

QUASITORIC FUNCTORS AND FINAL SPACES

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We introduce open quasitoric manifolds and their functorial properties, including complex bundle maps of their stable tangent bundles, and relate these new spaces to the standard constructions of toric topology: quasitoric manifolds, moment angle manifolds and polyhedral products.

We extend the domain of these constructions to countably infinite simplicial complexes, clarifying and generalising constructions of Davis and Januszkiewicz. In particular we describe final spaces in the categories of open quasitoric manifolds and quasitoric spaces, as well as in the categories of characteristic pairs and dicharacteristic pairs. We show how quasitoric manifolds can be constructed smoothly as pullbacks of the final spaces \mathcal{QT}_n for $n \geq 1$, and how stably complex structure also arises this way.

We calculate the integral cohomology of quasitoric spaces over Cohen-Macaulay simplicial complexes, including the final spaces \mathcal{QT}_n as a special case. We describe a procedure for calculating the Chern numbers of a quasitoric manifold M^{2n} and, relating this to our cohomology calculations, show how it may be interpreted in terms of the simplicial homology of \mathcal{H}_n , the simplicial complex underlying \mathcal{QT}_n .

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Chapter 1

Introduction

Historical background

The subject now known as toric topology began with Davis and Januszkiewicz's definition and study of toric manifolds (now called quasitoric manifolds, to distinguish them from the smooth projective toric varieties of algebraic geometry) in their 1991 paper [10]. They defined a *quasitoric manifold* as a topological manifold M^{2n} with action of the n -torus $T^n = (S^1)^n$ satisfying local and global conditions. Specifically, M^{2n} should be locally isomorphic to the standard representation $T^n \circlearrowleft \mathbf{C}^n$ and the orbit space should take the form of a *simple polytope* P^n .

They showed that a quasitoric manifold could be reconstructed up to equivariant homeomorphism from suitable combinatorial data, remarking that such a manifold should admit an equivariant smooth structure. The combinatorial data consists of the polytope P^n , defined up to combinatorial equivalence, and a *characteristic function* $\bar{\ell}$ (our notation) which assigns to each facet of P^n a one-dimensional subtorus $T^1 \leq T^n$, subject to a compatibility condition. They observed that $\bar{\ell}$ could be interpreted as a non-degenerate simplicial map $K_P \rightarrow \mathcal{J}_n$ where K_P is the $(n-1)$ -dimensional simplicial complex dual to P^n and \mathcal{J}_n is a certain infinite $(n-1)$ -dimensional complex.

In fact they showed how to construct a *quasitoric space* $\overline{\mathcal{X}}(K, \bar{\ell})$ from any simplicial complex K admitting a characteristic function, by replacing the simple polytope P^n with a cubical complex $\mathcal{I}(K)$ dual to K . This applies in particular to the complex \mathcal{J}_n with its canonical characteristic function $\bar{\mu}$. Then given any non-degenerate simplicial map $K \rightarrow \mathcal{J}_n$ we have an induced characteristic function $\bar{\ell}$ on K and a map $\mathcal{I}(K) \rightarrow \mathcal{I}(\mathcal{J}_n)$, and the quasitoric space $\overline{\mathcal{X}}(K, \bar{\ell})$ over $\mathcal{I}(K)$

is obtained as a pullback from the quasitoric space over $\mathcal{I}(\mathcal{J}_n)$. However, these quasitoric spaces over cubical complexes are not generally manifolds.

Subsequently, [5] and [8] have given an explicit construction of quasitoric manifolds as smooth manifolds and have shown that any quasitoric manifold M^{2n} admits a number of equivariant stably complex structures, obtained by choosing an *omniorientation* for M^{2n} . This is equivalent to a choice of orientation for P^n and for the one-dimensional subtorus of T^n assigned to each facet; thus we obtain the notion of a directed characteristic function or *dicharacteristic*, ℓ . The omniorientation yields an orientation for the normal bundle of each *facial submanifold* which makes each such bundle into a complex line bundle. It is the sum of these complex line bundles, called *facial bundles*, that forms the stably complex structure. Davis and Januszkiewicz discussed these structures, but did not make clear what choices needed to be made. As a consequence of their work on omniorientations, [8] showed that, for $n > 1$, every complex cobordism class in dimension $2n$ contains an omnioriented quasitoric manifold.

An important role in the study of quasitoric manifolds is played by the *moment angle manifold* Z_P associated to P^n . The moment angle manifold is homeomorphic to the *moment angle complex* $\mathcal{Z}(K_P)$, where $\mathcal{Z}(-)$ is a covariant functor from the category of simplicial complexes to topological spaces. The study of moment angle complexes, and their generalisations, is now a central theme in toric topology. The most common generalisation is the *polyhedral product* $(X, A)^K$ defined for a pair of spaces (X, A) and simplicial complex K . In this context we have $\mathcal{Z}(K) = (D^2, S^1)^K$ and the cubical complex $\mathcal{I}(K) = (I, \{1\})^K$.

The contents of this thesis

The motivation for this thesis is to develop further the construction of quasitoric manifolds as pullbacks. Davis and Januszkiewicz discussed pullbacks of the *linear model* $\mathbf{C}^n \rightarrow \mathbf{R}_{\geq}^n$ in [10], before describing the simplicial complex \mathcal{J}_n with canonical characteristic function $\bar{\mu}$ and the quasitoric space $\bar{\mathcal{X}}(\mathcal{J}_n, \bar{\mu}, \cdot)$ over $\mathcal{I}(\mathcal{J}_n)$ so that any quasitoric space is a pullback of the diagram:

$$\begin{array}{ccc} \bar{\mathcal{X}}(K, \bar{\ell}) & \dashrightarrow & \bar{\mathcal{X}}(\mathcal{J}_n, \bar{\mu}) \\ \vdots \downarrow & & \downarrow \\ \mathcal{I}(K) & \longrightarrow & \mathcal{I}(\mathcal{J}_n) \end{array}$$

The space $\overline{\mathcal{X}}(\mathcal{J}_n, \bar{\mu})$ is not a manifold. Although we know that $\overline{\mathcal{X}}(K, \bar{\ell})$ is equivariantly homeomorphic to the stably complex quasitoric manifold $M^{2n}(P^n, \ell)$ in the case that K is the dual to the simple polytope P^n (and $\bar{\ell}$ is the undirected characteristic function underlying the dicharacteristic ℓ), the pullback construction does not yield the smooth manifold structure, nor the stably complex structure.

In Chapter 5 we present a new picture where:

1. We replace the simplicial complex \mathcal{J}_n by a related complex \mathcal{H}_n with canonical dicharacteristic ν . This allows us to deal with pairs (K, ℓ) , where ℓ is a dicharacteristic on the simplicial complex K .
2. We replace the construction $\overline{\mathcal{X}}(K, \bar{\ell})$ by a new construction $\mathcal{Y}(K, \ell)$ which yields a stably complex smooth T^n -manifold with orbit space $\mathcal{R}(K)$ a smooth manifold with corners (both infinite dimensional in the case of $\mathcal{Y}(\mathcal{H}_n, \nu)$, which we denote \mathcal{QT}_n with orbit space \mathcal{Q}_n).
3. We obtain $M^{2n}(P^n, \ell)$ as a pullback of \mathcal{QT}_n over a smooth map $P^n \rightarrow \mathcal{Q}_n$ and the pullback construction yields the smooth and stably complex structures on $M^{2n}(P^n, \ell)$.

The pullback construction leads to Theorem 5.1.7 and Corollary 5.1.8 while the realisation of the stably complex structure leads to Theorem 5.2.1.

These results are based on foundations laid down in Chapters 3 and 4. In Chapter 3 we define the new functors $\mathcal{Y}(-)$ and $\mathcal{R}(-)$ from the category of finite dicharacteristics pairs (of dimension n) to the category of stably complex T^n -manifolds; and from the category of finite simplicial complexes (with non-degenerate simplicial maps) to the category of smooth manifolds with corners, respectively. We refer to the manifolds $\mathcal{Y}(K, \ell)$ as *open quasitoric manifolds*. The fact that these constructions are functors follows from a new construction that makes the polyhedral product $(X, A)^K$ functorial in K in the case that (X, A) is a pair of topological monoids (Proposition 3.1.1). This makes various known constructions including $\mathcal{U}(K)$ (open moment angle complexes) and $\mathcal{X}(K, \ell)$ (quasitoric spaces) functorial too.

There is a smooth embedding $M^{2n}(P^n, \ell) \rightarrow \mathcal{Y}(K, \ell)$ when K is dual to P^n , and we study the behaviour of the complex line bundles over $M^{2n}(P^n, \ell)$ and $\mathcal{Y}(K, \ell)$ under this map, and also the map $\mathcal{Y}(f): \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ given a non-degenerate simplicial map $f: K_1 \rightarrow K_2$. There turns out to be a simple

formula for the pullback $\mathcal{Y}(f)^*(\xi_w)$ in terms of the map f , where ξ_w is the complex line bundle over $\mathcal{Y}(K_2, \ell_2)$ relating to vertex w of K_2 . We summarise this and related results in Theorem 3.5.2.

In Chapter 4 we extend these construction to include finite dimensional simplicial complexes on countably infinite vertex sets. We define these by extending the definition of the polyhedral product functor as the colimit of a $\text{CAT}(\mathbf{K})$ diagram. The extension to the countably infinite case is fairly straightforward, but nevertheless is not found in the literature. We use some of the theory of \mathbf{R}^∞ and \mathbf{C}^∞ smooth manifolds to describe infinite dimensional open quasitoric manifolds. We are then ready to describe the final object \mathcal{QT}_n in the category of open quasitoric manifolds, with properties given in Theorems 4.5.2 and 4.5.3.

Having described the infinite dimensional manifold \mathcal{QT}_n and its stably complex structure, we are honour bound to calculate its cohomology. This we do in Chapter 6, first presenting some necessary background material from commutative algebra. Davis and Januszkiewicz calculated the integral cohomology of quasitoric manifolds in [10], using the initial observation (obtained by demonstrating a particular CW-structure) that their cohomology is concentrated in even degree. They also calculated the *rational* cohomology of quasitoric spaces over finite Cohen-Macaulay simplicial complexes.

We calculate the integral cohomology of quasitoric spaces (and hence open quasitoric manifolds) over finite or countably infinite, finite dimensional Cohen-Macaulay complexes, giving the new result in Theorem 6.4.6. The special case of \mathcal{QT}_n is presented as Theorem 6.4.8. The progression from field coefficients to integer coefficients is not immediate, and we require Lemma 6.4.3; although a roundabout argument could possibly be constructed by calculating over \mathbf{Z}_p for each prime p . The extension to the countably infinite case requires some more machinery: we are working with non-Noetherian rings, and there is no single analogue of the Cohen-Macaulay condition for these.

As in the cases calculated previously, we find that the cohomology of quasitoric spaces (over Cohen-Macaulay complexes) is generated by the first Chern classes of the facial line bundles, so given a map of such quasitoric spaces we are able to reference Theorem 4.5.3 to describe the map in cohomology.

In Chapter 7, we address the open problem of determining when two stably complex quasitoric manifolds are complex cobordant. We give an answer to the

problem by giving a procedure for calculating the Chern numbers of any omnioriented quasitoric manifold in terms of the underlying combinatorial data; this is a result of Corollary 7.1.6. This does not depend on the material in the rest of the thesis, involving only calculations in the cohomology of a given quasitoric manifold. However, we tie things together by giving a combinatorial interpretation in terms of the simplicial homology of the complex \mathcal{H}_n , in Theorem 7.2.3: we show that the Chern numbers of $M^{2n}(P^n, \ell)$ are determined by the image of the map $\ell_*: H_{n-1}(K) \rightarrow H_{n-1}(\mathcal{H}_n)$ in simplicial homology, where K is the simplicial complex dual to P^n . This interpretation does depend on previous chapters, using in particular the cohomology of \mathcal{QT}_n as calculated in Theorem 6.4.8 and the description of the map $H^*(\mathcal{QT}_n) \rightarrow H^*(M^{2n})$ in cohomology.

Chapter 2

Background

In this chapter we present an overview of the central topics of toric topology, including the necessary background material for subsequent chapters. Much of the material here can be found in the book [6].

A convention on multiple indices

When working with polytopes, simplicial complexes, and generally indexed subsets of indexed sets, we often have to work with multiple indices. For example, a simplicial complex may have vertices v_1, \dots, v_m or may simply have vertex set V . A subcomplex may have vertex set $\{v_j \mid j \in J\}$ for some indexing set J , and J may have n elements, which it is convenient to label j_1, \dots, j_n . Rather than write v_{j_1}, \dots, v_{j_n} for the corresponding elements of V , we will write v_1, \dots, v_n where this cannot cause confusion. It should be clear from the context that we do not mean the first n elements of the ordered set V , but rather the elements indexed by the ordered set J .

2.1 Simple polytopes

We begin with some classical background on simple polytopes, simplicial complexes and cubical complexes, which can be found in [6].

A *polytope* is the convex hull of a finitely many points in \mathbf{R}^n . Equivalently, a convex polyhedron is an intersection of finitely many half-spaces in \mathbf{R}^n :

$$P = \{x \in \mathbf{R}^n \mid a_i \cdot x + b_i \geq 0, i = 1, \dots, m\} \tag{2.1}$$

where $a_i \in (\mathbf{R}^n)^*$ and $b_i \in \mathbf{R}$. Then a polytope is a bounded convex polyhedron. This second definition will prove more useful.

The dimension of a polytope is defined to be the dimension of its affine hull, that is, the smallest affine subspace of \mathbf{R}^n that contains P . By choosing coordinates for that subspace, we may assume that the dimension of P is equal to the dimension of the ambient space \mathbf{R}^n . We will also insist that none of the inequalities is redundant. Thus m is the number of *facets* (codimension one faces) of P^n .

The faces of a polytope form a poset under inclusion, called the face poset, and two polytopes are said to be *combinatorially equivalent* if their face posets are isomorphic. We shall generally be interested in polytopes up to combinatorial equivalence; we refer to an equivalence class as a *combinatorial polytope*. By contrast, the convex hull of a set of points in \mathbf{R}^n is a *geometric polytope*.

We will be interested in *simple* polytopes. This is a combinatorial property:

Definition 2.1.1. A polytope P^n is *simple* if exactly n facets meet at each vertex.

2.1.1 Simplicial duals and cubical complexes

To any polytope P^n there is an associated simplicial complex:

Definition 2.1.2. Let P^n be a polytope with facets F_1, \dots, F_m . Then the *simplicial dual* or *dual complex* of P^n is the complex K_P whose vertex set is $V = \{v_1, \dots, v_m\}$ and where $v_{i_1} \dots v_{i_r}$ is a simplex of K_P if and only if the intersection $F_{i_1} \cap \dots \cap F_{i_r}$ is non-empty.

It is a standard result that P^n is simple if and only if K_P is an $(n - 1)$ -dimensional simplicial sphere. Indeed, if P^n is simple then K_P is the boundary of the n -dimensional simplicial polytope P^* described in [6, Chapter 1]. On the other hand, if P^n is not simple then more than n facets meet at some vertex, so the dimension of K_P is greater than $(n - 1)$.

To any finite simplicial complex, there are associated two *cubical complexes*. Informally, a finite cubical complex C consists of a finite collection of cubes $U = \{I_j^{q_j}\}$ glued along common faces $I^r = I_j^{q_j} \cap I_k^{q_k}$, such that every face of a cube in U is in U . For a formal definition see [6]. The cubes in U are the *faces* of C .

The n -cube $I^n = [0, 1]^n \subset \mathbf{R}^n$ is a cubical complex, with faces

$$C_{\sigma\subset\tau} = \{x \in I^n \mid x_i = 0 \text{ for } i \in \sigma, x_i = 1 \text{ for } i \notin \tau\}$$

for each pair of subsets $\sigma \subset \tau \subset \{1, \dots, n\}$.

Example 2.1.3. Consider the 3-cube I^3 . Let $\sigma = \{1\}$ and $\tau = \{1, 3\}$. Then the face $C_{\sigma\subset\tau} \subset I^3$ is the edge given by $x_1 = 0, x_2 = 1$.

In contrast with the elementary theorem that says any finite simplicial complex K may be embedded as a subcomplex of Δ^N for some large N , it is *not* true that any finite cubical complex can be embedded as a subcomplex of some I^N . Nevertheless, the complexes we are interested in are subcomplexes of I^m , where m is the number of vertices of K .

Definition 2.1.4. Let K be a simplicial complex on vertex set $V = \{v_1, \dots, v_m\}$. The cubical complex $\text{cub}(K)$ is the subcomplex of I^m consisting of the faces $C_{\sigma\subset\tau}$ for $\emptyset \neq \sigma \subset \tau \subset K$.

Definition 2.1.5. Let K be a simplicial complex on vertex set $V = \{v_1, \dots, v_m\}$. The cubical complex $\text{cc}(K)$ is the subcomplex of I^m consisting of the faces $C_{\sigma\subset\tau}$ for $\sigma \subset \tau \subset K$.

Note that, since the empty set is excluded from the choice of σ in 2.1.4, the complex $\text{cub}(K)$ consists of precisely those points of $\text{cc}(K)$ where at least one coordinate is equal to zero.

Example 2.1.6. Consider the 1-dimensional complex K_Δ dual to the 2-dimensional simple polytope Δ^2 . Then $K_\Delta = \partial\Delta^2$, the boundary of a 2-simplex. It has vertex set $V = \{v_1, v_2, v_3\}$ and has simplices corresponding to all subsets of V except for V itself.

Since K_Δ does not contain the 2-simplex $\tau' = v_1v_2v_3$, the cubical complex $\text{cc}(K_\Delta)$ does not contain any faces of the form $C_{\sigma\subset\tau'}$. As a result, $\text{cc}(K_\Delta)$ is the subset of I^3 where at least one coordinate is equal to one, and the complex $\text{cub}(K_\Delta)$ is the subcomplex of $\text{cc}(K_\Delta)$ where at least one coordinate is equal to zero.

The preceding example illustrates the following *cubical decomposition* of a simple polytope, which is explained in detail in section 4.2 of [6]:

Proposition 2.1.7. *Let P^n be a simple polytope with facets F_1, \dots, F_m and simplicial dual K_P . There is a piecewise linear (PL) homeomorphism $i_P: P^n \rightarrow cc(K_P) \subset I^m$. Moreover, for each facet F_i we have $F_i = i_P^{-1}(\{x \in I^m \mid x_i = 0\})$ and hence the restriction $i_P|_{\partial P^n}$ is a PL homeomorphism $\partial P^n \cong cub(K_P) \subset I^m$.*

Sketch proof. Choose a point x_F in the interior of each face F of P^n (including P^n itself and the vertices). Each vertex v of P^n is the intersection of n facets F_{i_1}, \dots, F_{i_n} and there are 2^n faces of P^n which contain v . The points x_F corresponding to these faces form the vertices of a combinatorial n -cube C_v^n , and the map i_P is chosen to take C_v^n piecewise linearly to the cube $C_{\emptyset \subset \tau} \subset I^n$, where $\tau = \{i_1, \dots, i_n\}$. Moreover, i_P takes the intersection $C_v^n \cap F$ to the cube $C_{\sigma \subset \tau}$ for each face $F = \bigcap_{i \in \sigma} F_i$. \square

Example 2.1.8. Consider again the 2-simplex Δ^2 , with simplicial dual $K_\Delta = \partial\Delta^2$. The cubical complex $cc(K_\Delta)$ is the subset of I^3 where at least one coordinate is equal to one, and the complex $cub(K_\Delta)$ is the subcomplex of $cc(\Delta)$ where at least one coordinate is equal to zero.

A map i_P satisfying the conditions of 2.1.7 takes a chosen point in the interior of Δ^2 to $(1, 1, 1)$, the vertices of Δ^2 to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and a chosen point on each edge of Δ^2 to each of the points $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$.

2.2 Quasitoric manifolds

We now review the definition and construction of quasitoric manifolds, as described in [10].

The n -torus T^n is the product of n copies of the circle group $T^n = (S^1)^n$. It is usual to think of S^1 as the multiplicative group of unit complex numbers, and the *standard action* or *standard representation* of T^n on \mathbf{C}^n is obtained by mapping T^n isomorphically to the subgroup of $U(n)$ consisting of diagonal matrices. Explicitly,

$$T^n \times \mathbf{C}^n \rightarrow \mathbf{C}^n: (t_1, \dots, t_n) \times (z_1, \dots, z_n) \rightarrow (t_1 z_1, \dots, t_n z_n)$$

Nevertheless, it is sometimes useful to consider instead the isomorphism $T^n = \mathbf{R}^n / \mathbf{Z}^n$. There is an explicit isomorphism $\phi: \mathbf{R}^n / \mathbf{Z}^n \rightarrow (S^1)^n$ given by $\phi(x_1, \dots, x_n) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$. Moreover, any homomorphism $\Theta: T^k \rightarrow T^n$ is induced by a linear map $A: \mathbf{R}^k \rightarrow \mathbf{R}^n$, whose matrix \mathbf{A} must have integer coefficients in order to

descend to a map of the quotients $\mathbf{R}^k/\mathbf{Z}^k \rightarrow \mathbf{R}^n/\mathbf{Z}^n$. The following observation has been used implicitly in the literature since [10], where it motivates condition (*):

Lemma 2.2.1. *The homomorphism Θ is injective if and only if the columns of \mathbf{A} form part of a \mathbf{Z} -basis for \mathbf{Z}^n .*

Proof. Let $\langle A \rangle$ denote the submodule of \mathbf{Z}^n spanned over \mathbf{Z} by the columns of \mathbf{A} . We note that the columns of \mathbf{A} form part of a \mathbf{Z} -basis for \mathbf{Z}^n if and only if \mathbf{A} is non-singular and $\langle A \rangle$ is a direct summand of \mathbf{Z}^n ; the latter being true if and only if $ay \in \langle A \rangle \Rightarrow y \in \langle A \rangle$ for $a \in \mathbf{Z}$ and $y \in \mathbf{Z}^n$. We note also that Θ is injective if and only if $A(x) \in \mathbf{Z}^n \Rightarrow x \in \mathbf{Z}^k$

Suppose Θ is not injective. Then there is some $x \in \mathbf{R}^k \setminus \mathbf{Z}^k$ with $A(x) \in \mathbf{Z}^n$. We may assume \mathbf{A} is non-singular, or it is trivially false that the columns of \mathbf{A} form part of a \mathbf{Z} -basis for \mathbf{Z}^n . So A is injective and hence $y = A(x)$ is not in $\langle A \rangle$. Since \mathbf{A} is non-singular and has integer coefficients, x has rational co-ordinates and so some integer multiple of x lies in \mathbf{Z}^k and some integer multiple of y lies in $\langle A \rangle$. Thus $\langle A \rangle$ is not a direct summand of \mathbf{Z}^n .

On the other hand, suppose the columns of \mathbf{A} form part of a \mathbf{Z} -basis for \mathbf{Z}^n . Consider some $x \in \mathbf{R}^k$ for which $y = A(x)$ lies in \mathbf{Z}^n . Some integer multiple of y lies in $\langle A \rangle$, and so $y \in \langle A \rangle$ because $\langle A \rangle$ is a direct summand of \mathbf{Z}^n . So $y = A(x')$ for some integer vector $x' \in \mathbf{R}^k$, but A is injective so $x' = x$. \square

We will make great use of this observation, particularly in the cases $k = n$, where Θ is an automorphism if and only if $\det \mathbf{A} = \pm 1$; and $k = 1$, where \mathbf{A} is an $(n \times 1)$ integer vector whose coordinates have least common multiple equal to one if and only if Θ is injective. We shall call such a vector *primitive*. In the case $k = 1$, we will write T_v for the 1-dimensional image of the homomorphism $\Theta: T^1 \rightarrow T^n$ determined by the vector v .

A *standard chart* for a T^n -manifold is a T^n stable open set U along with a homeomorphism $\varphi: U \rightarrow V \subset \mathbf{C}^n$ and automorphism $\Theta: T^n \rightarrow T^n$ such that $\varphi(t \cdot u) = \Theta(t) \cdot \varphi(u)$ for all $u \in U$ and $t \in T^n$. In other words, φ is a Θ -equivariant homeomorphism. We say an action of T^n on a manifold M^{2n} is *locally standard* if every point $x \in M^{2n}$ lies in some standard chart.

We are ready to give the definition of quasitoric manifold, as in [10]:

Definition 2.2.2. *A quasitoric manifold over the combinatorial simple polytope P^n is a T^n -manifold M^{2n} with locally standard action, along with a projection*

map $\pi: M^{2n} \rightarrow P^n$ such that the preimage of a point in the interior of any codimension- k face of P^n is a k -dimensional orbit of the T^n action.

Note that the polytope P^n is integral to the definition. The term *quasitoric* is used to distinguish these manifolds from the non-singular toric varieties of toric geometry, which are known as *toric manifolds*. Every toric manifold is topologically a quasitoric manifold, but the converse does not hold. This was demonstrated by [10], although they retained the term toric manifold for the topological construction.

Example 2.2.3. The most familiar example of a quasitoric manifold is complex projective space $\mathbf{C}P^n$, obtained as a quotient space $\mathbf{C}P^n = (\mathbf{C}^{n+1} \setminus 0)/\mathbf{C}_*$ where $\mathbf{C}_* = \mathbf{C} \setminus \{0\}$ acts on \mathbf{C}^{n+1} by coordinate-wise multiplication $\lambda \times (z_1, \dots, z_{n+1}) \mapsto (\lambda z_1, \dots, \lambda z_{n+1})$. Alternatively, first dividing out the action of the positive reals, $\mathbf{C}P^n = S^{2n+1}/S^1$ where S^{2n+1} is the unit sphere in \mathbf{C}^{n+1} and S^1 is the group of unit complex numbers.

We represent points of $\mathbf{C}P^n$ in homogenous coordinates $(z_1 : \dots : z_{n+1})$. There is a well-defined T^n action on $\mathbf{C}P^n$ given by

$$(t_1, \dots, t_n) \times (z_1 : \dots : z_{n+1}) \mapsto (t_1 z_1 : \dots : t_n z_n : z_{n+1})$$

which makes $\mathbf{C}P^n$ into a quasitoric manifold with base polytope Δ^n , the n -simplex. Realising Δ^n as

$$\Delta^n = \{x \in \mathbf{R}^{n+1} \mid x_1 + \dots + x_{n+1} = 1, x_i \geq 0 \text{ for all } i\},$$

the projection map $\pi: \mathbf{C}P^n \rightarrow \Delta^n$ may be taken to be

$$\pi(z_1 : \dots : z_{n+1}) = \frac{1}{|z_1|^2 + \dots + |z_{n+1}|^2} (|z_1|^2, \dots, |z_{n+1}|^2).$$

A trivial example of a T^n action that is *not* locally standard is obtained by replacing the T^n action on $\mathbf{C}P^n$ by the action

$$(t_1, \dots, t_n) \times (z_1 : \dots : z_{n+1}) \mapsto (t_1^2 z_1 : \dots : t_n z_n : z_{n+1})$$

whereby every point of $\mathbf{C}P^n$ is fixed by $t = (-1, 1, \dots, 1)$. A more interesting class of such examples is given by the weighted projective spaces (see e.g [16]).

These are quotients of S^{2n+1} under a twisted action of S^1 given by

$$\lambda \times (z_1, \dots, z_{n+1}) \mapsto (\lambda^{\chi_1} z_1, \dots, \lambda^{\chi_{n+1}} z_{n+1})$$

for an integer weight vector $(\chi_1, \dots, \chi_{n+1})$. The resulting spaces are smooth T^n -orbifolds, but not generally smooth T^n -manifolds.

2.2.1 The characteristic function

Now let F_1, \dots, F_m be the facets of P^n . For each F_i , points in the preimage $\pi^{-1}(\text{int } F_i)$ share a common 1-dimensional isotropy group $T(F_i) \subset T^n$ and the preimage $\pi^{-1}(F_i)$ is itself a quasitoric manifold $M_i^{2(n-1)}$ over the simple polytope F_i under the action of $T^n/T(F_i) \cong T^{n-1}$, known as the *facial submanifold* corresponding to F_i ([10], lemma 1.3).

We may choose vectors $\lambda_i \in \mathbf{Z}^n$ such that the homomorphism $\Theta_{\lambda_i}: T^1 \rightarrow T^n$ represented by each λ_i is injective, with image $T(F_i)$. The vectors λ_i are primitive, and are determined up to sign. If we do not wish to specify a sign, we may represent $\pm\lambda_i$ by the line $\bar{\lambda}_i = \langle \lambda_i \rangle$ in \mathbf{Z}^n . The function $\bar{\ell}: F_i \mapsto \bar{\lambda}_i$ is called the *characteristic function* of M^{2n} .

It is common to identify $\bar{\ell}$ with the function $F_i \mapsto T(F_i)$, equating the line $\bar{\lambda}_i$ with the subgroup $T(F_i) < T^n$. This can be a source of confusion, so we will clearly distinguish the two functions by using the notation $\bar{\lambda}_i^T$ whenever we wish to refer to the group $T(F_i)$ rather than the line $\bar{\lambda}_i$.

Because the T^n action is locally standard, the matrix with columns $\lambda_1, \dots, \lambda_k$ defines a monomorphism $\Theta: T^k \rightarrow T^n$ whenever $F_1 \cap \dots \cap F_k \neq \emptyset$, so (e.g. by lemma 2.2.1) the isotropy groups of intersecting facets must satisfy the following condition, which does not depend on the choice of sign for each λ_i :

Condition 2.2.4 (Condition (*) of [10]). Whenever n facets F_1, \dots, F_n have non-empty intersection, the matrix with columns $\lambda_1, \dots, \lambda_n$ must have determinant ± 1 . Equivalently, whenever k facets F_1, \dots, F_k have non-empty intersection, the vectors $\lambda_1, \dots, \lambda_k$ must form part of a \mathbf{Z} -basis for \mathbf{Z}^n .

Example 2.2.5. We continue the case of $\mathbf{C}P^n$ from Example 2.2.3. The polytope P^n is

$$\Delta^n = \{x \in \mathbf{R}^{n+1} \mid x_1 + \dots + x_{n+1} = 1, x_i \geq 0 \text{ for all } i\}$$

with facets F_1, \dots, F_{n+1} given by the equalities $x_i = 0$ for $i = 1, \dots, n+1$. For each facet we have

$$\pi^{-1}(F_i) = \{(z_1 : \dots : z_{n+1}) \in \mathbf{C}P^n \mid z_i = 0\}$$

and

$$\pi^{-1}(\text{int } F_i) = \{(z_1 : \dots : z_{n+1}) \in \mathbf{C}P^n \mid z_i = 0, z_j \neq 0 \text{ for } i \neq j\}.$$

Then for $i = 1, \dots, n$ we have $T(F_i) = T_i$, the i -th coordinate torus, with $\lambda_i = \pm e_i$, the i -th standard basis vector. For $i = n+1$ we have $T(F_{n+1}) = T_\delta$, the diagonal torus $\{(t, \dots, t) \in T^n\}$, and $\lambda_{n+1} = \pm(e_1 + \dots + e_n)$.

2.2.2 Construction of quasitoric manifolds

A quasitoric manifold M^{2n} may be reconstructed up to equivariant homeomorphism from the polytope P^n and characteristic function $\bar{\ell}$ ([10], proposition 1.8). Indeed, let P^n be any simple polytope and $\bar{\ell}$ a map which assigns a line in \mathbf{Z}^n to each facet of P^n . If $\bar{\ell}$ satisfies condition 2.2.4 then it is called a characteristic function for P^n and the pair $(P^n, \bar{\ell})$ is sometimes called a *characteristic pair*. For any point $x \in P^n$ let $\bar{\ell}(x) = \prod_{x \in F_i} \bar{\lambda}_i^T$, the product in T^n of the subgroups $\bar{\lambda}_i^T$ where $\bar{\lambda}_i = \bar{\ell}(F_i)$. We construct a quasitoric manifold from $(P^n, \bar{\ell})$ as in [10] by forming:

$$M^{2n}(P^n, \bar{\ell}) = (T^n \times P^n) / \sim$$

where the equivalence relation \sim is defined by $(t_1, x_1) \sim (t_2, x_2)$ if and only if $x_1 = x_2$ and $t_1 t_2^{-1} \in \bar{\ell}(x_1)$.

The result is indeed a quasitoric manifold over P^n with characteristic function $\bar{\ell}$, and any two quasitoric manifolds over P^n with the same characteristic function are equivariantly homeomorphic. Thus we have a classification of quasitoric manifolds up to equivariant homeomorphism in terms of characteristic pairs, due to [10]. To be precise, let $\varphi: T^n \rightarrow T^n$ be any automorphism of the torus. Then a φ -equivariant homeomorphism of quasitoric manifolds M_1^{2n} and M_2^{2n} is a homeomorphism $f: M_1^{2n} \rightarrow M_2^{2n}$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$. The automorphism φ induces an automorphism of the lattice of toric subgroups of T^n , and the resulting change of characteristic function is called a φ -translation of the pair $(P^n, \bar{\ell})$. Then there is a bijection between φ -equivariant homeomorphism classes of quasitoric manifolds and φ -translation classes of characteristic

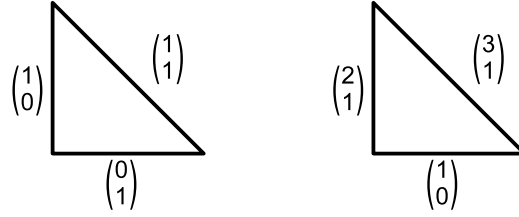


Figure 2.1: Two characteristic functions on the triangle

pairs. In particular, restricting to the case where φ is the identity, equivariant homeomorphism classes of quasitoric manifolds are in one-to-one correspondence with characteristic pairs.

Example 2.2.6. We may construct $\mathbf{C}P^n$ up to equivariant homeomorphism as $T^n \times \Delta^n / \sim$ where $(t_1, x) \sim (t_2, x)$ if and only if $t_1 t_2^{-1} \in \prod_{x_i=0} \bar{\lambda}_i^T$ where the isotropy groups $\bar{\lambda}_i^T$ are

$$\begin{aligned} \bar{\lambda}_i^T &= T_i \text{ for } i = 1, \dots, n \\ \bar{\lambda}_{n+1}^T &= T_\delta \end{aligned}$$

as described in example 2.2.5. The left-most triangle in figure 2.1 shows the case $n = 2$ with the facets of the triangle Δ^2 labelled by the characteristic function $\bar{\ell}$ (with choice of sign for each vector). An automorphism $\varphi: T^2 \rightarrow T^2$ represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

yields the alternative characteristic function shown on the right-most triangle. The quasitoric manifold M^4 constructed with the second characteristic function is φ -equivariantly homeomorphic to $\mathbf{C}P^2$.

2.2.3 Smooth construction

The construction above does not yield an explicit smooth structure on the quasitoric manifold M^{2n} . A more geometric construction was given by [8], as follows.

We begin by taking a realisation of the simple polytope P^n as an intersection

of half-spaces in \mathbf{R}^n as in (2.1):

$$P^n = \{x \in \mathbf{R}^n \mid a_i \cdot x + b_i \geq 0, i = 1, \dots, m\}$$

where $a_i \in (\mathbf{R}^n)^*$ and $b_i \in \mathbf{R}$. Then we define the injection $\iota: \mathbf{R}^n \rightarrow \mathbf{R}^m$ by $\iota(x)_i = a_i \cdot x + b_i$. This maps points of P^n into \mathbf{R}_{\geq}^m while sending points of $\mathbf{R}^n \setminus P^n$ elsewhere, so we have $P^n = \iota^{-1}(\mathbf{R}_{\geq}^m)$. There is a projection $\rho: \mathbf{C}^m \rightarrow \mathbf{R}^m$ given by $\rho(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$ with image \mathbf{R}_{\geq}^m , and we define the subspace $Z_P \subset \mathbf{C}^m$ by $Z_P = \rho^{-1}(\iota(P^n))$.

In fact Z_P is a smooth, compact, $(m+n)$ -dimensional submanifold of \mathbf{C}^m . It inherits a smooth T^m action from \mathbf{C}^m , and can be shown to be independent of the choice of realisation of P^n , up to equivariant diffeomorphism ([6], section 6.1). What's more, Z_P^{m+n} is equivariantly framed in \mathbf{C}^m ; that is, there is a real vector bundle isomorphism

$$\tau(Z_P) \oplus \mathbf{R}^{m-n} \cong \tau(\mathbf{C}^m)|_{Z_P^{m+n}}$$

where the \mathbf{R}^{m-n} component is a trivial real bundle with trivial T^m action. This has been proved in various ways; perhaps most elegantly in [7] where Z_P^{m+n} is shown to be a transverse intersection of real quadratic hypersurfaces. The smooth manifold Z_P^{m+n} is known as the *moment-angle* manifold of P^n .

Example 2.2.7. We consider the n -simplex realised as

$$\Delta^n = \{x \in \mathbf{R}^n \mid x_i \geq 0 \text{ for } i = 1, \dots, n, x_1 + \dots + x_n \leq 1\}$$

with facets F_1, \dots, F_n given by the equalities $x_i = 0$ for $i = 1, \dots, n$ and facet F_{n+1} given by $x_1 + \dots + x_n = 1$.

