INVARIANT MEASURES FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS AND SPLITTING-UP METHOD FOR STOCHASTIC FLOWS

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Contents

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This thesis consists of two parts. We start with some background theory that will be used throughout the thesis. Then, in the first part, we investigate the existence and uniqueness of the solution of the stochastic partial differential equation with two reflecting walls. Then we establish the existence and uniqueness of invariant measure of this equation under some reasonable conditions.

In the second part, we study the splitting-up method for approximating the solutions of stochastic Stokes equations using resolvent method.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Notations

Through this thesis, I numbered equations, lemmas, theorems, etc., separately per chapter. Cross-reference between chapters explicitly mention chapter numbers.

Table of Symbols

a.s., a.e.	almost surely, almost everywhere
w.r.t	with respect to
$\emptyset$	the empty set
$\sum$	summation
$\prod$	product
A^c	the complement of A
$\subseteq$	contained in
$\cap$, $\cup$	intersection, union
z^+, z^-	$\max(z, 0)$, $-\min(z, 0)$
sup	supermum
$:=$	equal to by definition
$P(\cdot), E(\cdot)$	probability, expectation
$\mathcal{L}(\cdot)$	the law of a random variable
$\mathcal{N}(\mu, \sigma^2)$	the normal distribution with mean μ, covariance σ^2

I	the identity operator
$\mathrm{Tr}\ T$	the trace of the operator T
$\mathbb{R}^d$	the d-dimensional Euclidean space
$\mathbb{R}_+$	the set of all the positive real numbers
$\mathcal{B}(H)$	the σ-field of all the Borel subsets of H
$\mathcal{B}_b(H)$	the set of all the real bounded Borel functions on H
$\mathcal{M}(H)$	the set of all probability measures defined on $(H,\ \mathcal{B}(H))$
$\mathcal{L}(U;H)$	the space of all the bounded operators from U to H
$\mathcal{L}_2(U;H)$	the space of all the Hilbert-Schmidt operators from U to H.
$C(X;Y)$	the set of all the continuous mappings from X to Y
$C(X)$	the set of all the real continuous functions defined on X
$C_b^k(X)$	the set of all the bounded functions with continuous derivatives up to order k on X
C_0	the set of all the continuous functions with compact support
$f', f'', \nabla f$	first, second derivatives, and gradient of f
1_B	the indicator function of set B

I close all the proof with the symbol $\square$. I give all the references by number enclosed within square brackets.

Statements

The work contained in Chapter 2 has been published in *Infinite Dimensional Analysis, Quantum Probability and Related Topics* under the title *White noise driven SPDEs with two reflecting walls* with the co-authors Tusheng Zhang.

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Dedication

To My Parents and Luke

Chapter 1

Introduction

1.1 Motivation and Summary

In this thesis, we are concerned with the existence and uniqueness of solutions of some stochastic partial differential equations (SPDEs in short) with two reflecting smooth walls h^1 and h^2 in infinite dimensions and their invariant measures and strong Feller property.

For stochastic differential equations (SDEs in short) of Skorohod type, the existence and uniqueness of the solutions had been discussed in Tanka [37] which is on a convex domain and in Lions and Sznitman [24] for more general domains.

The stochastic problem can be translated to a deterministic problem with reflection along an irregular boundary function. Such deterministic problems with reflection are called 'inequations' and have been widely studied by several authors, see Bensoussan and Lions [9], Haussmann and Pardoux [21], Mignot and Puel [25]. Parabolic SPDEs with reflection are natural extension of the widely studied deterministic parabolic obstacle problems. They also can be used to model fluctuations of an interface near a wall, see Funaki and Olla [19]. When the interface touches the wall, it will be repulsed.

In Chapter 2, we will study the reflected stochastic partial differential equations

(SPDEs) of the following type:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(x,t,u\big) + \sigma\big(x,t,u\big)\dot{W}(x,t) \\ &\quad + \eta(x,t) - \xi(x,t) \qquad (1.1)\end{aligned}$$

in $(x,t) \in Q := [0,1] \times \mathbb{R}_+$ while $h^1(x,t) \leq u(x,t) \leq h^2(x,t)$. In recent years, there is a growing interest on the study of SPDEs with reflection. Several works are devoted to the existence and uniqueness of the solutions. In the case of a constant diffusion coefficient and a single reflecting barrier $h^1 = 0$, Naulart and Pardoux [27] proved the existence and uniqueness of the solutions. In the case of a non-constant diffusion coefficient and a single reflecting barrier $h^1 = 0$, the existence of a minimal solution was obtained by Donati-Martin and Pardoux [14]. The existence and particularly the uniqueness of the solutions for a fully non-linear SPDE with reflecting barrier 0 have been established by Xu and Zhang [40]. In the case of double reflecting barriers, Otobe [29] obtained the existence and uniqueness of the solutions of a SPDE driven by an additive white noise. We will establish the existence and uniqueness of the solutions of a fully non-linear SPDE with two reflecting walls (1.1).

Invariant measures, as an essential part of the ergodic theory, are of great interest. Invariant measures for stochastic evolution equations driven by Wiener processes and infinite dimensional diffusion processes in general have been studied by many people, we refer readers to Da Prato and Zabczyk [16] and references therein. For white noise driven SPDEs without reflection, the existence and uniqueness of invariant measures has been studied by many people, see Mueller [26], Sowers [36]. For SPDEs with reflection, when the diffusion coefficient σ is a constant, existence and uniqueness of invariant measures was obtained by Otobe [28], [29].

The strong Feller plays an important role in the probabilistic potential theory in infinite dimensions (see Carmona [10] and Gross [20]) and also in obtaining the uniqueness of invariant measure for transition semigroup (see Doob [11] and Khas'minski [22]). The strong Feller property of SDEs and SPDEs has been studied by several authors, see Da Prato and Zabczyk [17], Peszat and Zabczyk [31] and references therein. The strong Feller property of SPDEs with reflection at 0 was first proved in Zhang [42].

In Chapter 3, we study the existence and uniqueness of invariant measures, as well as the strong Feller property of the following fully non-linear SPDEs with two reflecting walls:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}(x,t) \\ \qquad\qquad +\eta(x,t) - \xi(x,t); \\ u(x,0) = u_0(x) \in C(S^1); \\ h^1(x) \le u(x,t) \le h^2(x), \quad \text{for } (x,t) \in Q^1, \end{cases} \tag{1.2}$$

where $Q^1 := S^1 \times \mathbb{R}_+$, $S^1 := \mathbb{R}(mod 2\pi)$. In addition to the invariant measure and Feller property, various other properties of the solution have been studied by several authors, see Dalang et al. [12], Donati-Martin and Pardoux [15], Zambotti [41].

The mathematical theory and numerical techniques of the deterministic differential equations and stochastic differential equations have been considered in a large amount of literatures. For classical SDEs, Kloeden and Platen [23] present approximating schemes. We concentrate in Chapter 4 on the splitting up method to approximate the solutions of stochastic Stokes equations. The splitting studied here is an underlying principle for certain numerical methods for the solutions of SDEs. The idea of splitting up comes from the Lie-Trotter formula, see Barbu [2] and Teman [38]. In Beale and Greengard [6] and Popa [33], the authors dealt with the approximation of the solutions of deterministic Navier-Stokes equations by splitting up them into two partial steps: the Euler equations with the tangential boundary condition and the Stokes equations with the no-slip boundary condition on sufficiently small time intervals. As a preparation to approximate the solutions of stochastic Navier-Stokes equations, we use splitting up methods to approximate the solutions of stochastic Stokes equations. Then we will try to extend the result of Beale and Greengard [6] to stochastic Navier-Stokes equations.

For some types of stochastic differential equations, by using splitting up method, the convergence results were obtained, for example, in Asiminoaei and Rascanu [1], Bensoussan, Glowinski and Rascanu [7] and [8]. We also mention that the splitting up method for solving Hamitton-Jacobi equations and, implicitly, for calculating the value function was initiated by Barbu in [2], [3], and developed by him and Popa, separately, in [4], [32], [34].

1.2 Background Theory

In this section, we recall some background material which will be used in the following chapters.

1.2.1 The Stochastic Integral with Respect to White Noises

Let $(E,\ \mathcal{E},\ \nu)$ be a σ-finite measure space. A white noise based on ν is a random set function W on $\mathcal{E}$ such that

(i) $W(A)$ is a $N(0, \nu(A))$ random variable;

(ii) if $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

We see that it is a mean-zero Gaussian process with covariance function

$$\mathbb{E}\{W(A)\ W(B)\} = \nu(A \cap B).$$

Normally, we use $E := \mathbb{R}^2_+$ and ν: Lebesgue measure on E.

A process $\{W(x,t)\}_{(x,t)} \in E$ defined by $W(x,t) := W((0,x] \times (0,t])$ is called Brownian sheet. If we may expect the existence of the Ito integral, $\dot{W}(\phi)$ must be $\int \int \phi(x,t) W(dx, dt)$.

In the classical case, one constructs the stochastic integral as a process rather than as a random variable. One can then say that the integral is a martingale, for instance. The analogue of martingale in our setting is martingale measure. Accordingly, we will define our stochastic integral as a martingale measure.

Let $\{\mathcal{F}_t\}$ be a right continuous filtration. A process $\{M_t(A),\ \mathcal{F}_t, t \geq 0,\ A \in \mathcal{A}\}$ is a martingale measure if

(i) $M_0(A) = 0$;

(ii) if $t > 0$, M_t is a σ-finite L^2-valued measure;

(iii) $\{M_t(A),\ \mathcal{F}_t, t \geq 0\}$ is a martingale.

A martingale measure M is worthy if there exists a random σ-finite measure $K(A \times B \times C,\ \omega)$, $A \times B \times C \in \mathcal{E} \times \mathcal{E} \times \mathcal{B},\ \omega \in \Omega$ such that

(i) K is positive definite and symmetric in A and B;

(ii) for fixed A and B, $\{K(A \times B \times (0,t]), t > 0\}$ is predictable;

(iii) $\mathbb{E}\{K(E \times E \times [0,T])\} < \infty$;

(iv) for any rectangle D, $|C(D)| \leq K(D)$,

where C is the covariance functional of M:

$$C(A \times B \times (s,t]) =< M(A), M(B) >_t - < M(A), M(B) >_s .$$

We call K the dominating measure of M.

A function $f:\ E \times [0,\infty) \times \Omega \to \mathbb{R}$ is elementary if it is of the form $f(x,s,\omega) = X(\omega)I_{(a,b]}(s)I_A(x)$, where $0 \leq a < t$, X is bounded, $\mathcal{F}_a$-measurable and $A \in \mathcal{E}$. f is simple if it is a finite sum of elementary functions, we denote the class of simple functions by $\mathcal{S}$. Define a martingale measure $f \cdot M$ by

$$f \cdot M_t(B) = X(\omega)(M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B)).$$

A function is predictable if it is $\mathcal{P}$-measurable, where the predictable σ-field $\mathcal{P}$ on $\Omega \times E \times R_+$ is the σ-field generated by $\mathcal{S}$. $\mathcal{P}_M$ be the class of all predictable f for which $\|f\|_M := [\mathbb{E}(\int_{E\times E\times R_+} |f(x,s)|^2 K(dx,dy,ds))]^{\frac{1}{2}} < \infty$. $\mathcal{S}$ is dense in $\mathcal{P}_M$. From

Walsh [39], if $f \in \mathcal{P}_M$, then $f \cdot M$ is a worthy martingale measure. Its covariance and dominating measures respectively are given by

$$C_{f\cdot M}(dx, dy, ds) = f(x, s)f(y, s)C_M(dx, dy, ds);$$

$$K_{f\cdot M}(dx, dy, ds) = |f(x, s)f(y, s)|K_M(dx, dy, ds).$$

Moreover, if $g \in \mathcal{P}_M$ and $A,\ B \in \mathcal{E}$, then

$$< f \cdot M(A), g \cdot M(B) >_t = \int_{A\times B\times[0,t]} f(x, s)f(y, s)C_M(dx, dy, ds);$$

$$\mathbb{E}[(f \cdot M_t(A))^2] \leq \|f\|_M^2.$$

M is an orthogonal martingale measure if for, any two disjoint sets A and B, martingales $\{M_t(A),\ t \geq 0\}$ and $\{M_t(B),\ t \geq 0\}$ are orthogonal (i.e. zero quadratic variance).

Proposition 1.2.1. *Let M be an orthogonal martingale measure, and suppose that for each $A \in \mathcal{E}$, $t \to M_t(A)$ is continuous. Then M is a white noise if and only if its covariance measure C is deterministic.*

1.2.2 Markovian Semigroups and Invariant Measures

Let $\{S(t)|\ t \in \mathbb{R}_+\}$ be a family of the bounded, linear operators on a Banach space $(E,\ |\cdot|_E)$. $\{S(t)|\ t \in \mathbb{R}_+\}$ is called a semigroup on E, if $S(s)S(t) = S(s+t),\ \forall s, t \geq 0$. We call semigroup $\{S(t)|\ t \in \mathbb{R}_+\}$ a strongly continuous semigroup if it satisfies

(i) $S(0) = I$;

(ii) $\lim_{t\to 0} |S(t)x - x|_E = 0,\ \forall x \in E$.

Let $\{S(t)|\ t \in \mathbb{R}_+\}$ be a strongly continuous semigroup on E. Define the infinitesimal generator A of $S(t)$ by $Ax := \lim_{t\to 0} t^{-1}(S(t)x - x)$, for $x \in D(A) = \{x \in E :$ $\lim_{t\to 0} t^{-1}(S(t)x - x)$ exists in $E\}$. Then A is a closed operator whose domain $D(A)$ is dense in E. We also have, for every $x \in D(A)$, $\frac{dS(t)x}{dt} = AS(t)x = S(t)Ax$. We say that $P_t(x,\Gamma)$, $t \geq 0$, $\Gamma \in \mathcal{B}(E)$, is a Markovian transition function (in short a transition funciton) on E, if

(i) $P_t(x,\cdot)$ is a probability measure on $(E,\mathcal{B}(E))$ for each $t \geq 0,\ x \in E$;

(ii) $P_t(\cdot,\Gamma)$ is a $\mathcal{B}(E)$-measurable function for each $t \geq 0,\ \Gamma \in \mathcal{B}(E)$;

(iii) $P_{t+s}(x,\Gamma) = \int_E P_t(x,dy)P_s(y,\Gamma)$, for each $t,\ s \geq 0,\ x \in E,\ \Gamma \in \mathcal{B}(E)$;

(iv) $P_0(x,\Gamma) = \mathbb{1}_\Gamma(x)$, for each $x \in E$, $\Gamma \in \mathcal{B}(E)$.

Denote by $\mathcal{B}_b(E)$ the set of all the bounded Borel functions on E and by $\mathcal{M}(E)$ the set of all probability measures defined on $(E,\ \mathcal{B}(E))$.

Definition 1.2.1. *Let $P_t(x,\Gamma),\ t \geq 0,\ x \in E,\ \Gamma \in \mathcal{B}(E)$ be the transition function. We define a semigroup of linear operators $P_t,\ t \geq 0$, on the space $\mathcal{B}_b(E)$ by*

$$P_t\phi(x) := \int_E P_t(x,dy)\phi(y),\ t \geq 0,\ x \in E,\ \phi \in \mathcal{B}_b(E).$$

Then $P_t,\ t \geq 0$ is called the Markovian transition semigroup associated to the transition function $P_t(x,\Gamma),\ t \geq 0,\ x \in E,\ \Gamma \in \mathcal{B}(E)$. We also call P_t, $t \geq 0$, a Markovian semigroup or a transition semigroup for short.

For $\mu \in \mathcal{M}(E)$ we set

$$P_t^*\mu(\Gamma) := \int_E P_t(x,\Gamma)\mu(dx),$$

where $t \geq 0,\ \Gamma \in \mathcal{B}(E)$.

Definition 1.2.2. *A probability measure $\mu \in \mathcal{M}(E)$ is said to be invariant or stationary with respect to $P_t,\ t \geq 0$, if and only if $P_t^*\mu = \mu$ for each $t \geq 0$, where $P_t,\ t \geq 0$ is the Markovian semigroup on $\mathcal{B}_b(E)$.*

1.2.3 The Stochastic Integral with Respect to U-Valued Wiener Processes

Let $(H, \ |\cdot|_H, \ <\cdot,\cdot>_H)$ and $(U, \ |\cdot|_U, \ <\cdot,\cdot>_U)$ be two separable Hilbert space. A bounded linear operator Q over U is said to be in the trace class if for some (and hence all) orthonormal bases $\{e_k\}_k$ of U the sum of positive terms $Tr|Q| := \sum_k <(Q*Q)^{\frac{1}{2}}e_k, \ e_k>_U$ is finite. Here let Q be a symmetric nonnegative trace class operator on U. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \geq 0\})$ be a complete filtered probability space. Denote by $\mathcal{P}_t, \ t \in [0,T]$ the $\mathcal{F}_t$-predictable σ-algebra of subsets of $[0,T] \times \Omega$.

Definition 1.2.3. *An U-valued stochastic process $W(t), t \geq 0$ is called a Q-Wiener process w.r.t. $\{\mathcal{F}_t, t \geq 0\}$ if*

(i) $W(0) = 0$, a.s.;

(ii) W has continuous trajectories;

(iii) $W(t)$ is $\mathcal{F}_t$-measurable;

(iv) $W(t) - W(s)$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t$;

(v) For every nonzero $h \in U$, $\mathcal{L}(<W(t) - W(s), h>_U) = \mathcal{N}(0, (t-s) <Qh, h>_U)$.

To define the stochastic integral w.r.t. Q-Wiener process, we start with the simple process. A simple process can be written in the form

$$\Phi(t,\omega) = \sum_{k=0}^{n-1} 1_{(t_k,t_{k+1}]}(s)\Phi_k(\omega),$$

where n is a natural number, $0 = t_0 < t_1 < \cdots < t_n = T$ and Φ_k is a $\mathcal{L}(U;H)$-valued random variable such that Φ_k is $\mathcal{F}_{t_k}$-measurable, $k = 0,1,2,\cdots,n-1$. Define the stochastic integral by the formula

$$\int_0^t \Phi(s)dW(s) = \sum_{k=0}^{n-1} \Phi_k(W(t_{k+1} \wedge t) - W(t_k \wedge t)), \quad t \in [0,T] \tag{1.3}$$

and denote it by $\Phi \cdot W(t)$.

