(F)$$

and

$$H^\pm(\text{Harm}_{X_M}^*(M), \delta_{X_M}) \cong H_G^*(M, \partial M)/\mathfrak{m}_X H_G^*(M, \partial M) \cong H^*(F, \partial F).$$

Other perhaps interesting conclusion is that: Applying theorem 6.1.1 on the manifold F with its boundary ∂F and then using theorem 6.3.1 we obtain the following theorem

Theorem 6.3.2 *If every component of F has a boundary, then*

$$H^\pm(\text{Harm}_{X_M}^*(M), d_{X_M}) \cong H^\pm(\text{Harm}^*(F), d).$$

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