The map

$$\iota: \mathbf{R}^n \rightarrow \mathbf{R}^{n+1}$$

is given by

$$\begin{aligned} \iota_i(x) &= x_i \text{ for } i = 1, \dots, n \\ \iota_{n+1}(x) &= 1 - (x_1 + \dots + x_n) \end{aligned}$$

and $Z_P^{2n+1} = \rho^{-1}(\iota(\Delta^n))$ is given by the equations

$$\begin{aligned} |z_i|^2 &= x_i \text{ for } i = 1, \dots, n \\ |z_{n+1}|^2 &= 1 - (x_1 + \dots + x_n) \end{aligned}$$

which may be rewritten as

$$|z_1|^2 + \dots + |z_{n+1}|^2 = 1$$

so Z_P^{2n+1} is the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$

We now assume that P^n is equipped with a characteristic function $\bar{\ell}$ and choose, for each facet F_i of P^n , an orientation of $T(F_i)$. This is equivalent to choosing one of two monomorphisms $T^1 \rightarrow T^n$ with image $T(F_i)$, which means choosing a sign for the vector λ_i . We call the resulting map $\ell: F_i \mapsto \lambda_i$ a *directed characteristic function* or *dicharacteristic* for P^n . The pair (P^n, ℓ) is sometimes called a *dicharacteristic pair*.

The choice of dicharacteristic yields a homomorphism $\Theta_\ell: T^m \rightarrow T^n$ represented by the matrix Λ whose columns are the vectors λ_i . The kernel κ of Θ_ℓ is an $(m - n)$ -torus and, because ℓ satisfies condition 2.2.4, κ acts freely on Z_P^{m+n} . Therefore the quotient Z_P^{m+n}/κ is a smooth $2n$ -dimensional manifold with action of $T^m/\kappa \cong T^n$. In fact Z_P^{m+n}/κ is equivariantly homeomorphic to $M^{2n}(P^n, \bar{\ell})$, where we identify T^n with T^m/κ via a section $s: T^n \hookrightarrow T^m$ such that the composition $\Theta_\ell \circ s$ is the identity. To see this, we note that $Z_P^{m+n} \cong (T^m \times P^n)/\approx$ where $(t_1, x) \approx (t_2, x)$ if and only if $t_1 t_2^{-1} \in \prod_{x \in F_i} T_i^1$, the product of coordinate subtori corresponding to the facets of P^n that contain x . Then we may check that $Z_P^{m+n}/\kappa \cong (T^n \times P^n)/\sim \cong M^{2n}$ as defined in the previous section.

Example 2.2.8. Consider the n -simplex Δ^n as realised in example 2.2.7 with the characteristic function of example 2.2.5. Choosing a sign for each vector λ_i we have a matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

This yields the homomorphism $\Theta_\ell: T^{n+1} \rightarrow T^n$ given by

$$\Theta_\ell(t_1, \dots, t_{n+1}) = (t_1 t_{n+1}^{-1}, \dots, t_n t_{n+1}^{-1})$$

with kernel $\kappa = T_\delta$, the diagonal torus. We know from example 2.2.7 that Z_P^{2n+1} is the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$, so we have $M^{2n}(\Delta^n, \bar{\ell}) = S^{2n+1}/T_\delta$ and we have reconstructed the smooth T^n -manifold $\mathbf{C}P^n$.

Example 2.2.9. An alternative choice of sign for each vector λ_i in example 2.2.8 would give

$$\Lambda' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

which yields the homomorphism

$$\Theta_{\ell'}(t_1, \dots, t_{n+1}) = (t_1 t_{n+1}, \dots, t_n t_{n+1})$$

with kernel $\kappa' = \{(t, \dots, t, t^{-1}) \in T^{n+1}\}$. This gives an alternative S^1 action on S^{2n+1} , but we have the diffeomorphism $g: S^{2n+1} \rightarrow S^{2n+1}$ given by

$$g(z_1, \dots, z_n, z_{n+1}) = (z_1, \dots, z_n, \overline{z_{n+1}})$$

and the automorphism $\varphi: T^{n+1} \rightarrow T^{n+1}$ given by

$$\varphi(t_1, \dots, t_n, t_{n+1}) = (t_1, \dots, t_n, t_{n+1}^{-1})$$

which satisfy $\varphi(\kappa) = \kappa'$ and $g(t \cdot z) = \varphi(t) \cdot g(z)$ (that is, g is a φ -equivariant diffeomorphism). So the T^n -manifolds S^{2n+1}/κ and S^{2n+1}/κ' are equivariantly diffeomorphic.

As illustrated by example 2.2.9, although the smooth construction of M^{2n} uses a choice of dicharacteristic ℓ the resulting smooth manifold depends only on the underlying undirected characteristic function $\bar{\ell}$ [8].

2.2.4 Stably complex structures

The existence of stably complex structures on quasitoric manifolds was noted in [10], but the dependence on the dicharacteristic was made explicit in [8]. The following geometrical exposition most closely follows [7].

The choice of dicharacteristic required for the smooth construction of M^{2n} is essentially arbitrary: the resulting smooth manifold depends only on the undirected characteristic function. By contrast, a particular choice of dicharacteristic is necessary to define a stably complex structure on M^{2n} : different choices of dicharacteristic corresponding to the same undirected characteristic function will generally give rise to different stably complex structures.

Recall that the moment angle manifold Z_P^{m+n} is equivariantly framed in \mathbf{C}^n , so we have a real vector bundle isomorphism

$$\tau(Z_P^{m+n}) \oplus \mathbf{R}^{m-n} \cong \tau(\mathbf{C}^m)|_{Z_P^{m+n}}$$

where the T^m action on the \mathbf{R}^{m-n} component is trivial. But $\tau(\mathbf{C}^m)$ is trivial, so we have

$$\tau(Z_P^{m+n}) \oplus \mathbf{R}^{m-n} \cong \mathbf{C}^m$$

as real bundles over Z_P^{m+n} . We may divide by the action of the torus κ to obtain

$$\tau(M^{2n}) \oplus \mathbf{R}^{m-n} \oplus \mathbf{R}^{m-n} \cong \xi^m = Z_P^{m+n} \times_{\kappa} \mathbf{C}^m \quad (2.2)$$

which is an isomorphism of T^n vector bundles over M^{2n} . The left hand side comes from a theorem of Szczarba [25] which allows us to identify the quotient of $\tau(Z_P^{m+n})$ by the free action of $\kappa \cong T^{m-n}$ with the bundle $\tau(Z_P^{m+n}/\kappa) \oplus \tau_e(\kappa)$ because κ is Abelian. Here $\tau_e(\kappa)$ is the tangent space of κ at the identity, e . The right hand side of (2.2) is the bundle with total space $Z_P^{m+n} \times_{\kappa} \mathbf{C}^m = (Z_P^{m+n} \times \mathbf{C}^m)/\kappa$ which may be decomposed as a sum $\xi = \bigoplus_{i=1}^m \xi_i$ where ξ_i is the bundle with total space $Z_P^{m+n} \times_{\kappa} \mathbf{C}_i$ obtained from the i -th coordinate sub-bundle of $Z_P^{m+n} \times \mathbf{C}^m$.

These bundles depend on the action of the torus κ on the bundle $Z_P^{m+n} \times \mathbf{C}_i$. In particular, since κ is the kernel of the homomorphism $\Theta_{\ell}: T^m \rightarrow T^n$ obtained by choosing an orientation for each isotropy group $T(F_i)$, the action of κ on $Z_P^{m+n} \times \mathbf{C}_j$ depends on the choice of orientation for $T(F_j)$ and choosing the opposite orientation for $T(F_j)$ has the effect of reversing the action on $Z_P^{m+n} \times \mathbf{C}_j$, which

is equivalent to replacing $Z_P^{m+n} \times \mathbf{C}_j$ with the complex conjugate $Z_P^{m+n} \times \overline{(\mathbf{C}_j)}$. Thus choosing the opposite orientation for $T(F_j)$ has the effect of replacing ξ_j with $\bar{\xi}_j$.

The isomorphism

$$\tau(M^{2n}) \oplus \mathbf{R}^{2(m-n)} \cong \xi^m \quad (2.3)$$

gives an *equivariant stably complex structure* on M^{2n} , that is, an isomorphism of real T^n bundles between the stable tangent bundle of M^{2n} and a complex vector bundle.

Example 2.2.10. In example 2.2.8, the choice of dicharacteristic for Δ^n yields the homomorphism

$$\Theta_\ell(t_1, \dots, t_{n+1}) = (t_1 t_{n+1}^{-1}, \dots, t_n t_{n+1}^{-1})$$

with kernel the diagonal torus $\kappa = T_\delta < T^{n+1}$. Identifying S^1 with T_δ via $t \mapsto (t, \dots, t)$, the action of S^1 on $Z_P = S^{2n+1} \subset \mathbf{C}^{n+1}$ is the classical coordinatewise action, yielding the Hopf bundle

$$S^1 \rightarrow S^{2n+1} \rightarrow \mathbf{C}P^n$$

and the stably complex structure

$$\tau(\mathbf{C}P^n) \oplus \mathbf{R}^2 \cong S^{2n+1} \times_\kappa \mathbf{C}^{n+1}$$

obtained on $\mathbf{C}P^n$ is the classical one.

On the other hand, the dicharacteristic chosen in example 2.2.9 yields the alternative action

$$t(z_1, \dots, z_n, z_{n+1}) = (tz_1, \dots, tz_n, t^{-1}z_{n+1})$$

and the alternative stably complex structure

$$\tau(\mathbf{C}P^n) \oplus \mathbf{R}^2 \cong S^{2n+1} \times_{\kappa'} \mathbf{C}^{n+1}.$$

2.2.5 Omniorientations

The isomorphism (2.3) depends on the choice of basis for the trivial bundle $\mathbf{R}^{2(m-n)}$, but for two stably complex structures to be equivalent it is only necessary for the trivial components to have the same orientation. Thus the stably complex structure depends on a chosen orientation for $\mathbf{R}^{2(m-n)}$. This is equivalent to a choice of orientation for M^{2n} , because the orientation of the sum $\tau(M^{2n}) \oplus \mathbf{R}^{2(m-n)}$ is fixed: it must agree with the orientation of the complex vector bundle ξ^m . An orientation for M^{2n} is induced by choosing an orientation for the polytope P^n (or the ambient \mathbf{R}^n), because the torus T^n is canonically oriented.

Therefore, the stably complex structure on M^{2n} is determined by choosing a dicharacteristic ℓ compatible with the characteristic function $\bar{\ell}$ and an orientation for P^n . The choice of dicharacteristic can be reinterpreted in terms of the facial submanifolds $M_i^{2(n-1)}$: There is an action of $T(F_i)$ on $\eta(M_i^{2(n-1)})$, the normal bundle of $M_i^{2(n-1)}$ in M^{2n} and so a choice of orientation for $T(F_i)$ gives an action of S^1 on $\eta(M_i^{2(n-1)})$, which equips it with an orientation and hence the structure of a complex line bundle. Given an orientation for M^{2n} , the choice of orientation for $T(F_i)$ is therefore equivalent to a choice of orientation for $M_i^{2(n-1)}$.

Definition 2.2.11 ([8], section 3). An *omniorientation* for the quasitoric manifold M^{2n} is a choice of orientation for M^{2n} and for each of the facial submanifolds $M_i^{2(n-1)}$.

A choice of omniorientation for $M^{2n}(P^n, \bar{\ell})$ is equivalent to a choice of orientation for P^n and dicharacteristic ℓ compatible with $\bar{\ell}$. A choice of omniorientation for M^{2n} determines a stably complex structure.

What's more, the normal bundle $\eta(M_i^{2(n-1)})$ with its chosen complex structure is isomorphic, as a complex line bundle, to the restriction to $M_i^{2(n-1)}$ of the bundle ξ_i constructed above; while the restriction of ξ_i to the complement $M^{2n} \setminus M_i^{2(n-1)}$ is trivial ([7], (3.8)). For this reason, ξ_i is referred to as the *i*-th *facial line bundle*, and can be seen to be defined independently of (2.2).

2.3 Polyhedral products

Given a simple polytope P^n , the moment angle manifold Z_P is equivariantly homeomorphic to the space $\mathcal{Z}(K_P)$ defined as follows, where K_P is the $(n - 1)$ -dimensional simplicial sphere dual to P^n :

$$\mathcal{Z}(K_P) = \bigcup_{\sigma \in K_P} \mathcal{Z}(\sigma) \subset (D^2)^m$$

where $\mathcal{Z}(\sigma) = Y_1 \times \cdots \times Y_m \subset (D^2)^m$ with $Y_i = D^2$ for $i \in \sigma$ and $Y_i = S^1 \subset D^2$ for $i \notin \sigma$ [10, section 6.1]. This construction can be extended to any finite simplicial complex K on vertex set V , and we call $\mathcal{Z}(K)$ the *moment angle complex* of K . There is a canonical action of T^V on $\mathcal{Z}(K)$, arising out of the action of S^1 on D^2 , where we take D^2 to be the unit disc in \mathbf{C} [10, section 6.2].

The moment angle complex can be seen as a special case of a more general construction called the *polyhedral product* (introduced in [6], where it was denoted $K_\bullet(X, A)$). Before giving the definition of the polyhedral product, we will fix some terminology.

2.3.1 Some category theory

We do not intend to give an introduction to category theory here; an ample background is available in [18]. However, we remark that the subject suffers from a clash of terminology, especially around the terms limit, colimit, direct limit and inverse limit.

Recall that a *diagram* in a category \mathbf{C} is nothing but a covariant functor $\mathbf{B} \rightarrow \mathbf{C}$ from another category \mathbf{B} to \mathbf{C} . If we wish to be explicit about the source category \mathbf{B} , we refer to a \mathbf{B} *diagram in \mathbf{C}* , or we may say that the diagram is *indexed by \mathbf{B}* . Any diagram may have a *colimit*. If the diagram is indexed by the natural numbers (as an ordered set), then the colimit may be called a *direct limit*. For example, the direct limit of the following diagram in \mathbf{Top} , where each subsequent morphism is the inclusion of S^{n-1} as the equator in S^n , is the contractible space S^∞ :

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \cdots$$

(Taking each S^n to be the unit sphere in \mathbf{R}^{n+1} , it is usual to identify S^∞ with the unit sphere in \mathbf{R}^∞ , the topological vector space of finite sequences, with the

colimit topology).

Dually, any diagram may have a *limit*, and if the diagram is indexed by an ordered set then the limit may be called an *inverse limit*.

We will make use of two categories associated to a finite-dimensional simplicial complex K , on a countable vertex set. First, we have the category $\text{CAT}(K)$ (see e.g. [22], section 3) whose objects are the simplices of $\sigma_i \in K$ (including the empty simplex) and whose morphisms are the inclusions $\sigma_1 \hookrightarrow \sigma_2$ between them.

Example 2.3.1. Let K be the one-simplex Δ^1 , on vertices v_1 and v_2 . Then $\text{CAT}(K)$ has objects \emptyset, v_1, v_2 and v_1v_2 and non-trivial morphisms $\emptyset \rightarrow v_1, \emptyset \rightarrow v_2, \emptyset \rightarrow v_1v_2, v_1 \rightarrow v_1v_2$ and $v_2 \rightarrow v_1v_2$.

Second, we introduce the new category $\text{SUB}(K)$ whose objects are the finite subcomplexes of K , and whose morphisms are the inclusions among them, making $\text{SUB}(K)$ a poset. In the case that K itself is finite, $\text{SUB}(K)$ has a final object (or greatest element, in the language of posets), which is K . When K is not finite, we may choose any ordering of the vertices $\{v_1, \dots\}$ of K and let K_n be the restriction of K to vertex set $\{v_1, \dots, v_n\}$. Then the sequence $K_1 \hookrightarrow K_2 \hookrightarrow \dots$ is a *cofinal* subset of the poset $\text{SUB}(K)$: for any $K' \in \text{SUB}(K)$ we have $K' \subset K_n$ for some n .

2.3.2 Colimits of $\text{CAT}(K)$ diagrams

We can now define the polyhedral product functor, as in [6]:

Definition 2.3.2. For a finite simplicial complex K on vertex set V , and a pair of spaces (X, A) , the *polyhedral product* $(X, A)^K$ is the colimit of the $\text{CAT}(K)$ diagram that assigns to each simplex $\sigma \in K$ the space $(X, A)^\sigma \subset X^V$ and to each morphism $\sigma \hookrightarrow \sigma'$ the inclusion $(X, A)^\sigma \hookrightarrow (X, A)^{\sigma'}$; where $(X, A)^\sigma$ is defined by

$$(X, A)^\sigma = Y_{v_1} \times \dots \times Y_{v_m} \subset X^V$$

with $Y_{v_i} = X$ for $v_i \in \sigma$ and $Y_{v_i} = A$ for $v_i \notin \sigma$.

We immediately have:

Proposition 2.3.3. *Given a pair of spaces (X, A) and finite simplicial complex K on vertex set V , the polyhedral product $(X, A)^K$ is the union of the subspaces $(X, A)^\sigma \subset X^V$ for $\sigma \in K$.*

In Chapter 4 we shall extend definition 2.3.2 to the case where K is a finite dimensional simplicial complex on countable vertex set.

When (X, A) is a CW-pair the inclusions in the $\text{CAT}(K)$ diagram that defines $(X, A)^K$ are all cofibrations, so the colimit $(X, A)^K$ realises the *homotopy colimit* [26].

In particular, the moment angle complex is a polyhedral product: $\mathcal{Z}(K) \cong (D^2, S^1)^K$.

The polyhedral product is a covariant functor both of K (in the category of finite simplicial complexes with inclusions as morphisms) and of the pair (X, A) .

We make the trivial observation that, given an action of the group G on X with A a stable subspace, we obtain a coordinatewise action of G^V on $(X, A)^K$ with orbit space $(X/G, A/G)^K$. In particular, the T^V action on $\mathcal{Z}(K) = (D^2, S^1)^K$ has orbit space $\mathcal{I}(K) = (I, 1)^K$, which by definition is the cubical complex $cc(K)$ (see section 2.1.1). In the case that $K = K_P$ is dual to a simple polytope P^n , the space $\mathcal{I}(K)$ is a cubical decomposition of P^n .

Given a point $x' \in X$ which is not in A , we may define a *facial structure* on $(X, A)^K$ as follows. For each vertex $v \in V$, the *facet* $F_v \subset (X, A)^K$ is the set of points $x \in (X, A)^K \subset X^V$ for which the v -th coordinate of x is equal to x' . Then the *faces* of $(X, A)^K$ are the intersections of the facets. We will refer to x' as the *facial point*.

Example 2.3.4. We define the faces of $\mathcal{I}(K)$ by taking $0 \in I$ as the facial point. Let P^n be a simple polytope. Under the piecewise linear homeomorphism $i_P: P^n \rightarrow cc(K_P) = \mathcal{I}(K_P)$ of proposition 2.1.7, the faces of P^n correspond to the faces of $\mathcal{I}(K_P)$.

We will also be interested in $(\mathbf{C}, \mathbf{C}^*)^K$, denoted $\mathcal{U}(K)$, where \mathbf{C}^* is the group of non-zero complex numbers. There is a $(\mathbf{C}^*)^V$ action on $\mathcal{U}(K)$, and the action of the compact subgroup $T^V \subset (\mathbf{C}^*)^V$ has orbit space $\mathcal{R}(K) = (\mathbf{R}_{\geq}, \mathbf{R}_{>})^K$.

2.4 Generalised toric spaces

In [10] Davis and Januszkiewicz extend their construction of quasitoric manifolds as quotients $T^n \times P^n / \sim$ to arbitrary simplicial complexes K by making use of the cubical complex $\mathcal{I}(K)$, which they call a simple polyhedral complex, as follows.

They define a facial structure on $\mathcal{I}(K)$ which amounts to that in Example 2.3.4. That is, in our terminology, they define the faces of $\mathcal{I}(K) = (I, 1)^K$ by

choosing $0 \in I$ as the facial point, which is consistent with the definition of the faces of P^n in the case that K is dual to the simple polytope P^n .

A characteristic function of dimension n is defined on $\mathcal{I}(K)$ in the same way as in polytope case (see section 2.2.2). That is, $\bar{\ell}$ assigns a line in \mathbf{Z}^n to each facet of $\mathcal{I}(K)$, subject to condition 2.2.4. Then the *quasitoric space* $\bar{\mathcal{X}}(K, \bar{\ell})$ is constructed as $T^n \times \mathcal{I}(K) / \sim$ where $(t_1, x) \sim (t_2, x)$ if and only if $t_1 t_2^{-1} \in \bar{\ell}(x) = \prod_{x \in F_i} \bar{\ell}(F_i)$, the product in T^n of the subgroups $\bar{\ell}(F_i)$.

Note that n is not explicitly determined by the dimension of K . However, in order for condition 2.2.4 to be satisfied, n must be at least $\dim K + 1$.

2.4.1 Characteristic functions on simplicial complexes

Since the facets of $\mathcal{I}(K)$ (or of P^n when K is the simplicial dual K_P) are in one-to-one correspondence with the vertices of K , a characteristic function $\bar{\ell}$ on $\mathcal{I}(K)$ may be considered a function on the vertices of K . To be precise, we have the following, essentially due to [10]:

Definition 2.4.1. Let K be a simplicial complex with vertex set V . A characteristic function of dimension n on K is a map $\bar{\ell}: V \rightarrow \{\text{lines in } \mathbf{Z}^n\}$ satisfying the following condition:

Condition 2.4.2. If $v_1 \dots v_k$ is a simplex of K then $\langle \bar{\ell}(v_1), \dots, \bar{\ell}(v_k) \rangle$ is a direct summand of \mathbf{Z}^n .

The pair $(K, \bar{\ell})$ will be called a characteristic pair (of dimension n).

For dicharacteristics (see section 2.2.3) we make the equivalent definition:

Definition 2.4.3. Let K be a simplicial complex with vertex set V . A dicharacteristic of dimension n on K is a map $\ell: V \rightarrow \{\text{primitive vectors in } \mathbf{Z}^n\}$ satisfying the following condition:

Condition 2.4.4. If $v_1 \dots v_k$ is a simplex of K then $\{\ell(v_1), \dots, \ell(v_k)\}$ forms a basis for a direct summand of \mathbf{Z}^n .

The pair (K, ℓ) will be called a dicharacteristic pair (of dimension n).

In the case that K_P is the dual of a simple polytope P^n , [7] use similar terminology: They refer to the pair (P^n, ℓ) as a *combinatorial pair*.

Given two simplicial complexes K_1 and K_2 , a map $f: K_1 \rightarrow K_2$ is *non-degenerate* if the restriction $f|_\sigma$ of f to any simplex $\sigma \in K_1$ is injective.

If K_1 and K_2 are equipped with n -dimensional characteristic functions $\bar{\ell}_1$ and $\bar{\ell}_2$, we will say that a simplicial map $f: K_1 \rightarrow K_2$ *preserves the characteristic function* if $\bar{\ell}_1 = \bar{\ell}_2 \circ f$. We may refer to $f: (K_1, \bar{\ell}_1) \rightarrow (K_2, \bar{\ell}_2)$ as a *map of characteristic pairs*. Similarly, if K_1 and K_2 are equipped with n -dimensional dicharacteristics ℓ_1 and ℓ_2 then a simplicial map $f: K_1 \rightarrow K_2$ *preserves the dicharacteristic* if $\ell_1 = \ell_2 \circ f$. We may refer to $f: (K_1, \ell_1) \rightarrow (K_2, \ell_2)$ as a *map of dicharacteristic pairs*.

Any map K_1 and K_2 that preserves a characteristic function must be non-degenerate. Conversely, given any non-degenerate map $f: K_1 \rightarrow K_2$ and a characteristic function $\bar{\ell}_2$ on K_2 , there is a unique characteristic function on K_1 such that f preserves the characteristic function. This is the pullback $\bar{\ell}_1 = f^*(\bar{\ell}_2) = \bar{\ell}_2 \circ f$. The same is true for dicharacteristics.

These observations allow us to introduce two categories for each positive integer n :

1. The category of characteristic pairs $(K, \bar{\ell})$ of dimension n , with maps of characteristic pairs as morphisms;
2. The category of dicharacteristic pairs (K, ℓ) of dimension n , with maps of dicharacteristic pairs as morphisms.

Davis and Januszkiewicz observed in [10] that, given a map $f: (K_1, \bar{\ell}_1) \rightarrow (K_2, \bar{\ell}_2)$ of characteristic pairs of dimension n , there exists a T^n equivariant map $\bar{\mathcal{X}}(K_1, \bar{\ell}_1) \rightarrow \bar{\mathcal{X}}(K_2, \bar{\ell}_2)$ and in fact $\bar{\mathcal{X}}(K_1, \bar{\ell}_1)$ is equivariantly homeomorphic to the pullback of the diagram:

$$\begin{array}{ccc} \bar{\mathcal{X}}(K_1, \bar{\ell}_1) & \longrightarrow & \bar{\mathcal{X}}(K_2, \bar{\ell}_2) \\ \downarrow & & \downarrow \\ \mathcal{I}(K_1) & \longrightarrow & \mathcal{I}(K_2) \end{array}$$

In particular, as described in [10], there is an $(n - 1)$ -dimensional complex \mathcal{J}_n whose vertex set is the set of lines in \mathbf{Z}^n and whose maximal simplices are n -tuples of lines whose span is \mathbf{Z}^n . There is a canonical characteristic function $\bar{\nu}$ on \mathcal{J}_n , which associates to each vertex the line $\langle v \rangle$ defining it. For $n > 1$, the complex \mathcal{J}_n is countably infinite.

We shall see in Chapter 4 how to construct the cubical complex dual to \mathcal{J}_n , and therefore the quasitoric space $\overline{\mathcal{X}}(\mathcal{J}_n, \bar{\nu})$. Then, as observed in [10], any simplicial complex K with characteristic function $\bar{\ell}$ admits a unique non-degenerate map $K_1 \rightarrow \mathcal{J}_n$ which preserves the characteristic function, and $\mathcal{X}(K, \ell)$ may be obtained from $\overline{\mathcal{X}}(\mathcal{J}_n, \bar{\nu})$ as a pullback.

Chapter 3

Quasitoric spaces and maps

In this chapter we introduce open quasitoric manifolds $\mathcal{Y}(K, \ell)$ and describe their functorial properties, along with the functorial properties of open moment angle complexes, moment angle complexes and quasitoric spaces. We relate the properties of these spaces to the properties of quasitoric manifolds and quasitoric spaces described in the previous chapter, which allows us to describe the behaviour of the facial line bundles given a non-degenerate equivariant map of quasitoric manifolds $M^{2n}(P_1^n, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$, or a map $M^{2n}(P_1^n, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$

3.1 Properties of the polyhedral product

We recall the polyhedral product $(X, A)^K$ from Definition 2.3.2, where (X, A) is a pair of spaces and K is a finite simplicial complex on vertex set V . By proposition 2.3.3 we have

$$(X, A)^K = \bigcup_{\sigma \in K} (X, A)^\sigma \subset X^V$$

where

$$(X, A)^\sigma = Y_{v_1} \times \cdots \times Y_{v_m} \subset X^V$$

with $Y_{v_i} = X$ for $v_i \in \sigma$ and $Y_{v_i} = A$ for $v_i \notin \sigma$.

We shall be interested in the case where X is a commutative topological monoid with unit $*$, and A a submonoid. We shall call such a pair (X, A) a *monoidal pair* and denote by MonPair the category of monoidal pairs, with maps of topological pairs that are monoid homomorphisms as morphisms.

Now given a commutative topological monoid X and *any* map of finite simplicial complexes $f: K_1 \rightarrow K_2$, where K_1 and K_2 have vertex sets V and W

respectively, there is a homomorphism $X_f: X^V \rightarrow X^W$ defined coordinatewise by

$$(X_f)_w(x) = \prod_{f(v)=w} x_v \quad (3.1)$$

where $(X_f)_w(x)$ denotes the w -th coordinate of $X_f(x) \in X^W$, and x_v denotes the v -th coordinate of $x \in X^V$.

Denoting by SimpC the category of finite simplicial complexes and simplicial maps, and by Top^* the category of basepointed topological spaces, we now prove:

Proposition 3.1.1. *The polyhedral product $(X, A)^K$ is a bifunctor $\text{MonPair} \times \text{SimpC} \rightarrow \text{Top}^*$, where for a simplicial map $f: K_1 \rightarrow K_2$ the map*

$$(X, A)^f: (X, A)^{K_1} \rightarrow (X, A)^{K_2}$$

is the restriction of X_f to $(X, A)^{K_1}$.

Proof. A basepoint of $(X, A)^K$ is obtained by taking $x_v = *$ for all coordinates $v \in V$. The restriction of X_f maps $(X, A)^{K_1}$ into $(X, A)^{K_2}$, and it is clear from (3.1) that

$$(X, A)^{(f_1 \circ f_2)} = (X, A)^{f_1} \circ (X, A)^{f_2}$$

Given maps $f: K_1 \rightarrow K_2$ and $h: (X, A) \rightarrow (Y, B)$ the commutative diagram

$$\begin{array}{ccc} X^V & \xrightarrow{X_f} & X^W \\ \downarrow h^V & & \downarrow h^W \\ Y^V & \xrightarrow{Y_f} & Y^W \end{array}$$

restricts to

$$\begin{array}{ccc} (X, A)^{K_1} & \xrightarrow{(X, A)^f} & (X, A)^{K_2} \\ \downarrow h^{K_1} & & \downarrow h^{K_2} \\ (Y, B)^{K_1} & \xrightarrow{(Y, B)^f} & (Y, B)^{K_2} \end{array}$$

□

Example 3.1.2. The special case $(X, A) = (\text{cone}(G), G)$, where G is a commutative topological group and $\text{cone}(G)$ is the cone over G with obvious G action,

is discussed in [6]. The group G^V acts coordinatewise on $(\text{cone}(G), G)^K$. Taking $G = S^1$ we obtain the moment angle complex $\mathcal{Z}(K)$ with $T^V = (S^1)^V$ action.

Example 3.1.3. Taking $K_1 = \Delta^{n-1}$, the full simplicial complex on n vertices, we have $(X, A)^{K_1} = X^n$. Now taking K_2 to be a single point, $(X, A)^{K_2} = X$ and the collapse $f: \Delta^{n-1} \rightarrow pt$ yields $(X, A)^f: X^n \rightarrow X$ which is the usual multiplication map.

If the commutative group G acts on X via a monoid homomorphism $G \rightarrow A$, then G^V acts on $(X, A)^K$ coordinatewise. The quotient X/G inherits the structure of a topological monoid with submonoid A/G , and the quotient map $(X, A)^K \rightarrow (X, A)^K/G^V = (X/G, A/G)^K$ is precisely the map induced by the homomorphism of pairs $(X, A) \rightarrow (X/G, A/G)$. Given a simplicial map $f: K_1 \rightarrow K_2$, we have $\Theta_f: G^V \rightarrow G^W$ defined by 3.1 and the map $(X, A)^f: (X, A)^{K_1} \rightarrow (X, A)^{K_2}$ is Θ_f -equivariant.

We will be interested exclusively in the case $G = S^1$, where the inclusion of D^2 in \mathbf{C} as the unit disc induces a map of monoidal pairs $(D^2, S^1) \rightarrow (\mathbf{C}, \mathbf{C}^*)$, each with S^1 action, and we have a commutative diagram of monoidal pairs

$$\begin{array}{ccc} (D^2, S^1) & \hookrightarrow & (\mathbf{C}, \mathbf{C}^*) \\ \downarrow /S^1 & & \downarrow /S^1 \\ (I, 1) & \hookrightarrow & (\mathbf{R}_{\geq}, \mathbf{R}_{>}) \end{array}$$

3.2 Open moment angle complexes

In [6], the following special case of the polyhedral product is considered:

Definition 3.2.1. For a finite simplicial complex K , the *open moment-angle complex* of K is $\mathcal{U}(K) = (\mathbf{C}, \mathbf{C}^*)^K$ where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$.

By proposition 3.1.1, $\mathcal{U}(K)$ is a functor $\mathcal{U}: \text{SimpC} \rightarrow \text{Top}^*$.

The spaces $\mathcal{U}(K)$ are complements of coordinate subspace arrangements, and are of independent interest (see e.g. [13]). The following description follows [6]: If the complex K has vertices v_1, \dots, v_m then for each simplex $\sigma = v_{i_1} \dots v_{i_r}$ we have the coordinate subspace $L_\sigma = \{(z_1, \dots, z_m) \in \mathbf{C}^m \mid z_{i_1} = \dots = z_{i_r} = 0\}$. The coordinate subspace determined by K is $\mathcal{CA}(K) = \{L_\sigma \mid \sigma \notin K\}$ and its complement is $\mathcal{U}(K) = \mathbf{C}^m \setminus \bigcup_{\sigma \notin K} L_\sigma$.

Example 3.2.2. Let K be the boundary of the n -simplex $\delta\Delta^n$, that is, K has vertices v_1, \dots, v_{n+1} and the only missing face is $v_1 \dots v_{n+1}$. Then $\mathcal{CA}(\delta\Delta^n) = \{0\}$ and $\mathcal{U}(\delta\Delta^n) = \mathbf{C}^{n+1} \setminus \{0\}$.

There is a canonical complex manifold structure on $\mathcal{U}(K)$, induced from the standard structure on \mathbf{C}^V . Moreover, $\mathcal{U}(K)$ admits a canonical smooth, holomorphic action of the torus T^V via the inclusion $S^1 \hookrightarrow \mathbf{C}^*$, with orbit space $(\mathbf{R}_{\geq}, \mathbf{R}_{>})^K$. We introduce notation for this space:

Definition 3.2.3. We will denote by $\mathcal{R}(K)$ the polyhedral product $(\mathbf{R}_{\geq}, \mathbf{R}_{>})^K$.

This makes \mathcal{R} into a functor $\mathcal{R}: \text{SimpC} \rightarrow \text{Top}^*$. It is useful to have an explicit orbit map $\mathcal{U}(K) \rightarrow \mathcal{R}(K)$, so we choose the restriction of the map $\pi: \mathbf{C}^V \rightarrow \mathbf{R}_{\geq}^V$ taking each coordinate to the square of its modulus. This gives a natural transformation $\pi: \mathcal{U} \rightarrow \mathcal{R}$, since by proposition 3.1.1 the polyhedral product is a bifunctor $\text{MonPair} \times \text{SimpC} \rightarrow \text{Top}^*$, and π is the natural transformation $\text{Func}(\text{SimpC}, \text{Top}^*) \rightarrow \text{Func}(\text{SimpC}, \text{Top}^*)$ given by the map of monoidal pairs $(\mathbf{C}, \mathbf{C}^*) \rightarrow (\mathbf{R}_{\geq}, \mathbf{R}_{>})$. There is a section $s: \mathcal{R} \rightarrow \mathcal{U}$ obtained by sending each coordinate to its positive square root.

The orbit space $\mathcal{R}(K)$ admits a facial structure, with facets $F_v, v \in V$ the coordinate subspaces $F_v = \{z \in (\mathbf{R}_{\geq}, \mathbf{R}_{>})^K \mid z_v = 0\}$. Facets $\{F_v \mid v \in \sigma \subset V\}$ meet in a non-empty face F_σ if and only if σ is a simplex of K . In fact, $\mathcal{R}(K)$ has the structure of a smooth manifold with corners with the facial structure described.

We shall call an equivariant map *non-degenerate* if it preserves orbit types. That is, given a G -space X and H -space Y with a homomorphism $\Theta: G \rightarrow H$, a Θ -equivariant map $f: X \rightarrow Y$ is non-degenerate if whenever $\Theta(g)(f(x)) = f(x)$ we have $g(x) = x$.

Recall from section 2.4.1 that a simplicial map $K_1 \rightarrow K_2$ is called non-degenerate if it takes the vertices of any simplex of K_1 to distinct vertices of K_2 . Now given a non-degenerate simplicial map $f: K_1 \rightarrow K_2$, the map of open moment angle complexes $\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ is a non-degenerate Θ_f -equivariant map where $\Theta_f: T^V \rightarrow T^W$ is given by (3.1). The induced map of orbit spaces $\mathcal{R}(f): \mathcal{R}(K_1) \rightarrow \mathcal{R}(K_2)$ takes a face F_σ to $F_{f(\sigma)}$.

Moreover, $\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ is holomorphic, being the restriction to

$\mathcal{U}(K_1)$ of the holomorphic map $\mathbf{C}_f: \mathbf{C}^V \rightarrow \mathbf{C}^W$ given coordinatewise by:

$$(\mathbf{C}_f)_w(z) = \prod_{f(v)=w} z_v.$$

Example 3.2.4. Let K_1 be the simplicial complex dual to the hexagon, with vertices v_1, \dots, v_6 and edges $v_1v_2, v_2v_3, v_3v_5, v_4v_5, v_5v_6, v_6v_1$. Let K_2 be the boundary of the 2-simplex, $\partial\Delta^2$, with vertices w_1, w_2, w_3 and edges w_1w_2, w_2w_3, w_3w_1 . We have $\mathcal{U}(K_2) = \mathbf{C}^W \setminus \{0\}$.

The non-degenerate map $f: K_1 \rightarrow K_2$ given by

$$\begin{aligned} f(v_1) &= f(v_4) = w_1 \\ f(v_2) &= f(v_5) = w_2 \\ f(v_3) &= f(v_6) = w_3 \end{aligned}$$

induces $\mathbf{C}_f: \mathbf{C}^V \rightarrow \mathbf{C}^W$ given by

$$\mathbf{C}_f(z_1, z_2, z_3, z_4, z_5, z_6) = (z_1z_4, z_2z_5, z_3z_6). \quad (3.2)$$

Since at most two of z_1, \dots, z_6 can be simultaneously zero in $\mathcal{U}(K_1) \subset \mathbf{C}^V$, the point $(0, 0, 0)$ is not in the image of the restriction of \mathbf{C}_f to $\mathcal{U}(K_1)$, so (3.2) defines a map $\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$.