Introduce the subspace of U: $U_0 := Q^{\frac{1}{2}}U$. $(U_0, |\cdot|_0, \ <\cdot,\cdot>_0)$ is an Hilbert space with $< u, v >_0 := < Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v >_U$, where $Q^{-\frac{1}{2}}u = y$ with $|y|_U = \inf\{|x|_U;\ Q^{\frac{1}{2}}x = u\}$. Q is a trace class operator, so the identity mapping from U_0 to U is a Hilbert-Schmidt. Denoting by $\mathcal{L}_2(U_0; H)$ the separable Hilbert space of Hilbert-schmidt operators equipped with the norm $|\Phi|^2_{\mathcal{L}_2(U_0;H)}$. The space $\mathcal{L}_2(U_0; H)$ will play a significant role in the construction of the stochastic integral for more general processes.

Proposition 1.2.2. *If a process Φ is simple, the process $\Phi \cdot W(t), t \in [0, T]$ is a continuous, square integrable martingale in H and*

$$E\left|\int_0^T \Phi(s)dW(s)\right|^2 = E\int_0^T |\Phi(s)|^2_{\mathcal{L}_2(U_0;H)}ds.$$

One can show that the simple processes form a dense subspace of $L^2([0, T] \times \Omega, \mathcal{P}_T, \lambda \times P; \mathcal{L}_2(U_0; H))$, where λ is the Lebesgue measure on $\mathcal{B}([0, T])$, $\lambda \times P$ is the product measure of λ and P restricted to $\mathcal{P}_T$. Then Proposition 1.2.2 implies that the stochastic integral is an isometric transformation from a dense subspace of $L^2([0, T] \times \Omega, \mathcal{P}_T, \lambda \times P; \mathcal{L}_2(U_0; H))$ into the space of continuous square integrable martingales.

By extending the domain of the transformation, we can define the stochastic integral of an arbitrary process in $L^2([0, T] \times \Omega, \mathcal{P}_T, \lambda \times P; \mathcal{L}_2(U_0; H))$. Finally, we extend the stochastic integral to $\mathcal{L}_2(U_0; H)$-valued predictable processes Φ satisfying the even weaker condition

$$P\left(\int_0^T |\Phi(s)|^2_{\mathcal{L}_2(U_0;H)}ds < \infty\right) = 1 \tag{1.4}$$

by the so-called localization procedure (see Lemma 4.9 in Da Prato and Zabczyk [17]).

1.2.4 Useful Lemmas

In this subsection, we list some lemmas which will be used frequently in the following chapters.

Lemma 1.2.1 (Gronwall Inequality)**.** *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.*
(1)If β is non-negative and if u satisfies the integral inequality

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)ds,\ \ t \in I,$$

then

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds,\ \ t \in I.$$

(2) If, in addition, the function α is non-decreasing, then

$$u(t) \le \alpha(t)e^{\int_a^t \beta(r)dr},\ \ t \in I.$$

The following lemma is an elegant result from Garsia, Rademich and Rumsey (see Corollary 1.2, P.273 in lemma [39]).

Lemma 1.2.2. *Let $\mathcal{R}$ be a unit cube in $\mathbb{R}^n$ and $\{X_\alpha, \alpha \in \mathcal{R}\}$ be a real valued stochastic process. Suppose that there exist constants $k>1, K>0, \varepsilon>0$ such that*

$$E|X_\alpha - X_\beta|^k \le K|\alpha-\beta|^{n+\varepsilon},$$

then (1) *X has a continuous version.*
(2) *there exist constants a,γ depending only on n,k and ε and a random variable Y such that a.s. for all $(\alpha,\beta) \in \mathcal{R}^2$,*

$$|X_\alpha - X_\beta| \le Y|\alpha-\beta|^{\frac{\varepsilon}{k}}\Big(\log\big(\frac{\gamma}{|\alpha-\beta|}\big)\Big)^{\frac{2}{k}}$$

and $EY^k \le aK$.

Chapter 2

Existence and Uniqueness of the Solutions of SPDEs with Two Reflecting Walls

2.1 Introduction

Consider the following SPDE with two reflecting walls:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t,u) + \sigma(x,t,u)\dot{W} + \eta - \xi; \\ u(0,t) = u(1,t) = 0, \quad \text{for } t \geq 0; \\ u(x,0) = u_0(x) \in C([0,1]); \\ h^1(x,t) \leq u(x,t) \leq h^2(x,t), \quad \text{for } (x,t) \in Q, \end{cases} \tag{2.1}$$

where $Q = [0,1] \times \mathbb{R}_+$ and $\dot{W}$ is a space-time white noise on $(Q, \mathcal{B}(Q), \nu)$, where ν is lebesgue measure on Q. More precisely, we are looking for a continuous random field $\{u(x,t), 0 \leq x \leq 1, t \geq 0\}$ which is the solution of equation (2.1) satisfying $h^1(x,t) \leq u(x,t) \leq h^2(x,t)$. When $u(x,t)$ hits $h^1(x,t)$ or $h^2(x,t)$, the additional forces are added to prevent u from leaving $[h^1,\ h^2]$. These forces are expressed by random measures ξ and η in equation (2.1) which play a similar role as the local time in the usual Skorokhod equation constructing Brownian motions with reflecting barriers.

We assume that the reflecting walls $h^1(x,t)$, $h^2(x,t)$ are continuous functions satisfying $h^1(0,t)$, $h^1(1,t) \leq 0$, $h^2(0,t)$, $h^2(1,t) \geq 0$, and

(H1) $h^1(x,t) < h^2(x,t)$ for $x \in (0,1)$ and $t \in \mathbb{R}_+$;

(H2) $\frac{\partial h^i}{\partial t} + \frac{\partial^2 h^i}{\partial x^2} \in L^2([0,1]\times[0,T])$, where $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ are interpreted in a distributional sense;

(H3) $\frac{\partial}{\partial t} h^i(0,t) = \frac{\partial}{\partial t} h^i(1,t) = 0$ for $t \geq 0$;

(H4) $\frac{\partial}{\partial t}(h^2 - h^1) \geq 0$.

We also assume that the coefficients: $f, \sigma : [0,1] \times \mathbb{R}_+ \times C(Q) \to \mathbb{R}$ satisfy for every $T \geq 0$,

(F1) $f(\cdot,\cdot;0), \sigma(\cdot,\cdot;0) \in L^2([0,1]\times[0,T])$;

(F2) for every $M > 0$, there exists $C_{T,M} > 0$ such that

$$|f(x,t;u) - f(x,t;\hat{u})| + |\sigma(x,t;u) - \sigma(x,t;\hat{u})| \leq C_{T,M} \sup_{y\in[0,1],s\in[0,t]} |u(y,s) - \hat{u}(y,s)|,$$

for every $x \in [0,1]$ and $t \in [0,T]$, $u, \hat{u} \in C(Q)$ satisfying

$\sup_{x\in[0,1],t\in[0,T]} |u(x,t)| \leq M, \quad \sup_{x\in[0,1],t\in[0,T]} |\hat{u}(x,t)| \leq M;$

(F3) there exists $C_T > 0$ such that

$$|f(x,t;u)| + |\sigma(x,t;u)| \leq C_T(1 + \sup_{y\in[0,1],s\in[0,t]} |u(y,s)|)$$

for every $x \in [0,1]$ and $t \in [0,T]$.

From the arguments in Bensoussan and Lious [9], under assumption (H1)-(H4), (F1)-(F3) and $u(x,0) \in H^2(0,1)$ satisfy $h^1(x,0) \leq u(x,0) \leq h^2(x,0)$, $u(0,0) = u(1,0) = 0$, there exists a unique $u \in C([0,T]; C([0,1]) \cap L^2(0,T; H^2(0,1))$, $\frac{du}{dt} \in L^2((0,T); L^2(0,1))$, $u(0,t) = u(1,t) = 0$ such that, for every $t \in (0,T)$,

$$(\frac{\partial u}{\partial t}, v - u) + \frac{1}{2}(\frac{\partial u}{\partial x}, \frac{\partial (v-u)}{\partial x}) + (f(u), v-u) \geq 0$$

is satisfied for every $v \in H^1$ with $h^1(x,t) \leq v(x) \leq h^2(x,t)$, $v(0) = v(1) = 0$, where

$(,)$ denotes the inner product in $L^2([0,1])$.

The following is the definition of the solution of a SPDE with the two reflecting walls $h^1,\ h^2$.

Definition 2.1.1. *A triplet* (u,η,ξ) *defined on a filtered probability space* $(\Omega, P, \mathcal{F}; \{\mathcal{F}_t;\ t\geq 0\})$ *is a solution to the SPDE (2.1), denoted by* $(u_0;0,0;f,\sigma;h^1,h^2)$, *if*

(i) $u = \{u(x,t);(x,t)\in Q\}$ *is a continuous, adapted random field (i.e.,* $u(x,t)$ *is* $\mathcal{F}_t$*-measurable* $\forall\ t\geq 0, x\in[0,1]$*) satisfying* $h^1(x,t)\leq u(x,t)\leq h^2(x,t)$ *and* $u(0,t)=u(1,t)=0$, *a.s;*

(ii) $\eta(dx,dt)$ *and* $\xi(dx,dt)$ *are positive and adapted (i.e.* $\eta(B)$ *and* $\xi(B)$ *is* $\mathcal{F}_t$*-measurable if* $B\subset(0,1)\times[0,t]$*) random measures on* $(0,1)\times\mathbb{R}_+$ *satisfying*

$$\eta\big((\theta,1-\theta)\times[0,T]\big)<\infty,\ \xi\big((\theta,1-\theta)\times[0,T]\big)<\infty$$

for $0<\theta<\frac{1}{2}$ *and* $T>0$;

(iii) for all $t\geq 0$ *and* $\phi\in C_0^\infty(0,1)$ *(the set of smooth functions with compact supports) we have*

$$\begin{aligned}&\big(u(t),\phi\big)-\int_0^t(u(s),\phi'')ds-\int_0^t\big(f(y,s,u),\phi\big)ds-\int_0^t\int_0^1\phi\sigma(y,s,u)W(dx,ds)\\ =\;&\big(u_0,\phi\big)+\int_0^t\int_0^1\phi\eta(dx,ds)-\int_0^t\int_0^1\phi\xi(dx,ds),\ a.s,\end{aligned}\tag{2.2}$$

where $u(t)$ *denotes* $u(\cdot,t)$;

(iv)

$$\int_Q\big(u(x,t)-h^1(x,t)\big)\eta(dx,dt)=\int_Q\big(h^2(x,t)-u(x,t)\big)\xi(dx,dt)=0.$$

The purpose of this chapter is to establish the existence and uniqueness of the solutions of a fully non-linear SPDE with two reflecting walls. Our approach is similar to that in Xu and Zhang [40]. However, we need to separate the convergence of two sequences of random measures which correspond to the two reflecting walls respectively.

The chapter is organized as follows. The definition and assumptions will be given in Section 2.2. Section 2.3 is devoted to the existence and uniqueness of the solutions of deterministic obstacle problems. In Section 2.4, we establish the existence and uniqueness of the solutions of the SPDE with two reflecting walls.

2.2 Deterministic Obstacle Problems

Let h^1, h^2 be as in Section 2.1 and $u_0 \in C([0,1])$ with $h^1(x,0) \le u_0(x) \le h^2(x,0)$. Let $v(x,t)$ (sometimes denote by v) $\in C(Q)$, $v(x,0) = u_0(x)$ and $v(0,t) = v(1,t) = 0$. Consider a deterministic PDE with two reflecting walls:

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} - \frac{\partial^2 z(x,t)}{\partial x^2} = \eta(x,t) - \xi(x,t); \\ z(0,t) = z(1,t) = 0, \quad \text{for } t \ge 0; \\ z(x,0) = 0, \quad \text{for } x \in [0,1]; \\ h^1(x,t) \le z(x,t) + v(x,t) \le h^2(x,t), \quad \text{for } (x,t) \in Q. \end{cases} \tag{2.3}$$

We first present a precise definition of the solution for equation (2.3).

Definition 2.2.1. *A triplet* (z, η, ξ) *is called a solution to the PDE (2.3) if*

(i) $z = z(x,t); (x,t) \in Q$ *is a continuous function satisfying* $h^1(x,t) \le z(x,t) + v(x,t) \le h^2(x,t)$, $z(x,0) = 0$, $z(0,t) = z(1,t) = 0$*;*

(ii) $\eta(dx, dt)$ *and* $\xi(dx, dt)$ *are measures on* $(0,1) \times \mathbb{R}_+$ *satisfying*

$$\eta\big((\theta, 1-\theta) \times [0,T]\big) < \infty, \ \xi\big((\theta, 1-\theta) \times [0,T]\big) < \infty$$

for every $0 < \theta < \frac{1}{2}$ *and* $T > 0$*;*

(iii) for all $t \geq 0$ and $\phi \in C_0^\infty(0,1)$ we have

$$\begin{aligned} & \big(z(t),\phi\big) - \int_0^t (z(s),\phi^{''})ds \\ = & \int_0^t\int_0^1 \phi\eta(dx,ds) - \int_0^t\int_0^1 \phi\xi(dx,ds), \end{aligned} \tag{2.4}$$

where $z(t)$ denotes $z(\cdot,t)$;

(iv)

$$\begin{aligned} & \int_Q \big(z(x,t)+v(x,t)-h^1(x,t)\big)\eta(dx,dt) \\ = & \int_Q \big(h^2(x,t)-z(x,t)-v(x,t)\big)\xi(dx,dt) \\ = & \ 0. \end{aligned}$$

To find a solution to equation (2.3), we first consider a single reflecting barrier problem, denoted by $(0;0,0;f;h^1)$:

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f(x,t,z) + \eta; \\ z(0,t) = z(1,t) = 0, \quad \text{for } t \geq 0; \\ z(x,0) = 0, \quad \text{for } x \in [0,1]; \\ z(x,t) + v(x,t) \geq h^1(x,t), \quad \text{for } (x,t) \in Q, \end{cases} \tag{2.5}$$

where the coefficient f is as in Section 2.1 and satisfies the Lipschitz condition. The existence and uniqueness of the solutions of $(0;0,0;f;h^1)$ follows from Nualart and Pardoux [27] by replacing the function $v(x,t)$ there by $v(x,t) - h^1(x,t)$. Next lemma is a comparison theorem for the PDE with reflection.

Lemma 2.2.1 (Comparison)**.** *Let two functions f and $\hat{f}$ satisfy (F1) and Lipschitz condition with $f \leq \hat{f}$. We denote by z (resp.$\hat{z}$) the solution of equation (2.5) corresponding to f (resp. $\hat{f}$) with the same initial condition. Then we have $z \leq \hat{z}$.*

Proof. Consider the penalized problem:

$$\begin{cases} \frac{\partial z^\delta}{\partial t} = \frac{\partial^2 z^\delta}{\partial x^2} + f(z^\delta) + \frac{1}{\delta}(z^\delta + v - h^1)^-; \\ z^\delta(0,t) = z^\delta(1,t) = 0, \quad \text{for } t \geq 0; \\ z^\delta(x,0) = 0, \quad \text{for } x \in [0,1]. \end{cases} \tag{2.6}$$

Denote by $\hat{z}^\delta$ the solution of equation (2.6) replacing f by $\hat{f}$ (denoted by equation (2.6)$'$). Equation (2.6) is a PDE without reflection.

Multiply (2.6) by $\nu^+ = (z^\delta - \hat{z}^\delta)^+$, if $\nu := (z^\delta - \hat{z}^\delta)$ and multiply (2.6)$'$ by $-\nu^+$; adding, we obtain

$$\begin{aligned} \int_0^T (\frac{\partial \nu_t}{\partial t}, \nu_t^+) dt &= \int_0^T (\frac{\partial^2 \nu_t}{\partial x^2}, \nu_t^+) dt + \int_0^T (f(z^\delta) - \hat{f}(\hat{z}^\delta), \nu_t^+) dt \\ &+ \int_0^T \frac{1}{\delta}\big((z^\delta + v - h^1)^- - (\hat{z}^\delta + v - h^1)^-, \nu_t^+\big) dt. \end{aligned}$$

We know

$$\int_0^T (\frac{\partial \nu_t}{\partial t}, \nu_t^+) dt = \int_0^T (\frac{\partial \nu_t^+}{\partial t}, \nu_t^+) dt = \frac{1}{2} |\nu_T^+|_{L^2}^2,$$

$$\int_0^T (\frac{\partial^2 \nu_t}{\partial x^2}, \nu_t^+) dt = -\int_0^T (\frac{\partial \nu_t^+}{\partial x}, \frac{\partial \nu_t^+}{\partial x}) dt = -\int_0^T |\frac{\partial \nu_t^+}{\partial x}|_{L^2}^2 dt,$$

and $\int_0^T \frac{1}{\delta}\big((z^\delta + v - h^1)^- - (\hat{z}^\delta + v - h^1)^-, \nu_t^+\big) dt \leq 0$ because on $\{(x,t); \nu(x,t) > 0\}$, $z^\delta \geq \hat{z}^\delta$. Moreover,

$$\begin{aligned} \int_0^T \big(f(z^\delta) - \hat{f}(\hat{z}^\delta), \nu_t^+\big) dt &= \int_0^T \big(f(z^\delta) - \hat{f}(z^\delta) + \hat{f}(z^\delta) - \hat{f}(\hat{z}^\delta), \nu_t^+\big) dt \\ &\leq \int_0^T \big(\hat{f}(z^\delta) - \hat{f}(\hat{z}^\delta), \nu_t^+\big) dt \\ &\leq C \int_0^T (z^\delta - \hat{z}^\delta, \nu_t^+) dt \\ &= C \int_0^T |\nu_t^+|_{L^2}^2 dt. \end{aligned}$$

Then we have $\frac{1}{2}|\nu_T^+|_{L^2}^2 \leq C \int_0^T |\nu_t^+|_{L^2}^2 dt$. By Gronwall inequality, we immediately get to $|\nu_T^+|_{L^2}^2 = 0$. Hence $\nu_t^+ = 0$. Therefore, we have $z^\delta \leq \hat{z}^\delta$. According to Nualart and

Pardoux [27], $z^\delta \to z$ and $\hat{z}^\delta \to \hat{z}$ as $\delta \to 0$. Hence, we deduce $z \leq \hat{z}$. □

The following result is the existence and uniqueness of the solutions of the PDE with two reflecting walls.

Theorem 2.2.1. *Equation (2.3) admits a unique solution* (z, η, ξ).

