NEUMANN PROBLEMS FOR SECOND ORDER ELLIPTIC OPERATORS WITH SINGULAR COEFFICIENTS

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Engineering and Physical Sciences

2012

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Contents

The University of Manchester

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Doctor of Philosophy
Neumann Problems for Second Order Elliptic Operators with Singular Coefficients
May 18, 2012

In this thesis, we prove the existence and uniqueness of the solution to a Neumann boundary problem for an elliptic differential operator with singular coefficients, and reveal the relationship between the solution to the partial differential equation (PDE in abbreviation) and the solution to a kind of backward stochastic differential equations (BSDE in abbreviation).

This study is motivated by the research on the Dirichlet problem for an elliptic operator ([42]). But it turns out that different methods are needed to deal with the reflecting diffusion on a bounded domain. For example, the integral with respect to the boundary local time, which is a nondecreasing process associated with the reflecting diffusion, needs to be estimated. This leads us to a detailed study of the reflecting diffusion. As a result, two-sided estimates on the heat kernels are established.

We introduce a new type of backward differential equations with infinity horizon and prove the existence and uniqueness of both L^2 and L^1 solutions of the BSDEs. In this thesis, we use the BSDE to solve the semilinear Neumann boundary problem. However, this research on the BSDEs has its independent interest.

Under certain conditions on both the "singular" coefficient of the elliptic operator and the "semilinear coefficient " in the deterministic differential equation, we find an explicit probabilistic solution to the Neumann problem, which supplies a L^2 solution of a BSDE with infinite horizon. We also show that, less restrictive conditions on the coefficients are needed if the solution to the Neumann boundary problem only provides a L^1 solution to the BSDE.

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Publications

[1] Weiguo Yang and Xue Yang, "A note on strong limit theorems for arbitrary stochastic sequences", Statist. Probab. Lett. 78 (2008).

[2] Xinfang Han, Li Ma and Xue Yang, "Perturbation of generalized Dirichlet forms by signed smooth measures and the associated Markov processes", (Chinese) Acta Math. Sci. Ser. A Chin. Ed. 30 (2010).

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[4] Xue Yang and T.S. Zhang, "The estimates of heat kernels with Neumann boundary conditions", (Accepted for publication by Potential Analysis)

Pre-publication articles:

[5] Xue Yang and T.S. Zhang, "A probabilistic approach to Neumann problems of semilinear elliptic PDEs with singular coefficients". (Submitted)

[6] Xue Yang and T.S. Zhang, "Absolute continuity of the law of solutions of SPDEs driven by white noise with coefficients depending on the past of the solutions ". (Preprint)

[7] Xue Yang and T.S. Zhang, "A probabilistic approach to Dirichlet problems with non-linear divergence term ". (Preprint)

Acknowledgements

First of all, I would like to express my most sincere thanks to my supervisor Prof. Tusheng Zhang for his help, guidance and patience throughout my three-year Ph.D study. Under his supervision, I have learned so much, not only the professional knowledge, but also the boundless dedication and braveness. His influence on all aspects of my work has been immeasurable.

Furthermore, I am very grateful to Prof. Zhi-ming Ma at the Chinese Academy of Sciences, for showing me the beauty of Mathematics and welcoming me into this research field. I also would like to thank Prof. Zhao Dong and Prof. Fuzhou Gong for their continuous help and encouragement over the years. I am grateful to Prof. Niels Jacob and Dr.Kees Van Schaik for reading this thesis so carefully and giving me so many helpful and inspiring suggestions.

I am indebted to my parents, Chenglin Yang and Furong Shi. It is their love, support and constant encouragement that give me the solid foundation upon which everything else can be built. I also would like to dedicate this thesis to my fiancé, Xinxin Shu, because of his enormous love, understanding and care. I am grateful to him for sharing my happiness and sadness even though we are living at the opposite sides of this world during the long time I have been studying in the United Kingdom.

I am fortunate to have joined the Marie Curie initial training network, not only for the financial support, but also for the precious chance to communicate with so many excellent scholars in my research fields. Last but not least, a big thank you goes to all of the friendly faculty and students at the school of mathematics. It is them who make the time at Manchester the most memorable in my life.

Dedication

To My Parents and Fiancé

Chapter 1

Introduction

1.1 Motivation and Contribution

This thesis is devoted to the study of the semilinear Neumann boundary problem for an elliptic differential operator with singular coefficients, and culminates in an explicit probabilistic solution of this problem. This study is motivated by the previous research on the reflecting diffusions and the Dirichlet problems for such kind of operators.

The theory of reflecting diffusion on a bounded domain plays an important role in this thesis. Reflecting Brownian motion (RBM in abbreviation) on a bounded domain has been studied in different ways. For example, in the view of Skorohod equations (see [3], [21]), RBM (X_t) on bounded domain D can be decomposed as a semimartingale

$$X_t = X_0 + B_t + \frac{1}{2}\int_0^t \vec{n}(X_s)\, dL_s,$$

where (B_t) is a standard Brownian motion, $\vec{n}(x)$ is the unit inward normal vector at $x \in \partial D$ and L_t is a continuous increasing process but increases only when $X_t \in \partial D$. Furthermore, considering the RBM in the framework of Dirichlet form, we know that

RBM is a diffusion process associated with the regular Dirichlet form:

$$\mathcal{E}(u,v)=\frac{1}{2}\sum_{i,j}\int_D \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx.$$

The generator of RBM is $G=\frac{1}{2}\triangle$ equipped with the Neumann boundary condition $\frac{\partial}{\partial\vec{n}}=0$ on ∂D.

Reflecting diffusion is a generalization of RBM, by adding the diffusion matrix $A(x)=(a_{ij}(x))$ and the drift term $\sum_i b_i\frac{\partial}{\partial x_i}$ (see [5],[26]). In general, the reflecting diffusion (X_t) has a decomposition in the following form:

$$X_t=X_0+\int_0^t\sigma(X_s)\,dB_s+\int_0^t b(X_s)\,ds+\frac{1}{2}\int_0^t A\vec{n}(X_s)\,dL_s,$$

where the matrix $\sigma(x)$ is the positive definite symmetric square root of the matrix $A(x)$. Inspired by the method in [3],[21], we know that estimates for the local time, from which the integrability of the semigroup is derived, are necessary for the probabilistic solution to a Neumann boundary problem. To this end, a detailed study on the reflecting diffusion is needed.

The operator we consider in the thesis is the following

$$\begin{aligned}L &= \frac{1}{2}\nabla\cdot(A\nabla)+B\cdot\nabla-div(\hat{B}\cdot)+q\\ &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{d}B_i(x)\frac{\partial}{\partial x_i}-\sum_{i=1}^{d}\frac{\partial\hat{B}_i\cdot}{\partial x_i}+q(x).\end{aligned}$$

L acts on the functions defined on a smooth bounded domain D and the mixed boundary condition

$$\frac{1}{2}<A\nabla u,\vec{n}>-<\hat{B}u,\vec{n}>=\Phi$$

on ∂D is required. The precise description of L is given in Section 3.1. Please note that in the following discussion, when we say a operator is defined on a domain D, it actually means that the operator acts on the functions defined on domain D.

The Dirichlet problem to an elliptic differential operator with singular coefficients

$$\begin{cases} Lu(x) = -F(x, u(x), \nabla u(x)), & \text{on } D \\ u(x) = \Phi(x) & \text{on } \partial D \end{cases} \tag{1.1}$$

has been studied ([7],[8] and [42]). The method dealing with the "bad" term $div(\hat{B}\cdot)$ called "time-reversal", which is the intrinsic motivation of our research, will be used in this thesis.

This thesis mainly studies the following three problems.

(1) Two-sided estimates for the heat kernels associated with the operator L equipped with mixed boundary condition.

Although there has been a great amount of literature on the estimates for heat kernels with Dirichlet boundary conditions (see [7] [33] [40], [41] and references therein), there is not so much work on estimates of heat kernels with Neumann boundary conditions. Here we mention three papers. Two-sided estimates of the heat kernel of reflecting Brownian motion ($A = I$, $B = \hat{B} = 0$) on Lipschitz domains are obtained in [3]. When the coefficients A and B are smooth and $\hat{B} = 0$, the heat kernels under mixed boundary conditions are constructed in [22] and [34], but the Gaussian bounds are not established for the heat kernel there. Using the estimates on heat kernels established by us, we get the integrability of the semigroup associated with the operator L.

(2) Existence and uniqueness of the solutions to BSDEs with infinite horizon.

Thanks to the development of the BSDEs in recent years, it is possible to represent the solutions of the nonlinear PDEs by the solutions of certain BSDEs associated with a diffusion process generated by some linear operator $\mathcal{A}$. The first result on a probabilistic interpretation for solutions of semilinear parabolic PDEs is obtained by Peng in [32] and subsequently in [31], in both of which the terminal time of the BSDE is finite. But in our situation, considering the reflecting diffusion, we have to solve the BSDEs with infinite horizon. The integrability of the solution to the BSDE becomes crucial and makes the problem much harder.

Since the term $\int_0^t A\vec{n}(X(s))dL_s$ is involved in the decomposition of reflecting diffusion process X, the BSDE, which we use to solve the nonlinear Neumann problem, also involves an integral with respect to the local time L_t. This is a new type of BSDE. The research on such a kind of BSDE has an independent interest.

(3) Probabilistic solution of the Neumann boundary problem associated with the operator L.

Based on the first two topics, we use probabilistic methods to solve the mixed boundary value problem for semilinear second order elliptic partial differential equations in the following form:

$$\begin{cases} Lu(x) = -F(x, u(x), \nabla u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) - \widehat{B} \cdot n(x)u(x) = \Phi(x) & \text{on } \partial D \end{cases} \tag{1.2}$$

Probabilistic approaches to boundary value problems of second order differential operators have been adopted by many authors and the earliest work went back as early as 1944 in [24]. So far there has been a lot of studies on the Dirichlet boundary problem (see [8],[16], [36] and [42]). However, there are not many articles on the probabilistic approach to the Neumann boundary problem. Here we only mention one reference. When $A = I$, $B = 0$ and $\hat{B} = 0$, the following Neumann boundary problem

$$\begin{cases} \frac{1}{2}\triangle u(x) + qu(x) = 0, & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial n}(x) = \phi(x) & \text{on } \partial D \end{cases}$$

is solved in [3], which gives the solution the following representation:

$$u(x) = E_x[\int_0^\infty e^{\int_0^t q(\bar{B}_u)du}\phi(\bar{B}_t)dL_t^0].$$

Here $(\bar{B}_t)_{t>0}$ defined on the probability space $(\Omega, E_x, x \in D)$ is the reflecting Brownian motion associated with the infinitesimal generator $G = \frac{1}{2}\triangle$, and L_t^0, $t > 0$ is the boundary local time satisfying $L_t^0 = \int_0^t I_{\partial D}(\bar{B}_s)dL_s^0$.

There are two essential difficulties in the third topic. One lies in the divergence term $div(\hat{B}\cdot)$ of the operator L. The term $div(\hat{B}\cdot)$ is hard to deal with because the divergence does not exist as $\hat{B}$ is only a measurable vector field. It should be interpreted in the distributional sense. Since $\hat{B}$ is not differentiable, the term $\nabla\cdot(\hat{B}\cdot)$ can not be handled by Girsanov transform or Feynman-Kac transform. Therefore, the "time reversal" method is used here. The other difficulty lies in the boundary local time in the decomposition of the reflecting diffusion. The method dealing with the boundary local time is inspired by the paper [19]. However the equation considered in [19] is linear

$$\begin{cases} (\frac{1}{2}\triangle - \nu)u(x) = 0 & \text{on } D \\ \frac{\partial u}{\partial n} = \phi & \text{on } \partial D, \end{cases}$$

and only a probabilistic interpretation of the solution to the Neumann problem is given.

In conclusion, the analysis of the reflecting diffusion in the first topic helps to build the BSDE of new type in the second topic. The two topics handled first are the basis of the third one. Note that, every topic has its own interest.

1.2 Structure of the Thesis

This thesis is organized in five chapters. The first chapter is a brief survey of the literature. We summarize the motivation and contribution of this thesis, and indicate the difficulties we meet in the course of our study as well.

Chapter 2 introduces the basic theories of Dirichlet form, backward stochastic differential equations and some inequalities used in the following chapters.

Chapter 3 provides both upper and lower bound estimates on the heat kernel of Gaussian type associated with operator L equipped with the Neumann boundary conditions.

Chapter 4 considers the existence and uniqueness of the solutions (Y, Z) to the following BSDE with infinite horizon:

$$\begin{aligned} Y_x(t) &= Y_x(T) + \int_t^T F(X(s), Y_x(s), Z_x(s))ds - \int_t^T e^{\int_0^s q(X(u))dt} \Phi(X(s))dL_s \\ &\quad - \int_t^T \langle Z_x(s), dM_x(s) \rangle, \quad for \quad t < T, \end{aligned}$$

where $M_x(t)$ is the martingale of the reflecting diffusion $X(t)$. Actually, both L^1 and L^2 solutions of the BSDE are obtained in this chapter. By using them, we get the solutions of the Neumann boundary problem, which require different conditions on the operator L.

Chapter 5 discusses a future work we are interested in. We want to consider that the divergence term could also be nonlinear, for example, in the form of $div(\hat{B}(x, u(x)))$.

Chapter 2

Background Theory

In this chapter, we recall some background material which will be used in the following chapters.

2.1 Regular Symmetric Dirichlet Forms

Let E be a locally compact separable Hausdorff space. m is a Radon measure with support on E. $L^2(E, m)$ denotes the space of functions defined on E square integrable with respect to the measure m. Let $(\cdot, \cdot)$ denote the inner product on $L^2(E, m)$.

Definition 2.1.1. *A family of linear bounded operators $\{T_t, t > 0\}$ with domain $\mathcal{D}(T_t) = L^2(E, m)$ is called a symmetric contraction semigroup, if the following conditions are satisfied:*
(1) (Symmetry) $\int_E T_t g(x) f(x) m(dx) = \int_E g(x) T_t f(x) m(dx)$, for any $t > 0$ and $f, g \in L^2(E, m)$;
(2) (Semigroup property) $T_t T_s = T_{t+s}$, $t, s > 0$;
(3) (Contraction property) The norm of the operator satisfies $\|T_t\| \leq 1$.
Moreover, $\{T_t, t > 0\}$ is said to be strongly continuous if
(4) $(T_t u - u, T_t u - u) \to 0$, as $t \to 0$, for $u \in L^2(E, m)$.

Definition 2.1.2. *A family of linear bounded operators $\{G_\alpha, \alpha > 0\}$ with domain*

$\mathcal{D}(G_\alpha) = L^2(E,m)$ *is called a symmetric contraction resolvent, if the following conditions are satisfied:*

(1) (Symmetry) $\int_E G_\alpha g(x)f(x)m(dx) = \int_E g(x)G_\alpha f(x)m(dx)$, *for any* $\alpha > 0$ *and* $f, g \in L^2(E,m)$*;*

(2) (Resolvent equation) $G_\alpha - G_\beta + (\alpha-\beta)G_\alpha G_\beta = 0$*;*

(3) (Contraction property) The norm of the operator satisfies $\|G_\alpha\| \le \alpha^{-1}$, *for any* $\alpha > 0$.

Moreover, $\{G_\alpha, \alpha > 0\}$ *is said to be strongly continuous if*

(4) $(\alpha G_\alpha u - u, \alpha G_\alpha u - u) \to 0$, *as* $\alpha \to \infty$, *for* $u \in L^2(E,m)$.

Definition 2.1.3. *For a strongly continuous semigroup* $\{T_t, t > 0\}$ *on* $L^2(E,m)$*, the operator* $(A, \mathcal{D}(A))$ *defined as follows,*

$$\begin{aligned} \mathcal{D}(A) &:= \{f \in L^2(E,m) | \lim_{t\to 0} \frac{T_t f - f}{t} \quad \textit{exits}\}, \\ Af &:= \lim_{t\to 0} \frac{T_t f - f}{t}, \quad \textit{for} \quad f \in \mathcal{D}(A), \end{aligned}$$

is called the generator of the semigroup.

The following theorem reveals the relationship between the generator, semigroup and resolvent.

Theorem 2.1.4. *([38]Hille-Yosida's Theorem)*

(1) For a strongly continuous semigroup $\{T_t, t > 0\}$ *on* $L^2(E,m)$*, define*

$$G_\alpha u = \int_0^\infty e^{-\alpha t} T_t u \, dt,$$

then $\{G_\alpha, \alpha > 0\}$ *is a strongly continuous resolvent and every* T_t*,* $t > 0$ *has the following representation:*

$$T_t := \lim_{\alpha\to\infty} e^{\alpha(\alpha G_\alpha - 1)t}.$$

(2) For the linear densely defined self-adjoint operator A *such that, for any* $\alpha > 0$*, the inverse operator* $(\alpha - A)^{-1}$ *exists and is a linear bounded operator on* $L^2(E,m)$

satisfying $\|\alpha(\alpha - A)^{-1}\| \leq 1$. *We define*

$$G_\alpha := (\alpha - A)^{-1}.$$

Then $\{G_\alpha, \alpha > 0\}$ *is a resolvent and the operator can be expressed as* $A = \alpha - G_\alpha^{-1}$.

Definition 2.1.5. *Suppose* $\mathcal{D}(\mathcal{E})$ *is a dense linear subset in* $L^2(E, m)$ *and* $\mathcal{E}(\cdot, \cdot)$ *is a symmetric bilinear form defined on* $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ *with values in* R. *Then* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is called a symmetric Dirichlet form if the following conditions are satisfied:*
(i) $\mathcal{D}(\mathcal{E})$ *is a Hilbert space equipped with the inner product* $\mathcal{E}_1(,) := \mathcal{E}(,) + (,)$.
(ii) (Markovian) For any $\varepsilon > 0$, *there exists a real function* $\phi_\varepsilon(t)$, $t \in R^1$, *satisfying*

$$\begin{aligned} &\phi_\varepsilon(t) = t, \quad \forall t \in [0,1], \quad -\varepsilon \leq \phi_\varepsilon(t) \leq \varepsilon + 1, \quad \forall t \in R^1, \quad \textit{and} \\ &0 \leq \phi_\varepsilon(t') - \phi_\varepsilon(t) \leq t' - t \quad \textit{whenever} \quad t < t', \end{aligned}$$

such that

$$u \in \mathcal{D}(\mathcal{E}) \Rightarrow \phi_\varepsilon(u) \in \mathcal{D}(\mathcal{E}), \quad \mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \leq \mathcal{E}(u, u).$$

Moreover, the measure m *is called the reference measure of the Dirichlet form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

The following theorem reveals the relationship between the Dirichlet form, generator and resolvent.

Theorem 2.1.6. *([15], [30])*
(1) Given a symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *defined on* $L^2(E, m)$, *we can define a corresponding non-positive self-adjoint operator in the following way: define*

$$\mathcal{D}(A) \;:=\; \{f \in \mathcal{D}(\mathcal{E}) : g \mapsto \mathcal{E}(f, g) \quad \textit{is a continuous linear functional on} \quad L^2(E, m)\},$$

then for every $f \in \mathcal{D}(A)$, *let* Af *denote the unique element in* $L^2(E, m)$ *such that* $(-Af, g) = \mathcal{E}(f, g)$ *for all* $g \in \mathcal{D}(\mathcal{E})$.

(2) Let $\{G_\alpha, \alpha > 0\}$ *be the resolvent which is associated with the operator* A *defined as in (1), then*

$$\mathcal{E}_\alpha(G_\alpha u, v) := \mathcal{E}(G_\alpha u, v) + \alpha(G_\alpha u, v) = (u, v).$$

Moreover, for $u \in L^2$, $u \in \mathcal{D}(\mathcal{E})$ *if and only if* $\lim_{\alpha\to\infty} \alpha(u - \alpha G_\alpha u, u)$ *exists. In this case,*

$$\mathcal{E}(u, u) = \lim_{\alpha\to\infty} \alpha(u - \alpha G_\alpha u, u).$$

The Dirichlet forms are also strongly related to a class of Markov processes, so that it is possible to apply the analytic theories to deal with the stochastic processes.

A Markov process $(\Omega, \mathcal{F}, X_t, \mathcal{F}_t, P_x)$ with the state space E is called a Hunt process if $\{X_t\}_{t\geq 0}$ is strong Markovian and quasi-continuous with respect to the $\sigma-$filtration $\{\mathcal{F}_t\}_{t\geq 0}$. θ_t and γ_t are the shift and reverse operators on Ω respectively, defined by

$$\begin{aligned} X_s(\theta_t(\omega)) &= X_{t+s}(\omega), s, t \geq 0, \\ X_s(\gamma_t(\omega)) &= X_{t-s}(\omega), s \leq t. \end{aligned}$$

Set $\mathcal{B}_b := \{\text{Borel measurable and bounded function on} \quad E\}$.

Denote by p_t the Markov transition function, i.e. $p_t f(x) = E_x[f(X_t)]$, for any $f \in \mathcal{B}_b$. Then $\{X_t\}_{t\geq 0}$ is called m-symmetric if

$$(u, p_t v) = (p_t u, v),$$

for any $u, v \in \mathcal{B}_b$.

Theorem 2.1.7. *([15])*
Every symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *with reference measure* m *is associated with a m-symmetric Hunt process* $(\Omega, \mathcal{F})$*, in the sense that* $p_t f = T_t f$ $m-a.e.$*, for* $f \in \mathcal{B}_b \bigcap L^2(E, m)$.

In the rest of the section, we introduce several kinds of functionals associated with the Hunt process and Dirichlet form.

The reference measure m now is not "fine" enough as long as the Hunt processes are considered, so that a kind of Choquet capacity is used to describe the "small" set. For an open set $G \in E$, the capacity of G is defined as:

$$Cap(G) = \inf\{\mathcal{E}_1(u,u) | \quad u \in \mathcal{D}(\mathcal{E}), u \geq 1 \quad \text{on} \quad G\}$$

and $Cap(G) = \infty$ if G is an empty set.

For any subset $B \subset E$, the set function:

$$Cap(B) = \inf\{Cap(G) | \quad G \supset B \quad \text{is a open set}\}$$

can be proved to be a Choquet capacity (in [15]).

A set N, which is a Borel set (i.e. $N \in \mathcal{B}(E)$), is called an exceptional set if $Cap(N) = 0$. It is proved in [15] that, N is an exceptional set if and only if $P_x(h(N) < \infty) = 0$, $m - a.e.$, where $h(N)$ is the hitting time for the set N of the process $\{X_t\}_{t>0}$, that is, $h(N) := \inf\{t \geq 0 | \quad X_t \in N\}$.

A function f is said to be quasi-continuous if for any $\varepsilon > 0$ there is an open subset $G \subset E$ with $Cap(G) < \varepsilon$ such that the restriction of f on $E - G$, $f|_{E-G}$ is continuous. It is proved in [15] that every function $h \in \mathcal{D}(\mathcal{E})$ has a quasi-continuous version, denoted by $\tilde{h}$.

Definition 2.1.8. *An extended real valued process $\{A_t\}_{t\geq 0}$ defined on Ω is an additive functional (AF in abbreviation) if the following conditions are satisfied:*

(1) A_t is $\mathcal{F}_t$-measurable for any $t > 0$;

(2) there exists a set $\Lambda \in \mathcal{F}_\infty := \bigcup_t \mathcal{F}_t$ and an exceptional set $N \subset D$ with $Cap(N) = 0$ such that $P_x(\Lambda) = 1$ for $x \in E \cap N^c$ and $\theta_t \Lambda \in \Lambda$ for all $t \geq 0$. Moreover, for any $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous and has left limit in $t \in [0, \infty]$ with $A_0(\omega) = 0$ and $|A_t(\omega)| < \infty$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$.

If $t \mapsto A_t(\omega)$ is positive and continuous, then $\{A_t\}_{t\geq 0}$ is a positive continuous *additive functional (PCAF in abbreviation).*

In the following discussion, we use one capital letter A to denote the process $\{A_t\}_{t\geq 0}$ for convenience.

Definition 2.1.9. *A positive Borel measure ν is smooth if the following conditions are satisfied:*
(1) $\nu(N) = 0$ if $Cap(N) = 0$;
(2) there exists an increasing sequence $\{F_n\}$ of closed sets satisfying $\lim_{n\to\infty} \nu(K - F_n) = 0$ for any compact set K, such that $\nu(F_n) < \infty$ and $\nu(E - \cup_n F_n) = 0$.

It is proved (in [15]) that there is a one-to-one correspondence between the set of smooth measures and the set of PCAFs:

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}(A_t) := \lim_{t\downarrow 0} \frac{1}{t} \int_E h(x) E_x(A_t) m(dx) = \int_E \tilde{h}(x)\nu(dx), \tag{2.1}$$

for any positive function $h \in \mathcal{D}(\mathcal{E})$.

Define

$$\begin{aligned} \mathcal{M} : &= \{M| \quad M \quad \text{is an AF with exceptional set N}, \forall t > 0, \quad E_x M_t^2 < \infty, \\ &\qquad E_x M_t = 0, \quad for \quad x \in E - N\}. \end{aligned}$$

$\mathcal{M}$ is called the set of martingale additive functionals (MAF in abbreviation). For $M \in \mathcal{M}$, we define

$$e(M) := \sup_{t>0} \frac{1}{2t} E_m M_t^2 (\leq \infty).$$

$e(M)$ is called the energy of M. A MAF M is said to have finite energy if $M \in \overset{\circ}{\mathcal{M}} := \{M \in \mathcal{M}| \quad e(M) < \infty\}$.

By the definition of MAF, it is known that $\{M_t, \mathcal{F}_t, P_x\}_{t\geq 0}$ is a square integrable martingale for $x \in E - N$. Let $< M >$ be the sharp bracket process of M ([18]), then $< M >$ is a PCAF. We denote by $\mu_{<M>}$ the smooth measure associated with the PCAF $< M >$ and we call it the energy measure of MAF M. By simple calculation, it can be shown that

$$e(M) = \frac{1}{2}\mu_{<M>}(E), \quad M \in \mathcal{M}.$$

In fact, by (2.1), it follows that

$$e(M) = \sup_{t>0} \frac{1}{2t} E_m M_t^2 = \frac{1}{2} \sup_{t>0} \frac{1}{t} E_m < M >_t = \frac{1}{2}\mu_{<M>}(E).$$

A continuous AF (CAF in abbreviation) $\{N_t\}_{t\geq 0}$ is called a CAF of zero energy, if N_t belongs to the following set,

$$\begin{aligned} \mathcal{N}_c : &= \{N| \quad N \quad \text{is CAF with exception set Z}, \quad e(N) = 0, \\ & \forall t > 0, \quad E_x N_t < \infty, \quad x \in E - Z\}. \end{aligned}$$

Theorem 2.1.10. *([15] Fukushima's decomposition)*
For $u \in \mathcal{D}(\mathcal{E})$, an AF $\{u(X)\}_{t\geq 0}$ can be decomposed as

$$u(X_t) = u(X_0) + M_t^u + N_t^u, \quad t \geq 0,$$

where $M^u \in \overset{\circ}{\mathcal{M}}$ and $N^u \in \mathcal{N}_c$.

Moreover, if N^u is a CAF of bounded variation on $[0, t]$ for any $t > 0$, and μ is the signed measure associated with N^u satisfying $|\mu|(E) < \infty$, then for any bounded $v \in \mathcal{D}(\mathcal{E})$,

$$\mathcal{E}(u, v) = -\int_E v(x)\, \mu(dx).$$

The following example reveals a relationship between the Markov process, Dirichlet form and the various kinds of functionals introduced above.

Example 1. *D is a d-dimensional smooth bounded Euclidean domain, dx is the d-dimensional Lebesgue measure and $\lambda(dx)$ is the (d-1)-dimensional Lebesgue measure on ∂D.*

Let $A(x) = (a_{ij})_{1\leq ij\leq d}$: $R^d \to R^d \otimes R^d$ be a smooth, symmetric matrix-valued function and assume that A is uniformly elliptic. That is, there is a constant $\lambda > 1$

such that

$$\frac{1}{\lambda} I_{d\times d} \leq A(\cdot) \leq \lambda I_{d\times d}.$$

Consider the bilinear form:

$$\begin{cases} \mathcal{E}_0(u,v) = \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx, \\ D(\mathcal{E}_0) = W^{1,2}(D) := \{u : u \in L^2(D), \frac{\partial u}{\partial x_i} \in L^2(D), i = 1, ..., d\}. \end{cases}$$

It is easy to verify that $(\mathcal{E}_0, D(\mathcal{E}_0))$ *is a regular symmetric Dirichlet form and it is associated with a Hunt process* $\{X\}_{t\geq 0}$.