The map $\Theta_f: T^V \rightarrow T^W$ is given by

$$\Theta_f(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1t_4, t_2t_5, t_3t_6)$$

and we have

$$\begin{aligned} & \Theta_f(t_1, t_2, t_3, t_4, t_5, t_6) \cdot \mathcal{U}(f)(z_1, z_2, z_3, z_4, z_5, z_6) \\ &= (t_1t_4, t_2t_5, t_3t_6) \cdot (z_1z_4, z_2z_5, z_3z_6) \\ &= (t_1z_1t_4z_4, t_2z_2t_5z_5, t_3z_3t_6z_6) \\ &= \mathcal{U}(f)(t_1z_1, t_2z_2, t_3z_3, t_4z_4, t_5z_5, t_6z_6) \\ &= \mathcal{U}(f)((t_1, t_2, t_3, t_4, t_5, t_6) \cdot (z_1, z_2, z_3, z_4, z_5, z_6)) \end{aligned}$$

We prove the following:

Proposition 3.2.5. *Given a non-degenerate simplicial map $f: K_1 \rightarrow K_2$, any*

non-degenerate Θ_f -equivariant map $g: \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ is equivariantly homotopic to $\mathcal{U}(f)$.

Proof. We have the section $s(K_1): \mathcal{R}(K_1) \rightarrow \mathcal{U}(K_1)$ sending each coordinate to its positive square root, and similarly $s(K_2): \mathcal{R}(K_2) \rightarrow \mathcal{U}(K_2)$. A map $g: \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ satisfying the conditions is determined by a map $h: \mathcal{R}(K_1) \rightarrow \mathcal{R}(K_2)$ taking each F_σ to $F_{f(\sigma)}$ and a map $j: \mathcal{R}(K_1) \rightarrow T^W$, whence g is given on the image of $s(K_1)$ by $g(s(K_1)(x)) = j(x) \circ s(K_2)(h(x))$, and at other points by the equivariance condition. For example, in the case $g = \mathcal{U}(f)$, the map $h: \mathcal{R}(K_1) \rightarrow \mathcal{R}(K_2)$ is multiplication of coordinates and the map $j: \mathcal{R}(K_1) \rightarrow T^W$ takes every point to the identity.

We now show that any two maps $g_1, g_2: \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ satisfying the conditions are equivariantly homotopic. Let $h_1, h_2: \mathcal{R}(K_1) \rightarrow \mathcal{R}(K_2)$ and $j_1, j_2: \mathcal{R}(K_1) \rightarrow T^W$ be the corresponding maps as described above. The homotopy $H: \mathcal{R}(K_1) \times I \rightarrow \mathcal{R}(K_2)$ from h_1 to h_2 given by $H(x, t) = (1 - t)h_1(x) + th_2(x)$ has the property that $H_t = H(-, t): \mathcal{R}(K_1) \rightarrow \mathcal{R}(K_2)$ takes each F_σ to $F_{f(\sigma)}$ for all values of t ; and there exists a homotopy $J: \mathcal{R}(K_1) \times I \rightarrow T^W$ because $\mathcal{R}(K_1)$ is contractible and T^W is path connected. We obtain an equivariant homotopy $G: \mathcal{U}(K_1) \times I \rightarrow \mathcal{U}(K_2)$ by setting $G(s(K_1)(x), t) = J(x, t) \circ s(K_2)(H(x, t))$ and extending G to other values equivariantly. \square

3.2.1 Stable bundle maps

Since $\mathcal{U}(f)$ is holomorphic, the derivative $D\mathcal{U}(f): \tau(\mathcal{U}(K_1)) \rightarrow \tau(\mathcal{U}(K_2))$ is a map of complex vector bundles. The tangent bundles are trivial, and the inclusions $\mathcal{U}(K_1) \subset \mathbf{C}^V$ and $\mathcal{U}(K_2) \subset \mathbf{C}^W$ yield identifications of the tangent spaces at each point with \mathbf{C}^V and \mathbf{C}^W respectively. We see from (3.1) that there is a decomposition:

$$D\mathcal{U}(f) = \bigoplus_{w \in W} (D\mathcal{U}(f)_w: \bigoplus_{f(v)=w} \mathbf{C}_v \rightarrow \mathbf{C}_w)$$

That is, for each $w \in W$, the derivative $D\mathcal{U}(f)$ takes the subbundle $\bigoplus_{f(v)=w} \mathbf{C}_v \subset \tau(\mathcal{U}(K_1))$ into $\mathbf{C}_w \subset \tau(\mathcal{U}(K_2))$. We now focus attention on the restriction $D\mathcal{U}(f)_w: \bigoplus_{f(v)=w} \mathbf{C}_v \rightarrow \mathbf{C}_w$, for a given $w \in W$.

Lemma 3.2.6. *If $f^{-1}(w)$ is nonempty, the kernel of $D\mathcal{U}(f)_w$ has dimension $|f^{-1}(w)| - 1$ at every point of $\mathcal{U}(K)$, and the resulting subbundle of $\tau(\mathcal{U}(K_1))$ is*

equivariantly framed.

Proof. At a point $z \in \mathcal{U}(K_1) \subset \mathbf{C}^V$, the restriction $D\mathcal{U}(f)_w$ is equal to the derivative of the multiplication map $\bigoplus_{f(v)=w} \mathbf{C}_v \rightarrow \mathbf{C}_w$ given by $(z_{v_1}, \dots, z_{v_k}) \mapsto z_{v_1} \dots z_{v_k}$ where $\{v_1, \dots, v_k\} = f^{-1}(w)$, and this derivative is given by the $(1 \times k)$ matrix

$$(z_{v_2} \dots z_{v_k}, \dots, z_{v_1} \dots \hat{z}_{v_i} \dots z_{v_k}, \dots, z_{v_1} \dots z_{v_{k-1}})$$

where \hat{z}_{v_i} indicates that in the i -th column the term z_{v_i} is omitted. Since the simplicial map f is non-degenerate, not more than one coordinate z_{v_i} is zero and so basis for the kernel is given by the vectors $(z_{v_1}, 0, \dots, 0, -z_{v_i}, 0, \dots)$ for $i = 2, \dots, k$. \square

On the other hand, $D\mathcal{U}(f)_w: \bigoplus_{f(v)=w} \mathbf{C}_v \rightarrow \mathbf{C}_w$ is clearly surjective, so for each $w \in W$ with $f^{-1}(w)$ non-empty the decompositions

$$\bigoplus_{f(v)=w} \mathbf{C}_v = \ker(D\mathcal{U}(f)_w) \oplus \text{im}(D\mathcal{U}(f)_w)$$

at each point of $\mathcal{U}(K_1)$ yield a Θ_f -equivariant bundle map

$$\widetilde{(\mathcal{U}(f))}_w: \bigoplus_{f(v)=w} \mathbf{C}_v \rightarrow \mathbf{C}_w \oplus \mathbf{C}^{|f^{-1}(w)|-1}$$

covering $\mathcal{U}(f)$, where the component $\mathbf{C}^{|f^{-1}(w)|-1}$ is a trivial bundle over $\mathcal{U}(K_2)$.

In the case that $f^{-1}(w)$ is empty, \mathbf{C}_w is equivariantly framed on the image of $\mathcal{U}(f)$ (the framing vector is given by z_w , which is non-zero on the image of $\mathcal{U}(f)$) and we have the bundle map

$$\widetilde{(\mathcal{U}(f))}_w: \mathbf{C} \rightarrow \mathbf{C}_w$$

covering $\mathcal{U}(f)$, where the additional summand $\mathbf{C} = \mathcal{U}(f)^*(\mathbf{C}_w)$ is a trivial bundle over $\mathcal{U}(K_1)$.

Putting these bundle maps together for all $w \in W$, we have established:

Proposition 3.2.7. *Let $f: K_1 \rightarrow K_2$ be a non-degenerate simplicial map. The map $\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ is covered by a complex bundle map of the stable*

tangent bundles:

$$\widetilde{\mathcal{U}(f)}: \tau(\mathcal{U}(K_1)) \oplus \bigoplus_{f^{-1}(w)=\emptyset} \mathbf{C} \rightarrow \tau(\mathcal{U}(K_2)) \oplus \bigoplus_{f^{-1}(w)\neq\emptyset} \mathbf{C}^{|f^{-1}(w)|-1}$$

3.3 Open quasitoric manifolds

For $n \geq 1$ we now introduce the open quasitoric manifold functor $\mathcal{Y}(-)$ from the category of dicharacteristic pairs of dimension n to the category of T^n -equivariant stably complex manifolds.

Let (K, ℓ) be a dicharacteristic pair of dimension n ; so K is equipped with a dicharacteristic of dimension n , a map $\ell: V \rightarrow \{\text{primitive vectors in } \mathbf{Z}^n\}$ satisfying condition 2.4.4 (see definition 2.4.3). We construct an *open quasitoric manifold* as follows. As in section 2.2.3, there is a homomorphism $\Theta_\ell: T^V \rightarrow T^n$ whose restriction to the v -th coordinate torus is the map $S^1 \rightarrow T^n$ determined by $\ell(v)$ and condition 2.4.4 ensures that the kernel κ of Θ_ℓ acts freely on $\mathcal{U}(K)$. The action of κ is also smooth, so $\mathcal{U}(K)/\kappa$ is a smooth manifold of dimension $|V| + n$ with T^V/κ action.

Definition 3.3.1. Given a finite simplicial complex K and dicharacteristic ℓ , the open quasitoric manifold $\mathcal{Y}(K, \ell)$ is the smooth manifold $\mathcal{U}(K)/\kappa$.

If Θ_ℓ is surjective, which must happen if $\dim K = (n - 1)$ by 2.4.4, then $\mathcal{Y}(K, \ell)$ admits a T^n action via the isomorphism $T^n \cong T^V/\kappa$, which may also be realised by a choice of section $T^n \hookrightarrow T^V$. If Θ_ℓ is not surjective then there is an isomorphism $\text{im}(\Theta_\ell) \cong T^V/\kappa$ yielding an $\text{im}(\Theta_\ell)$ action on $\mathcal{Y}(K, \ell)$, which may be extended to a T^n action by choosing a decomposition $T^n \cong \text{im}(\Theta_\ell) \oplus \text{coker}(\Theta_\ell)$ and allowing the cokernel to act trivially.

There is a T^n -equivariant stably complex structure on $\mathcal{Y}(K, \ell)$, for the tangent bundle of $\mathcal{U}(K)$ is:

$$\tau(\mathcal{U}(K)) = \mathcal{U}(K) \times \mathbf{C}^V$$

and after factoring out the free action of κ , according to Szczarba [25] we have

$$\tau(\mathcal{Y}(K, \ell)) \oplus \tau_e(\kappa) \cong \mathcal{U}(K) \times_{\kappa} \mathbf{C}^V$$

as bundles over $\mathcal{Y}(K, \ell)$, where $\tau_e(\kappa)$ is a trivial real bundle and the right hand side is a complex vector bundle $\zeta^{|V|}$. Writing the bundle $\mathcal{U}(K) \times \mathbf{C}^V$ as $\mathcal{U}(K) \times \bigoplus_{v \in V} \mathbf{C}_v$ we see that $\zeta^{|V|}$ is a sum of line bundles of the form $\zeta_v = \mathcal{U}(K) \times_{\kappa} \mathbf{C}_v$.

Remark 3.3.2. Note that the stably complex structure depends on the orientations of the one dimensional subtori $\ell(F_v) \leq T^n$, since it depends on the homomorphism $\Theta_\ell: T^V \rightarrow T^n$. Specifically, the complex structure on ζ_v depends on the v -th coordinate of κ , which depends on the orientation of $\ell(F_v)$. We will see in chapter 6 that for different choices of dicharacteristic ℓ_1 and ℓ_2 with the same underlying undirected characteristic function $\bar{\ell}$, the bundles $\mathcal{U}(K) \times_{\kappa_1} \mathbf{C}^V$ and $\mathcal{U}(K) \times_{\kappa_2} \mathbf{C}^V$ are not equivariantly isomorphic, so the dicharacteristic is determined by the isomorphism class of $\zeta^{|V|}$ and hence is a function of the stably complex T^n manifold $\mathcal{Y}(K, \ell)$.

Example 3.3.3. Let K^{n-1} be the boundary of the n -simplex $\partial\Delta^n$, the simplicial complex dual to the simple polytope Δ^n . Define the dicharacteristic ℓ on the vertex set $V = \{v_1, \dots, v_{n+1}\}$ by

$$\begin{aligned}\ell(v_i) &= e_i, \text{ for } i = 1, \dots, n \\ \ell(v_{n+1}) &= -(e_1 + \dots + e_n)\end{aligned}$$

where e_i is the i -th standard basis vector in \mathbf{Z}^n . Then $\mathcal{U}(K) = \mathbf{C}^{n+1} \setminus \{0\}$ and the homomorphism $\Theta_\ell: T^V \rightarrow T^n$ is given by

$$\Theta_\ell(t_1, \dots, t_{n+1}) = (t_1 t_{n+1}^{-1}, \dots, t_n t_{n+1}^{-1})$$

with kernel $\kappa = T_\delta$, the diagonal torus. We know from examples 2.2.8 and 2.2.10 that the quasitoric manifold constructed from the polytope Δ^n with dicharacteristic ℓ is $\mathbf{C}P^n = S^{2n+1}/T_\delta$ with stably complex structure

$$\tau(\mathbf{C}P^n) \oplus \mathbf{R}^2 \cong S^{2n+1} \times_{T_\delta} \mathbf{C}^{n+1}.$$

The open quasitoric manifold $\mathcal{Y}(K, \ell)$ is the $(2n+1)$ -dimensional T^n -manifold $(\mathbf{C}^{n+1} \setminus \{0\})/T_\delta$ with stably complex structure

$$\tau(\mathcal{Y}(K, \ell)) \oplus \mathbf{R} \cong (\mathbf{C}^{n+1} \setminus \{0\}) \times_{T_\delta} \mathbf{C}^{n+1}.$$

3.3.1 Maps of open quasitoric manifolds

Given two dicharacteristic pairs (K_1, ℓ_1) and (K_2, ℓ_2) and a non-degenerate simplicial map $f: K_1 \rightarrow K_2$ that preserves the dicharacteristic, the homomorphism

$\Theta_{\ell_1}: T^V \rightarrow T^n$ satisfies $\Theta_{\ell_1} = \Theta_{\ell_2} \circ \Theta_f$. The kernel κ_1 of Θ_{ℓ_1} is taken into the kernel κ_2 of Θ_{ℓ_2} by Θ_f . Factoring out the free torus actions the map $\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ yields a T^n -equivariant map $\mathcal{Y}(f): \mathcal{U}(K_1)/\kappa_1 \rightarrow \mathcal{U}(K_2)/\kappa_1 \rightarrow \mathcal{U}(K_2)/\kappa_2$, that is:

$$\mathcal{Y}(f): \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$$

Proposition 3.3.4. *If $f: K_1 \rightarrow K_2$ preserves the dicharacteristic then the map $\mathcal{Y}(f): \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ described above is covered by a complex bundle map of the stable tangent bundles, whose restriction to the unstable tangent bundles is the derivative $D\mathcal{Y}(f)$.*

Proof. For the map $\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$, the kernel of the derivative $D\mathcal{U}(f)$ is equivariantly framed as a complex vector bundle of dimension $|V| - |f(V)|$ and the complement of $\text{im } D\mathcal{U}(f)$ in $\tau(\mathcal{U}(K_2))$ is equivariantly framed in complex dimension $W \setminus f(V)$ on the image of $\mathcal{U}(f)$, so the map

$$D\mathcal{U}(f): \tau(\mathcal{U}(K_1)) \rightarrow \tau(\mathcal{U}(K_2))$$

is the restriction of a bundle map

$$\widetilde{\mathcal{U}(f)}: \tau(\mathcal{U}(K_1)) \oplus \mathbf{C}^{W \setminus f(V)} \rightarrow \tau(\mathcal{U}(K_2)) \oplus \mathbf{C}^{|V| - |f(V)|}$$

defined at a point $z \in \mathcal{U}(K_1)$ by

$$\widetilde{\mathcal{U}(f)}(z, \xi, \zeta) = (\mathcal{U}(f)(z), D_z \mathcal{U}(f)(\xi') + \zeta, \xi - \xi')$$

where ξ is a tangent vector at z , $\zeta \in \text{coker}(D_z \mathcal{U}(f))$ and $\xi = \xi' + (\xi - \xi')$ corresponds to a decomposition $\tau(\mathcal{U}(K_1)) = \text{im}(D\mathcal{U}(f)) \oplus \text{ker}(D\mathcal{U}(f))$.

We may divide out the free action of κ_1 to yield

$$\widetilde{\mathcal{Y}(f)'}: \tau(\mathcal{Y}(K_1, \ell_1)) \oplus \tau_e(\kappa_1) \oplus \mathbf{C}^{W \setminus f(V)} \rightarrow \tau(\mathcal{U}(K_2)/\Theta_f(\kappa_1)) \oplus \tau(\Theta_f(\kappa_1)) \oplus \mathbf{C}^{|V| - |f(V)|}$$

covering the map $\mathcal{Y}(f)': \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{U}(K_2)/\Theta_f(\kappa_1)$. Now dividing out the action of $\kappa_2/\Theta_f(\kappa_1)$ on the right hand side we have a bundle map

$$\tau(\mathcal{U}(K_2)/\Theta_f(\kappa_1)) \rightarrow \tau(\mathcal{Y}(K_2, \ell_2)) \oplus \tau(\kappa_2/\Theta_f(\kappa_1))$$

covering the quotient $\mathcal{U}(K_2)/\Theta_f(\kappa_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ and so we obtain the bundle

map

$$\widetilde{\mathcal{Y}}(f): \tau(\mathcal{Y}(K_1, \ell_1)) \oplus \tau_e(\kappa_1) \oplus \mathbf{C}^{W \setminus f(V)} \rightarrow \tau(\mathcal{Y}(K_2, \ell_2)) \oplus \tau_e(\kappa_2) \oplus \mathbf{C}^{|V| - |f(V)|}$$

whose restriction to the tangent bundles is the derivative $D\mathcal{Y}(f)$.

But the map $\widetilde{\mathcal{Y}}(f)$ is precisely the map

$$\zeta^{|V|} = \mathcal{U}(K_1) \times_{\kappa_1} \mathbf{C}^V \rightarrow \mathcal{U}(K_2) \times_{\kappa_2} \mathbf{C}^W = \xi^{|W|}$$

of complex vector bundles. □

We can describe this complex bundle map further. Let $\mathcal{U}(K_1) \times_{\kappa_1} \mathbf{C}^V = \bigoplus \zeta_v$ and $\mathcal{U}(K_2) \times_{\kappa_2} \mathbf{C}^W = \bigoplus \zeta_w$ be the complex line bundles described above. With the same hypotheses we have:

Proposition 3.3.5. *The complex bundle map of Proposition 3.3.4 covering $\mathcal{Y}(f)$ is composed of complex bundle maps $\widetilde{\mathcal{Y}}(f)_w: \bigoplus_{f(v)=w} \zeta_v \rightarrow \zeta_w \oplus \mathbf{C}^{|f^{-1}(w)|-1}$, or in the case where $f^{-1}(w)$ is empty, $\widetilde{\mathcal{Y}}(f)_w: \mathbf{C} \rightarrow \zeta_w$.*

Proof. We treat the first case, the second being trivial. Covering

$$\mathcal{U}(f): \mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$$

we have a Θ_f -equivariant bundle map

$$\widetilde{\mathcal{U}}(f)_w: \mathcal{U}(K_1) \times \left(\bigoplus_{f(v)=w} \mathbf{C}_v \right) \rightarrow \mathcal{U}(K_2) \times (\mathbf{C}_w \oplus \mathbf{C}^{|f^{-1}(w)|-1})$$

with the $\mathbf{C}^{|f^{-1}(w)|-1}$ summand equivariantly framed. After factoring out the freely acting tori κ_1 and κ_2 we obtain

$$\widetilde{\mathcal{Y}}(f)_w: \mathcal{U}(K_1) \times_{\kappa_1} \left(\bigoplus_{f(v)=w} \mathbf{C}_v \right) \rightarrow \mathcal{U}(K_2) \times_{\kappa_2} (\mathbf{C}_w \oplus \mathbf{C}^{|f^{-1}(w)|-1}).$$

But this is

$$\widetilde{\mathcal{Y}}(f)_w: \bigoplus_{f(v)=w} \zeta_v \rightarrow \zeta_w \oplus \mathbf{C}^{|f^{-1}(w)|-1}.$$

□

Corollary 3.3.6. $\mathcal{Y}(f)^*(\xi_w)$ is equivariantly stably isomorphic to $\bigoplus_{f(v)=w} \zeta_v$ (as complex vector bundles).

The map $\widetilde{\mathcal{Y}}(f)$ makes \mathcal{Y} a functor from the category of dicharacteristic pairs with non-degenerate simplicial maps that preserve the dicharacteristic, to the category of T^n -equivariant stably complex manifolds with equivariant stable bundle maps.

Given a map of dicharacteristic pairs $f: (K_1, \ell_1) \rightarrow (K_2, \ell_2)$, the map $\mathcal{Y}(f)$ is essentially unique. To make this precise, by a map of base spaces corresponding to f , we will mean a map $\mathcal{R}(K_1) \rightarrow \mathcal{R}(K_2)$ taking each face F_σ of $\mathcal{R}(K_1)$ to face $F_{f(\sigma)}$ of $\mathcal{R}(K_2)$. Then by the same argument used to prove proposition 3.2.5 we have:

Proposition 3.3.7. *Let $f: (K_1, \ell_1) \rightarrow (K_2, \ell_2)$ be a map of dicharacteristic pairs. Any T^n -equivariant map $g: \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ which induces a map of base spaces corresponding to $f: K_1 \rightarrow K_2$ is equivariantly homotopic to the map $\mathcal{Y}(f)$.*

Conversely, any non-degenerate equivariant map $g: \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ must induce a map of base spaces corresponding to some non-degenerate map $f: K_1 \rightarrow K_2$ that preserves the underlying *undirected* characteristic function. To work with stably complex structures, we must insist that f preserve the dicharacteristic. In this case, we shall call the map g *directed*. The undirected case will be discussed in more detail below.

3.4 Quasitoric spaces and quasitoric manifolds

In this section we relate the properties of open quasitoric manifolds $\mathcal{Y}(K, \ell)$ to the established quasitoric spaces $\mathcal{X}(K, \ell)$ and quasitoric manifolds $M^{2n}(P^n, \ell)$.

We begin by considering the moment angle complex $\mathcal{Z}(K) = (D^2, S^1)^K$ (see section 2.3). This is canonically embedded in $\mathcal{U}(K)$ via the inclusion $(D^2, S^1) \hookrightarrow (\mathbf{C}, \mathbf{C}^*)$ and the orbit space $\mathcal{I}(K) = (I, 1)^K$ is canonically embedded in $\mathcal{R}(K)$. We can define a facial structure on $\mathcal{I}(K)$ as in example 2.3.4, by taking face F_σ to be the set of points with v -th coordinate zero for $v \in \sigma$. If K has a dicharacteristic ℓ , the kernel κ of $\Theta_\ell: T^V \rightarrow T^n$ acts freely on $\mathcal{Z}(K)$ and we obtain the *closed quasitoric space* $\mathcal{X}(K, \ell) = \mathcal{Z}(K)/\kappa$.

The inclusion of $\mathcal{Z}(K)$ in $\mathcal{U}(K)$ induces an inclusion of $\mathcal{X}(K, \ell)$ in $\mathcal{Y}(K, \ell)$ covering the inclusion of $\mathcal{I}(K)$ in $\mathcal{R}(K)$. We obtain a complex vector bundle over $\mathcal{Z}(K)$ as the restriction of $\tau(\mathcal{U}(K)) = \mathbf{C}^V$ and a complex vector bundle $\xi|$ on $\mathcal{X}(K, \ell)$ as the restriction of $\tau(\mathcal{Y}(K, \ell)) \oplus \tau_e(\kappa) = \mathcal{U}(K) \times_\kappa \mathbf{C}^V = \xi$.

There is an equivariant deformation retract $r: \mathcal{R}(K) \rightarrow \mathcal{I}(K)$ preserving the facial structures which is covered by equivariant deformation retracts $\mathcal{U}(K) \rightarrow \mathcal{Z}(K)$ and $\mathcal{Y}(K, \ell) \rightarrow \mathcal{X}(K, \ell)$ (see [6], Theorem 8.9. This is non-trivial, as there is no map of pairs $(\mathbf{R}_{\geq}, \mathbf{R}_{>}) \rightarrow (I, 1)$). By the same argument used to prove Proposition 3.2.5 we have

Proposition 3.4.1. *Any two maps $\mathcal{I}(K) \rightarrow \mathcal{R}(K)$ which preserve the face lattices are homotopic. Any two T^V -equivariant maps $\mathcal{Z}(K) \rightarrow \mathcal{U}(K)$ which preserve orbit types are equivariantly homotopic. Any two T^n -equivariant maps $\mathcal{X}(K, \ell) \rightarrow \mathcal{Y}(K, \ell)$ which cover maps $\mathcal{I}(K) \rightarrow \mathcal{R}(K)$ that preserve the face lattice are equivariantly homotopic.*

3.4.1 Quasitoric manifolds

Recall from sections 2.1.1 and 2.3.2 that when K is the dual of a simple polytope P^n , the cubical complex $\mathcal{I}(K) = cc(K)$ gives a cubical decomposition of P^n . Given a realisation of $P^n \subset \mathbf{R}^n$ as an intersection of half spaces

$$P^n = \{x \in \mathbf{R}^n \mid a_v \cdot x + b_v \geq 0, v \in V\} \quad (3.3)$$

we have the smooth embedding $\iota: P^n \hookrightarrow \mathcal{R}(K)$ described in section 2.2.3, given coordinate-wise by $\iota(x)_v = a_v \cdot x + b_v$, the perpendicular distance between x and the hyperplane defining facet F_v . Then the *moment angle manifold* Z_P is constructed as $Z_P = \rho^{-1}(\iota(P^n))$ where ρ is the projection $\rho: \mathbf{C}^V \rightarrow \mathbf{R}^V$ taking each coordinate to the square of its modulus, and Z_P is equivariantly framed in \mathbf{C}^V [7]. Recall from section 2.3 that Z_P is equivariantly homeomorphic to $\mathcal{Z}(K_P)$.

We therefore obtain a smooth embedding of Z_P in $\mathcal{U}(K)$ and this embedding is equivariantly framed in codimension $(m - n)$, $m = |V|$. Thus, given a dicharacteristic ℓ , we also obtain a smooth embedding of the *quasitoric manifold* $M^{2n}(P^n, \ell)$ in $\mathcal{Y}(K, \ell)$ which is again equivariantly framed in codimension $(m - n)$. In fact there is a *smooth* equivariant retraction $\mathcal{U}(K) \rightarrow Z_P$ [3], and therefore there are smooth retractions $\mathcal{Y}(K, \ell) \rightarrow M^{2n}$ and $\mathcal{R}(K) \rightarrow P^n$.

We may now obtain equivariant stably complex structures on Z_P and M^{2n} in a similar way to that described in section 2.2.4 (see also [7]). For Z_P we have

$$\tau(Z_P) \oplus \mathbf{R}^{m-n} \cong \tau(\mathcal{U}(K))|_{Z_P} = \mathbf{C}^V$$

For M^{2n} we have

$$\tau(M^{2n}) \oplus \mathbf{R}^{m-n} \cong \tau(\mathcal{Y}(K, \ell))$$

and hence

$$\tau(M^{2n}) \oplus \tau_e(\kappa) \oplus \mathbf{R}^{m-n} \cong (\mathcal{U}(K) \times_{\kappa} \mathbf{C}^V)|_{M^{2n}} = Z_P \times_{\kappa} \mathbf{C}^V = \zeta|_M \quad (3.4)$$

where in each case the component \mathbf{R}^{m-n} is equivariantly framed.

We make the trivial observation that for the embedding $\iota: M^{2n} \rightarrow \mathcal{Y}(K, \ell)$, $\iota^*(\zeta) = \zeta|_M$ and for each component line bundle $\iota^*(\zeta_v) = \zeta_v|_M$.

These constructions do not depend on the choice of realisation of $P^n \subset \mathbf{R}^n$, since any two are related by a diffeomorphism ϕ of \mathbf{R}^n and the two embeddings ι_1 and $(\iota_2 \circ \phi): P^n \hookrightarrow \mathcal{R}(K)$ are smoothly homotopic (via linear interpolation between $\iota_1(x)$ and $\iota_2 \cdot \phi(x)$ in $\mathcal{R}(K) \subset \mathbf{R}^V$, and this homotopy yields smooth equivariant homotopies between the embeddings of Z_P in $\mathcal{U}(K)$ and of $M^{2n}(P^n, \ell)$ in $\mathcal{Y}(K, \ell)$. As long as the same orientation is chosen for the trivial component \mathbf{R}^{m-n} , the same homotopies yield equivalences of stably complex structures

$$\tau(Z_P) \oplus \mathbf{R}^{m-n} \cong \tau(\mathcal{U}(K))|_{Z_P}$$

and

$$\tau(M^{2n}) \oplus \tau_e(\kappa) \oplus \mathbf{R}^{m-n} \cong \zeta|_M.$$

3.4.2 Orientations

Before describing maps of quasitoric manifolds, it is worth making some comments about orientations. The manifold $\mathcal{U}(K)$ is canonically oriented as a full dimensional submanifold of \mathbf{C}^V , but T^V is not canonically oriented (an orientation would be induced by ordering the vertices V of K , as in [7]). A choice of orientation of T^V induces an orientation of the orbit space $\mathcal{R}(K)$.

Given a dicharacteristic ℓ , an orientation of T^V induces an orientation of the kernel κ of $\Theta_{\ell}: T^V \rightarrow T^n$, because T^n is canonically oriented. This in turn induces an orientation on $\mathcal{Y}(K, \ell)$ via $\tau(\mathcal{Y}(K, \ell)) \oplus \tau(\kappa) = \mathbf{C}^V$. This orientation is consistent with the orientations on T^n and the orbit space $\mathcal{R}(K)$.

In the case of a quasitoric manifold, suppose we are given an orientation for P^n (arising from an orientation of \mathbf{R}^n). This determines an orientation for M^{2n} because T^n is canonically oriented. Now Z_P is equivariantly framed in $\mathcal{U}(K)$

by $\mathbf{R}^{|V|-n}$ and this framing projects to a framing of P^n in $\mathcal{R}(K)$, so a choice of orientation for $\mathbf{R}^{|V|-n}$ is equivalent to a choice of orientation for $\mathcal{R}(K)$ and hence for T^V . So we may freely choose an orientation for $\mathbf{R}^{|V|-n}$, inducing orientations on T^V , κ , $\mathcal{R}(K)$ and $\mathcal{Y}(K, \ell)$. The quasitoric manifold M^{2n} is framed by $\mathbf{R}^{|V|-n}$ in $\mathcal{Y}(K, \ell)$ and the orientations of $\mathbf{R}^{|V|-n}$ and $\mathcal{Y}(K, \ell)$ are consistent with the orientation of M^{2n} . In the stably complex structure

$$\tau(M^{2n}) \oplus \mathbf{R}^{|V|-n} \oplus \tau_e(\kappa) = \zeta|_M$$

the orientation of either $\mathbf{R}^{|V|-n}$ or $\tau_e(\kappa)$ may be chosen freely, inducing an orientation on the other so that the orientation of the sum $\mathbf{R}^{|V|-n} \oplus \tau_e(\kappa)$ is determined by the orientation of M^{2n} .

3.5 Maps of quasitoric manifolds

Any non-degenerate equivariant map $M^{2n}(P_1^n, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ must cover a map of orbit spaces that preserves the face structures, corresponding to a non-degenerate simplicial map $K_1 \rightarrow K_2$ that preserves the undirected characteristic function. The same is true for maps $M^{2n}(P_1^n, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$ or $\mathcal{Y}(K_1, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$. By the argument used to prove proposition 3.2.5 we now have the following result (see also proposition 3.3.7):

Proposition 3.5.1. *Any two non-degenerate equivariant maps*

- $M^{2n}(P_1^n, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$
- $M^{2n}(P_1^n, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$, or
- $\mathcal{Y}(K_1, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$

corresponding to the same simplicial map $K_1 \rightarrow K_2$ are equivariantly homotopic.

We may insist that such maps preserve the dicharacteristic. Then we prove:

Theorem 3.5.2. *Let $f: (K_1, \ell_1) \rightarrow (K_2, \ell_2)$ be a non-degenerate simplicial map that preserves the dicharacteristic. Let $\zeta = \bigoplus_v \zeta_v$ and $\xi = \bigoplus_w \xi_w$ denote the complex structures on the stable tangent bundles of $\mathcal{Y}(K_1, \ell_1)$ and $\mathcal{Y}(K_2, \ell_2)$ respectively, and $\zeta| = \bigoplus_v \zeta_v|$ and $\xi| = \bigoplus_w \xi_w|$ the corresponding structures on $M^{2n}(P_1^n, \ell_1)$ and $M^{2n}(P_2^n, \ell_2)$.*

1. If $g: \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;
2. If $g: M^{2n}(P_1^n, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;
3. If $g: M^{2n}(P_1^n, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;
4. If $g: \mathcal{Y}(K_1, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;

where the relation \simeq_s denotes equivariant stable isomorphism.

Proof. We have 1 from 3.3.6 and 3.3.7. For 2, the map g is homotopic to $\mathcal{Y}(f) \circ \iota_{K_1}$, where $\mathcal{Y}(f)$ is the map $\mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ described in proposition 3.3.5 and ι_1 is the inclusion $M^{2n}(P_1^n, \ell_1) \hookrightarrow \mathcal{Y}(K_1, \ell_1)$. So $g^*(\xi_w) \simeq (\iota_1)^* \mathcal{Y}(f)^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$ as required. For 3, the map $\iota_2 \circ g: M^{2n}(P_1^n, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ satisfies $(\iota_2 \circ g)^*(\xi_w) = (g)^*(\iota_2)^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$; but $(\iota_2)^*(\xi_w) \simeq_s \xi_w$. For 4, we consider the composite $\iota_2 \circ g$. \square

Similar statements hold for the complex bundles over $\mathcal{X}(K_1, \ell_1)$ and $\mathcal{X}(K_2, \ell_2)$.

Remark 3.5.3. These results only hold for maps that preserve the dicharacteristic. For example, let $\mathbf{C}P^1$ and S^2 denote the quasitoric manifolds over the interval I^1 with dicharacteristics $(1, -1)$ and $(1, 1)$ respectively (the first dicharacteristic yields the classical stably complex structure on $\mathbf{C}P^1$, while the second yields a trivial structure). There is a non-degenerate T^1 -equivariant map $g: S^2 \rightarrow \mathbf{C}P^1$ covering the identity map of I^1 . We know from [10] and shall see later that $H^2(\mathbf{C}P^1)$ is generated by the first Chern classes of the line bundles ξ_1 and ξ_2 , with $c_1(\xi_1) = c_1(\xi_2)$. Similarly $H^2(S^2)$ is generated by $c_1(\zeta_1)$ and $c_1(\zeta_2)$, but $c_1(\zeta_1) = -c_1(\zeta_2)$. Now $c_1(\xi_1 \oplus \xi_2) = c_1(\xi_1) + c_1(\xi_2)$ is twice a generator of $H^2(\mathbf{C}P^1)$, so $c_1(g^*(\xi_1 \oplus \xi_2)) = g^*(c_1(\xi_1 \oplus \xi_2))$ is twice a generator of $H^2(S^2)$ because g has degree one. But $c_1(\zeta_1 \oplus \zeta_2) = c_1(\zeta_1) + c_1(\zeta_2) = 0 \in H^2(S^2)$, so it is certainly not true that $g^*(\xi_1 \oplus \xi_2) \simeq_s \zeta_1 \oplus \zeta_2$.

Example 3.5.4. On the other hand, there is a map $g: S^2 \rightarrow \mathbf{C}P^1$ that *does* preserve the dicharacteristic. The induced map of base spaces $h: I \rightarrow I$ may be taken to be $h(x) = x(1 - x)$, which in particular gives $h(0) = h(1) = 0$. Then

by Proposition 3.5.2 $g^*(\xi_1) \simeq_s \zeta_1 \oplus \zeta_2$ and $g^*(\xi_2) \simeq_s 0$. So $g^*(c_1(\xi_1)) = c_1(\zeta_1) + c_1(\zeta_2) = 0 = g^*(c_1(\xi_2))$ and the induced map $g^*: H^2(\mathbf{C}P^1) \rightarrow H^2(S^2)$ is zero. But we knew this already, because g is null-homotopic (by lifting a homotopy between $h: I \rightarrow I$ and the trivial map $I \rightarrow \{0\}$, for example $H(x, t) = tx(1-x)$).

3.6 The undirected case

To highlight the role of the dicharacteristic, we shall prove the following, which is well known in the case of quasitoric manifolds [10]:

Proposition 3.6.1. *The equivariant diffeomorphism type of the open quasitoric manifold $\mathcal{Y}(K, \ell)$ depends only on the undirected characteristic function $\bar{\ell}$ underlying the dicharacteristic ℓ .*

Proof. Suppose the simplicial complex K is endowed with two dicharacteristics ℓ_1 and ℓ_2 that differ only in the orientations of the one-dimensional subtori assigned to each vertex. In particular, let V_0 denote the set of vertices of K for which $\ell_1(v) = \ell_2(v)$ and let V_1 denote the set of vertices for which $\ell_1(v) = -\ell_2(v)$, with $V = V_0 \cup V_1$. The map $g: \mathcal{U}(K) \rightarrow \mathcal{U}(K)$ given by

$$g(z)_v = \begin{cases} z_v & \text{if } v \in V_0 \\ \bar{z}_v & \text{if } v \in V_1 \end{cases}$$

is a diffeomorphism of real manifolds (but it does not preserve the complex structure if V_1 is non-empty). The corresponding automorphism $\Theta_g: T^V \rightarrow T^V$ takes κ_1 to κ_2 , where these are the kernels of Θ_{ℓ_1} and $\Theta_{\ell_2}: T^V \rightarrow T^n$ respectively. After dividing out the actions of κ_1 and κ_2 and choosing sections s_1 and $s_2: T^n \rightarrow T^V$ so that $s_2 = \Theta_g \circ s_1$ we obtain a T^n -equivariant diffeomorphism $\check{g}: \mathcal{Y}(K, \ell_1) \rightarrow \mathcal{Y}(K, \ell_2)$ which covers the identity on the orbit space $\mathcal{R}(K)$. \square

The same construction yields an equivariant homeomorphism $\mathcal{X}(K, \ell_1) \rightarrow \mathcal{X}(K, \ell_2)$, and an equivariant diffeomorphism $M^{2n}(K, \ell_1) \rightarrow M^{2n}(K, \ell_2)$ in the case that K is the dual of a simple polytope, both covering the identity on the orbit space.