Proof. Existence: Consider the penalized PDE:

$$\begin{cases} \frac{\partial z^{\varepsilon,\delta}}{\partial t} = \frac{\partial^2 z^{\varepsilon,\delta}}{\partial x^2} + \frac{1}{\delta}(z^{\varepsilon,\delta} + v - h^1)^- - \frac{1}{\varepsilon}(z^{\varepsilon,\delta} + v - h^2)^+; \\ z^{\varepsilon,\delta}(0,t) = z^{\varepsilon,\delta}(1,t) = 0; \\ z^{\varepsilon,\delta}(x,0) = 0. \end{cases} \tag{2.7}$$

Denote by $z^{\varepsilon,\delta}$ the solution of equation (2.7) and $\hat{z}^{\varepsilon,\delta}$ is the solution to equation (2.7) replacing v by $\hat{v}$.

Define $k := \|v - \hat{v}\|_\infty^T$, where $\|v\|_\infty^T := \sup_{x\in[0,1],t\in[0,T]} |v(x,t)|$ and $\omega := (z^{\varepsilon,\delta} - \hat{z}^{\varepsilon,\delta}) - k$. Then ω satisfies the following PDE:

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \frac{\partial^2 \omega}{\partial x^2} + \frac{1}{\delta}\big((z^{\varepsilon,\delta} + v - h^1)^- - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h^1)^-\big) \\ &\quad - \frac{1}{\varepsilon}\big((z^{\varepsilon,\delta} + v - h^2)^+ - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h^2)^+\big). \end{aligned}$$

Multiplying the above equation by ω^+ yeilds

$$\begin{aligned} \int_0^T \big(\frac{\partial \omega}{\partial t}, \omega^+\big) dt &= \int_0^T \big(\frac{\partial^2 \omega}{\partial x^2}, \omega^+\big) dt + \int_0^T \frac{1}{\delta}\big((z^{\varepsilon,\delta} + v - h^1)^- \\ &\quad - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h^1)^-, \omega^+\big) dt \\ &\quad - \int_0^T \frac{1}{\varepsilon}\big((z^{\varepsilon,\delta} + v - h^2)^+ - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h^2)^+, \omega^+\big) dt \end{aligned}$$

which means

$$\frac{1}{2}|\omega_T^+|_{L^2}^2 \leq -\int_0^T |\frac{\partial \omega^+}{\partial x}|_{L^2}^2 dt,$$

because $\frac{1}{\delta}\big((z^{\varepsilon,\delta} + v - h^1)^- - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h^1)^-, \omega^+\big) \leq 0$, $\frac{1}{\varepsilon}\big((z^{\varepsilon,\delta} + v - h^2)^+ - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h^2)^+, \omega^+\big) \geq 0$. Hence, we get $\omega_t^+ = 0$, $z^{\varepsilon,\delta} - \hat{z}^{\varepsilon,\delta} \leq k$. By symmetry, $\hat{z}^{\varepsilon,\delta} - z^{\varepsilon,\delta} \leq k$.

Then we have for any $T > 0$,

$$\|z^{\varepsilon,\delta} - \hat{z}^{\varepsilon,\delta}\|_\infty^T \le \|v - \hat{v}\|_\infty^T. \tag{2.8}$$

Let us consider a single reflection problem $(0; 0, 0; -\frac{(\cdot+v-h^2)^+}{\varepsilon}; h^1)$ and denote by $(z^\varepsilon, \eta^\varepsilon)$ its unique solution. From Nualart and Pardoux [27], it is know that z^ε is the limit of $z^{\varepsilon,\delta}$ when $\delta \downarrow 0$. By virtue of the comparison theorem (Lemma 2.2.1), z^ε is a decreasing sequence as ε decreases. For any $(x,t) \in Q$, define $z(x,t) := \lim_{\varepsilon,\delta\downarrow 0} z^{\varepsilon,\delta}$. we want to show $z \in C(Q)$. Let $\{v_n, n \in \mathbb{N}\} \subset C_0^\infty\big((0,1)\times\mathbb{R}_+\big)$ satisfy $v_n(x,t) \Rightarrow v(x,t)$ uniformly on compact subsets of Q. Let $z_n^{\varepsilon,\delta}$ denote the unique solution to (2.7) replacing v by v_n. Then denote $z_n := \lim_{\varepsilon,\delta\downarrow 0} z_n^{\varepsilon,\delta}$ exists. From the arguments in Bensoussan and Lious [9], we get z_n is continuous and a unique solution to the following evolutionary variational inequality with two obstacles:

$$\begin{cases} \big(\frac{\partial z_n}{\partial t} - \frac{\partial^2 z_n}{\partial x^2}, y - z_n\big) \ge 0; \\ z_n(x,0) \in H^2(0,1),\ z_n(0,0) = z_n(1,0) = 0; \\ h^1(x,0) \le z_n(x,0) + v_n(x,0) \le h^2(x,0); \\ \frac{\partial}{\partial t}(h^2 - h^1) \ge 0 \end{cases}$$

is satisfied for every $y \in H^1$ with $h^1(x,t) \le y(x,t) + v_n(x,t) \le h^2(x,t)$, $y(0,t) = y(1,t) = 0$. From inequality (2.8), we know $\|z_n^{\varepsilon,\delta} - z^{\varepsilon,\delta}\|_\infty^T \le \|v_n - v\|_\infty^T$. Hence let $\varepsilon,\ \delta \downarrow 0$, we conclude that $\|z_n - z\|_\infty^T \le \|v_n - v\|_\infty^T$. Then $z \in C(Q)$, by letting $n \to \infty$.

We will show that the function $z(x,t)$ is a solution to equation (2.7) i.e., it satisfies the conditions (i)-(iv) of Definition 2.2.1. Let $\phi(x,t) \in C_0^\infty((0,1)\times\mathbb{R}_+)$. From the definition of the solution of the single reflecting barrier problem $(0; 0, 0; -\frac{(\cdot+v-h^2)^+}{\varepsilon}; h^1)$:

$$\begin{aligned} &(z^\varepsilon(t), \phi) - \int_0^t (z^\varepsilon(s), \frac{\partial \phi}{\partial t})ds - \int_0^t (z^\varepsilon(s), \frac{\partial^2 \phi}{\partial x^2})ds \\ = &\int_0^t \int_0^1 \phi\big(\eta^\varepsilon(dx, ds) - \xi^\varepsilon(dx, ds)\big), \end{aligned} \tag{2.9}$$

where $\xi^\varepsilon(dx,dt) := \frac{(z^\varepsilon+v-h^2)^+}{\varepsilon}dxdt$ and $\eta^\varepsilon(dx,dt) := \lim\limits_{\delta\downarrow 0}\frac{(z^{\varepsilon,\delta}+v-h^1)^-}{\delta}dxdt$. The limit of the left hand side exists as $\varepsilon \to 0$. Therefore, $\lim\limits_{\varepsilon\downarrow 0}(\eta^\varepsilon - \xi^\varepsilon)$ exists in the space of distributions. Next we want to show that both $\lim\limits_{\varepsilon\downarrow 0}\eta^\varepsilon$ and $\lim\limits_{\varepsilon\downarrow 0}\xi^\varepsilon$ exists. As $h^1(x,t) < h^2(x,t)$ in $(x,t) \in (0,1)\times\mathbb{R}_+$, for any compact set K of $(0,1)\times\mathbb{R}_+$, there exists θ_K such that $h^2(x,t) - h^1(x,t) \geq \theta_K > 0$. The continuous functions $z^\varepsilon(x,t)$ decreases to a continuous function $z(x,t)$ on Q. By Dini theorem we know that $z^\varepsilon(x,t) \Rightarrow z(x,t)$ uniformly on compact subsets of Q. For $\phi(x,t) \in C_0^\infty((0,1)\times\mathbb{R}_+)$, denote by $K = \text{supp}\phi$ the compact support of ϕ. Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $|z^\varepsilon(x,t) - z(x,t)| < \frac{\theta_K}{4}$ on K.

Let θ_K be chosen as above. Since

$$\text{supp}\eta^\varepsilon \subseteq \{(x,t); z^\varepsilon(x,t) + v(x,t) = h^1(x,t)\}$$

and

$$\text{supp}\xi^\varepsilon = \{(x,t); z^\varepsilon(x,t) + v(x,t) \geq h^2(x,t)\},$$

we have

$$\text{supp}\eta^\varepsilon \cap K \subseteq \{(x,t); z(x,t) - \frac{\theta_K}{4} + v(x,t) \leq h^1(x,t)\} \cap K =: A_K$$

and

$$\text{supp}\xi^\varepsilon \cap K \subseteq \{(x,t); z(x,t) + \frac{\theta_K}{4} + v(x,t) \geq h^2(x,t)\} \cap K =: B_K,$$

for $\varepsilon < \varepsilon_0$. By the choice of θ_K, it is easy to see $A_K \cap B_K = \emptyset$. Thus, we can find $\tilde{\phi}(x,t) \in C_0^\infty((0,1)\times\mathbb{R}_+)$ such that $\tilde{\phi} = \phi$ on A_K, $\text{supp}\tilde{\phi} \cap B_K = \emptyset$ and $\text{supp}\tilde{\phi} \cap \text{supp}\xi^\varepsilon = \emptyset$ for $\varepsilon < \varepsilon_0$. Hence,

$$\begin{aligned}
& \lim_{\varepsilon\downarrow 0}\int_0^t\int_0^1 \phi(x,s)\eta^\varepsilon(dx,ds) \\
= & \lim_{\varepsilon\downarrow 0}\int_0^t\int_0^1 \tilde{\phi}(x,s)\eta^\varepsilon(dx,ds) \\
= & \lim_{\varepsilon\downarrow 0}\Big(\int_0^t\int_0^1 \tilde{\phi}(x,s)\eta^\varepsilon(dx,ds) - \int_0^t\int_0^1 \tilde{\phi}(x,s)\xi^\varepsilon(dx,ds)\Big)
\end{aligned}$$

exists. Therefore, $\eta^\varepsilon \Rightarrow \eta$ in the space of distributions. Similarly, $\xi^\varepsilon \Rightarrow \xi$. Let $\varepsilon \to 0$ in equation (2.9) to see that (z, η, ξ) satisfies condition (iii) of the definition 2.2.1.

Multiplying (2.9) by ε and letting $\varepsilon \downarrow 0$, we get $\int_0^t \int_0^1 \phi(z+v-h^2)^+ dxdt = 0$ for any $\phi \in C_0^\infty\big((0,1)\times\mathbb{R}_+\big)$. This implies $z+v-h^2 \le 0$ a.s. and $z+v-h^2$ being a continuous function, so we obtain $z+v \le h^2$. Since $h^1 \le z^\varepsilon + v$, we see that $h^1 \le z+v$. So $h^1(x,t) \le z(x,t)+v(x,t) \le h^2(x,t)$.

Now let us show that (iv) holds in the definition 2.2.1. By the definition of ξ^ε, $\int_0^\infty \int_0^1 \big(z^\varepsilon(x,t)+v(x,t)-h^2(x,t)\big)\phi(x,t)\xi^\varepsilon(dx,dt) \ge 0$ for every $\phi(x,t) \in C_0^\infty((0,1)\times\mathbb{R}_+)$ with $\phi(x,t)\ge 0$. Letting $\varepsilon \downarrow 0$, we have

$$\int_0^T \int_0^1 \big(z(x,t)+v(x,t)-h^2(x,t)\big)\phi(x,t)\xi(dx,dt) \ge 0.$$

Hence, we must have $\int_0^\infty \int_0^1 \big(z(x,t)+v(x,t)-h^2(x,t)\big)\xi(dx,dt) = 0$. From the single reflecting barrier problem $(0;0,0;-\frac{(\cdot+v-h^2)^+}{\varepsilon};h^1)$, we know $\int_0^\infty \int_0^1 \big(z^\varepsilon(x,t)+v(x,t)-h^1(x,t)\big)\phi\eta^\varepsilon(dx,dt) = 0$ for every $\phi(x,t)\in C_0^\infty((0,1)\times\mathbb{R}_+)$. Then letting $\varepsilon\downarrow 0$ we get $\int_0^\infty \int_0^1 \big(z(x,t)+v(x,t)-h^1(x,t)\big)\eta(dx,dt) = 0$.

It only remains to check condition (ii) in the definition 2.2.1. Taking a non-negative function $\phi \in C_0^\infty\big((0,1)\times\mathbb{R}_+\big)$ such that $\phi(x,t)=1$ on $(\text{supp}\eta)\cap([\theta,\ 1-\theta]\times[0,T])$ and $\phi(x,t)=0$ on $\text{supp}\xi\cap\text{supp}\phi$. Hence, in view of (2.4)

$$\begin{aligned} & \eta\big((\theta,\ 1-\theta)\times[0,T]\big) \\ \le\ & \int_0^T\int_0^1 \phi(x,t)\eta(dx,ds) - \int_0^T\int_0^1 \phi(x,t)\xi(dx,ds) \\ <\ & \infty, \end{aligned}$$

for all $0<\theta<\frac{1}{2}$ and $T>0$. Similarly, we get $\xi\big((\theta,\ 1-\theta)\times[0,T]\big) < \infty$ for all $T>0$.

Uniqueness: Let (z_1,η_1,ξ_1) and (z_2,η_2,ξ_2) are two solutions to equation (2.3). For any $\phi \in C_0^\infty((0,1) \times \mathbb{R}_+)$, we have

$$\begin{aligned} &(z_1(t) - z_2(t), \phi) - \int_0^t (z_1(s) - z_2(s), \frac{\partial \phi}{\partial s})ds - \int_0^t (z_1(s) - z_2(s), \frac{\partial^2 \phi}{\partial x^2})ds \\ = &\int_0^t \int_0^1 \phi\big(\eta_1(dx,ds) - \eta_2(dx,ds)\big) - \int_0^t \int_0^1 \phi\big(\xi_1(dx,ds) - \xi_2(dx,ds)\big) \\ &. \end{aligned} \tag{2.10}$$

Putting $\omega(x,t) := z_1(x,t) - z_2(x,t)$, Then for any $\psi \in C_0^\infty(Q)$ and its support contained in $[\delta,\ 1-\delta] \times \mathbb{R}_+$ (for some $\delta > 0$), we have

$$\begin{aligned} (\omega_t, \psi_t) \;\; = \;\; &\int_0^t (\omega_s, \frac{\partial}{\partial s}\psi_s)ds + \int_0^t (\omega_s, \frac{\partial^2}{\partial x^2}\psi_s)ds + \int_0^t \int_0^1 \psi_s\big(\eta_1(dx,ds) - \eta_2(dx,ds)\big) \\ &- \int_0^t \int_0^1 \psi_s\big(\xi_1(dx,ds) - \xi_2(dx,ds)\big). \end{aligned}$$

Fix δ and let $\varphi \in C_0^\infty(0,1)$ and its support contained in $[\delta,\ 1-\delta]$. To mollifier ω, let $\varepsilon(x) \in C_0^\infty(\mathbb{R})$, which is symmetric, its support is contained in $[-1,\ 1]$, $\int_{-1}^1 \varepsilon(x)dx = 1$ and it is nonnegative definite. Then we introduce $\varepsilon_n(x) := n\varepsilon(nx)$ which support contained in $[-\frac{1}{n},\ \frac{1}{n}]$ and $\int_{-\frac{1}{n}}^{\frac{1}{n}} \varepsilon_n(x)dx = 1$. Define $\varepsilon_{n,m}(x) = \varepsilon_n(x)\varepsilon_m(x)$ and $\psi_{n,m} = [(\omega\varphi) * \varepsilon_{n,m}]\varphi$, i.e.

$$\psi_{n,m}(x,t) = \Big(\int_{(t-\frac{1}{n})^+}^{t+\frac{1}{n}} \int_0^1 \omega(y,s)\varphi(y)\varepsilon_n(t-s)\varepsilon_m(x-y)dyds\Big)\varphi(x),$$

where assuming $\frac{1}{n} < \delta$. Then we get

$$\lim_{n,m\to\infty} (\psi_{n,m}(t), \omega(t)) = |\omega(t)\varphi|_{L^2}^2,$$

$$\lim_{n,m\to\infty} \int_0^t (\frac{\partial \psi_{n,m}(s)}{\partial s}, \omega(s))ds = 0,$$

and

$$\begin{aligned}
&\lim_{n,m\to\infty}\Big(\int_0^t\int_0^1 \psi_{n,m}(x,s)(\eta_1(dx,ds)-\eta_2(dx,ds))\Big) \\
&\quad -\int_0^t\int_0^1 \psi_{n,m}(x,s)(\xi_1(dx,ds)-\xi_2(dx,ds))\Big) \\
= &\int_0^t\int_0^1 \omega(x,s)\varphi(x)^2(\eta_1(dx,ds)-\eta_2(dx,ds)) \\
&-\int_0^t\int_0^1 \omega(x,s)\varphi(x)^2(\xi_1(dx,ds)-\xi_2(dx,ds)) \\
= &-\int_0^t\int_0^1 \big((z_1+v-h^1)\varphi^2\eta_2(dx,ds)+(z_2+v-h^1)\varphi^2\eta_1(dx,ds)\big) \\
&+\int_0^t\int_0^1 \big((z_1+v-h^2)\varphi^2\xi_2(dx,ds)+(z_2+v-h^2)\varphi^2\xi_1(dx,ds)\big) \\
\leq &\ 0,
\end{aligned}$$

as well as

$$\liminf_{m\to\infty}\lim_{n\to\infty}\int_0^t(\frac{\partial^2\psi_{n,m}(s)}{\partial x^2},\omega(s))ds \leq \frac{1}{2}\int_0^t\int_0^1 \omega(x,s)^2(\varphi^2)''(x)dxds.$$

Consequently, we deduce

$$\int_0^1 \omega(x,t)^2\varphi(x)^2dx \leq \frac{1}{2}\int_0^t\int_0^1 \omega(x,s)^2(\varphi^2)''(x)dxds,$$

for all $t>0$ and any function $\varphi\in C_0^\infty((0,1))$. It can be seen that the above inequality still holds for $(\varphi^2)''(\in C_0((0,1)))$ being a measure. So choose

$$\varphi(x)^2=\begin{cases}\frac{1}{\varepsilon}(x-a), & a<x\leq a+\varepsilon\\ 1, & a+\varepsilon<x\leq b\\ \frac{1}{\varepsilon}(b+\varepsilon-x), & b<x\leq b+\varepsilon\\ 0, & x\in(a,\ b+\varepsilon)^c,\end{cases}$$

where $0 < a < a+\varepsilon \leq b < b+\varepsilon < 1$. Then define $\beta(x) = \int_0^t \omega(x,s)^2 dt$, we know

$$\begin{aligned}
& \int_0^1 \omega(x,t)^2 \varphi(x)^2 dx \\
\leq\ & \frac{1}{2\varepsilon} \int_0^t \int_0^1 \omega(x,s)^2 [\delta(x-a) - \delta(x-a-\varepsilon) - \delta(x-b) + \delta(x-b-\varepsilon)] dx ds \\
\leq\ & \frac{1}{2\varepsilon} \int_0^t [\omega(a,s)^2 - \omega(a+\varepsilon,s)^2 - \omega(b,s)^2 + \omega(b+\varepsilon,s)^2] ds \\
\leq\ & \frac{1}{2\varepsilon} [\beta(a) - \beta(a+\varepsilon) - \beta(b) + \beta(b+\varepsilon)].
\end{aligned}$$

By taking $\varepsilon = b - a$, we get $\beta(a+2\varepsilon) - \beta(a+\varepsilon) \geq \beta(a+\varepsilon) - \beta(a)$. With $\beta(1) = 0$ and the continuous of β, we obtain $\beta(x) = 0$ for all $x \in [0,1]$. It shows $\omega_t = 0$ for all $t > 0$.