For any bounded functions u, $f \in D(\mathcal{E}_0)$, *by Fukushima's decomposition, it follows that*

$$\begin{aligned} \int_D f(x)\,\mu_{<M^u>}(dx) &= \lim_{t\downarrow 0}\frac{1}{t}E_{f\cdot m}(u(X_t)-u(X_0))^2 \\ &= \lim_{t\downarrow 0}\frac{1}{t}\int_D (p_tu^2(x) - 2u(x)p_tu(x) + u^2(x))f(x)\,dx \\ &= \lim_{t\downarrow 0}\frac{2}{t}\int_D u(x)f(x)(u(x)-p_tu(x))\,dx \\ &\quad -\lim_{t\downarrow 0}\frac{1}{t}\int_D u^2(x)(f(x)-p_tf(x))\,dx \\ &= 2\mathcal{E}_0(uf,u) - \mathcal{E}_0(u^2,f) \\ &= \sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}f(x)\,dx. \end{aligned}$$

Therefore by the fact that $\mu_{<M^u>}(dx) = \sum_{i,j} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx$, *we obtain*

$$< M^u >_t = \int_0^t \sum_{i,j} a_{ij}(X_s)\frac{\partial u}{\partial x_i}(X_s)\frac{\partial u}{\partial x_j}(X_s)\,ds,$$

and

$$M_t^u = \sum_{i,j}\int_0^t \sigma_{ij}(X_s)\frac{\partial u}{\partial x_i}(X_s)\,dB_s^j = \int_0^t < \sigma\nabla u(X_s), dB_s > .$$

Here the matrix $\sigma(x)$ *is the positive definite symmetric square root of the matrix* $A(x)$. $\{B_t\}_{t>0}$ *is a d-dimensional standard Brownian motion.*

For a positive integer number i, $1 \leq i \leq d$, *we set* $u_0(x) = x_i$ *on* D, *where* x_i *is the* ith *coordinate of* x. *By Fukushima's decomposition, it follows that*

$$X_t^i = X_0^i + \sum_j \int_0^t \sigma_{ij}(X_s)\, dB_s^j + N_t^{u_0}, \quad t > 0.$$

By further calculation, we get that, for any bounded function $f \in D(\mathcal{E}_0)$,

$$\begin{aligned} \mathcal{E}_0(u_0, f) &= \frac{1}{2}\int_D \sum_j \int_D a_{ij}(x) \frac{\partial f}{\partial x_j}\, dx \\ &= -\frac{1}{2}\int_D (\sum_j \frac{\partial a_{ij}}{\partial x_j}) f(x)\, dx - \frac{1}{2}\int_{\partial D} (\sum_j a_{ij} n^j)(x) f(x)\, \lambda(dx). \quad (2.2) \end{aligned}$$

Here $\vec{n}$ *denotes the inward normal vector to the boundary* ∂D . *For any* $\xi \in \partial D$, $U(\xi)$ *denotes a neighborhood of* ξ. *If there exists a smooth function* ψ *such that* $\partial D \cap U(\xi) = \{x : \psi(x_1, ..., x_d) = 0\}$, *and* $D \cap U(\xi) = \{x : \psi(x_1, ..., x_d) > 0\}$, *then* $\vec{n}$ *is given by locally, i.e., for* $x \in \partial D \cap U(\xi)$,

$$\vec{n}(x) = (n^1(x), ..., n^d(x)) = (\frac{\partial \psi}{\partial x_1}(x), ..., \frac{\partial \psi}{\partial x_d}(x)) / (\sum_{i=1}^d (\frac{\partial \psi}{\partial x_i}(x))^2)^{\frac{1}{2}}.$$

Denote by $\{L_t\}_{t \geq 0}$ *the PCAF associated with the measure* $\frac{1}{2}\lambda$. *By* (2.2), *we know that* N^{u_0} *is associated with the smooth measure*

$$\nu(dx) = \frac{1}{2}(\sum_j \frac{\partial a_{ij}}{\partial x_j})\, dx + \frac{1}{2}(\sum_j a_{ij} n^j)(x)\, \lambda(dx).$$

Therefore,

$$N_t^{u_0} = \frac{1}{2}\int_0^t (\sum_j \frac{\partial a_{ij}}{\partial x_j})(X_s)\, ds + (\sum_j a_{ij} n^j)(X_s)\, dL_t.$$

Now we get the decomposition of the Hunt process associated with the Dirichlet *form* $(\mathcal{E}_0, D(\mathcal{E}_0))$*:*

$$X_t^i = X_0^i + \sum_j \int_0^t \sigma_{ij}(X_s)\, dB_s^j + \frac{1}{2}\int_0^t (\sum_j \frac{\partial a_{ij}}{\partial x_j})(X_s)\, ds + \int_0^t (\sum_j a_{ij} n^j)(X_s)\, dL_s,$$
$$i = 1, ..., d,$$

which can be written simply as follows:

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \frac{1}{2}\int_0^t \nabla A(X_s)\, ds + \int_0^t A\vec{n}(X_s)\, dL_s.$$

2.2 Backward Stochastic Differential Equations

Given a probability space $(\Omega, \mathcal{F}, P)$, we denote by E the expectation under the measure P. $\{W_t\}_{t\geq 0}$ is a d-dimensional standard Brownian motion. The $\sigma-$filtration $(\mathcal{F}_t)$ is generated by $\{W_t\}_{t\geq 0}$,

$$\mathcal{F}_t := \mathcal{N} \vee \sigma\{W_s; 0 \leq s \leq t\} := \sigma\{\mathcal{N}, \quad \sigma\{W_s; 0 \leq s \leq t\}\},$$

where $\mathcal{N}$ is the set of P-null sets in $\mathcal{F}_\infty = \sigma\{W_s; 0 \leq s < \infty\}$.

Let $< \cdot, \cdot >$ denote the scalar product in the Euclid space R^n and $|\cdot|$ the length of a vector in R^n. For any $T > 0$, define the set of all of the $\mathcal{F}_t$-adapted, square integrable processes on $[0, T]$ as follows

$$\mathcal{M}(0, T; R^n) := \{\{v_t\}| \quad \{v_t\} \text{ is } R^n\text{-valued, } \mathcal{F}_t\text{-adapted and } E\int_0^T |v_t|^2\, dt < \infty\}.$$

Suppose a function

$$g = g(\omega, t, y, z) : \Omega \times [0, T] \times R^n \times R^{n\times d} \to R^n$$

satisfies the following conditions:

(1) for any $(y, z) \in R^n \times R^{n\times d}$, $g(\cdot, y, z)$ is R^n-valued, $\mathcal{F}_t$-adapted;

(2) (Lipschitz condition) there is a constant $C > 0$, such that for any $y, y' \in R^n$ and $z, z' \in R^{n \times d}$,

$$|g(t, y, z) - g(t, y', z')| \leq C(|y - y'| + |z - z'|);$$

(3) $\int_0^T |g(\cdot, 0, 0)| ds \in L^2(\Omega, \mathcal{F}_T, P; R)$.

Consider the following backward stochastic differential equation:

$$\begin{aligned} Y_t &= \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \\ \xi &\in \mathcal{F}_T. \end{aligned} \tag{2.3}$$

Here the processes Y and Z are unknown, and a pair of processes (Y, Z) satisfying (2.3) is called a solution of the BSDE.

The following theorem is a classical result of the existence and uniqueness of the solution (Y, Z).

Theorem 2.2.1. *([31])*
Suppose that the function g satisfies the above conditions (1)-(3). Then for any terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^n)$, the BSDE (2.3) *has a unique solution $(Y, Z) \in \mathcal{M}(0, T; R^n \times R^{n \times d})$. Moreover, Y_0 and Z_0 are constants.*

The following example comes from the Chapter 2 in [12] and gives a method to solve a kind of BSDE, and it reveals a relationship between the BSDEs and the PDEs.

Example 2. *Set the initial condition $\zeta \in L^2(\Omega, \mathcal{F}_t, P; R^n)$. Suppose the coefficients b and σ satisfy the Lipschitz condition: there exists a constant $C > 0$ such that for any $t \in [0, T]$ and $x, x' \in R$,*

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|.$$

Denote by $(X^{t,\zeta})$ *the solution of the following stochastic differential equation:*

$$\begin{aligned} dX_s^{t,\zeta} &= b(s,X_s^{t,\zeta})ds+\sigma(s,X_s^{t,\zeta})\,dW_s, \quad s\in[t,T], \\ X_t^{t,\zeta} &= \zeta. \end{aligned}$$

By Ito's formula, it follows that $(X^{t,\zeta})$ *is associated with the generator:*

$$L=\frac{1}{2}\sum_{i,j}a_{ij}\frac{\partial^2\cdot}{\partial x_i\partial x_j}+\sum_i b_i\frac{\partial}{\partial x_i}.$$

Suppose that $u:[0,T]\times R^n\to R^n$ *is the solution of the following partial differential equation:*

$$\begin{aligned} &\partial_t u(t,x)+Lu(t,x)+f(t,x,u,\sigma\nabla u)=0, \\ &u(T,x)=\Phi(x). \end{aligned}$$

Suppose that the coefficients f *and* Φ *satisfy the following conditions,*

(1) (Lipschitz condition) $f(t,x,y,z)$ *is Lipschitz continuous with respect to the variables* y *and* z*;*

(2) (α *Hölder condition)* $\alpha\in(0,1)$, $\forall(x,y,z),(x',y',z')\in R^n\times R\times R^{1\times d}$*, it follows that*

$$|\Phi(x)-\Phi(x')|+|f(t,x,y,z)-f(t,x',y'z')|\le C(|y-y'|+|z-z'|+|x-x'|^\alpha);$$

(3) (Linear growth) $|f(t,x,0,0)|+|\Phi(x)|\le C(1+|x|)$,
then $(u,\sigma\nabla u)(s,X_s^{t,\zeta})$ *is a solution of the following BSDE:*

$$Y_s^{t,\zeta}=\Phi(X_T^{t,\zeta})+\int_s^T f(u,X_u^{t,\zeta},Y_u^{t,\zeta},Z_u^{t,\zeta})\,du-\int_s^T Z_u^{t,\zeta}\,dW_u, \quad s\in[t,T].$$

2.3 Useful Inequalities and Lemmas

Lemma 2.3.1 ([10] **Gronwall's Inequality**). *Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.*

(1) If β is non-negative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\,ds, \quad t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}\,ds, \quad t \in I.$$

(2) If, in addition, the function α is non-decreasing, then

$$u(t) \leq \alpha(t)e^{\int_a^t \beta(r)\,dr}, \quad t \in I.$$

Lemma 2.3.2 ([4], [18] **Doob's Inequality**). *$M = (M_t, t > 0)$ is a continuous martingale on a probability space (Ω, P). Set $M_t^* = \sup_{0 \leq s \leq t} |M_s|$. For any $p \in (1, \infty)$, the following inequality holds,*

$$E(M_t^*)^p \leq (\frac{p}{p-1})^p E(|M_t|^p).$$

For any $p \in (0, 1)$, the following inequality holds,

$$E(M_t^*)^p \leq \frac{1}{1-p}(E(|M_t|))^p.$$

Lemma 2.3.3 ([18] **Burkholder-Davis-Gundy Inequality**). *For any continuous martingale $\{X_t\}_{t \geq 0}$ with $X_0 = 0$, any stopping time τ and any $0 < p < \infty$, the* *following inequality holds,*

$$c_pE(< X >_\tau^{\frac{p}{2}}) \leq E(\sup_{0\leq s\leq \tau} |X_s|^p) \leq C_pE(< X >_\tau^{\frac{p}{2}}),$$

where the constants c_p and C_p only depend on the choice of p.

Assume D is a domain in R^d with smooth boundary. Define the Sobolev space

$$W^{1,p}(D) := \{f \in L^p(D)| \quad \text{weak derivative} \quad \frac{\partial f}{\partial x_i} \in L^p(D), \quad i = 1, ..., d\},$$

equipped with the norm $\|f\|_{W^{1,p}} = (\int_D (|f|^p + |\nabla f|^p)\ dx)^{\frac{1}{p}}$.

Define the Hölder space: for $\gamma \in (0, 1)$,

$$C^\gamma(D) := \{f| \sup_{x,y\in D;x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty\},$$

with the Hölder coefficient $\|f\|_{C^\gamma} = \sup_{x,y\in D;x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$.

Lemma 2.3.4 ([20] **Sobolev's Embedding Theorem**). *Let $p > d$ and $\gamma = 1 - \frac{d}{p}$. Suppose $f \in W^{1,p}(D)$, then it holds that $f \in C^\gamma(D)$.*

Moreover, there is a constant $C > 0$, such that for any $f \in W^{1,p}(D)$,

$$\|f\|_{C^\gamma} \leq C\|f\|_{W^{1,p}}.$$

Lemma 2.3.5 ([20] **Poincaré Inequality**). *Assume that $1 \leq p \leq \infty$ and that D is a bounded connected open subset of the d-dimensional Euclidean space R^d with a Lipschitz boundary. Then there exists a constant C, depending only on D and p, such that for every function u in the Sobolev space $W^{1,p}(D)$:*

$$\|u - \bar{u}\|_{L^p} \leq C\|\nabla u\|_{L^p},$$

where $\bar{u} = \frac{1}{m(D)} \int_D u(x)dx$ and $m(D)$ is the Lebesgue measure of the domain D.

Chapter 3

Two-sided Estimates on the Heat Kernels

3.1 Introduction

Consider an elliptic operator as follows,

$$\begin{aligned} L &= \frac{1}{2}\nabla\cdot(A\nabla)+B\cdot\nabla-\nabla\cdot(\hat{B}\cdot)+Q \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{d}B_i(x)\frac{\partial}{\partial x_i}-\sum_{i}\frac{\partial}{\partial x_i}(\hat{B}_i(x)\cdot)+Q(x) \quad (3.1) \end{aligned}$$

in a d-dimensional smooth bounded Euclidean domain D.

$A(\cdot)=(a_{ij})_{1\leq ij\leq d}$: $R^d\to\ R^d\otimes R^d$ is a smooth, symmetric matrix-valued function and we assume that A is uniformly elliptic. That is, there is a constant $\lambda>1$ such that

$$\frac{1}{\lambda}I_{d\times d}\leq A(\cdot)\leq\lambda I_{d\times d}. \quad (3.2)$$

Here $B=(B_1,...,B_d)$ and $\hat{B}=(\hat{B}_1,...,\hat{B}_d):\ R^d\to\ R^d$ are Borel measurable

R^d-valued functions, and Q is a Borel measurable function on R^d such that:

$$I_D(|B|^2 + |\hat{B}|^2 + |Q|) \in L^p(D)$$

for some $p > d$.

Satisfying the following mixed boundary condition

$$\frac{1}{2} < A\nabla u, \vec{n} > - < \hat{B}, \vec{n}\rangle u = 0, \text{on} \partial D, \tag{3.3}$$

the operator L determines a quadratic form :

$$\begin{aligned} \mathcal{Q}(u,v) = (-Lu, v) &= \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx - \sum_i \int_D B_i(x)\frac{\partial u}{\partial x_i}v(x)\,dx \\ &\quad -\sum_i \int_D \hat{B}_i(x)\frac{\partial v}{\partial x_i}u(x)\,dx - \int_D Q(x)u(x)v(x)\,dx, \end{aligned}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(D)$ and $\vec{n}$ denotes the inward normal vector to the boundary ∂D . For any $\xi \in \partial D$, $U(\xi)$ denotes a neighborhood of ξ. For every $\xi \in \partial D$ there exists a neighborhood $U(\xi)$ and a smooth function ψ such that $\partial D \cap U(\xi) = \{x : \psi(x_1, ..., x_d) = 0\}$, and $D \cap U(\xi) = \{x : \psi(x_1, ..., x_d) > 0\}$. Then $\vec{n}$ has the following expression in this local coordinates: for $x \in \partial D \cap U(\xi)$,

$$\vec{n}(x) = (\frac{\partial \psi}{\partial x_1}(x), ..., \frac{\partial \psi}{\partial x_d}(x))/(\sum_{i=1}^{d}(\frac{\partial \psi}{\partial x_i}(x))^2)^{\frac{1}{2}}.$$

Set $\vec{\gamma}(x) = A(x)\vec{n}(x)$ and denote $\frac{\partial u}{\partial \vec{\gamma}} :=< A\nabla u, \vec{n} >$.

The domain of the quadratic form is

$$\mathcal{D}(\mathcal{E}) = W^{1,2}(D) := \{u : u \in L^2(D), \frac{\partial u}{\partial x_i} \in L^2(D), i = 1, ..., d\}.$$

We use $\{T_t,\, t \geq 0\}$ to denote the semigroup generated by L, and we will prove in the following discussion that there exists a function $l(t, x, y)$ which is the heat kernel associated with the semigroup T_t in the sense: $T_t g(x) = \int_D l(t, x, y)g(y)dy$.

The purpose of this chapter is to provide both upper and lower bound estimates for the heat kernel $l(t,x,y)$ associated with operator L equipped with the mixed boundary conditions (3.3). By a "time reverse" technique (see Section 3.2) introduced in [7], we transform the problem of estimating the heat kernels with mixed boundary condition associated with the general operator L in (3.1) to a problem of estimating the fundamental solution $p(t,x,y)$ of the following simpler problem:

$$\begin{cases} \frac{\partial u}{\partial t} = Gu & x \in D, \\ < A\nabla u, \vec{n} >= 0 & x \in \partial D. \end{cases} \tag{3.4}$$

Here

$$G = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q$$

for an appropriate vector field b and a function q. The precise expressions of b and q will be given in Section 3.2.

For the upper bound, we use parametrix and perturbation methods. For the lower bound, we need to assume that the domain is convex. Our method is inspired by the one in [3].

This chapter is organized as follows. In Section 3.2, some preliminary results are proved. The reduction of estimating the heat kernel associated with the operator L to the estimate of the heat kernel associated with the operator G is explained. In Section 3.3, we obtain the upper bound for the heat kernel $p_2(t,x,y)$ associated with the operator

$$G_2 = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla$$

The lower bound for $p_2(t,x,y)$ is given in Section 3.4. Finally, the two sided estimates of the heat kernel associated with the general operator L are proved in

Section 3.5.

3.2 A Reduction Method

Consider the following regular Dirichlet form

$$\begin{cases}\mathcal{E}_0(u,v) = \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx,\\ D(\mathcal{E}_0) = W^{1,2}(D).\end{cases}$$

Denoting the associated reflecting diffusion process by $(\Omega, \mathcal{F}_t, X_t, \theta_t, \gamma_t, P^x)$ by the discussion in Example 1, Section 2.1, we know the following decomposition holds:

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dB_s + \frac{1}{2}\int_0^t \nabla A(X_s)\,ds + \frac{1}{2}\int_0^t A\vec{n}(X_s)\,dL_s. \quad P^x - a.s. \tag{3.5}$$

Here the square integrable martingale $M_t := \int_0^t \sigma(X_s)\,dB_s$ has the property:

$$< M^i, M^j >_t = \int_0^t a_{ij}(X_s)\,ds. \tag{3.6}$$

The following probabilistic representation of the semigroup $\{T_t\}_{t\geq 0}$ associated with the operator L was proved in [6]

$$\begin{aligned} T_t f(x) = \quad & E^x[f(X_t)\exp(\int_0^t (A^{-1}B)^*(X_s)dM_s + (\int_0^t (A^{-1}\hat{B})^*(X_s)dM_s)\circ\gamma_t - \\ & -\frac{1}{2}\int_0^t (B-\hat{B})^* A^{-1}(B-\hat{B})(X_s)ds + \int_0^t Q(X_s)ds)], \end{aligned}$$

where B, $\hat{B}$ and Q are the coefficients of operator L in (3.1).

Here E^x denotes the expectation under P^x and we denote by x^* the transpose of the vector x.

The following result plays an important role in the thesis. The proof will be given after some preparations.

Proposition 3.2.1. *Let $f = (f_1, ..., f_d)$ be a vector-valued function defined on the* *domain D satisfying that $|f| \in L^p(D)$ for $p > d$. If $u \in W^{1,2}(D)$ satisfies*

$$\int_D \sum_{ij} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_i}\, dx = \int_D \sum_i f_i \frac{\partial \psi}{\partial x_i}\, dx.$$

for any function $\psi \in W^{1,2}(D)$, then $u \in W^{1,p}(D)$.

Set

$$C^\infty_\gamma(D) := \{\phi \in C^\infty(D) | \frac{\partial \phi}{\partial \gamma} = 0 \quad on \quad \partial D\},$$

where $\gamma(x) = A(x)\vec{n}(x)$.

Let two positive numbers q and q satisfying $\frac{1}{q} + \frac{1}{p} = 1$. $W^{-1,p}(D)$ denotes the dual space of $W^{1,q}(D)$.

Remark 1. *Let f be the function as in Proposition* 3.2.1*, then we know $div(f) \in W^{-1,p}(D)$.*
In fact, for any $\phi \in W^{1,q}(D)$,

$$|\int_D < f, \nabla \phi > (x)dx| \leq \|f\|_{L^p} \cdot \|\nabla \phi\|_{L^q(D)} \leq \|f\|_{L^p} \cdot \|\phi\|_{W^{1,q}(D)},$$

which implies that

$${}_{W^{-1,p}(D)} < div(f), \phi >_{W^{1,q}(D)} = \int_D < f, \nabla \phi > (x)dx.$$

Moreover, we have $\|div(f)\|_{W^{-1,p}(D)} \leq \|f\|_{L^p}$.

For the uniformly elliptic diffusion matrix $A = (a_{ij})$, we construct a matrix $\tilde{A} = (\tilde{a}_{ij})$ whose inverse matrix $\tilde{A}^{-1} = (\tilde{a}^{ij})$ satisfies $\sqrt{det\tilde{A}^{-1}}\tilde{A} = A$, where $det\tilde{A}^{-1}$ is the determinant of the matrix $\tilde{A}^{-1}$. Denote $a = det\tilde{A}^{-1}$.

In fact, by calculation, we know that if we set $\tilde{A} = (det(A^{-1}))^{\frac{1}{d-2}}A$, then $det(\tilde{A}^{-1}) = (detA^{-1})^{-\frac{2}{d-2}}$. Therefore, $\sqrt{det\tilde{A}^{-1}}\tilde{A} = A$ satisfies.

We know that $\tilde{A}$ forms a symmetric strictly positive definite covariant tensor of order 2. $(D, \tilde{A})$ can be seen as a compact Riemannian manifold with the global

Euclidean coordinate system. The volume form dV on D is $dV = \sqrt{a}dx$. As usual, the metric tensor $\tilde{A}$, in Euclidean coordinates $(x_1, ...x_d)$ has the following representation

$$< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >_m = \tilde{a}^{ij},$$

where $<, >_m$ stands for the inner product of the tangent vectors in $(D, \tilde{A})$.

The gradient vector of a function f is

$$grad_m f = ((\sum_j \tilde{a}_{1j} \frac{\partial f}{\partial x_j}), ..., (\sum_j \tilde{a}_{dj} \frac{\partial f}{\partial x_j})).$$

Let $X = (X_1, ..., X_d)$ be a smooth vector field on $(D, \tilde{A})$. The divergence of X is

$$div_m(X) = \frac{1}{\sqrt{a}} \sum_i \frac{\partial}{\partial x_i}(\sqrt{a} X_i),$$

and the Laplace operator is given by

$$\triangle_m f := div_m grad_m f = \frac{1}{\sqrt{a}} \sum_{ij} \frac{\partial}{\partial x_j}(\tilde{a}_{ij} \sqrt{a} \frac{\partial f}{\partial x_i}).$$

So far using the above discussion, we know the following relationship between the integral on the manifold $(D, \tilde{A})$ with respect to the measure dV and the integral on the Euclidean domain D with respect to the Lebesgue measure dx under the global Euclidean coordination system ,

$$\int_D \triangle_m f(x) g(x) \, dV(x) = \int_D \nabla(A \nabla f)(x) g(x) \, dx.$$

Therefore, with the above Riemannian structure, we are able to apply Theorem 2.3' [39] to obtain the following lemma.

Lemma 3.2.2. *Let $g \in W^{-1,p}(D)$. Suppose that $v \in L^2(D)$ satisfies*

$$\int_D < v, \nabla(A \nabla \phi) > (x) \, dx = {}_{W^{-1,p}(D)} < g, \phi >_{W^{1,p}(D)}, \tag{3.7}$$

for any $\phi \in C_\gamma^\infty(D)$*. Then* $v \in W^{1,p}(D)$ *and moreover, there exists a constant* C *such that,*

$$\|v\|_{W^{1,p}(D)} \leq C(\|g\|_{W^{-1,p}(D)} + |\int_D v(x)\,dx|).$$

Proof of Proposition 3.2.1:

Proof. (1) Firstly we prove that, if $u \in W^{1,2}(D)$ satisfies that

$$\int_D \sum_{ij} < A\nabla u, \nabla\psi > \,dx = \int_D \sum_i < f, \nabla\psi > \,dx,$$

for any function $\psi \in W^{1,2}(D)$, and $\int_D u(x)\,dx = 0$, then $u \in W^{1,p}(D)$ and moreover, there exists a constant $C_1 > 0$ such that

$$\|\nabla u\|_{L^p} \leq C_1 \|f\|_{L^p}.$$

There exists $f^k = (f_1^k, ..., f_d^k)$ such that $f_i^k \in C_0^\infty(D)$, $i = 1, ..., d$ and

$$\lim_{k\to\infty} \|f^k - f\|_{L^p} = 0.$$

Note that in this case, $div(f^k) = \sum_i \frac{\partial f_i^k}{\partial x_i} \in L^p(D)$.

By [39], for every k, there exists $v_k \in W^{1,2}(D)$, $\int_D v_k(x)dx = 0$ such that

$$\int_D \sum_{ij} a_{ij} \frac{\partial v_k}{\partial x_i} \frac{\partial \psi}{\partial x_i}\,dx = \int_D \sum_i f_i^k \frac{\partial \psi}{\partial x_i}\,dx,$$

for $\psi \in W^{1,2}(D)$.

By Green's identity, and the fact that $f^k = 0$ on $\partial D = 0$, we see that v_k satisfies the formula (3.7) with $g = div f^k$, i.e.

$$\int_D v_k(x)\nabla(A\nabla\phi)(x)dx = \int_D div(f_k)\phi(x)dx \tag{3.8}$$

for every $\phi \in C_\gamma^\infty(D)$.

Therefore, v_k satisfies the conditions in Lemma 3.2.2, so we know that $v_k \in W^{1,p}(D)$ and $\|\nabla v_k\|_{W^{1,p}(D)} \leq C\|div(f^k)\|_{W^{-1,p}(D)} \leq C\|f^k\|_{L^p}$.

Set positive number p', such that $\frac{1}{p'} + \frac{1}{p} = 1$. So $p' < 2 < p$ implies that $L^p(D) \subset L^2(D) \subset L^{p'}(D)$.

For any $\phi \in W^{1,2}(D)$,

$$\begin{aligned} |\int_D < f^k - f, \nabla\phi > (x)\, dx| &\leq \int_D |f^k - f| \cdot |\nabla\phi|\, dx \\ &\leq \|f^k - f\|_{L^p} \cdot \|\nabla\phi\|_{L^{p'}} \\ &\leq C_1\|f^k - f\|_{L^p} \cdot \|\nabla\phi\|_{L^2}. \end{aligned}$$

So that

$$\lim_{k\to\infty} \int_D < f^k, \nabla\phi > dx = \int_D < f, \nabla\phi > dx.$$

It follows that

$$\begin{aligned} \lim_{k\to\infty} \int_D < A\nabla v_k, \nabla\phi > dx &= \lim_{k\to\infty} \int_D < f^k, \nabla\phi > dx \\ &= \int_D < f, \nabla\phi > dx = \int_D < A\nabla u, \nabla\phi > dx. \end{aligned}$$

On the other hand, we know that $\|\nabla v_k - \nabla v_{k'}\|_{L^p} \leq C\|f^k - f^{k'}\|_{L^p}$.

By the Poincaré inequality and the fact that $\int_D v_l\, dx = 0$, for $l \geq 1$, it follows that

$$\|v_k - v_{k'}\|_{L^p} \leq M\|\nabla v_k - \nabla v_{k'}\|_{L^p} \leq CM\|f^k - f^{k'}\|_{L^p}.$$

So that $\{v_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $W^{1,p}(D)$. Therefore, there exists $\bar{v} \in W^{1,p}(D)$, such that $\|v_k - \bar{v}\|_{W^{1,p}(D)} \to 0$ as $k \to \infty$.

For any $\phi \in W^{1,2}(D)$, it follows that

$$\lim_{k\to\infty} \int_D < A\nabla v_k, \nabla\phi > dx = \int_D < A\nabla\bar{v}, \nabla\phi > dx, \quad \forall\phi \in W^{1,2}(D),$$

which implies that

$$\int_D < A\nabla(\bar{v}-u), \nabla\phi > dx = 0.$$

Then $\bar{v}-u$ satisfies Lemma 3.2.2 with $g \equiv 0$. Hence $\bar{v}-u \in W^{1,p}(D)$ and $\|\bar{v}-u\|_{W^{1,p}(D)} = 0$.

(2) In the general case, setting $\bar{u} = u - \int_D u(x)dx$, we know that the function $\bar{u}$ satisfies the conditions in part (1). Therefore, we find that $\bar{u} \in W^{1,p}(D)$. Then the proposition is proved. □

As a conclusion of this section, a reduction method is given as follows.

From [7], it follows that there exists a function $v \in \mathcal{D}(\mathcal{E}_0)$ satisfying

$$\begin{aligned} & (\int_0^t (A^{-1}\hat{B})^*(X_s)\, dM_s) \circ \gamma_t \\ = & -\int_0^t \nabla v(X_s) dM_s + v(X_t) - v(X_0) - \int_0^t (A^{-1}\hat{B})^*(X_s)\, dM_s, \end{aligned} \tag{3.9}$$

and moreover,

$$\begin{cases} div(A\nabla v) = -div(\hat{B}) & \text{on } D, \\ \frac{\partial v}{\partial \gamma} = -2 < \hat{B}, \vec{n} > & \text{on } \partial D. \end{cases} \tag{3.10}$$

Therefore, by Proposition 3.2.1, we know that $v \in W^{1,p}(D)$. In particular, by Sobolev's embedding theorem, v is bounded and continuous.

Thus the representation of T_t becomes:

$$\begin{aligned} T_t f(x) &= e^{-v(x)} E^x[f(X_t) e^{v(X_t)} \exp(\int_0^t (A^{-1}(B-\hat{B}-A\nabla v))^* dM_s \\ & \quad -\frac{1}{2}\int_0^t (B-\hat{B}-A\nabla v)^* A^{-1}(B-\hat{B}-A\nabla v)(X_s) ds \\ & \quad +\int_0^t (Q + \frac{1}{2}(\nabla v)^* A(\nabla v) - < B-\hat{B}, \nabla v >)(X_s) ds)] \\ &= e^{-v(x)} S_t[f e^v](x) \end{aligned} \tag{3.11}$$

Here, S_t is the semigroup generated by the following operator equipped with the boundary condition $< A\nabla u, \vec{n} >= 0$, $x \in \partial D$:

$$\begin{aligned} G &= \frac{1}{2}\nabla\cdot(A\nabla) + (B - \hat{B} - (A\nabla v))\cdot\nabla + (Q + \frac{1}{2}(\nabla v)A(\nabla v)^{*} - < B - \hat{B}, \nabla v >) \\ &= \frac{1}{2}\nabla\cdot(A\nabla) + b\cdot\nabla + q. \end{aligned} \tag{3.12}$$

Here we set $b = B - \hat{B} - (A\nabla v)$ and $q = Q + \frac{1}{2}(\nabla v)A(\nabla v)^{*} - < B - \hat{B}, \nabla v >$.

In the following discussion, we will construct the heat kernel denoted by $p(t,x,y)$ and associated with the semigroup S_t. For any $f \in C^{\infty}(D)$, we have

$$T_t f(x) = e^{-v(x)} \int_D e^{v(y)} p(t,x,y) f(y) dy. \tag{3.13}$$

Hence

$$l(t,x,y) = p(t,x,y)e^{v(y)-v(x)}. \tag{3.14}$$

Thus, the estimates of $l(t,x,y)$ will follow from that of $p(t,x,y)$. Due to (3.10), it is easy to see that the corresponding boundary condition also holds:

$$\frac{1}{2}\frac{\partial l}{\partial \gamma}(t,x,y) - l(t,x,y) < \hat{B}, \vec{n} >= 0, \quad on \quad \partial D.$$

The rest of the chapter will be devoted to the estimates on $p(t,x,y)$.