So $\mathcal{Y}(K, \ell)$, $\mathcal{X}(K, \ell)$ and $M^{2n}(P^n, \ell)$ depend only on the undirected characteristic function $\bar{\ell}$ up to equivariant diffeomorphism (or homeomorphism in the case of $\mathcal{X}(K, \ell)$), as spaces *without* additional complex structure. We may denote

these constructions $\overline{\mathcal{Y}}(K, \bar{\ell})$, $\overline{\mathcal{X}}(K, \bar{\ell})$ and $\overline{M}^{2n}(K, \bar{\ell})$ to distinguish them from the stably complex constructions. On the other hand, the equivariant stably complex structures on $\mathcal{Y}(K, \ell)$ and $M^{2n}(P^n, \ell)$ depend on the dicharacteristic ℓ and are *not* determined by the underlying undirected characteristic function: different choices of dicharacteristic yield stably complex structures that are not equivariantly isomorphic and they may not even be non-equivariantly isomorphic, as seen in Remark 3.5.3 for example where the two stably complex structures described on S^2 have different first Chern classes.

Now given simplicial complexes K_1 and K_2 with undirected characteristic functions $\bar{\ell}_1$ and $\bar{\ell}_2$, we may have a non-degenerate simplicial map $\bar{f}: (K_1, \bar{\ell}_1) \rightarrow (K_2, \bar{\ell}_2)$ that preserves the characteristic function, i.e. $\ell_2 \circ \bar{f} = \bar{\ell}_1$. Evidently, for any choice of dicharacteristic ℓ_2 for K_2 compatible with $\bar{\ell}_2$ there will be a unique choice of ℓ_1 compatible with $\bar{\ell}_1$ such that \bar{f} lifts to a map f of dicharacteristic pairs. Then we will have an equivariant map $\mathcal{Y}(f): \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ as described in Section 3.3. For any other choices ℓ_1' and ℓ_2' compatible with $\bar{\ell}_1$ and $\bar{\ell}_2$ we will have the composite $(g_2)^{-1} \circ \mathcal{Y}(f) \circ g_1: \mathcal{Y}(K_1, \ell_1') \rightarrow \mathcal{Y}(K_2, \ell_2')$ where $g_1: \mathcal{Y}(K_1, \ell_1') \rightarrow \mathcal{Y}(K_1, \ell_1)$ and $g_2: \mathcal{Y}(K_2, \ell_2') \rightarrow \mathcal{Y}(K_2, \ell_2)$ are equivariant diffeomorphisms covering the identity on the orbit space, as described in the proof of Proposition 3.6.1. If ℓ_1' and ℓ_2' are also such that \bar{f} lifts to $f': (K_1, \ell_1') \rightarrow (K_2, \ell_2')$ preserving the dicharacteristic, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}(K_1, \ell_1) & \xrightarrow{\mathcal{Y}(f)} & \mathcal{Y}(K_2, \ell_2) \\ g_1 \uparrow \cong & & g_2 \uparrow \cong \\ \mathcal{Y}(K_1, \ell_1') & \xrightarrow{\mathcal{Y}(f)'} & \mathcal{Y}(K_2, \ell_2') \end{array}$$

We may therefore think of $\overline{\mathcal{Y}}(K, \bar{\ell})$ as an equivalence class of open quasitoric manifolds, with equivariant diffeomorphism of smooth T^n -manifolds without additional structure as the equivalence relation, and these remarks also apply to $\overline{\mathcal{X}}(K, \bar{\ell})$ and $\overline{M}^{2n}(K, \bar{\ell})$.

Chapter 4

Infinite complexes and final spaces

For $n > 1$, the category of finite simplicial complexes with dimension n dicharacteristic does not contain a final object but, as we shall see, a final object exists in the larger category which includes both finite and countably infinite simplicial complexes with dimension n dicharacteristic. In this chapter we introduce the necessary machinery to work with quasitoric spaces over countably infinite simplicial complexes. This will allow us to construct quasitoric spaces associated to the final objects in these categories. We begin with an overview of some properties of infinite dimensional smooth manifolds.

4.1 Direct limit smooth manifolds

We will be interested in smooth manifolds modelled on the vector spaces \mathbf{R}^∞ and \mathbf{C}^∞ of finite sequences in \mathbf{R} and \mathbf{C} respectively, equipped with the colimit topology. That is, we consider \mathbf{K}^∞ ($\mathbf{K} = \mathbf{R}$ or \mathbf{C}) as the colimit of the diagram $\mathbf{K}^1 \hookrightarrow \mathbf{K}^2 \hookrightarrow \dots$ and define a set $U \in \mathbf{K}^\infty$ to be open if and only if $U \cap \mathbf{K}^n$ is open for each n . Here we give a brief summary of the theory we require, which is laid out in [12] and [11], while [17] contains some familiar examples from algebraic topology, such as infinite spheres and Grassmannians.

There are at least two approaches to calculus over large vector spaces in the literature: the convenient differential calculus of [17] and the general differential calculus of [2]. However, in the case of smooth \mathbf{R}^∞ and \mathbf{C}^∞ manifolds, the theories agree. We present appropriate definitions of derivatives and smooth maps in this

context, taken from [12]. The definitions are given in the context of sequentially complete, locally convex (s.c.l.c.) vector spaces; for our purposes it is sufficient to note that these include \mathbf{R}^s and \mathbf{C}^s for $s \in \mathbf{N}_0 \cup \{\infty\}$. Note also that the definitions coincides with the usual definitions in the case $s < \infty$.

Definition 4.1.1. Let X and Y be (s.c.l.c.) topological vector spaces, U be an open subset of X , and $f: U \rightarrow Y$ be a continuous map. Given $x \in U$ and $h \in X$, the *derivative* of f at x in the direction h is defined as

$$df(x)(h) := \lim_{t \rightarrow 0} t^{-1}(f(x + th) - f(x))$$

whenever the limit exists.

We say that f is *differentiable* at x if $df(x)(h)$ exists for all $h \in X$; it is \mathcal{C}^1 if it is differentiable at all $x \in U$ and $df: U \times X \rightarrow Y$, $(x, h) \mapsto df(x)(h)$ is continuous.

Definition 4.1.2. Higher derivatives are defined recursively by

$$d^n f(x)(h_1, \dots, h_n) := \lim_{t \rightarrow 0} t^{-1}(d^{n-1} f(x + th_n)(h_1, \dots, h_{n-1}) - d^{n-1} f(x)(h_1, \dots, h_{n-1}))$$

provided that all limits involved exist. The function f is said to be of class \mathcal{C}^n if $d^n f: U \times X^n \rightarrow Y$ is continuous; it is of class \mathcal{C}^∞ (or *smooth*) if it is \mathcal{C}^n for all n .

In the cases of \mathbf{R}^s and \mathbf{C}^s for $s \in \mathbf{N}_0 \cup \{\infty\}$ we also have the following, also from [12]: Suppose that $V_1 \leq V_2 \leq \dots$ is a sequence of finite-dimensional subspaces such that $X = \cup_{i \in \mathbf{N}} V_i$. Then setting $U_i = U \cap V_i$ all derivatives of f of a given order exist if and only if this holds for the derivatives of $f|_{U_i}$ for all i . Then the function $d^n f$ is continuous if and only if $d^n(f|_{U_i})$ is continuous for all i .

Composites of \mathcal{C}^r maps are \mathcal{C}^r , so smooth manifolds modelled on s.c.l.c. topological vector spaces can be defined in the usual way, as can vector bundles and in particular the tangent bundle, and the definitions are consistent with the usual definitions in the cases of \mathbf{R}^s and \mathbf{C}^s for $s < \infty$.

There are two results from [11] that we will make particular use of. Let \mathbf{K} denote either \mathbf{R} or \mathbf{C} . Let $S := ((M_i)_{i \in I}, (\lambda_{i,j})_{i \geq j})$ be a direct system of finite dimensional smooth \mathbf{K} -manifolds and embeddings $\lambda_{i,j}: M_j \hookrightarrow M_i$. That is, I is a directed set (a poset in which every pair of elements has an upper bound) and the embeddings $\lambda_{i,j}$ for each pair $i \geq j$ satisfy $\lambda_{i,i} = id_{M_i}$ and $\lambda_{i,j} \circ \lambda_{j,k} = \lambda_{i,k}$ for

$i \geq j \geq k$. Let $(M, (\lambda_i)_{i \in I})$ be the colimit of S in the category of topological spaces (with maps $\lambda_i: M_i \rightarrow M$), and abbreviate $s := \sup\{\dim(M_i) \mid i \in I\} \in \mathbf{N}_0 \cup \{\infty\}$.

Theorem 4.1.3. [11, Theorem 3.1] *There exists a uniquely determined smooth manifold structure on M , modelled on \mathbf{K}^s , which makes $\lambda_i: M_i \rightarrow M$ a smooth map for each $i \in I$, and such that $(M, (\lambda_i)_{i \in I}) = \text{colim } S$ in the category of smooth manifolds modelled on topological \mathbf{K} -vector spaces. For each $i \in I$ and $x \in M_i$, the differential $D_x(\lambda_i): \tau_x(M_i) \rightarrow \tau_{\lambda_i(x)}(M)$ is injective.*

Proposition 4.1.4. [11, Proposition 3.4] *Assume that $f: X \rightarrow M$ is a \mathcal{C}^r -map, where $r \in \mathbf{N}_0 \cup \{\infty\}$ and X is a \mathcal{C}^r -manifold modelled on a metrizable s.c.l.c. topological \mathbf{K} -vector space E . Then every $x \in X$ has an open neighbourhood S such that $f(S) \subset \lambda_i(M_i)$ for some $i \in \mathbf{N}$ and such that $\lambda_i^{-1} \circ f|_S: S \rightarrow M_i$ is \mathcal{C}^r .*

A metrizable topological space is one whose topology is the induced topology of some metric. In particular, \mathbf{R}^n and \mathbf{C}^n are metrizable for finite n , but \mathbf{R}^∞ and \mathbf{C}^∞ (with the colimit topology) are not.

4.2 Infinite polyhedral products

The literature on polyhedral products is restricted to the case of finite simplicial complexes (usually by specifying a finite vertex set). We now extend the scope of the theory by defining the polyhedral product $(X, A)^K$ for a finite dimensional simplicial complex on countably infinite vertex set as the colimit of a $\text{CAT}(\mathbf{K})$ -diagram. Recall from section 2.3.1 that, for a simplicial complex K , the category $\text{CAT}(\mathbf{K})$ has as objects the simplices of K and as morphisms the inclusions of those simplices. Then if K is a finite simplicial complex on vertex set V the polyhedral product $(X, A)^K$ is the colimit of the $\text{CAT}(\mathbf{K})$ -diagram in Top that assigns to each simplex $\sigma \in K$ the space

$$(X, A)^\sigma = \{(x_v) \in X^V \mid x_v \in A \text{ for } v \notin \sigma\} \quad (4.1)$$

and to each morphism $\sigma \hookrightarrow \sigma'$ the inclusion $(X, A)^\sigma \hookrightarrow (X, A)^{\sigma'}$. To extend this definition to the case of countably infinite K we must choose an appropriate space to play the role of X^V . To that end, we make the following definition:

Definition 4.2.1. Given a topological space X and countable set V , we denote by $X^{(V)}$ the colimit in Top of the diagram whose objects are the spaces X^W for

finite subsets $W \subset V$ and whose morphisms are the inclusions $X^W \hookrightarrow X^{W'}$ for $W \subset W'$.

Note that $X^{(V)}$ is a topological space, with the colimit topology. That is, a set $U \subset X^{(V)}$ is open if and only if $\iota_W^{-1}(U)$ is open in X^W for every finite subset $W \subset V$, where ι_W is the final map $\iota_W: X^W \rightarrow X^{(V)}$ from X^W to the colimit.

It will be useful to interpret $X^{(V)}$ as a topologization of the subset of the cartesian product X^V where all but finitely many coordinates are equal to x_0 , for some chosen basepoint $x_0 \in X$. Each X^W for finite $W \subset V$ may be then be identified with a subspace of $X^{(V)}$ by taking missing coordinates to be equal to x_0 , and $X^{(V)}$ is the union of the X^W . Then a set $U \subset X^{(V)}$ is open if and only if $U \cap X^W$ is open in X^W for every finite subset $W \subset V$. We shall use the notation $(X, x_0)^{(V)}$ when we wish to specify the basepoint x_0 .

Remark 4.2.2. The construction of $X^{(V)}$ is familiar from other categories, for example the categories of abelian groups and vector spaces, where it forms the coproduct, or direct sum. In particular if G is a commutative topological group with identity e then $(G, e)^{(V)}$ is the direct sum of copies of G indexed by V , and in the more general case that X is a commutative topological monoid with identity e then $(X, e)^{(V)}$ is the direct sum of commutative topological monoids.

Example 4.2.3. Let V be \mathbf{N}^+ , the set of non-zero natural numbers. Then with $X = \mathbf{R}$ and $x_0 = 0$, the colimit $(\mathbf{R}, 0)^{(V)}$ can be identified with \mathbf{R}^∞ , the vector space of sequences that are eventually zero. This is a topological vector space; the colimit of the increasing series of topological vector spaces $\mathbf{R}^1 \hookrightarrow \mathbf{R}^2 \hookrightarrow \dots \hookrightarrow \mathbf{R}^k \hookrightarrow \dots$.

We now wish to define the subspace $(X, A)^\sigma \subset X^{(V)}$ for a simplex $\sigma \in K$. We can define this, too, as a colimit:

Definition 4.2.4. Given a pair (X, A) and a finite subset $\sigma \in V$, we denote by $(X, A)^{\sigma|V}$ the colimit of the diagram whose objects are the spaces

$$(X, A)^{\sigma|W} = \{(x_w) \in X^W \mid x_w \in A \text{ for } w \notin \sigma\}$$

for finite subsets $W \subset V$ and whose morphisms are the inclusions $(X, A)^{\sigma|W} \hookrightarrow (X, A)^{\sigma|W'}$ for $W \subset W'$.

This definition of $(X, A)^{\sigma|V}$ agrees with (4.1) when V is finite. Since for each W we have $(X, A)^{\sigma|W} \subset X^W$, there is a natural transformation from the diagram

defining $(X, A)^{\sigma|V}$ to the diagram defining $X^{(V)}$ in which each map is an inclusion. Therefore there is an inclusion of the colimits, i.e. $(X, A)^{\sigma|V}$ is a subspace of $X^{(V)}$. We have immediately:

Proposition 4.2.5. *When $X^{(V)}$ is identified with $(X, x_0)^{(V)}$ for $x_0 \in A \subset X$, by setting missing coordinates equal to x_0 , the subspace $(X, A)^{\sigma|V}$ is equal to*

$$(X, A, x_0)^{\sigma|V} = \{(x_v) \in X^{(V)} \mid x_v \in A \text{ for } v \notin \sigma\}$$

We are now in a position to define the polyhedral product, in a way consistent with the finite case:

Definition 4.2.6. Given a pair (X, A) and a finite dimensional simplicial complex K on countable vertex set V , the polyhedral product $(X, A)^K$ is the colimit of the CAT(K)-diagram in Top that assigns to each simplex $\sigma \in K$ the space

$$(X, A)^{\sigma|V} \subset X^{(V)}$$

and to each morphism $\sigma \hookrightarrow \sigma'$ the inclusion $(X, A)^{\sigma|V} \hookrightarrow (X, A)^{\sigma'|V}$.

As a consequence of Proposition 4.2.5 we have an alternative description, also familiar from the finite case:

Proposition 4.2.7. *The polyhedral product $(X, A)^K$ may be identified with the subspace*

$$\begin{aligned} (X, A, x_0)^K &= \bigcup_{\sigma \in K} (X, A, x_0)^{\sigma|V} \\ &= \{(x_v) \in X^{(V)} \mid \{v \in V \mid x_v \notin A\} \in K\} \end{aligned}$$

for a chosen basepoint $x_0 \in A$

Example 4.2.8. Let $\Delta(V)$ be the simplicial complex consisting of all finite subsets of V . Then $(X, A)^{\Delta(V)} = X^{(V)}$.

Example 4.2.9. The space $(\mathbf{C}, \mathbf{C}^*, 1)^K$ is the complement in $(\mathbf{C}, 1)^{(V)}$ of the subspace arrangement $\bigcup_{\sigma \notin K} L_\sigma$ where

$$L_\sigma = \{(z_v) \in (\mathbf{C}, 1)^{(V)} \mid z_v = 0 \text{ for } v \in \sigma\}$$

In order to use the machinery of section 4.1 we will give one more description of $(X, A)^K$. In section 2.3.1 we defined $\text{SUB}(K)$ as the poset of finite subcomplexes of K under inclusion. Any ordering of the vertex set V gives a cofinal sequence $K_1 \rightarrow K_2 \rightarrow \dots$ in $\text{SUB}(K)$, where K_n is the restriction of K to vertex set $(v_1 \dots, v_n)$.

Proposition 4.2.10. *For any choice of basepoint $x_0 \in A$, the polyhedral product $(X, A)^K$ is homeomorphic to the colimit of the $\text{SUB}(K)$ -diagram that assigns to each finite subcomplex $K' \subset K$ the polyhedral product $(X, A)^{K'}$, and to each inclusion $K' \hookrightarrow K''$ the inclusion $(X, A)^{K'} \hookrightarrow (X, A)^{K''}$, where missing coordinates are set equal to x_0 .*

Proof. The result is immediate when the $(X, A)^{K'}$ are considered as subspaces of $X^{(V)}$. \square

4.2.1 CW structure

We note that if (X, A) is a CW-pair with X and A locally compact, then X^W has a product CW structure for any finite $W \subset V$ and $(X, A)^W$ is a CW subcomplex of X^W . Then all the morphisms in the diagrams in definition 4.2.6 and proposition 4.2.10 are cofibrations, so we note that the colimit $(X, A)^V$ of each diagram is also the *homotopy colimit* [26]. Moreover, for finite or countably infinite V we prove the following:

Proposition 4.2.11. *Let (X, A) be a CW-pair with X and A locally compact and $x_0 \in A$ a 0-cell, and let V be a countable set. Then $(X, x_0)^{(V)}$ is a CW-complex and so is $(X, A, x_0)^K$ for any finite dimensional simplicial complex K on vertex set V . Moreover, $(X, A, x_0)^{K'}$ is a CW-subcomplex of $(X, A, x_0)^K$ for any subcomplex $K' \subset K$.*

Proof. For the cells of $(X, x_0)^{(V)}$ we take a 0-cell for the basepoint $x_0 \times \dots \times x_0$, and for every V -tuple $(e_v \subset X \mid v \in V)$ of cells with only finitely many not equal to the 0-cell x_0 we have a cell $\prod_{v \in V} e_v \subset (X, x_0)^{(V)}$. Each cell lies in some finite product X^W for a $W \subset V$ (where X^W is identified as a subspace of $(X, x_0)^{(V)}$ by setting missing coordinates equal to x_0). The attaching map for the cell $\prod_{v \in V} e_v$ in $(X, x_0)^{(V)}$ is equal to its attaching map in any such X^W , under the standard product CW structure. The attaching map thus defined is independent of the choice of W

Let Y denote the CW-complex with cells and attaching maps as described. We will show that Y is homeomorphic to $(X, x_0)^{(V)}$.

The space Y is the colimit of the diagram of spaces whose objects are the finite subcomplexes of Y and whose morphisms are their inclusions. On the other hand, the space $(X, x_0)^{(V)}$ is the colimit of the diagram whose objects are the finite products X^W and whose morphisms are the inclusions $X^W \subset X^{W'}$, while each X^W is the colimit of the diagram of its finite subcomplexes under the standard product CW structure. Each finite subcomplex of Y is contained in some X^W , so $(X, x_0)^{(V)}$ is the colimit of the diagram of finite subcomplexes of Y , and hence is homeomorphic to Y .

It now follows from Proposition 4.2.7 that $(X, A, x_0)^K$ is a CW-subcomplex of $(X, x_0)^{(V)}$ for any finite dimensional simplicial complex K on vertex set V , and that $(X, A, x_0)^{K'}$ is a CW-subcomplex of $(X, A, x_0)^K$ for any subcomplex $K' \subset K$. \square

4.3 Moment angle complexes

We now extend the definitions of the open moment angle complex $\mathcal{U}(K)$ and the moment angle complex $\mathcal{Z}(K)$ to the case where K is a finite-dimensional simplicial complex on countable vertex set, and observe their properties.

As in the finite case, we define the open moment angle complex $\mathcal{U}(K)$ to be the polyhedral product $(\mathbf{C}, \mathbf{C}^*)^K$, and identify this as a subspace of $(\mathbf{C}, 1)^{(V)}$. There is a canonical action of $T^{(V)}$ with orbit space $(\mathbf{R}_{\geq}, \mathbf{R}_{>})^K$. By Proposition 4.2.10, $\mathcal{U}(K)$ is the colimit of a direct system of finite dimensional smooth complex manifolds and therefore by Theorem 4.1.3 $\mathcal{U}(K)$ has a unique structure as a smooth complex manifold making the inclusion $\mathcal{U}(K') \hookrightarrow \mathcal{U}(K)$ smooth for each finite $K' \subset K$.

Proposition 4.3.1. *The tangent bundle $\tau(\mathcal{U}(K))$ is diffeomorphic to $\mathcal{U}(K) \times \mathbf{C}^{(V)}$.*

Proof. The tangent bundle $\tau(\mathcal{U}(K))$ is the colimit of the SUB(K) diagram of vector bundles $\tau(\mathcal{U}(K'))$ with derivatives $\tau(\mathcal{U}(K')) \hookrightarrow \tau(\mathcal{U}(K''))$ as morphisms. Each $\tau(\mathcal{U}(K'))$ is diffeomorphic to $(\mathbf{C} \times \mathbf{C}, \mathbf{C}^* \times \mathbf{C})^{K'} = (\mathbf{C}, \mathbf{C}^*)^{K'} \times \mathbf{C}^{K'}$ and the derivatives are the inclusions $(\mathbf{C} \times \mathbf{C}, \mathbf{C}^* \times \mathbf{C})^{K'} \hookrightarrow (\mathbf{C} \times \mathbf{C}, \mathbf{C}^* \times \mathbf{C})^{K''}$, so $\tau(\mathcal{U}(K))$ is homeomorphic to $(\mathbf{C} \times \mathbf{C}, \mathbf{C}^* \times \mathbf{C})^K = \mathcal{U}(K) \times \mathbf{C}^{(V)}$.

By Theorem 4.1.3 there is a unique smooth complex manifold structure on $\tau(\mathcal{U}(K))$ making each inclusion $(\mathbf{C} \times \mathbf{C}, \mathbf{C}^* \times \mathbf{C})^{K'} \hookrightarrow \tau(\mathcal{U}(K))$ smooth, and this can only be the product structure on $\mathcal{U}(K) \times \mathbf{C}^{(V)}$. \square

Similarly, the closed moment angle complex $\mathcal{Z}(K)$ is defined as $(D^2, S^1)^K$, and we have the inclusion $\mathcal{Z}(K) \hookrightarrow \mathcal{U}(K)$ which is a homotopy equivalence. The restriction of the complex vector bundle $\mathcal{U}(K) \times \mathbf{C}^{(V)}$ to $\mathcal{Z}(K)$ is $(D^2 \times \mathbf{C}, S^1 \times \mathbf{C})^K = \mathcal{Z}(K) \times \mathbf{C}^{(V)}$. The action of $T^{(V)}$ on $\mathcal{Z}(K)$ has orbit space $(I, 1)^K$.

The arguments of Section 3.2 remain valid for countably infinite K , and hence our results on maps between moment angle complexes carry over from the finite case. Specifically, given a non-degenerate simplicial map $f: K_1 \rightarrow K_2$ we have the homomorphism $\Theta_f: T^{(V)} \rightarrow T^{(W)}$ and any Θ_f -equivariant map $\mathcal{U}(K_1) \rightarrow \mathcal{U}(K_2)$ is equivariantly homotopic to the multiplication map

$$\mathcal{U}(f)_w(z) = \prod_{f(v)=w} z_v$$

We may describe the derivative $D\mathcal{U}(f): \tau(\mathcal{U}(K_1)) \rightarrow \tau(\mathcal{U}(K_2))$ as in Lemma 3.2.6, observing that $f^{-1}(w)$ may be infinite, in which case $\mathbf{C}^{|f^{-1}(w)|-1}$ should be interpreted as an equivariantly framed \mathbf{C}^∞ summand. We therefore deduce that $\mathcal{U}(f)$ is covered by equivariant bundle maps

$$\widetilde{(\mathcal{U}(f))}_w: \bigoplus_{f(v)=w} \mathbf{C}_v \rightarrow \mathbf{C}_w \oplus \mathbf{C}^{|f^{-1}(w)|-1}$$

or in the case that $f^{-1}(w)$ is empty

$$\widetilde{(\mathcal{U}(f))}_w: \mathbf{C} \rightarrow \mathbf{C}_w$$

where the additional summands $\mathbf{C}^{|f^{-1}(w)|-1}$ and \mathbf{C} are equivariantly framed. We will mostly be interested in the case where K_1 is a finite complex and K_2 is infinite, in which case $f^{-1}(w)$ will always be finite.

4.4 Quasitoric spaces

We continue by extending the definitions of the open quasitoric manifold functor $\mathcal{Y}(-)$ and the quasitoric space functor $\mathcal{X}(-)$ to the countably infinite case.

Given a dicharacteristic pair (K, ℓ) of dimension n where K is a finite-dimensional simplicial complex on countable vertex set, we may form the open quasitoric space $\mathcal{Y}(K, \ell) = \mathcal{U}(K)/\kappa$, as in Definition 3.3.1 in the finite case. That is, the dicharacteristic ℓ determines a homomorphism $\Theta_\ell: T^{(V)} \rightarrow T^n$ whose restriction to the v -th coordinate torus of $T^{(V)}$ is the homomorphism $S^1 \rightarrow T^n$ determined by $\ell(v)$. The kernel κ of Θ_ℓ acts freely and smoothly on $\mathcal{U}(K)$ and we define:

Definition 4.4.1. The open quasitoric manifold $\mathcal{Y}(K, \ell)$ is the quotient $\mathcal{U}(K)/\kappa$.

Proposition 4.4.2. $\mathcal{Y}(K, \ell)$ admits a canonical smooth T^n -manifold structure, as the colimit of the SUB(K) diagram of smooth T^n -manifolds $\mathcal{Y}(K', \ell|_{K'})$ for finite subcomplexes $K' \subset K$.

Proof. For finite subcomplexes $K_1 \subset K_2 \subset K$ the T^n -equivariant inclusion $\iota: \mathcal{Y}(K_1, \ell|_{K_1}) \hookrightarrow \mathcal{Y}(K_2, \ell|_{K_2})$ is an embedding of smooth T^n -manifolds. The space $\mathcal{Y}(K, \ell)$ is the colimit of the corresponding SUB(K) diagram, and SUB(K) is filtered so $\mathcal{Y}(K, \ell)$ is the direct limit of a cofinal sequence of smooth T^n -manifolds. By theorem 4.1.3, $\mathcal{Y}(K, \ell)$ admits a unique compatible smooth manifold structure. The T^n action on $\mathcal{Y}(K, \ell)$ is smooth, because its restriction to each $\mathcal{Y}(K', \ell|_{K'})$ is smooth. \square

Corollary 4.4.3. The tangent bundle $\tau(\mathcal{Y}(K, \ell))$ is the colimit of the SUB(K) diagram of vector bundles $\tau(\mathcal{Y}(K', \ell|_{K'}))$ where K' runs over all finite subcomplexes of K .

To define a stably complex structure on $\tau(\mathcal{Y}(K, \ell))$ we cannot make direct use of Szczarba's result [25] as in the finite case, (in particular, the kernel κ of Θ_ℓ is not compact). We may instead proceed as follows:

Proposition 4.4.4. There is an isomorphism of real vector bundles

$$\tau(\mathcal{Y}(K, \ell)) \oplus \tau_e(\kappa) \cong \mathcal{U}(K) \times_\kappa \mathbf{C}^{(V)}$$

where $\tau_e(\kappa)$ is a trivial real bundle and the right hand side is a complex vector bundle ζ over $\mathcal{Y}(K, \ell)$, a sum of line bundles $\bigoplus_{v \in V} \zeta_v$.

Note that if K is infinite, both $\tau(\mathcal{Y}(K, \ell))$ and $\tau_e(\kappa)$ are infinite dimensional.

Proof. As K' runs over finite subcomplexes the isomorphisms of real bundles $\tau(\mathcal{Y}(K', \ell|_{K'})) \oplus \tau_e(\kappa|_{K'}) \cong \mathcal{U}(K') \times_{\kappa|_{K'}} \mathbf{C}^{(V)}$ yield a natural isomorphism of

SUB(K) diagrams of real vector bundles, so we have an isomorphism of their colimits. \square

The quasitoric space $X(K, \ell)$ is defined as the quotient $\mathcal{Z}(K)/\kappa$ and the inclusion $X(K, \ell) \hookrightarrow \mathcal{Y}(K, \ell)$ is a homotopy equivalence, as in the finite case.

The following result carries over from the finite case, as it depends only on linear interpolation between two maps into $(\mathbf{R}_{\geq}, \mathbf{R}_{>})^{K_2} \subset \mathbf{R}^{(W)}$ and the facts that $(\mathbf{R}_{\geq}, \mathbf{R}_{>})^{K_1}$ is contractible and T^n is path connected:

Proposition 4.4.5. *Any two non-degenerate equivariant maps $\mathcal{X}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$, $\mathcal{X}(K_1, \ell_1) \rightarrow \mathcal{X}(K_2, \ell_2)$ or $\mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{X}(K_2, \ell_2)$ corresponding to the same simplicial map $K_1 \rightarrow K_2$, where K_1 and K_2 may be countably infinite, are equivariantly homotopic.*

Then diagram chasing as in Section 3.5 yields:

Proposition 4.4.6. *Let $f: (K_1, \ell_1) \rightarrow (K_2, \ell_2)$ be a non-degenerate simplicial map that preserves the dicharacteristic, where K_1 and K_2 may be countably infinite. Let $\zeta = \bigoplus_v \zeta_v$ and $\xi = \bigoplus_w \xi_w$ denote the complex structures on the stable tangent bundles of $\mathcal{Y}(K_1, \ell_1)$ and $\mathcal{Y}(K_2, \ell_2)$ respectively, and $\zeta| = \bigoplus_v \zeta_v|$ and $\xi| = \bigoplus_w \xi_w|$ the corresponding bundles over $\mathcal{X}(K_1, \ell_1)$ or $M^{2n}(P_1^n, \ell_1)$ and $\mathcal{X}(K_2, \ell_2)$ or $M^{2n}(P_2^n, \ell_2)$.*

1. *If $g: \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;*
2. *If $g: \mathcal{X}(K_1, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v|$;*
3. *If $g: \mathcal{X}(K_1, \ell_1) \rightarrow \mathcal{X}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w|) \simeq_s \bigoplus_{f(v)=w} \zeta_v|$;*
4. *If $g: \mathcal{Y}(K_1, \ell_1) \rightarrow \mathcal{X}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w|) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;*
5. *If $g: M^{2n}(P_1^n, \ell_1) \rightarrow \mathcal{Y}(K_2, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w) \simeq_s \bigoplus_{f(v)=w} \zeta_v|$;*
6. *If $g: \mathcal{Y}(K_1, \ell_1) \rightarrow M^{2n}(P_2^n, \ell_2)$ is a non-degenerate equivariant map corresponding to f , we have $g^*(\xi_w|) \simeq_s \bigoplus_{f(v)=w} \zeta_v$;*

4.5 Final quasitoric spaces

Our motivation for developing the machinery of Sections 4.3 and 4.4 is the wish to describe quasitoric spaces corresponding to the final objects in categories of characteristic pairs and dicharacteristic pairs of dimension n , for $n > 1$, and to describe the maps of quasitoric spaces corresponding to final maps in these categories. We can now do so.

Recall that an dicharacteristic ℓ of dimension n for a simplicial complex K must satisfy Condition 2.4.4. That is, if $\sigma = \{v_1, \dots, v_r\}$ is a simplex of K , the vectors $\lambda(v_1), \dots, \lambda(v_r)$ such that $\ell(v_i) = \lambda(v_i)$ form part of a \mathbf{Z} -basis for \mathbf{Z}^n . The underlying undirected characteristic function $\bar{\ell}$ satisfies the condition that the lines $\bar{\lambda}(v_1), \dots, \bar{\lambda}(v_r)$, where $\bar{\lambda}(v_i) = \langle \lambda(v_i) \rangle$, span a summand of \mathbf{Z}^n .

In [10] Davis and Januszkiewicz introduce the simplicial complex \mathcal{J}_n whose vertices $j \in \mathcal{J}_n$ are lines in \mathbf{Z}^n and whose maximal simplices are all n -tuples of lines j_1, \dots, j_n whose span is the whole of \mathbf{Z}^n . Then an undirected characteristic function $\bar{\ell}$ for a simplicial complex K is equivalent to a non-degenerate simplicial map $\bar{\ell}: K \rightarrow \mathcal{J}_n$. There is a canonical characteristic function $\bar{\mu}$ on \mathcal{J}_n given by the identity map $\mathcal{J}_n \rightarrow \mathcal{J}_n$. In [10] the authors then consider the undirected quasitoric space $\bar{\mathcal{X}}(\mathcal{J}_n, \bar{\mu})$, and make the observation that any quasitoric space can be obtained as the pullback of a map of orbit spaces $\mathcal{I}(K) \rightarrow \mathcal{I}(\mathcal{J}_n)$ corresponding to the simplicial map $\bar{\ell}: K \rightarrow \mathcal{J}_n$. We will also consider the open quasitoric space $\bar{\mathcal{Y}}(\mathcal{J}_n, \bar{\mu})$, which we denote $\bar{\mathcal{Q}\mathcal{T}}_n$. We denote the orbit space $\mathcal{R}(\mathcal{J}_n)$ by $\bar{\mathcal{Q}}_n$.

In order to work with directed characteristic functions we introduce a related complex defined as follows:

Definition 4.5.1. We will denote by \mathcal{H}_n the complex whose vertices $h \in \mathcal{H}_n$ are primitive vectors in \mathbf{Z}^n and whose maximal simplices are bases for \mathbf{Z}^n .

Then a dicharacteristic ℓ for K is equivalent to a non-degenerate simplicial map $\ell: K \rightarrow \mathcal{H}_n$. We consider \mathcal{H}_n itself to have a canonical dicharacteristic ν , given by the identity map $\mathcal{H}_n \rightarrow \mathcal{H}_n$. We denote the open quasitoric space $\mathcal{Y}(\mathcal{H}_n, \nu)$ by $\mathcal{Q}\mathcal{T}_n$ and its orbit space $\mathcal{R}(\mathcal{H}_n)$ by \mathcal{Q}_n . We shall denote the underlying undirected space $\bar{\mathcal{Y}}(\mathcal{H}_n, \bar{\nu})$ by $\overline{(\mathcal{Q}\mathcal{T}_n)}$, to distinguish it from $\bar{\mathcal{Q}\mathcal{T}}_n = \bar{\mathcal{Y}}(\mathcal{J}_n, \bar{\mu})$.

There is a forgetful map $\mathcal{H}_n \rightarrow \mathcal{J}_n$ which induces an equivariant map of undirected quasitoric spaces $\overline{(\mathcal{Q}\mathcal{T}_n)} \rightarrow \bar{\mathcal{Q}\mathcal{T}}_n$, unique up to equivariant homotopy, covering a map of base spaces $\mathcal{Q}_n \rightarrow \bar{\mathcal{Q}}_n$.

The following statements are immediate from the definitions:

Theorem 4.5.2. *Let K be a simplicial complex with undirected characteristic function $\bar{\ell}$. Then there exists a unique, non-degenerate simplicial map $\bar{f}(\bar{\ell}): (K, \bar{\ell}) \rightarrow (\mathcal{J}_n, \bar{\mu})$ that preserves the characteristic function and a non-degenerate T^n -equivariant map $\bar{\mathcal{Y}}(\bar{\ell}): \bar{\mathcal{Y}}(K, \bar{\ell}) \rightarrow \overline{\mathcal{QT}}_n$, unique up to equivariant homotopy.*

A choice of dicharacteristic ℓ compatible with $\bar{\ell}$ yields a unique lift of $\bar{f}(\bar{\ell})$ to $f(\ell): (K, \ell) \rightarrow (\mathcal{H}_n, \nu)$ and a unique directed map of quasitoric spaces $\mathcal{Y}(\ell): \mathcal{Y}(K, \ell) \rightarrow \mathcal{QT}_n$, up to equivariant homotopy, which is covered by a bundle map of equivariant stably complex structures. The underlying map of undirected spaces gives an equivariant homotopy class of lifts

$$\begin{array}{ccc} & & \overline{(\mathcal{QT}_n)} \\ & \nearrow \bar{\mathcal{Y}}(\bar{\ell}) & \downarrow \\ \bar{\mathcal{Y}}(K, \bar{\ell}) & \xrightarrow{\bar{\mathcal{Y}}(\bar{\ell})} & \overline{\mathcal{QT}}_n \end{array}$$

Equivalent statements apply to the quasitoric spaces $\mathcal{X}(K, \ell)$ and $\bar{\mathcal{X}}(K, \lambda)$.