Recall that

$$\text{supp}\eta_1,\ \text{supp}\eta_2 \subseteq \{(x,t); z_1(x,t) + v(x,t) = h^1(x,t)\} =: A$$

and

$$\text{supp}\xi_1,\ \text{supp}\xi_2 \subseteq \{(x,t); z_1(x,t) + v(x,t) = h^2(x,t)\} =: B.$$

Because $A \cap B \cap ((0,1) \times \mathbb{R}_+) = \emptyset$, for any $\phi \in C_0^\infty((0,1) \times \mathbb{R}_+)$ with $\text{supp}\phi \subset (\text{supp}\eta_1 \cup \text{supp}\eta_2)$, it holds that $\text{supp}\phi \cap \text{supp}\xi_1 = \emptyset$ and $\text{supp}\phi \cap \text{supp}\xi_2 = \emptyset$. Applying equation (2.10) to such functions ϕ, we deduce that $\eta_1 = \eta_2$. Similarly, $\xi_1 = \xi_2$. Then the uniqueness is proved. □

Remark 2.2.1. *Let $\hat{z}$ be the solution to equation (2.3) replacing v by $\hat{v}$, where $\hat{v}(x,t)$ is another continuous function on Q. We see that*

$$\|z - \hat{z}\|_\infty^T \leq \|v - \hat{v}\|_\infty^T, \tag{2.11}$$

for any $T > 0$.

2.3 SPDEs with Two Reflecting Walls

As in Section 2.1, consider the following SPDE with two reflecting walls:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t,u) + \sigma(x,t,u)\dot{W} + \eta - \xi; \\ u(0,t) = u(1,t) = 0, \quad \text{for } t \geq 0; \\ u(x,0) = u_0(x) \in C([0,1]); \\ h^1(x,t) \leq u(x,t) \leq h^2(x,t), \quad \text{for } (x,t) \in Q. \end{cases} \tag{2.12}$$

Theorem 2.3.1. *Let* $u_0 \in C([0,1])$ *satisfy* $h^1(x,0) \leq u_0(x) \leq h^2(x,0)$, $u_0(0) = u_0(1) = 0$. *Under the hypotheses (H1)-(H4), (F1)-(F3), there exists a unique solution to the SPDE with two reflecting walls* $(u_0; 0, 0; f, \sigma; h^1, h^2)$.

Proof. Let

$$\begin{aligned} v_1(x,t) &= \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(y,s;u_0)dyds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,s;u_0)W(dy,ds), \end{aligned}$$

where $G_t(x,y)$ is Green's function of the heat equation. As in Walsh [39] we see that $v_1(x,t)$ satisfies

$$\begin{cases} \frac{\partial v_1(x,t)}{\partial t} = \frac{\partial^2 v_1(x,t)}{\partial x^2} + f(x,t;u_0) + \sigma(x,t;u_0)\dot{W}(x,t); \\ v_1(0,t) = v_1(1,t) = 0; \\ v_1(x,0) = u_0(x). \end{cases} \tag{2.13}$$

Let (z_1, η_1, ξ_1) be the unique solution of equation (2.3) with $v = v_1$ (denote by equation (2.3)'). Then (2.13)+(2.3)' satisfy

$$\begin{cases} \frac{\partial (v_1+z_1)(x,t)}{\partial t} = \frac{\partial^2 (v_1+z_1)(x,t)}{\partial x^2} + f(x,t;u_0) + \sigma(x,t;u_0)\dot{W}(x,t) + \eta_1 - \xi_1; \\ (v_1+z_1)(0,t) = (v_1+z_1)(1,t) = 0; \\ (v_1+z_1)(x,0) = u_0(x); \\ h^1(x,t) \leq (v_1+z_1)(x,t) \leq h^2(x,t). \end{cases}$$

Set $u_1 = z_1 + v_1$, then (u_1, η_1, ξ_1) is the unique (from the uniqueness of z_1 and v_1) solution of the following reflected SPDE with two walls:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = \frac{\partial^2 u_1(x,t)}{\partial x^2} + f(x,t;u_0) + \sigma(x,t;u_0)\dot{W}(x,t) + \eta_1 - \xi_1; \\ u_1(0,t) = u_1(1,t) = 0; \\ u_1(x,0) = u_0(x); \\ h^1(x,t) \le u_1(x,t) \le h^2(x,t). \end{cases}$$

Iterating this procedure, suppose u_{n-1} has been defined. Let

$$\begin{aligned} v_n(x,t) &= \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(y,s;u_{n-1})dyds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,s;u_{n-1})W(dy,ds). \end{aligned} \tag{2.14}$$

Denote by (z_n, η_n, ξ_n) the unique solution of equation (2.3) replacing v by v_n. Set $u_n = z_n + v_n$, Then (u_n, η_n, ξ_n) is the unique solution of the following reflected SPDE with two walls:

$$\begin{cases} \frac{\partial u_n(x,t)}{\partial t} = \frac{\partial^2 u_n(x,t)}{\partial x^2} + f(x,t;u_{n-1}) + \sigma(x,t;u_{n-1})\dot{W}(x,t) + \eta_n - \xi_n; \\ u_n(0,t) = u_n(1,t) = 0; \\ u_n(x,0) = u_0(x); \\ h^1(x,t) \le u_n(x,t) \le h^2(x,t). \end{cases}$$

By inequality (2.11), we know that

$$\|z_n - z_{n-1}\|_\infty^T \le \|v_n - v_{n-1}\|_\infty^T. \tag{2.15}$$

Hence,

$$\|u_n - u_{n-1}\|_\infty^T \le 2\|v_n - v_{n-1}\|_\infty^T. \tag{2.16}$$

The following claim provides the uniform control of the k-th moment of stochastic convolution and it was proved in Donati-Martin and Pardoux [14] and also in Xu and Zhang [40].

Claim 2.3.1. *Let* $F(x,t) = \int_0^t \int_0^1 G_{t-s}(x,y)H(y,s)W(dy,ds)$. *Then it holds that*

$$\mathbb{E}\big(\sup_{0\le x\le 1, 0\le s\le t} |F(x,s)|\big)^k \le C_1(k,T)\int_0^t \sup_{0\le y\le 1, 0\le s\le u} \mathbb{E}\big(|H(y,s)|^k\big)du,$$

for $t \le T$, *where* $C_1(k,T)$ *is a constant depending on* k, T.

Applying the above claim, we see that

$$\begin{aligned}
& \mathbb{E}\big(\sup_{0\le x\le 1, 0\le u\le t} |\int_0^u \int_0^1 G_{u-s}(x,y)\big(\sigma(y,s,u_{n-1}) - \sigma(y,s,u_{n-2})\big)W(dy,ds)|\big)^k \\
\le\ & C_1(k,T)\int_0^t \sup_{0\le y\le 1, 0\le s\le u} \mathbb{E}\big(|u_{n-1}(y,s) - u_{n-2}(y,s)|^k\big)du \\
=\ & C_1(k,T)\mathbb{E}\int_0^t \big(\|u_{n-1} - u_{n-2}\|_\infty^u\big)^k du.
\end{aligned}$$

It is also easy to see that

$$\begin{aligned}
& \mathbb{E}\big(\sup_{0\le x\le 1, 0\le u\le t} |\int_0^u \int_0^1 G_{u-s}(x,y)\big(f(y,s,u_{n-1}) - f(y,s,u_{n-2})\big)dy, ds|\big)^k \\
\le\ & C_2(k,T)\mathbb{E}\int_0^t \big(\|u_{n-1} - u_{n-2}\|_\infty^u\big)^k du.
\end{aligned}$$

Using inequality (2.16), (F2), we obtain

$$\begin{aligned}
\mathbb{E}\big(\|u_n - u_{n-1}\|_\infty^T\big)^k &\le 2^k\mathbb{E}\big(\|v_n - v_{n-1}\|_\infty^T\big)^k \\
&\le C(k,T)\mathbb{E}\int_0^T \big(\|u_n - u_{n-1}\|_\infty^t\big)^k dt \\
&\le C^{k-1}(k,T)\mathbb{E}\big(\|u_1 - u_0\|_\infty^T\big)^k \frac{T^{n-1}}{(n-1)!} \qquad (2.17)
\end{aligned}$$

where $C(k,T)$ is a constant depending on k, T. This implies that $\lim_{n,m\to\infty} \mathbb{E}\big(\|u_n - u_m\|_\infty^T\big)^k = 0$. Thus, there exists a random field $u(\cdot,\cdot) \in C(Q)$ such that $\lim_{n\to\infty} \mathbb{E}\big(\|u_n -$

$u\|_\infty^T\big)^k = 0$ for $T > 0$. So we obtain $\lim\limits_{n\to\infty} \mathbb{E}\big(\|v_n - v\|_\infty^T\big)^k = 0$, where

$$\begin{aligned} v(x,t) &= \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(y,s;u)dyds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,s;u)W(dy,ds). \end{aligned} \tag{2.18}$$

Thus, for any $\phi \in C_0^\infty((0,1)\times\mathbb{R}_+)$

$$\begin{aligned} &\big(v(t),\phi\big) - \big(u_0,\phi\big) - \int_0^t \big(v(s), \frac{\partial\phi}{\partial s}\big)ds - \int_0^t \big(v(s), \frac{\partial^2\phi}{\partial x^2}\big)ds \\ = &\int_0^t \big(f(y,s,u),\phi\big)ds + \int_0^t\int_0^1 \phi\sigma(y,s,u)W(dy,ds). \end{aligned}$$

From inequality (2.15), there exists a continuous random field $z(\cdot,\cdot)$ on Q such that $\lim\limits_{n\to\infty} \mathbb{E}\big(\|z_n - z\|_\infty^T\big)^k = 0$ for $T > 0$. So $z_n \Rightarrow z$ uniformly on $[0,1]\times[0,T]$ a.s. (If necessary we take the subsequence of $\{z_n\}_{n\geq 1}$). Similar to the proof of Theorem 2.2.1, we can show that $\eta(dx,ds) := \lim\limits_{n\to\infty} \eta_n(dx,ds)$, $\xi(dx,ds) := \lim\limits_{n\to\infty} \xi_n(dx,ds)$ exists almost surely and (z,η,ξ) is the solution of equation (2.3). Put $u(x,t) = z(x,t) + v(x,t)$. It is easy to verify (u,η,ξ) is a solution to the SPDE (2.12) with two reflecting walls.

Uniqueness: suppose (u_1,η_1,ξ_1) and (u_2,η_2,ξ_2) are two solutions to equation (2.12). Introduce

$$\begin{aligned} v_i(x,t) &:= \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(y,s;u_i)dyds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(y,s;u_i)W(dy,ds), \quad i = 1,2. \end{aligned}$$

Then $(z_i := u_i - v_i, \eta_i, \xi_i)$ is the solution to equation (2.3) with $v = v_i$ a.s. Hence, $\|z_1 - z_2\|_\infty^T \leq \|v_1 - v_2\|_\infty^T$ and $\|u_1 - u_2\|_\infty^T \leq 2\|v_1 - v_2\|_\infty^T$. By the similar proof as that for inequality (2.17), we have

$$\mathbb{E}\big(\|u_1 - u_2\|_\infty^T\big)^k \leq C(k,T)\mathbb{E}\int_0^T \big(\|u_1 - u_2\|_\infty^t\big)^k dt.$$

By Gronwall inequality, we have $\mathbb{E}\big(\|u_1 - u_2\|_\infty^T\big) = 0$. Hence, $u_1 = u_2$ a.s. Note that

$$\text{supp}\eta_1,\ \text{supp}\eta_2 \subseteq \{(x,t); u_1(x,t) = h^1(x,t)\}$$

and

$$\text{supp}\xi_1,\ \text{supp}\xi_2 \subseteq \{(x,t); u_1(x,t) = h^2(x,t)\}.$$

Using

$$\begin{aligned} & \big(u_1(t) - u_2(t), \phi\big) - \int_0^t \big(u_1(s) - u_2(s), \frac{\partial \phi}{\partial s}\big) ds - \int_0^t \big(u_1(s) - u_2(s), \frac{\partial^2 \phi}{\partial x^2}\big) ds \\ = & \int_0^t \big(f(y,s,u_1) - f(y,s,u_2), \phi\big) ds + \int_0^t \int_0^1 \phi\big(\sigma(y,s,u_1) - \sigma(y,s,u_2)\big) W(dy,ds) \\ & + \int_0^t \int_0^1 \phi\big(\eta_1(dy,ds) - \eta_2(dy,ds)\big) - \int_0^t \int_0^1 \phi\big(\xi_1(dy,ds) - \xi_2(dy,ds)\big), \end{aligned}$$

by choosing appropriate test functions as at the end of the proof of Theorem 2.2.1, we obtain that $\eta_1 = \eta_2$ and $\xi_1 = \xi_2$. $\square$

Chapter 3

Existence and Uniqueness of Invariant Measures for SPDEs with Two Reflecting Walls

3.1 Introduction

Consider the following stochastic partial differential equations (SPDEs) with two reflecting walls

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) + \sigma(u)\dot{W} + \eta - \xi; \\ u(x,0) = u_0(x) \in C(S^1); \\ u(0,t) = u(2\pi,t),\ u_x(0,t) = u_x(2\pi,t) \ \text{ for } t \geq 0; \\ h^1(x) \leq u(x,t) \leq h^2(x), \ \text{ for } (x,t) \in Q^1, \end{cases} \tag{3.1}$$

where $Q^1 = S^1 \times \mathbb{R}_+$, $S^1 = \mathbb{R}(mod 2\pi)$. Let $\{e^{i\theta};\ \theta \in \mathbb{R}\}$ denotes a circular ring. We can make a natural identification of S^1 with the interval $[0, 2\pi]$: the random field $W(x,t) := W(\{e^{i\theta};\ 0 \leq \theta \leq x\} \times [0,t])$ is a regular Brownian sheet defined on a filtered probability space $(\Omega, P, \mathcal{F}; \{\mathcal{F}_t; t \geq 0\})$. Here the hard wall is not depending on time and the space variable takes values on the unit circle S^1. ξ and η effect the same as that in equation (2.1).

We assume that the reflecting walls $h^1(x),\ h^2(x)$ are continuous functions satisfying

(H5) $h^1(x) < h^2(x)$ for $x \in S^1$;

(H6) $\frac{\partial^2 h^i}{\partial x^2} \in L^2(S^1)$, where $\frac{\partial^2}{\partial x^2}$ is interpreted in a distributional sense.

We also assume that the coefficients: $f, \sigma : \mathbb{R} \to \mathbb{R}$ satisfy

(F4) there exists $L > 0$ such that

$$|f(z_1) - f(z_2)| + |\sigma(z_1) - \sigma(z_2)| \le L|z_1 - z_2|,\ z_1,\ z_2 \in \mathbb{R};$$

(F5) there exists $C > 0$ such that

$$|f(z)| + |\sigma(z)| \le C(1 + |z|),\ z \in \mathbb{R}.$$

The following is the definition of a solution of a SPDE with two reflecting walls $h^1,\ h^2$ which is similar as Definition 2.1.1.

Definition 3.1.1. *A triplet (u, η, ξ) is a solution to the SPDE (3.1) if*

(i) $u = \{u(x,t); (x,t) \in Q^1\}$ is a continuous, adapted random field satisfying $h^1(x) \le u(x,t) \le h^2(x)$, a.s;

(ii) $\eta(dx, dt)$ and $\xi(dx, dt)$ are positive and adapted random measures on Q^1 satisfying

$$\eta\big(S^1 \times [0,T]\big) < \infty,\ \xi\big(S^1 \times [0,T]\big) < \infty$$

for $T > 0$;

(iii) for all $t \ge 0$ and $\phi \in C^\infty(S^1)$ we have

$$\begin{aligned} & \big(u(t), \phi\big) - \int_0^t (u(s), \phi'')ds - \int_0^t \big(f(u(s)), \phi\big)ds - \int_0^t \int_{S^1} \phi(x)\sigma(u(x,s))W(dx,ds) \\ = \ & \big(u_0, \phi(x)\big) + \int_0^t \int_{S^1} \phi(x)\eta(dx,ds) - \int_0^t \int_{S^1} \phi(x)\xi(dx,ds),\ a.s, \end{aligned} \tag{3.2}$$

where $(,)$ denotes the inner product in $L^2(S^1)$;

(iv)

$$\int_{Q^1} \big(u(x,t) - h^1(x)\big)\eta(dx,dt) = \int_{Q^1} \big(h^2(x) - u(x,t)\big)\xi(dx,dt) = 0.$$

The existence and uniqueness of the solution of equation (3.1) is established similar as that done in Chapter 2. The aim of this paper is to establish the existence and uniqueness of invariant measures, as well as the strong Feller property of fully non-linear SPDEs with two reflecting walls (3.1).

For the existence of invariant measures, our approach is to use Krylov-Bogolyubov theorem. To this end, the continuity of the solution with respect to the solutions of some random obstacle problems plays an important role. For the uniqueness, we adapted a coupling method used by Mueller [26]. Because of the reflection, we need to establish a kind of uniform coupling for approximating solutions. The strong Feller property of SPDEs with two reflecting walls will be obtained in a similar way as that that in Zhang [42].