3.3 Upper Bound Estimates

In this section, we consider the operator of the following form,

$$G_2 = \frac{1}{2}\nabla\cdot(A\nabla) + b\cdot\nabla.$$

Let $p_2(t,x,y)$ denote the heat kernel associated with G_2 on D equipped with the boundary condition $< A\nabla u, \vec{n} >= 0$. We aim to establish an upper estimate for $p_2(t,x,y)$. To this end, we first consider the heat kernel $p_1(t,x,y)$ associated with

G_1:

$$G_1 = \frac{1}{2}\nabla\cdot(A\nabla).$$

3.3.1 Upper Bounds for Heat Kernels Associated with G_1

Local coordinate transformations will be used in this section to deal with the boundary of the domain. Rewrite the operator G_1 in the following form:

$$\begin{aligned} G_1 &= \frac{1}{2}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(\sum_{j=1}^{d} a_{ij}\frac{\partial}{\partial x_j}) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{2}\sum_{j=1}^{d}(\sum_{i=1}^{d}\frac{\partial a_{ij}}{\partial x_i})\frac{\partial}{\partial x_j}. \end{aligned}$$

Note that the expression of G_1 depends on the choice of the coordinate system $x = (x_1, ..., x_d)$. For convenience, we denote the global Euclidean coordinate mapping by $\sigma_0 : D \to R^d$ with $\sigma_0(x) = (x_1, ..., x_d)$.

If we consider another coordinate system $\sigma(x) = (\bar{x_1}, ..., \bar{x_d}) : D \to R^n$ with $\bar{x_i} = \bar{x_i}(x_1, ..., x_d)$, $i = 1, .., d$, being smooth, then an easy calculation yields that

$$\begin{aligned} G_1 &= \frac{1}{2}\sum_{k.l=1}^{d}(\sum_{i,j=1}^{d} a_{ij}\frac{\partial \bar{x_k}}{\partial x_i}\frac{\partial \bar{x_l}}{\partial x_j})\frac{\partial^2}{\partial \bar{x_l}\partial \bar{x_k}} + \frac{1}{2}\sum_{k=1}^{d}(\sum_{l=1}^{d}\frac{\partial}{\partial \bar{x_l}}(\sum_{i,j=1}^{d} a_{ij}\frac{\partial \bar{x_k}}{\partial x_i}\frac{\partial \bar{x_l}}{\partial x_j}))\frac{\partial}{\partial \bar{x_k}} \\ &= \frac{1}{2}\sum_{k,l=1}^{d}\bar{a}_{kl}\frac{\partial^2}{\partial \bar{x_k}\partial \bar{x_l}} + \frac{1}{2}\sum_{k=1}^{d}(\sum_{l=1}^{d}\frac{\partial \bar{a}_{kl}}{\partial \bar{x_l}})\frac{\partial}{\partial \bar{x_k}}. \qquad (3.15) \end{aligned}$$

here $\bar{a}_{kl} = \sum_{i,j=1}^{d} a_{ij}\frac{\partial \bar{x_k}}{\partial x_i}\frac{\partial \bar{x_l}}{\partial x_j}$.

Therefore G_1 under the new coordinate system $\sigma(x) = (\bar{x_1}, ..., \bar{x_d})$ has the same form as under the Euclidean coordinate system σ_0 with a diffusion matrix $\bar{A} = (\bar{a}_{kl})_{1\le k,l\le d}$.

Based on this observation, we see that the diffusion matrix $(a_{ij})_{1\le i,j\le d}$ is transformed between two local coordinates in the following way. Suppose that $U_1(\xi)$ and $U_2(\xi)$ are two neighborhoods of the point $\xi \in \bar{D}$ and the mappings $\sigma_1 : U_1(\xi) \to R^d$

and $\sigma_2 : U_2(\xi) \to R^d$ are the coordinate systems on $U_1(\xi)$ and $U_2(\xi)$ respectively. For any $z \in U_1(\xi) \bigcap U_2(\xi) \bigcap \bar{D}$ with coordinates $\sigma_1(z) = (z_1, ..., z_d)$ and $\sigma_2(z) = (\bar{z}_1, ..., \bar{z}_d)$, we have

$$\bar{a}_{ij} = \sum_{kl} \frac{\partial \bar{z}_i}{\partial z_k} \frac{\partial \bar{z}_j}{\partial z_l} a_{kl} \tag{3.16}$$

here $\bar{a}_{ij}(z)$ and $a_{ij}(\bar{z})$ denote the diffusion matrix associated with the coordinate systems σ_2 and σ_1 respectively.

The Neumann boundary condition

$$\frac{\partial f}{\partial \gamma} :=< A\vec{n}, \nabla f >= 0, \quad \text{on} \quad \partial D$$

which is described precisely in Section 3.1, actually has different expressions under different local coordinate systems. Suppose that $U(\xi_0)$ is a neighborhood of the point $\xi_0 \in \partial D$ with coordinate mapping $\sigma(\xi) = (\xi_1, ..., \xi_d)$ for $\xi \in U(\xi_0) \cap D$. Recall that there exists a smooth function ϕ such that $U(\xi_0) \cap \partial D = \{\xi, \phi(\xi_1, ..., \xi_d) = 0\}$ and $U(\xi_0) \cap D = \{\xi, \phi(\xi_1, ..., \xi_d) > 0\}$.

Set

$$\gamma(\xi) := (\sum_i a_{1i}(\xi) \frac{\partial \phi(\xi)}{\partial \xi_i}, ..., \sum_i a_{di}(\xi) \frac{\partial \phi(\xi)}{\partial \xi_i}) / (\sum_{ij} a_{ij} \frac{\partial \phi(\xi)}{\partial \xi_i} \frac{\partial \phi(\xi)}{\partial \xi_j})^{\frac{1}{2}}$$

and

$$\frac{\partial f}{\partial \gamma}(\xi) := \sum_{ij} a_{ij}(\xi) \frac{\partial \phi(\xi)}{\partial \xi_i} \frac{\partial f}{\partial \xi_j} / (\sum_{ij} a_{ij} \frac{\partial \phi(\xi)}{\partial \xi_i} \frac{\partial \phi(\xi)}{\partial \xi_j})^{\frac{1}{2}},$$

where f is the smooth function on $\bar{D}$.

The operator G_1 satisfying the Neumann boundary condition means that for a function u in the domain of G_1, u must satisfy $\frac{\partial u}{\partial \gamma}(\xi) = 0$ for $\xi \in \partial D \cap U(\xi_0)$.

$A^{-1}(x) = (a^{ij}(x))_{ij}$ denotes the inverse matrix of $A(x)$, which forms a symmetric strictly positive definite covariant tensor of order 2. A^{-1} is used to define the length of a piecewise C^1 smooth curve as follows.

Suppose a curve C is defined by $z : \theta \in [0,1] \to z(\theta) \in \bar{D}$. Then the length of C, calculated in the coordinate system $z = (z_1, ..., z_d)$, is

$$L(C) = \int_{[0,1]} (\sum_{ij} a^{ij}(z(\theta)) \frac{dz_i}{d\theta} \frac{dz_j}{d\theta})^{\frac{1}{2}} d\theta.$$

Remark 2. *Set* $a = \sqrt{detA^{-1}}$, *where* $detA^{-1}$ *is the determinant of the matrix* A^{-1}. *By the transformation (3.16), it is easy to verify that the length of the curve is independent of the choice of the local coordinate system.*

Define the distance between $x, y \in \bar{D}$, $d(x,y)$, as the infimum of the length of all smooth curves contained in $\bar{D}$ which connect x and y. And write $d(x, \partial D) = \inf_{y \in \partial D} d(x,y)$.

Lemma 3.3.1. *Fix* $\xi \in \bar{D}$. *Let* $U(\xi)$ *be a convex neighborhood of* ξ *with coordinate system* $\sigma(z) = (z_1, z_2, ..., z_d)$ *for* $z \in U(\xi) \cap \bar{D}$. *Then there exist two constants* $K_1, K_2 > 0$, *such that for* $z^1, z^2 \in U(\xi) \cap \bar{D}$

$$K_2 (\sum_{i=1}^{d} |z_i^1 - z_i^2|^2)^{\frac{1}{2}} \le d(z^1, z^2) \le K_1 (\sum_{i=1}^{d} |z_i^1 - z_i^2|^2)^{\frac{1}{2}} \tag{3.17}$$

Proof. (1) Let C be a C^1 piecewise smooth curve, given by :

$$C : \theta \in [0,1] \to \ c(\theta) \in \overline{D}$$

such that $c(0) = z^1$ and $c(1) = z^2$. Then we obtain

$$l(C) = \int_{[0,1]} (\sum_{i,j} a^{ij}(c(\theta)) \frac{dc_i(\theta)}{d\theta} \frac{dc_j(\theta)}{d\theta})^{\frac{1}{2}} \, d\theta \ge \frac{1}{K} \sum_i \int_0^1 |\frac{dc_i(\theta)}{d\theta}| \, d\theta \ge \frac{1}{K} |z_i^1 - z_i^2|,$$

where the first inequality comes from the uniformly ellipticity of the matrix $(a^{ij})_{1 \le i,j \le d}$.

This implies

$$l(C) \ge \frac{1}{K\sqrt{d}} (\sum_{i=1}^{d} |z_i^1 - z_i^2|^2)^{\frac{1}{2}}.$$

As C is arbitrary, we have $d(z^1, z^2) \ge \frac{1}{K\sqrt{d}} (\sum_{i=1}^{d} |z_i^1 - z_i^2|^2)^{\frac{1}{2}}$.

Here, the constant K is independent of the choice of z but dependent on the choice of the local coordinate system. Setting $K_2 = \frac{1}{K\sqrt{d}}$, we have proved the first half of (3.17).

(2) Define a curve P from z^1 to z^2 by

$$P : \theta \in [0,1] \to \ p(\theta) = \sigma^{-1}(z_1^1 + \theta(z_1^2 - z_1^1), ..., z_d^1 + \theta(z_d^2 - z_d^1))$$

then we get that

$$d(z^1, z^2) \leq l(P) = \int_0^1 (\sum_{i,j} a^{ij}(p(\theta))(z_i^2 - z_i^1)(z_j^2 - z_j^1))^{\frac{1}{2}} d\theta \leq K_1 (\sum_{i=1}^d |z_i^2 - z_i^1|^2)^{\frac{1}{2}}$$

□

Remark 3. *Because of this Lemma, for two neighborhoods $U_1(\xi), U_2(\xi)$ of ξ with coordinate systems $\sigma_1(z) = (z_1, \cdots, z_d), \sigma_2(z) = (\bar{z}_1, \cdots, \bar{z}_d)$ respectively, there are positive numbers C_1, C_2, such that $C_2(\sum_{i=1}^d |\bar{z}_i^1 - \bar{z}_i^2|^2)^{\frac{1}{2}} \leq (\sum_{i=1}^d |z_i^1 - z_i^2|^2)^{\frac{1}{2}} \leq C_1(\sum_{i=1}^d |\bar{z}_i^1 - \bar{z}_i^2|^2)^{\frac{1}{2}}$, for $z^1, z^2 \in U_1(\xi) \cap U_2(\xi)$.*

The following result is from [22].

Lemma 3.3.2. *Fix any $x_0 \in \partial D$, there is a neighborhood U_0 of x_0 with coordinate system $(\bar{x}_1, \cdots, \bar{x}_d)$, such that*
(1) $\partial D \cap U_0 = \{x : \bar{x}_d = 0, x \in \bar{D} \cap U_0\}$, $D \cap U_0 = \{x : \bar{x}_d > 0, x \in \bar{D} \cap U_0\}$.
(2) For $x \in U_0 \cap \partial D$, $\bar{a}_{id}(x) = \bar{a}_{di}(x) = \delta_{id}$. Here $\{\bar{a}_{ij}\}$ denotes the diffusion matrix A associated with the local coordinate $(\bar{x}_i)$ and δ_{ij} denotes Kronecker's delta.

From now on we call this coordinate system the canonical coordinate neighborhood of $x_0 \in \partial D$.

Theorem 3.3.3. *There exist a positive constant T_1 and a function $p_1(t, x, y)$ defined* *on* $[0,T_1]\times\bar{D}\times\bar{D}$ *which solves the following equation:*

$$\begin{cases}\frac{\partial p_1}{\partial t}(t,x,y)=G_1p_1(t,x,y) & (t,x,y)\in(0,T_1]\times D\times D,\\ \frac{\partial p_1}{\partial\gamma}(t,x,y)=0 & x\in\partial D.\end{cases}\tag{3.18}$$

And moreover $p_1(t,x,y)$ *admits an upper bound of Gaussian type:*

$$p_1(t,x,y)\le C_1t^{-\frac{d}{2}}\exp(-C_2t^{-1}|x-y|^2),\quad t\le T_1,\tag{3.19}$$

here C_1 *and* C_2 *are positive constants.*

Proof. From [34], we can choose a finite number of canonical coordinate neighborhoods U_i, $1\le i\le M$, open subsets B_{ij}, $1\le j\le M_i$, of U_i and non-negative functions λ_{ij} in $C^2(\bar{D})$ with supports contained in B_{ij}, satisfying the following conditions: $\{B_{ij},1\le i\le M,1\le j\le M_i\}$ is a covering of $\bar{D}$; if B_{ij} intersects $B_{i'j'}$, then $\bar{B}_{i'j'}\subset U_i$; $\sum_{ij}\lambda_{ij}(x)^2=1$ for $x\in\bar{D}$; $\frac{\partial\lambda_{ij}}{\partial\vec{n}}(x)=0$ for $x\in\partial D$.

Suppose that U_i contains boundary points for $1\le i\le i_0$, while U_i $(i_0+1\le i\le M)$ not. Let $\sigma_i(x)=(x^1_{(i)},\cdots,x^d_{(i)})$ be the canonical coordinate system in $U_i(1\le i\le i_0)$. We will use the Euclidean coordinate system $\sigma_0(x)=(x_1,\cdots,x_d)$ in U_i $(i_0+1\le i\le M)$.

From the appendix in [34], we know that there is a smooth function $q(t,x,y)$ defined on $[0,T]\times\bar{D}\times\bar{D}$, where T is an arbitrarily fixed positive number, satisfying the reflecting boundary condition $\frac{\partial q}{\partial\gamma}(t,x,y)=0$, $x\in\partial D$ and $\lim_{t\to0}q(t,x,y)=\delta_x(y)$. Moreover, q satisfies the following upper bound: for some constants $K_1,K_2>0$, $x,y\in\bar{D}$:

$$|q(t,x,y)|\le K_1\sum_{ij}\lambda_{ij}(x)\lambda_{ij}(y)t^{-\frac{d}{2}}\exp(-K_2t^{-1}\sum_{k=1}^{d}|x^k_{(i)}-y^k_{(i)}|^2),\quad x,y\in\bar{D}.$$

By the Remark 3, there is a constant $C>0$ independent of x and coordinate neighborhood U_i such that $\sum_{k=1}^d|x^k_{(i)}-y^k_{(i)}|^2\ge C\sum_{k=1}^d|x_k-y_k|^2$, where the right side denotes the corresponding Euclidean coordinate. And because $|\lambda_{ij}| \leq 1$, we get

$$|q(t,x,y)| \leq M_1 t^{-\frac{d}{2}} \exp(-M_2 t^{-1} \sum_{k=1}^{d} |x_k - y_k|^2) \tag{3.20}$$

where $M_1, M_2 > 0$ are constants.

Let $f(t,x,y)$ be a solution of the following integral equation

$$f(t,x,y) = (G_1 - \frac{\partial}{\partial t})q(t,x,y) + \int_0^t ds \int_{\bar{D}} (G_1 - \frac{\partial}{\partial t})q(t-s,x,z)f(s,z,y)\,dz$$

where dz denotes the Lebesgue measure on the domain D. This is an integral equation of Volterra type. We will follow the method in [22] and [21] to solve this equation by the method of iteration in the following discussion.

Define $p_1(t,x,y)$ by

$$p_1(t,x,y) = q(t,x,y) + \int_0^t ds \int_{\bar{D}} q(t-s,x,z)f(s,z,y)\,dz. \tag{3.21}$$

It is easy to verify that $p_1(t,x,y)$ satisfies the equation (3.18).

To obtain the upper bound of $p_1(t,x,y)$, we first establish an upper bound for $f(t,x,y)$. For this, we write $f(t,x,y)$ as a series.

Set

$$e_0(t,x,y) = (G_1 - \frac{\partial}{\partial t})q(t,x,y),$$

$$e_{n+1}(t,x,y) = \int_0^t ds \int_{\bar{D}} e_0(t-s,x,z)e_n(s,z,y)\,dz.$$

Then the following equation holds if the series is convergent,

$$f(t,x,y) = \sum_{n=0}^{\infty} e_n(t,x,y)$$

In fact, by [34], there exists a constant M_3, such that $e_0(t,x,y)$ can be chosen to satisfy

$$|e_0(t,x,y)| \leq M_3 t^{-\frac{d+1}{2}} \exp(-M_2 t^{-1} \sum_{i.j=1}^{d} |x_i - y_i|^2).$$

Let $|x-y|^2 = \sum_{i=1}^{d} |x_i - y_i|^2$. Then it follows, for $t \in [0,T]$,

$$\begin{aligned}
|e_1(t,x,y)| &\leq \int_0^t ds \int_{\bar{D}} |e_0(t-s,x,z)e_0(s,z,y)|\, dz \\
&\leq (M_3)^2 \int_0^t ds \int_{\bar{D}} (t-s)^{-\frac{d+1}{2}} \exp(-M_2(t-s)^{-1}|x-z|^2) s^{-\frac{d+1}{2}} \\
&\qquad \times \exp(-M_2 s^{-1}|z-y|^2)\, dz \\
&= (M_3)^2 (2\pi)^d \int_0^t \frac{ds}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \int_{\bar{D}} \frac{\exp(-\frac{M_2|x-z|^2}{t-s})}{(2\pi(t-s))^{\frac{d}{2}}} \frac{\exp(-\frac{M_2|z-y|^2}{s})}{(2\pi(s))^{\frac{d}{2}}}\, dz \\
&\leq (M_3)^2 (2\pi)^{\frac{d}{2}} t^{-\frac{d}{2}} \exp(-\frac{M_2|x-y|^2}{t}) \sqrt{\frac{2}{t}} \int_0^{\frac{t}{2}} \frac{ds}{s^{\frac{1}{2}}} \\
&\leq (M_3)^2 (2\pi)^{\frac{d}{2}} 2\sqrt{T} t^{-\frac{d+1}{2}} \exp(-\frac{M_2|x-y|^2}{t}). \qquad (3.22)
\end{aligned}$$

Iterating this calculation, we get, for $t \in [0,T]$,

$$|e_k(t,x,y)| \leq {M_3}^{k+1} (2\pi)^{k\frac{d}{2}} 2^k T^{\frac{k}{2}} t^{-\frac{d+1}{2}} \exp(-\frac{M_2|x-y|^2}{t}).$$

Therefore, there exist positive numbers T_1 and M_4, such that, for any $t \in [0,T_1]$,

$$|f(t,x,y)| \leq \sum_{n=0}^{\infty} |e_n(t,x,y)| \leq M_4 t^{-\frac{d+1}{2}} \exp(-\frac{M_2|x-y|^2}{t}). \qquad (3.23)$$

Combining (3.20), (3.21) and (3.23), we get:

$$
\begin{aligned}
|p_1(t,x,y)| &\le M_1 t^{-\frac{d}{2}} \exp(-M_2 t^{-1}|x-y|^2) + \int_0^t ds \int_{\bar{D}} M_1 (t-s)^{-\frac{d}{2}} \times \\
&\qquad \times \exp(-M_2(t-s)^{-1}|x-z|^2) \times M_4 s^{-\frac{d+1}{2}} \exp(-M_2 s^{-1}|z-y|^2)\, dz \\
&\le M_1 t^{-\frac{d}{2}} \exp(-M_2 t^{-1}|x-y|^2) \\
&\qquad + M_1 M_4 t^{-\frac{d}{2}} \exp(-M_2 t^{-1}|x-y|^2)(2\pi)^{\frac{d}{2}} \int_0^t \frac{ds}{\sqrt{s}} \\
&\le M_5 t^{-\frac{d}{2}} \exp(-M_2 t^{-1}|x-y|^2), \qquad (3.24)
\end{aligned}
$$

as $t \in [0, T_1]$. Here M_5 is a positive number, depending on M_1, M_4 and T_1. □

Lemma 3.3.4. *Let $p_1(t,x,y)$ be defined as in Theorem 3.3.3. There exist two constants $\lambda_1, \lambda_2 > 0$ such that*

$$
|\nabla p_1| \le \lambda_1 t^{-\frac{d+1}{2}} \exp(-\lambda_2 t^{-1}|x-y|^2), t \in [0, T_1], \quad x, y \in D. \qquad (3.25)
$$

Proof. Let U_i, B_{ij} and λ_{ij} be the same as in the Theorem 3.3.3. By the proof of lemma 2.2 in [34], we have, for $x, y \in U_i$,

$$
|\frac{\partial q}{\partial x^l_{(i)}}(t,x,y)| \le M_6 t^{-\frac{d+1}{2}} \exp(-M_7 t^{-1} \sum_{k=1}^d |y^k_{(i)} - x^k_{(i)}|^2).
$$

Recall that coordinate system $\sigma_0(x) = (x_1, \cdots, x_d)$ denotes the global Euclidean coordinate system. Assume $x \in U_i$. Because the mapping $\sigma_i \circ \sigma_0^{-1} : (x_1, \cdots, x_d) \in R^d \to (x^1_{(i)}, ..., x^d_{(i)}) \in R^d$ is continuous and bounded, there exists a constant $K_3 > 0$ independent of x, k and j, such that

$$
\begin{aligned}
|\frac{\partial q}{\partial x_j}| &= |\sum_{k=1}^d \frac{\partial q}{\partial x^k_{(i)}} \frac{\partial x^k_{(i)}}{x_j}| \\
&\le K_3 \sum_k |\frac{\partial q}{\partial x^k_{(i)}}(t,x,y)| \\
&\le M_8 t^{-\frac{d+1}{2}} \exp(-M_7 t^{-1}|y-x|^2) \qquad (3.26)
\end{aligned}
$$

The last equality is due to the fact $\sum_{k=1}^{d}|y_{(i)}^{k}-x_{(j)}^{k}|^2\leq C\sum_{k=1}^{d}|y_k-x_k|^2$. Thus there exist constants $M_9, M_{10}>0$ such that

$$\begin{aligned}
|\frac{\partial p_1}{\partial x_i}| &\leq |\frac{\partial q}{\partial x_i}|+\int_0^t\int_{\bar{D}}|\frac{\partial q(t-s,x,z)}{\partial x_i}|\cdot|f(s,z,y)|\,dzds\\
&\leq M_8t^{-\frac{d+1}{2}}\exp(-M_7t^{-1}|y-x|^2)\\
&\quad +M_8M_4\int_0^t\frac{ds}{(t-s)^{\frac{1}{2}}s^{\frac{1}{2}}}\int_{\bar{D}}\frac{\exp(-\frac{M_7|x-z|^2}{t-s})}{(t-s)^{\frac{d}{2}}}\frac{\exp(-\frac{M_2|z-y|^2}{s})}{(s)^{\frac{d}{2}}}\,dz\\
&\leq M_9t^{-\frac{d+1}{2}}\exp(-M_{10}t^{-1}|x-y|^2),
\end{aligned}$$

for $t\in[0,T_1]$. □

3.3.2 Upper Bounds for Heat Kernels Associated with G_2

Recall that $G_2=G_1+b\cdot\nabla$, where b was defined in (3.12). The following theorem is the main result of this section.

Theorem 3.3.5. *There exist a constant $T_2>0$ and a continuous function $p_2(t,x,y)$, which is the heat kernel associated with the operator G_2. $p_2(t,x,y)$ admits an upper bound of Gaussian type. That is, there exist some constants $C_1, C_2>0$, such that*

$$|p_2(t,x,y)|\leq C_1t^{-\frac{d}{2}}\exp(-C_2t^{-1}|x-y|^2) \tag{3.27}$$

for $t\in[0,T_2]$, $x,y\in D$.

Let $B: D\to R^d$ is a vector-valued measurable function. Set

$$N_h^{\alpha}(B):=\sup_{x\in D}\int_0^h\int_D|B(y)|s^{-\frac{d+1}{2}}\exp(-\alpha\frac{|x-y|^2}{s})\,dyds \tag{3.28}$$

Definition 3.3.6. *We say that B satisfies condition K if*

$$\lim_{h\to 0}N_h^{\alpha}(B)=0, for\quad all\quad \alpha>0. \tag{3.29}$$

Lemma 3.3.7. *If $|B|\in L^p(D)$ $(p>d)$, then B satisfies condition K.*

Proof. For $|B| \in L^p(D)$ and $p > d$, we have $|B|$ is of the Kato class K_{d+1}, i.e., $\lim_{\delta\to 0}\sup_{x\in D}\int_{\{|y-x|\le\delta\}} \frac{|B(y)|}{|y-x|^{d-1}}\,dy = 0$.

In fact, let $q > 0$, $\frac{1}{q}+\frac{1}{p} = 1$, then for any $\delta > 0$

$$\begin{aligned}
&\int_{\{|y-x|\le\delta\}} \frac{|B(y)|}{|y-x|^{d-1}}\,dy \\
\le\ & (\int_D |B(y)|^p dy)^{\frac{1}{p}}(\int_{\{|y-x|\le\delta\}} \frac{1}{|y-x|^{(d-1)q}}\,dy)^{\frac{1}{q}} \\
\le\ & C\|B\|_{L^p}(\int_0^\delta \frac{1}{r^{(d-1)(q-1)}}\,dr)^{\frac{1}{q}}.
\end{aligned}$$

Because $d < p$, we know that $(d-1)(q-1) < 1$. Then

$$sup_x \int_{\{|y-x|\le\delta\}} \frac{|B(y)|}{|y-x|^{d-1}}\,dy \le \tilde{C}\|B\|_{L^p}\delta^{1-(d-1)(q-1)}.$$

That is, for any $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$sup_x \int_{\{|y-x|\le\delta\}} \frac{|B(y)|}{|y-x|^{d-1}}\,dy \le \varepsilon.$$

For the above δ,

$$\begin{aligned}
&\int_D\int_0^h |B(y)|s^{-\frac{d+1}{2}}\exp(-\alpha\frac{|x-y|^2}{s})\,dyds \\
\le\ & \underbrace{\int_{\{|y-x|\le\delta\}} \cdots\,dy}_{I} + \underbrace{\int_{\{|y-x|\ge\delta\}} \cdots\,dy}_{II}.
\end{aligned}$$

First, we get

$$\begin{aligned}
(I) \le\ & \int_{\{|y-x|\le\delta\}} |B(y)|\int_0^h s^{-\frac{d+1}{2}}\exp(-\alpha\frac{|x-y|^2}{s})\,dyds \\
\le\ & \int_{\{|y-x|\le\delta\}} |B(y)|\int_0^\infty l^{\frac{d+1}{2}-2}\exp(-\alpha|x-y|^2 l)\,dldy \\
\le\ & \int_{\{|y-x|\le\delta\}} |B(y)|\frac{1}{\alpha^{\frac{d-1}{2}}|x-y|^{d-1}}\int_0^\infty l^{\frac{d-1}{2}-1}e^{-l}\,dldy \\
\le\ & C\int_{\{|y-x|\le\delta\}} \frac{|B(y)|}{|y-x|^{d-1}}\,dy \le C\varepsilon.
\end{aligned}$$

By the Hölder's inequality, it follows that

$$
\begin{aligned}
(II) \quad \leq \quad & \left(\int_{\{|y-x|\geq\delta\}} |B(y)|^p \left(\int_0^h \frac{\exp(-\alpha\frac{|x-y|^2}{s})}{s^{\frac{d+1}{2}}}\, ds\right) dy\right)^{\frac{1}{p}} \\
& \left(\int_{\{|y-x|\geq\delta\}} \left(\int_0^h \frac{\exp(-\alpha\frac{|x-y|^2}{s})}{s^{\frac{d+1}{2}}}\, ds\right) dy\right)^{\frac{1}{q}}.
\end{aligned}
$$

On the set $\{|y-x|\geq\delta\}$, we get

$$
\begin{aligned}
\int_0^h \frac{\exp(-\alpha\frac{|x-y|^2}{s})}{s^{\frac{d+1}{2}}}\, ds \quad & \leq \quad \int_0^h \frac{\exp(-\alpha\frac{\delta^2}{s})}{s^{\frac{d+1}{2}}}\, ds = \int_{\frac{1}{h}}^\infty e^{-\alpha\delta^2 s} s^{\frac{d+1}{2}-1} s^{-1}\, ds \\
& \leq \quad h\int_0^\infty e^{-\alpha\delta^2 s} s^{\frac{d+1}{2}-1}\, ds = \frac{1}{\alpha^{\frac{d+1}{s}}\delta^{d+1}}\Gamma(\frac{d+1}{2})h,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(II) \quad & \leq \quad \left(\int_{\{|y-x|\geq\delta\}} |B(y)|^p \left(\frac{C_2}{\delta^{d+1}}h\right) dy\right)^{\frac{1}{p}} \left(\int_0^h \frac{1}{s^{\frac{1}{2}}}\left(\int \frac{\exp(-\alpha\frac{|x-y|^2}{s})}{s^{\frac{d}{2}}}\, dy\right) ds\right)^{\frac{1}{q}} \\
& \leq \quad \frac{C_3}{\delta^{\frac{d+1}{p}}}\|B\|_{L^p} h^{\frac{1}{p}+\frac{1}{2q}}.
\end{aligned}
$$

Therefore, for arbitrary $\varepsilon_0 > 0$, there exists a positive number $\delta_0 > 0$ such that for any $h < \delta_0$, $N_h^\alpha(B) < \varepsilon$. That is $\lim_{h\to 0} N_h^\alpha(B) = 0$. □

Lemma 3.3.8. *([41])*

Suppose $0 < a < b$, *there exist positive constants* $C_{a,b}$ *and* c *depending only on* a *and* b *such that*

$$
\begin{aligned}
& \int_0^t \int_D \frac{\exp(-a(t-s)^{-1}|x-z|^2)}{(t-s)^{\frac{d}{2}}}|B(z)|\frac{\exp(-bs^{-1}|z-y|^2)}{s^{\frac{d+1}{2}}}\, dzds \\
\leq \quad & C_{a,b}N_h^c(B)\frac{\exp(-at^{-1}|x-y|^2)}{t^{\frac{d}{2}}} \qquad (3.30)
\end{aligned}
$$

and

$$\begin{aligned}&\int_0^t\int_D \frac{\exp(-a(t-s)^{-1}|x-z|^2)}{(t-s)^{\frac{d+1}{2}}}|B(z)|\frac{\exp(-bs^{-1}|z-y|^2)}{s^{\frac{d+1}{2}}}\,dzds\\ \le\ & C_{a,b}N_h^c(B)\frac{\exp(-at^{-1}|x-y|^2)}{t^{\frac{d+1}{2}}}\end{aligned} \tag{3.31}$$

for $t < h$.