4.5.1 Quasitoric manifolds

We now explain the relation of the final spaces $\overline{\mathcal{QT}}_n$ and \mathcal{QT}_n to quasitoric manifolds.

In the case of a simple polytope P^n with simplicial dual K_P , we may choose a realisation of P^n as an intersection of half spaces:

$$P^n = \{x \in \mathbf{R}^n \mid Ax + b \geq 0\} \quad (4.2)$$

for a $V \times n$ real matrix A and $b \in \mathbf{R}^V$, where $m = |V|$ is the number of facets of P^n . Any point of \mathbf{R}^n is uniquely determined by its signed distance from any n intersecting facets, so the affine map $\iota_P: \mathbf{R}^n \rightarrow \mathbf{R}^V$ given by $(\iota(x)_P)_v = A_v x + b_v$ gives an immersion of \mathbf{R}^n in \mathbf{R}^V , with $P^n = \iota_P^{-1}(\mathcal{R}(K_P))$, and this is covered by the smooth T^n -equivariant embedding $\iota: M^{2n}(P^n, \ell) \hookrightarrow \mathcal{Y}(K, \ell)$ which does not depend on the realisation (4.2) up to equivariant smooth homotopy, as described in section 3.4.1.

Then composition with the final map $\mathcal{Y}(\ell): \mathcal{Y}(K, \ell) \rightarrow \mathcal{QT}_n$ gives an equivariant smooth map $\mathcal{Y}(\ell) \circ \iota: M^{2n}(P^n, \ell) \rightarrow \mathcal{QT}_n$. The same is true using the undirected spaces $\bar{M}^{2n}(P^n, \bar{\ell})$ and $\bar{\mathcal{Y}}(K, \bar{\ell})$ with the map $\bar{\mathcal{Y}}(\bar{\ell}): \bar{\mathcal{Y}}(K, \bar{\ell}) \rightarrow \overline{\mathcal{QT}}_n$ and so we have:

Theorem 4.5.3. *Let P^n be a simple polytope with undirected characteristic function $\bar{\ell}$ and simplicial dual K . Then there exists a unique, non-degenerate simplicial map $\bar{f}(\bar{\ell}): (K, \bar{\ell}) \rightarrow (\mathcal{J}_n, \bar{\mu})$ that preserves the characteristic function and a non-degenerate, T^n -equivariant smooth map $\bar{M}(\bar{\ell}): \bar{M}^{2n}(P^n, \bar{\ell}) \rightarrow \overline{\mathcal{QT}}_n$, unique up to equivariant smooth homotopy.*

A choice of dicharacteristic ℓ compatible with $\bar{\ell}$ yields a unique lift of $\bar{f}(\bar{\ell})$ to $f(\ell): (K, \ell) \rightarrow (\mathcal{H}_n, \nu)$ and a unique directed map of quasitoric spaces

$$M(\ell): M^{2n}(P^n, \ell) \rightarrow \mathcal{QT}_n,$$

which is covered by a bundle map of equivariant stably complex structures. The underlying map of undirected spaces gives an equivariant homotopy class of lifts

$$\begin{array}{ccc} & & \overline{(\mathcal{QT}_n)} \\ & \nearrow^{\bar{M}(\bar{\ell})} & \downarrow \\ \bar{M}^{2n}(P^n, \bar{\ell}) & \xrightarrow{\bar{M}(\bar{\ell})} & \overline{\mathcal{QT}}_n \end{array}$$

Chapter 5

Quasitoric manifolds as pullbacks

Davis and Januszkiewicz commented in [10] that any quasitoric space $\overline{\mathcal{X}}(K, \bar{\ell})$ can be constructed as a pullback of the diagram

$$\begin{array}{ccc} \overline{\mathcal{X}}(K, \bar{\ell}) & \cdots\cdots\cdots\rightarrow & \overline{\mathcal{X}}(\mathcal{J}_n, \bar{\mu}) \\ \downarrow & & \downarrow \\ \mathcal{I}(K) & \longrightarrow & \mathcal{I}(\mathcal{J}_n) \end{array}$$

where the horizontal map $\mathcal{I}(K) \rightarrow \mathcal{I}(\mathcal{J})$ is determined by the characteristic function $\bar{\ell}$, which yields the simplicial map $\bar{\ell}: K \rightarrow \mathcal{J}_n$; but $\overline{\mathcal{X}}(\mathcal{J}, \bar{\mu})$ is not a manifold so their construction does not yield a smooth structure on $M^{2n}(P^n, \ell)$ in the case that K is dual to the simple polytope P^n .

In this chapter we will show how a quasitoric manifold $M^{2n}(P^n, \ell)$ can be constructed as a smooth T^n -manifold as the pullback of a diagram

$$\begin{array}{ccc} M^{2n} & \cdots\cdots\cdots\rightarrow & \mathcal{QT}_n \\ \downarrow & & \downarrow \\ P^n & \longrightarrow & \mathcal{Q}_n \end{array}$$

where the map $P^n \rightarrow \mathcal{Q}_n$ is determined by the dicharacteristic ℓ . The equivariant smooth structure obtained on $M^{2n}(P^n, \ell)$ is isomorphic to that constructed by [8] and described in section 2.2.3. What's more, we will show that the stably complex structure on $M^{2n}(P^n, \ell)$ described by [8] can be recovered up to equivariant stable isomorphism by pulling back the stably complex structure on \mathcal{QT}_n described in Proposition 4.4.4.

In fact, rather than working with infinite dimensional smooth manifolds and vector bundles, we will see that we can always restrict attention to a finite dimensional submanifold of \mathcal{QT}_n .

5.1 Pullback construction

Suppose then that (P^n, ℓ) is a dicharacteristic pair, where P^n is a simple polytope with simplicial dual K . As in Section 4.5.1, we may choose a realisation of P^n in \mathbf{R}^n as an intersection of half spaces:

$$P^n = \{x \in \mathbf{R}^n \mid Ax + b \geq 0\}$$

for some $V \times n$ real matrix A and $b \in \mathbf{R}^V$, giving an embedding of \mathbf{R}^n in \mathbf{R}^V via $(\iota_P(x))_v = A_v x + b_v$. We know from section 3.4.1 that this is covered by a T^n -equivariant embedding $\iota: M^{2n}(P^n, \ell) \hookrightarrow \mathcal{Y}(K, \ell)$, but we will proceed to show how $M^{2n}(P^n, \ell)$ can be constructed from \mathcal{QT}_n as a pullback.

The dicharacteristic ℓ yields the simplicial map $\ell: K \rightarrow \mathcal{H}_n$ and so we have a composite map $\mathcal{R}(\ell): \mathcal{R}(K) \rightarrow \mathcal{Q}_n \subset (\mathbf{R}, 1)^{(H_n)}$, where \mathcal{Q}_n is the orbit space of \mathcal{QT}_n as described in Section 4.5.

The map $\mathcal{R}(\ell)$ is given by

$$(\mathcal{R}(\ell))_w(y) = \prod_{\ell(v)=w} y_v$$

which extends (by applying the same formula) to a map $\mathbf{R}^V \rightarrow (\mathbf{R}, 1)^{(H_n)}$. Thus we have a smooth composite map $F(\ell): \mathbf{R}^n \rightarrow (\mathbf{R}, 1)^{(H_n)}$.

Example 5.1.1. The case $n = 1$ is the only case where the simplicial complex \mathcal{H}_n is finite: \mathcal{H}_1 consists of two distinct vertices corresponding to the vectors (1) and (-1) respectively. So \mathcal{Q}_1 is $\mathbf{R}_{\geq}^2 \setminus \{0\}$.

Consider the 1-simplex Δ^1 , presented by:

$$\Delta^1 = \left\{ x \in \mathbf{R}^1 \mid \begin{pmatrix} 1 \\ -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

That is, $x \geq 0$ and $x \leq 1$. The facets of Δ^1 are v_1 and v_2 , corresponding to the

points $x = 0$ and $x = 1$ respectively, and the map

$$\iota_{\Delta}: \mathbf{R}^1 \rightarrow \mathbf{R}^V$$

is given by

$$(\iota_{\Delta})_1(x) = x$$

$$(\iota_{\Delta})_2(x) = 1 - x,$$

where we identify \mathbf{R}^V with \mathbf{R}^2 in the obvious way.

Now suppose the dicharacteristic ℓ is given as $\ell(v_1) = (1)$, $\ell(v_2) = (1)$. The space $(\mathbf{R}, 1)^{(H_1)}$ is \mathbf{R}^2 , with coordinates corresponding to the vectors (1) and (-1) respectively, and the map

$$\mathcal{R}(\ell): \mathcal{R}(K) \rightarrow \mathcal{Q}_1 \subset (\mathbf{R}, 1)^{(H_1)}$$

is given by

$$(\mathcal{R}(\ell))_{(1)}(y) = y_1 y_2$$

$$(\mathcal{R}(\ell))_{(-1)}(y) = 1.$$

This extends to a map $\mathbf{R}^V \rightarrow (\mathbf{R}, 1)^{(H_1)}$ and the composite

$$F(\ell): \mathbf{R}^1 \rightarrow (\mathbf{R}, 1)^{(H_1)}$$

is given by

$$F(\ell)_{(1)}(x) = x(1 - x)$$

$$F(\ell)_{(-1)}(x) = 1.$$

Example 5.1.2. In Example 5.1.1, suppose instead that the dicharacteristic ℓ is given by $\ell(v_1) = (1)$, $\ell(v_2) = (-1)$. Then the map $\mathcal{R}(\ell)$ is given by

$$(\mathcal{R}(\ell))_{(1)}(y) = y_1$$

$$(\mathcal{R}(\ell))_{(-1)}(y) = y_2$$

and the composite map

$$F(\ell): \mathbf{R}^1 \rightarrow (\mathbf{R}, 1)^{(H_1)}$$

is given by

$$\begin{aligned} F(\ell)_{(1)}(x) &= x \\ F(\ell)_{(-1)}(x) &= (1 - x). \end{aligned}$$

Example 5.1.3. Consider the square I^2 , presented by:

$$\Delta^2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2 \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \geq 0 \right\}$$

That is, $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 1$ and $x_2 \leq 1$. Let v_1, \dots, v_4 be the facets of I^2 corresponding to these respective inequalities. The map

$$\iota_I: \mathbf{R}^2 \rightarrow \mathbf{R}^V$$

is given by

$$\begin{aligned} (\iota_I)_1(x) &= x_1 \\ (\iota_I)_2(x) &= x_2 \\ (\iota_I)_3(x) &= 1 - x_1 \\ (\iota_I)_4(x) &= 1 - x_2, \end{aligned}$$

identifying \mathbf{R}^V with \mathbf{R}^4 .

Now suppose the dicharacteristic ℓ is given as $\ell(v_1) = (1, 0)$, $\ell(v_2) = (0, 1)$, $\ell(v_3) = (1, 0)$, $\ell(v_4) = (1, -1)$. Let h_1, h_2 and h_3 be the coordinates of $(\mathbf{R}, 1)^{(H_2)}$ corresponding to the vectors $(1, 0)$, $(0, 1)$ and $(1, -1)$ respectively. The map

$$\mathcal{R}(\ell): \mathcal{R}(K) \rightarrow \mathcal{Q}_2 \subset (\mathbf{R}, 1)^{(H_2)}$$

is given by

$$\begin{aligned} (\mathcal{R}(\ell))_1(y) &= y_1 y_3 \\ (\mathcal{R}(\ell))_2(y) &= y_2 \\ (\mathcal{R}(\ell))_3(y) &= y_4 \\ (\mathcal{R}(\ell))_h(y) &= 1 \text{ for } h \notin \{h_1, h_2, h_3\} \end{aligned}$$

and the composite

$$F(\ell): \mathbf{R}^2 \rightarrow (\mathbf{R}, 1)^{(H_2)}$$

is given by

$$F(\ell)_1(x) = x_1(1 - x_1)$$

$$F(\ell)_2(x) = x_2$$

$$F(\ell)_3(x) = 1 - x_2$$

$$F(\ell)_h(x) = 1 \text{ for } h \notin \{h_1, h_2, h_3\}.$$

As these examples illustrate, each coordinate of $F(\ell): \mathbf{R}^n \rightarrow (\mathbf{R}, 1)^{(H_n)}$ is given by a product of affine functions $\mathbf{R}^n \rightarrow \mathbf{R}$. These affine terms measure, up to a scalar multiple, the perpendicular distance of a point $x \in \mathbf{R}^n$ from each facet of P^n . The dicharacteristic ℓ cannot take the same value on two intersecting facets, so for any point $x \in P^n$ and any vector $h \in H_n$ at most one of the affine terms in $F(\ell)_h(x)$ can be equal to zero. The geometric properties of $F(\ell)$ are described by the following proposition:

Proposition 5.1.4. *The map $F(\ell): \mathbf{R}^n \rightarrow (\mathbf{R}, 1)^{(H_n)}$ has the following properties:*

1. $F(\ell)$ takes the interior of P^n to the interior of $\mathcal{Q}_n \subset (\mathbf{R}, 1)^{(H_n)}$, and takes the interior of each face F_σ of P^n to the interior of face $F_{\ell(\sigma)}$ of \mathcal{Q}_n ;
2. $F(\ell)$ is transverse to each face of \mathcal{Q}_n . That is, at a point x in face F_σ of P^n , the derivative $D_x F(\ell): \tau(\mathbf{R}^n) \rightarrow \tau((\mathbf{R}, 1)^{(H_n)})$ gives a surjective map from $\tau(\mathbf{R}^n)$ to the normal space $\nu(F_{\ell(\sigma)}) = \tau((\mathbf{R}, 1)^{(H_n)}) / \tau(F_{\ell(\sigma)})$;
3. for some open neighbourhood $\epsilon(P^n)$ of P^n in \mathbf{R}^n , $F(\ell)^{-1}(\mathcal{Q}_n) \cap \epsilon(P^n) = P^n$.

Moreover, any other (smooth) map from a neighbourhood $\epsilon'(P^n)$ of P^n to $(\mathbf{R}, 1)^{(H_n)}$ satisfying these conditions is (smoothly) homotopic to $F(\ell)$ on the neighbourhood $\epsilon(P^n) \cap \epsilon'(P^n)$ via a homotopy through such maps.

Proof. Property 1 follows immediately from the definitions, because the interior of F_σ is precisely the set of points of P^n satisfying $(\iota_P(x))_v = 0$ for $v \in \sigma$ and $(\iota_P(x))_v \neq 0$ for $v \notin \sigma$.

For 2, we know from property 1 that $D_x F(\ell)$ takes $\tau_x(F_\sigma)$ into $\tau(F_{\ell(\sigma)})$ when x is a point in the interior of F_σ . We shall show that $D_x F(\ell)^{-1}(\tau(F_{\ell(\sigma)}))$ is precisely

$\tau_x(F_\sigma)$. Then since the codimension of F_σ in \mathbf{R}^n is equal to the codimension of $F_{\ell(\sigma)}$ in $(\mathbf{R}, 1)^{(H_n)}$ it will follow that the map $\tau_x(\mathbf{R}^n) \rightarrow \nu(F_{\ell(\sigma)})$ defined by $D_x F(\ell)$ is surjective and $F(\ell)$ is transverse to $F_{\ell(\sigma)}$.

For this, we note that $F(\ell)$ is given coordinatewise by

$$F(\ell)_h(x) = \prod_{\ell(v)=h} y_v(x)$$

where

$$y_v(x) = A_v x + b_v$$

is the scaled perpendicular distance of x from facet F_v . A vector u in $\tau_x(F_\sigma)$ satisfying $D_x F(\ell)(u) \in \tau(F_{\ell(\sigma)})$ must satisfy $D_x F(\ell)_h(u) = 0$ for all $h \in \ell(\sigma)$. Now ℓ is a non-degenerate simplicial map so for each $v \in \sigma$ there is precisely one $h \in \ell(\sigma)$ such that $\ell(v) = h$ and we have $y_v(x) = 0$ because $x \in F_v$. We can write the derivative $D_x F(\ell)_h$ as

$$D_x F(\ell)_h(u) = \sum_{\ell(v')=h} (D_x y_{v'}(u) \prod_{\substack{\ell(v'')=h \\ v'' \neq v'}} y_{v''}(x))$$

which reduces to

$$D_x F(\ell)_h(u) = D_x y_v(u) \prod_{\substack{\ell(v'')=h \\ v'' \neq v}} y_{v''}(x)$$

because $y_v(x) = 0$. The product is non-zero, being a product of scaled perpendicular distances from facets of P^n that do not contain x , so for the derivative to be zero we must have $D_x y_v(u) = 0$, or in other words the vector u is parallel to the facet F_v .

For property 3, we choose a neighbourhood $\epsilon(P^n)$ small enough that, for two facets F_{v_1} and F_{v_2} with $\ell(v_1) = \ell(v_2)$, the distances y_{v_1} and y_{v_2} cannot both be negative. This is possible because facets F_{v_1} and F_{v_2} do not intersect.

For the last part, the required homotopy between $F(\ell): \epsilon(P^n) \rightarrow (\mathbf{R}, 1)^{(H_n)}$ and another map $G: \epsilon'(P^n) \rightarrow (\mathbf{R}, 1)^{(H_n)}$ satisfying the conditions is given by linear interpolation in $(\mathbf{R}, 1)^{(H_n)}$. That is, we can take the homotopy $H: \epsilon(P^n) \cap \epsilon'(P^n) \times [0, 1] \rightarrow (\mathbf{R}, 1)^{(H_n)}$ given by $H(x, t) = (1 - t)F(\ell)(x) + tG(x)$. \square

Now let $F: \epsilon(P^n) \rightarrow (\mathbf{R}, 1)^{(H_n)}$ be any smooth map satisfying the conditions

of proposition 5.1.4, and consider the diagram

$$\begin{array}{ccc}
 F^*(\mathcal{QT}_n) & \xrightarrow{\quad} & \mathcal{QT}_n \\
 \downarrow & \searrow & \downarrow \\
 P^n & \xrightarrow{F|_P} & \mathcal{Q}_n \\
 \downarrow & \searrow & \downarrow \pi \\
 \epsilon(P^n) & \xrightarrow{F} & (\mathbf{R}, 1)^{(H_n)}
 \end{array}$$

where $F^*(\mathcal{QT}_n)$ is the pullback

$$F^*(\mathcal{QT}_n) = \{(x, z) \in \epsilon(P^n) \times \mathcal{QT}_n \mid F(x) = \pi(z)\}$$

which lies in $P^n \times \mathcal{QT}_n \subset \epsilon(P^n) \times \mathcal{QT}_n$ by condition 3 of proposition 5.1.4.

Lemma 5.1.5. *We may choose a smaller neighbourhood $\epsilon'(P^n)$ such that $F(\epsilon'(P^n))$ lies in $(\mathbf{R}, 1)^{(H')} \subset (\mathbf{R}, 1)^{(H_n)}$ for some finite subset $H' \subset H_n$.*

Proof. Choose $\epsilon'(P^n)$ such that $\epsilon'(P^n) \subset X \subset \epsilon(P^n)$ for some compact set X . F is smooth, so by proposition 4.1.4 each point $x \in X$, has an open neighbourhood S_x such that $F(S_x) \subset (\mathbf{R}, 1)^{(H_x)}$ for a finite subset $H_x \subset H_n$. Because X is compact, we may take an open cover $X = \bigcup_{x \in I} S_x$ by finitely many of the sets S_x , so $F(X) \subset (\mathbf{R}, 1)^{(H')}$ where $H' = \bigcup_{x \in I} H_x$ is finite. \square

In the case that $F = F(\ell)$, the subset $H' \subset H_n$ is precisely the image $\ell(V)$ of the dicharacteristic ℓ . For other choices of F the subset H' must contain $\ell(V)$ but could be strictly larger.

Now let \mathcal{H}' denote the restriction of the final simplicial complex \mathcal{H}_n to vertex set H' and let $\nu|_{H'}$ denote the restriction of the canonical characteristic function ν on \mathcal{H}_n . Let $\mathcal{QT}_n|_{H'}$ denote the open quasitoric manifold $\mathcal{Y}(\mathcal{H}', \nu|_{H'}) \subset \mathcal{QT}_n$. Then we have the following corollary of lemma 5.1.5:

Corollary 5.1.6. *The pullback $F^*(\mathcal{QT}_n) \subset P^n \times \mathcal{QT}_n$ lies in the subspace $P^n \times \mathcal{QT}_n|_{H'} \subset P^n \times \mathcal{QT}_n$ and is equal to the pullback*

$$F^*(\mathcal{QT}_n|_{H'}) = \{(x, z) \in \epsilon(P^n) \times \mathcal{QT}_n|_{H'} \mid F(x) = \pi(z)\}$$

of the diagram

$$\begin{array}{ccccc}
 F^*(\mathcal{QT}_n) & \xlongequal{\quad} & F^*(\mathcal{QT}_n|_{H'}) & \longrightarrow & \mathcal{QT}_n|_{H'} \\
 & & \downarrow & \searrow & \downarrow \\
 & & P^n & \xrightarrow{F|_P} & \mathcal{Q}_n|_{H'} \\
 & & & \searrow & \downarrow \pi \\
 & & & & (\mathbf{R}, 1)^{(H')} \\
 & & \swarrow & \nearrow & \\
 & & \epsilon(P^n) & \xrightarrow{F} & (\mathbf{R}, 1)^{(H')}
 \end{array}$$

We now prove that the pullback $F^*(\mathcal{QT}_n)$ is indeed the quasitoric manifold $M^{2n}(P^n, \ell)$:

Theorem 5.1.7. *$F^*(\mathcal{QT}_n)$ is an equivariantly framed T^n -invariant submanifold of $\epsilon(P^n) \times \mathcal{QT}_n$, and is equivariantly diffeomorphic to $M^{2n}(P^n, \ell)$, with the T^n -equivariant smooth structure described in [8] (see section 2.2.3).*

Proof. Since $\mathcal{Q}_n|_{H'}$ is a subset of $(\mathbf{R}, 1)^{(H')}$ we may rewrite the condition $F(x) = \pi(z)$ as $g(x, z) = 0$ where $g: \epsilon(P^n) \times \mathcal{QT}_n|_{H'} \rightarrow \mathbf{R}^{H'}$ is the difference $g(x, z) = F(x) - \pi(z)$. Then $F^*(\mathcal{QT}_n) = F^*(\mathcal{QT}_n|_{H'}) = g^{-1}(0)$ and we will show that zero is a regular value of the smooth map g .

Suppose that $g(x, z) = 0$ for some point $(x, z) \in \epsilon(P^n) \times \mathcal{QT}_n|_{H'}$. We must show that the derivative $D_{(x,z)}g: \tau_x(\epsilon(P^n)) \oplus \tau_z(\mathcal{QT}_n|_{H'}) \rightarrow \mathbf{R}^{H'}$ is surjective. For each $h \in H'$ we will show that e_h , the h -th standard basis vector, is in the image of $D_{(x,z)}g$.

If $\pi(z)_h \neq 0$ then we may choose $\tilde{z} \in \mathcal{U}(\mathcal{H}')$ whose image in $\mathcal{QT}_n|_{H'}$ is z , and we know that $\tilde{z}_h \neq 0$. Now let ζ be the image in $\tau_z(\mathcal{QT}_n|_{H'})$ of $\tilde{\zeta} \in \tau_{\tilde{z}}(\mathcal{U}(\mathcal{H}'))$, a unit tangent vector in the direction of \tilde{z}_h . Then $D_{(x,z)}g(0, \zeta) = D_z\pi(\zeta) = 2|\tilde{z}_h|e_h$ because the projection from $\mathcal{U}(\mathcal{H}')$ to $\mathbf{R}^{H'}$ takes the squared modulus of each coordinate. We know $|\tilde{z}_h| \neq 0$ so e_h is in the image of $D_{(x,z)}g$.

On the other hand, if $\pi(z)_h = 0$ then we must have $F(x)_h = 0$ and since F is transverse to the faces of $\mathcal{Q}_n|_{H'}$ by condition 2 of proposition 5.1.4, there is some vector ξ such that $D(x, z)g(\xi, 0) = D_xF(\xi) = e_h + u$ where $u = a_1e_{h_1} + \dots + a_ke_{h_k}$ for coordinates h_i with $\pi(z)_{h_i} \neq 0$. So u is in the image of $D_{(x,z)}g$ and hence so is e_h .

So zero is a regular value of g and $F^*(\mathcal{QT}_n|_{H'}) = g^{-1}(0)$ is a smooth submanifold of $\epsilon(P^n) \times \mathcal{QT}_n|_{H'}$, framed by pulling back a basis for $\tau_0(\mathbf{R}^{H'}) = \mathbf{R}^{H'}$. Moreover g is T^n -invariant, where T^n acts on the second factor of $\epsilon(P^n) \times \mathcal{QT}_n|_{H'}$, so the submanifold $F^*(\mathcal{QT}_n|_{H'})$ and the framing are T^n -equivariant.

To see that $F^*(\mathcal{QT}_n|_{H'})$ is equivariantly diffeomorphic to $M^{2n}(P^n, \ell)$ we will construct an embedding $M^{2n}(P^n, \ell) \hookrightarrow \epsilon(P^n) \times \mathcal{QT}_n|_{H'}$ whose image is $F^*(\mathcal{QT}_n|_{H'})$. We begin by constructing a map $\Phi: M^{2n}(P^n, \ell) \rightarrow \mathcal{QT}_n|_{H'}$ that covers $F|_P: P^n \rightarrow \mathcal{Q}_n|_{H'}$. This can be done by first defining $\tilde{\Phi}: Z_P \rightarrow \mathcal{U}(\mathcal{H}')$ by letting $\tilde{\Phi}(z)$ be the unique non-negative real point covering $F \circ \pi'(z)$ when z is real and non-negative, where π' is the smooth projection $\pi': Z_P \rightarrow P^n$; all other values are determined by insisting that $\tilde{\Phi}$ is equivariant. Then $\tilde{\Phi}$ is smooth because F is transverse to the coordinate hyperplanes, and we obtain $\Phi: M^{2n}(P^n, \ell) \rightarrow \mathcal{QT}_n|_{H'}$ as the induced map of quotient spaces.

So we have $\Phi: M^{2n}(P^n, \ell) \rightarrow \mathcal{QT}_n|_{H'}$ which covers $F|_P: P^n \rightarrow \mathcal{Q}_n|_{H'}$ (the restriction of F to P^n). Now define $\Psi: M^{2n}(P^n, \ell) \rightarrow \epsilon(P^n) \times \mathcal{QT}_n|_{H'}$ by $\Psi(z) = (\pi(z), \Phi(z))$. Since Ψ is a non-degenerate T^n -equivariant map (because Φ is), and is clearly smooth, it is sufficient to show that the induced map of orbit spaces $\bar{\Psi}: P^n \rightarrow \epsilon(P^n) \times \mathcal{Q}_n|_{H'}$ takes P^n bijectively to the orbit space of $F^*(\mathcal{QT}_n|_{H'})$, which is $\{(x, y) \in \epsilon(P^n) \times \mathcal{Q}_n|_{H'} \mid F(x) = y\}$. But $\bar{\Psi}(x, y) = (x, F(x))$ so this is trivial. □

The framing of $F^*(\mathcal{QT}_n)$ would appear to depend on the choice of finite subset $H' \subset H_n$, since we pull back a basis for $\mathbf{R}^{H'}$ at zero along the map $g: \epsilon(P^n) \times \mathcal{QT}_n|_{H'} \rightarrow \mathbf{R}^{H'}$. In fact, the framing of $F^*(\mathcal{QT}_n)$ in $\epsilon(P^n) \times \mathcal{QT}_n|_{H'}$ by $\mathbf{R}^{H'}$ extends to a framing in $\epsilon(P^n) \times \mathcal{QT}_n$ by \mathbf{R}^{H_n} , obtained by pulling back along a map $g': \epsilon(P^n) \times \mathcal{QT}_n \rightarrow \mathbf{R}^{H_n}$. This framing in $\epsilon(P^n) \times \mathcal{QT}_n$ does not depend on the choice of $H' \subset H_n$, as we now show:

Corollary 5.1.8. *$F^*(\mathcal{QT}_n)$ is an equivariantly framed T^n -invariant submanifold of $\epsilon(P^n) \times \mathcal{QT}_n$, equivariantly diffeomorphic to $M^{2n}(P^n, \ell)$, and the framing does not depend on the choice of subset $H' \subset H_n$ for which $F(\epsilon(P^n)) \subset (\mathbf{R}, 1)^{(H')}$.*

Proof. Consider the smooth projection $\mathcal{QT}_n \rightarrow \mathcal{Q}_n \subset (\mathbf{R}, 1)^{(H_n)}$. Subtracting one from each coordinate and composing with the coordinate projection $\mathbf{R}^{H_n} \rightarrow \mathbf{R}^{H_n \setminus H'}$ we have a smooth map $\mathcal{QT}_n \rightarrow \mathbf{R}^{H_n \setminus H'}$ and $\mathcal{QT}_n|_{H'}$ is the inverse image of the regular value 0. So the embedding $\mathcal{QT}_n|_{H'} \hookrightarrow \mathcal{QT}_n$ is equivariantly framed by $\mathbf{R}^{H_n \setminus H'}$. This yields a framing of $\epsilon(P^n) \times \mathcal{QT}_n|_{H'}$ in $\epsilon(P^n) \times \mathcal{QT}_n$ by $\mathbf{R}^{H_n \setminus H'}$ and the framing of $F^*(\mathcal{QT}_n)$ in $\epsilon(P^n) \times \mathcal{QT}_n|_{H'}$ by $\mathbf{R}^{H'}$ gives a framing of $F^*(\mathcal{QT}_n)$ in $\epsilon(P^n) \times \mathcal{QT}_n$ by $\mathbf{R}^{H'} \oplus \mathbf{R}^{H_n \setminus H'} = \mathbf{R}^{H_n}$.

Regardless of the choice of H' , the resulting framing of $F^*(\mathcal{QT}_n)$ in $\epsilon(P^n) \times \mathcal{QT}_n$ is obtained by pulling back a basis for \mathbf{R}^{H_n} along the map $g'(x, z) = F(x) - \pi(z)$ from $\epsilon(P^n) \times \mathcal{QT}_n$ to \mathbf{R}^{H_n} , where $F^*(\mathcal{QT}_n) = g'^{-1}(0)$ and zero is a regular value of g' . \square

Example 5.1.9 (Pullbacks of the linear model). Suppose ℓ takes values in $H' = \{e_1, \dots, e_n\}$, the standard basis for \mathbf{Z}^n , then the folding map $F(\ell)$ takes \mathbf{R}^n into $(\mathbf{R}, 1)^{(H')}$ which is a copy of \mathbf{R}^n embedded in $(\mathbf{R}, 1)^{(H_n)}$. The restriction of \mathcal{H}_n to H' is the $(n-1)$ -simplex, so $\mathcal{QT}_n|_{H'} = \mathcal{U}(\mathcal{H}') = \mathbf{C}^n$ (because the homomorphism $\Theta_{H'}: T^n \rightarrow T^n$ is the identity), and there is a smooth projection onto $\mathcal{Q}_n|_{H'} = \mathbf{R}_{\geq}^n$. This is referred to in [10] as the *linear model*.

The pullback construction yields an equivariant framing of $M^{2n}(P^n, \ell)$ in $\mathbf{R}^n \times \mathbf{C}^n \cong \mathbf{R}^{3n}$. This framing makes $M^{2n}(P^n, \ell)$ into an equivariant framed boundary. To see this, we observe that the map $g: \epsilon(P^n) \times \mathbf{C}^n \rightarrow \mathbf{R}^n$ for which $M^{2n}(P^n, \ell) = g^{-1}(0)$ is equivariantly homotopic to a map g' for which $g'^{-1}(0)$ is empty: By compactness of P^n each coordinate $g_i(x, z)$ is bounded above on $\epsilon(P^n)$ and so for sufficiently large $a \in \mathbf{R}$ we may set $g'_i(x, z) = g_i(x, z) - a$ coordinate-wise. Then the equivariant homotopy $G_i(x, z, t) = g_i(x, z) - ta$ gives a null-cobordism of $M^{2n}(P^n, \ell)$ via the Pontryagin construction (see e.g. [20]).

5.2 Stably complex structure

The framing of $M^{2n}(P^n, \ell)$ in $\mathbf{R}^n \times \mathcal{QT}_n|_{H'}$ yields an isomorphism of vector bundles

$$\tau(M^{2n}(P^n, \ell)) \oplus \mathbf{R}^{H'} \cong \tau(\mathbf{R}^n \times \mathcal{QT}_n|_{H'})|_{M^{2n}(P^n, \ell)}$$

Meanwhile we have

$$\tau(\mathbf{R}^n \times \mathcal{QT}_n|_{H'}) \cong \tau(\mathbf{R}^n) \oplus \tau(\mathcal{QT}_n|_{H'}) \cong \mathbf{R}^n \oplus \tau(\mathcal{QT}_n|_{H'})$$

and the equivariant stably complex structure on $\mathcal{QT}_n|_{H'}$

$$\tau(\mathcal{QT}_n|_{H'}) \oplus \tau_e(\kappa') = \xi'$$

from section 3.3, so we can place an equivariant stably complex structure on $\mathbf{R}^n \times \mathcal{QT}_n|_{H'}$ by introducing an extra \mathbf{R}^n summand:

$$\tau(\mathbf{R}^n \times \mathcal{QT}_n|_{H'}) \oplus \tau_e(\kappa') \oplus \mathbf{R}^n = \xi' \oplus \mathbf{C}^n$$

This yields an equivariant stably complex structure γ' on $M^{2n}(P^n, \ell)$, explicitly

$$\tau(M^{2n}(P^n, \ell)) \oplus \tau_e(\kappa') \oplus \mathbf{R}^{H'} \oplus \mathbf{R}^n \cong (\xi' \oplus \mathbf{C}^n)|_{M^{2n}} = \gamma' \quad (5.1)$$

The stably complex structure thus produced does not depend on the choice of H' , and in fact is stably isomorphic to the structure described by [8], as we now show:

Theorem 5.2.1. *The stably complex structure γ' on $M^{2n}(P^n, \ell)$ induced from the inclusion in $\mathbf{R}^n \times \mathcal{QT}_n|_{H'}$ is stably isomorphic to the structure ζ coming from $\mathcal{U}(K)$ given by (3.4).*

Proof. We have the two equivariant stably complex structures:

$$\tau(M^{2n}) \oplus \tau_e(\kappa'') \oplus \mathbf{R}^{m-n} \cong \zeta|$$

and

$$\tau(M^{2n}) \oplus \tau_e(\kappa') \oplus \mathbf{R}^{H'} \oplus \mathbf{R}^n \cong \gamma'$$

and it suffices to show that the bundles $\zeta|$ and γ' are stably isomorphic.

Consider the commutative diagram:

$$\begin{array}{ccc} M^{2n}(P^n, \ell) & \xrightarrow{\Psi} & \mathbf{R}^n \times \mathcal{QT}_n|_{H'} \\ & \searrow \Phi & \downarrow \pi \\ & & \mathcal{QT}_n|_{H'} \end{array}$$

We have

$$\zeta| \simeq_s \Phi^*(\xi)$$

where \simeq_s denotes stable isomorphism by proposition 3.3.4, while

$$\Phi^*(\xi) \simeq_s \Psi^*\pi^*(\xi)$$

and $\gamma' \simeq_s \Psi^*\pi^*(\xi)$ by construction. □

We may carry out the same construction in $\mathbf{R}^n \times \mathcal{QT}_n$, without making use of the finite dimensional submanifold $\mathcal{QT}_n|_{H'}$. The framing of $M^{2n}(P^n, \ell)$ in $\mathbf{R}^n \times \mathcal{QT}_n$ yields

$$\tau(M^{2n}(P^n, \ell)) \oplus \mathbf{R}^{H_n} \cong \mathbf{R}^n \oplus \tau(\mathcal{QT}_n)$$

and we have the equivariant stably complex structure

$$\tau(\mathcal{QT}_n) \oplus \tau_e(\kappa) = \xi$$

giving

$$\tau(M^{2n}(P^n, \ell)) \oplus \tau_e(\kappa) \oplus \mathbf{R}^{H_n} \oplus \mathbf{R}^n \cong (\xi \oplus \mathbf{C}^n)|_{M^{2n}} = \gamma \quad (5.2)$$

which makes use of the infinite dimensional stabilising bundle \mathbf{R}^{H_n} . By the same argument used in the proof of Proposition 5.2.1, this structure is stably isomorphic to the structure described in [8], and hence also stably isomorphic to (5.1) for any choice of H' .

Example 5.2.2. We consider pullbacks of the linear model again. Recall that the pullback construction yields an equivariant framing of $M^{2n}(P^n, \ell)$ in $\mathbf{R}^n \times \mathbf{C}^n \cong \mathbf{R}^{3n}$. The equivariant stably complex structure

$$\tau(M^{2n}) \oplus \tau_e(\kappa) \oplus \mathbf{R}^{H'} \oplus \mathbf{R}^n \cong \gamma = (\xi \oplus \mathbf{C}^n)|_{M^{2n}}$$

therefore reduces to

$$\tau(M^{2n}) \oplus \tau_e(\kappa) \oplus \mathbf{R}^{H'} \oplus \mathbf{R}^n \cong \mathbf{C}^n \oplus \tau(\mathbf{C}^n)|_{M^{2n}}$$

The right hand side $\mathbf{C}^n \oplus \tau(\mathbf{C}^n)|_{M^{2n}}$ is trivial as a non-equivariant complex vector bundle, but not necessarily as a T^n -equivariant bundle.