The rest of this chapter is organized as follows. In Section 3.2, we give the proof of the existence of invariant measures and the uniqueness shall be obtained in Section 3.3. Section 3.4 establishes the strong Feller property.

3.2 Existence of Invariant Measures

Denote by $\mathcal{B}(C(S^1))$ the σ-field of all Borel subsets of $C(S^1)$ and by $\mathcal{M}(C(S^1))$ the set of all probability measures defined on $(C(S^1),\ \mathcal{B}(C(S^1)))$. We denote by $u(x,t,u_0)$ the solution of equation (3.1) and by $P_t(u_0,\cdot)$ the corresponding transition function

$$P_t(u_0,\Gamma) = P(u(\cdot,t,u_0)\in\Gamma),\ \Gamma\in\mathcal{B}(C(S^1)),\ t>0,$$

where u_0 is the initial condition. The initial condition $u_0(x)$ satisfies

(F6) $u_0(x)\in C(S^1)$ satisfy $h^1(x)\le u_0(x)\le h^2(x)$, for $x\in S^1$.

Theorem 3.2.1. *Suppose the hypotheses (H5)-(H6), (F4)-(F6) hold. Then there exists an invariant measure to equation (3.1) on $C(S^1)$.*

Proof. According to Krylov-Bogolyubov theorem (see Da Prato and Zabczyk [17]), if the family $P_t(u_0,\cdot),\ t\geq 1$ is uniformly tight, then there exists an invariant measure for equation (1.1). So we need to show that for any $\varepsilon>0$ there is a compact set $K\subset C(S^1)$ such that

$$P(u(t)\in K)\geq 1-\varepsilon,\quad \text{for any } t\geq 1. \tag{3.3}$$

where $u(t)=u(t,u_0)=u(\cdot,t,u_0)$. On the other hand, for any $t\geq 1$, we have by the Markov property

$$P(u(t)\in K) \;=\; \mathbb{E}\big(P_1(u(t-1),K)\big). \tag{3.4}$$

Thus it is enough to show $P\big(u(1,u(t-1))\in K\big)\geq 1-\varepsilon$, for any $t\geq 1$. As $h^1(\cdot)\leq u(t-1)(\cdot)\leq h^2(\cdot)$, it suffices to find a compact subset $K\subset C(S^1)$ such that

$$P_1(g,K)\geq 1-\varepsilon,\quad \textit{for all } g\in C(S^1) \textit{ with } h^1\leq g\leq h^2. \tag{3.5}$$

Put

$$\begin{aligned} v(x,t,g) \;=\; & \int_0^t\int_{S^1} G_{t-s}(x,y)f(u(y,s,g))dyds \\ & +\int_0^t\int_{S^1} G_{t-s}(x,y)\sigma(u(y,s,g))W(dy,ds), \end{aligned} \tag{3.6}$$

where $G_t(x,y)$ is the Green's function of the heat equation on S^1. Then u can be written in the form

$$\begin{aligned} u(x,t,g)-\int_{S^1} G_t(x,y)g(y)dy \;=\; & v(x,t,g)+\int_0^t\int_{S^1} G_{t-s}(x,y)\eta(g)(dx,dt) \\ & -\int_0^t\int_{S^1} G_{t-s}(x,y)\xi(g)(dx,dt), \end{aligned}$$

where $\eta(g),\ \xi(g)$ indicates the dependence of the random measures on the initial condition g. Put

$$\bar{u}(x,t,g) = u(x,t,g) - \int_{S^1} G_t(x,y)g(y)dy$$

Then $(\bar{u},\eta,\xi)$ solves a random obstacle problem. From the relationship between $\bar{u}$ and v proved in inequality (2.16), we have the following inequality

$$\|\bar{u}(g) - \bar{u}(\hat{g})\|_\infty^1 \leq 2\|v(g) - v(\hat{g})\|_\infty^1,$$

where $\|u\|_\infty^1 := \sup_{x\in S^1, t\in[0,1]} |u(x,t)|$. So $\bar{u}$ is a continuous functional of v and denoted by $u = \Phi(v)$, where $\Phi(\cdot):\ C(S^1\times[0,1]) \to C(S^1\times[0,1])$ is continuous. In particular, $\bar{u}(\cdot,1,g)$ is also a continuous functional of v, from $C(S^1\times[0,1])$ to $C(S^1)$. We denote this functional by Φ_1, i.e. $\bar{u}(\cdot,1,g) = \Phi_1(v(\cdot,g))$, where $v(\cdot,g) = v(\cdot,\cdot,g)$. If K'' is a compact subset of $C(S^1\times[0,1]))$, then $K' = \Phi_1(K'')$ is a compact subset in $C(S^1)$ and

$$\begin{aligned} P(\bar{u}(\cdot,1,g)\in K') &= P(\bar{u}(\cdot,1,g)\in\Phi_1(K'')) \\ &\geq P(v(\cdot,g)\in K''). \end{aligned} \tag{3.7}$$

Next, we want to find a compact set $K''(\subset C(S^1\times[0,1])$ such that

$$P(v(\cdot,g)\in K'') \geq 1-\varepsilon, \quad for\ all\ g\in C(S^1)\ with\ h^1\leq g\leq h^2. \tag{3.8}$$

For $0<\alpha<\frac{1}{4}$ and $\kappa>0$, from Proposition A.1 in Sowers [35] and using a similar proof to that of Corollary 3.4 in Walsh [39], there exists a random variable $Y(g)$ such that with probability one, for all $x,y\in S^1$ and $s,t\in(0,1]$,

$$|v(x,t,g) - v(y,s,g)| \leq Y(g)(d((x,t),(y,s)))^{\alpha-\kappa}\ and\ \mathbb{E}(Y(g))^{\frac{1}{\kappa}} \leq C_0, \tag{3.9}$$

where $d((x,t),(y,s)) := \left(r^2(x,y)+(t-s)^2\right)^{\frac{1}{2}}$ with $r(x,y)$ is the distance between x and y on S^1.

Define

$$\|v\|_\alpha = \sup \quad \{ \quad \frac{|v(x,t)-v(y,s)|}{d^\alpha((x,t),(y,s))}; \\ (x,t),(y,s) \in S^1 \times [0,1], (x,t) \neq (y,s)\}, \ \ for\ \alpha < \frac{1}{4}.$$

By the Arzela-Ascoli theorem, for all $r > 0$, $K_r := \{v; \|v\|_\alpha \leq r\}$ is a compact subset of $C(S^1 \times [0,1])$. In view of (3.9), we see that for given $\varepsilon > 0$, there exists r_0 such that

$$P(v(\cdot, g) \in K^c_{r_0}) \leq \varepsilon, \quad for\ all\ g\ with\ h^1 \leq g \leq h^2.$$

Choosing $K'' = K_{r_0}$, we obtain (3.8). Hence $P(\bar{u}(\cdot,1,g) \in K') \geq 1-\varepsilon$ for all $g \in C(S^1)$ with $h^1 \leq g \leq h^2$. On the other hand, it is easy to see that there is a compact subset $K_0 \subset C(S^1)$ such that

$$\{\int_{S^1} G_1(x,y)g(y)dy; \quad h^1 \leq g \leq h^2\} \subset K_0$$

Define $K = K' + K_0$. We have

$$P_1(g,K) = P(u(\cdot,1,g) \in K) \geq P(\bar{u}(\cdot,1,g) \in K') \geq 1-\varepsilon,$$

for all $g \in C(S^1)$ with $h^1 \leq g \leq h^2$. This finishes the proof. □

3.3 Uniqueness of Invariant Measures

For the uniqueness of invariant measures, we need the following proposition. For simplicity, we put $u(x,t) = u(x,t,u_0)$.

Proposition 3.3.1. *Under the assumption in Theorem 3.2.1, for any* $p \geq 1$, $T > 0$, $\sup_{\varepsilon,\delta} \mathbb{E}(\|u^{\varepsilon,\delta}\|^T_\infty)^p < \infty$ *and* $u^{\varepsilon,\delta}$ *converges uniformly on* $S^1 \times [0,T]$ *to* u *as* $\varepsilon, \delta \to 0$ *a.s,* *where u, $u^{\varepsilon,\delta}$ are the solutions of equation (3.1) and the penalized SPDEs*

$$\begin{cases} \frac{\partial u^{\varepsilon,\delta}(x,t)}{\partial t} = \frac{\partial^2 u^{\varepsilon,\delta}(x,t)}{\partial x^2} + f(u^{\varepsilon,\delta}(x,t)) + \sigma(u^{\varepsilon,\delta}(x,t))\dot{W}(x,t) \\ \qquad\qquad + \frac{1}{\delta}(u^{\varepsilon,\delta}(x,t) - h^1(x))^- - \frac{1}{\varepsilon}(u^{\varepsilon,\delta}(x,t) - h^2(x))^+; \\ u^{\varepsilon,\delta}(x,0) = u_0(x). \end{cases}$$

Proof. Let $v^{\varepsilon,\delta}$ be the solution of equation

$$\begin{cases} \frac{\partial v^{\varepsilon,\delta}(x,t)}{\partial t} = \frac{\partial^2 v^{\varepsilon,\delta}(x,t)}{\partial x^2} + f(u^{\varepsilon,\delta}(x,t)) + \sigma(u^{\varepsilon,\delta}(x,t))\dot{W}(x,t); \\ v^{\varepsilon,\delta}(x,0) = u_0(x). \end{cases} \tag{3.10}$$

Set $\bar{\Phi}^{\varepsilon,\delta}(t) = \sup_{s\le t, y\in S^1}(v^{\varepsilon,\delta}(y,s) - h^2(y))^+$. Note that $\bar{\Phi}^{\varepsilon,\delta}(t)$ is increasing w.r.t. t and $v^{\varepsilon,\delta} - \bar{\Phi}^{\varepsilon,\delta} \le h^2$. $\bar{z}^{\varepsilon,\delta}(x,t) := v^{\varepsilon,\delta}(x,t) - \bar{\Phi}^{\varepsilon,\delta}(t) - u^{\varepsilon,\delta}(x,t)$ is a solution of equation

$$\begin{cases} \frac{\partial \bar{z}^{\varepsilon,\delta}}{\partial t} + \frac{\partial \bar{\Phi}^{\varepsilon,\delta}}{\partial t} = \frac{\partial^2 \bar{z}^{\varepsilon,\delta}}{\partial x^2} - \frac{1}{\delta}(u^{\varepsilon,\delta} - h^1)^- + \frac{1}{\varepsilon}(u^{\varepsilon,\delta} - h^2)^+; \\ \bar{z}^{\varepsilon,\delta}(x,0) = 0. \end{cases} \tag{3.11}$$

Multiplying (3.11) by $(\bar{z}^{\varepsilon,\delta})^+$ and using $((u^{\varepsilon,\delta} - h^2)^+,\ (\bar{z}^{\varepsilon,\delta})^+) = 0$ we get $(\bar{z}^{\varepsilon,\delta})^+ = 0$. Hence,

$$u^{\varepsilon,\delta} \ge v^{\varepsilon,\delta} - \bar{\Phi}^{\varepsilon,\delta}.$$

Similarly, setting $\bar{z}^{\varepsilon,\delta}(x,t) = u^{\varepsilon,\delta}(x,t) - v^{\varepsilon,\delta}(x,t) - \sup_{s\le t, y\in S^1}(v^{\varepsilon,\delta}(y,s) - h^1(s))^-$, we can show that

$$u^{\varepsilon,\delta} \le v^{\varepsilon,\delta} + \sup_{s\le t, y\in S^1}(v^{\varepsilon,\delta} - h^1)^-.$$

As $\sup_{\varepsilon,\delta}\mathbb{E}(\|v^{\varepsilon,\delta}\|_\infty^T)^p < \infty$, the above two inequalities implies

$$\sup_{\varepsilon,\delta}\mathbb{E}(\|u^{\varepsilon,\delta}\|_\infty^T)^p < \infty.$$

Since $u^{\varepsilon,\delta}$ is increasing in δ by the comparison theorem of SPDEs (see Donati-Martin and Pardoux [14]), we can show $u^\varepsilon := \lim_{\delta\downarrow 0} u^{\varepsilon,\delta}$ exists a.s. and u^ε solves

$$\begin{cases} \frac{\partial u^\varepsilon(x,t)}{\partial t} = \frac{\partial^2 u^\varepsilon(x,t)}{\partial x^2} + f(u^\varepsilon(x,t)) + \sigma(u^\varepsilon(x,t))\dot{W}(x,t) \\ \qquad\qquad +\eta^\varepsilon(x,t) - \frac{1}{\varepsilon}(u^\varepsilon(x,t) - h^2(x))^+; \\ u^\varepsilon(x,0) \geq h^1(x); \\ u^\varepsilon(x,0) = u_0(x), \end{cases} \tag{3.12}$$

where $\eta^\varepsilon(dx,dt) := \lim_{\delta\downarrow 0} \frac{(u^{\varepsilon,\delta}(x,t)-h^1(x))^+}{\delta} dxdt$. Also, by comparison, we know that u^ε is decreasing as $\varepsilon \downarrow 0$. Let v^ε be the solution of equation (3.10) replacing $u^{\varepsilon,\delta}$ by u^ε. Setting $\bar{z}^\varepsilon(x,t) = u^\varepsilon(x,t) - v^\varepsilon(x,t) - \sup_{s\leq t, y\in S^1}(v^\varepsilon(y,s) - h^1(y))^-$, we can show

$$u^\varepsilon \leq v^\varepsilon + \sup_{s\leq t, y\in S^1}(v^\varepsilon - h^1)^-.$$

In addition, by the definition of u^ε, $u^\varepsilon \geq h^1$. Hence, $u := \lim_{\varepsilon\downarrow 0} u^\varepsilon = \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} u^{\varepsilon,\delta}$ exists a.s.

The continuity of u can be proved similarly as in Theorem 4.1 in Donati-Martin and Pardoux [14]. The uniform convergence of $u^{\varepsilon,\delta}$ w.r.t. (x,t) follows from Dini's theorem. □

The following result is the uniqueness of invariant measures.

Theorem 3.3.1. *Under the assumptions in Theorem 3.2.1 and that $|\sigma| \geq L_0$ for some constant $L_0 > 0$, there is a unique invariant measure for the equation (3.1).*

Proof. We will adopt the coupling method used in Mueller [26] to SPDEs with reflection. Let $u^1(x,0)$ and $u^2(x,0)$ be two initial values having distributions given by two invariant probabilities μ_1 and μ_2. Then $u^1(x,t)$ and $u^2(x,t)$ also have these distributions for any $t > 0$. Thus

$$Var(\mu_1 - \mu_2) \leq P\big(\sup_{x\in S^1} |u^1(x,t) - u^2(x,t)| \neq 0\big).$$

Thus, for given two initial functions $u^1(x,0)$ and $u^2(x,0)$, it is sufficient to construct two coupled processes $u^1(x,t)$, $u^2(x,t)$ satisfying equation (3.1), driven by different white noises on a probability space $(\Omega, \mathcal{F}, P)$, such that

$$\lim_{t\to\infty} P\Big(\sup_{x\in S^1} |u^1(x,t) - u^2(x,t)| \neq 0\Big) = 0. \tag{3.13}$$

We first assume $u^1(x,0) \geq u^2(x,0)$, $x \in S^1$. We want to construct two independent space-time white noises $W_1(x,t)$, $W_2(x,t)$ defined on a probability space $(\Omega, \mathcal{F}, P)$, and a solution u, v of the following SPDEs with two reflecting walls

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}_1(x,t) \\
&\quad + \eta_1(x,t) - \xi_1(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + f\big(v(x,t)\big) + \eta_2(x,t) - \xi_2(x,t) \\
&\quad + \sigma\big(v(x,t)\big)\big[(1 - |u-v| \wedge 1)^{\frac{1}{2}}\dot{W}_1(x,t) + (|u-v|\wedge 1)^{\frac{1}{2}}\dot{W}_2(x,t)\big], \\
u(x,0) &= u^1(x,0), \quad v(x,0) = u^2(x,0).
\end{aligned} \tag{3.14}$$

Note that the coefficients in the second equation in (3.14) is not Lipschitz. The existence of a solution of equation (3.14) is not automatic. In the following, using a similar method as that in the paper Mueller [26], we will give a construction of a solution on some probability space. The construction will also be used to prove the successful coupling

$$\lim_{t\to\infty} P\Big(\sup_{x\in S^1} |u(x,t) - v(x,t)| \neq 0\Big) = 0.$$

Lemma 3.3.1. *Under the assumptions in Theorem 3.2.1, there exists a solution of equation (3.14).*

Proof. For $0 \leq z \leq 1$, set

$$\begin{aligned}
f_n(z) &= \big(z + \frac{1}{n}\big)^{\frac{1}{2}} - \big(\frac{1}{n}\big)^{\frac{1}{2}}, \\
g_n(x) &= \big(1 - f_n(z)^2\big)^{\frac{1}{2}}.
\end{aligned}$$

We have $f_n(z)^2 + g_n(z)^2 = 1$ and that $f_n(z) \to z^{\frac{1}{2}}$, $g_n(z) \to (1-z)^{\frac{1}{2}}$ uniformly as $n \to \infty$, for $z \in S^1$.