Proof of Theorem 3.3.5:

By the standard parametrix method ([13]), the fundamental solution of the parabolic equation associated with G_2 has the following expression:

$$p_2(t,x,y) = p_1(t,x,y) + \int_0^t\int_D p_1(t-s,x,z)\Phi(s,z,y)\,dzds,$$

where

$$\Phi(t,x,y) = b(x)\cdot\nabla_x p_1(t,x,y) + \int_0^t\int_D b(x)\cdot\nabla_x p_1(t-s,x,z)\Phi(s,z,y)\,dzds,$$

and $p_1(t,x,y)$ is as in section 3.1.

Let

$$f_1(t,x,y) = b(x)\cdot\nabla_x p_1(t,x,y),$$

$$f_{n+1}(t,x,y) = \int_0^t\int_D b(x)\cdot\nabla_x p_1(t-s,x,z)f_n(s,z,y)\,dzds.$$

Then

$$\Phi(t,x,y) = \sum_{n=1}^{\infty} f_n(t,x,y).$$

By the upper estimates on p_1, ∇p_1 and Lemma 3.3.8, we have

$$\begin{aligned}
& |\int_0^t \int_D p_1(t-s,x,z) f_1(s,z,y)\, dzds| \\
= \; & |\int_0^t \int_D p_1(t-s,x,z) b(z)\cdot \nabla_z p_1(s,z,y)\, dzds| \\
\leq \; & M_{11} N_h^{\tilde{M}_{12}}(b) \frac{\exp(-M_{12}t^{-1}|x-y|^2)}{t^{\frac{d}{2}}}.
\end{aligned} \tag{3.32}$$

Let $g(t,x,y) = \int_0^t \int_D p_1(t-s,x,z)b(z)\cdot\nabla_z p_1(s,z,y)\, dzds$. Therefore

$$\begin{aligned}
& |\int_0^t \int_D p_1(t-s,x,z) f_{n+1}(s,z,y)\, dzds| \\
= \; & |\int_0^t \int_D p_1(t-s,x,z)\{\int_0^s \int_D b(z)\cdot\nabla_z p_1(s-l,z,\omega) f_n(l,\omega,y) m(d\omega) dl\}\, dzds| \\
= \; & |\int_0^t \int_D \{\int_0^{t-l} \int_D p_1(t-l-s,x,z) b(z)\cdot\nabla_z p_1(s,z,\omega)\, dzds\} f_n(l,\omega,y) m(dw)\, dl| \\
= \; & |\int_0^t \int_D g(t-l,x,\omega) f_n(l,\omega,y)\, d\omega ds|.
\end{aligned}$$

By the estimates (3.32) for $g(t,x,y)$ and the Lemma 3.3.8, we get:

$$\begin{aligned}
& |\int_0^t \int_D p_1(t-s,x,z) f_{n+1}(s,z,y)\, dzds| \\
\leq \; & (N_h^{\tilde{M}_{12}}(b) M_{11})^n t^{-\frac{d}{2}} \exp(-M_{12}t^{-1}|x-y|^2).
\end{aligned}$$

Choosing h small enough, it follows that there exists $T_2 > 0$ such that

$$p_2(t,x,y) \leq M_{13} t^{-\frac{d}{2}} \exp(-M_{12}t^{-1}|x-y|^2) \quad on \quad [0,T_2]\times\bar{D}\times\bar{D}.$$

$\square$

3.4 Lower Bound Estimates

In this section, we establish the lower bound for the heat kernel of operator G_2 equipped with the Neumann boundary condition. Recall that $G_2 = \frac{1}{2}\nabla\cdot(A\nabla) + b\cdot\nabla$.

We first establish the lower bound for the heat kernel associated with the diffusion operator $G_1 = \frac{1}{2}\nabla \cdot (A\nabla)$.

3.4.1 Lower Bounds for Heat Kernels Associated with G_1

Recall that the reflecting diffusion $\{X_t, P^x\}_{t\geq 0}$ associated with the regular Dirichlet form $\mathcal{E}_0(\cdot,\cdot)$ has the generator $G_1 = \frac{1}{2}\nabla \cdot (A\nabla)$. And we have shown that the heat kernel $p_1(t,x,y)$ of the operator G_1 equipped with Neumann boundary condition, has upper bounds of Gaussian type:

$$p_1(t,x,y) \leq C_1 t^{-\frac{d}{2}} \exp(-\frac{C_2|x-y|^2}{t})$$

for $t \in [0,T_1]$ $x,y \in D$, where C_1, C_2 and T_1 are positive constants.

Proposition 3.4.1. *There are positive numbers C_3 and C_4 such that, for $t \in [0,T_1]$, $x \in D$ and any $\varepsilon > 0$,*

$$P^x[\sup_{s\leq t}|X_s - x| \geq \varepsilon] \leq C_3 \exp(-\frac{C_4\varepsilon^2}{t}) \tag{3.33}$$

Proof. Note that

$$\begin{aligned} P^x[|X_t - x| \geq \varepsilon] &= \int_{\{|x-y|\geq\varepsilon\}\cap D} p_1(t,x,y)\, dy \\ &\leq C_1 t^{-\frac{d}{2}} \int_{\{|y-x|\geq\varepsilon\}\cap D} e^{-\frac{C_2|x-y|^2}{t}}\, dy \\ &\leq \widetilde{C_1} \exp(-\frac{C_2\varepsilon^2}{2t}). \end{aligned}$$

Define the stopping time $\tau = \inf\{t > 0, |X_t - X_0| \geq \varepsilon\}$.

Then we have

$$\begin{aligned}
& P^x[\sup_{s\le t}|X_s - x| \ge \varepsilon] \\
= \ & P^x[\tau \le t] \\
= \ & P^x[|X_t - X_0| \ge \frac{\varepsilon}{2}; \tau \le t] + P^x[|X_t - X_0| \le \frac{\varepsilon}{2}; \tau \le t] \\
\le \ & P^x[|X_t - X_0| \ge \frac{\varepsilon}{2}] + P^x[|X_t - X_\tau| \ge \frac{\varepsilon}{2}; \tau \le t] \\
= \ & P^x[|X_t - X_0| \ge \frac{\varepsilon}{2}] + E^x[E^{X_\tau}[|X_{t-\tau} - X_0| \ge \frac{\varepsilon}{2}], \tau \le t].
\end{aligned}$$

Note that on $\{\tau \le t\}$,

$$E^{X_\tau}[|X_{t-\tau} - X_0| \ge \frac{\varepsilon}{2}] \le \sup_{x\in D, s\le t} P^x[|X_s - x| \ge \frac{\varepsilon}{2}] \le \widetilde{C_1}\exp(-\frac{C_2\varepsilon^2}{8t}),$$

and (3.33) is proved. □

Set

$$L_r = \frac{1}{2}\nabla\cdot(A_r\nabla) \quad on \quad D_r := \{rx : x \in D\}, \tag{3.34}$$

here $A_r(x) = (a^r_{ij}(x))_{1\le i,j\le d} := (a_{ij}(x/r))_{1\le i,j\le d}$.

Proposition 3.4.2. *For any $x \in D_r$, if we define the probability measure $P^x_r = P^{x/r}$, then the process $\{rX_{t/r^2}, P^x_r\}_{t\ge 0}$ is a reflecting diffusion on D_r associated with the generator operator L_r equipped with the Neumann boundary condition.*

Proof. Firstly, we prove that the heat kernel $p_r(t, x, y)$ of the process $\{rX_{t/r^2}, P^x_r\}_{t\ge 0}$ is equal to $\frac{1}{r^d}p_1(\frac{t}{r^2}, \frac{x}{r}, \frac{y}{r})$, where $p_1(t, x, y)$ is the heat kernel associated with the operator G_1 on D.

Let $Y_t = rX_{t/r^2}$. For any function $f \in C^\infty(D_r)$, define the semigroup associated with (Y_t, P^x_r) as $T^r_t f(x) = E^x_r[f(Y_t)]$. On the one hand, we have, for $x \in D_r$, $T^r_t f(x) = \int_{D_r} f(y)p_r(t, x, y)\, dy$.

On the other hand, we have, for $x \in D_r$,

$$\begin{aligned} T_t^r f(x) = E_r^x[f(Y_t)] = E^{x/r}[f(rX_{t/r^2})] &= \int_D f(ry)p_1(\frac{t}{r^2}, \frac{x}{r}, y)\, dy \\ &= \frac{1}{r^d}\int_{D_r} f(y)p_1(\frac{t}{r^2}, \frac{x}{r}, \frac{y}{r})\, dy. \end{aligned}$$

This implies

$$p_r(t,x,y) = \frac{1}{r^d}p_1(\frac{t}{r^2}, \frac{x}{r}, \frac{y}{r}).$$

Secondly, we prove that the generator of the semigroup T_t^r is L_r. Fix any $f \in C^\infty(\bar{D}_r)$, then $\tilde{f}(z) := f(rz)$ for $z \in D$ belongs to $C^\infty(\bar{D})$. For any $x \in D_r$,

$$\begin{aligned} \lim_{t\to 0}\frac{f - T_t^r f}{t}(x) &= \lim_{t\to 0}\frac{1}{t}\left(f(x) - \frac{1}{r^d}\int_{D_r} f(y)p_1(\frac{t}{r^2}, \frac{x}{r}, \frac{y}{r})\, dy\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left(f(x) - \int_D f(ry)p_1(\frac{t}{r^2}, \frac{x}{r}, y)\, dy\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left(\tilde{f}(x/r) - \int_D \tilde{f}(y)p_1(\frac{t}{r^2}, \frac{x}{r}, y)\, dy\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left(\tilde{f} - T_{t/r^2}\tilde{f}\right)(x/r) = \frac{1}{r^2}(L_1\tilde{f})(x/r) \\ &= \frac{1}{2r^2}\nabla\cdot(A\nabla\tilde{f})(x/r) = \frac{1}{2}\nabla\cdot(A_r\nabla\ f)(x). \end{aligned}$$

The last equality follows from the fact that $\frac{\partial\tilde{f}}{\partial x_i}(x/r) = r\frac{\partial f}{\partial x_i}(x)$ and $r\frac{\partial a_{ij}^r}{\partial x_i}(x) = \frac{\partial a_{ij}}{\partial x_i}(x/r)$, for $x \in D_r$. Without loss of generalization, we assume that there is a smooth function ϕ such that $\partial D = \{x : \phi(x) = 0\}$. Defining $\tilde{\phi}(x) = \phi(x/r)$ for $x \in D_r$, we get $\partial D_r = \{x : \tilde{\phi}(x) = 0\}$. Therefore, for any $x \in \partial D_r$,

$$\begin{aligned} < A_r(x)\vec{n}(x), \nabla p_r(t,x,y) > &= \frac{1}{|\nabla\phi(x)|} < A_r(x)\nabla\tilde{\phi}(x), \frac{1}{r^{1+d}}\nabla p_1(t/r^2, x/r, y/r) > \\ &= \frac{1}{r^{2+d}|\nabla\phi(x)|} < A(x/r)\nabla\phi(x/r), \nabla p_1(t/r^2, x/r, y/r) > \\ &= 0 \end{aligned}$$

This proves that p_r satisfies the Neumann boundary condition. □

Proposition 3.4.3. *There exist two constants c and c' such that*

$$p_1(t,x,y) \geq \frac{c}{t^{\frac{d}{2}}} \qquad \textit{whenever} \qquad |x-y| \leq c'\sqrt{t}.$$

Proof. By [3] and Proposition 3.4.1, there are some constants $t_0 > 0$ and $\tilde{c}$ depending on the Lipschitz constant γ of the boundary of domain D and the ellipticity constant λ, such that

$$p_1(t_0,x,y) \geq \tilde{c} \qquad \textit{whenever} \qquad |x-y| \leq 1$$

Set $c' = 1/\sqrt{t_0}$. Fix any $t > 0$ and $x, y \in D$ with $|x-y| \leq c'\sqrt{t}$. Set $r = \sqrt{t_0/t}$. Since the domain D_r shares the same Lipschitz constant with D and $p_r(t,\cdot,\cdot)$ admits the same upper bound as $p_1(t,x,y)$, we know that $p_r(t_0,m,n) \geq \tilde{c}$ if $|m-n| \leq 1$.

Therefore,

$$|rx - ry| \leq rc'\sqrt{t} = c'\sqrt{t_0} = 1$$

implies that $p_r(t_0,rx,ry) \geq \tilde{c}$. So

$$p_r(t_0,rx,ry) = \frac{1}{r^d} p_1(t_0/r^2,x,y) = \frac{1}{r^d} p_1(t,x,y) \geq \tilde{c}.$$

Setting $c = \tilde{c} t_0^{\frac{d}{2}}$, we get $p_1(t,x,y) \geq \frac{c}{t^{\frac{d}{2}}}$ for $|x-y| \leq c'\sqrt{t}$. □

After the preparation of Proposition 3.4.3, the following result follows similarly as the proof of Theorem 2.7 in [14].

Theorem 3.4.4. *Suppose that domain D is convex. There exist constants $C_1, C_2 > 0$ and $T_3 > 0$, such that for $x, y \in D, t \in [0, T_3]$*

$$p_1(t,x,y) \geq C_1 t^{-\frac{d}{2}} \exp\left(-\frac{C_2|x-y|^2}{t}\right)$$

Proof. Fix $t > 0, x, y \in D$. If $|x-y| < c'\sqrt{t}$, by last proposition, we have $p_1(t,x,y) \geq ct^{-\frac{d}{2}} \geq ct^{-\frac{d}{2}} \exp(-\frac{|x-y|^2}{t})$. Therefore we suppose that $|x-y| \geq c'\sqrt{t}$.

Because ∂D is compact and smooth, there exist two constants $r_0 > 0$ and $\delta > 0$ only depending on D, such that $|D \cap B(x,r)| \geq \delta |B(x,r)|$ for $x \in \bar{D}$, $r \leq r_0$. Here $B(z,r) = \{x \in R^d : |x-z| < r\}$ and $| \cdot |$ means the volume of the set $\cdot$.

Since D is convex, the set $c(x,y) := \{(1-t)x + ty, t \in [0,1]\} \subset D$. Let k be the smallest positive integer dominating $\frac{9|x-y|^2}{c'^2 t}$ and set $s_i = x + \frac{i(y-x)}{k} (1 \leq i \leq k-1)$, $r = \frac{c'}{3}\sqrt{\frac{t}{k}}$. So that $r = \frac{c'}{3}\sqrt{\frac{t}{k}} \geq \frac{|x-y|}{k}$ and $r < r_0$.

Denote $S_i = B(s_i, r)$, where $B(z,r) = \{x \in R^d : |x-z| < r\}$. Then $\{S_i\}_{i=0,\dots,k}$ is a sequence of balls. For $\xi_i \in S_i$, $max\{|x-\xi_1|, |y-\xi_{k-1}|, |\xi_l - \xi_{l+1}|, 1 \leq l < k-1\} < c'\sqrt{\frac{t}{k}}$. Hence, by the last proposition,

$$\begin{aligned} p_1(t,x,y) &= \int_D \cdots \int_D p_1(\frac{t}{k}, x, \xi_1) p_1(\frac{t}{k}, \xi_1, \xi_2) \cdots p_1(\frac{t}{k}, \xi_{k-1}, y)\, d\xi_1 \cdots \xi_{k-1} \\ &\geq \int_{S_1 \cap D} \cdots \int_{S_{k-1} \cap D} p_1(\frac{t}{k}, x, \xi_1) p_1(\frac{t}{k}, \xi_1, \xi_2) \cdots p_1(\frac{t}{k}, \xi_{k-1}, y)\, d\xi_1 \cdots \xi_{k-1} \\ &\geq (c(\frac{t}{k})^{-d/2})^k |\delta S_1|^{k-1} \end{aligned}$$

Here, the volume of the ball $|S_1| = \omega_d r^d = \omega_d (\frac{c'}{3})^d (\frac{t}{k})^{d/2}$, and the constant ω_d only depends on d.

Therefore, we get that

$$p_1(t,x,y) \geq (\frac{3}{\omega_d c'}) t^{-d/2} k^{d/2} \delta^{k-1} (\frac{1}{3}\omega_d c' c)^k =: C_1 t^{-d/2} k^{d/2} C_2^k$$

Then, there is a constant $C_3 > 0$, such that

$$\log p_1(t,x,y) \geq \log C_1 t^{-d/2} + k \log C_2 \geq \log C_1 t^{-d/2} - C_3 k$$

Since $\frac{9|x-y|^2}{c'^2 t} \leq k < \frac{9|x-y|^2}{c'^2 t} + 1$, we know that

$$p_1(t,x,y) \geq C_4 t^{-\frac{d}{2}} e^{\frac{-C_6|x-y|^2}{t}}. \tag{3.35}$$

Here, the constants C_4, C_5, C_6 only depend on λ, T_3 and d.

□

3.4.2 Lower Bounds for Heat Kernels Associated with G_2

Recall that

$$G_2 = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla.$$

As in the proof of theorem 3.3.5, we set

$$f_1(t,x,y) = b(x) \cdot \nabla_x p_1(t,x,y),$$

$$f_{n+1}(t,x,y) = \int_0^t \int_D b(z) \cdot \nabla_x p_1(t-s,x,z) f_n(s,z,y)\, dzds.$$

Then by the construction of function $p_2(t,x,y)$ in Theorem 3.3.5, the following inequality holds: there are constants $\lambda_1 > 0, \lambda_2 > 0$, such that

$$\begin{aligned} |p_2(t,x,y) - p_1(t,x,y)| &= |\sum_{k=1}^{\infty} \int_0^t \int_D p_1(t-s,x,z) f_{n+1}(s,z,y)\, dzds| \\ &\le \lambda_1 N_h^{\lambda_2}(b) t^{-\frac{d}{2}} \exp(-\frac{\lambda_2 |x-y|^2}{t}), \end{aligned}$$

and consequently,

$$p_2(t,x,y) \ge p_1(t,x,y) - \lambda_1 N_h^{\lambda_2}(b) t^{-\frac{d}{2}} \exp(-\frac{\lambda_2 |x-y|^2}{t}) \tag{3.36}$$

Because $\lim_{h\to 0} N_h^{\lambda_2}(b) = 0$, we can chose h small enough so that

$$\begin{aligned} p_2(t,x,y) &\ge p_1(t,x,y) - \lambda_1 N_h^{\alpha}(b) t^{-\frac{d}{2}} \exp(-\frac{\lambda_2 |x-y|^2}{t}) \\ &\ge C_4 t^{-\frac{d}{2}} e^{\frac{-C_6|x-y|^2}{t}} - \lambda_1 N_h^{\alpha}(b) t^{-\frac{d}{2}} \exp(-\frac{\lambda_2 |x-y|^2}{t}) \\ &\ge \frac{1}{2} C_4 t^{-\frac{d}{2}} e^{\frac{-C_6 \vee \lambda_2 |x-y|^2}{t}}, \end{aligned} \tag{3.37}$$

for $t \le h$.

3.5 Two-sided Estimates for the Heat Kernel $p(t, x, y)$ of Operator G

Recall that $G = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q$ with b and q defined in (3.12). In this section, we establish both the upper and lower bounds of the heat kernel $p(t, x, y)$ associated with the operator G equipped with the Neumann boundary condition.

3.5.1 Upper Bounds for the Heat Kernel $p(t, x, y)$

Define the exponential martingale

$$Z_t = e^{\int_0^t A^{-1}b(X_s)dM_s - \frac{1}{2}\int_0^t <b,A^{-1}b>(X_s)ds},$$

where the martingale $\{M_t, t \geq 0\}$ was defined in (3.5) and (3.6).

As $|b| \in L^p$, $p > d$, then by [27], we know that Z_t is an exponential martingale on the probability space $\{\Omega, \mathcal{F}_t, P^x\}_{t\geq 0}$. Define a family of measures $(\tilde{P}^x, x \in D)$ on $\mathcal{F}_\infty$ by

$$\frac{d\tilde{P}^x}{dP^x}|_{\mathcal{F}_t} = Z_t.$$

Then, by the Feymann-Kac formula, if $f \in C^\infty(D)$, we have:

$$\begin{aligned} S_t f(x) &= E^x[Z_t e^{\int_0^t q(X_s)ds} f(X_t)] = \tilde{E}^x[e^{\int_0^t q(X_s)ds} f(X_t)] \\ &= \tilde{E}^x[f(X_t)\sum_{n=0}^\infty \frac{1}{n!}(\int_0^t q(X_s)\,ds)^n]. \end{aligned}$$

Here $\tilde{E}^x$ denotes the expectation under $\tilde{P}^x$.

Set

$$\begin{aligned} Q_0(t,x) &= \tilde{E}^x[f(X_t)] && (3.38) \\ Q_n(t,x) &= \tilde{E}^x[\int_0^t q(X_s)Q_{n-1}(t-s, X_s)\,ds] && (3.39) \end{aligned}$$

Then if the following series is convergent, we have

$$S_t f(x) = \sum_{n=0}^{\infty} Q_n(t,x)$$

Theorem 3.5.1. *The heat kernel $p(t,x,y)$ associated with the operator G equipped with Neumann boundary condition has the following upper bound of Gaussian type: for $K_1, K_2 > 0$,*

$$p(t,x,y) \le K_1 t^{-\frac{d}{2}} e^{-\frac{K_2|x-y|^2}{t}}, \quad for 0 \le t \le T_4, \tag{3.40}$$

where T_4 is a positive constant.

Proof. Firstly, we establish the upper bounds for $Q_n(t,x)$:

$$\begin{aligned} |Q_0(t,x)| &= |\int_D p_2(t,x,y)f(y)dy| \le \int_D M_{13} t^{-\frac{d}{2}} e^{-\frac{M_{12}|x-y|^2}{t}} |f(y)|\,dy, & (3.41)\\ |Q_1(t,x)| &= |\int_0^t \tilde{E}^x[q(X_s)Q_0(t-s,X_s)]\,ds| \\ &= |\int_0^t \int_D p_2(s,x,y)q(y)Q_0(t-s,y)\,dyds| \\ &\le \int_D |f(z)|(\int_0^t \int_D M_{13} s^{-\frac{d}{2}} e^{-\frac{M_{12}|x-y|^2}{s}} |q(y)| M_{13}(t-s)^{-\frac{d}{2}} e^{-\frac{M_{12}|y-z|^2}{t-s}}\,dyds)\,dz \\ &\le M_{13}^2 N_h^{M_{12}}(|q|) \int_D t^{-\frac{d}{2}} e^{-\frac{M_{12}|x-z|^2}{t}} |f(z)|\,dz. & (3.42) \end{aligned}$$

The last inequality follows from the fact that q satisfies condition K.

Iterating the calculation, we get

$$|Q_n(t,x)| \le M_{13}^{n+1} (N_h^{M_{12}}(|q|))^n \int_D t^{-\frac{d}{2}} e^{-\frac{M_{12}|x-z|^2}{t}} |f(z)|\,dz.$$

There exits $T_4 > 0$, such that if $h < T_4$, then $M_{13} N_h^{M_{12}}(|q|) < 1$. This implies

$$\sum_{n=0}^{\infty} Q_n(t,x) \le \frac{M_{13}}{1 - M_{13}^2 N_h^{M_{12}}(|q|)} \int_D t^{-\frac{d}{2}} e^{-\frac{M_{12}|x-z|^2}{t}} |f(z)|\,dz$$

for $t \le h \le T_4$.

Therefore, we conclude that there exists $K_1, K_2 > 0$ such that

$$|p(t,x,y)| \leq K_1 t^{-\frac{d}{2}} e^{-\frac{K_2|x-z|^2}{t}}, \quad \text{for} 0 \leq t \leq T_4.$$

□

3.5.2 Lower Bounds for the Heat Kernel $p(t,x,y)$

Theorem 3.5.2. *Given a positive function $f \in L^p$ for some $p > d$. Let $\xi(t,x,y)$ be the heat kernel of the operator $G_f := \frac{1}{2}\nabla(A\nabla) + b \cdot \nabla - f$, that is,*

$$\tilde{E}^x[\exp(-\int_0^t f(X_s)ds)\cdot] = \int_D \xi(t,x,y) \cdot dy.$$

Then there are two constants $k_1, k_2 > 0$ and $T > 0$, such that for $t \in [0,T]$

$$\xi(t,x,y) \geq k_1 t^{-\frac{d}{2}} e^{-\frac{k_2|x-y|^2}{t}}. \tag{3.43}$$

Proof. We set $p_f(t,x,y)$ to be the heat kernel such that

$$\tilde{E}^x[\exp(\int_0^t f(X_s)ds)\cdot] = \int_D p_f(t,x,y) \cdot dy.$$

Let $g \in C(\overline{D})$ be a non-negative function satisfying $\|g\|_{L^1(D)} = 1$. By Hölder's inequality, we get

$$\begin{aligned} |\int_D p_2(t,x,y)g(y)\,dy| = |\tilde{E}^x[g(X_t)]| &\leq \tilde{E}^x[e^{\int_0^t f(X_s)\,ds} g(X_t)]^{\frac{1}{2}} \tilde{E}^x[e^{-\int_0^t f(X_s)\,ds} g(X_t)]^{\frac{1}{2}} \\ &= (\int_D p_f(t,x,y)g(y)\,dy)^{\frac{1}{2}} (\int_D \xi(t,x,y)g(y)\,dy)^{\frac{1}{2}} \end{aligned}$$

Let $g_k, k \geq 0$ be a sequence of nonnegative continuous functions on D with $\|g_k\|_{L^1(D)} = 1$, such that g_k tends to the Dirac measure δ_{y_0} at y_0 . Then we have

$$p_2(t,x,y_0)^2 \leq p_f(t,x,y_0)\xi(t,x,y_0).$$

By the lower bound of $p_2(t,x,y)$ and upper bound of $p_f(t,x,y)$ established in Section 3.4.2 and Theorem 3.5.1 respectively, we get the required lower bounds in (3.43). $\square$

Decompose function q as follows:

$$q = q^+ - q^-,$$

where $q^+ = \max\{q, 0\}$ and $q^- = -min\{q, 0\}$.

Let f be a nonnegative continuous function on $\bar{D}$. Then we have

$$\begin{aligned} \int_D p(t,x,y)f(y)\,dy &= \tilde{E}^x[\exp\,(\int_0^t q(X_s)\,ds)f(X_t)] \\ &= \tilde{E}^x[\exp\,(\int_0^t q^+(X_s) - q^-(X_s)\,ds)f(X_t)] \\ &\geq \tilde{E}^x[\exp\,(\int_0^t -q^-(X_s)\,ds)f(X_t)] \\ &= \int_D \hat{\xi}(t,x,y)f(y)\,dy \end{aligned} \tag{3.44}$$

where $\hat{\xi}(t,x,y)$ is the heat kernel associated with the semigroup

$$\hat{S}_t\cdot := \tilde{E}^x[\exp\,(\int_0^t -q^-(X_s)\,ds)\cdot].$$

By Theorem 3.5.2, we know that there exist constants $\hat{k}_1, \hat{k}_2 > 0$ and $T_5 > 0$, such that the heat kernel $\hat{\xi}(t,x,y) \geq \hat{k}_1 t^{-\frac{d}{2}} e^{-\frac{\hat{k}_2|x-y|^2}{t}}$ for $t \leq T_5$.

Therefore, the heat kernel $p(t,x,y)$ associated with operator G also admits the following lower bound:

$$p(t,x,y) \geq \hat{\xi}(t,x,y) \geq \hat{k}_1 t^{-\frac{d}{2}} e^{-\frac{\hat{k}_2|x-y|^2}{t}}, \quad \text{for} 0 \leq t \leq T_5.$$

Chapter 4

Neumann Problems for Semilinear Elliptic PDEs

4.1 Introduction

In this chapter, our aim is to use probabilistic methods to solve the mixed boundary value problem for semilinear second order elliptic partial differential equations (PDEs in abbreviation) of the following form:

$$\begin{cases} Lu(x) = -F(x, u(x), \nabla u(x)) & \text{on } D, \\ \frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) - \widehat{B} \cdot \vec{n}(x) u(x) = \Phi(x) & \text{on } \partial D \ , \end{cases} \tag{4.1}$$

on the bounded convex domain D with smooth boundary.

Both the elliptic operator $L = \frac{1}{2}\nabla \cdot (A\nabla) + B \cdot \nabla - \nabla \cdot (\hat{B}\cdot) + Q$ and the quadratic form $\mathcal{Q}(\cdot, \cdot)$ associated with L are as in Chapter 3.

The function $F(\cdot, \cdot, \cdot)$ in (4.1) is a nonlinear function defined on $R^d \times R \times R^d$. Φ is a bounded measurable function defined on the boundary ∂D. Set $\vec{\gamma} = A\vec{n}$, where $\vec{n}$ denotes the inward normal vector field defined on the boundary ∂D.

As discussed in Chapter 3, the term $\nabla \cdot (\hat{B}\cdot)$ is tackled using the time-reversal of Girsanov transform with the symmetric diffusion $(\Omega, P_x^0, X_t^0, t \geq 0)$ associated with the operator

$$G_1 = \frac{1}{2}\nabla \cdot (A\nabla),$$

which is the symmetric part of L. The semigroup T_t associated with the operator L has the following form:

$$\begin{aligned} T_t f(x) &= E_x^0[f(X_t^0)\exp(\int_0^t (A^{-1}B)^*(X_s^0)dM_s^0 + (\int_0^t (A^{-1}\hat{B})^*(X_s^0)dM_s^0) \circ \gamma_t^0 \\ &\quad -\frac{1}{2}\int_0^t (B-\hat{B})^* A^{-1}(B-\hat{B})(X_s^0)ds + \int_0^t Q(X_s^0)ds)] \\ &= e^{-v(x)} S_t[fe^v](x), \end{aligned} \tag{4.2}$$

where v is got in the Section 3.2.