Chapter 6

Cohomology of quasitoric spaces

The goal of this chapter is the calculation of the cohomology ring $H^*(\mathcal{QT}_n, \mathbf{Z})$. The cohomology of quasitoric manifolds with integer coefficients was described by Davis and Januszkiewicz in [10]. They also calculated the rational cohomology of quasitoric spaces constructed over finite Cohen-Macaulay complexes. We review their work and extend it to calculate the integral cohomology of quasitoric spaces over Cohen-Macaulay complexes that may be countably infinite. This includes the case of $H^*(\mathcal{QT}_n, \mathbf{Z})$, which we describe explicitly.

Our methods depend on the cohomological properties of the Stanley-Reisner ring $\mathbf{Z}(K)$ of the complex K , which follow (in the finite case) from the fact that the complex K , and hence the ring $\mathbf{Z}(K)$, are Cohen-Macaulay. The extension to countably infinite complexes presents a significant obstacle: in this case $\mathbf{Z}(K)$ is non-Noetherian, and there is no single extension of the Cohen-Macaulay condition to the non-Noetherian setting. [9] provides a helpful result, but we proceed with a direct line of proof by close study of the Koszul and Čech cohomology of the Stanley-Reisner rings of an increasing sequence of subcomplexes of K , making use of the Mittag-Leffler condition to pass to the limit $\mathbf{Z}(K)$.

6.1 Commutative and homological algebra

We begin with a brief exposition of some commutative algebra, concerning Cohen-Macaulay rings, Koszul and Čech complexes and homology. A comprehensive reference is [4].

6.1.1 Systems of parameters and regular sequences

We begin this section in a general setting, following [4]. All rings considered will be commutative with unit.

Recall that $\text{Spec } R$ denotes the set of prime ideals of a ring R . For any $\mathfrak{p} \in \text{Spec } R$, the *height* of \mathfrak{p} , denoted $\text{ht } \mathfrak{p}$, is the supremum of the lengths t of strictly descending chains of prime ideals of R :

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_t$$

For an arbitrary ideal $I \triangleleft R$, the height of I is defined to be the infimum of the heights of all prime ideals containing I . If R is Noetherian, then every proper ideal in R has finite height.

Definition 6.1.1. The *Krull dimension* of a Noetherian ring R is

$$\dim R = \sup\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R\}$$

Recall that a *local* ring is a ring that has only one maximal ideal. The local ring R with maximal ideal \mathfrak{m} is often denoted (R, \mathfrak{m}) . If (R, \mathfrak{m}) is a Noetherian local ring it follows that $\dim R = \text{ht } \mathfrak{m}$.

Definition 6.1.2. A sequence of elements $\mathbf{x} = x_1, \dots, x_n$ in a Noetherian local ring (R, \mathfrak{m}) is called a *system of parameters* if $n = \dim R$ and $\dim(R/(\mathbf{x})) = 0$.

Recall that the *radical* of an ideal $I \triangleleft R$, denoted \sqrt{I} , is the ideal

$$\sqrt{I} = \{r \in R \mid r^j \in I \text{ for some } j \geq 0\}$$

So $\mathbf{x} = x_1, \dots, x_n$ is a system of parameters in the Noetherian local ring (R, \mathfrak{m}) if and only if $n = \dim R$ and $\sqrt{(\mathbf{x})} = \mathfrak{m}$.

A system of parameters for a Noetherian local ring is guaranteed to exist.

Example 6.1.3. Let k be a field. The polynomial ring $k[y_1, \dots, y_n]$ is a Noetherian local ring with maximal ideal $\mathfrak{m} = (y_1, \dots, y_n)$. The dimension of $k[y_1, \dots, y_n]$ is n , with a maximal chain of prime ideals given by

$$\mathfrak{m} = (y_1, \dots, y_n) \supset (y_1, \dots, y_{n-1}) \supset \cdots \supset (y_1)$$

The sequence of elements $\mathbf{y} = y_1, \dots, y_n$ is a system of parameters for $k[y_1, \dots, y_n]$, as is $\mathbf{y}' = (\mathbf{A}\mathbf{y})_1, \dots, (\mathbf{A}\mathbf{y})_n$ for any invertible $n \times n$ matrix \mathbf{A} .

Definition 6.1.4. A sequence $\mathbf{x} = x_1, \dots, x_k$ in a ring R is called a *regular sequence* if x_i is not a zero-divisor in $R/(x_1, \dots, x_{i-1})$ for each i , and $(\mathbf{x}) \neq R$.

If R is Noetherian, then all regular sequences contained within an ideal $I \triangleleft R$ may be extended to maximal regular sequences in I , and all such maximal sequences have the same length.

Definition 6.1.5. The common length of these maximal sequences is called the *grade* of I on R , denoted $\text{grade}(I, R)$. If (R, \mathfrak{m}) is a Noetherian local ring, then all regular sequences are contained within \mathfrak{m} and the grade of \mathfrak{m} is called the *depth* of R .

Example 6.1.6. Consider the polynomial ring $k[y_1, \dots, y_n]$ for a field k , as in Example 6.1.3. The sequence $\mathbf{y} = y_1, \dots, y_i$ is a regular sequence for $i = 1, \dots, n$ and is a maximal regular sequence for $i = n$. Thus, $\text{depth } k[y_1, \dots, y_n] = n$.

For a Noetherian local ring (R, \mathfrak{m}) , every regular sequence is part of a system of parameters and hence $\text{depth } R \leq \dim R$. The following definition is a cornerstone of commutative algebra:

Definition 6.1.7. A Noetherian local ring is called *Cohen-Macaulay* if its depth is equal to its dimension. An arbitrary Noetherian ring R is called *Cohen-Macaulay* if the localisation $R_{\mathfrak{m}}$ at each maximal ideal $\mathfrak{m} \triangleleft R$ is Cohen-Macaulay.

Then a standard result (see e.g. [4], Theorem 2.1.2) is:

Theorem 6.1.8. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring. Then a sequence \mathbf{x} in R is regular if and only if it forms part of a system of parameters.*

As a result, any system of parameters in a Cohen-Macaulay local ring is a maximal regular sequence; and any maximal regular sequence is a system of parameters.

Example 6.1.9. For a field k , the polynomial ring $k[y_1, \dots, y_n]$ is Cohen-Macaulay. Since the sequence $\mathbf{y}' = (\mathbf{A}\mathbf{y})_1, \dots, (\mathbf{A}\mathbf{y})_n$ for any invertible $n \times n$ matrix \mathbf{A} is a system of parameters, it is also a maximal regular sequence.

We also note (e.g. [4], Corollary 2.1.4):

Lemma 6.1.10. *For any Cohen-Macaulay ring R and ideal $I \triangleleft R$ with $I \neq R$ we have $\text{ht } I = \text{grade}(I, R)$.*

There are similar definitions of dimension, depth and Cohen-Macaulay-ness for R modules, but we will not make use of them here.

6.1.2 The Koszul complex

The *Koszul complex* $K_*(x)$ for a single element x of a ring R is defined to be the chain complex of R -modules

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

where the two copies of R are taken to be in homological dimensions 1 and 0, and the map is multiplication by x . Following [23], it is helpful to label the generator of the R in dimension one; call it e . Then we have a chain complex

$$0 \longrightarrow Re \xrightarrow{\partial_1} R \longrightarrow 0$$

with $\partial_1 e = x$.

The tensor product of chain complexes of R -modules is defined as follows (see e.g. [23], Definition 2.38): if $C = \{C_q, \partial_q\}$, i.e. C is the complex

$$\cdots \longrightarrow C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \longrightarrow \cdots$$

and $C' = \{C'_q, \partial'_q\}$ then $C \otimes_R C' = \{(C \otimes_R C')_r, \epsilon_r\}$ where

$$(C \otimes_R C')_r = \bigoplus_{i+j=r} C_i \otimes_R C'_j$$

and $\epsilon_r: (C \otimes_R C')_r \rightarrow (C \otimes_R C')_{r-1}$ is given by

$$\epsilon_r(c_i \otimes c'_j) = \partial_i c_i \otimes c'_j + (-1)^i c_i \otimes \partial'_j c'_j.$$

The following may be taken as a definition (as in [23], Definition 2.39; but see [4], Section 1.6 for a more general exposition):

Definition 6.1.11. For a sequence of elements $\mathbf{x} = x_1, \dots, x_n \in R$ the *Koszul*

complex of \mathbf{x} is the tensor product of chain complexes of R -modules

$$K_*(\mathbf{x}) = K_*(x_1) \otimes_R \cdots \otimes_R K_*(x_n).$$

If M is an R -module, the *Koszul complex of \mathbf{x} with coefficients in M* is the complex $K_*(\mathbf{x}, M) = K_*(\mathbf{x}) \otimes_R M$.

Note that, since $R \otimes_R M \cong M$ for any R -module M , we have $K_*(\mathbf{x}, R) = K_*(\mathbf{x})$.

Example 6.1.12. For a two element sequence $\mathbf{x} = x_1, x_2$ the Koszul complex $K_*(\mathbf{x})$ is

$$0 \longrightarrow R \xrightarrow{\partial_2} R^2 \xrightarrow{\partial_1} R \longrightarrow 0$$

where the map $\partial_2: R \rightarrow R^2$ is given by the matrix

$$\begin{pmatrix} x_1 & -x_2 \end{pmatrix}$$

and $\partial_1: R^2 \rightarrow R$ is given by

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

With coefficients in the R -module M , the Koszul complex $K_*(\mathbf{x}, M)$ is

$$0 \longrightarrow R \otimes_R M \xrightarrow{\partial_2 \otimes M} R^2 \otimes_R M \xrightarrow{\partial_1 \otimes M} R \otimes_R M \longrightarrow 0.$$

Returning to the general case of $n \geq 1$, when $M = R$ we may set $e_{i_1 \dots i_q} = u_1 \otimes \cdots \otimes u_n$ where u_i is the generator $e_i \in K_1(x_i)$ if $i \in \{i_1, \dots, i_q\}$ and $u_i = 1 \in K_0(x_i)$ otherwise. Then $K_q(\mathbf{x})$ is a free R -module with basis $\{e_{i_1 \dots i_q} \mid 1 \leq i_1 \leq \dots \leq i_q \leq n\}$ and

$$\partial_q(e_{i_1 \dots i_q}) = \sum_{j=1}^q (-1)^{j-1} x_{i_j} e_{i_1 \dots \hat{i}_j \dots i_q}$$

where i_j is missing from the subscript.

The Koszul homology is defined in a general setting in [4] (as Definition 1.6.3). Specialising to the case of a sequence of elements \mathbf{x} we may take:

Definition 6.1.13. The *Koszul homology* of the sequence \mathbf{x} , denoted $H_*(\mathbf{x})$, is the homology of $K_*(\mathbf{x})$. For an R -module M , the *Koszul homology of \mathbf{x} with coefficients in M* is the homology of $K_*(\mathbf{x}, M)$, denoted $H_*(\mathbf{x}, M)$.

For example, for a single element $x \in R$, we have $H_0(x, M) = M/xM$ and $H_1(x, M) = \{m \in M \mid xm = 0\}$, an ideal in M .

6.1.3 The graded case

We will make use of the following observation in our cohomology calculations: If R is graded (and M a graded R module), then the grading is inherited by the Koszul complex and hence by the modules $H_*(\mathbf{x}, M)$. If the elements of \mathbf{x} are homogeneous and of common degree i , then each differential in $K_*(\mathbf{x}, M)$ raises the degree by i , so each graded component $[H_i(\mathbf{x}, M)]_j$ is the homology of the subcomplex

$$[K_{i+1}(\mathbf{x}, M)]_{j-i} \rightarrow [K_i(\mathbf{x}, M)]_j \rightarrow [K_{i-1}(\mathbf{x}, M)]_{j+i}$$

6.1.4 Properties of Koszul homology

We will be interested in knowing when the Koszul homology is zero. The fundamental property we require is the following (e.g. [4], Theorem 1.6.17):

Theorem 6.1.14. *When R is a Noetherian ring and I is the ideal generated by $\mathbf{x} = x_1, \dots, x_n$, $H_*(\mathbf{x})$ vanishes in all dimensions if and only if $I = R$. If $I \neq R$ let*

$$h = \max\{i \mid H_i(\mathbf{x}) \neq 0\}$$

Then $n - h = \text{grade}(I, R)$.

We record the following well known consequence, which we will make use of later:

Corollary 6.1.15. *If R is a Cohen-Macaulay ring then $n - h = \text{ht } I$. If R is a Noetherian local ring and \mathbf{x} is a system of parameters for R then $H_i(\mathbf{x}) = 0$ for all $i > 0$ if and only if R is Cohen-Macaulay.*

Proof. By Lemma 6.1.10, if R is Cohen-Macaulay then $\text{grade}(I, R) = \text{ht } I$. If R is a Noetherian local ring and $\mathbf{x} = x_1, \dots, x_n$ is a system of parameters for R then $\dim R = n$ (see Definition 6.1.2) and $\text{depth } R = \text{grade}((\mathbf{x}), R)$ (Definition 6.1.5). Then $H_i(\mathbf{x}) = 0$ for all $i > 0$ if and only if $\dim R = \text{depth } R$, or in other words if R is Cohen-Macaulay (Definition 6.1.7). \square

In dealing with infinite simplicial complexes we will make use of a few more properties of Koszul homology.

Lemma 6.1.16 (e.g. [4], Proposition 1.6.11). *For a given sequence of elements $\mathbf{x} \in R$ the Koszul complex $K_*(\mathbf{x}, -)$ is an exact functor from the category of R -modules to the category of chain complexes of R -modules.*

That is, given a map of R -modules $M \rightarrow N$, tensoring with $K_*(\mathbf{x})$ gives a map $K_*(\mathbf{x}, M) \rightarrow K_*(\mathbf{x}, N)$, and given a short exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

we obtain a short exact sequence of chain complexes

$$0 \rightarrow K_*(\mathbf{x}, L) \rightarrow K_*(\mathbf{x}, M) \rightarrow K_*(\mathbf{x}, N) \rightarrow 0$$

and hence a long exact sequence in homology:

$$\cdots \rightarrow H_i(\mathbf{x}, L) \rightarrow H_i(\mathbf{x}, M) \rightarrow H_i(\mathbf{x}, N) \rightarrow H_{i-1}(\mathbf{x}, L) \rightarrow \cdots$$

We also have the following standard observation (e.g. [4], Proposition 1.6.7):

Lemma 6.1.17. *Given a ring homomorphism $\varphi: R \rightarrow S$ there is an isomorphism of R -modules $K_*(\mathbf{x}) \otimes_R S \cong K_*(\mathbf{x}, S) \cong K_*(\varphi(\mathbf{x}))$, where the last term is the Koszul complex of $\varphi(\mathbf{x})$ over S , considered as an R -module. If N is an S -module, then*

$$K_*(\mathbf{x}) \otimes_R N \cong (K_*(\mathbf{x}) \otimes_R S) \otimes_S N \cong K_*(\varphi(\mathbf{x})) \otimes_S N$$

as R modules, so there is an isomorphism of R -modules $H_(\mathbf{x}, N) \cong H_*(\varphi(\mathbf{x}), N)$, and this isomorphism is natural in N .*

We will be particularly interested in the composite map

$$H_*(\mathbf{x}, R) \rightarrow H_*(\mathbf{x}, S) \cong H_*(\varphi(\mathbf{x}), S)$$

induced by

$$K_*(\mathbf{x}, R) \rightarrow K_*(\mathbf{x}, S) \cong K_*(\varphi(\mathbf{x}), S)$$

6.1.5 Koszul cohomology and Čech cohomology

The *Koszul cohomology* is also defined. For a sequence $\mathbf{x} = x_1, \dots, x_n \in R$ and R module M , we set $K^*(\mathbf{x}, M) = \text{Hom}(K_*(\mathbf{x}), M)$, and then the Koszul cohomology $H^*(\mathbf{x}, M)$ is the homology of $K^*(\mathbf{x}, M)$. The complex $K^*(\mathbf{x})$ and cohomology $H^*(\mathbf{x})$ are defined by setting $M = R$.

In fact $K_*(\mathbf{x})$ and $K^*(\mathbf{x})$ are isomorphic, up to a dimensional shift. That is, there is a chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n(\mathbf{x}) & \xrightarrow{\partial_n} & \dots & \xrightarrow{\partial_1} & K_0(\mathbf{x}) \longrightarrow 0 \\ & & \downarrow \tau_n & & & & \downarrow \tau_0 \\ 0 & \longrightarrow & K^0(\mathbf{x}) & \xrightarrow{\delta_1} & \dots & \xrightarrow{\delta_n} & K^n(\mathbf{x}) \longrightarrow 0 \end{array}$$

which can be shown to be an isomorphism (see [4], Proposition 1.6.10). Then since there is a natural isomorphism $\text{Hom}(R, M) \cong R^* \otimes M$, there is an isomorphism $K_*(\mathbf{x}, M) \cong K^*(\mathbf{x}, M)$ and we have $H_i(\mathbf{x}, M) \cong H^{n-i}(\mathbf{x}, M)$ for all i .

The extended Čech complex

The *extended Čech complex*, C^* , for a sequence \mathbf{x} is the tensor product of the individual complexes

$$0 \longrightarrow R \xrightarrow{1} R_{x_i} \longrightarrow 0$$

where R_{x_i} is the localisation of R with respect to the ideal (x_i) , also denoted $R[1/x_i]$. The modules R and R_{x_i} are taken to be in homological dimensions zero and one respectively. The *Čech cohomology* of R with coefficients in an R -module M , denoted $\check{H}^*(\mathbf{x}; M)$, is the homology of $C^* \otimes_R M$.

The stable Koszul complex

The extended Čech complex is sometimes called the stable Koszul complex. This is because there is a sequence of maps of Koszul complexes

$$K^*(\mathbf{x}^0, M) \rightarrow K^*(\mathbf{x}^1, M) \rightarrow K^*(\mathbf{x}^2, M) \rightarrow K^*(\mathbf{x}^3, M) \rightarrow \dots$$

where \mathbf{x}^l is the sequence x_1^l, \dots, x_n^l , which for a single element x_i is given by

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \xrightarrow{1} & M & \longrightarrow & 0 \\
 & & \downarrow 1 & & \downarrow x_i & & \\
 0 & \longrightarrow & M & \xrightarrow{x_i} & M & \longrightarrow & 0 \\
 & & \downarrow 1 & & \downarrow x_i & & \\
 0 & \longrightarrow & M & \xrightarrow{x_i^2} & M & \longrightarrow & 0 \\
 & & \downarrow 1 & & \downarrow x_i & & \\
 & & \vdots & & \vdots & &
 \end{array}$$

The colimit of this sequence (in the category of chain complexes of R -modules) is denoted $K^*(\mathbf{x}^\infty, M)$, and is naturally isomorphic to the extended Čech complex $C^* \otimes M$ ([4], Proposition 3.5.5). The final map $K^*(\mathbf{x}^j, M) \rightarrow C^* \otimes M$ is given, for a single element x_i , by

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \xrightarrow{x_i^j} & M & \longrightarrow & 0 \\
 & & \downarrow 1 & & \downarrow 1/x_i^j & & \\
 0 & \longrightarrow & M & \xrightarrow{1} & M_{x_i} & \longrightarrow & 0
 \end{array}$$

Taking homology commutes with colimits in this category, so $\check{H}^*(\mathbf{x}, M) \cong H^*(\mathbf{x}^\infty, M) \cong \operatorname{colim} H^*(\mathbf{x}^l, M)$. There is a natural map $H^*(\mathbf{x}, M) \rightarrow \check{H}^*(\mathbf{x}, M)$ induced by the chain map $K^*(\mathbf{x}, M) \rightarrow K^*(\mathbf{x}^\infty, M)$. The following is shown in ([14], Proposition 2.7):

Proposition 6.1.18. *For any commutative ring R (Noetherian or otherwise) and R module M ,*

$$\begin{aligned}
 & \sup\{k \geq 0 \mid H_{n-i}(\mathbf{x}, M) = 0 \text{ for all } i < k\} \\
 = & \sup\{k \geq 0 \mid \check{H}^i(\mathbf{x}, M) = 0 \text{ for all } i < k\}
 \end{aligned}$$

We will summarise the functorial properties of Čech cohomology (following immediately from lemmas 6.1.16 and 6.1.17) in the following:

Proposition 6.1.19. *The Čech cohomology $\check{H}^*(\mathbf{x}, M)$ is covariant in the coefficient module M . Given a ring homomorphism $\varphi: R \rightarrow S$ there is an isomorphism of R -modules $\check{H}^*(\mathbf{x}, S) \cong \check{H}^*(\varphi(\mathbf{x}))$, and the chain map $K^*(\mathbf{x}, M) \rightarrow K^*(\mathbf{x}^\infty, M)$*

induces a commutative diagram

$$\begin{array}{ccccc}
 H^*(\mathbf{x}, R) & \longrightarrow & H^*(\mathbf{x}, S) & \xrightarrow{\cong} & H^*(\varphi(\mathbf{x}), S) \\
 \downarrow & & \downarrow & & \downarrow \\
 \check{H}^*(\mathbf{x}, R) & \longrightarrow & \check{H}^*(\mathbf{x}, S) & \xrightarrow{\cong} & \check{H}^*(\varphi(\mathbf{x}), S)
 \end{array}$$

Change of sequence

If R is Noetherian, the Čech cohomology depends only on the radical of the ideal (\mathbf{x}) . In fact there is a natural isomorphism $\check{H}^*(\mathbf{x}, M) \cong H_{(\mathbf{x})}^*(M)$, where $H_{(\mathbf{x})}^*(M)$ is the *local cohomology* functor (see e.g. [14]). We will only require the existence of a natural isomorphism $\check{H}^*(\mathbf{x}, M) \cong \check{H}^*(\mathbf{y}, M)$, the composition of the isomorphisms $\check{H}^*(\mathbf{x}, M) \cong H_{(\mathbf{x})}^*(M) \cong H_{(\mathbf{y})}^*(M) \cong \check{H}^*(\mathbf{y}, M)$, when \mathbf{y} is another sequence in R such that $\sqrt{(\mathbf{x})} = \sqrt{(\mathbf{y})}$.

6.1.6 Homology of inverse limits

We will make use of some results on limits of chain complexes. The results are given for towers of chain complexes of abelian groups $\{C_i\} = \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0$, i.e. specifically for inverse limits. All the material presented in this section can be found in section 3.5 of [27].

Given a tower of Abelian groups $\{A_i\}$, the map

$$\Delta: \prod_{i=0}^{\infty} A_i \rightarrow \prod_{i=0}^{\infty} A_i$$

is defined elementwise by

$$\Delta(\dots, a_i, \dots, a_0) \mapsto (\dots, a_i - \bar{a}_{i+1}, \dots, a_0 - \bar{a}_1)$$

where \bar{a}_i is the image of $a_i \in A_i$ in A_{i-1} . The kernel of Δ is $\lim A_i$ and there is a functor $\lim^1 A_i$ which can be defined to be the cokernel of Δ .

The functor \lim^1 is most useful when it is zero. The *Mittag-Leffler* condition ensures that this is true. A tower of Abelian groups $\{A_i\}$ is said to satisfy the Mittag-Leffler condition if for each k there exists a $j \geq k$ such that the image of A_i in A_k is equal to the image of A_j in A_k for all $i \geq j$. This is satisfied, for example, if each map $A_{i+1} \rightarrow A_i$ is surjective, or if for each k there exists a $j \geq k$

such that the image of A_j in A_k is zero. It can be shown that if $\{A_i\}$ satisfies the Mittag-Leffler condition, then $\lim^1 A_i = 0$.

The key result is the following:

Theorem 6.1.20 ([27], Theorem 3.5.8). *Let $\{C_i\}$ be a tower of chain complexes of Abelian groups satisfying the Mittag-Leffler condition. Then there is an exact sequence*

$$0 \rightarrow \lim^1 H_{q+1}(C_i) \rightarrow H_q(C) \rightarrow \lim H_q(C_i) \rightarrow 0$$

where $C = \lim C_i$. In particular, if the tower $\{H_{q+1}(C_i)\}$ also satisfies the Mittag-Leffler condition then $H_q(C) \cong \lim H_q(C_i)$.

Example 6.1.21. As an immediate application, suppose $X_0 \subset X_1 \subset X_2 \subset \dots$ is an increasing sequence of spaces, and let $X = \operatorname{colim} X_i$. The singular cochain complexes $C^*(X_i)$ form a tower where each map $C^*(X_i) \rightarrow C^*(X_{i-1})$ is surjective, since any singular cochain in X_{i-1} may be extended to a singular cochain in X_i . Therefore the tower $\{C^*(X_i)\}$ satisfies the Mittag-Leffler condition and we have an exact sequence

$$0 \rightarrow \lim^1 H^{q+1}(X_i) \rightarrow H^q(X) \rightarrow \lim H^q(X_i) \rightarrow 0$$

An identical statement holds for simplicial cohomology, replacing the spaces X_i with simplicial complexes and the singular cochain complexes $C^*(X_i)$ with simplicial cochain complexes.

6.2 Stanley-Reisner rings and Cohen-Macaulay complexes

In this section we review the definition of the Stanley-Reisner ring $R[K]$ of a finite simplicial complex K and its cohomological properties: namely, $R[K]$ is a Cohen-Macaulay ring if and only if K is a Cohen-Macaulay complex over R . We extend the definition of $R[K]$ to the case of a finite dimensional simplicial complex on countable vertex set. Although there is no immediate analogue of the Cohen-Macaulay condition for $R[K]$, because it is non-Noetherian, we establish the result we need in terms of the limit of the Čech cohomology of the rings $R[K']$ for finite subcomplexes $K' \subset K$.

We begin with the classical definition (see e.g. [4], Definition 5.1.2):

Definition 6.2.1. Let K be a finite simplicial complex with vertex set V , and let R be a commutative ring. The *Stanley-Reisner ring* $R[K]$ is the R -algebra

$$R[K] = R[x_v \mid v \in V]/I_K$$

where I_K is the ideal generated by all monomials of the form $x_1 \dots x_k$ where $v_1 \dots v_k \notin K$.

We observe that, given a map $f: K_1 \rightarrow K_2$, there is a map of Stanley-Reisner rings $f^*: R[K_2] \rightarrow R[K_1]$ sending $y_w \mapsto \sum_{f(v)=w} x_v$. So the Stanley Reisner ring is a contravariant functor from the category of finite simplicial complexes with simplicial maps to the category of finitely generated R -algebras.

Example 6.2.2. When K is the $(n-1)$ -simplex $\Delta^{(n-1)}$ on vertex set $\{v_1, \dots, v_n\}$, the ideal I_K is empty and the Stanley Reisner ring $R[K]$ is the polynomial ring $R[x_1, \dots, x_n]$.

Example 6.2.3. When K is the boundary of the $(n-1)$ -simplex, $\partial\Delta^{(n-1)}$, we have $I_K = (x_1 \dots x_n)$ and $R[K] = R[x_1, \dots, x_n]/(x_1 \dots x_n)$.

There is an alternative description of $R[K]$ as a limit. Recall from Chapter 2 that $\text{CAT}(K)$ is the category whose objects are the simplices of K and whose morphisms are the inclusions among them. Then, as made clear in [22], by construction we have:

Proposition 6.2.4. *The Stanley-Reisner ring $R[K]$ is the limit of the $\text{CAT}(K)^{op}$ diagram of R -algebras which assigns to each simplex $\sigma = v_1 \dots v_k$ of K the polynomial algebra $R[\sigma] = R[x_1, \dots, x_k]$ and to each inclusion $\sigma_1 \hookrightarrow \sigma_2$ the restriction $R[\sigma_2] \rightarrow R[\sigma_1]$ (sending $x_i \mapsto x_i$ for $x_i \in \sigma_1$ and $x_i \mapsto 0$ otherwise).*

We now give this as our definition of $R[K]$ in the infinite case:

Definition 6.2.5. Let K be a finite or countably infinite simplicial complex with vertex set V , and let R be a commutative ring. The Stanley-Reisner ring $R[K]$ is the limit of the $\text{CAT}(K)^{op}$ diagram of R -algebras which assigns to each simplex $\sigma = v_1 \dots v_k$ of K the polynomial algebra $R[\sigma] = R[x_1, \dots, x_k]$ and to each inclusion $\sigma_1 \hookrightarrow \sigma_2$ the restriction $R[\sigma_2] \rightarrow R[\sigma_1]$.

The following description will be useful for our cohomology calculations:

Proposition 6.2.6. *The Stanley-Reisner ring $R[K]$ is the limit of the $\text{SUB}(K)^{op}$ diagram that assigns to each finite subcomplex K' of K the Stanley-Reisner ring $R[K']$ and to each inclusion $\iota: K' \hookrightarrow K''$ the restriction $\iota^*: R[K''] \rightarrow R[K']$.*

Proof. Let $D(K)$ denote the $\text{CAT}(K)^{op}$ diagram of R -algebras of Definition 6.2.5. Let \mathcal{C} denote the category of all finite full subdiagrams of $D(K)$, and let \mathcal{C}^{op} denote the \mathcal{C}^{op} diagram which assigns to each full subdiagram D of $D(K)$ the limit of D , and to each inclusion of subdiagrams $D' \hookrightarrow D$ the unique morphism $\lim D \rightarrow \lim D'$ making the following diagram commute:

$$\begin{array}{ccc} \lim D & \longrightarrow & D \\ \downarrow & \searrow & \uparrow \\ \lim D' & \longrightarrow & D' \end{array}$$

Then the limit of $D(K)$ is equal to the limit of the \mathcal{C}^{op} diagram \mathcal{C} .

The $\text{SUB}(K)^{op}$ diagram that assigns to each finite subcomplex $K' \subset K$ the Stanley-Reisner ring $R[K']$ and to each inclusion $\iota: K' \hookrightarrow K''$ the restriction $\iota^*: R[K''] \rightarrow R[K']$ is a coinitial subdiagram of \mathcal{C} , so its limit is equal to $\lim \mathcal{C}$, as required. \square

We now give a more explicit description of $R[K]$, to clarify the structure the infinite case:

Proposition 6.2.7. *Let K be a finite or countably infinite simplicial complex with vertex set V , and let R be a commutative ring. As an R -module, the Stanley-Reisner ring $R[K]$ is isomorphic to the product*

$$\prod_{\sigma^k \in K} \bigoplus_{(\alpha_1, \dots, \alpha_k) \geq 1} R\langle x_1^{\alpha_1} \cdots x_k^{\alpha_k} \rangle$$

where the product is taken over all simplices of K , for a k -simplex σ^k with vertices v_1, \dots, v_k the sum is taken over all k -tuples $(\alpha_1, \dots, \alpha_k)$ of strictly positive integers, and $R\langle x_1^{\alpha_1} \cdots x_k^{\alpha_k} \rangle$ denotes the free R -module with generator $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$.

The multiplicative structure on $R[K]$ is given on the generators as follows. Let $x_1^{\alpha_1} \cdots x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \cdots x_k^{\alpha_k}$ and $x_1^{\beta_1} \cdots x_i^{\beta_i} x_{i+1}^{\beta_{i+1}} \cdots x_j^{\beta_j}$ be generators corresponding to simplices $\sigma^k = v_1 \cdots v_i v_{i+1} \cdots v_k$ and $\sigma'^j = v_1 \cdots v_i v'_{i+1} \cdots v'_j$ with common vertices v_1, \dots, v_i . Their product is the generator

$$x_1^{\alpha_1 + \beta_1} \cdots x_i^{\alpha_i + \beta_i} x_{i+1}^{\alpha_{i+1}} \cdots x_k^{\alpha_k} x_{i+1}^{\beta_{i+1}} \cdots x_j^{\beta_j}$$

if $v_1 \cdots v_i v_{i+1} \cdots v_k v'_{i+1} \cdots v'_j$ is a simplex of K ; zero otherwise.

For notational purposes, the order of the vertices in the term $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is unimportant: we denote the same generator by $x_{s(1)}^{\alpha_{s(1)}} \cdots x_{s(k)}^{\alpha_{s(k)}}$ for any permutation $s \in S_k$.

Proof. From definition 6.2.5, $R[K]$ is the limit of the $\text{CAT}(K)^{op}$ diagram of R -algebras which assigns to each simplex $\sigma = v_1 \cdots v_k$ of K the polynomial algebra $R[\sigma] = R[x_1, \dots, x_k]$ and to each inclusion $\iota: \sigma_1 \hookrightarrow \sigma_2$ the restriction $\iota^*: R[\sigma_2] \rightarrow R[\sigma_1]$. Thus $R[K]$ is the subalgebra of the product

$$\prod_{\sigma^k \in K} R[\sigma^k] \quad (6.1)$$

where $x_{\sigma_1} = \iota^*(x_{\sigma_2})$ for each inclusion $\iota: \sigma_1 \hookrightarrow \sigma_2$, with $x_\sigma \in R[\sigma]$ denoting the σ -th component of the product (6.1).

As an R -module, $R[\sigma^k]$ is the sum

$$\bigoplus_{\sigma^j \subseteq \sigma^k} \bigoplus_{(\alpha_1, \dots, \alpha_j) \geq 1} R\langle x_1^{\alpha_1} \cdots x_j^{\alpha_j} \rangle$$

of free R -modules. The restriction $x_{\sigma^j} = \iota^*(x_{\sigma^k})$ ensures that the coefficient of $R\langle x_1^{\alpha_1} \cdots x_j^{\alpha_j} \rangle$ in $R[\sigma^k]$ is equal to the coefficient in $R[\sigma^j]$, so $R[K]$ is the image of the monomorphism of R -modules

$$\phi: \prod_{\sigma^k \in K} \bigoplus_{(\alpha_1, \dots, \alpha_k) \geq 1} R\langle x_1^{\alpha_1} \cdots x_k^{\alpha_k} \rangle \rightarrow \prod_{\sigma^k \in K} R[\sigma^k] \quad (6.2)$$

given by

$$\phi(x_1^{\alpha_1} \cdots x_j^{\alpha_j}) = \bigoplus_{\sigma^j \subseteq \sigma^k} (x_1^{\alpha_1} \cdots x_j^{\alpha_j})_{\sigma^k}$$

where $(x_1^{\alpha_1} \cdots x_j^{\alpha_j})_{\sigma^k}$ denotes the generator $x_1^{\alpha_1} \cdots x_j^{\alpha_j}$ of $R[\sigma^k]$.

The multiplication is well defined: letting $R'[\sigma]$ denote the sum

$$R'[\sigma^k] = \bigoplus_{(\alpha_1, \dots, \alpha_k) \geq 1} R\langle x_1^{\alpha_1} \cdots x_k^{\alpha_k} \rangle,$$

we need only check that the product of two elements of $R[K]$ gives a well defined

element of each sum $R'[\sigma]$. But if

$$x_i = \prod_{\sigma} \bigoplus_{\alpha} \lambda_{i,\sigma,\alpha} x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

for $i = 1, 2$, with finitely many non-zero coefficients $\lambda_{i,\sigma,\alpha}$ for each σ , then

$$x_1 x_2 = \prod_{\sigma} \bigoplus_{\alpha} \lambda_{\sigma,\alpha} x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

with only finitely many non-zero coefficients $\lambda_{\sigma,\alpha}$ for each σ , since there are only finitely many pairs of simplices $\sigma_1, \sigma_2 \subseteq \sigma$.

Finally, with the multiplication described, the map (6.2) is an R -algebra homomorphism. □

The homological properties of the Stanley-Reisner ring are closely related to the cohomology of K . Recall that for a simplex $\sigma \in K$ the *link* of σ in K , denoted $\text{lk } \sigma$, is the subcomplex

$$\text{lk } \sigma = \{\sigma' \subset V \setminus \sigma \mid \sigma \cup \sigma' \in K\}$$

while the *star* of σ in K , denoted $\text{st } \sigma$, is the subcomplex

$$\text{st } \sigma = \{\sigma' \in K \mid \sigma \subset \sigma'\}$$

We write $\text{lk}_K \sigma$ and $\text{st}_K \sigma$ when the complex K is not obvious.

We have the following definition (see e.g. [23]):

Definition 6.2.8. A finite dimensional simplicial complex K is Cohen-Macaulay over R if and only if K is pure (of dimension n , say), $H^i(K; R) = 0$ for $i < n$ and for any simplex $\sigma^j \in K$, $H^i(\text{lk } \sigma; R) = 0$ for $i < (n - j - 1)$. (It follows that $\text{lk } \sigma$ is itself Cohen-Macaulay over R .) The complex K is *Cohen-Macaulay* if it is Cohen-Macaulay over \mathbf{Z} .

Note that the definition includes the case of a finite dimensional complex on countably infinite vertex set.

As the name suggests, it is a standard result that a *finite* simplicial complex K is Cohen-Macaulay over R if and only if the Stanley-Reisner ring $R[K]$ is Cohen-Macaulay (see e.g. [4], Corollary 5.3.9, where the condition on $R[K]$ is taken as

the definition). This follows from the following observation of Hochster, drawn from [23]:

Proposition 6.2.9. *Let K be a finite simplicial complex on vertex set v_1, \dots, v_m , and $R[K]$ be its Stanley Reisner ring with coefficients in R , with generators x_1, \dots, x_m . Then*

$$\check{H}^i((x_1, \dots, x_m)) = H^i((x_1, \dots, x_m)^\infty) \cong \bigoplus_{F \in K} \bigoplus_{\substack{\alpha \leq 0 \\ \text{supp } \alpha = F}} \check{H}^{i-|F|-1}(\text{lk } F)$$

where the second sum is over m -tuples $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{Z}^m$ where $\alpha_i \leq 0$ for each i and $\text{supp } \alpha = F$ where $\text{supp } \alpha = \{v_i \mid \alpha_i < 0\}$.