Let $\dot{\overline{W}}_1(x,t)$, $\dot{\overline{W}}_2(x,t)$ be two independent space-time white noises defined on a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. Let $\overline{u}, \overline{v}^n$ be the unique solution of the following SPDEs with two reflecting walls

$$\begin{aligned}
\frac{\partial \overline{u}(x,t)}{\partial t} &= \frac{\partial^2 \overline{u}(x,t)}{\partial x^2} + f\big(\overline{u}(x,t)\big) + \sigma\big(\overline{u}(x,t)\big)\dot{\overline{W}}_1(x,t) \\
&\quad + \overline{\eta}_1(x,t) - \overline{\xi}_1(x,t), \\
\frac{\partial \overline{v}^n(x,t)}{\partial t} &= \frac{\partial^2 \overline{v}^n(x,t)}{\partial x^2} + f\big(\overline{v}^n(x,t)\big) + \overline{\eta}_2^n(x,t) - \overline{\xi}_2^n(x,t) \\
&\quad + \sigma\big(\overline{v}^n(x,t)\big)\big[g_n(|\bar{u} - \overline{v}^n| \wedge 1)\dot{\overline{W}}_1(x,t) + f_n(|\overline{u} - \overline{v}^n| \wedge 1)\dot{\overline{W}}_2(x,t)\big], \\
\overline{u}(x,0) &= u^1(x,0), \quad \overline{v}(x,0) = u^2(x,0). \qquad (3.15)
\end{aligned}$$

The existence and uniqueness of $(\overline{u}, \overline{v}^n)$ is guaranteed because of the Lipschitz continuity of the coefficients. Put

$$\begin{aligned}
\hat{u}(x,t) &= \int_{S^1} G_t(x,y)u^1(y,0)dy + \int_0^t \int_{S^1} G_{t-s}(x,y)f(\overline{u}(y,s))dyds \\
&\quad + \int_0^t \int_{S^1} G_{t-s}(x,y)\sigma(\overline{u}(y,s))\overline{W}_1(dy,ds) \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
\hat{v}^n(x,t) &= \int_{S^1} G_t(x,y)u^2(y,0)dy + \int_0^t \int_{S^1} G_{t-s}(x,y)f(\overline{v}^n(y,s))dyds \\
&\quad + \int_0^t \int_{S^1} G_{t-s}(x,y)\sigma(\overline{v}^n(y,s))W^n(dy,ds), \qquad (3.17)
\end{aligned}$$

where

$$\dot{W}^n(x,t) = \big[g_n(|\overline{u} - \overline{v}^n| \wedge 1)\dot{\overline{W}}_1(x,t) + f_n(|\overline{u} - \overline{v}^n| \wedge 1)\dot{\overline{W}}_2(x,t)\big]$$

is another space-time white noise on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. From the proof of Theorem 3.2.1, it is known that there exists a continuous functional Φ such that $\bar{u} = \Phi(\hat{u})$ and

$\bar{v}^n = \Phi(\hat{v}^n)$. On the other hand, following the same proof of Lemma 3.1 in Mueller [26] and the inequality (3.3), we get that the sequence $\hat{u}, \hat{v}^n, n \geq 1$ is tight. As the images under the continuous map Φ, the vector $(\overline{u}, \overline{v}^n, \overline{W}_1, \overline{W}_2)$ is also tight. By Skorohod's representation theorem, there exist random fields (u, v^n, W_1, W_2), $n \geq 1$ on some probability space $(\Omega, \mathcal{F}, P)$ such that (u, v^n, W_1, W_2) has the same law as $(\overline{u}, \overline{v}^n, \overline{W}_1, \overline{W}_2)$ and that the following SPDEs with two reflecting walls hold

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}_1(x,t) \\
&\quad + \eta_1(x,t) - \xi_1(x,t), \\
\frac{\partial v^n(x,t)}{\partial t} &= \frac{\partial^2 v^n(x,t)}{\partial x^2} + f\big(v^n(x,t)\big) + \eta_2^n(x,t) - \xi_2^n(x,t) \\
&\quad + \sigma\big(v^n(x,t)\big)\big[g_n(|u-v^n| \wedge 1)\dot{W}_1(x,t) + f_n(|u-v^n| \wedge 1)\dot{W}_2(x,t)\big], \\
u(x,0) &= u^1(x,0), \quad v^n(x,0) = u^2(x,0). \qquad (3.18)
\end{aligned}$$

Furthermore, v^n convergence to v in $C(S^1 \times [0,T])$. Then $v^n \to v$ uniformly almost surely as $n \to \infty$. By a similar proof as that of Theorem 2.3.1, we can prove that the limit (u,v) satisfies (i)-(iv) of the definition. Then (u,v) is the solution of the following SPDEs with two reflecting walls

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u(x,t)\big) + \sigma\big(u(x,t)\big)\dot{W}_1(x,t) \\
&\quad + \eta_1(x,t) - \xi_1(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + f\big(v(x,t)\big) + \eta_2(x,t) - \xi_2(x,t) \\
&\quad + \sigma\big(v(x,t)\big)\big[(1-|u-v| \wedge 1)^{\frac{1}{2}}\dot{W}_1(x,t) + (|u-v| \wedge 1)^{\frac{1}{2}}\dot{W}_2(x,t)\big], \\
u(x,0) &= u^1(x,0), \quad v(x,0) = u^2(x,0). \qquad (3.19)
\end{aligned}$$

□

The next step is to show that u, v admits a successful coupling. To this end, consider the following approximating SPDEs

$$\begin{cases} \frac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{\partial^2 u^{\varepsilon,\delta}}{\partial x^2} + f(u^{\varepsilon,\delta}) + \frac{1}{\delta}(u^{\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(u^{\varepsilon,\delta} - h^2)^+ + \sigma(u^{\varepsilon,\delta})\dot{W}_1; \\ \frac{\partial v^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 v^{n,\varepsilon,\delta}}{\partial x^2} + f(v^{n,\varepsilon,\delta}) + \frac{1}{\delta}(v^{n,\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(v^{n,\varepsilon,\delta} - h^2)^+ \\ \qquad +\sigma(v^{n,\varepsilon,\delta})\big[g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_1(x,t) \\ \qquad +f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_2(x,t)\big]; \\ u^{\varepsilon,\delta}(x,0) = u^1(x,0),\ v^{n,\varepsilon,\delta}(x,0) = u^2(x,0). \end{cases} \tag{3.20}$$

We may and will assume that $f(u)$ is non-increasing. Otherwise, we consider $\tilde{u} := e^{-Lt}u$, $\tilde{v} := e^{-Lt}v$, where L is the Lipschitz constant in (F4), which satisfy

$$\begin{aligned} \frac{\partial \tilde{u}(x,t)}{\partial t} &= \frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + e^{-Lt} f\big(e^{Lt}\tilde{u}(x,t)\big) - L\tilde{u}(x,t) \\ &\quad +e^{-Lt}\sigma\big(e^{Lt}\tilde{u}(x,t)\big)\dot{W}_1(x,t) + \eta_3(x,t) - \xi_3(x,t), \\ \frac{\partial \tilde{v}^n(x,t)}{\partial t} &= \frac{\partial^2 \tilde{v}^n(x,t)}{\partial x^2} + e^{-Lt} f\big(e^{Lt}\tilde{v}(x,t)\big) - L\tilde{v}(x,t) + \eta_4^n(x,t) - \xi_4^n(x,t) \\ &\quad +e^{-Lt}\sigma\big(e^{Lt}\tilde{v}^n(x,t)\big)\big[g_n(|e^{Lt}\tilde{u} - e^{Lt}\tilde{v}^n| \wedge 1)\dot{W}_1(x,t) \\ &\quad +f_n(|e^{Lt}\tilde{u} - e^{Lt}\tilde{v}^n| \wedge 1)\dot{W}_2(x,t)\big], \\ u(x,0) &= u^1(x,0), \quad v^n(x,0) = u^2(x,0). \end{aligned}$$

The new drift $e^{-Lt}f(e^{Lt}x) - Lx$ is non-increasing. Also, if $\tilde{u},\ \tilde{v}$ satisfy a successful coupling, so does $u,\ v$. Note that all the coefficients in (3.20) are Lipschitz continuous.

Lemma 3.3.2. *$v^{n,\varepsilon,\delta}(x,t) \to v^n(x,t)$ in probability uniformly on $S^1 \times [0,T]$, for any $T > 0$, as $\varepsilon, \delta \to 0$.*

Proof. Consider the following SPDE:

$$\begin{cases} \frac{\partial \tilde{v}^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 \tilde{v}^{n,\varepsilon,\delta}}{\partial x^2} + f(\tilde{v}^{n,\varepsilon,\delta}) + \frac{1}{\delta}(\tilde{v}^{n,\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(\tilde{v}^{n,\varepsilon,\delta} - h^2)^+ \\ \qquad +\sigma(\tilde{v}^{n,\varepsilon,\delta})\big[g_n(|u - \tilde{v}^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_1(x,t) \\ \qquad +f_n(|u - \tilde{v}^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_2(x,t)\big]; \\ \tilde{v}^{n,\varepsilon,\delta}(x,0) = u^2(x,0). \end{cases}$$

Denote the solution by $\tilde{v}^{n,\varepsilon,\delta}(x,t)$. Define $w^{\varepsilon,\delta}(x,t)$ and $z^{\varepsilon,\delta}(x,t)$ satisfy, respectively,

$$\begin{cases} \frac{\partial w^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 w^{n,\varepsilon,\delta}}{\partial x^2} + f(v^{n,\varepsilon,\delta}) + \sigma(v^{n,\varepsilon,\delta})\big[g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_1(x,t) \\ \qquad\qquad +f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_2(x,t)\big]; \\ w^{n,\varepsilon,\delta}(x,0) = u^2(x,0) \end{cases}$$

and

$$\begin{cases} \frac{\partial z^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 z^{n,\varepsilon,\delta}}{\partial x^2} + \frac{1}{\delta}(z^{n,\varepsilon,\delta} + w^{n,\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(z^{n,\varepsilon,\delta} + w^{n,\varepsilon,\delta} - h^2)^+; \\ z^{n,\varepsilon,\delta}(x,0) = 0. \end{cases}$$

Then $v^{n,\varepsilon,\delta} = z^{n,\varepsilon,\delta} + w^{n,\varepsilon,\delta}$. Similarly, define $\tilde{w}^{\varepsilon,\delta}(x,t)$ and $\tilde{z}^{\varepsilon,\delta}(x,t)$ satisfy, respectively,

$$\begin{cases} \frac{\partial \tilde{w}^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 \tilde{w}^{n,\varepsilon,\delta}}{\partial x^2} + f(v^{n,\varepsilon,\delta}) + \sigma(v^{n,\varepsilon,\delta})\big[g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_1(x,t) \\ \qquad\qquad +f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1)\dot{W}_2(x,t)\big]; \\ \tilde{w}^{n,\varepsilon,\delta}(x,0) = u^2(x,0) \end{cases}$$

and

$$\begin{cases} \frac{\partial \tilde{z}^{n,\varepsilon,\delta}}{\partial t} = \frac{\partial^2 \tilde{z}^{n,\varepsilon,\delta}}{\partial x^2} + \frac{1}{\delta}(\tilde{z}^{n,\varepsilon,\delta} + \tilde{w}^{n,\varepsilon,\delta} - h^1)^- - \frac{1}{\varepsilon}(\tilde{z}^{n,\varepsilon,\delta} + \tilde{w}^{n,\varepsilon,\delta} - h^2)^+; \\ \tilde{z}^{n,\varepsilon,\delta}(x,0) = 0. \end{cases}$$

Therefore, $\tilde{v}^{n,\varepsilon,\delta} = \tilde{z}^{n,\varepsilon,\delta} + \tilde{w}^{n,\varepsilon,\delta}$. From the inequality (2.8),

$$\|z^{n,\varepsilon,\delta} - \tilde{z}^{n,\varepsilon,\delta}\|_\infty^T \leq \|w^{n,\varepsilon,\delta} - \tilde{w}^{n,\varepsilon,\delta}\|_\infty^T,$$

for any $T > 0$. Thus,

$$\|v^{n,\varepsilon,\delta} - \tilde{v}^{n,\varepsilon,\delta}\|_\infty^T \leq 2\|w^{n,\varepsilon,\delta} - \tilde{w}^{n,\varepsilon,\delta}\|_\infty^T,$$

for any $T > 0$. Now we will show that $\|v^{n,\varepsilon,\delta}(x,t) - \tilde{v}^{n,\varepsilon,\delta}\|_\infty^T \to 0$ in probability uniformly on $S^1 \times [0,T]$ as $\varepsilon, \delta \to 0$.

As the similar proof in Donati-Martin and Pardoux [14], it can be shown that $\sup_{\varepsilon,\delta} \mathbb{E}(\|v^{n,\varepsilon,\delta}\|_\infty^T)^p < \infty$, for arbitrarily large p and any $T > 0$. Set the stopping time $\tau_M := \inf\{t;\ \sup_{x\in S^1}|v^{n,\varepsilon,\delta}| > M\}$. For fixed $M > 0$, we have

$$
\begin{aligned}
& \mathbb{E}\big(\|w^{n,\varepsilon,\delta}(t\wedge\tau_M) - \tilde{w}^{n,\varepsilon,\delta}(t\wedge\tau_M)\|_\infty^T\big)^2 \\
\leq\ & C_1\int_0^T \mathbb{E}\big(\|f(v^{n,\varepsilon,\delta}) - f(\tilde{v}^{n,\varepsilon,\delta})\|_\infty^{t\wedge\tau_M}\big)^2 ds \\
& +C_2\int_0^T \mathbb{E}\big(\|\sigma(v^{n,\varepsilon,\delta})g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}|\wedge 1) - \sigma(\tilde{v}^{n,\varepsilon,\delta})g_n(|u - \tilde{v}^{n,\varepsilon,\delta}|\wedge 1))\|_\infty^{t\wedge\tau_M}\big)^2 ds \\
& +C_2\int_0^T \mathbb{E}\big(\|\sigma(v^{n,\varepsilon,\delta})f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}|\wedge 1) - \sigma(\tilde{v}^{n,\varepsilon,\delta})f_n(|u - \tilde{v}^{n,\varepsilon,\delta}|\wedge 1))\|_\infty^{t\wedge\tau_M}\big)^2 ds \\
\leq\ & C_1\int_0^T \mathbb{E}\big(\|w^{n,\varepsilon,\delta} - \tilde{w}^{n,\varepsilon,\delta}\|_\infty^{t\wedge\tau_M}\big)^2 ds \\
& +C_2\int_0^T \mathbb{E}\big(\|\sigma(v^{n,\varepsilon,\delta})\|_\infty^{t\wedge\tau_M}\cdot[\|u^{\varepsilon,\delta} - u\|_\infty^{t\wedge\tau_M} + \|w^{n,\varepsilon,\delta} - \tilde{w}^{n,\varepsilon,\delta}\|_\infty^{t\wedge\tau_M}]\big)^2 ds \\
\leq\ & C_1\epsilon + C_2\int_0^T \mathbb{E}\big(\|w^{n,\varepsilon,\delta} - \tilde{w}^{n,\varepsilon,\delta}\|_\infty^{t\wedge\tau_M}\big)^2 ds
\end{aligned}
$$

where the constants changes in each line and $\epsilon \to 0$ as $\varepsilon, \delta \to 0$ because of Proposition 3.3.1. Apply Gronwall inequality, we get $\mathbb{E}\big(\|w^{n,\varepsilon,\delta}(t\wedge\tau_M) - \tilde{w}^{n,\varepsilon,\delta}(t\wedge\tau_M)\|_\infty^T\big)^2 \to 0$ as $\varepsilon, \delta \to 0$. Hence, $\mathbb{E}\big(\|v^{n,\varepsilon,\delta}(t\wedge\tau_M) - \tilde{v}^{n,\varepsilon,\delta}(t\wedge\tau_M)\|_\infty^T\big)^2 \to 0$. So $\|v^{n,\varepsilon,\delta}(x,t) - \tilde{v}^{n,\varepsilon,\delta}\|_\infty^T \to 0$ in probability. In fact, for every $\epsilon_0 > 0$, set the event $A^{n,\varepsilon,\delta} := \{\|v^{n,\varepsilon,\delta} - \tilde{v}^{n,\varepsilon,\delta}\|_\infty^T > \epsilon_0\}$,

$$
\begin{aligned}
P(A^{n,\varepsilon,\delta}) &= P(A^{n,\varepsilon,\delta},\ \tau_M \leq T) + P(A^{n,\varepsilon,\delta},\ \tau_M > T) \\
&\leq P(\tau_M \leq T) + \frac{\mathbb{E}\big(\|w^{n,\varepsilon,\delta}(t\wedge\tau_M) - \tilde{w}^{n,\varepsilon,\delta}(t\wedge\tau_M)\|_\infty^T\big)^2}{{\epsilon_0}^2} \\
&\leq \frac{\mathbb{E}(\|v^{n,\varepsilon,\delta}\|_\infty^T)^2}{{\epsilon_0}^2} + \frac{\mathbb{E}\big(\|w^{n,\varepsilon,\delta}(t\wedge\tau_M) - \tilde{w}^{n,\varepsilon,\delta}(t\wedge\tau_M)\|_\infty^T\big)^2}{{\epsilon_0}^2} \\
&\to 0,
\end{aligned}
$$

as $\varepsilon, \delta \to 0$.