Here M^0 is the martingale part of the diffusion X^0, γ_t^0 is the reverse operator and $\{S_t\}$ is the semigroup associated with the operator $G = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q$.

Please note that in this chapter, for convenience, we will use $(\Omega, P_x^0, X_t^0, t \geq 0)$ to denote the process associated the operator G_1 and use $(\Omega, P_x, X(t), t > 0)$ to denote the process associated with $G_2 = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla$. This notation is different from that in Chapter 3.

Chapter 4 is organized as follows. In the next section, we obtain a pair of L^2 solutions (Y, Z) of the BSDEs with infinite horizon:

$$\begin{aligned} &dY(t) = -F(X(t), Y(t), Z(t))dt + e^{\int_0^t q(X(u))dt}\Phi(X(s))dL_t + < Z(t), dM(t) >, \\ &\lim_{t\to\infty} e^{\int_0^t d(X(u))\,du} Y_t = 0 \quad in \quad L^2(\Omega), \end{aligned} \tag{4.3}$$

where $\{X(t)\}_{t>0}$ is the reflecting diffusion associated with the infinitesimal generator $G_2 = \frac{1}{2}\nabla(A\nabla) + b \cdot \nabla$, $M(t)$ is the martingale part of $\{X(t)\}_{t>0}$ and function d is associated with the semilinear function $F(x, y, z)$ and the ellipticity constant λ. Here we call (Y, Z) the L^2 solution because we estimate the solution in $L^2(\Omega)$ in the whole Section 4.2. The crucial point in this section is that the BSDE to be considered has a term associated with the local time and needs to be solved on an infinite time interval.

In section 4.3, we solve the linear PDEs of the form:

$$\begin{cases} \frac{1}{2}\nabla(A\nabla u)(x) + b\cdot\nabla u(x) + qu(x) = F(x) & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial\vec{\gamma}}(x) = \phi(x) & \text{on } \partial D\ . \end{cases} \tag{4.4}$$

Some useful estimates, which are used in the subsequent discussions, are also proved in this section.

In section 4.4, we obtain the solution to the semilinear PDE:

$$\begin{cases} \frac{1}{2}\nabla(A\nabla u)(x) + b\cdot\nabla u(x) + qu(x) = G(x, u(x), \nabla u(x)) & \text{on } D, \\ \frac{1}{2}\frac{\partial u}{\partial\vec{\gamma}}(x) = \phi(x) & \text{on } \partial D\ . \end{cases} \tag{4.5}$$

Using the solution $(Y_x(t), Z_x(t))$ to the BSDE (4.3), we set $F(x) := G(x, u_0(x), v_0(x))$, where $u_0(x) = E_x[Y_x(0)]$ and $v_0(x) = E_x[Z_x(0)]$, so that we can transform the semilinear equation (4.5) to the linear equation (4.4). With the solution $u(x)$ of the linear case shown in Section 4.3, what we need to do is to prove that $u_0(x)$ coincides with $u(x)$. Then we will show that $u(x)$ is the solution to (4.5).

In section 4.5, we finally solve the non-linear equation:

$$\begin{cases} Lu(x) = -F(x, u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial\vec{\gamma}}(x) - \widehat{B}\cdot n(x)u(x) = \Phi(x) & \text{on } \partial D. \end{cases} \tag{4.6}$$

The relationship between the operator L and G is the crucial point to complete the proof. We apply the transformation introduced in [7] and Section 3.2 to transform the problem (4.6) to a similar problem as (4.5), then an inverse transformation will solve the final problem (4.6).

In section 4.6, we also obtain the L^1 solutions of the BSDEs (4.3). Then using the same methods introduced in Section 4.4 and 4.5, we solve the equations (4.5) and (4.6). Since all of the solutions are estimated in $L^1(\Omega)$ in this section, different conditions on the coefficients of operator L are obtained.

4.2 BSDEs with Singular Coefficients and Infinite Horizon

Consider the operator

$$G_2 = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i},$$

on the domian D equipped with the Neumann boundary condition:

$$\frac{\partial}{\partial \vec{\gamma}} :=< A\vec{n}, \nabla\cdot >= 0, \quad \text{on} \quad \partial D.$$

By [26], there exists a unique reflecting diffusion process denoted by $(\Omega, \mathcal{F}_t, X_x(t), P_x, \theta_t, x \in D)$ associated with the generator G_2. Here $\theta : \Omega \to \Omega$ is the shift operator defined as follows:

$$X_x(s)(\theta_t \cdot) = X_x(t+s), \quad s, t \geq 0.$$

Let E_x denote the expectation under the measure P_x.

Set $\tilde{b} = \{\tilde{b}_1, ..., \tilde{b}_d\}$, where $\tilde{b}_i = \frac{1}{2}\sum_j \frac{\partial a_{ij}}{\partial x_j} + b_i$.

Then the process $X_x(t)$ has the following decomposition:

$$X_x(t) = X_x(0) + M_x(t) + \int_0^t \tilde{b}(X_x(s))\, ds + \int_0^t A\vec{n}(X_x(s))\, dL_s, \quad P_x - a.s.. \quad (4.7)$$

Here M_x is a $\mathcal{F}_t$-measurable square integrable continuous martingale additive functional:

$$M_x(t) = \int_0^t \sigma(X_s) dB_s,$$

where the matrix $\sigma(x)$ is the positive definite symmetric square root of the matrix

$A(x)$ and $\{(B_t\}_{t\geq 0}$ is a standard Brownian motion.

L_t is a positive increasing continuous additive functional satisfying

$$L_t = \int_0^t I_{\{X_x(s)\in\partial D\}}\, dL_s.$$

We write $X_x(t)$ as $X(t)$ for short in the following discussion.

In this section, we will study the backward stochastic differential equations with singular coefficients and infinite horizon associated with the martingale part $M_x(t)$ and the local time L_t. The unique L^2 solution of such kind of BSDE is obtained.

Let $g(\omega, t, y, z) : \Omega \times R^+ \times R \times R^d \to R$ be a progressively measurable function. Consider the following conditions:

(A.1) $(y_1 - y_2)(g(t, y_1, z) - g(t, y_2, z)) \leq -a_1(t)|y_1 - y_2|^2$, *a.s.*,

(A.2) $|g(t, y, z_1) - g(t, y, z_2)| \leq a_2|z_1 - z_2|$, *a.s.*,

(A.3) $|g(t, y, z)| \leq |g(t, 0, 0)| + a_3(t)(1 + |y|)$, *a.s.*.

Here $a_1(t)$ and $a_3(t)$ are two progressively measurable processes and a_2 is a constant. Set $a(t) = -a_1(t) + \delta a_2^2$, for some constant $\delta > \frac{1}{2\lambda}$, where λ is the elliptic constant which appeared in (3.2).

Lemma 4.2.1. *Assume the conditions **(A.1)-(A.3)** hold and*

$$E_x[\int_0^\infty e^{2\int_0^t a(u)\,du}|g(t, 0, 0)|^2\, dt] < \infty.$$

Then there exists a unique solution $(Y_x(t), Z_x(t))$ to the following backward stochastic differential equation:

$$Y_x(t) = Y_x(T) + \int_t^T g(s, Y_x(s), Z_x(s))\, ds - \int_t^T < Z_x(s), dM_x(s) >, \quad for \quad 0 \leq t \leq T;$$
$$\lim_{t\to\infty} e^{\int_0^t a(u)\,du} Y_x(t) = 0, \quad in \quad L^2(\Omega). \tag{4.8}$$

Moreover,

$$E_x[\sup_t e^{2\int_0^t a(u)\,du}|Y_x(t)|^2] < \infty \quad \textit{and} \quad E_x[\int_0^\infty e^{2\int_0^s a(u)\,du}|Z_x(s)|^2\,ds] < \infty. \tag{4.9}$$

Proof. **Existence:**

The proof of this lemma is similar to that of Theorem 3.2 in [42], but the terminal conditions here are different. By Theorem 3.1 in [42], the following BSDE has a unique solution $(Y_x^n(t), Z_x^n(t))$:

$$Y_x^n(t) = \int_t^n g(s, Y_x^n(s), Z_x^n(s))\,ds - \int_t^n < Z_x^n(s), dM_x(s) >, \quad \text{for} \quad 0 \le t \le n, \tag{4.10}$$

and moreover,

$$Y_x^n(t) = 0, \quad Z_x^n(t) = 0, \quad t > n.$$

Fix $t > 0$ and $n > m > t$. It follows that

$$\begin{aligned}
& e^{2\int_0^t a(u)\,du}|Y_x^n(t) - Y_x^m(t)|^2 \\
& + \int_t^n e^{2\int_0^s a(u)\,du} < A(X(s))(Z_x^n(s) - Z_x^m(s)), (Z_x^n(s) - Z_x^m(s)) > \,ds \\
= \; & -2\int_t^n a(s)e^{2\int_0^s a(u)\,du}|Y_x^n(s) - Y_x^m(s)|^2\,ds \\
& +2\int_t^n e^{2\int_0^s a(u)\,du}(Y_x^n(s) - Y_x^m(s))(g(s, Y_x^n(s), Z_x^n(s)) - g(s, Y_x^m(s), Z_x^m(s)))\,ds \\
& +2\int_m^n e^{2\int_0^s a(u)\,du}(Y_x^n(s) - Y_x^m(s))g(s,0,0)\,ds \\
& -2\int_t^n e^{2\int_0^s a(u)\,du}(Y_x^n(s) - Y_x^m(s)) < Z_x^n(s) - Z_x^m(s), dM_x(s) > .
\end{aligned}$$

Choose two positive numbers δ_1 and δ_2 such that $\delta_1 > \frac{1}{2\lambda}$ and $\delta_1 + \delta_2 < \delta$.

Then by

$$\begin{aligned}
& 2\int_t^n e^{2\int_0^s a(u)\,du}(Y_x^n(s)-Y_x^m(s))(g(s,Y_x^n(s),Z_x^n(s))-g(s,Y_x^m(s),Z_x^m(s)))\,ds \\
\leq\ & -2\int_t^n a_1(s)e^{2\int_0^s a(u)\,du}|Y_x^n(s)-Y_x^m(s)|^2\,ds \\
& +2\delta_1 a_2^2\int_t^n e^{2\int_0^s a(u)\,du}|Y_x^n(s)-Y_x^m(s)|^2\,ds \\
& +\frac{1}{2\lambda\delta_1}\int_t^n e^{2\int_0^s a(u)\,du} < A(X(s))(Z_x^n(s)-Z_x^m(s)),(Z_x^n(s)-Z_x^m(s)) > \,ds
\end{aligned}$$

and

$$\begin{aligned}
& 2\int_m^n e^{2\int_0^s a(u)\,du}(Y_x^n(s)-Y_x^m(s))g(s,0,0)\,ds \\
\leq\ & 2\delta_2 a_2^2\int_m^n e^{2\int_0^s a(u)\,du}|Y_x^n(s)-Y_x^m(s)|^2\,ds+\frac{1}{2\delta_2 a_2^2}\int_m^n e^{2\int_0^s a(u)\,du}|g(s,0,0)|^2\,ds,
\end{aligned}$$

it follows that

$$\begin{aligned}
& E_x[e^{2\int_0^t a(u)\,du}|Y_x^n(t)-Y_x^m(t)|^2]+\frac{1}{\lambda}(1-\frac{1}{2\lambda\delta_1})E_x[\int_t^\infty e^{2\int_0^s a(u)\,du}|Z_x^n(s)-Z_x^m(s)|^2\,ds] \\
& \leq \frac{1}{2\delta_2 a_2^2}E_x[\int_m^n e^{2\int_0^s a(u)\,du}|g(s,0,0)|^2\,ds].
\end{aligned}$$

This implies that

$$E_x[\int_0^\infty e^{2\int_0^s a(u)\,du}|Z_x^n(s)-Z_x^m(s)|^2\,ds]\to 0, \quad as \quad m,n\to\infty.$$

Hence there exists $\tilde{Z}_x$ such that

$$\tilde{Z}_x=\lim_{n\to\infty} e^{\int_0^{\cdot} a(u)\,du}Z_x^n \quad \text{in} \quad L^2([0,\infty)\times\Omega).$$

At the same time, we also obtain the following estimate:

$$
\begin{aligned}
& \sup_t e^{2\int_0^t a(u)\,du} |Y_x^n(t) - Y_x^m(t)|^2 \\
\leq\ & \frac{1}{2\delta_2 a_2^2} \int_m^n e^{2\int_0^s a(u)\,du} |g(s,0,0)|^2\,ds \\
& +2\sup_t \left| \int_t^n e^{2\int_0^s a(u)\,du} (Y_x^n(s) - Y_x^m(s)) < Z_x^n(s) - Z_x^m(s), dM_x(s) > \right|.
\end{aligned}
$$

Taking expectation on both sides of the above inequality, by the Burkholder-Davis-Gundy inequality, we obtain

$$
\begin{aligned}
& E_x[\sup_t e^{2\int_0^t a(u)\,du} |Y_x^n(t) - Y_x^m(t)|^2] \\
\leq\ & \frac{1}{2\delta_2 a_2^2} E_x[\int_m^n e^{2\int_0^s a(u)\,du} |g(s,0,0)|^2\,ds] \\
& +C_1 E_x[\{\int_t^n e^{4\int_0^s a(u)\,du} |Y_x^n(s) - Y_x^m(s)|^2 |Z_x^n(s) - Z_x^m(s)|^2\,ds\}^{\frac{1}{2}}] \\
\leq\ & \frac{1}{2\delta_2 a_2^2} E_x[\int_m^n e^{2\int_0^s a(u)\,du} |g(s,0,0)|^2\,ds] + \frac{1}{2} E_x[\sup_t e^{2\int_0^t a(u)\,du} |Y_x^n(t) - Y_x^m(t)|^2] \\
& +C_2 E_x[\int_0^\infty e^{2\int_0^s a(u)\,du} |Z_x^n(s) - Z_x^m(s)|^2\,ds].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& E_x[\sup_t e^{2\int_0^t a(u)\,du} |Y_x^n(t) - Y_x^m(t)|^2] \\
\leq\ & \frac{1}{\delta_2 a_2^2} E_x[\int_m^n e^{2\int_0^s a(u)\,du} |g(s,0,0)|^2\,ds] + 2C_2 E_x[\int_0^\infty e^{2\int_0^s a(u)\,du} |Z_x^n(s) - Z_x^m(s)|^2\,ds] \\
\to\ & 0, \quad as \quad m,n \to \infty.
\end{aligned}
$$

Therefore, there exists $\{\tilde{Y}_x(t)\}$ such that

$$
\lim_{n\to\infty} E_x[\sup_t |\tilde{Y}_x(t) - e^{\int_0^t a(u)\,du} Y_x^n(t)|^2] = 0.
$$

For any $\varepsilon > 0$, there exist a positive number N such that for any $n \geq N$,

$$
E_x[\sup_t |\tilde{Y}_x(t) - e^{\int_0^t a(u)\,du} Y_x^n(t)|^2] < \frac{\varepsilon}{2}.
$$

For $t > N$, noticing $Y_x^N(t) = 0$, it follows that

$$\begin{aligned} E_x[|\tilde{Y}_x(t)|^2] &\leq 2E_x[|\tilde{Y}_x(t) - e^{\int_0^t a(u)\,du}|Y_x^N(t)|^2] + 2E_x[e^{2\int_0^t a(u)\,du}|Y_x^N(t)|^2] \\ &\leq 2E_x[\sup_t |\tilde{Y}_x(t) - e^{\int_0^t a(u)\,du}|Y_x^N(t)|^2] + 2E_x[e^{2\int_0^t a(u)\,du}|Y_x^N(t)|^2] \\ &< \varepsilon. \end{aligned}$$

Thus we have $\lim_{t\to\infty} E_x[|\tilde{Y}_x(t)|^2] = 0$.

By the chain rule, it is easy to see that, from (4.10),

$$Y_x(t) = e^{-\int_0^t a(u)\,du}\tilde{Y}_x(t) \quad and \quad Z_x(t) = e^{-\int_0^t a(u)\,du}\tilde{Z}_x(t)$$

satisfy the equation (4.8) and

$$\lim_{t\to\infty} E_x[e^{2\int_0^t a(u)\,du}|Y_x(t)|^2] = \lim_{t\to\infty} E_x[|\tilde{Y}_x(t)|^2] = 0.$$

According to the above proof, we also conclude that (4.9) holds.

Uniqueness:

Suppose that (Y_x^1, Z_x^1) and (Y_x^2, Z_x^2) are two solutions of the equation (4.8). Set $\bar{Y}_x(t) = Y_x^1(t) - Y_x^2(t)$ and $\bar{Z}_x(t) = Z_x^1(t) - Z_x^2(t)$, *a.s.*. Then

$$\begin{aligned} d(e^{\int_0^t a(u)\,du}\bar{Y}_x(t)) &= -e^{\int_0^t a(u)\,du}(g(t, Y_x^1(t), Z_x^1(t)) - g(t, Y_x^2(t), Z_x^2(t)))dt \\ &\quad + a(t)e^{\int_0^t a(u)\,du}\bar{Y}_x(t)dt \\ &\quad + e^{\int_0^t a(u)\,du} < \bar{Z}_x(t), dM_x(t) > . \end{aligned} \tag{4.11}$$

By Ito's formula, we get, for any $t < T$,

$$\begin{aligned}
& e^{2\int_0^t a(u)\,du}|\bar{Y}_x(t)|^2 + \int_t^T e^{2\int_0^t a(u)\,du} < A(X(s))\bar{Z}_x(s), \bar{Z}_x(s) > \,ds \\
= \; & e^{2\int_0^T a(u)\,du}|\bar{Y}_x(T)|^2 + 2\int_t^T e^{2\int_0^s a(u)\,du}\bar{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))\,ds \\
& -2\int_t^T a(s)e^{2\int_0^s a(u)\,du}|\bar{Y}_x(s)|^2\,ds \\
& -2\int_t^T a(s)e^{2\int_0^s a(u)\,du}\bar{Y}_x(s) < \bar{Z}_x(s), dM_x(s) > . \qquad (4.12)
\end{aligned}$$

By conditions **(A.1)** and **(A.2)**, we have

$$\begin{aligned}
& 2\int_t^T e^{2\int_0^s a(u)\,du}\bar{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))\,ds \\
= \; & 2\int_t^T e^{2\int_0^s a(u)\,du}\bar{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^1(s)))\,ds \\
& +2\int_t^T e^{2\int_0^s a(u)\,du}\bar{Y}_x(s)(g(s, Y_x^2(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))\,ds \\
\leq \; & -2\int_t^T a_1(s)e^{2\int_0^s a(u)\,du}|\bar{Y}_x(s)|^2\,ds + a_2\int_t^T e^{2\int_0^s a(u)\,du}\bar{Y}_x(s)|\bar{Z}_x(s)|\,ds \\
\leq \; & -2\int_t^T a_1(s)e^{2\int_0^s a(u)\,du}|\bar{Y}_x(s)|^2\,ds + c'a_2\int_t^T e^{2\int_0^s a(u)\,du}|\bar{Y}_x(s)|^2\,ds \\
& +a_2\frac{1}{c'\lambda}\int_t^T e^{2\int_0^s a(u)\,du}|\bar{Z}_x(s)|^2\,ds. \qquad (4.13)
\end{aligned}$$

Choosing $c' = 2\delta a_2$, we obtain

$$\begin{aligned}
& |e^{\int_0^t a(u)\,du}\bar{Y}_x(t)|^2 + (1 - \frac{1}{2\delta\lambda})\int_t^T e^{2\int_0^s a(u)\,du} < A(X(s))\bar{Z}_x(s), \bar{Z}_x(s) > \,ds \\
\leq \; & e^{2\int_0^T a(u)\,du}|\bar{Y}_x(T)|^2 - 2\int_t^T a(s)e^{2\int_0^s a(u)\,du}\bar{Y}_x(s) < \bar{Z}_x(s), dM_x(s) > \qquad (4.14)
\end{aligned}$$

Taking expectation on both sides of the above inequality, we get that, for any $t < T$,

$$E_x[e^{2\int_0^t a(u)\,du}|\bar{Y}_x(t)|^2] \leq E_x[e^{2\int_0^T a(u)\,du}|\bar{Y}_x(T)|^2].$$

Since both Y^1 and Y^2 satisfy the terminal condition in (4.8), we obtain that

$$\lim_{T\to\infty} E_x[e^{2\int_0^T a(u)\,du}|\bar{Y}_x(T)|^2] = 0,$$

which leads to $E_x[e^{2\int_0^t a(u)\,du}|\bar{Y}_x(t)|^2] = 0$.

We conclude that $Y_x^1(t) = Y_x^2(t)$ and $Z_x^1(t) = Z_x^2(t)$. □

We now want to apply Lemma 4.2.1 to a particular situation.

Let $F(x, y, z) : R^d \times R \times R^d \to R$ be a Borel measurable function. Consider the following conditions:

(D.1) $(y_1 - y_2)(F(x, y_1, z) - F(x, y_2, z)) \le -d_1(x)|y_1 - y_2|^2$,

(D.2) $|F(x, y, z_1) - F(x, y, z_2)| \le d_2|z_1 - z_2|$,

(D.3) $|F(x, y, z)| \le |F(x, 0, z)| + K(x)(1 + |y|)$.

Set $d(x) = -d_1(x) + \delta d_2^2$ for some constant $\delta > \frac{1}{2\lambda}$. The next result follows from Lemma 4.2.1.

Lemma 4.2.2. *Assume the conditions **(D.1)**-**(D.3)** hold and*

$$E_x[\int_0^\infty e^{2\int_0^t d(X(u))du}|F(X(t), 0, 0)|^2 dt] < \infty.$$

Then there exists a unique solution $(Y_x(t), Z_x(t))$ to the following equation:

$$\begin{aligned} Y_x(t) = Y_x(T) + \int_t^T & F(X(s), Y_x(s), Z_x(s))\,ds \\ & - \int_t^T < Z_x(s), dM_x(s) >, \quad for \quad 0 \le t \le T; \\ \lim_{t\to\infty} e^{\int_0^t d(X(u))\,du} & Y_x(t) = 0, \quad in \quad L^2(\Omega). \end{aligned} \tag{4.15}$$

Consider the following condition instead of **(D.3)**:

(D.3)′ $|F(X(t), y, z)| \le K(t)$, for any $y \in R$ and $z \in R^d$.

Let Φ be a bounded measurable function defined on ∂D, and $\tilde{q} \in L^p(D)$, for $p > d$. The following theorem is the main result in this section.

Theorem 4.2.3. *Assume the conditions **(D.1)**, **(D.2)** and **(D.3)**′ hold and that*

$$E_{x_0}[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))\,du}\,dL_s] < \infty \tag{4.16}$$

for some $x_0 \in D$ *and for* $x \in D$,

$$E_x[\int_0^\infty e^{2\int_0^t d(X(u))\,du}\{e^{2\int_0^t \tilde{q}(X(u))\,du} + |K(t)|^2\}\,dt] < \infty. \tag{4.17}$$

Then there exists a unique solution (Y_x, Z_x) *to the following BSDE:*

$$\begin{aligned} Y_x(t) &= Y_x(T) + \int_t^T F(X(s), Y_x(s), Z_x(s))\,ds - \int_t^T e^{\int_0^s \tilde{q}(X(u))dt}\Phi(X(s))\,dL_s \\ &\quad - \int_t^T < Z_x(s), dM_x(s) >, \quad \textit{for} \quad 0 \le t \le T, \end{aligned} \tag{4.18}$$

and

$$\lim_{t\to\infty} e^{\int_0^t d(X(u))\,du} Y_t = 0 \quad \textit{in} \quad L^2(\Omega). \tag{4.19}$$

Proof. **Uniqueness:**

Suppose that (Y_x^1, Z_x^1) and (Y_x^2, Z_x^2) are two solutions of the equation (4.18) satisfying (4.19).

Set $\bar{Y}_x(t) = Y_x^1(t) - Y_x^2(t)$ and $\bar{Z}_x(t) = Z_x^1(t) - Z_x^2(t)$. Then

$$\begin{aligned} & d(e^{\int_0^t d(X(u))\,du}\bar{Y}_x(t)) \\ = \;& -e^{\int_0^t d(X(u))\,du}(F(X(t), Y_x^1(t), Z_x^1(t)) - F(X(t), Y_x^2(t), Z_x^2(t)))dt \\ & + d(X(t))e^{\int_0^t d(X(u))\,du}\bar{Y}_x(t)dt + e^{\int_0^t d(X(u))\,du} < \bar{Z}_x(t), dM_x(t) > . \end{aligned}$$

By Ito's formula, we get, for any $t < T$,

$$\begin{aligned}
& e^{2\int_0^t d(X(u))\,du}|\bar{Y}_x(t)|^2 + \int_t^T e^{2\int_0^t d(X(u))\,du} < A(X(s))\bar{Z}_x(s), \bar{Z}_x(s) > ds \\
= \ & e^{2\int_0^T d(X(u))\,du}|\bar{Y}_x(T)|^2 \\
& +2\int_t^T e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s)(F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s)))\,ds \\
& -2\int_t^T d(X(s))e^{2\int_0^s d(X(u))\,du}|\bar{Y}_x(s)|^2\,ds \\
& -2\int_t^T d_1(X(s))e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s) < \bar{Z}_x(s), dM_x(s) > \qquad (4.20)
\end{aligned}$$

By **(D.1)** and **(D.2)**, we have

$$\begin{aligned}
& 2\int_t^T e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s)(F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s)))\,ds \\
= \ & 2\int_t^T e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s)(F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^1(s)))\,ds \\
& +2\int_t^T e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s)(F(X(s), Y_x^2(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s)))\,ds \\
\leq \ & -2\int_t^T d_1(X(s))e^{2\int_0^s d(X(u))\,du}|\bar{Y}_x(s)|^2\,ds + d_2\int_t^\infty e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s)|\bar{Z}_x(s)|\,ds \\
\leq \ & -2\int_t^T d_1(X(s))e^{2\int_0^s d(X(u))\,du}|\bar{Y}_x(s)|^2\,ds + cd_2\int_t^T e^{2\int_0^s d(X(u))\,du}|\bar{Y}_x(s)|^2\,ds \\
& +d_2\frac{1}{c\lambda}\int_t^T e^{2\int_0^s d(X(u))\,du}|\bar{Z}_x(s)|^2\,ds. \qquad (4.21)
\end{aligned}$$

Choosing $c = 2\delta d_2$, from (4.21) we obtain that

$$\begin{aligned}
& |e^{\int_0^t d(X(u))\,du}\bar{Y}_x(t)|^2 + (1 - \frac{1}{2\delta\lambda})\int_t^T e^{2\int_0^t d(X(u))\,du} < A(X(s))\bar{Z}_x(s), \bar{Z}_x(s) > ds \\
\leq \ & e^{2\int_0^T d(X(u))\,du}|\bar{Y}_x(T)|^2 - 2\int_t^T d(X(s))e^{2\int_0^s d(X(u))\,du}\bar{Y}_x(s) < \bar{Z}_x(s), dM_x(s) > . \\
& \qquad (4.22)
\end{aligned}$$

Taking expectation on both sides of the above inequality and letting T tend to ∞, we get that

$$E_x[e^{2\int_0^t d(X(u))\,du}|\bar{Y}_x(t)|^2] = 0$$

We conclude that $Y_x^1(t) = Y_x^2(t)$ and hence from (4.22), $Z_x^1(t) = Z_x^2(t)$.

Existence:

First of all, the assumption (4.16) and the following Lemma 4.4.3 imply that

$$\sup_x E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))\,du}\, dL_s] < \infty.$$

1°: There exists $(p_x(t), q_x(t))$ such that

$$dp_x(t) = e^{\int_0^t \tilde{q}(X(u))\,du}\Phi(X(t))dL_t + < q_x(t), dM_x(t) >, \tag{4.23}$$

and $e^{\int_0^t d(X(u))\,du} p_x(t) \to 0$ as $t \to \infty$, in $L^2(\Omega)$.

In fact, let

$$\begin{aligned} p_x(t) \quad &:= \quad -E_x[\int_t^\infty e^{\int_0^s \tilde{q}(X(u))\,du}\Phi(X(s))\, dL_s|\mathcal{F}_t] \\ &= \quad \int_0^t e^{\int_0^s \tilde{q}(X(u))\,du}\Phi(X(s))L_s - E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))\,du}\Phi(X(s))\, dL_s|\mathcal{F}_t]. \end{aligned} \tag{4.24}$$

By the martingale representation theorem in [42], there exists a process $q_x(t)$, such that

$$\begin{aligned} -E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))\,du}\Phi(X(s))\, dL_s|\mathcal{F}_t] = \quad &- \quad E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))\,du}\Phi(X(s))\, dL_s] \\ &+ \quad \int_0^t < q_x(s), dM_x(s) > . \end{aligned} \tag{4.25}$$

Then (p_x, q_x) satisfies the equation (4.23).