Proof. We put a *fine grading* on $R[K]$ by setting $\deg x_i = (0, \dots, 0, 1, 0, \dots, 0)$, the 1 appearing in the i -th position. We first describe the localisation $R[K]_F$, identifying a face F with the corresponding subset $F \subset \{x_1, \dots, x_n\}$.

$$R[K]_F = \begin{cases} R[\{x_i, x_i^{-1} \mid x_i \in F\} \cup \{x_j \mid x_j \in \text{lk } F\}] & , F \in K \\ 0 & , F \notin K \end{cases}$$

The grading on $R[K]$ induces a compatible grading on each $R[K]_F$ and so we may examine each graded component $H^i(\mathbf{x}^\infty)_\alpha$ separately. The r -th term in $K(\mathbf{x}^\infty)_\alpha$ is

$$\bigoplus_{\substack{F' \in K \\ |F'|=r}} (R[K]_{F'})_\alpha$$

Now letting $F = \{x_i \mid \alpha_i < 0\}$, $j = |F|$ and $G = \{x_i \mid \alpha_i > 0\}$ we can identify $K(\mathbf{x}^\infty)_\alpha^r$ with the free R module generated by all faces $F' \in K$ with $|F'| = r$ such that $F \subset F'$ and $F' \cup G \in K$. Each such F' corresponds to an $(r-j)$ dimensional face of $\text{lk}_{\text{st } G} F$ (the link of F in the star on G), and choosing appropriate signs for the generators we may identify $K(\mathbf{x}^\infty)_\alpha$ with the augmented simplicial cochain complex of $\text{lk}_{\text{st } G} F$, with dimension shifted by $j+1$.

We must sum over all $\alpha \in \mathbf{Z}^m$, but we note that if $G \neq 0$, $\text{lk}_{\text{st } G} F$ is a cone on G and so $\check{H}^*(\text{lk}_{\text{st } G} F) = 0$. We therefore need only consider the cases where $G = 0$, i.e. $\alpha \leq 0$. This yields the result. \square

There is no corresponding result in the countably infinite case: $R[K]$ is non-Noetherian, and there is no single extension of the Cohen-Macaulay condition

to non-Noetherian rings. Instead, we make the following observation about the isomorphism of Hochster (proposition 6.2.9):

Proposition 6.2.10. *Let K be a finite or countably infinite complex, and choose any ordering on the vertices. Denote by K_j the restriction of K to vertex set $V_j = \{v_1, \dots, v_j\}$, and let $\check{H}_{R[K_j]}^*((x_1, \dots, x_j))$ denote the Čech cohomology of the sequence (x_1, \dots, x_j) in the ring $R[K_j]$. Then the map $\check{H}_{R[K_j]}^i((x_1, \dots, x_j)) \rightarrow \check{H}_{R[K_{j-1}]}^i((x_1, \dots, x_{j-1}))$ induced by the restriction map $R[K_j] \rightarrow R[K_{j-1}]$ is the sum of the maps $\check{H}^{i-|F|-1}(\mathrm{lk}_{K_j} F) \rightarrow \check{H}^{i-|F|-1}(\mathrm{lk}_{K_{j-1}} F)$ for $F \in K_j$ induced by the inclusions $\mathrm{lk}_{K_{j-1}} F \hookrightarrow \mathrm{lk}_{K_j} F$, where we take $\mathrm{lk}_{K_{j-1}} F = 0$ if $F \notin K_{j-1}$.*

Proof. The map $K_{R[K_j]}^r((x_1, \dots, x_j))_\alpha \rightarrow K_{R[K_{j-1}]}^r((x_1, \dots, x_{j-1}))_\alpha$ decomposes as the sum of maps $(R[K_j]_{F'})_\alpha \rightarrow (R[K_j]_{F'} \otimes R[K_{j-1}])_\alpha$, each of which is zero if $x_j \in F'$ and is the induced map of simplicial cochain modules $C^r(\mathrm{lk}_{K_j} F) \rightarrow C^r(\mathrm{lk}_{K_{j-1}} F)$ with dimension shifted by $|F| + 1$ otherwise. \square

We now have the following:

Corollary 6.2.11. *Let K be an $(n - 1)$ -dimensional countably infinite Cohen-Macaulay complex over R , with ordered vertices v_1, v_2, \dots and let K_j denote the restriction of K to vertex set $\{v_1, \dots, v_j\}$. Then $\lim_{j \rightarrow \infty} \check{H}_{R[K_j]}^i(x_1, \dots, x_j) = 0$ and $\lim_{j \rightarrow \infty} \check{H}_{R[K_j]}^{i-1}(x_1, \dots, x_j) = 0$ for $i < n$.*

Proof. We have

$$\lim \check{H}_{R[K_j]}^i((x_1, \dots, x_j)) \cong \bigoplus_{F \in K} \bigoplus_{\substack{\alpha \leq 0 \\ \mathrm{supp} \alpha = F}} \lim \check{H}^{i-|F|-1}(\mathrm{lk}_{K_j} F)$$

and

$$\lim^1 \check{H}_{R[K_j]}^{i-1}((x_1, \dots, x_j)) \cong \bigoplus_{F \in K} \bigoplus_{\substack{\alpha \leq 0 \\ \mathrm{supp} \alpha = F}} \lim^1 \check{H}^{i-|F|-2}(\mathrm{lk}_{K_j} F)$$

and since K is Cohen-Macaulay, $\check{H}^{i-|F|-1}(\mathrm{lk}_K F) = 0$. But there is an exact sequence

$$0 \rightarrow \lim^1 \check{H}^{i-|F|-2}(\mathrm{lk}_{K_j} F) \rightarrow \check{H}^{i-|F|-1}(\mathrm{lk}_K F) \rightarrow \lim \check{H}^{i-|F|-1}(\mathrm{lk}_{K_j} F) \rightarrow 0$$

so \lim and \lim^1 are also zero. \square

6.2.1 The complexes \mathcal{J}_n and \mathcal{H}_n

We now consider the complexes \mathcal{J}_n and \mathcal{H}_n described in section 4.5. It is shown in [10] that the complex \mathcal{J}_n is Cohen-Macaulay for all n . We shall show that \mathcal{H}_n is also Cohen-Macaulay. We first set up some new machinery:

Definition 6.2.12. Let K be a simplicial complex on vertex set V . A *projective covering* of K is a non-degenerate simplicial map $\pi: K' \rightarrow K$ which is maximal in the following sense. First, for any vertex $v \in V$, $\pi^{-1}(v)$ is non-empty; and second, $v_1 \dots v_k$ is a simplex of K' if and only if $\pi(v_1) \dots \pi(v_k)$ is a $(k-1)$ -simplex of K .

It is clear that K' is determined up to isomorphism by the cardinality of the sets $\pi^{-1}(v)$ for each $v \in V$. We shall refer to K' as a *projective cover* of K . For example, for $r \in \mathbf{N}_{>}$ we may define the *r -fold cover* of K , denoted rK , to be the projective cover with $|\pi^{-1}(v)| = r$ for all v . It is easy to check that the $r(sK) = (rs)K$.

Example 6.2.13. Let K be the boundary of the triangle $\partial\Delta^2$, with vertex set $V = \{v_1, v_2, v_3\}$ and 1-simplices $\{v_1v_2, v_2v_3, v_1v_3\}$. An r -fold cover of K has vertex set $V' = \{v_1, \dots, v_{3r}\}$ and 1-simplices $\{v_iv_j \mid i \not\equiv j \pmod{3}\}$

Proposition 6.2.14. *A projective cover of a Cohen-Macaulay complex is Cohen-Macaulay.*

Proof. We proceed by induction on n , observing that the case $n = 0$ is trivial. So let $\pi: K' \rightarrow K$ be a projective cover with K Cohen-Macaulay of dimension $n > 0$.

First, for any simplex $\sigma^i \in K'$, $0 \leq i \leq (n-1)$, it is clear that $\text{lk}_{K'}\sigma$ is a projective cover of $\text{lk}_K\pi(\sigma)$. So by induction $\text{lk}_{K'}\sigma$ is Cohen-Macaulay and hence $(n-i-2)$ -connected as required.

Now we show that $\tilde{H}_k(K') = 0$ for $k < n$, so that K' is $(n-1)$ -connected. Let c be a k -cycle representing a homology class $[c] \in \tilde{H}_k(K')$. If c contains at most one vertex in the set $\pi^{-1}(v)$ for each vertex v of K , then we may choose a section $s: K \rightarrow K'$ such that $s \circ \pi(c) = c$. But then $[c] = s_*\pi_*[c] = 0$, because $\tilde{H}_k(K) = 0$. So we must show that any k -cycle c is homologous to a cycle c' containing at most one vertex in the set $\pi^{-1}(v)$ for each $v \in K$.

In fact we shall choose any section $s: K \rightarrow K'$ and show that any k -cycle in K' is homologous to a cycle containing only vertices from the set $s(V)$.

Let c' be the cycle obtained by replacing every occurrence of the vertex w in c by $\bar{w} = s \circ \pi(w)$. Now if c was of the form $\sum \lambda_i w w_{i_1} \dots w_{i_k}$ plus terms not involving w , then $c - c' = \sum \lambda_i (w w_{i_1} \dots w_{i_k} - \bar{w} w_{i_1} \dots w_{i_k})$. We will construct a $(k + 1)$ -cycle whose boundary is $c - c'$.

Note that $\sum \lambda_i w_{i_1} \dots w_{i_k}$ is a $(k - 1)$ -cycle in $\text{lk } w = \text{lk } \bar{w}$, since

$$\delta \sum \lambda_i w w_{i_1} \dots w_{i_k} = \sum \lambda_i w_{i_1} \dots w_{i_k}$$

plus terms involving w , and the terms involving w must sum to zero because c is a cycle. But $\text{lk } w$ is a projective cover of the link of $\pi(w)$ in K , which is Cohen-Macaulay. So $\text{lk } w$ is Cohen-Macaulay by induction on n and hence $(k - 1)$ -connected. Choose a chain $c'' = \sum \mu_i w_{i_1} \dots w_{i_{k+1}}$ in $\text{lk } w$ whose boundary is $\sum \lambda_i w_{i_1} \dots w_{i_k}$. Then $C = \sum \mu_i (\bar{w} w_{i_1} \dots w_{i_{k+1}} - w w_{i_1} \dots w_{i_{k+1}})$ has boundary $c - c'$.

So $\tilde{H}_k(K') = 0$ for $k < n$ and K' is $(n - 1)$ -connected, hence Cohen-Macaulay. \square

The result for \mathcal{H}_n now follows immediately:

Corollary 6.2.15. \mathcal{H}_n is Cohen-Macaulay for all n .

Proof. The map $\mathcal{H}_n \rightarrow \mathcal{J}_n$ that sends a primitive vector in \mathbf{Z}^n to the line it generates is a 2-fold projective covering, and \mathcal{J}_n is Cohen-Macaulay [10]. \square

6.3 Davis-Januszkiewicz space

We return to toric topology by reviewing the definition and properties of the Davis-Januszkiewicz space $DJ(K)$ introduced in [10], and extending it to the countably infinite case.

Following section 6.5 of [6], we begin with the space $BT^V = ET^V/T^V$, where ET^V is a contractible space with a free T^V action. The quotient BT^V is the classifying space for principle T^V bundles, which is homotopy equivalent to the product of $|V|$ copies of $\mathbf{C}P^\infty$ in the finite case. A similar statement is true in the case that V is countably infinite: $(S^\infty)^{(V)}$ is contractible and admits a free action of T^V , with orbit space $(\mathbf{C}P^\infty)^{(V)}$.

In both the finite case (as in [6]) and the infinite case, we define the space $B_{T^V} X = (ET^V \times X)/T^V$ for any T^V -space X , which is often denoted $ET^V \times_{T^V} X$.

This is the classical *Borel construction*. Projection onto the first factor of $ET^V \times X$ yields a fibre bundle $X \rightarrow B_{T^V} X \rightarrow BT^V$ with fibre X . On the other hand, there is a fibre bundle $T^V \rightarrow ET^V \times X \rightarrow B_{T^V} X$ with total space homotopy equivalent to X . We will be particularly interested in $B_{T^V} \mathcal{Z}(K)$ for the T^V -space $\mathcal{Z}(K)$ and $B_{T^n} \mathcal{X}(K, \ell)$ for the T^n -space $\mathcal{X}(K, \ell)$.

For finite K , the *Davis-Januszkiewicz space* is defined as $DJ(K) = (\mathbf{C}P^\infty, *)^K$, a subspace of BT^V , and we will take this as the definition of $DJ(K)$ in the infinite case also. In the finite case, it is shown in [6] that $B_{T^V} \mathcal{Z}(K)$ and $DJ(K)$ are homotopy equivalent. Specifically it is shown that there is a deformation retract $B_{T^V} \mathcal{Z}(K) \rightarrow DJ(K)$ which makes the following diagram commute:

$$\begin{array}{ccc} B_{T^V} \mathcal{Z}(K) & \xrightarrow{p} & BT^V \\ \downarrow & & \parallel \\ DJ(K) & \xrightarrow{i} & BT^V \end{array}$$

Here p is the projection described above and i is the inclusion of the subspace $DJ(K)$ in BT^V . Moreover, the retraction is functorial and so extends to the case that V is countably infinite, by passing to colimits of SUB(K) diagrams.

It is clear, at least in the finite case, that $H^*(DJ(K); R) = R[K]$, the Stanley-Reisner ring of K with generators $\{x_v \mid v \in V\}$ in degree two corresponding to the coordinate subtori of T^V . Then the inclusion $i: DJ(K) \rightarrow BT^V$ and projection $p: B_{T^V} \mathcal{Z}(K) \rightarrow BT^V$ induce the quotient map

$$i^*, p^*: R[x_1, \dots, x_m] \rightarrow R[x_1, \dots, x_m]/I_K = R[K]$$

in cohomology [6]. Moreover, given an inclusion $\iota: K_1 \rightarrow K_2$ the induced map

$$DJ(\iota)^*: H^*(DJ(K_2); R) \rightarrow H^*(DJ(K_1); R)$$

is the map of Stanley-Reisner rings $\iota^*: R[K_2] \rightarrow R[K_1]$.

In the case that K is finite dimensional and countably infinite, we may take any ordering of the vertices of K to obtain an increasing sequence of complexes

$$\cdots \subset K_i \subset K_{i+1} \subset \cdots$$

yielding an increasing sequence of spaces

$$\cdots \subset DJ(K_i) \subset DJ(K_{i+1}) \subset \cdots$$

of which $DJ(K)$ is the colimit. By Theorem 6.1.20 we have the exact sequence

$$0 \rightarrow \lim^1 H^{q+1}(DJ(K_i)) \rightarrow H^q(DJ(K)) \rightarrow \lim H^q(DJ(K_i)) \rightarrow 0$$

and since each map $H^q(DJ(K_i)) \rightarrow H^q(DJ(K_{i-1}))$ is surjective we have

$$H^q(DJ(K)) \cong \lim H^q(DJ(K_i)) = \lim R[K_i] = R[K]$$

6.3.1 Quasitoric spaces

Given a simplicial complex K with dicharacteristic ℓ we have the homomorphism $\Theta_\ell: T^V \rightarrow T^n$ with kernel κ and may choose a section $s: T^n \rightarrow T^V$ such that the composite $\Theta_\ell \circ s$ is the identity. Then, following section 4 of [10], we have a series of homotopy equivalences $B_{T^V} \mathcal{Z}(K) \sim (ET^V \times \mathcal{Z}(K))/T^V \sim (ET^V \times (\mathcal{Z}(K)/\kappa))/(T^V/\kappa) \sim (ET^V \times \mathcal{X}(K, \ell))/(T^V/\kappa)$ since κ acts freely on both components. Then we can identify T^V/κ with T^n via the section $s: T^n \rightarrow T^V$ which gives a free T^n action on ET^V and we have $(ET^V \times \mathcal{X}(K, \ell))/(T^V/\kappa) \sim (ET^n \times \mathcal{X}(K, \ell))/T^n \sim B_{T^n} \mathcal{X}(K, \ell)$, since ET^V is a contractible space with free T^n action, and therefore a valid choice of ET^n . Thus we have an explicit homotopy equivalence $B_{T^V} \mathcal{Z}(K) \sim B_{T^n} \mathcal{X}(K, \ell)$, a special case of the classical result $B_G X \sim B_{(G/H)}(X/H)$ for a G -space X where $H \triangleleft G$ acts freely.

We will deviate from the approach of [10], who carry out their cohomology calculations by considering the fibration

$$\mathcal{X}(K, \ell) \rightarrow B_{T^n} \mathcal{X}(K, \ell) \rightarrow BT^n.$$

We will instead use the fibrations

$$T^V \rightarrow ET^V \times \mathcal{Z}(K) \rightarrow B_{T^V} \mathcal{Z}(K)$$

and

$$T^n \rightarrow ET^n \times \mathcal{X}(K, \ell) \rightarrow B_{T^n} \mathcal{X}(K, \ell).$$

To this end, we note that the principal bundles $T^V \rightarrow ET^V \times \mathcal{Z}(K) \rightarrow B_{T^V} \mathcal{Z}(K)$

and $T^n \rightarrow ET^n \times \mathcal{X}(K, \ell) \rightarrow B_{T^n} \mathcal{X}(K, \ell)$ fit into a homotopy commutative diagram

$$\begin{array}{ccccc}
 T^V & \xrightarrow{\Theta_\ell} & T^n & & \\
 \downarrow & & \downarrow & & \\
 ET^V \times \mathcal{Z}(K) & \xrightarrow{/\kappa} & ET^n \times \mathcal{X}(K, \ell) & & \\
 \downarrow & & \downarrow & & \\
 B_{T^V} \mathcal{Z}(K) & \xrightarrow{\sim} & B_{T^n} \mathcal{X}(K, \ell) & \xrightarrow{\sim} & DJ(K)
 \end{array}$$

and projection onto the first factor gives the diagram

$$\begin{array}{ccccccc}
 & & T^V & \xrightarrow{\Theta_\ell} & T^n & & \\
 & \swarrow & \downarrow & & \downarrow & \searrow & \\
 ET^V \times \mathcal{Z}(K) & \longrightarrow & ET^V & \xrightarrow{/\kappa} & ET^n & \longleftarrow & ET^n \times \mathcal{X}(K, \ell) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 B_{T^V} \mathcal{Z}(K) & \longrightarrow & BT^V & \xrightarrow{B\ell} & BT^n & \longleftarrow & B_{T^n} \mathcal{X}(K, \ell) \\
 & \searrow & \downarrow & \nearrow & \downarrow & \swarrow & \\
 & & DJ(K) & & & &
 \end{array}$$

$\begin{array}{ccc} r & i & B\ell \circ i \\ & \nearrow & \searrow \\ & & DJ(K) \end{array}$

where the bottom maps r are deformation retracts.

6.3.2 Vector bundles over Davis-Januszkiewicz space

Given any G -space X , the construction $B_G X = EG \times_G X$ is the classical *Borel construction*, and the cohomology of $B_G X$ is the *Borel cohomology* or *G -equivariant cohomology* of X , $H_G^*(X)$. So we have seen that $H_{T^n}^*(\mathcal{X}(K, \ell); R) \cong H_{T^V}^*(\mathcal{Z}(K); R) \cong R[K]$ for any ring R .

Now given a G -equivariant (real or complex) vector bundle ξ over X , we may construct a vector bundle over $B_G X$ by first taking the G -equivariant bundle $EG \times \xi$ over $EG \times X$, then dividing out the action of G to yield the (non-equivariant) bundle $B_G \xi = EG \times_G \xi$.

The original equivariant bundle over X can be recovered up to equivariant isomorphism by pulling back along the composite

$$X \xrightarrow{i} EG \times X \xrightarrow{/G} B_G X \tag{6.3}$$

so it is clear that two equivariant bundles ξ_1 and ξ_2 are equivariantly isomorphic if and only if the bundles $B_G\xi_1$ and $B_G\xi_2$ are isomorphic. This is well known.

The following is used implicitly in [10]; we give a full explanation here:

In the case of $\mathcal{X}(K, \ell)$, recall that the v -th line bundle ζ_v over $\mathcal{X}(K, \ell)$ is obtained from the v -th coordinate bundle $\tilde{\zeta}_v = \mathcal{Z}(K) \times \mathbf{C}_v$ (the restriction to $\mathcal{Z}(K)$ of the v -th coordinate of the tangent bundle $\tau(\mathcal{U}(K)) = \mathcal{U}(K) \times \mathbf{C}^V$) by dividing out the action of the torus κ . We have the homotopy commutative diagram:

$$\begin{array}{ccc}
 \mathcal{Z}(K) & \longrightarrow & ET^V \times \mathcal{Z}(K) \\
 \downarrow / \kappa & & \downarrow / \kappa \\
 \mathcal{X}(K, \ell) & \longrightarrow & ET^n \times \mathcal{X}(K, \ell) \\
 & & \downarrow / T^n \\
 & & B_{T^n} \mathcal{X}(K, \ell) \sim B_{T^V} \mathcal{Z}(K)
 \end{array}$$

covered by a diagram of complex line bundles (by 6.3):

$$\begin{array}{ccc}
 \tilde{\zeta}_v & \longrightarrow & ET^V \times \tilde{\zeta}_v \\
 \downarrow / \kappa & & \downarrow / \kappa \\
 \zeta_v & \longrightarrow & ET^n \times \zeta_v \\
 & & \downarrow / T^n \\
 & & B_{T^n} \zeta_v \sim B_{T^V} \tilde{\zeta}_v
 \end{array}$$

Now the projection $B_{T^V} \mathcal{Z}(K) \rightarrow BT^V$ is covered by a bundle map $B_{T^V} \tilde{\zeta}_v \rightarrow ET^V \times_{T^V} \mathbf{C}_v$, so the first Chern class of $B_{T^V} \tilde{\zeta}_v$ is $c_1(B_{T^V} \tilde{\zeta}_v) = x_v \in H^*(B_{T^V} \mathcal{Z}(K)) = H^*(DJ(K))$. The first Chern class of ζ_v will be the image $i^* \pi^*(x_v) \in H^*(\mathcal{X}(K, \ell))$, where we have the maps

$$\mathcal{X}(K, \ell) \xrightarrow{i} ET^n \times \mathcal{X}(K, \ell) \xrightarrow{\pi} B_{T^n} X \sim B_{T^V} \mathcal{Z}(K) \sim DJ(K)$$

Recall that the T^n manifold without stably complex structure underlying $\mathcal{Y}(K, \ell)$ and depending only on the undirected characteristic function $\bar{\ell}$ is denoted $\bar{\mathcal{Y}}(K, \bar{\ell})$. As an immediate application of the above ideas, we prove the following:

Proposition 6.3.1. *Let K be a simplicial complex with undirected characteristic function $\bar{\ell}$. Different choices of dicharacteristic ℓ compatible with $\bar{\ell}$ for K*

yield non-equivalent equivariant stably complex structures on $\overline{\mathcal{Y}}(K, \bar{\ell})$, and on $\overline{M}^{2n}(K, \bar{\ell})$ when K is dual to P^n .

Proof. In the notation of section 3.6, the T^n equivariant diffeomorphism $\check{g}: \mathcal{Y}(K, \ell_1) \rightarrow \mathcal{Y}(K, \ell_2)$ is covered by a bundle maps $\xi_v \rightarrow \zeta_v$ or $\xi_v \rightarrow \bar{\zeta}_v$ according to whether $\ell_1(v) = \pm \ell_2(v)$, where $\xi = \oplus_v \xi_v$ and $\zeta = \oplus_v \zeta_v$ are the stably complex structures on $\mathcal{Y}(K, \ell_1)$ and $\mathcal{Y}(K, \ell_2)$ respectively. The bundles ξ and ζ are equivariantly stably isomorphic over $\overline{\mathcal{Y}}(K, \bar{\ell})$ if and only if the bundles $B_{T^n}\xi$ and $B_{T^n}\zeta$ are stably isomorphic over $DJ(K)$, but in the cohomology of $B_{T^n}\mathcal{Y}(K, \ell_1)$ we have $c_1(B_{T^n}\xi) = \sum_v x_v$ while $c_1(B_{T^n}\check{g}^*(\zeta)) = \sum_v \pm x_v$ and these are equal only if the dicharacteristics are identical. \square

Note that for two equivariant stably complex structures on $\overline{\mathcal{Y}}(K, \bar{\ell})$ to be equivariantly isomorphic we require an equivariant stable bundle map covering the identity map on $\overline{\mathcal{Y}}(K, \bar{\ell})$. There are many examples where different choices of ℓ yield stably complex T^n manifolds $\mathcal{Y}(K, \ell_1)$ and $\mathcal{Y}(K, \ell_2)$ which admit an equivariant stable bundle map covering an equivariant diffeomorphism $\mathcal{Y}(K, \ell_1) \rightarrow \mathcal{Y}(K, \ell_2)$ that is *not* equivariantly homotopic to the identity. We present one here:

Example 6.3.2. Consider the two dicharacteristics on the square given by labelling the edges, in order:

$$\ell_1: (1, 0), (0, 1), (-1, 0), (0, -1)$$

$$\ell_2: (-1, 0), (0, -1), (1, 0), (0, 1)$$

There is an evident equivariant diffeomorphism $\mathcal{Y}(K, \ell_1) \rightarrow \mathcal{Y}(K, \ell_2)$ covering a rotation of the square through 180 degrees, which is covered by an equivariant bundle map. In other words, $\mathcal{Y}(K, \ell_1)$ and $\mathcal{Y}(K, \ell_2)$ are isomorphic in the category of stably complex T^n manifolds. However, the equivariant stably complex structures on the underlying T^n manifold $\overline{\mathcal{Y}}(K, \bar{\ell})$ are *not* isomorphic, that is, $\mathcal{Y}(K, \ell_1)$ and $\mathcal{Y}(K, \ell_2)$ are not isomorphic over $\overline{\mathcal{Y}}(K, \bar{\ell})$.

6.4 Cohomology calculations

In this section we calculate the integral cohomology of quasitoric spaces $\mathcal{X}(K, \ell)$ for finite and countably infinite complexes K with dicharacteristic ℓ , of fixed

dimension n . We make repeated use of the Leray-Serre spectral sequence in cohomology. A comprehensive explanation is given in [15], see also [19]. In all the cases we consider, the base space will have trivial fundamental group.

Davis and Januszkiewicz calculated the rational cohomology of quasitoric spaces over finite simplicial complexes in [10] by considering the Leray-Serre spectral sequence for the fibration

$$\mathcal{X}(K, \ell) \rightarrow B_{T^n} \mathcal{X}(K, \ell) \rightarrow BT^n.$$

In our calculations, the fibrations

$$T^V \rightarrow ET^V \times \mathcal{Z}(K) \rightarrow B_{T^V} \mathcal{Z}(K)$$

and

$$T^n \rightarrow ET^n \times \mathcal{X}(K, \ell) \rightarrow B_{T^n} \mathcal{X}(K, \ell)$$

will instead play the lead role.

We begin by considering the Leray-Serre spectral sequence for the fibration $T^n \rightarrow ET^n \rightarrow BT^n$. The E_2 page is $H^*(BT^n) \otimes H^*(T^n)$. For each coordinate torus T_i , $i = 1, \dots, n$ there is a map of fibrations

$$h_i: \begin{array}{ccc} T_i & \longrightarrow & T^n \\ \downarrow & & \downarrow \\ ET_i & \longrightarrow & ET^n \\ \downarrow & & \downarrow \\ BT_i & \longrightarrow & BT^n \end{array}$$

Let $(h_i)^*$ be the induced map of E_2 . Then if τ_j is the generator of $H^*(T^n)$ representing the j -th coordinate subtorus taken with positive orientation, lying in $E_2^{0,1}$ for the fibration $T^n \rightarrow ET^n \rightarrow BT^n$, then $(h_i)^*(\tau_j)$ is zero for $i \neq j$ and equals the generator τ'_i of $H^*(T_i)$ when $i = j$. Let $t_j = d_2 \tau_j$, then t_j is the generator of $H^*(BT^n)$, lying in $E_2^{2,0}$, such that $(h_i)^*(t_j)$ is zero for $i \neq j$ and $(h_i)^*(t_i) = t'_i = d_2 \tau'_i$, the generator of BT_i when $i = j$.

It is straightforward to check that the spectral sequence for $T^n \rightarrow ET^n \rightarrow BT^n$ collapses at the E_3 page, that is, $E_3 = E_\infty$ and $E_3^{0,0} = \mathbf{Z}$ is the only non-zero term. Indeed, the E_2 page is the Koszul complex of $H^*(BT^n) = \mathbf{Z}[t_1, \dots, t_n]$ with respect to the sequence $\mathbf{t} = t_1, \dots, t_n$. Here $\mathbf{Z}[t_1, \dots, t_n]$ is a graded ring with

generators in degree 2 and $E_2^{p,q}$ is $[K_q(\mathbf{t})]_p$, the p -th graded component of $K_q(\mathbf{t})$. On the E_3 page we therefore have $E_3^{p,q} = [H_q(\mathbf{t})]_p$ which is zero for $q \neq 0$.

We now consider the diagram

$$f: \begin{array}{ccc} T^V & \xrightarrow{\Theta_\ell} & T^n \\ \downarrow & & \downarrow \\ ET^V & \xrightarrow{/\kappa} & ET^n \\ \downarrow & & \downarrow \\ BT^V & \xrightarrow{B\ell} & BT^n \end{array} \quad (6.4)$$

from the previous section, where V may be finite or countably infinite. Recall that, given the dicharacteristic ℓ on K , $\lambda(v)$ is defined by identifying the oriented subtorus $\ell(v) \leq T^n$ with the map $T^1 \hookrightarrow T^n$ given by $t \mapsto (t^{\lambda_1(v)}, \dots, t^{\lambda_n(v)})$. Alternatively, $\lambda_i(v) = L_{i,v}$ where the integer matrix L corresponds to the linear map $\mathbf{R}^V \rightarrow \mathbf{R}^n$ which induces $\Theta_\ell: T^V \rightarrow T^n$ under the identification $T^V = \mathbf{R}^V/\mathbf{Z}^V$, $T^n = \mathbf{R}^n/\mathbf{Z}^n$.

The following statement appears in [10]:

Lemma 6.4.1. *Let t_1, \dots, t_n be the generators of $H^*(BT^n)$ as defined above and let $\{x_v \mid v \in V\}$ be generators of $H^*(BT^V)$ defined similarly, corresponding to the positively oriented coordinate subtori. Then the map $(B\ell)^*: H^*(BT^n) \rightarrow H^*(BT^V)$ takes t_i to $\sum_{v \in V} \lambda_i(v)x_v$.*

Proof. Denote by f^* the map of E_2 pages of the spectral sequences for the map of fibrations 6.4. Let τ_v be the generator of $H^*(T^V)$ which represents the v -th coordinate subtorus taken with positive orientation, so that $d_2(\tau_v) = x_v$. It is easy to see that $f^*(\tau_i) = \Theta_\ell^*(\tau_i) = \sum_{v \in V} \lambda_i(v)\tau_v$, so $(B\ell)^*(t_i) = f^*(t_i) = d_2f^*(\tau_i) = \sum_{v \in V} \lambda_i(v)x_v$. \square

Recalling that the inclusion $i: DJ(K) \hookrightarrow BT^V = DJ(\Delta(V))$ induces the quotient map $\mathbf{Z}[x_v] \rightarrow \mathbf{Z}[K] = \mathbf{Z}[x_v]/I_K$ we have

Corollary 6.4.2. *The map $i^*(B\ell)^*: H^*(BT^n) \rightarrow H^*(DJ(K))$ sends t_i to $\theta_i = \sum_{v \in V} \lambda_i(v)x_v \in \mathbf{Z}[K]$.*

We will require the following identification of the radical $\sqrt{(\theta)}$ for the sequence $\theta = \theta_1, \dots, \theta_n$, which we prove here:

Lemma 6.4.3. *For any dicharacteristic pair (K^{n-1}, ℓ) and any ring R we have $\sqrt{(\theta)} = (x_v \mid v \in V)$ in $R[K]$.*

Proof. We shall show that any monomial of degree $n + 1$ in the generators $\{x_v\}$ is in (θ) . The result will follow because $(x_v \mid v \in V)$ is clearly its own radical. We proceed by induction on n .

For the case $n = 1$, K^0 is a disjoint union of vertices and ℓ assigns either 1 or -1 to each. So $\theta_1 = \sum_{v \in V} \pm x_v$. Then $x_v^2 = \pm x_v \theta_1$ for all v , all products $x_v x_w$ for $v \neq w$ being trivial.

Now given $K = K^{n-1}$ for $n > 1$, let v be an arbitrary vertex of K , and x the corresponding generator of $R[K]$. We may apply any invertible linear transformation to $\theta_1, \dots, \theta_n$ without changing the ideal (θ) , and $\ell(v)$ is a primitive vector in \mathbf{Z}^n so there exists some invertible (over \mathbf{Z}) $(n \times n)$ matrix \mathbf{A} with $\mathbf{A}\ell(v) = (0, \dots, 0, 1)^T$. So without loss of generality we may assume $\ell(v) = (0, \dots, 0, 1)^T$, that is, the coefficient of x in θ_i is 0 for $i < n$, and 1 for $i = n$. Now consider $\text{lk } v$, which admits a dicharacteristic ℓ' of degree $n - 1$ obtained by deleting the n -th coordinate of ℓ . The sequence θ' corresponding to ℓ' is the image of $\theta_1, \dots, \theta_{n-1}$ in $R[\text{lk } v]$. By induction, any monomial $y = x_1^{\alpha_1} \dots x_r^{\alpha_r}$ of degree n in $R[\text{lk } v]$ lies in $(\theta_1, \dots, \theta_{n-1})R[\text{lk } v]$. Hence the monomial $xx_1^{\alpha_1} \dots x_r^{\alpha_r}$ lies in $(\theta_1, \dots, \theta_{n-1})R[K]$.

We have shown that any monomial of degree $n + 1$ in $R[K]$ in which the index of x is one lies in $(\theta_1, \dots, \theta_{n-1})$, hence in (θ) . We complete the proof by induction on the index of x . For given $x^\alpha x_1^{\alpha_1} \dots x_r^{\alpha_r} \in R[K]$, we may write $x = \theta_n - (\text{other generators})$, in which case $x^\alpha x_1^{\alpha_1} \dots x_r^{\alpha_r} = x^{\alpha-1} x_1^{\alpha_1} \dots x_r^{\alpha_r} \theta_n - (\text{monomials with } x \text{ in index } \alpha - 1)$. So $x^\alpha x_1^{\alpha_1} \dots x_r^{\alpha_r} \in (\theta)$ by induction on α . □

To perform the cohomology calculations, we will make use of the homotopy commutative diagram

$$\begin{array}{ccc}
 T^n & & \\
 \downarrow & \searrow & \\
 ET^n \times \mathcal{X}(K, \ell) & \xrightarrow{f} & ET^n \\
 \downarrow & & \downarrow \\
 B_{T^n} \mathcal{X}(K, \ell) & \longrightarrow & BT^n \\
 \sim \downarrow r & \nearrow B\ell \circ i & \\
 DJ(K) & &
 \end{array}$$

where the horizontal maps give a map of fibrations f induced by projection onto the first factor of $ET^n \times \mathcal{X}(K, \ell)$. Recall that $B_{T^n} \mathcal{X}(K, \ell) = B_{T^V} \mathcal{Z}(K)$ which deformation retracts onto $DJ(K)$. We will abuse notation mildly by denoting $r^*(x_v)$ by x_v and $r^*(\theta_i)$ by θ_i . Let f^* denote the induced map of Leray-Serre spectral sequences. The differentials on the E_2 page of the spectral sequence for $T^n \rightarrow ET^n \times \mathcal{X}(K, \ell) \rightarrow B_{T^n} \mathcal{X}(K, \ell)$ are determined by $d_2(t_i) = \theta_i$, $d_2(x_v) = 0$, and so the E_2 page is the Koszul complex of $\mathbf{Z}[K]$ with respect to the sequence $\theta = \theta_1, \dots, \theta_n$.

We give our result for the finite case first:

Proposition 6.4.4. *If K is a finite Cohen-Macaulay complex, then $H^*(\mathcal{X}(K, \ell)) \cong \mathbf{Z}[K]/(\theta)$.*

Proof. Since $\sqrt{(\theta)} = \sqrt{(x_v)}$ we have $\check{H}^i(\theta) = \check{H}^i(\mathbf{x}) = 0$ for $i < n$, the latter holding because K is Cohen-Macaulay. Therefore $H_{n-i}(\theta) = H^i(\theta) = 0$ for $i < n$ by proposition 6.1.18, and the only non-zero entries on the E_3 page are on the horizontal axis. Hence $E_\infty = E_3$ and $H^*(\mathcal{X}(K, \ell)) = H_0(\theta) = \mathbf{Z}[K]/(\theta)$. \square

Without using the Čech cohomology, one could prove 6.4.4 by the argument that since $\mathbf{Z}[K]$ is Cohen-Macaulay, $\text{depth}(\theta) = \text{ht}(\theta) = n$ so $H_{n-i}(\theta) = 0$ for $i < n$.

If K is countably infinite, $\mathbf{Z}[K]$ is non-Noetherian so the Cohen-Macaulay property cannot be used directly. The sequence $\mathbf{x} = \{x_v \mid v \in V\}$ is also infinite, so we cannot use $\check{H}^i(\mathbf{x})$. However, we may proceed by using a limit argument to obtain:

Proposition 6.4.5. *If K is a countably infinite Cohen-Macaulay complex, then $H^*(\mathcal{X}(K, \ell)) \cong \mathbf{Z}[K]/(\theta)$.*

Proof. We shall show that $\check{H}^i(\theta) = 0$ for $i < n$; we may then proceed as above.