Then we can apply Proposition 3.3.1 to get $\tilde{v}^{n,\varepsilon,\delta}(x,t) \to v^n(x,t)$ uniformly on

$S^1 \times [0,T]$ as $\varepsilon, \delta \to 0$. Therefore, $v^{n,\varepsilon,\delta}(x,t) \to v^n(x,t)$ in probability on $S^1 \times [0,T]$ as $\varepsilon, \delta \to 0$. □

As $u^1(x,0) \geq u^2(x,0)$, as Lemma 3.1 in Mueller [26], we can show that $u^{\varepsilon,\delta} \geq v^{n,\varepsilon,\delta}$. Let

$$U^{n,\varepsilon,\delta}(t) = \int_{S^1} (u^{\varepsilon,\delta}(x,t) - v^{n,\varepsilon,\delta}(x,t))dx. \tag{3.21}$$

It follows from the above equation that

$$U^{n,\varepsilon,\delta}(t) = \int_{S^1} (u_1(x,0) - u_2(x,0))dx + \int_0^t C^{n,\varepsilon,\delta}(s)ds + M^{n,\varepsilon,\delta}(t), \tag{3.22}$$

where

$$\begin{aligned} C^{n,\varepsilon,\delta}(t) &= \int_{S^1} \big\{ f(u^{\varepsilon,\delta}) - f(v^{n,\varepsilon,\delta}) + \frac{1}{\delta}(u^{\varepsilon,\delta} - h^1)^-(x,t) - \frac{1}{\delta}(v^{n,\varepsilon,\delta} - h^1)^-(x,t) \\ &\quad -\big(\frac{1}{\varepsilon}(u^{\varepsilon,\delta} - h^2)^+(x,t) - \frac{1}{\varepsilon}(v^{n,\varepsilon,\delta} - h^2)^+(x,t)\big)\big\} dx \\ &\leq 0, \end{aligned}$$

$$\begin{aligned} M^{n,\varepsilon,\delta}(t) &= \int_0^t \int_{S^1} \sigma(u^{\varepsilon,\delta}(x,s)) W_1(dx,ds) \\ &\quad - \int_0^t \int_{S^1} \sigma(v^{n,\varepsilon,\delta}(x,s)) g_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1) W_1(dx,ds) \\ &\quad - \int_0^t \int_{S^1} \sigma(v^{n,\varepsilon,\delta}(x,s)) f_n(|u^{\varepsilon,\delta} - v^{n,\varepsilon,\delta}| \wedge 1) W_2(dx,ds). \end{aligned}$$

Observe that

$$\begin{aligned} &\lim_{\varepsilon,\delta \to 0} U^{n,\varepsilon,\delta}(t) \\ &= U^n(t) := \int_{S^1} (u(x,t) - v^n(x,t))dx, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} & \lim_{\varepsilon,\delta\to 0} M^{n,\varepsilon,\delta}(t) \\ = \; & M^n(t) := \int_0^t \int_{S^1} \sigma(u(x,s)) W_1(dx,ds) \\ & - \int_0^t \int_{S^1} \sigma(v^n(x,s)) g_n(|u-v^n| \wedge 1) W_1(dx,ds) \\ & - \int_0^t \int_{S^1} \sigma(v^n(x,s)) f_n(|u-v^n| \wedge 1) W_2(dx,ds). \end{aligned} \tag{3.24}$$

Letting $\varepsilon, \delta \to 0$ in (3.22) we see that

$$U^n(t) = \int_{S^1} (u_1(x,0) - u_2(x,0))dx + A^n(t) + M^n(t), \tag{3.25}$$

where $A^n(t) = \lim_{\varepsilon,\delta\to 0} \int_0^t C^{n,\varepsilon,\delta}(s)ds$ is a continuous, adapted non-increasing process. Now, sending n to ∞ we obtain

$$U(t) = \int_{S^1} (u_1(x,0) - u_2(x,0))dx + A(t) + M(t), \tag{3.26}$$

where

$$U(t) = \int_{S^1} (u(x,t) - v(x,t))dx,$$

$$\begin{aligned} M(t) \; = \; & \int_0^t \int_{S^1} \sigma(u(x,s)) W_1(dx,ds) \\ & - \int_0^t \int_{S^1} \sigma(v(x,s))(1 - |u-v| \wedge 1)^{\frac{1}{2}} W_1(dx,ds) \\ & - \int_0^t \int_{S^1} \sigma(v(x,s))(|u-v| \wedge 1)^{\frac{1}{2}} W_2(dx,ds), \end{aligned}$$

and $A(t) = \lim_{n\to\infty} A^n(t)$ a continuous, adapted non-increasing process. The existence of the limits of A^n follows from the existence of the limit of U^n and M^n. Now we will modify the proof in Mueller [26] to obtain the successful coupling of u and v. In view of the assumption on σ and the boundedness of the walls h^1, h^2, it is easy to verify that

$$\frac{d < M > (t)}{dt} \geq C_0 U(t) \tag{3.27}$$

for some positive constant C_0. Thus, there exists a non-negative adapted process $V(t)$ such that

$$\frac{d < M > (t)}{dt} = U(t)V(t), \qquad V(t) \geq C_0.$$

Let

$$\begin{aligned} \phi(t) &= \int_0^t V(s)ds, \\ X(t) &= U(\phi^{-1}(t)). \end{aligned} \tag{3.28}$$

Then the time-changed process X satisfies the following equation

$$X(t) = U(0) + \tilde{A}(t) + \int_0^t X^{\frac{1}{2}}(s)dB(s), \tag{3.29}$$

where B is a Brownian motion and $\tilde{A}$ is an adapted non-increasing process. Let $Y(t) = 2X^{\frac{1}{2}}(t)$. Applying Ito's formula (before Y hits 0) we obtain

$$Y(t) = Y(0) + 2\int_0^t \frac{1}{Y(s)} d\tilde{A}(s) - \frac{1}{2}\int_0^t \frac{1}{Y(s)} ds + B(t). \tag{3.30}$$

As $\tilde{A}$ is non-increasing, it follows that

$$0 \leq Y(t) \leq Y(0) + B(t). \tag{3.31}$$

The property of one dimensional Brownian motion implies that Y hits 0 with probability 1. Hence

$$\lim_{t\to\infty} P\Big(\sup_{x\in S^1} |u(x,t) - v(x,t)| \neq 0\Big) = 0.$$

Next let us consider the general case, i.e. we do not assume $u^1(x,0) \geq u^2(x,0)$,

$x \in S^1$. Consider a solution v, u^1, u^2 of the following SPDEs with two reflecting walls

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + f\big(v(x,t)\big) + \sigma\big(v(x,t)\big)\dot{W}_1(x,t) \\
&\quad + \eta_v(x,t) - \xi_v(x,t), \\
\frac{\partial u^i(x,t)}{\partial t} &= \frac{\partial^2 u^i(x,t)}{\partial x^2} + f\big(u^i(x,t)\big) + \eta_{u^i}(x,t) - \xi_{u^i}(x,t) \\
&\quad + \sigma\big(u^i(x,t)\big)\big[(1 - |v - u^i| \wedge 1)^{\frac{1}{2}}\dot{W}_1(x,t) + (|v - u^i| \wedge 1)^{\frac{1}{2}}\dot{W}_2(x,t)\big], \\
v(x,0) &= \max_{i=1,2}\{u^i(x,0)\}.
\end{aligned}$$

By following the arguments in the first part, we have

$$\lim_{t\to\infty} P\big(\sup_{x\in S^1} |v(x,t) - u^i(x,t)| \neq 0\big) = 0, \; i = 1,2.$$

The inequality

$$0 \leq \sup_{x\in S^1} |u^1(x,t) - u^2(x,t)| \leq \sum_{i=1}^{2}\big(\sup_{x\in S^1} |v(x,t) - u^i(x,t)|\big)$$

implies

$$\lim_{t\to\infty} P\big(\sup_{x\in S^1} |u^1(x,t) - u^2(x,t)| \neq 0\big) = 0.$$

□

3.4 Strong Feller Property

In this section, we consider the strong Feller property of the solution of equation (3.1). Let $H = L^2(S^1)$. If $\varphi \in B_b(H)$ (the Banach space of all real bounded Borel functions, endowed with the sup norm), we define, for $x \in S^1$, $0 \leq t \leq T$ and $g \in H$,

$$P_t\varphi(g) = \mathbb{E}\varphi(u(x,t,g)).$$

Definition 3.4.1. *The family $\{P_t\}$ is called strong Feller if for arbitrary $\varphi \in B_b(H)$,* *the function $P_t\varphi(\cdot)$ is continuous for all $t > 0$.*

Theorem 3.4.1. *Under the hypotheses (H5)-(H6), (F4)-(F6) and that $p_1 \le |\sigma(\cdot)| \le p_2$ for some constants $p_1,\ p_2 > 0$, then for any $T > 0$ there exists a constant C'_T such that for all $\varphi \in B_b(H)$ and $t \in (0, T]$,*

$$|P_t\varphi(u_0^1) - P_t\varphi(u_0^2)| \le \frac{C'_T}{\sqrt{t}}\|\varphi\|_\infty |u_0^1 - u_0^2|_H, \tag{3.32}$$

for $u_0^1,\ u_0^2 \in H$ with $h^1(x) \le u_0^1(x),\ u_0^2(x) \le h^2(x)$, where $\|\varphi\|_\infty = \sup_{u_0}|\varphi(u_0)|$. In particular, $P_t,\ t > 0$, is strong Feller.

Proof. Choose a non-negative function $\phi \in C_0^\infty(R)$ with $\int_R \phi(x) = 1$ and denote

$$f_n(\zeta) = n\int_R \phi\big(n(\zeta - y)\big) f(y)dy,$$

$$\sigma_n(\zeta) = n\int_R \phi\big(n(\zeta - y)\big)\sigma(y)dy,$$

$$k_n(\zeta, x) = n\int_R \phi\big(n(\zeta - y)\big)(y - h^1(x))^- dy,$$

$$l_n(\zeta, x) = n\int_R \phi\big(n(\zeta - y)\big)(y - h^2(x))^+ dy.$$

So $f_n,\ \sigma_n,\ k_n,\ l_n$ are smooth w.r.t. ζ. Let

$$\begin{aligned} u_n^{\varepsilon,\delta}(x, t, u_0) = & \int_{S^1} G_t(x, y)u_0(y)dy + \int_0^t\int_{S^1} G_{t-s}(x, y)f_n\big(u_n^{\varepsilon,\delta}(y, s, u_0)\big)dyds \\ & + \int_0^t\int_{S^1} G_{t-s}(x, y)\sigma_n\big(u_n^{\varepsilon,\delta}(y, s, u_0)\big)W(dy, ds) \\ & + \frac{1}{\delta}\int_0^t\int_{S^1} G_{t-s}(x, y)k_n\big(u_n^{\varepsilon,\delta}(y, s, u_0), y\big)dyds \\ & - \frac{1}{\varepsilon}\int_0^t\int_{S^1} G_{t-s}(x, y)l_n\big(u_n^{\varepsilon,\delta}(y, s, u_0), y\big)dyds. \end{aligned}$$

Since $f_n(\zeta) \to f(\zeta)$, $\sigma_n(\zeta) \to \sigma(\zeta)$, $k_n(\zeta, x) \to (\zeta - h^1(x))^-$ and $l_n(\zeta, x) \to (\zeta - h^2(x))^+$

as $n \to \infty$, we can show that for any fixed $\varepsilon,\ \delta$ and $p \geq 1$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big(|u_n^{\varepsilon,\delta}(\cdot,t,u_0) - u^{\varepsilon,\delta}(\cdot,t,u_0)|_H^p\big) = 0.$$

By Lemma 7.1.5 in Da Prato and Zabczyk [17] and Proposition 3.3.1, it is enough to prove that there exists a constant C'_T, independent of $\varepsilon,\ \delta$ and n, such that

$$|P_t^{n,\varepsilon,\delta}\varphi(u_0^1) - P_t^{n,\varepsilon,\delta}\varphi(u_0^2)| \ \leq\ \frac{C'_T}{\sqrt{t}}\|\varphi\|_\infty |u_0^1 - u_0^2|_H, \tag{3.33}$$

where $P_t^{n,\varepsilon,\delta}\varphi(u_0) := \mathbb{E}\big(\varphi(u_n^{\varepsilon,\delta}(\cdot,\cdot,u_0))\big)$ and $u_0^1,\, u_0^2 \in H,\ \varphi \in C_b^2(H)$.

From Theorem 5.4.1 in Da Prato and Zabczyk [17], $u_n^{\varepsilon,\delta}(\cdot,\cdot,u_0)$ is continuously differentiable w.r.t. u_0. Denote by $X_n^{\varepsilon,\delta}(x,t) := \big(Du_n^{\varepsilon,\delta}(\cdot,\cdot,u_0)(\bar{u}_0)\big)(x,t)$ the directional derivative of $u_n^{\varepsilon,\delta}(\cdot,\cdot,u_0)$ at u_0 in the direction of $\bar{u}_0$ and it satisfies the mild form of a SPDE

$$\begin{aligned} X_n^{\varepsilon,\delta}(x,t) \ =\ & \int_{S^1} G_t(x,y)\bar{u}_0(y)dy + \int_0^t\int_{S^1} G_{t-s}(x,y) f'_n\big(u_n^{\varepsilon,\delta}(y,s,u_0)\big)X_n^{\varepsilon,\delta}(y,s)dyds \\ & + \int_0^t\int_{S^1} G_{t-s}(x,y)\sigma'_n\big(u_n^{\varepsilon,\delta}(y,s,u_0)\big)X_n^{\varepsilon,\delta}(y,s)W(dy,ds) \\ & + \frac{1}{\delta}\int_0^t\int_{S^1} G_{t-s}(x,y)\frac{\partial}{\partial\zeta}k_n\big(u_n^{\varepsilon,\delta}(y,s,u_0),y\big)X_n^{\varepsilon,\delta}(y,s)dyds \\ & - \frac{1}{\varepsilon}\int_0^t\int_{S^1} G_{t-s}(x,y)\frac{\partial}{\partial\zeta}l_n\big(u_n^{\varepsilon,\delta}(y,s,u_0),y\big)X_n^{\varepsilon,\delta}(y,s)dyds. \end{aligned}$$

Since $\frac{\partial}{\partial\zeta}k_n\big(u_n^{\varepsilon,\delta}(y,s,u_0),y\big) \leq 0$, $\frac{\partial}{\partial\zeta}l_n\big(u_n^{\varepsilon,\delta}(y,s,u_0),y\big) \geq 0$, we use the similar arguments as that in Zhang [42] and to get

$$\sup_{\varepsilon,\delta\geq 0, t\in[0,T]} \mathbb{E}\Big(\int_{S^1}(X_n^{\varepsilon,\delta}(y,t))^2 dy\Big) \ \leq\ C|\bar{u}_0|_H^2,$$

where C is a constant. By Elworthy-Li formula (Lemma 7.1.3 in Da Prato and Zabczyk [17]), we obtain

$$|\ \langle DP_t\varphi(u_0), \bar{u}_0\rangle\ |^2 \ \leq\ \frac{C}{p_1^2 t}\|\varphi\|_\infty^2 |\bar{u}_0|_H^2.$$

This implies inequality (3.33) which completes the proof. □

Chapter 4

Approximation Scheme for Stochastic Flows by Splitting-Up Method

4.1 Introduction

Let D is a bounded domain in $\mathbb{R}^d$ with smooth boundary ∂D, where $d = 2$ or $d = 3$. Denote by $v : D \to \mathbb{R}^d$ the velocity vector field and $\pi : D \to \mathbb{R}$ the scalar pressure. The stochastic Stokes equations describe the time evolution of incompressible fluid flow and are given as follows

$$\frac{\partial v}{\partial t} - \Delta v + \nabla \pi = \tilde{\sigma}(v)\dot{W}_t \text{ in } D \times (0,T); \tag{4.1}$$

$$\nabla \cdot v = 0 \text{ in } D \times (0,T) \tag{4.2}$$

with no-slip boundary condition

$$v = 0 \text{ on } \partial D \times [0,T], \tag{4.3}$$

where $W(t)$ is a U-valued Wiener process in a given real separable Hilbert space $(U,\ |\cdot|_U,\ <\cdot,\ \cdot>_U)$. Here the viscosity coefficient is assumed to be 1.

To formulate the stochastic Stokes equations, we need the usual Sobolev spaces $H^{m,2}$ (m is an integer) in which derivatives of functions up to order m belong to L^2. Denoted by $H_0^{1,2}$ the completion of C_0^∞ (the set of smooth functions with compact supports) w.r.t. norm of $H^{1,2}$. We introduce the space H_2 for stochastic Stokes equations:

$$H_2 = \{u \in (L^2(D))^d :\ div\ u = 0\ in\ D\ \ and\ u \cdot N = 0\ on\ \partial D\},$$

where N is the exterior normal vector field. H_2 is the closure in $(L^2(D))^d$ of the space of divergence-free vector field and tangential boundary.

Define the Stokes operator $A_2\ :\ D(A_2) \to H_2$ by

$$A_2 = -P_2\Delta,$$

where $P_2\ :\ (L^2(D))^d \to H_2$ is Leray projection and $D(A_2) = (H^{2,2})^d \cap (H_0^{1,2})^d \cap H_2$. In the following discussion, we may denote $(L^2(D))^d$ by L^2, $(H_0^{1,2}(D))^d$ by $H_0^{1,2}$ and $(H^{2,2}(D))^d$ by $H^{2,2}$ and write P, A instead of P_2, A_2 for the sake of simplicity if there is no confusion.

Applying the operator P on each term of equation (4.1), we rewrite the stochastic operatorial Stokes equations:

$$v' + Av \ \ = \ \ \sigma(v)\dot{W} \ \text{ in } (0,T) \text{ on space } H_2, \tag{4.4}$$

Where $\sigma = P\tilde{\sigma}$. We consider a fixed complete stochastic basis $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t, t \geq 0\})$. Let Q be a symmetric nonnegative trace class operator on U. Then there exists a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in U and a real numbers α_i such that

$$Qe_i = \alpha_i e_i,$$

where $\alpha_i \geq 0$ and $\sum_i \alpha_i < \infty$. For arbitrary t, W has the expansion

$$W(t) = \sum_i \sqrt{\alpha_i}\beta_i(t)e_i,$$

where

$$\beta_i(t) = \frac{1}{\sqrt{\alpha_i}} < W(t),\ e_i >_U, \quad i = 1, 2, \cdots .$$

For convenience, we denote $(H^{2,2} \cap H_2)^d$ and H_2 by H^2 and H^0. Define

$$|G|^2_{Q,H} = Tr(GQG^*) = \sum_i \alpha_i |Ge_i|^2 \ \text{ for } \mathrm{G} \in \mathcal{L}(\mathrm{U}, \mathrm{H}).$$

We assume that $\sigma(\cdot):\ H^s \to \mathcal{L}(U, H^s)$, $s = 0, 2$ satisfies

(H7) $|\sigma(u) - \sigma(v)|^2_{Q,H^s} \leq L|u - v|^2_{H^s}$;

(H8) $|\sigma(u)|^2_{Q,H^s} \leq b(1 + |u|^2_{H^s})$.

The framework of the scheme is explained below. One can decompose a complicated stochastic differential equation to a deterministic equation and a stochastic equation which are simpler to handle than the original problem. Let an initial value v_0 be given. On a fixed time interval $[0, T]$, this interval is divided into n subintervals and each of size $\varepsilon = \frac{T}{n}$. The splitting scheme defines an approximate solution of the SDE at the times $m\varepsilon$, for integral m, $0 \leq m \leq n$. This solution, denoted by y_ε, is defined recursively. Set $y_\varepsilon(0) = v_0$. Let $u_\varepsilon((m+1)\varepsilon)$ be the solution of the deterministic equations at the end of a time interval of size ε, with initial condition $y_\varepsilon(m\varepsilon)$. Then $y_\varepsilon((m+1)\varepsilon)$ is the solution of the SDE with initial condition $u_\varepsilon((m+1)\varepsilon)$, after time ε has elapsed.