Moreover,

$$\begin{aligned}
p_x(t) &= -E_x[\int_t^{\infty} e^{\int_0^s \tilde{q}(X(u))\,du}\Phi(X(s))\,dL_s|\mathcal{F}_t] \\
&= -e^{\int_0^t \tilde{q}(X(u))\,du}E_x[\int_t^{\infty} e^{\int_t^s \tilde{q}(X(u))\,du}\Phi(X(s))\,dL_s|\mathcal{F}_t] \\
&= -e^{\int_0^t \tilde{q}(X(u))\,du}E_x[\int_0^{\infty} e^{\int_t^{s+t} \tilde{q}(X(u))\,du}\Phi(X(s+t))\,dL_{s+t}|\mathcal{F}_t] \\
&= -e^{\int_0^t \tilde{q}(X(u))\,du}E_x[\int_0^{\infty} e^{\int_0^s \tilde{q}(X(u+t))\,du}\Phi(X(s+t))\,dL_{s+t}|\mathcal{F}_t] \\
&= -e^{\int_0^t \tilde{q}(X(u))\,du}E_{X(t)}[\int_0^{\infty} e^{\int_0^l \tilde{q}(X(u))\,du}\Phi(X(l))\,dL_l]. \qquad (4.26)
\end{aligned}$$

The last equality follows from the fact that $L_{t+s} = L_t + L_s \circ \theta_t$. Therefore,

$$\sup_x |p_x(t)| \leq e^{\int_0^t \tilde{q}(X(u))\,du}\sup_{x\in D}|\Phi(x)| \cdot \sup_{x\in\bar{D}} E_x[\int_0^{\infty} e^{\int_0^t \tilde{q}(X(u))\,du}dL_t].$$

Set $M = \sup_{x\in D}|\Phi(x)| \cdot \sup_{x\in\bar{D}} E_x[\int_0^{\infty} e^{\int_0^t \tilde{q}(X(u))\,du}dL_t]$.
By the following Lemma 4.4.3 and condition (4.16), it follows that $M < \infty$.
In view of (4.17), we have

$$\lim_{t\to\infty} e^{\int_0^t (d+\tilde{q})(X(u))\,du} = 0$$

in $L^2(\Omega)$. Hence,

$$e^{\int_0^t d(X(u))\,du}p_x(t) \leq Me^{\int_0^t (d+\tilde{q})(X(u))\,du} \to 0 \quad as \quad t\to\infty, \quad in \quad L^2(\Omega). \qquad (4.27)$$

2° : Set $g(t,y,z) = F(X(t), p_x(t)+y, q_x(t)+z)$. Then

$$\begin{aligned}
& (y_1-y_2)(g(t,y_1,z)-g(t,y_2,z)) \\
=\ & (y_1-y_2)(F(X(t),p_x(t)+y_1,q_x(t)+z)-F(X(t),p_x(t)+y_2,q_x(t)+z)) \\
\leq\ & -d_1(X(t))|y_1-y_2|^2, \qquad (4.28)
\end{aligned}$$

and

$$\begin{aligned}
& |g(t,y,z_1)-g(t,y,z_2)| \\
= \ & |F(X(t),p_x(t)+y,q_x(t)+z_1)-F(X(t),p_x(t)+y,q_x(t)+z_1)| \\
\leq \ & d_2|z_1-z_2|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& E_x[\int_0^\infty e^{2\int_0^t d(X(u))\,du}|g(X(t),0,0)|^2\,dt] \\
\leq \ & E_x[\int_0^\infty e^{2\int_0^t d(X(u))\,du}|F(X(t),p_x(t),q_x(t))|^2\,dt] \\
\leq \ & E_x[\int_0^\infty e^{2\int_0^t d(X(u))\,du}|K(t)|^2\,dt] \\
< \ & \infty. \qquad (4.29)
\end{aligned}$$

Therefore, g satisfies all the conditions of the Lemma 4.2.1. So there exists a pair of processes (k_x, l_x) such that

$$dk_x(t) = -g(t,k_x(t),l_x(t))\,dt + < l_x(t), dM_x(t) >,$$

and

$$e^{\int_0^t d(X(u))\,du}k_x(t) \to 0,$$

as $t \to \infty$.

Putting $Y_x(t) = p_x(t) + k_x(t)$ and $Z_x(t) = q_x(t) + l_x(t)$, we find that $(Y_x(t), Z_x(t))$ satisfies the following equation

$$dY_x(t) = e^{\int_0^t \tilde{q}(X(u))\,du}\phi(X(t))\,dL_t - F(t,Y_x(t),Z_x(t))\,dt + < Z_x(t), dM_x >$$

and

$$\lim_{t\to\infty} e^{\int_0^t d(X(u))\,du}Y_t = 0.$$

□

Corollary 4.2.4. *Assume all of the assumptions in Theorem* 4.2.3 *are satisfied. If*

$$\sup_x E_x[\int_0^\infty e^{2\int_0^s d(X(u))\,du} K^2(s)\,ds < \infty, \tag{4.30}$$

then it holds that

$$\sup_{x\in D} |Y_x(0)| < \infty.$$

Proof. As shown in the proof of theorem, $Y_x(t)$ has the decomposition: $Y_x(t) = p_x(t) + k_x(t)$.

Firstly, set $t = 0$ in (4.26), it follows that

$$\begin{aligned} |p_x(0)| &\leq E_{X(t)}[|\int_0^\infty e^{\int_0^l \tilde{q}(X(u))\,du}\Phi(X(l))dL_l|] \\ &\leq \sup_x \|\Phi\|_\infty \sup_x E_x[\int_0^\infty e^{\int_0^l \tilde{q}(X(u))\,du} dL_l] \\ &< \infty. \end{aligned} \tag{4.31}$$

Secondly, by Ito's formula, we obtain

$$\begin{aligned} de^{2\int_0^t d(X(u))\,du}|k_x(t)|^2 &= -2e^{2\int_0^t d(X(u))\,du} k_x(t) g(t, k_x(t), l_x(t))\,dt \\ &\quad +2e^{2\int_0^t d(X(u))\,du} k_x(t) d(X(t))\,dt \\ &\quad +2e^{2\int_0^t d(X(u))\,du} k_x(t) < l_x(t), dM_x(t) > \\ &\quad +e^{2\int_0^t d(X(u))\,du} < A(X(t)) l_x(t), l_x(t) > dt. \end{aligned}$$

By further calculation and choosing two positive numbers δ_1 and δ_2 such that $\delta_1 > \frac{1}{2\lambda}$ and $\delta_1 + \delta_2 < \delta$, we obtain that, for any $t < T$,

$$\begin{aligned} &E_x[e^{2\int_0^t d(X(u))\,du}|k_x(t)|^2] + \frac{1}{\lambda}(1 - \frac{1}{2\lambda\delta_1})E_x[\int_t^T e^{2\int_0^s d(X(u))\,du}|l_x(s)|^2\,ds] \\ \leq\ & E_x[e^{2\int_0^T d(X(u))\,du}|k_x(T)|^2] + \frac{1}{2\delta_2 d_2^2} E_x[\int_t^T e^{2\int_0^s d(X(u))\,du}|g(s,0,0)|^2\,ds]. \end{aligned}$$

Setting $t = 0$, we get

$$\begin{aligned} |k_x(0)|^2 = E_x[|k_x(0)|^2] &\leq E_x[e^{2\int_0^T d(X(u))\,du}|k_x(T)|^2] \\ &+ \frac{1}{2\delta_2 d_2^2} E_x[\int_0^T e^{2\int_0^s d(X(u))\,du}|g(s,0,0)|^2\,ds]. \end{aligned}$$

Because $T > t$ is chosen arbitrary, it follows that

$$|k_x(0)|^2 \leq \frac{1}{2\delta_2 d_2^2} E_x[\int_0^\infty e^{2\int_0^s d(X(u))\,du}|g(s,0,0)|^2\,ds].$$

Therefore, by condition (4.30), it follows that

$$\sup_x |k_x(0)| \leq \left(\frac{1}{2\delta_2 d_2^2} \sup_x E_x[\int_0^\infty e^{2\int_0^s d(X(u))\,du}|g(s,0,0)|^2\,ds]\right)^{\frac{1}{2}} < \infty.$$

So that $\sup_x |Y_x(0)| \leq \sup_x |p_x(0)| + \sup_x |k_x(0)| < \infty$. □

4.3 Linear PDEs

Given an operator

$$G = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q,$$

satisfying the Neumann boundary condition and with the domain $\mathcal{D}(G) \subset W^{1,2}(D)$ densely, where D is a bounded smooth domain in R^d.

Here $b = (b_1, ..., b_d)$ is a R^d-valued Borel measurable function, and q is a Borel measurable function on R^d such that:

$$I_D(|b|^2 + |q|) \in L^p(D), \quad p > d.$$

In this section, we solve the following linear boundary value problem:

$$\begin{cases} \frac{1}{2}\nabla \cdot (A\nabla u)(x) + b \cdot \nabla u(x) + q(x)u(x) = F(x) & \text{on } D, \\ \frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) = \phi & \text{on } \partial D\ , \end{cases} \tag{4.32}$$

where F and ϕ are bounded measurable functions on D and ∂D respectively.

By Green's identity, it is known that operator G defined on a bounded domain D with Neumann boundary condition $\frac{\partial \cdot}{\partial \vec{\gamma}}(x) = 0$ is associated with a quadratic form ([15], [30]):

$$\begin{aligned}\mathcal{E}(f,g) : &= -\int_D Gf(x)g(x)dx \\ &= \frac{1}{2}\int_D < A\nabla f, \nabla g > dx - \int_D b\cdot\nabla f(x)g(x)dx - \int_D q(x)f(x)g(x)dx.\end{aligned}$$

Definition 4.3.1. *A bounded continuous function u defined on D is a weak solution of the problem* (4.32) *if $u \in W^{1,2}(D)$, and for any $g \in C^\infty(\overline{D})$,*

$$\mathcal{E}(u,g) = -\int_{\partial D}\phi(x)g(x)\lambda(dx) - \int_D F(x)g(x)dx,$$

where λ denotes the $d-1$ dimensional Lebesgue measure on ∂D.

Consider the operator

$$G_1 = \frac{1}{2}\nabla\cdot(A\nabla u) \tag{4.33}$$

on domain D with boundary condition $\frac{\partial u}{\partial \vec{\gamma}} = 0$ on ∂D.

G_1 is associated with a reflecting diffusion process (X^0, P_x^0). By [26] and Example 1, X^0 has the following decomposition:

$$\begin{aligned} dX_t^0 &= \sigma(X_t^0)\,dW_t + \frac{1}{2}\nabla A(X_t^0)\,dt + \vec{\gamma}(X_t^0)\,dL_t^0, \\ L_t^0 &= \int_0^t I_{\{X_s^0\in\partial D\}}\,dL_s^0, \end{aligned} \tag{4.34}$$

where the matrix $\sigma(x)$ is the positive definite symmetric square root of the matrix $A(x)$ and $\{W_t\}_{t\geq 0}$ is a d-dimensional standard Brownian motion.

It is well known that operator G_1 is associated with the regular Dirichlet form

([15]):

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int_D a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx$$

and the domain of $\mathcal{E}^0$ is $W^{1,2}(D)$.

The following lemma can be proved similarly as the Corollary 3.8 in [21] using the heat kernel estimates on $p_1(t,x,y)$ associated with G_1 in Chapter 3.

Lemma 4.3.2. *There exists a constant* $K > 0$*, such that*

$$\sup_{x\in\bar{D}} E_x^0[L_t^0] \le K\sqrt{t} \quad \text{and} \quad \inf_{x\in\bar{D}} E_x^0[L_t^0] > 0.$$

Moreover, for all positive integer n*, we have* $\sup_{x\in\bar{D}} E_x^0[(L_t^0)^n] \le K_n t^{\frac{n}{2}}$*, for some constant* $K_n > 0$*.*

Set $M_t^0 = \int_0^t \sigma(X_s^0)\,dW_s$ and

$$Z_t = e^{\int_0^t <A^{-1}b(X_s^0),dM_s^0> - \frac{1}{2}\int_0^t b^* A^{-1}b(X_s^0)\,ds + \int_0^t q(X_s^0)\,ds}, \tag{4.35}$$

where b^* is the transpose of the row vector b.

The proof of the following two Lemmas are inspired by that of the Lemma 2.1 and Theorem 2.2 in [21].

Lemma 4.3.3. *For* $t > 0$*, there are two strictly positive functions* $M_1(t)$ *and* $M_2(t)$ *such that, for any* $x \in \overline{D}$*,* $M_1(t) \le E_x^0[\int_0^t Z_s\,dL_s^0] \le M_2(t)$*. Furthermore,* $M_2(t) \to 0$ *as* $t \to 0$*.*

Proof. 1°: Put

$$\tilde{M}(t) = e^{\int_0^t <A^{-1}b(X_s^0),dM_s^0> - \frac{1}{2}\int_0^t b^* A^{-1}b(X_s^0)\,ds}, \tag{4.36}$$

$$e_q(t) = e^{\int_0^t q(X_s^0)\,ds}.$$

Then by martingale inequality and the fact that L^0 is increasing and continuous, we have

$$\begin{aligned}
\sup_{x\in\overline{D}} E_x^0\Big[\int_0^t Z_s\, dL_s^0\Big] &= \sup_{x\in\overline{D}} E_x^0\Big[\int_0^t \tilde{M}(s)e_q(s)\ dL_s^0\Big] \\
&\le \sup_{x\in\overline{D}} E_x^0\big[\max_{0\le s\le t} |\tilde{M}(s)|^2\big]^{\frac{1}{2}} \cdot \sup_{x\in\overline{D}} E_x^0\big[e_{2|q|}(t)(L_t^0)^2\big]^{\frac{1}{2}} \\
&\le C \underbrace{\sup_{x\in\overline{D}} E_x^0\big[|\tilde{M}_t|^2\big]^{\frac{1}{2}}}_{(I)} \cdot \underbrace{\sup_{x\in\overline{D}} E_x^0\big[e_{4|q|}(t)\big]^{\frac{1}{4}}}_{(II)} \cdot \underbrace{\sup_{x\in\overline{D}} E_x^0\big[(L_t^0)^4\big]^{\frac{1}{4}}}_{(III)}
\end{aligned}$$

By the fact $|b|^2, |q| \in L^p$ and Theorem 2.1 in [27], (I) and (II) are bounded if t belongs to a bounded interval.

Because of $E_x^0[(L_t^0)^n] \le K_n t^{\frac{n}{2}}$, we see that $M_2(t) := K(I)(II)\sqrt{t}$ is the required upper bound.

2°: Since

$$E_x^0[L_t^0]^2 \le E_x^0\Big[\int_0^t \tilde{M}^{-1}(s)e_{-q}(s)\, dL_s^0\Big] \cdot E_x^0\Big[\int_0^t \tilde{M}(s)e_q(s)\, dL_s^0\Big], \tag{4.37}$$

we obtain

$$E_x^0\Big[\int_0^t \tilde{M}(s)e_q(s)\, dL_s^0\Big] \ge \frac{E_x^0[L_t^0]^2}{E_x^0[\int_0^t \tilde{M}^{-1}(s)e_{-q}(s)\, dL_s^0]}. \tag{4.38}$$

Here

$$\begin{aligned}
\tilde{M}^{-1}(t) &= e^{-\int_0^t <A^{-1}b(X_s^0), dM_s^0> + \frac{1}{2}\int_0^t b^*A^{-1}b(X_s^0)\, ds} \\
&= e^{-\int_0^t <A^{-1}b(X_s), dM_s^0> - \frac{1}{2}\int_0^t b^*A^{-1}b(X_s^0)\, ds} \cdot e^{\int_0^t b^*A^{-1}b(X_s^0)\, ds} \\
&:= N(t)\cdot e^{\int_0^t b^*A^{-1}b(X_s^0)\, ds}
\end{aligned} \tag{4.39}$$

By the proof of the first part, replacing $\tilde{M}_t$, q by N_t and $b^*A^{-1}b - q$ respectively, it can be seen that there exists $K(t) > 0$ such that $\sup_{x\in\overline{D}} E_x^0[\int_0^t \tilde{M}^{-1}(s)e_{-q}(s)\, dL_s^0] \le K(t)$.

As $\inf_{x\in\overline{D}} E_x^0[L_t^0] > 0$, we complete the proof of the lemma by setting $M_1(t) = \frac{\inf_{x\in\overline{D}} E_x^0[L_t^0]^2}{K(t)}$. □

Set $L(x) := E_x^0[\int_0^\infty Z_s\, dL_s^0]$.

Lemma 4.3.4. *If there is a point $x_0 \in \overline{D}$, such that $L(x_0) < \infty$, then there are two positive constants K and β such that* $\sup_{x\in\overline{D}} E_x^0[Z_t] \leq Ke^{-\beta t}$.

Proof. By Girsanov Theorem and Feymann-Kac formula, $G = \frac{1}{2}\nabla\cdot(A\nabla)+b\cdot\nabla+q$ is associated with the semigroup $\{S_t\}_{t>0}$, where $S_t f(x) = E_x^0[Z_t f(X_t^0)]$ for $f \in L^2(D)$.

By the upper and lower bound estimates of the heat kernel $p(t,x,y)$ associated with S_t in [37] and Chapter 3, without losing generality, we assume that

$$m_1^{-1}e^{-m_3|x-y|^2} \leq p(1,x,y) \leq m_1 e^{-m_2|x-y|^2}.$$

So that we have

$$p(1,x,y) \leq m_1 e^{-m_2|x-y|^2} \leq m_1,$$

and

$$p(1,x,y) \geq m_1^{-1}e^{-m_3|x-y|^2} \geq m_1^{-1}e^{-m_3(\sup_{x,y\in D}|x-y|^2)}.$$

The above two estimates imply that, for any positive function $f \in L^2(D)$,

$$c^{-1}\int_D f(x)dx \leq E_x^0[Z_1 f(X_1^0)] = \int_D p(1,x,y)f(y)dy \leq c\int_D f(x)dx, \qquad (4.40)$$

where c is a positive constant. Since

$$L(x) = \sum_{n=0}^{\infty} E_x^0[Z_n E_{X_n^0}^0[\int_0^1 Z_s L^0(ds)]] \geq M_1(1)\sum_{n=0}^{\infty} E_x^0[Z_n]$$

and $L(x_0) < \infty$, there is a positive integer number N such that

$$\frac{1}{2c^2} \geq E_{x_0}^0[Z_N] = E_{x_0}^0[Z_1 E_{X_1}^0[Z_{N-1}]] \geq c^{-1}\int_D E_x^0[Z_{N-1}]m(dx).$$

This implies

$$\int_D E_x^0[Z_{N-1}]m(dx) \le \frac{1}{2c}.$$

Thus

$$\sup_{x\in\overline{D}} E_x^0[Z_N] = \sup_{x\in\overline{D}} E_x^0[Z_1 E_{X_1}^0[Z_{N-1}]] \le c\int_D E_x^0[Z_{N-1}]m(dx) \le \frac{1}{2}. \tag{4.41}$$

For any $t > 0$, there exists a positive number n such that $\frac{t}{N} \in [n-1, n)$. Then by (4.41), it follows that

$$\begin{aligned} E_x^0[Z_t] \le \frac{1}{2^{n-1}} E_x^0[Z_{t-N(n-1)}] \ &\le \ \left(\sup_{x\in D, 0\le t\le N} E_x^0[Z_t]\right)\frac{1}{2^{n-1}} \\ &\le \ 2 \sup_{x\in D, 0\le t\le N} E_x^0[Z_t] e^{-\frac{ln2}{N}t}. \end{aligned}$$

□

Theorem 4.3.5. *If there exists $x_0 \in \overline{D}$ such that $L(x_0) < \infty$, then there exists a unique bounded continuous weak solution of the problem* (4.32).

Proof. Existence :

Due to Theorem 3.2 in [8], there exists a unique, bounded, continuous weak solution u_2 of the following problem:

$$\begin{cases} Gu_2(x) = 0, & \text{on } D, \\ \frac{1}{2}\frac{\partial u_2}{\partial \vec{\gamma}}(x) = \phi, & \text{on } \partial D \ . \end{cases} \tag{4.42}$$

Thus by the linearity of the problem (4.32), we only need to show that the following problem has a bounded continuous weak solution:

$$\begin{cases} Gu_1(x) = F(x), & \text{on } D, \\ \frac{\partial u_1}{\partial \vec{\gamma}}(x) = 0, & \text{on } \partial D \ . \end{cases} \tag{4.43}$$

The semigroup associated with operator G is $\{S_t, t > 0\}$. By Lemma 4.3.4, we have

$$\sup_{x\in D}|S_tF(x)| = \sup_{x\in D}|E_x^0[Z_tF(X_t^0)]| \leq Ke^{-\beta t}\|F\|_\infty.$$

Then

$$u_1(x) := \int_0^\infty S_tF(x)dt$$

is well defined and has the following bound:

$$\sup_{x\in D}|u_1(x)| \leq \frac{K}{\beta}\|F\|_\infty.$$

Moreover, the function $u_1(x)$ is also continuous on D. In fact, fixing any $x \in D$ and $\epsilon > 0$, we can firstly choose a constant $t_0 > 0$, such that $\sup_{z\in D}|\int_0^{t_0} S_sF(z)\,ds| < \frac{\epsilon}{3}$. And because $S_{t_0}u_1(x)$ is continuous, there exists a constant $\delta > 0$, such that for any y with $|y-x| < \delta$, $|S_{t_0}u_1(x) - S_{t_0}u_1(y)| \leq \frac{\epsilon}{3}$.

We find that

$$\begin{aligned} S_tu_1(x) = E_x^0[Z_tu_1(X_t^0)] &= E_x^0[Z_t\int_0^\infty E_{X_t^0}[Z_sF(X_s^0)]\,ds] \\ &= \int_0^\infty E_x^0[Z_{t+s}F(X_{t+s}^0)]\,ds \\ &= \int_t^\infty S_sF(x)\,ds \\ &= u_1(x) - \int_0^t S_sF(x)\,ds. \end{aligned} \tag{4.44}$$

For any y satisfying $|y-x| < \delta$, it follows that

$$|u_1(x) - u_1(y)| \leq |S_{t_0}u_1(x) - S_{t_0}u_1(y)| + |\int_0^{t_0} S_sF(x)\,ds| + |\int_0^{t_0} S_sF(y)\,ds| \leq \epsilon.$$

This implies that the function u_1 is continuous on domain D.

Denote the resolvent associated with operator G by $\{G_\beta, \beta > 0\}$. Note that

$$\begin{aligned}
G_\beta u_1(x) &= \int_0^\infty e^{-\beta t} S_t u_1(x) dt \\
&= \int_0^\infty e^{-\beta t} u_1(x) dt - \int_0^\infty e^{-\beta t} \int_0^t S_s F(x)\, ds dt \\
&= \frac{1}{\beta} u_1(x) - \int_0^\infty \int_0^t e^{-\beta t} S_s F(x)\, ds dt \\
&= \frac{1}{\beta} u_1(x) - \int_0^\infty S_s F(x) (\int_s^\infty e^{-\beta t} dt)\, ds \\
&= \frac{1}{\beta} u_1(x) - \frac{1}{\beta} G_\beta F(x). \qquad (4.45)
\end{aligned}$$

We have

$$\beta(u_1(x) - \beta G_\beta u_1(x)) = \beta G_\beta F(x).$$

Therefore,

$$\begin{aligned}
\lim_{\beta\to\infty} \int_D \beta(u_1(x) - \beta G_\beta u_1(x)) u_1(x) dx &= \lim_{\beta\to\infty} \int_D \beta G_\beta F(x) u_1(x) dx \\
&= \int_D F(x) u_1(x) dx < \infty.
\end{aligned}$$

This implies that $u_1 \in D(\mathcal{E})$ (see [30]) and u_1 is a weak solution of equation (4.43). By the linearity, $u = u_1 + u_2$ is a bounded continuous weak solution of equation (4.32).

Uniqueness :

Let v_1 and v_2 be two bounded continuous weak solutions of the equation (4.32). Then $v_1 - v_2$ is the solution of equation (4.42) with $\phi = 0$. Then by the uniqueness of solutions to the equation (4.42) proved in [8], we know that $v_1 = v_2$. □

4.4 Semilinear PDEs

In this section, we solve the following semilinear boundary value problem:

$$\begin{cases} \frac{1}{2}\nabla \cdot (A\nabla u)(x) + b \cdot \nabla u(x) + q(x)u(x) = -H(x, u(x), \nabla u(x)), & \text{on } D, \\ \frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) = \phi(x) & \text{on } \partial D. \end{cases} \qquad (4.46)$$

Recall $\mathcal{E}(\cdot,\cdot)$ is the quadratic form associated with the operator $G = \frac{1}{2}\nabla\cdot(A\nabla) + b\cdot\nabla + q$. Then

$$\mathcal{E}(u,v) = \frac{1}{2}\int_D < A\nabla u, \nabla v > dx - \int_D < b, \nabla u > vdx - \int_D quvdx.$$

Definition 4.4.1. *A bounded continuous function $u(x)$ defined on D is called a weak solution of the equation* (4.46) *if $u \in W^{1,2}(D)$, and for any $g \in C^\infty(\overline{D})$,*

$$\mathcal{E}(u,g) = \int_{\partial D} \phi(x)g(x)\lambda(dx) + \int_D H(x,u(x),\nabla u(x))g(x)dx.$$

Recall that L_t is the boundary local time of $X(t)$ defined in (4.7) and L_t^0 is the boundary local time of X_t^0 in (4.34).

As a consequence of the Girsanov theorem, we have the following lemma.

Lemma 4.4.2. *Assume the function f satisfies $E_x[\int_0^T e^{\int_0^t f(X(u))\,du} dL_t] < \infty$. Then it holds that*

$$E_x[\int_0^T e^{\int_0^t f(X(u))\,du} dL_t] = E_x^0[\int_0^T \tilde{M}_t e^{\int_0^t f(X_u^0)\,du} dL_t^0],$$

where $\tilde{M}_t$ was defined in (4.36).

Proof. Because the boundary ∂D is smooth, it is not difficult to see that there exists a function $g \in C^2(\bar{D})$ with $\frac{\partial g}{\partial \vec{\gamma}}(x) = 1$ if $x \in \partial D$. By Ito's formula, we get

$$g(X_T^0) = g(X_0^0) + \int_0^T < \nabla g(X_s^0), dM_s^0 > + \int_0^T (G_1 g)(X_s^0)\,ds + L_T^0.$$

And by the chain rule, we obtain

$$\begin{aligned} e^{\int_0^T f(X_u^0)\,du} g(X_T^0) &= g(X_0^0) + \int_0^T e^{\int_0^s f(X_u^0)\,du} < \nabla g(X_s^0), dM_s^0 > \\ &\quad + \int_0^T e^{\int_0^s f(X_u^0)\,du} (fg + G_1 g)(X_s^0)\,ds + \int_0^T e^{\int_0^s f(X_u^0)\,du}\, dL_s^0. \end{aligned}$$

Finally, it follows that

$$\begin{aligned}
& \tilde{M}_T e^{\int_0^T f(X_u^0)\,du} g(X_T^0) \\
= \;& g(X_0^0) + \int_0^T \tilde{M}_s e^{\int_0^s f(X_u^0)\,du} < \nabla g(X_s^0), dM_s^0 > \\
& + \int_0^T \tilde{M}_s e^{\int_0^s f(X_u^0)\,du} (fg + G_1 g)(X_s^0)\,ds + \int_0^T \tilde{M}_s e^{\int_0^s f(X_u^0)\,du}\,dL_s^0 \\
& + \int_0^T e^{\int_0^s f(X_u^0)\,du} g(X_s^0) d\tilde{M}_s + \int_0^T \tilde{M}_s e^{\int_0^s f(X_u^0)\,du} < b, \nabla g > (X_s^0)\,ds.
\end{aligned}$$

Taking expectations on both sides of the above equation, we obtain

$$\begin{aligned}
E_x^0[\tilde{M}_T e^{\int_0^T f(X_u^0)\,du} g(X_T^0)] \;=\;& g(x) + E_x^0[\int_0^T \tilde{M}_s e^{\int_0^s f(X_u^0)\,du} (fg + G_2 g)(X_s^0)\,ds] \\
& + E_x^0[\int_0^T \tilde{M}_s e^{\int_0^s f(X_u^0)\,du}\,dL_s^0]. \qquad (4.47)
\end{aligned}$$

On the other hand, by the Girsanov theorem and Ito's formula, we obtain

$$\begin{aligned}
& E_x^0[\tilde{M}_T e^{\int_0^T f(X_u^0)\,du} g(X_T^0)] \\
= \;& E_x[e^{\int_0^T f(X(u))\,du} g(X(T))] \\
= \;& g(x) + E_x[\int_0^T e^{\int_0^s f(X(u))\,du} (fg + G_2 g)(X(s))\,ds] \\
& + E_x[\int_0^T e^{\int_0^s f(X(u))\,du}\,dL_s] \qquad (4.48)
\end{aligned}$$

Comparing the formulas (4.47) and (4.48), we get the final result in this lemma.

□

Lemma 4.4.3. *Suppose that the function $\tilde{q} \in L^p(D)$ and $p > d$. If there exists some point $x_0 \in D$, such that*

$$E_{x_0}[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))\,du} dL_t] < \infty, \qquad (4.49)$$

it holds that

$$\sup_{x\in D} E_x[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))\,du} dL_t] < \infty.$$

Proof. Due to the Theorem 3.2 in [8] and the condition (4.49), there is a continuous bounded function $f \in W^{1,2}(D)$, such that

$$\frac{1}{2}\nabla\cdot(A\nabla f) + b\cdot\nabla f + \tilde{q} f = 0$$

on domain D and $\frac{1}{2}\frac{\partial f}{\partial \tilde{\gamma}} = 1$ on ∂D.

Moreover f has the following expression:

$$f(x) = E_x^0[\int_0^\infty e^{\int_0^t < A^{-1}b(X^0(u)), dM_u^0 > - \frac{1}{2}\int_0^t b^* A^{-1} b(X_s^0)\,ds + \int_0^t \tilde{q}(X_u^0)\,du} dL_t^0].$$

By the Lemma 4.4.2, it follows that

$$f(x) = E_x[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))\,du} dL_t].$$

Then the lemma is proved, for f is bounded on the domain $\bar{D}$. □

Let $H : R^d \times R \times R^d \to R$ be a bounded Borel measurable function. Introduce the following conditions:

(H.1) $(y_1 - y_2)(H(x, y_1, z) - H(x, y_2, z)) \le -h_1(x)|y_1 - y_2|^2$,

(H.2) $|H(x, y, z_1) - H(x, y, z_2)| \le h_2|z_1 - z_2|$.

Set $h(t) = -h_1(X(t)) + \delta h_2^2 + q(X(t))$ and $\tilde{h}(t) = -h_1(X(t)) + \delta h_2^2$ for some constant $\delta > \frac{1}{2\lambda}$.

Theorem 4.4.4. *Suppose that the conditions **(H.1)** and **(H.2)** are satisfied. Assume*

$$E_{x_1}[\int_0^\infty e^{2\int_0^t (q(X(u)) + \tilde{h}(u))\,du} dt] < \infty, \quad for \quad some \quad x_1 \in D, \tag{4.50}$$

and there exists some point $x_0 \in D$*, such that*

$$E_{x_0}[\int_0^\infty e^{\int_0^t q(X(u))\,du} dL_t] < \infty. \tag{4.51}$$

Then the semilinear Neumann boundary value problem (4.46) *has a unique continuous weak solution.*

Proof. Set

$$\tilde{H}(t,x,y,z) := e^{\int_0^t q(X(u))dt} H(x, e^{-\int_0^t q(X(u))dt} y, e^{-\int_0^t q(X(u))dt} z).$$

Then

$$(y_1 - y_2)(\tilde{H}(t, X(t), y_1, z) - \tilde{H}(t, X(t), y_2, z)) \leq -h_1(X(t))|y_1 - y_2|^2 \tag{4.52}$$

and

$$|\tilde{H}(t, X(t), y, z_1) - \tilde{H}(t, X(t), y, z_2)| \leq h_2 |z_1 - z_2|. \tag{4.53}$$

Using the fact that H is bounded, we can show that

$$\tilde{H}(t, X(t), y, z) \leq e^{\int_0^t q(X(u))dt} \|H\|_\infty.$$

By Theorem 4.2.3, there exists a unique process $(\hat{Y}_x, \hat{Z}_x)$ satisfying

$$d\hat{Y}_x(t) = -\tilde{H}(t, X(t), \hat{Y}_x(t), \hat{Z}_x(t))dt + e^{\int_0^t q(X(u))\,du} \phi(X(t)) dL(t) + < \hat{Z}_x(t), dM_x(t) >$$
$$e^{\int_0^t \tilde{h}(u)\,du} \hat{Y}_x(t) \to 0 \quad as \quad t \to \infty.$$

Furthermore, Corollary 4.2.4 implies that $\sup_x \hat{Y}_x(0) < \infty$.