Choose any ordering of the vertices of K , and denote by K_j the restriction of K to vertex set $V_j = \{v_1, \dots, v_j\}$. The inclusions $K_j \hookrightarrow K_{j+1}$ induce homomorphisms $\varphi_j: \mathbf{Z}[K_{j+1}] \rightarrow \mathbf{Z}[K_j] = \mathbf{Z}[K_{j+1}]/(x_{j+1})$ and we have $\mathbf{Z}[K] = \lim \mathbf{Z}[K_j]$. Denote by φ'_j the homomorphism $\varphi'_j: \mathbf{Z}[K] \rightarrow \mathbf{Z}[K_j]$. There are commutative diagrams

$$\begin{array}{ccc} \check{H}^*(\varphi'_{j+1}(\theta)) & \longrightarrow & \check{H}^*(\varphi'_j(\theta)) \\ \downarrow \cong & & \downarrow \cong \\ \check{H}^*((x_1, \dots, x_{j+1})) & \longrightarrow & \check{H}^*((x_1, \dots, x_j)) \end{array}$$

because the radicals of the ideals are equal by lemma 6.4.3.

Now the stable Koszul complex $K^*(\theta^\infty)$ for $\mathbf{Z}[K]$ is the limit of the tower of complexes $K^*(\varphi'_j(\theta)^\infty)$ and this tower satisfies the Mittag-Leffler condition because the homomorphisms $\mathbf{Z}[K_{j+1}] \rightarrow \mathbf{Z}[K_j]$ and $\mathbf{Z}[K_{j+1}]_\theta \rightarrow \mathbf{Z}[K_j]_\theta$ are surjective. Therefore there is an exact sequence

$$0 \rightarrow \lim^1 \check{H}^{i-1}(\varphi'_j(\theta)) \rightarrow \check{H}^i(\theta) \rightarrow \lim \check{H}^i(\varphi'_j(\theta)) \rightarrow 0$$

But the outer terms are isomorphic to $\lim^1 \check{H}^{i-1}(x_1, \dots, x_j)$ and $\lim \check{H}^i(x_1, \dots, x_j)$ respectively, which are both zero for $i < n$ by corollary 6.2.11 because K is Cohen-Macaulay. □

Since the first Chern class of ζ_v , the v -th facial line bundle over $\mathcal{X}(K, \ell)$ is the image of x_v in $H^*(\mathcal{X}(K, \ell))$ we have proved

Theorem 6.4.6. *If K is a finite or countably infinite Cohen-Macaulay complex of dimension $(n - 1)$, with dicharacteristic ℓ of degree n , then*

$$H^*(\mathcal{X}(K, \ell)) \cong \mathbf{Z}[K]/(\theta)$$

where the generator x_v corresponding to vertex v of K is the first Chern class of the complex line bundle ζ_v .

Corollary 6.4.7. *The same is true for the open quasitoric space $\mathcal{Y}(K, \ell)$.*

In particular, we have calculated the cohomology of the final space \mathcal{QT}_n :

Theorem 6.4.8. *The cohomology of \mathcal{QT}_n is given by*

$$H^*(\mathcal{QT}_n) = \mathbf{Z}[\mathcal{H}_n]/(\theta_1, \dots, \theta_n)$$

where $\mathbf{Z}[\mathcal{H}_n]$ is the Stanley Reisner ring of \mathcal{H}_n with generators $\{w_h \mid h \in H_n\}$, where H_n is the set of primitive vectors in \mathbf{Z}^n ; and the linear term θ_i is the sum $\theta_i = \sum_h h^i w_h$ where h^i is the i -th coordinate of h . Moreover, $w_h = c_1(\xi_h)$ where ξ_h is the complex line bundle over \mathcal{QT}_n corresponding to vertex h of \mathcal{H}_n .

Chapter 7

Calculations in complex cobordism

We know from [8] that every complex cobordism class in dimension greater than two contains a stably complex quasitoric manifold. The key to the theorem is replacing the Milnor hypersurfaces $H_{i,j}$, which form a multiplicative generating set for Ω_*^U along with the complex projective spaces $\mathbf{C}P^j$, with quasitoric manifolds $B_{i,j}$. Each $B_{i,j}$ is a smooth $\mathbf{C}P^{j-1}$ bundle over the *bounded flag manifold* B_i . Then $B_{i,j}$ is a quasitoric manifold over $I^i \times \Delta^{j-1}$, while each $\mathbf{C}P^j$ is a quasitoric manifold over Δ^j .

In this chapter we address a related problem: Given two combinatorial pairs (P_1^n, ℓ_1) and (P_2^n, ℓ_2) , can we determine whether $M^{2n}(P_1^n, \ell_1)$ and $M^{2n}(P_2^n, \ell_2)$ are complex cobordant? Since the complex cobordism class of a stably complex manifold is determined by its Chern numbers, the problem may be reformulated: Given a combinatorial pair (P^n, ℓ) , calculate the Chern numbers of $M^{2n}(P^n, \ell)$.

7.1 Chern numbers

For any positive integer n , a *partition* of n is a sequence of positive integers i_1, \dots, i_r , up to order, such that $\sum_{k=1}^r i_k = n$. For example, the sequences 1, 1, 2 and 1, 2, 1 constitute the same partition of 4, whereas 2, 2 is a different partition.

For any partition $p = i_1, \dots, i_r$ of n , the characteristic class c_p is defined as the product $\prod_{k=1}^r c_{i_k}$ of Chern classes (see e.g. [21]). Then the p -th *Chern number* $c_p(M^{2n})$ of a stably complex manifold M^{2n} is the value of the Kronecker product $\langle c_p(\zeta), \mu \rangle$ where $\mu \in H_{2n}(M; \mathbf{Z})$ is the fundamental class of M and $\tau(M) \oplus$

$\mathbf{R}^{2(m-n)} \cong \zeta^m$ is the stably complex structure.

It is a classical result (see e.g. [24], [21]) that the Chern numbers $c_p(M^{2n})$ as p runs over partitions of n form a complete set of invariants for complex cobordism. That is, the complex cobordism class of M^{2n} is determined by its Chern numbers (whereas any stably complex manifold of odd dimension is null-cobordant).

When M^{2n} is a quasitoric manifold, we may write explicit formulae for the Chern classes c_k in terms of the first Chern classes of the facial bundles, $x_v = c_1(\zeta_v)$, which generate the cohomology ring $H^*(M^{2n})$. The complex vector bundle ζ^m is the sum of line bundles $\bigoplus_{v \in V} \zeta_v$ and so the total Chern class is $c(\zeta) = \prod_v (1 + x_v)$ (see [6], section 5.4). This yields

$$c_k(\zeta) = \sum x_1 \dots x_k \quad (7.1)$$

where the sum is over all square-free monomials of degree k in the generators $\{x_v \mid v \in V\}$. Now a product $x_{v_1} \dots x_{v_k}$ will be zero unless $v_1 \dots v_k$ is a simplex of K , the simplicial complex dual to P^n , so we may rewrite the expression as

$$c_k(\zeta) = \sum_{\sigma^{(k-1)} \in K} x_\sigma$$

where $x_\sigma = x_{v_1} \dots x_{v_k}$ for $\sigma = v_1 \dots v_k$.

Example 7.1.1. Let P^3 be the cube I^3 , with facets F_1, \dots, F_6 numbered as on a die, so that F_i is opposite F_j when $i + j = 7$. The corresponding vertices of the dual K_P are v_1, \dots, v_6 . Then for any quasitoric manifold M^6 over I^3 we have generators $x_i = x_{v_i} = c_1(\zeta_i)$ for $H^*(M)$, where $i = 1, \dots, 6$. We have

$$\begin{aligned} c_1(\zeta) &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ c_2(\zeta) &= x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 \\ &\quad + x_2x_3 + x_2x_4 + x_2x_6 + x_3x_5 \\ &\quad + x_3x_6 + x_4x_5 + x_5x_6 + x_5x_6 \\ c_3(\zeta) &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_5 + x_1x_4x_5 \\ &\quad + x_2x_3x_6 + x_2x_4x_6 + x_3x_5x_6 + x_4x_5x_6 \end{aligned}$$

7.1.1 The sign of a vertex

Returning to the general case, for $M^{2n}(P^n, \ell)$ we can write the n -th Chern number as

$$c_n(M^{2n}) = \sum_{\sigma^{(n-1)} \in K} \langle x_\sigma, \mu \rangle$$

Now each $\sigma^{(n-1)} \in K$ is dual to a vertex σ^* of P , and we will show that $\langle x^\sigma, \mu \rangle = \text{sign } \sigma = \text{sign } \sigma^*$ where $\text{sign } \sigma^* = \pm 1$ is defined as follows (see [6], section 5.4.1):

The vertex $v = \sigma^*$ can be expressed as the intersection of n facets $F_1 \cap \dots \cap F_n$ of P , and for each facet F_k there is a unique edge E_k such that $F_k \cap E_k = v$. Let e_k be a vector along E_k with direction away from v ¹. We may reorder the F_k so that the ordered basis e_1, \dots, e_n agrees with the orientation of P^n ; that is, if the orientation of P^n is induced by inclusion in \mathbf{R}^n , then e_1, \dots, e_n is a positively oriented basis for \mathbf{R}^n .

On the other hand, the vectors $\lambda(F_1), \dots, \lambda(F_n)$ form an ordered basis for \mathbf{Z}^n which is either positively or negatively oriented. We may take $\text{sign } v = \det(\lambda(F_1), \dots, \lambda(F_n)) = \pm 1$.

The sign may also be understood geometrically. The tangent space $\tau_v(M^{2n})$ is the direct sum of the facial bundles ζ_k , and so the orientations of the facial bundles induce an orientation on $\tau_v(M^{2n})$. This may or may not agree with the chosen orientation on M^{2n} , and $\text{sign } v$ is $+1$ or -1 accordingly.

We will give one more interpretation. We may choose chart U containing v which is θ -equivariantly homeomorphic to a neighbourhood of zero in \mathbf{C}^n with the standard T^n action, for some automorphism θ of T^n . If we insist that the homeomorphism is orientation preserving, then $\text{sign } v = +1$ if and only if θ is also orientation preserving.

As noted in [7], the following is the motivating property of the sign:

Proposition 7.1.2. *The Kronecker product $\langle x_\sigma, \mu \rangle$ is given by $\langle x_\sigma, \mu \rangle = \text{sign } \sigma$.*

Proof. Recall that $x_\sigma = x_1 \dots x_n$. Now each x_i is Poincaré dual to the homology class given by the inclusion of the facial submanifold $M_i^{2(n-1)}$ ([6]), with orientation induced by the orientation of M^{2n} and of the normal bundle $\nu(M_i^{2(n-1)}) = \zeta_i|_{M_i}$. Thus the product x_σ is Poincaré dual to the transverse intersection $M_1 \cap \dots \cap M_n$ which is given by the inclusion of the point $v = \sigma^*$,

¹Alternatively we may take e_k to be an inward pointing normal vector to F_k .

with orientation induced by the orientation of M^{2n} and of the normal bundle $\nu(v) = \bigoplus_i \zeta_i|_v$. So x_σ is equal to $\pm\mu^*$ according to whether the orientations of the facial bundles ζ_i agree with the orientation of M^{2n} . \square

We immediately have the following (see e.g. (5.15) in [6]):

Corollary 7.1.3. *The n -th Chern number is given by $c_n(M^{2n}) = \sum_{\sigma \in K} \text{sign } \sigma$.*

Example 7.1.4. Let P^2 be a hexagon, oriented by inclusion in \mathbf{R}^2 with facets F_1, \dots, F_6 in anticlockwise order. Define a dicharacteristic ℓ by:

$$\begin{aligned} \lambda(F_1) = \lambda(F_4) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda(F_2) = \lambda(F_5) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda(F_3) &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} \text{ and } \lambda(F_6) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Then the sign at each vertex is $\det(\lambda(F_i), \lambda(F_{i+1}))$ and there are four positive and two negative vertices. Hence $c_2(M^4(P^2, \ell)) = 2$.

7.1.2 Calculations in $H^*(M^{2n})$

The sign of a vertex is easy to compute. Proposition 7.1.2 therefore shows that if we can express the characteristic classes $c_p(\zeta)$ in terms of the monomials $x_{v_1} \dots x_{v_n} = x_{\sigma^{(n-1)}}$, we can evaluate $c_p(M^{2n})$ and hence the complex cobordism invariants of M^{2n} .

We therefore turn to the task of expressing $c_p(\zeta)$ for a partition $p = i_1, \dots, i_k$ as a sum of terms $x_{\sigma^{(n-1)}}$. From the definition of c_p above, it may be written explicitly as a sum of monomials

$$c_p(\zeta) = \sum x_{\sigma_1} x_{\sigma_2} \dots x_{\sigma_r}$$

where each $\sigma_k = v_1 \dots v_{i_k}$ is an $(i_k - 1)$ -simplex of K , and we take the sum over all possible choices of $\sigma_1, \dots, \sigma_r$ with $|\sigma_k| = i_k - 1$. In fact we may as well write

$$c_p = \sum_{\{U_k \subset V: |U_k|=i_k\}} x_{U_1} \dots x_{U_r} \tag{7.2}$$

with $x_{U_k} = \prod_{v \in U_k} x_v$, taking the sum over all choices of (not necessarily disjoint) subsets $U_1, \dots, U_r \subset V$, since the product x_{U_k} will be zero when U_k is not a simplex of K .

Representations of Chern classes

Let $x_1^{a_1} \dots x_s^{a_s}$ be a monomial where $v_1 \dots v_s$ are distinct vertices of K , $x_i = x_{v_i}$ and $a_1 + \dots + a_s = n$. The number of occurrences of $x_1^{a_1} \dots x_s^{a_s}$ in (7.2) depends only on the a_i . Indeed, as described in [21], there exist polynomials s_I for each partition $I = a_1, \dots, a_s$ of n satisfying $s_I(c_1, \dots, c_n) = \sum x_1^{a_1} \dots x_s^{a_s}$ where the sum is over all distinct monomials with the form $t_1^{a_1} \dots t_s^{a_s}$ and $c_i = \sum x_1 \dots x_i$ as in (7.1). For example,

$$\begin{aligned} s_1(c_1) &= \sum x_1 = c_1 \\ s_2(c_1, c_2) &= \sum x_1^2 = c_1^2 - 2c_2 \\ s_{1,1}(c_1, c_2) &= \sum x_1 x_2 = c_2 \end{aligned}$$

The $s_I(c_1, \dots, c_n)$ form an alternative basis for the \mathbf{Z} -module generated by $\{c_p\}$ for partitions p of n , so to express each c_p as a sum of $x_{\sigma(n-1)}$ it is sufficient to express each sum $\sum x_1^{a_1} \dots x_s^{a_s}$ in such terms. We may then recover expressions for the Chern numbers, for example:

$$\begin{aligned} c_1 &= s_1 \\ c_{1,1} &= s_2 + 2s_{1,1} \\ c_2 &= s_{1,1} \end{aligned}$$

In [1], Adams gives an alternative formulation. He defines the classes c_α , as α runs through sequences of non-negative integers of which only finitely many are non-zero, as follows. Recall that the Chern classes are elements of the cohomology ring $H^*(BU)$. The cohomology of $BU(1) \cong \mathbf{C}P^\infty$ is the polynomial ring on one generator x , so $H^*(BU(1))$ has an additive basis $1, x, x^2, \dots$. We may denote the dual basis of $H_*(BU(1))$ by b_0, b_1, b_2, \dots so that b_i is dual to x^i . We have the inclusion $BU(1) \hookrightarrow BU$ and there is a multiplication $BU \times BU \rightarrow BU$ given by the Whitney sum map; this yields a multiplication on $H_*(BU)$. Then $H_*(BU)$ has an additive basis consisting of monomials

$$b_1^{\alpha_1} b_2^{\alpha_2} \dots$$

as well as the multiplicative identity b_0 . The dual basis of $H^*(BU)$ consists of the elements $c_\alpha = c_{(\alpha_1, \alpha_2, \dots)}$.

Adams' basis for $H^*(BU)$ is closely related to the basis $\{s_I\}$: If $I = a_1, \dots, a_s$, a partition of n , then $s_I = c_\alpha$ where α_i is the number of occurrences of the integer i among the terms a_1, \dots, a_s .

We will perform our calculations using the classes s_I , but a translation to Adams' basis is straightforward.

Expression in terms of square free monomials

Define the *length* of a monomial $x_1^{a_1} \dots x_s^{a_s}$ to be the integer s . Then since a_1, \dots, a_n is a partition of n , the length is equal to n precisely when $a_1 = \dots = a_n = 1$, in other words when the monomial is square free. A square free monomial $x_1 \dots x_n \in H^*(M^{2n})$ is either equal to zero or $x_{\sigma^{(n-1)}}$ for some $\sigma^{(n-1)} \in K$. We prove:

Lemma 7.1.5. *If $x = x_1^{a_1} \dots x_s^{a_s}$ is a degree n monomial in $H^{2n}(M^{2n}(P^n, \ell))$ of length $s < n$ then x may be expressed as a linear combination of monomials of length $s + 1$.*

Proof. If $v_1 \dots v_s$ is not a simplex of K , then $x = 0$. Otherwise, there exist some v_{s+1}, \dots, v_n such that $v_1 \dots v_n$ is an $(n-1)$ -simplex, in which case $\lambda(v_1), \dots, \lambda(v_n)$ is a basis for \mathbf{Z}^n . Denote by A the $n \times n$ matrix with columns $(\lambda(v_1), \dots, \lambda(v_n))$.

The linear relations among the generators x_v , $v \in K$ may be expressed by the matrix equation $Lx = 0$ for $n \times V$ matrix L and vector $x = (x_v \mid v \in K)$, where the v -th column of L is the vector $\lambda(v) \in \mathbf{Z}^n$. We may premultiply by A^{-1} to obtain

$$A^{-1}Lx = 0 \tag{7.3}$$

where now the v_i -th column of $A^{-1}L$ is the standard basis vector e_i , for $i = 1, \dots, n$. Choose any $j \in \{1, \dots, s\}$ such that $a_j > 1$. The j -th row of (7.3) gives an expression $\sum_v \lambda_v x_v = 0$ where $\lambda_{v_j} = 1$ and $\lambda_{v_i} = 0$ for $i = 1, \dots, n$, $i \neq j$. This gives the expression $x_j = \sum_{v \neq v_1, \dots, v_n} (-\lambda_v) x_v$ and hence $x = \sum_{v \neq v_1, \dots, v_n} (-\lambda_v) x_1^{a_1} \dots x_j^{a_j-1} \dots x_s^{a_s} x_v$, a sum of monomials of length $s + 1$. \square

By induction on $(n - s)$ we have:

Corollary 7.1.6. *Every sum $\sum x_1^{a_1} \dots x_s^{a_s}$ may be expressed as a linear combination of square free monomials $\sum_{\sigma^{(n-1)} \in K} \lambda_\sigma x_\sigma$.*

Corollary 7.1.6 essentially gives an algorithm for expressing the terms $s_I(c_1, \dots, c_n)$, and hence the classes $c_p(\zeta)$, as linear combinations of square free monomials in the generators x_v , $v \in K$. The following example illustrates the method.

Example 7.1.7. The complex manifold $\mathbf{C}P^3$ is a toric manifold over the simplex Δ^3 , which we may consider as a quasitoric manifold. The standard stably complex structure is induced by a dicharacteristic taking values

$$\lambda(v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda(v_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda(v_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda(v_4) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

with the facets numbered in such a manner that the sign at each vertex is $+1$. We have linear relations

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \quad (7.4)$$

It is immediately clear in this case that we have $x_1 = x_2 = x_3 = x_4$. Nevertheless, it is instructive to proceed systematically. Let us calculate the Chern numbers. First,

$$\begin{aligned} c_3(\mathbf{C}P^3) &= \langle x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \mu \rangle \\ &= \text{sign } v_1v_2v_3 + \text{sign } v_1v_2v_4 + \text{sign } v_1v_3v_4 + \text{sign } v_2v_3v_4 \\ &= +4 \end{aligned}$$

For $c_{1,2}$ we have $c_1c_2 = s_{1,2} + 3s_{1,1,1} = s_{1,2} + 3c_3$, so we calculate

$$\begin{aligned} s_{1,2}(\zeta) &= x_1^2x_2 + x_1^2x_3 + x_1^2x_4 \\ &+ x_2^2x_1 + x_2^2x_3 + x_2^2x_4 \\ &+ x_3^2x_1 + x_3^2x_2 + x_3^2x_4 \\ &+ x_4^2x_1 + x_4^2x_2 + x_4^2x_3 \end{aligned}$$

Now we may rewrite $x_1^2x_2 = x_1x_2x_4$ since the first row of (7.4) gives $x_1 = x_4$. It is less trivial to rewrite $x_1^2x_4$ as a square free monomial, because no individual row gives an expression for x_1 that does not involve x_4 . However, we note that $v_1v_2v_4$ is a simplex of K and set

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Multiplying (7.4) by A^{-1} yields

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

so we obtain $x_1 = x_3$ and $x_1^2x_4 = x_1x_3x_4$. In fact we could obtain these relations by subtracting the third row from the first row of (7.4). In practice it will often be easier to use elementary row operations rather than to calculate A^{-1} explicitly.

Continuing in this vein we arrive at

$$s_{1,2}(\zeta) = x_1x_2x_4 + x_1x_3x_4 + x_1x_3x_4 \quad (7.5)$$

$$+ x_1x_2x_4 + x_2x_3x_4 + x_2x_3x_4 \quad (7.6)$$

$$+ x_1x_3x_4 + x_2x_3x_4 + x_2x_3x_4 \quad (7.7)$$

$$+ x_1x_3x_4 + x_2x_3x_4 + x_2x_3x_4 \quad (7.8)$$

Counting the number of occurrences of each monomial, we see

$$\begin{aligned} s_{1,2}(\mathbf{CP}^3) &= 0 \operatorname{sign} v_1v_2v_3 + 2 \operatorname{sign} v_1v_2v_4 + 4 \operatorname{sign} v_1v_3v_4 + 6 \operatorname{sign} v_2v_3v_4 \\ &= +12 \end{aligned}$$

So $c_{1,2}(\mathbf{CP}^3) = 12 + 3 \times 4 = 24$. To calculate $c_{1,1,1}$ we need

$$s_3(\zeta) = x_1^3 + x_2^3 + x_3^3 + x_4^3$$

We can write, for example, $x_1^3 = x_1^2x_4 = x_1x_3x_4$ and obtain

$$\begin{aligned} s_3(\zeta) &= x_1x_3x_4 + x_1x_2x_4 + x_1x_3x_4 + x_1x_2x_4 \\ s_3(\mathbf{CP}^3) &= 2 \operatorname{sign} v_1v_2v_4 + 2 \operatorname{sign} v_1v_3v_4 \\ &= +4 \end{aligned}$$

and using $c_{1,1,1} = s_3 + 3s_{1,2} + 6s_{1,1,1}$ we have $c_{1,1,1}(\mathbf{CP}^3) = 4 + 3 \times 12 + 6 \times 4 = 64$.

Remark 7.1.8. As noted, the formula in (7.5) is by no means the only possible expression for $s_{1,2}(\zeta)$; in fact the linear relations yield $x_1 = x_2 = x_3 = x_4$. The procedure described in Lemma 7.1.5 is highly redundant, but is guaranteed to yield expressions for the Chern numbers in terms of square free monomials x_σ .

Remark 7.1.9. In general, \mathbf{CP}^n is a quasitoric manifold over the n -simplex with dicharacteristic taking values $e_1, \dots, e_n, e_1 + \dots + e_n$ and all vertices having positive sign. We will always have relations $x_1 = \dots = x_{n+1}$ in $H^*(\mathbf{CP}^n)$ so will be able to rewrite any monomial in the generators as a single square free monomial with unchanged coefficient.

Therefore, the Chern number $c_p(\mathbf{CP}^n)$ will be equal to the number of monomials in the expansion $c_p = c_{i_1} \dots c_{i_k}$ where $c_i = \sum x_1 \dots x_i$, the sum of all square free monomials of degree i in the generators. Thus we have the standard result:

$$c_p(\mathbf{CP}^n) = \prod_{i_k \in p} \binom{n+1}{i_k}$$

Example 7.1.10. Consider a quasitoric manifold M^4 over the square, with facets numbered anticlockwise and dicharacteristic $\ell(F_1) = (1, 0)$, $\ell(F_2) = (2, 1)$, $\ell(F_3) = (3, 2)$, $\ell(F_4) = (2, 1)$. We have

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

To calculate $c_{1,1}$ we need to evaluate

$$s_{1,1} = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

We need an expression for x_3 in terms of x_1, x_2, x_4 with integer coefficients, and we obtain this by subtracting the second row of the matrix from the first, to yield

$$x_1 + x_2 + x_3 + x_4 = 0$$

and we can now write

$$\begin{aligned} s_{1,1} &= x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ &= x_1(-2x_2 - 3x_3 - 2x_4) + x_2(-2x_3 - x_4) + \\ &\quad x_3(-x_1 - x_2 - x_4) + x_4(-x_2 - 2x_3) \\ &= -2x_1x_2 - 4x_1x_3 - 2x_1x_4 - 3x_2x_3 - 2x_2x_4 - 3x_3x_4 \\ &= -2x_1x_2 - 2x_1x_4 - 3x_2x_3 - 3x_3x_4 \\ s_{1,1}(M^4) &= -2 \operatorname{sign} v_1v_2 - 2 \operatorname{sign} v_1v_4 - 3 \operatorname{sign} v_2v_3 - 3 \operatorname{sign} v_3v_4 \\ &= -2(+1) - 2(-1) - 3(+1) - 3(-1) \\ &= 0 \end{aligned}$$

What's more

$$\begin{aligned} s_2(M^4) &= \operatorname{sign} v_1v_2 + \operatorname{sign} v_2v_3 + \operatorname{sign} v_3v_4 + \operatorname{sign} v_1v_4 \\ &= 0 \end{aligned}$$

so M^4 is null-cobordant.

7.2 Calculation in \mathcal{QT}_n

In this section, we shall arrive at a combinatorial representation of the Chern numbers of $M^{2n}(P^n, \ell)$ in terms of the map $\ell_*: H_{n-1}(K) \rightarrow H_{n-1}(\mathcal{H}_n)$ in simplicial homology induced by the map $\ell: K \rightarrow \mathcal{H}_n$ defining the dicharacteristic ℓ .

We know, since the final map $\mathcal{Y}(\ell): M^{2n}(P^n, \ell) \rightarrow \mathcal{QT}_n$ is covered by a complex bundle map of the stable tangent bundles, that $c_p(M^{2n}) = (\mathcal{Y}(\ell))^*(c_p(\mathcal{QT}_n))$. It makes sense, therefore, to perform the calculations described in the previous section in $H^*(\mathcal{QT}_n)$ before pulling back to $H^*(M^{2n})$. We may avoid any concerns about the characteristic classes of an infinite dimensional vector bundle either by restricting attention to a finite dimensional submanifold of \mathcal{QT}_n corresponding

to a finite subcomplex of \mathcal{H}_n containing the image of ℓ (as in Chapter 5), or equivalently by working directly with the first Chern classes of the line bundles $w_h = c_1(\xi_h) \in H^2(\mathcal{QT}_n, \mathbf{Z})$. We shall take the latter approach here.

We therefore define the elements $c_i \in H^{2k}(\mathcal{QT}_n)$, $i = 1, \dots, n$ and $c_p \in H^{2n}(\mathcal{QT}_n)$ for p a partition of n as follows:

$$c_i = \sum_{\sigma^{(i-1)} \in \mathcal{H}_n} w_\sigma, \text{ where } w_\sigma = \prod_{h \in \sigma} w_h$$

$$c_p = \prod_{k=1}^r c_{i_k}, \text{ where } p = i_1, \dots, i_r$$

We will again make use of the polynomials s_I for each partition $I = a_1, \dots, a_s$ of n satisfying $s_I(c_1, \dots, c_n) = \sum w_1^{a_1} \dots w_s^{a_s}$, where the sum is now over all distinct monomials of the form $t_1^{a_1} \dots t_s^{a_s}$ in the generators $w_h \in H^2(\mathcal{QT}_n)$.

By theorem 6.4.6 and its corollary, the map $(\mathcal{Y}(\ell))^*: H^*(\mathcal{QT}_n) \rightarrow H^*(M^{2n})$ is characterized by

$$(\mathcal{Y}(\ell))^*(w_\sigma) = \sum_{\ell(\sigma')=\sigma} x_{\sigma'}$$

and so we have immediately:

Lemma 7.2.1. *For the stably complex structure ζ on M^{2n} :*

- $c_i(\zeta) = (\mathcal{Y}(\ell))^*(c_i)$, $i = 1, \dots, n$;
- $c_p(\zeta) = (\mathcal{Y}(\ell))^*(c_p)$ for each partition p of n ; and
- $s_I(\zeta) = (\mathcal{Y}(\ell))^*(s_I)$ for each partition I of n .

We will write each $s_I \in H^{2n}(\mathcal{QT}_n)$ as a linear combination of the square free monomials $w_{\sigma^{(n-1)}}$, as we did in $H^{2n}(M)$, but we must take care because we are working with infinite sums.

Proposition 7.2.2. *Every sum $\sum w_1^{a_1} \dots w_s^{a_s}$ may be expressed as a linear combination of square free monomials $\sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \lambda_\sigma w_\sigma$.*

Proof. As in $H^*(M^{2n})$, we may express any monomial of length $s < n$ as a linear combination of monomials of length $s + 1$. Explicitly, if $w = w_1^{a_1} \dots w_s^{a_s}$ is non-zero, then there exist some h_{s+1}, \dots, h_n such that $h_1 \dots h_n$ is an $(n - 1)$ simplex of \mathcal{H}_n , in which case h_1, \dots, h_n is a basis for \mathbf{Z}^n . We obtain a suitable expression

for w by multiplying the linear relations in $H^2(\mathcal{QT}_n)$ by A^{-1} , where A is the $n \times n$ matrix with columns (h_1, \dots, h_n) .

Now each length $s + 1$ monomial $(w'_1)^{a'_1} \dots (w'_{s+1})^{a'_{s+1}}$ can only occur with non-zero coefficient in the linear expression for a finite number of length s monomials $w_1^{a_1} \dots w_s^{a_s}$, since $\{w_1, \dots, w_s\}$ must be a subset of $\{w'_1, \dots, w'_{s+1}\}$. Therefore we obtain a well defined expression for any sum of length s monomials as a linear combination of length $s + 1$ monomials. By induction on $(n - s)$, the result is proved. \square

Let us choose some expression

$$s_I = \sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \lambda_\sigma^I w_\sigma \tag{7.9}$$

for each partition I of n , where $\{\lambda_\sigma^I\}$ are integer coefficients. Then it is clear from the formula for $(\mathcal{Y}(\ell))^*: H^*(\mathcal{QT}_n) \rightarrow H^*(M^{2n})$ that for any quasitoric manifold $M^{2n}(P^n, \ell)$,

$$s_I(M^{2n}) = \sum_{\sigma^{(n-1)} \in K} \lambda_{\ell(\sigma)}^I \text{sign } \sigma$$

We may express the classes c_p as linear combinations of the s_I , obtaining

$$c_p = \sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \chi_\sigma^p w_\sigma \tag{7.10}$$

$$c_p(M^{2n}) = \sum_{\sigma^{(n-1)} \in K} \chi_{\ell(\sigma)}^p \text{sign } \sigma \tag{7.11}$$

where now $\{\chi_\sigma^p\}$ are integer coefficients expressing c_p in terms of the w_σ .

7.2.1 Representation in simplicial homology

We can now give a description of the Chern numbers in terms of simplicial homology. The map $\ell: K \rightarrow \mathcal{H}_n$ induces a map of simplicial chain complexes $\ell_\#: C_*(K) \rightarrow C_*(\mathcal{H}_n)$, and we will restrict attention to the map $\ell_\#: C_{n-1}(K) \rightarrow C_{n-1}(\mathcal{H}_n)$. In order to describe this map in terms of the generators we must choose an orientation for each $(n - 1)$ -simplex σ^{n-1} of K and \mathcal{H}_n ; this is given by an equivalence class of orderings of the vertices of σ^{n-1} . We will use $[\sigma]$ to denote the simplex σ with its given orientation.

We may assume that P^n is oriented by a chosen realisation in \mathbf{R}^n . Then for

a simplex σ^{n-1} of K , we choose the equivalence class of orderings $[\sigma] = v_1 \dots v_n$ such that the ordered basis e_1, \dots, e_n is a positively oriented basis for \mathbf{R}^n , where e_k is an inward pointing normal vector to facet F_k of P^n . For example, if a polygon P^2 is oriented by inclusion in \mathbf{R}^2 , then a positive orientation for each 1-simplex $v_i v_j$ is obtained by taking the faces of P^2 in anticlockwise order.

For a simplex σ^{n-1} of \mathcal{H}_n , we choose $[\sigma] = h_1 \dots h_n$ whenever $\det(h_1, \dots, h_n) = +1$.

Now the simplicial map $\ell_\#: C_{n-1}(K) \rightarrow C_{n-1}(\mathcal{H}_n)$ is given by

$$\ell_\#([\sigma]) = \text{sign } \sigma \cdot [\ell(\sigma)]$$

and this map induces the map $\ell_*: H_{n-1}(K) \rightarrow H_{n-1}(\mathcal{H}_n)$ in simplicial homology. We may take expressions for the characteristic class c_p or s_I in $H^{2n}(\mathcal{QT}_n)$, as in (7.9) and (7.10):

$$\begin{aligned} c_p &= \sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \chi_\sigma^p w_\sigma \\ s_I &= \sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \lambda_\sigma^I w_\sigma \end{aligned}$$

We define the following cocycles in $H^{n-1}(\mathcal{H}_n)$:

$$\begin{aligned} \hat{c}_p &= \sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \chi_\sigma^p [\sigma]^* \\ \hat{s}_I &= \sum_{\sigma^{(n-1)} \in \mathcal{H}_n} \lambda_\sigma^I [\sigma]^* \end{aligned}$$

The orientation of P^n gives a fundamental class

$$\hat{\mu} = \sum_{\sigma^{n-1} \in K} [\sigma] \in H_{n-1}(K)$$

and we have

Theorem 7.2.3. *For a quasitoric manifold $M^{2n}(P^n, \ell)$, the p -th Chern number $c_p(M^{2n})$ is given by $c_p(M^{2n}) = \langle \hat{c}_p, \ell_*(\hat{\mu}) \rangle$. We also have $s_I(M^{2n}) = \langle \hat{s}_I, \ell_*(\hat{\mu}) \rangle$.*

Proof. We have

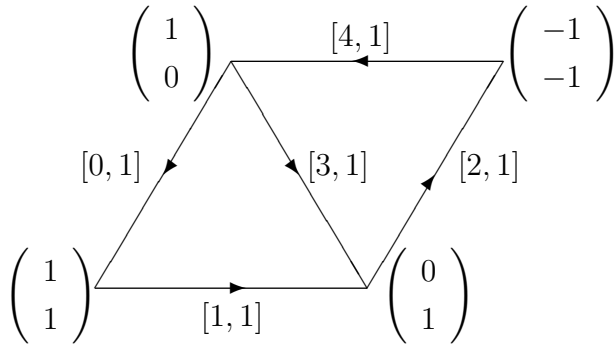
$$\begin{aligned}
 c_p(M^{2n}) &= \langle c_p(\zeta), \mu \rangle \\
 &= \sum_{\sigma^{n-1} \in K} \chi_{\ell(\sigma)}^p \text{sign } \sigma \\
 &= \sum_{\sigma^{n-1} \in K} \chi_{\ell(\sigma)}^p \langle [\ell(\sigma)]^*, \ell_{\#}([\sigma]) \rangle \\
 &= \sum_{\sigma^{n-1} \in K} \langle \hat{c}_p, \ell_{\#}([\sigma]) \rangle \\
 &= \langle \hat{c}_p, \ell_*(\hat{\mu}) \rangle
 \end{aligned}$$

and similarly for s_I . □

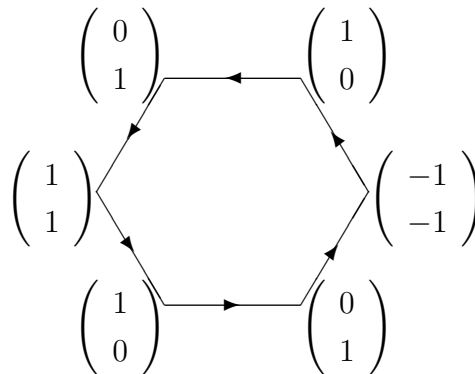
Example 7.2.4. The following diagram shows the restriction of \mathcal{H}_2 to the vertex set

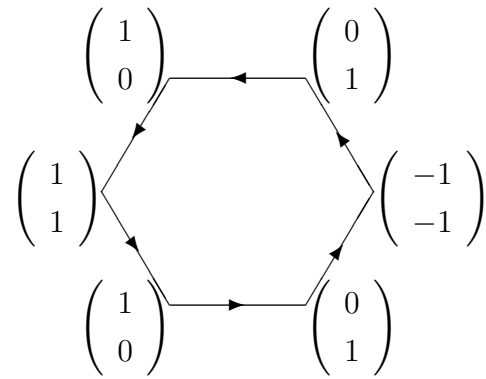
$$H' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$$

Arrows show the orientation of the 1-simplices, and the bracket $[\chi_{\sigma}^{1,1}, \chi_{\sigma}^2]$ attached to each 1-simplex shows the coefficient of that simplex in chosen representations of $\hat{c}_{1,1}$ and \hat{c}_2 respectively.



Now consider the following two dicharacteristics on the dual of the hexagon:





In the first case, we can calculate $[c_{1,1}, c_2](M^4(P^2, \ell)) = -[0, 1] + [3, 1] + [2, 1] + [4, 1] + [3, 1] - [1, 1] = [11, 2]$, while in the second case the image of the fundamental class of K in \mathcal{H}_2 is null-homologous, so $[c_{1,1}, c_2](M^4(P^2, \ell)) = [0, 0]$ and the stably complex manifold M^4 is null-cobordant.

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