From Ito's formula, it follows that

$$\begin{aligned}
& de^{-\int_0^t q(X(u))\,du}\hat{Y}_x(t) \\
= \;& -q(X(t))e^{-\int_0^t q(X(u))\,du}\hat{Y}_x(t)dt - e^{-\int_0^t q(X(u))\,du}\tilde{H}(t,X(t),\hat{Y}_x(t),\hat{Z}_x(t))dt \\
& +\phi(X(t))dL_t + < e^{-\int_0^t q(X(u))\,du}\hat{Z}_x(t), dM_x(t) > .
\end{aligned}$$

Setting $Y_x(t) := e^{-\int_0^t q(X(u))\,du}\hat{Y}_x(t)$ and $Z_x(t) := e^{-\int_0^t q(X(u))\,du}\hat{Z}_x(t)$, we obtain

$$\begin{aligned}
& dY_x(t) \\
= \;& -(q(X(t))Y_x(t) + H(X(t),Y_x(t),Z_x(t)))dt + \phi(X(t))dL_t + < Z_x(t), dM_x(t) > .
\end{aligned}$$

Moreover,

$$e^{\int_0^t h(u)\,du}Y_x(t) = e^{\int_0^t h(u)\,du}e^{-\int_0^t q(X(u))\,du}\hat{Y}_x(t) = e^{\int_0^t \tilde{h}(X(u))\,du}\hat{Y}_x(t) \to 0 \quad as \quad t \to \infty.$$

So by Ito's formula, we have that, for any $t < T$,

$$\begin{aligned}
& e^{\int_0^t h(u)\,du}Y_x(t) \\
= \;& e^{\int_0^T h(u)\,du}Y_x(T) + \int_t^T e^{\int_0^s h(u)\,du}\left(H(X(s),Y_x(s),Z_x(t)) + q(X(s))Y_x(s)\right)ds \\
& - \int_t^T e^{\int_0^s h(u)\,du}\phi(X(s))\,dL_s - \int_t^T h(s)e^{\int_0^s h(u)\,du}Y_x(s)\,ds \\
& - \int_t^T e^{\int_0^s h(u)\,du} < Z_x(t), dM_x(t) > . \qquad (4.54)
\end{aligned}$$

Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$.

Since $Y_x(0) = \hat{Y}_x(0)$, we know that u_0 is a bounded function on domain D. By the Markov property of X and the uniqueness of (Y_x, Z_x) , it is easy to see that

$$Y_x(t) = u_0(X(t)), \quad Z_x(t) = v_0(X(t)).$$

So that $\sup_{x\in D,t>0} |Y_x(t)| \leq \|u_0\|_\infty < \infty$.

Now consider the following problem:

$$\begin{cases} L_2 u(x) = -H(x, u_0(x), v_0(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) = \phi(x) & \text{on } \partial D. \end{cases} \tag{4.55}$$

By Theorem 4.3.5, equation (4.55) has a unique continuous weak solution u. Next we will show that $u = u_0$.

Since u belongs to the domain of the Dirichlet form associated with the process $X(t)$, it follows from the Fukushima's decomposition:

$$\begin{aligned} & du(X(t)) \\ = & -[H(X(t), u_0(X(t)), v_0(X(t))) + q(X(t))u(X(t))]dt \\ & +\phi(X(t))dL(t) + < \nabla u(X(t)), dM_x(t) > \\ = & -[H(X(t), Y_x(t), Z_x(t)) + q(X(t))u(X(t))] \\ & +\phi(X(t))dL(t) + < \nabla u(X(t)), dM_x(t) > . \end{aligned}$$

From the condition (4.50) and the boundedness of $u(x)$, it follows that

$$\lim_{t\to\infty} E_x[e^{2\int_0^t h(u)\,du} u^2(X(t))] \le \|u\|_\infty^2 \lim_{t\to\infty} E_x[e^{2\int_0^t (\tilde{h}+q)(u)\,du}] = 0.$$

By Ito's formula, it follows that, for any $t < T$,

$$\begin{aligned} & e^{\int_0^t h(u)\,du} u(X(t)) \\ = & \; e^{\int_0^T h(u)\,du} u(X(T)) + \int_t^T e^{\int_0^s h(u)\,du}[H(X(s), Y_x(s), Z_x(s)) + q(X(s))u(X(s))]\,ds \\ & - \int_t^T e^{\int_0^s h(u)\,du}\phi(X(s))dL(s) - \int_t^T h(s)e^{\int_0^s h(u)\,du} u(X(s))\,ds \\ & - \int_t^T e^{\int_0^s h(u)\,du} < \nabla u(X(t)), dM_x(t) > . \end{aligned} \tag{4.56}$$

Set

$$v_x(t) = u(X(t)) - Y_x(t) \quad \textit{and} \quad R_x(t) = \nabla u(X(t)) - Z_x(t).$$

Comparing the equations (4.54) and (4.56), we obtain the following equation: for any $t < T$,

$$\begin{aligned}
& e^{\int_0^t h(u)\,du} v(X(t)) \\
= \; & e^{\int_0^T h(u)\,du} v(X(T)) + \int_t^\infty (q(X(u)) - h(u)) e^{\int_0^s h(u)\,du} v(X(s))\,ds \\
& - \int_t^\infty e^{\int_0^s h(u)\,du} < R_x(t),, dM_x(t) > \\
= \; & e^{\int_0^T h(u)\,du} v(X(T)) - \int_t^T \tilde{h}(s) e^{\int_0^s h(u)\,du} v(X(s))\,ds \\
& - \int_t^T e^{\int_0^s h(u)\,du} < R_x(t), dM_x(t) > . \qquad (4.57)
\end{aligned}$$

Set $g(t) = e^{\int_0^t h(u)\,du} v(t)$. Taking conditional expectations on both sides of (4.57), we find that

$$\begin{aligned}
g(t) &= E_x[g(T) - \int_t^T \tilde{h}(s) g(s)\,ds | \mathcal{F}_t] \\
&= E_x[g(T)(1 - \int_t^T \tilde{h}(s)\,ds) + \int_t^T \int_s^T \tilde{h}(s)\tilde{h}(s_1) g(s_1)\,ds_1 ds | \mathcal{F}_t] \\
&= E_x[g(T)(1 - \int_t^T \tilde{h}(s)\,ds + \frac{1}{2}(\int_t^T \tilde{h}(s)\,ds)^2) \\
&\quad + (-1)^3 \int_t^T \int_s^T \int_{s_1}^T \tilde{h}(s)\tilde{h}(s_1)\tilde{h}(s_2) g(s_2)\,ds_2 ds_1 ds | \mathcal{F}_t].
\end{aligned}$$

Keeping iterating in the way above for n times, we obtain

$$\begin{aligned}
g(t) &= E_x[g(T)(\sum_{k=0}^n \frac{(-\int_t^T \tilde{h}(s)\,ds)^n}{n!}) \\
&\quad + (-1)^{n+1} \int_t^T \int_s^T \int_{s_1}^T \ldots \int_{s_{n-1}}^T \tilde{h}(s)\tilde{h}(s_1)\ldots\tilde{h}(s_n) g(s_n)\,ds_n \ldots ds_1 ds | \mathcal{F}_t].
\end{aligned}$$

It follows that

$$|g(t)| \leq E_x[|g(T)| e^{-\int_t^T \tilde{h}(s)\,ds} | \mathcal{F}_t].$$

Then by $g(t) = e^{\int_0^t h(u)\,du} v(t)$, we obtain,

$$|v(t)| \leq E_x[|v(T)|e^{\int_t^T (h(s)-\tilde{h}(s))\,ds}|\mathcal{F}_t] \leq (\|u_0\|_\infty + \|u\|_\infty)E_x[e^{\int_t^T q(X(s))\,ds}|\mathcal{F}_t]. \quad (4.58)$$

Hence, it follows that

$$0 \leq E_x[e^{\int_0^t q(X(s))\,ds}|v(t)|] \leq (\|u_0\|_\infty + \|u\|_\infty)E_x[e^{\int_0^T q(X(s))\,ds}],$$

for any $T > t$.

Since the condition (4.51) and Lemma 4.3.4 imply

$$\lim_{t\to\infty} E_x[e^{\int_0^t q(X(s))\,ds}] = 0,$$

we know that $E_x[e^{\int_0^t q(X(s))\,ds}|v(t)|] = 0$. This implies that $v(t) = 0$, $P_x - a.s.$.

Therefore, for any $t > 0$, we have $u(X(t)) = Y_x(t)$. In particular, $u(x) = E_x[u(X(0))] = E_x[Y_x(0)] = u_0(x)$. This shows that $u(x)$ is a weak solution of the equation (4.46).

If $\tilde{u}$ is another solution of the problem (4.46). Then the processes $\tilde{Y}_x(t) := \tilde{u}(X(t))$ and $\tilde{Z}_x(t) := \nabla\tilde{u}(X(t))$ satisfy the following equation

$$d\tilde{Y}_x(t) = -H(X(t), \tilde{Y}_x(t), \tilde{Z}_x(t))dt - \phi(X(t))dL_t + < \tilde{Z}_x(t), dM_x(t) > . \quad (4.59)$$

Set $\bar{Y}_x(t) = e^{\int_0^t q(X(u))\,du}\tilde{Y}_x(t)$ and $\bar{Z}_x(t) = e^{\int_0^t q(X(u))\,du}\tilde{Z}_x(t)$.

By the chain rule, it follows that

$$d\bar{Y}_x(t) = -\tilde{H}(X(t), \bar{Y}_x(t), \bar{Z}_x(t))dt + e^{\int_0^t q(X(u))\,du}\phi(X(t))dL(t) + < \bar{Z}_x(t), dM_x(t) > .$$

Moreover, because $\tilde{u}$ is bounded,

$$\lim_{t\to\infty} e^{\int_0^t \tilde{h}(u)\,du}\bar{Y}_x(t) = \lim_{t\to\infty} e^{\int_0^t h(u)\,du}\tilde{u}(X(t)) = 0$$

is also satisfied. Therefore, from the uniqueness of the solution of the BSDE in Theorem 4.2.3, we have

$$\tilde{Y}_x(t) = Y_x(t) \quad \text{and} \quad \tilde{Z}_x(t) = Z_x(t).$$

In particular,

$$\tilde{u}(x) = E_x[\tilde{Y}_x(t)] = E_x[Y_x(t)] = u(x).$$

□

4.5 Semilinear Elliptic PDEs with Singular Coefficients

Recall the operator

$$L = \frac{1}{2}\nabla \cdot (A\nabla) + B \cdot \nabla - div(\hat{B}\cdot) + Q$$

on the domain D equipped with the mixed boundary condition on ∂D:

$$\frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) - < \hat{B}, n > u(x) = 0.$$

$\mathcal{Q}(u, v)$ is the quadratic form associated with L, with the domain $\mathcal{D}(\mathcal{Q}) = W^{1,2}(D)$. $\{T_t, \, t \geq 0\}$ denotes the semigroup generated by L.

In this section, our main aim is to solve the following equation:

$$\begin{cases} Lf(x) = -F(x, f(x)), & \text{on } D, \\ \frac{1}{2}\frac{\partial f}{\partial \vec{\gamma}}(x) - < \widehat{B}, n > (x) f(x) = \Phi(x), & \text{on } \partial D. \end{cases} \tag{4.60}$$

Definition 4.5.1. *A bounded continuous function u defined on D is called a weak* *solution of the equation* (4.60) *if* $u \in W^{1,2}$*, and for any* $g \in C^\infty(\bar{D})$,

$$\mathcal{Q}(u,g) = \int_{\partial D} \Phi(x) g(x) \lambda(dx) + \int_D F(x, u(x)) g(x) dx.$$

Here the function $F : R^d \times R \to R$ is a bounded measurable function and satisfies the following condition:

(E.1) $(y_1 - y_2)(F(x, y_1) - F(x, y_2)) \le -r_1(x)|y_1 - y_2|^2$.

Set

$$\begin{aligned} \hat{Z}_t &= \exp(\int_0^t (A^{-1}B)^*(X_s^0) dM_s^0 + (\int_0^t (A^{-1}\hat{B})^*(X_s^0) dM_s^0) \circ \gamma_t^0 \\ &\quad - \frac{1}{2} \int_0^t (B - \hat{B})^* A^{-1} (B - \hat{B})(X_s^0)\, ds + \int_0^t Q(X_s^0)\, ds). \end{aligned} \tag{4.61}$$

By the reduction method introduced in Section 3.2, there exists a bounded, continuous functions $v \in W^{1,p}(D)$ satisfying

$$T_t f(x) = e^{-v(x)} S_t[f e^v](x).$$

Here, S_t is the semigroup generated by the operator: $G = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q$ equipped with the boundary condition $< A\nabla u, n >= 0$, where $b := B - \hat{B} - (A\nabla v)$ and $q := Q + \frac{1}{2}(\nabla v)^* A (\nabla v) - < B - \hat{B}, \nabla v >$.

In this section, we will stick to this particular choice of b and q. Recall that

$$\tilde{M}(t) = e^{\int_0^t < A^{-1} b(X_s^0), dM_s^0 > - \frac{1}{2} \int_0^t b^* A^{-1} b(X_s^0)\, ds}$$

and set $Z_t = \tilde{M}(t) e^{\int_0^t q(X_s^0)\, ds}$.

Then from (3.9), it follows that

$$\hat{Z}(t) = Z_t e^{v(X_t^0) - v(X_0^0)}.$$

The process (X, P_x) associated with operator G_2 has the following decomposition:

$$X(t) = x + \int_0^t \sigma(X(s))dB_s + \int_0^t (\frac{1}{2}\nabla A + b)(X(s))\, ds + \int_0^t \vec{\gamma}(X(s))\, dL_s, \quad P_x - a.s.$$

where $\{B_t\}_{t\geq 0}$ is a d-dimensional Brownian motion and L_t is the local time satisfying $L_t = \int_0^t I_{\partial D}(X(s))\, dL_s$. It is known from [29] that the processes (X, P_x) and (X^0, P_x^0) are related as follows,

$$dP_x|_{\mathcal{F}_t} = \tilde{M}_t dP_x^0|_{\mathcal{F}_t}.$$

Since we can not set up conditions on the functions q and b, which are actually the "intermediates", the following lemma gives us an important condition to prove the existence of the solution to equation (4.60).

Lemma 4.5.2. *Assume that there exists $x_0 \in D$, such that*

$$E_{x_0}^0[\int_0^\infty |\hat{Z}_t|^2 e^{\int_0^t (2Q-4r_1)(X_u^0)\, du} dL_t^0] < \infty. \tag{4.62}$$

Then there exists a positive number $\varepsilon > 0$, such that if $\|\hat{B}\|_{L^p} \leq \varepsilon$, the following inequality holds:

$$\sup_{x\in D} E_x[\int_0^\infty e^{2\int_0^t (-r_1+q)(X(u))\, du} dt] < \infty. \tag{4.63}$$

Proof. First note that,

$$\begin{aligned}
E_x[e^{2\int_0^t (-r_1+q)(X(u))\, du}] &= E_x^0[\tilde{M}(t) e^{2\int_0^t (-r_1+q)(X_u^0)\, du}] \\
&= E_x^0[Z(t) e^{\int_0^t (-2r_1+q)(X(u))\, du}] \\
&\leq C_1 E_x^0[\hat{Z}(t) e^{-2\int_0^t (r_1(X(u))\, du} e^{\int_0^t (Q+\frac{1}{2}<A\nabla v-2(B-\hat{B}),\nabla v>)(X_u^0)\, du}] \\
&\leq C_1 E_x^0[\hat{Z}^2(t) e^{2\int_0^t (Q-2r_1)(X_u^0)\, du}]^{\frac{1}{2}} \cdot E_x^0[e^{\int_0^t <A\nabla v-2(B-\hat{B}),\nabla v>(X_u^0)\, du}]^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 4.3.4 and condition (4.62), there exist two constants $c_2, \beta > 0$ such that

$$\sup_{x\in D} E_x[\hat{Z}^2(t)e^{2\int_0^t (Q-2r_1)(X_u^0)\,du}] < c_2 e^{-\beta t}.$$

Moreover, for $p > d$, by the Theorem 2.1 in [27], there exist two positive constants c_3 and c_4 such that

$$E_x^0[e^{\int_0^t <A\nabla v - 2(B-\hat{B}),\nabla v>(X_u^0)\,du}] \leq c_3 e^{c_4 t},$$

where $c_4 = c\| < A\nabla v - 2(B-\hat{B}), \nabla v > \|_{L^{p/2}}$.

Since $\|\nabla v\|_{L^p} \leq C\|\hat{B}\|_{L^p(D)}$ (see [37]), there exists $\varepsilon > 0$, such that $\|\hat{B}\|_{L^p(D)} \leq \varepsilon$ implies $c_4 < \beta$. Thus (4.63) holds. □

Theorem 4.5.3. *Assume* (4.62). *Then for some point* $x_0 \in D$

$$E_{x_0}^0[\int_0^\infty \hat{Z}_s\,dL_s^0] < \infty \tag{4.64}$$

and moreover, $\|\hat{B}\|_{L^p} \leq \varepsilon$, *where* ε *is as in Lemma* 4.5.2. *Then the problem* (4.60) *has a unique, bounded, continuous weak solution* $u(x)$.

Proof. Existence:

Set $\tilde{F}(x,y) = e^{v(x)}F(x, e^{-v(x)}y)$ and $\phi(x) = e^{v(x)}\Phi(x)$.

From the boundedness of v , $\tilde{F}$ is also bounded and satisfies

$$(y_1 - y_2)(\tilde{F}(x,y_1) - \tilde{F}(x,y_2)) \leq -r_1(x)|y_1 - y_2|^2.$$

Moreover, there is a constant $c > 0$, such that

$$\begin{aligned}\infty > E_{x_0}^0[\int_0^\infty \hat{Z}_s\,dL_s^0] &= E_{x_0}^0[\int_0^\infty Z_s e^{v(X_s^0)-v(X_0^0)}\,dL_s^0] \\ &\geq cE_{x_0}^0[\int_0^\infty Z_s\,dL_s^0] \\ &= cE_{x_0}^0[\int_0^\infty \tilde{M}_s e^{\int_0^s q(X_u^0)\,du}\,dL_s^0] \\ &= cE_{x_0}[\int_0^\infty e^{\int_0^s q(X_u)\,du}\,dL_s]. \end{aligned} \tag{4.65}$$

The last equality above is due to the Lemma 4.4.2.

Furthermore, by Lemma 4.4.3, it follows that

$$\sup_x E_x[\int_0^\infty e^{\int_0^t q(X(u))\,du} dL_t] < \infty. \tag{4.66}$$

By Lemma 4.5.2, the following condition is satisfied :

$$E_x[\int_0^\infty e^{2\int_0^t (q-r_1)(X(u))\,du} dt] < \infty, \tag{4.67}$$

So $\tilde{F}$ satisfies all of the conditions in Theorem 4.4.4 replacing G by $\tilde{F}$.

Thus the following problem

$$\begin{cases} Gu(x) = -\tilde{F}(x, u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) = \phi & \text{on } \partial D \end{cases} \tag{4.68}$$

has a unique bounded continuous weak solution u.

Set $f(x) = e^{-v(x)}u(x)$. Then we claim the function $f(x)$ is the weak solution of the equation (4.60).

Because function v is continuous and bounded, $f(x)$ is also continuous. From the fact that function u is the weak solution of the problem (4.68), we obtain, for any function $\psi \in C^\infty(D)$,

$$\begin{aligned} \mathcal{E}(u, e^{-v}\psi) &= \frac{1}{2}\int_D \big(< A\nabla u, \nabla(e^{-v}\psi) > - < b, \nabla u > e^{-v}\psi - e^{-v}qu\psi\big)\, dx \\ = & \int_{\partial D} e^{-v}\phi\psi\, d\lambda + \int_D \tilde{F}(x, u(x))e^{-v}\psi\, dx. \end{aligned} \tag{4.69}$$

As in the proof of Theorem 5.1 in [42], we can show that the left side of the equation (4.69) is equal to

$$\mathcal{Q}(f, \psi) = \frac{1}{2}\int_D [< A\nabla f, \nabla\psi > - < B, \nabla u > \psi - < \hat{B}, \nabla\psi > f - Qf\psi]\, dx.$$

At the same time, by the definition of the function ϕ and $\tilde{F}$, the right side of the equation (4.69) is equal to

$$\int_{\partial D} \Phi\psi \, d\lambda + \int_D F(x, f(x))\psi \, dx.$$

Thus it follows that, for any $\psi \in C^\infty(D)$,

$$\mathcal{Q}(f, \psi) = \int_{\partial D} \Phi\psi d\lambda + \int_D F(x, f(x))\psi \, dx,$$

which proves that function f is a weak solution of the problem (4.60).

Uniqueness:

If $\bar{f}$ is another solution of the problem (4.60), then $\bar{u} := e^v f$ can be shown to be the solution of the equation (4.68). Then by the uniqueness of the problem (4.68) proved in the Theorem 4.4.4, we find $\bar{u} = u$. Therefore, $f = \bar{f}$. $\square$

4.6 L^1 Solutions to the BSDEs and Semilinear PDEs

In this section, we consider the L^1 solutions of the BSDEs and use this result to solve the nonlinear elliptic partial differential equation with the mixed boundary condition.

Let $f : \Omega \times R^+ \times R \to R$ be progressively measurable. Consider the following conditions:

(I.1) $(y - y')(f(t, y) - f(t, y')) \le d(t)|y - y'|^2$ $-a.s.$, where $d(t)$ is a progressively measurable process;

(I.2)$E[\int_0^\infty e^{\int_0^s d(u)\,du}|f(s, 0)|\,ds] < \infty$;

(I.3) $P_x - a.s.$, for any $t > 0$, $y \to f(t, y)$ is continuous;

(I.4) $\forall r > 0, \quad T > 0$, $\psi_r(t) := \sup_{|y| \le r} |f(t, y) - f(t, 0)| \in L^1([0, T] \times \Omega, dt \times P_x)$.

The following lemma is deduced from Corollary 2.3 in [4].

Lemma 4.6.1. *Suppose a pair of progressively measurable processes* (Y, Z) *with values in* $R \times R^d$ *such that* $t \to Z_t$ *belongs to* $L^2([0,T])$ *and* $t \to f(t, Y_t)$ *belongs to* $L^1([0,T])$, $P_x - a.s.$.
If

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr - \int_t^T < Z_r, dM_r >, \tag{4.70}$$

then the following inequality holds, for $0 \le t < u \le T$,

$$|Y_t| \le |Y_u| + \int_t^u \hat{Y}_s f(s, Y_s)\, ds - \int_t^u \hat{Y}_s < Z_r, dM_r > .$$

where $\hat{y} = \frac{y}{|y|} I_{\{y \neq 0\}}$.

The following lemma can be proved by modifying the proof of Proposition 6.4 in [4].

Lemma 4.6.2. *Assume the conditions **(I.1)-(I.4)** with* $d(t) \equiv 0$. *Then there exists a unique solution* (Y, Z) *of the BSDE*

$$Y_t = \int_t^T f(r, Y_r)\, dr - \int_t^T < Z_r, dM_r >, \quad for \quad 0 \le t \le T. \tag{4.71}$$

Moreover, for each $\beta \in (0,1)$, $E[\sup_{t \le T} |Y_t|^\beta] + E[(\int_0^T |Z_r|^2\, dr)^{\frac{\beta}{2}}] < \infty$.

Suppose $\beta \in (0, 1)$.

$\mathcal{S}^\beta$ denotes the set of real-valued, adapted and continuous processes $\{Y_t\}_{t \ge 0}$ such that

$$\|Y\|^\beta := E[\sup_{t>0} |Y_t|^\beta] < \infty.$$

It is known that $\| \cdot \|^\beta$ induces a complete metric on the space of real-valued continuous processes ([4]).

M^β denotes the set of R^d-valued predictable processes $\{Z_t\}$ such that

$$\|Z\|_{M^\beta} := E[(\int_0^\infty |Z_t|^2 dt)^{\frac{\beta}{2}}] < \infty.$$

M^β is also a complete metric space with the distance deduced by $\|\cdot\|_{M^\beta}$.

Lemma 4.6.3. *Assume that conditions **(I.1)-(I.4)** with $d(t) \equiv 0$, then there exists a unique solution (Y, Z) of the BSDE*

$$\begin{aligned} & Y_t = Y_T + \int_t^T f(r, Y_r)\, dr - \int_t^T < Z_r, dM_r >, \quad \textit{for} \quad 0 \le t \le T; \\ & \lim_{t\to\infty} Y_t = 0, \quad P - a.s.. \end{aligned} \tag{4.72}$$

Proof. **Existence:**

By Lemma 4.6.2, there exists (Y^n, Z^n) such that, for $0 \le t \le n$,

$$Y_t^n = \int_t^n f(r, Y_r^n)\, dr - \int_t^n < Z_r^n, dM_r >,$$

and $Y_t^n = Z_t^n = 0$, for $t \ge n$.

Fix $t > 0$ and $t < n < n + i$, then

$$\begin{aligned} Y_t^{n+i} - Y_t^n &= \int_t^{n+i} (f(r, Y_r^{n+i}) - f(r, Y_r^n))\, dr - \int_t^{n+i} < (Z_r^{n+i} - Z_r^n), dM_r > \\ &\quad + \int_n^{n+i} f(r, 0)\, dr. \end{aligned}$$

Set

$$F^n(r, y) = f(r, y + Y_r^n) - f(r, Y_r^n) + f(r, 0) I_{\{r>n\}},$$

$$y_t^n = Y_t^{n+i} - Y_t^n \quad \text{and} \quad z_t^n = Z_t^{n+i} - Z_t^n.$$

Then (y_t^n, z_t^n) is the solution of the following BSDE:

$$y_t^n = \int_t^{n+i} F^n(r, y_r^n)\, dr - \int_t^{n+i} < z_r^n, dM_r > . \tag{4.73}$$

So that by the condition **(I.1)** with $d(t) \equiv 0$ and Lemma 4.6.1, it follows that

$$\begin{aligned}
|y_t^n| &\le \int_t^{n+i} < \hat{y}_r^n, F^n(r, y_r^n) > \, dr - \int_t^{n+i} \hat{y}_r^n < z_r^n, dM_r > \\
&\le \int_t^{n+i} \frac{I_{\{y_r^n \neq 0\}}}{|y_r^n|} < y_r^n, f(r, y_r^n + Y_r^n) - f(r, Y_r^n) > \, dr + \int_n^{n+i} |f(s,0)| \, ds \\
&\quad - \int_t^{n+i} \hat{y}_r^n < z_r^n, dM_r > \\
&\le \int_n^{n+i} |f(s,0)| \, ds - \int_t^{n+i} \hat{y}_r^n < z_r^n, dM_r > . \qquad (4.74)
\end{aligned}$$

Taking conditional expectations on both sides of the inequality, we get

$$|y_t^n| \le E[\int_n^{n+i} |f(s,0)| \, ds \quad | \mathcal{F}_t] := M_t^n,$$

where M_t^n is a martingale. Then by the Doob's inequality and condition **(I.2)**, it follows that, for $\beta \in (0,1)$,

$$\begin{aligned}
E[\sup_t |y_t^n|^\beta] \le E[\sup_t (M_t^n)^\beta] &\le \frac{1}{1-\beta} (E[\int_n^{n+i} |f(s,0)| \, ds])^\beta \\
&\to 0, \quad as \quad n \to \infty. \qquad (4.75)
\end{aligned}$$

Therefore we know that $\{Y^n\}$ is a Cauchy sequence under the norm $\| \cdot \|_\infty^\beta$. Hence there is a process Y such that $E[\sup_t |Y_t - Y_t^n|^\beta] \to 0$.

This also implies that $Y_t \to 0$, as $t \to \infty$, $P_x - a.s.$.

Moreover, by the equation (4.73), Ito's formula and the condition **(I.1)**, it follows that

$$\begin{aligned}
&|y_t^n|^2 + \int_t^{n+i} < A(X(r)) z_r^n, z_r^n > \, dr \\
= \;& 2 \int_t^{n+i} y_r^n F^n(r, y_r^n) \, dr - 2 \int_t^{n+i} y_r^n < z_r^n, dM_r > \\
\le \;& 2 \int_n^{n+i} y_r^n f(r, 0) \, dr + 2 | \int_t^{n+i} y_r^n < z_r^n, dM_r > | \\
\le \;& \sup_r |y_r^n|^2 + (\int_n^{n+i} |f(r,0)| \, dr)^2 + 2 | \int_t^\infty y_r^n < z_r^n, dM_r > |,
\end{aligned}$$

and thus that

$$(\int_t^{n+i} |z_r^n|^2\,dr)^{\frac{\beta}{2}} \le c_1[\sup_r |y_r^n|^\beta + (\int_n^{n+i} |f(r,0)|\,dr)^\beta + |\int_t^{n+i} y_r^n < z_r^n, dM_r > |^{\frac{\beta}{2}}].$$

Taking expectations on both sides of the inequality and applying theBurkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned} & E[(\int_t^{n+i} |z_r^n|^2\,dr)^{\frac{\beta}{2}}] \\ \le\ & c_1 E[\sup_r |y_r^n|^\beta] + c_1(E[\int_n^{n+i} |f(r,0)|\,dr])^\beta + c_2 E[(\int_t^{n+i} |y_r^n|^2 |z_r^n|^2\,dr)^{\frac{\beta}{4}}] \\ \le\ & c_1 E[\sup_r |y_r^n|^\beta] + c_1(E[\int_n^{n+i} |f(r,0)|\,dr])^\beta + c_2 E[(\sup_r |y_r^n|^{\frac{\beta}{2}} \int_n^{n+i} |z_r^n|^2\,dr)^{\frac{\beta}{4}}] \\ \le\ & (c_1 + \frac{c_2^2}{2}) E[\sup_r |y_r^n|^\beta] + c_1(E[\int_n^{n+i} |f(r,0)|\,dr])^\beta + \frac{1}{2} E[(\int_t^{n+i} |z_r^n|^2\,dr)^{\frac{\beta}{2}}]. \end{aligned}$$

Therefore, taking $t = 0$, we know that there is a constant $C > 0$, such that

$$\begin{aligned} & E[(\int_0^\infty |z_s^n|^2\,ds)^{\frac{\beta}{2}}] \\ \le\ & C E[\sup_t |y_t^n|^\beta] + C(E[\int_n^{n+i} |f(r,0)|\,dr])^\beta \\ \to\ & 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence $\{Z_t^n\}$ is a Cauchy sequence in M^β. Let Z denote the limit of $\{Z^n\}$.

Finally, by the condition **(I.3)**, we find that

$$\int_0^T f(t, Y_t^n)dt \to \int_0^T f(t, Y_t)dt, \quad P_x - a.s.. \tag{4.76}$$

Therefore, (Y,Z) is a solution of the BSDE (4.72).

Uniqueness:

Suppose (Y, Z) and (Y', Z') are two solutions to (4.72). Then by the estimate in

Lemma 4.6.1 and the fact $d(t) \equiv 0$, it follows that, for $t < T$,

$$\begin{aligned} |Y_t - Y'_t| \;\leq\; & |Y_T - Y'_T| + \int_t^T \frac{I_{Y_r \neq Y'_r}}{|Y_r - Y'_r|}(Y_r - Y'_r)(f(r, Y_r) - f(r, Y'_r))\, dr \\ & - \int_t^T < Z_r - Z'_r, dM_r >, \end{aligned}$$

which implies that

$$E|Y_t - Y'_t| \leq E|Y_T - Y'_T| \to 0, \quad as \quad T \to \infty.$$

Therefore, it holds that

$$\forall t > 0, \quad |Y_t - Y'_t| = 0, \quad P - a.s..$$

$\square$

(I.5) The process $d(t)$ is a progressively measurable process satisfying the following condition:

$$d(\cdot) \in L^1[[0,T] \times \Omega, dt \otimes P], \quad for \quad any \quad T > 0.$$

Theorem 4.6.4. *Assume the conditions* ***(I.1)-(I.5)*** *hold. Then there exists a unique process* (Y, Z) *such that,*

$$\begin{aligned} & Y_t = Y_T + \int_t^T f(r, Y_r)\, dr - \int_t^T < Z_r, dM_r >, \quad for \quad 0 \leq t \leq T; \\ & \lim_{t \to \infty} e^{\int_0^t d(u)\, du} Y_t = 0, \quad P - a.s. \end{aligned} \tag{4.77}$$

Proof. **Existence:**

Set $\hat{f}(t, y) = e^{\int_0^t d(u)\, du} f(t, e^{-\int_0^t d(u)\, du} y) - d(t)y$. Then

(1) $(y - y')(\hat{f}(t, y) - \hat{f}(t, y')) \leq 0$;

(2) $\hat{f}(t, 0) = e^{\int_0^t d(u)\, du} f(t, 0)$. So

$$E[\int_0^\infty |\hat{f}(s, 0)|\, ds] = E[\int_0^\infty e^{\int_0^s d(u)\, du} |f(s, 0)|\, ds] < \infty.$$

(3) $\sup_{|y|\leq r}|\hat{f}(t,y)-\hat{f}(t,0)| \leq \psi_r(t)+|d(t)|r$, where the process satisfies

$$\psi_r(t)+|d(t)|r \in L^1([0,T]\times\Omega, dt\otimes P),$$

for $T>0$.

Therefore, $\hat{f}$ satisfies all of the conditions of the Lemma 4.6.3. So there exists a pair of process $(\hat{Y},\hat{Z})$ satisfying the equation:

$$\hat{Y}_t = \hat{Y}_T + \int_t^T \hat{f}(r,\hat{Y}_r)\,dr - \int_t^T < \hat{Z}_r, dM_r >,$$

and obviously $\lim_{t\to\infty}\hat{Y}_t = 0$.

By the chain rule and the definition of the function $\hat{f}$, it follows that

$$de^{-\int_0^t d(u)\,du}\hat{Y}_t = -f(t, e^{-\int_0^t d(u)\,du}\hat{Y}_t)dt + < e^{-\int_0^t d(u)\,du}\hat{Z}_t, dM_t > .$$

Set $Y_t = e^{-\int_0^t d(u)\,du}\hat{Y}_t$ and $Z_t = e^{-\int_0^t d(u)\,du}\hat{Z}_t$. Then the process (Y,Z) is the solution to the equation (4.77).

Uniqueness:

The uniqueness of the solution to (4.77) follows from the uniqueness of the solution to equation (4.72). □

Let $H : R^d\times R\to R$ be a bounded Borel measurable function. Consider the following conditions:

(H.1)$'$ $(y_1-y_2)(H(x,y_1)-H(x,y_2)) \leq -h_1(x)|y_1-y_2|^2$, where $h_1\in L^p(D)$ for $p>\frac{d}{2}$.

(H.2)$'$ $y\to H(x,y)$ is continuous.

Theorem 4.6.5. *Assume the Conditions* ***(H.1)***$'$ *and* ***(H.2)***$'$ *hold and that there is some point* $x_0\in D$*, such that*

$$E_{x_0}[\int_0^\infty e^{\int_0^s(-h_1+q)(X(u))\,du}\,dL_s] < \infty, \tag{4.78}$$

and for some point $x_1 \in D$,

$$E_{x_1}[\int_0^\infty e^{\int_0^s q(X(u))\,du}\,dL_s] < \infty. \tag{4.79}$$

Then the semilinear Neumann boundary value problem

$$\begin{cases} Gu(x) = -H(x, u(x)), & on\ D \\ \frac{\partial u}{\partial \vec{\gamma}}(x) = \phi(x) & on\ \partial D \end{cases} \tag{4.80}$$

has a unique continuous weak solution.

Proof. Step 1

Set $\tilde{H}(t, x, y) = e^{\int_0^t q(X(u))dt} H(x, e^{-\int_0^t q(X(u))dt} y)$, then there exists a unique solution $(\hat{Y}_x, \hat{Z}_x)$ to the following BSDE: for any $T > 0$ and $0 < t < T$,

$$\begin{aligned} \hat{Y}_x(t) &= \hat{Y}_x(T) + \int_t^T \tilde{H}(s, X(s), \hat{Y}_x(s))\,ds - \int_t^T e^{\int_0^s q(X(u))dt} \phi(X(s))\,dL_s \\ &\quad - \int_t^T < \hat{Z}_x(s), dM_x(s) > \end{aligned}$$

and

$$\lim_{t\to\infty} e^{-\int_0^t h_1(X(u))\,du} \hat{Y}_t = 0 \quad P_x - a.s.$$

The uniqueness follows from the uniqueness proved in the last Theorem. Only the existence of solution $(\hat{Y}_x, \hat{Z}_x)$ needs to be proved:

(a) Similarly as the proof of Theorem 4.2.3, we can show that there exists $(p_x(t), q_x(t))$ such that

$$\begin{aligned} & dp_x(t) = e^{\int_0^t q(X(u))\,du} \phi(X(t)) dL_t + < q_x(t), dM_x(t) >, \\ & e^{-\int_0^t h_1(X(u))\,du} p_x(t) \to 0, \quad as \quad t \to \infty, \quad P_x - a.s.. \end{aligned} \tag{4.81}$$

(b) Set $g(t,x,y)=\tilde{H}(t,x,y+p_x(t))$. Then it follows that

$$(y-y')(g(t,X(t),y)-g(t,X(t),y'))\leq -h_1(X(t))|y-y'|^2.$$

The condition (4.78) and Lemma 4.3.4 imply, for $x\in D$,

$$E_x[\int_0^\infty e^{\int_0^s(-h_1+q)(X(u))\,du}\,ds]<\infty.$$

Furthermore, as the function H is bounded, we know that condition **(I.2)** is satisfied:

$$\begin{aligned}
& E_x[\int_0^\infty e^{-\int_0^s h_1(X(u))\,du}|g(s,X(s),0)|\,ds] \\
= \; & E_x[\int_0^\infty e^{-\int_0^s h_1(X(u))\,du}|\tilde{H}(s,X(s),p_x(s))|\,ds] \\
= \; & E_x[\int_0^\infty e^{\int_0^s(-h_1+q)(X(u))\,du}|H(X(s),e^{-\int_0^s q(X(u))\,du}p_x(s))|\,ds] \\
\leq \; & \|H\|_\infty E_x[\int_0^\infty e^{\int_0^t(-h_1+q)(X(u))\,du}dt] \\
< \; & \infty. \qquad (4.82)
\end{aligned}$$

Since $y\to g(x,y)$ is continuous, the condition **(I.3)** is satisfied.

Moreover, the condition **(I.4)** is also satisfied. In fact, for any $r>0$,

$$\psi_r(t)=\sup_r|\tilde{H}(t,X(t),y)-\tilde{H}(t,X(t),0)|\leq 2\|H\|_\infty e^{\int_0^t q(X_t)dt},$$

and for any $T>0$, by the fact that $q\in L^p(D)$ with $p>d$ and Theorem 2.1 in [27], $E_x[\int_0^T e^{\int_0^t q(X_u)\,du}dt]<\infty$.

Therefore, the function g satisfies all of the conditions of Theorem 4.6.4 . There exists a pair of process $(y_x(t),z_x(t))$ such that for any $T>0$ and $0<t<T$,

$$y_x(t)=y_x(T)+\int_t^T g(X(s),y_x(s))\,ds-\int_t^T <z_x(s),dM_x(s)> \qquad (4.83)$$

and

$$\lim_{t\to\infty} e^{-\int_0^t h_1(X(u))\,du} y_x(t) = 0 \quad P_x - a.s.. \tag{4.84}$$

Put $\hat{Y}_x(t) = p_x(t) + y_x(t)$ and $\hat{Z}_x(t) = q_x(t) + z_x(t)$. It follows that $(\hat{Y}_x(t), \hat{Z}_x(t))$ satisfies the following equation

$$d\hat{Y}_x(t) = e^{\int_0^t q(X(u))\,du}\phi(X(t))dL_t - \tilde{H}(t, X(t), \hat{Y}_x(t))dt + < \hat{Z}_x(t), dM_x >,$$

$$\lim_{t\to\infty} e^{-\int_0^t h_1(X(u))\,du}\hat{Y}_t = 0 \quad P_x - a.s..$$

Step 2.

Put $Y_x(t) := e^{-\int_0^t q(X(u))\,du}\hat{Y}_x(t)$ and $Z_x(t) := e^{-\int_0^t q(X(u))\,du}\hat{Z}_x(t)$, we have

$$dY_x(t) = -F(X(t), Y_x(t)) + \phi(X(t))dL_t + < Z_x(t), dM_x(t) >,$$

where $F(x, y) = q(x)y + H(x, y)$.

Moreover,

$$\begin{aligned} e^{\int_0^t(-h_1+q)(X(u))\,du}Y_x(t) &= e^{\int_0^t(-h_1+q)(X(u))\,du}e^{-\int_0^t q(X(u))dt}\hat{Y}_x(t) \\ &= e^{-\int_0^t h_1(X(u))\,du}\hat{Y}_x(t) \to 0 \quad as \quad t\to\infty. \end{aligned}$$

Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$.

Now as in Theorem 4.4.4, we can solve the following equation

$$\begin{cases} Gu(x) = -H(x, u_0(x)), & \text{on } D, \\ \frac{1}{2}\frac{\partial u}{\partial \vec{\gamma}}(x) = \phi(x) & \text{on } \partial D\ , \end{cases} \tag{4.85}$$

and prove that the solution u coincides with $u_0(x)$. This completes the proof of the whole theorem. □

Recall the operators L and G introduced in Section 4.5. Suppose that $F : R^d \times$

$R \to R$ is a bounded measurable function and $r_1 \in L^p(D)$. Consider the following conditions :

(E.1) $(y_1 - y_2)(F(x, y_1) - F(x, y_2)) \leq -r_1(x)|y_1 - y_2|^2$;

(E.2) $y \to F(x, y)$ is continuous.

Now following the same proof as that of Theorem 4.5.3, we obtain

Theorem 4.6.6. *Suppose that the function F satisfies the condition* ***(E.1)*** *and* ***(E.2)****, and there exist $x_0, x_1 \in D$ such that*

$$E^0_{x_0}[\int_0^\infty e^{-\int_0^t h_1(X^0_u)\,du} \hat{Z}_t dL^0_t] < \infty, \tag{4.86}$$

and

$$E^0_{x_1}[\int_0^\infty \hat{Z}_t dL^0_t] < \infty. \tag{4.87}$$

Then the following problem

$$\begin{cases} Lu(x) = -F(x, u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \bar{\gamma}}(x) - < \widehat{B}, n > (x)u(x) = \Phi(x), & \text{on } \partial D \end{cases} \tag{4.88}$$

has a unique, bounded, continuous weak solution.

Chapter 5

Future Studies

5.1 An Inspiring Example

In this chapter, we consider the operator

$$L = \frac{1}{2}\nabla(A\nabla) + < b, \nabla >,$$

which is associated with the process $\{\Omega, \mathcal{F}_t, X_t, P_x\}$, semigroup $\{P_t\}$ and the transition density function $p(t, x, y)$.

It is known that the adjoint operator L^* of L, satisfying, for $u \in D(L)$, $v \in D(L^*)$:

$$\int L(u)vdm = \int u(L^*v)dm,$$

and has the following expression:

$$L^* = \frac{1}{2}\nabla(A\nabla\cdot) - div(b\cdot). \tag{5.1}$$

In this sense, the term in the form $div(b\cdot)$ leads us to study the adjoint operator and adjoint process. In the following discussion, we fix time $t > 0$ and then define the backward filtration $\{\overleftarrow{\mathcal{F}}^t_s, s < t\}$ as $\overleftarrow{\mathcal{F}}^t_s = \overline{\sigma(X_r, r \in [s, t])}$. In [9] and [23], the reverse process $\{Y_s = X_s \circ r_t : s \in [0, t]\}$ is a Markov process with respect to the filtration

$\{\overleftarrow{\mathcal{F}}^t_s : s \in [0,t]\}$. L^* is associated with the semigroup P^* and the transition density function $p^*(t,x,y) = p(t,y,x)$.

Notice that the adjoint process Y is not homogenous under the measure P_x. Based on this notice, we apply the "time-reversal" method introduced before to deal with the divergence term. But we also find that, if we keep calculating the semigroup of the adjoint process Y under the probability P_x, something interesting will happen.

We try to study the simplest Dirichlet problem in this section:

$$\begin{cases} L^*u = \frac{1}{2}\nabla(A\nabla u)(x) - div(b(x)u(x)) = -F(x,u(x),\nabla u(x)), & \text{on } D \\ u = \Phi, & \text{on } \partial D\ . \end{cases} \tag{5.2}$$

Calculating rigorously, if $u \in \mathcal{D}(L) \cap \mathcal{D}(L^*)$, we get, for $s < t$

$$\begin{aligned} & u(X_t) - u(X_s) \\ = & \int_s^t \nabla u(X_r) dM_r + \int_s^t Lu(X_r)\, dr \\ = & \int_s^t \nabla u(X_r) dM_r + \int_s^t L^*u(X_r)\, dr + \int_s^t div(bu)(X_r)\, dr + \int_s^t < b, \nabla u > (X_r)\, dr. \end{aligned} \tag{5.3}$$

But the term $\int_s^t div(bu)(X_r)\, dr$ does not have a real meaning, because b is just measurable. Inspired by [28] and [35], we will try to transform $\int_s^t div(bu)(X_r)\, dr$ in the following discussion.

Fix x_0 and set $p_t(x) := p(t, x_0, x)$, for every $u \in W^{1,2}$, we define

$$\begin{aligned} \alpha^u(s,t) &:= \int_s^t < A\nabla \ln p_t, \nabla u > (X_r)\, dr, \\ \beta^u(s,t) &:= \int_s^t < b, \nabla u > (X_r)\, dr. \end{aligned}$$

Denote the semigroup associate with the adjoint process under P_{x_0} by

$$\begin{aligned} S_{s,t}f(x) := E_{x_0}[f(X_s)|X_t = x] &= \int \frac{p(s,x_0,y)p(t-s,y,x)}{p(t,x_0,x)} f(y)m(dy) \\ &= \frac{1}{p_t(x)} P^*_{t-s}(p_s f)(x), \quad s < t. \end{aligned}$$

By the following fact

$$\begin{aligned} & L^* P^*_u(p_{t-u}f) = P^*_u L^*(p_{t-u}f) \\ = \; & P^*_u[(L^* p_{t-u})f + < A\nabla p_{t-u}, \nabla f > + p_{t-u}(\frac{1}{2}\nabla(A\nabla f) - < b, \nabla f >)], \end{aligned}$$

we get

$$\begin{aligned} & S_{s,t}f(x) - f(x) \\ = \; & \frac{1}{p_t(x)}[P^*_{t-s}(p_s f)(x) - p_t f(x)] \\ = \; & \frac{1}{p_t(x)} \int_0^{t-s} \partial_u[P^*_u(p_{t-u}f)(x)]du \\ = \; & \frac{1}{p_t(x)} \int_0^{t-s} (\partial_u P^*_u)(p_{t-u}f)(x) - P^*_u(\partial_u p_{t-u}f)(x))du \\ = \; & \frac{1}{p_t(x)} \int_0^{t-s} (L^* P^*_u)(p_{t-u}f)(x) - P^*_u[(L^* p_{t-u})f](x))du \\ = \; & \int_0^{t-s} \frac{1}{p_t(x)} P^*_u[< A\nabla p_{t-u}, \nabla f > + p_{t-u}(\frac{1}{2}\nabla(A\nabla f) - < b, \nabla f >)]du \\ = \; & \int_s^t \frac{1}{p_t(x)} P^*_{t-u}(p_u < A\nabla \ln p_u, \nabla f >)du \\ & + \int_s^t \frac{1}{p_t(x)} P^*_{t-u}(p_u < \frac{1}{2}\nabla(A\nabla f) - < b, \nabla f >)du \\ = \; & \int_s^t S_{r,t}(< A\nabla \ln p_u, \nabla f > + \frac{1}{2}\nabla(A\nabla f) - < b, \nabla f >)(x)du. \end{aligned}$$

By the Markovian property of the adjoint process, it follows that

$$\begin{aligned}
& E_{x_0}[f(X_s) - f(X_t) - \alpha^f(s,t) - \frac{1}{2}\int_s^t \nabla(A\nabla f)(X_r)\,dr + \beta^f(s,t)|\overleftarrow{\mathcal{F}}_s^t] \\
= \; & E_{x_0}[f(X_s) - f(X_t) - \alpha^f(s,t) - \frac{1}{2}\int_s^t \nabla(A\nabla f)(X_r)\,dr + \beta^f(s,t)|X_t] \\
= \; & 0.
\end{aligned} \tag{5.4}$$

So far we know, when the time $t > 0$ is fixed, the process

$$\overleftarrow{M}^f(s,t) := f(X_s) - f(X_t) - \frac{1}{2}\int_s^t \nabla(A\nabla f)(X_r)\,dr - \alpha^f(s,t) + \beta^f(s,t), \quad s < t \tag{5.5}$$

is a martingale with respect to the filtration $\{\overleftarrow{\mathcal{F}}_s^t, s < t\}$. And it is not difficult to see the sharp process of the backward martingale $\overleftarrow{M}^f(s,t)$ is

$$< \overleftarrow{M}^f(\cdot,t), \overleftarrow{M}^f(\cdot,t) >_s^t = \int_s^t < A\nabla f, \nabla f > (X_r)\,dr.$$

Since the coordinate of point x is $x = (x_1, ..., x_d)$, we set $f(x) = x_i \in W^{1,2}$. Then it follows

$$\overleftarrow{M}^i(s,t) = X_s^i - X_t^i - \frac{1}{2}\sum_j \int_s^t \partial_j(a_{ij})(X_r)\,dr - \alpha^i(s,t) + \beta^i(s,u).$$

We write the backward martingale as following for short:

$$\overleftarrow{M}(s,t) = X_s - X_t - \frac{1}{2}\int_s^t \nabla A(X_r)\,dr - \int_s^t A\nabla(\ln p_r)(X_r)\,dr + \int_s^t b(X_r)\,dr.$$

Define the stochastic integral with respect to the backward martingale $\overleftarrow{M}^i(s,t)$ as follows, for a function g,

$$\int_s^t g(X_r)d\overleftarrow{M}^i(r,t) = \lim_{\|\triangle\|\to 0}\sum_{j=0}^k g(X_{t_{j+1}})(\overleftarrow{M}^i(t_j,t) - \overleftarrow{M}^i(t_{j+1},t)),$$

where $\triangle : s = t_0 < t_1 < ... < t_k = t$ is the partition of the interval $[s,t]$ and

$\|\triangle\| = \max_j(t_{j+1} - t_j)$.

Similarly as the "forward" process, we conclude that

$$\int_s^t \nabla u(X_r) d\overleftarrow{M}(r,t) = \overleftarrow{M}^u(s,t). \tag{5.6}$$

In fact,

$$\begin{aligned}
&\int_s^t \nabla u(X_r) d\overleftarrow{M}(r,t) \\
= \;& \lim_{\|\triangle\|\to 0} \sum_{j=0}^{k} \nabla u(X_{t_{j+1}})(X_{t_j} - X_{t_{j+1}}) - \frac{1}{2}\int_s^t < \nabla A, \nabla u > (X_r)\, dr \\
& - \int_s^t < A\nabla(\ln p_r), \nabla u > (X_r)\, dr + \int_s^t < b, \nabla u > (X_r)\, dr.
\end{aligned} \tag{5.7}$$

Set the first term in the right side of the equation (5.7) as (I), then it follows

$$\begin{aligned}
&(I) \\
= \;& \lim_{\|\triangle\|\to 0} \sum_{j=0}^{k} \nabla u(X_{t_j})(X_{t_j} - X_{t_{j+1}}) + \lim_{\|\triangle\|\to 0} \sum_{j=0}^{k} (\nabla u(X_{t_{j+1}} - \nabla u(X_{t_j}))(X_{t_j} - X_{t_{j+1}}) \\
= \;& -\int_s^t \nabla u(X_r) dM_r - \int_s^t < \frac{1}{2}\nabla A + b, \nabla u > (X_r)\, dr - \int_s^t \sum_{ij} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(X_r)\, dr \\
= \;& u(X_s) - u(X_t) + \int_s^t Lu(X_r)\, dr - \int_s^t < \frac{1}{2}\nabla A + b, \nabla u > (X_r)\, dr \\
& - \int_s^t \sum_{ij} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(X_r)\, dr.
\end{aligned}$$

By the above expression of (I), we get the conclusion (5.6).

Therefore, by (5.3) and (5.5), we conclude the following two results.

(1)For any $f \in W^{1,2}$, it follows

$$f(X_t) - f(X_s) = \frac{1}{2}\int_s^t \nabla f(X_r) dM_r - \frac{1}{2}\int_s^t \nabla f(X_r) d\overleftarrow{M}(r,t) - \frac{1}{2}\alpha^f(s,t) + \beta^f(s,t). \tag{5.8}$$

(2)For any function $f \in \mathcal{D}(L)$, it follows

$$\begin{aligned} & -\int_s^t \nabla(A\nabla f)(X_r)\,dr \\ = & \int_s^t \nabla f(X_r) dM_r + \int_s^t \nabla f(X_r) d\overleftarrow{M}(r,t) + \int_s^t < A\nabla \ln p_r, \nabla f > (X_r)\,dr. \end{aligned}$$

We use the notation in [35]. For function $g = (g_1, ..., g_d)$, define

$$\begin{aligned} & \int_s^t g(X_r) * dX_r \\ = & \sum_i \Big(\int_s^t g_i(X_r) dM_r^i + \int_s^t g_i(X_r) d\overleftarrow{M}^i(r,t) \Big) + \int_s^t < A\nabla \ln p_r, g > (X_r)\,dr \end{aligned}$$

We can generalize the second conclusion (2) as follows, which is inspired by [35].

Lemma 5.1.1. *Assume $g \in L^2(R^d, R^d)$, $f \in L^2(R^d)$ and $div(Ag) = f$ in the weak sense. Then it holds that*

$$-\int_s^t f(X_r)\,dr = \int_s^t g(X_r) * dX_r.$$

The function $u \in W^{1,2}$ is the weak solution of the equation (5.2), which can be rewritten as $div(A(\frac{1}{2}\nabla u - A^{-1}b)) = -F(x, u(x))$ in the weak sense. By Lemma 5.1.1, we have

$$\frac{1}{2}\int_s^t \nabla u(X_r) * dX_r - \int_s^t A^{-1}b(X_r) * dX_r = \int_s^t F(X_r, u(X_r)\,dr.$$

By (5.8), we know that

$$\begin{aligned} & \frac{1}{2}\int_s^t \nabla u(X_r) * dX_r \\ = & \frac{1}{2}\int_s^t \nabla u(X_r) dM_r + \frac{1}{2}\int_s^t \nabla u(X_r) d\overleftarrow{M}(r,t) + \int_s^t < A\nabla \ln p_r, \nabla u > (X_r)\,dr \\ = & -u(X_t) + u(X_s) + \int_s^t \nabla u(X_r) dM_r + \int_s^t < b, \nabla u > (X_r)\,dr. \end{aligned}$$

Then

$$\begin{aligned}\int_s^t F(X_r, u(X_r))\,dr &= -u(X_t) + u(X_s) + \int_s^t \nabla u(X_r) dM_r + \int_s^t < b, \nabla u > (X_r)\,dr \\ &\quad - \int_s^t A^{-1} b(X_r) * dX_r.\end{aligned}$$

Therefore, we have the following expression of $u(X)$,

$$\begin{aligned}u(X_t) - u(X_s) &= \int_s^t \nabla u(X_r) dM_r - \int_s^t F(X_r, u(X_r))\,dr \\ &\quad + \int_s^t < b, \nabla u > (X_r)\,dr - \int_s^t A^{-1}\hat{b}(X_r) * dX_r. \qquad (5.9)\end{aligned}$$

As a conclusion of this example, we say that the equation (5.9) supplies a candidate of the solution to the BSDE . Suppose u is the solution of the equation (5.2) and substitute the fixed time t for the first hitting time τ of the boundary ∂D. Then it follows

$$\begin{aligned}u(X_s) &= \Phi(X_\tau) + \int_s^\tau F(X_r, u(X_r))\,dr - \int_s^\tau < b, \nabla u > (X_r)\,dr \\ &\quad + \int_s^t A^{-1}\hat{b}(X_r) * dX_r - \int_s^\tau \nabla u(X_r) dM_r.\end{aligned}$$

5.2 Future Studies

Set

$$\begin{aligned}H_T &:= \{u \in L^2([0,T] \times R^d) : t \mapsto u(t, \cdot) \quad \text{is continuous in} \quad L^2(R^d) \quad on \quad [0,T], \\ &\quad u(t,\cdot) \in H^1(R^d) \quad and \quad \int_0^T (A\nabla u(t,\cdot), \nabla u(t,\cdot))dt < \infty\}.\end{aligned}$$

The following theorem is the main result in [35], which is the essential inspiration of our future study.

Theorem 5.2.1. *If $\Phi \in L^2(R^d)$ and $f : [0,T] \times R^d \times R \times R^d \to R$, $g : [0,T] \times R^d \times R \times R^d \to R$ satisfy the conditions,* *(1)* $f(\cdot,\cdot,0,0) \in L^2([0,T]\times R^d)$ *and* $g(\cdot,\cdot,0,0) \in L^2([0,T]\times R^d; R^d)$*,*

(2) $|f(t,x,y,z) - f(t,x,y',z')| \leq C(|y-y'| + |z-z'|)$*,*

(3) $|g(t,x,y,z) - g(t,x,y',z')| \leq C(|y-y'| + \alpha|z-z'|)$*,*

with some constant $C > 0$ *and* $\alpha \in (0,1)$*. Then there exists a unique determined solution* $u \in H_T$ *of the following equation:*

$$\begin{aligned}
&(\partial_t + \nabla(A\nabla))u(t,x) + f(t,x,u(t,x),\frac{1}{\sqrt{2}}\nabla u(t,x)\sigma(x)) \\
&\quad -div(Ag)(t,x,u(t,x),\frac{1}{\sqrt{2}}\nabla u(t,x)\sigma(x)) = 0. \\
&u(T,x) = \Phi(x)
\end{aligned} \tag{5.10}$$

Moreover, set $Y_t = u(t,X_t)$ *and* $Z_t = (\nabla u)(t,X_t)$*, then* (Y,Z) *is a solution of the following BSDE,*

$$\begin{aligned}
Y_t &= \Phi(X_T) - \int_t^T < Z_r, dM_r > + \int_t^T f(r,X_r,Y_r,\frac{1}{\sqrt{2}}Z_r\sigma(X_r))\,dr \\
&\quad + \frac{1}{2}\int_t^T g(r,X_r,Y_r,\frac{1}{\sqrt{2}}Z_r\sigma(X_r)) * dX,
\end{aligned} \tag{5.11}$$

for any $0 \leq t \leq T$*,* $P^m - a.s.$*.*

We notice that, the processes Y and Z as a pair of solutions to the BSDE (5.11) must be two functions u and ∇u composed of X_t respectively, i.e. $Y = u(X)$ and $Z = \nabla u(X)$. But as a general solution to the BSDE, this is not easy to be satisfied.

Last but not least, we list three problems for our future studies as the conclusion of the whole thesis.

(1) Study the BSDE (5.11), not only in the PDE (5.10) point of view, but also its own properties as a backward differential equation.

(2) Consider the following Dirichlet boundary problem:

$$\begin{cases}
\frac{1}{2}\nabla(A\nabla u)(x) + b\cdot\nabla u(x) - div\hat{b}(x,u(x)) + q(x)u(x) = -F(x,u(x),\nabla u(x)), & \text{on } D \\
u = \Phi, & \text{on } \partial D.
\end{cases}$$

(3) Consider the following Neumann boundary problem:

$$\begin{cases} \frac{1}{2}\nabla(A\nabla u)(x) + b\cdot\nabla u(x) - div\hat{b}(x,u(x)) + q(x)u(x) = -F(x,u(x),\nabla u(x)), & \text{on } D \\ \frac{1}{2} < A\nabla u, \vec{n} >=< \hat{b}(x,u(x)), \vec{n} >, & \text{on } \partial D. \end{cases}$$

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