Remark 4.1.1. *Denoted by* $v(t) = T_1(t, s)\xi_1,\ u(t) = T_2(t, s)\xi_2,\ y(t) = T_3(t, s)\xi_3$ *the solutions of equations*

$$\begin{cases} v'(t) + Av(t) = \sigma(v(t))\dot{W}, \quad t \in (s, T]; \\ v(s) = \xi_1, \end{cases}$$

$$\begin{cases} u'(t) + Au(t) = 0, \quad t \in (s, T]; \\ u(s) = \xi_2, \end{cases}$$

and

$$\begin{cases} y'(t) = \sigma(y(t))\dot{W}, \quad t \in (s, T]; \\ y(s) = \xi_3. \end{cases}$$

Then the convergence result

$$\lim_{\varepsilon \to 0} y_\varepsilon(t) = v(t)$$

is equivalent to Lie-Trotter type formula:

$$\lim_{n\to\infty} \prod_{i=0}^{n-1} T_3(\frac{n-i}{n}t, \frac{n-1-i}{n}t) T_2(\frac{n-i}{n}t, \frac{n-1-i}{n}t)\xi_1 = T_1(t,0)\xi_1.$$

The aim of this paper is to adopt splitting up method to approximate the solutions of stochastic Stokes equations. To the best of our knowledge, it is the first time to use resolvent for approximating the endpoints of every small interval of the deterministic equations. That is, we replace $u_\varepsilon((m+1)\varepsilon) = e^{-\varepsilon A} y_\varepsilon(m\varepsilon)$ with operator $u_\varepsilon((m+1)\varepsilon) = (I+\varepsilon A)^{-1} y_\varepsilon(m\varepsilon)$. Regarding to effective and practical computations, resolvent method is more convenient.

This chapter will be arranged in the following way. In section 4.2, we shall study the approximation scheme and give the main convergence result. Section 4.3 is devoted to the proof of difference between approximation solution and the exact solution.

4.2 The Scheme

In this section, we consider stochastic Stokes equations on H_2:

$$\begin{cases} v'(t) + Av(t) = \sigma(v(t))\dot{W}, \quad t \in (0, T]; \\ v(0) = v_0. \end{cases} \tag{4.5}$$

Let us denote by $M_W^2((0,T);H)$ the space of all adapted processes in $L^2((0,T);L^2(\Omega;H))$, where H is any separable Hilbert space, and denote by $C_W([0,T];H)$ its subspace of processes in $C([0,T];\ L^2(\Omega;H))$. As the similar proof in Theorem 3.1 in Flandoli [18], we have the following existence and uniqueness of weak and strong solutions.

Proposition 4.2.1. *Let $v_0 \in L^2(\Omega;H^s)$, $s=0,2$, under corresponding assumptions (H7) and (H8), then there exists an unique solution to equation (4.5) which means $v \in M_W^2((0,T);\ H^{s+1}) \cap C_W([0,T];\ H^s)$.*

To approximate the solution of equation (4.5), we adopt the idea of splitting up method by considering a deterministic equation and a stochastic equation

$$\bar{u}'(t) + A\bar{u}(t) = 0$$

and

$$y'(t) = \sigma(y(t))\dot{W}.$$

In general, by giving a sequence of differential initial values, we obtain a sequence of equations. We apply Trotter scheme to get the approximation of the solution of equation (4.5).

Let $n \in \mathbb{N}$, $\varepsilon = \frac{T}{n}$. Denote $t_m = m\varepsilon$, $m = 1, 2, \cdots, n-1$. Consider the deterministic equation in the interval $(0,\varepsilon]$ with initial value v_0:

$$\begin{cases} \bar{u}_\varepsilon'(t) + A\bar{u}_\varepsilon(t) = 0, \quad t \in (0,\varepsilon]; \\ \bar{u}_\varepsilon(0+) = v_0. \end{cases}$$

The mild form of solution at point ε is $\bar{u}_\varepsilon(\varepsilon) = e^{-\varepsilon A}v_0$. Suppose ε is quite small. In view of the exponential formula (see Theorem 8.3 in Chapter 1 in Pazy [30]), we replace $e^{-\varepsilon A}$ with operator $(I+\varepsilon A)^{-1}$. So we shall consider $u_1 := u_\varepsilon(\varepsilon) := (I+\varepsilon A)^{-1}v_0$ instead of $\bar{u}_\varepsilon(\varepsilon) = e^{-\varepsilon A}v_0$.

Then consider the stochastic equation:

$$\begin{cases} y'_\varepsilon(t) = \sigma(y_\varepsilon(t))\dot{W}, \quad t \in (0, \varepsilon]; \\ y_\varepsilon(0+) = u_1. \end{cases}$$

Therefore,

$$y_\varepsilon(t) = (I + \varepsilon A)^{-1} v_0 + \int_0^t \sigma(y_\varepsilon(s)) dW_s, \ t \in (0, \varepsilon].$$

We do similarly in the interval $(\varepsilon, 2\varepsilon]$ with the initial value $y_\varepsilon(\varepsilon)$:

$$u_2 := u_\varepsilon(2\varepsilon) \ := \ (I + \varepsilon A)^{-1} y_\varepsilon(\varepsilon);$$

$$\begin{aligned} y_\varepsilon(t) &= u_2 + \int_\varepsilon^t \sigma(y_\varepsilon(s)) dW_s \\ &= (I + \varepsilon A)^{-2} v_0 + \int_0^\varepsilon (I + \varepsilon A)^{-1} \sigma(y_\varepsilon(s)) dW_s + \int_\varepsilon^t \sigma(y_\varepsilon(s)) dW_s. \end{aligned}$$

By induction, we have the scheme in the interval $(m\varepsilon, (m+1)\varepsilon]$, $m = 0, 1, \cdots, n-1$, with the initial condition $y_\varepsilon(m\varepsilon)$:

$$u_{m+1} := u_\varepsilon((m+1)\varepsilon) \ := \ (I + \varepsilon A)^{-1} y_\varepsilon(m\varepsilon); \tag{4.6}$$

$$\begin{aligned} y_\varepsilon(t) &= u_{m+1} + \int_{m\varepsilon}^t \sigma(y_\varepsilon(s)) dW_s \\ &= (I + \varepsilon A)^{-(m+1)} v_0 + \sum_{i=1}^m \int_{(i-1)\varepsilon}^{i\varepsilon} (I + \varepsilon A)^{-(m+1-i)} \sigma(y_\varepsilon(s)) dW_s \\ &\quad + \int_{m\varepsilon}^t \sigma(y_\varepsilon(s)) dW_t. \end{aligned} \tag{4.7}$$

Introduce the notation

$$d(n,t) = [\frac{t}{\varepsilon}]\varepsilon, \quad t \in [0, T]$$

and $d^*(n,t) = d(n,t) + \varepsilon$. Then $d(n,t) \le t < d^*(n,t)$ for every $0 \le t \le T$. Thus,

$$\begin{aligned} y_\varepsilon(t) &= (I+\varepsilon A)^{-(\frac{d(n,t)}{\varepsilon}+1)} v_0 + \int_0^{d(n,t)} (I+\varepsilon A)^{-\frac{d(n,t)-d(n,s)}{\varepsilon}} \sigma(y_\varepsilon(s)) dW_s \\ &\quad + \int_{d(n,t)}^t \sigma(y_\varepsilon(s)) dW_t, \ t \in (0,T]. \end{aligned} \tag{4.8}$$

Remark 4.2.1. *We see that u_{m+1}, $m = 0, 1, \cdots, n-1$, are $\mathcal{F}_{m\varepsilon}$-measurable. $y_\varepsilon(t)$ are left continuous and with limit to right. Their discontinuity points are $0,\ \varepsilon,\ 2\varepsilon, \cdots, (n-1)\varepsilon$.*

Now we present one of our main convergence results.

Theorem 4.2.1. *Assume that (H7), (H8) holds. $v(t)$ is the solution of equation (4.5).*
(i) If the initial value $v_0 \in L^2(\Omega;\ L^2)$,

$$y_\varepsilon(t) \to v(t) \text{ in } L^2(\Omega;\ L^2), \text{ for } t \in [0,T]; \tag{4.9}$$

(ii) If $v_0 \in D(A)$,

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{L^2}] \le Ce^{4Ct\varepsilon} \cdot \varepsilon. \tag{4.10}$$

4.3 The Proof of Main Convergence Result

In this section, we prove our convergence result Theorem 4.2.1.

Lemma 4.3.1. *Assume that (H7), (H8) holds and the initial value $v_0 \in L^2(\Omega;\ H^s)$, $s = 0, 2$. Then*

$$\mathbb{E}[|y_\varepsilon(t)|^2_H] \le 2e^{2bt}(\mathbb{E}[|v_0|^2_H] + bt). \tag{4.11}$$

If there is not confusion, we denote H^s, $s = 0, 2$ by H.

Proof. Before proving the boundedness of the second moment of $y_\varepsilon(t)$, we give the estimation from [6]: there exist a constants $\varepsilon_0 > 0$ such that for all complex ε with $Re(\varepsilon) \geq -\varepsilon_0$,

$$\|(A+\varepsilon I)^{-1}\| \leq \frac{C}{|\varepsilon|+1}, \tag{4.12}$$

Therefore,

$$\|(I+\varepsilon A)^{-1}\| \leq \frac{1}{|\varepsilon|+1}. \tag{4.13}$$

From the approximation (4.7), we get, for $t \in (m\varepsilon, (m+1)\varepsilon]$,

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] \leq\ & 2\mathbb{E}[|(I+\varepsilon A)^{-(m+1)}v_0|_H^2] \\ & +2\sum_{i=1}^{m}\mathbb{E}[|\int_{t_{i-1}}^{t_i}(I+\varepsilon A)^{-(m+1-i)}\sigma(y_\varepsilon(s))dW_s|_H^2] \\ & +2\mathbb{E}[|\int_{t_m}^{t}\sigma(y_\varepsilon(s))dW_s|_H^2].\end{aligned}$$

Using inequality (4.13) and the following property of martingale

$$\mathbb{E}[|\int_0^t \sigma(y_\varepsilon(s))dW_s|_H^2] = \mathbb{E}[\int_0^t |\sigma(y_\varepsilon(s))|_{Q,H}^2 ds],$$

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] \leq\ & 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2\sum_{i=1}^{m}\mathbb{E}[\int_{t_{i-1}}^{t_i}|(I+\varepsilon A)^{-(m+1-i)}\sigma(y_\varepsilon(s))|_{Q,H}^2 ds] \\ & +2\mathbb{E}[\int_{t_m}^{t}|\sigma(y_\varepsilon(s))|_{Q,H}^2] \\ \leq\ & 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2\sum_{i=1}^{m}(\frac{1}{1+\varepsilon})^{2(m+1-i)}\mathbb{E}[\int_{t_{i-1}}^{t_i}|\sigma(y_\varepsilon(s))|_{Q,H}^2 ds] \\ & +2\mathbb{E}[\int_{t_m}^{t}|\sigma(y_\varepsilon(s))|_{Q,H}^2 ds].\end{aligned}$$

Then applying linear growth of σ, one obtains

$$\begin{aligned}
\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2b\sum_{i=1}^{m}(\frac{1}{1+\varepsilon})^{2(m+1-i)}\int_{t_{i-1}}^{t_i}(1+\mathbb{E}[|y_\varepsilon(s)|_H^2])ds \\
&\quad +2b\int_{t_m}^{t}(1+\mathbb{E}[|y_\varepsilon(s)|_H^2])ds \\
&\leq 2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2b\varepsilon\sum_{i=1}^{m}(\frac{1}{1+\varepsilon})^{2i} + 2b(t-t_m) \\
&\quad +2b\int_0^t(\frac{1}{1+\varepsilon})^{2(m-\frac{d(n,s)}{\varepsilon})}E[|y_\varepsilon(s)|_H^2]ds.
\end{aligned}$$

Then Gronwall inequality implies that

$$\begin{aligned}
\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq (2(\frac{1}{1+\varepsilon})^{2(m+1)}\mathbb{E}[|v_0|_H^2] + 2b\varepsilon\sum_{i=1}^{m}(\frac{1}{1+\varepsilon})^{2i} + 2b(t-t_m)) \\
&\quad \times\exp\{2b\varepsilon\sum_{i=1}^{m}(\frac{1}{1+\varepsilon})^{2i} + 2b(t-t_m))\}.
\end{aligned}$$

Therefore, we get the result by using inequality (4.13). □

Applying Ito formula to $|y_\varepsilon(t)|_H^2$, one obtains a little better result from equations (4.6) and (4.7). The detail will be given in the following lines. For $t \in (m\varepsilon, (m+1)\varepsilon]$,

$$|y_\varepsilon(t)|_H^2 = |u_\varepsilon((m+1)\varepsilon)|_H^2 + 2\int_{m\varepsilon}^{t}(y_\varepsilon(s),\ \sigma(y_\varepsilon(s))dW_s)_H + \int_{m\varepsilon}^{t}|\sigma(y_\varepsilon(s))|_{Q,H}^2.$$

Then,

$$\begin{aligned}
\mathbb{E}[|y_\varepsilon(t)|_H^2] &= \mathbb{E}[|u_{m+1}|_H^2] + \mathbb{E}[\int_{m\varepsilon}^{t}|\sigma(y_\varepsilon(s))|_{Q,H}^2 ds] \\
&= \mathbb{E}[|(I+\varepsilon A)^{-1}y_\varepsilon(m\varepsilon)|_H^2] + \int_{m\varepsilon}^{t}\mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds \\
&\leq (1+\varepsilon)^{-2}\mathbb{E}[|y_\varepsilon(m\varepsilon)|_H^2] + \int_{m\varepsilon}^{t}\mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds.
\end{aligned}$$

By induction,

$$\begin{aligned}\mathbb{E}[|y_\varepsilon(t)|_H^2] &\leq (1+\varepsilon)^{-2(m+1)}\mathbb{E}[|v_0|_H^2] + \sum_{i=1}^{m}(1+\varepsilon)^{-2(m+1-i)}\int_{t_{i-1}}^{t_i}\mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds \\ &\quad + \int_{t_m}^{t}\mathbb{E}[|\sigma(y_\varepsilon(s))|_{Q,H}^2]ds.\end{aligned}$$

Processing as in Lemma 4.3.1, we have

$$\mathbb{E}[|y_\varepsilon(t)|_H^2] \leq e^{bt}(\mathbb{E}[|v_0|_H^2] + bt). \tag{4.14}$$

The following inequality play an important role in proving the convergence with rate.

Proposition 4.3.1. *For $t,\ r \in (0,T]$. Let $t = j\varepsilon$ and $r \in ((j-1)\varepsilon, j\varepsilon)$, $j(\geq 2)$ is an integer. Then,*

$$|(I+\varepsilon A)^{-j}g - e^{-rA}g|_{L^2} \leq C\cdot\varepsilon^{\frac{1}{2}}\cdot|g|_{H^2}, \quad \text{for } g \in D(A). \tag{4.15}$$

Proof. The difference can be separated in the following way:

$$\begin{aligned}(I+\varepsilon A)^{-j}g - e^{-rA}g &= [(I+\varepsilon A)^{-j}g - e^{-tA}g] + [e^{-tA}g - e^{-rA}g] \\ &= [(I+\varepsilon A)^{-j}g - e^{-tA}g] + e^{-rA}[e^{-(t-r)A} - I]g.\end{aligned}$$

We have

$$\begin{aligned}|e^{-rA}[e^{-(t-r)A} - I]g|_{L^2} &= |\int_0^{t-r} e^{-rA}e^{-sA}Ag|_{L^2} \\ &\leq C_1\cdot\varepsilon\cdot|g|_{H^2}, \quad \text{for } g \in D(A), \end{aligned} \tag{4.16}$$

where C_1 is a constant.

So we only have to evaluate the distance $(I+\varepsilon A)^{-j}g - e^{-tA}g$. Denoted by $J_\varepsilon =$

$(I+\varepsilon A)^{-1}$. It satisfies $\|J_\varepsilon\| \le \frac{1}{1+\varepsilon}$ and

$$\|J_\varepsilon^k\| \le (\frac{1}{1+\varepsilon})^k < 1, \text{ for } k = 1, 2, \cdots .$$

Because, for $g \in D(A)$,

$$|J_\varepsilon g - g|_{L^2} = |\varepsilon A J_\varepsilon g|_{L^2} = |\varepsilon J_\varepsilon A g|_{L^2} \quad \le \quad \varepsilon |Ag|_{L^2}, \tag{4.17}$$

the operators $-A_\varepsilon := -\varepsilon^{-1}(I - J_\varepsilon)$ are bounded and they are the infinitesimal generators of uniformly continuous semigroups $S_\varepsilon(t)$ which satisfies

$$\|S_\varepsilon(t)\| = \|e^{-tA_\varepsilon}\| = \|e^{-\frac{t}{\varepsilon}} \sum_{k=0}^{\infty} (-\frac{t}{\varepsilon})^k \frac{1}{k!} J_\varepsilon^k\| \le e^{-\frac{t}{\varepsilon}} \sum_{k=0}^{\infty} (-\frac{t}{\varepsilon})^k \frac{1}{k!} \|J_\varepsilon^k\| \le 1.$$

Here we need the result from Lemma 5.1 in Chapter 3 in [30].

Lemma 4.3.2. *Let S be a bounded linear operator satisfying*

$$\|S^k\| \le MN^k, \text{ for } k = 1, 2, \cdots, \ N \ge 1,$$

Then for every $n \ge 0$, we have

$$|e^{(S-I)n} g - S^n g| \le MN^{n-1} e^{(N-1)n} [n^2(N-1)^2 + nN]^{\frac{1}{2}} |g - Sg|.$$

Therefore,

$$\begin{aligned} |S_\varepsilon(t) g - J_\varepsilon^j g|_{L^2} &\le j^{\frac{1}{2}} \cdot |g - J_\varepsilon g|_{L^2} \\ &= j^{\frac{1}{2}} \cdot \frac{t}{j} \cdot |\frac{g - J_\varepsilon g}{\varepsilon}|_{L^2}, \quad \text{for } g \in D(A). \end{aligned}$$

From (4.17),

$$|S_\varepsilon(t) g - J_\varepsilon^j g|_{L^2} \quad \le \quad C_2 \varepsilon^{\frac{1}{2}} \cdot |Ag|_{L^2}, \quad \text{for } g \in D(A). \tag{4.18}$$

Next, consider the estimation $|S_\varepsilon(t)g - e^{-tA}g|_{L^2}$. $S_\varepsilon(t)$ can be written formally $e^{-tA(I+\varepsilon A)^{-1}}$. Introduce the functional

$$f(s) = e^{-tA(I+sA)^{-1}}g, \text{ for fixed } g \in D(A).$$

We can see $f(0) = e^{-tA}g$. Moreover, $f:\ [0,1] \to L^2$ is continuous on $[0,1]$, differentiable on $(0,1)$. So Lagrange mean value theorem gives

$$|f(s) - f(0)|_{L^2} \quad \leq \quad s \cdot \sup_{r \in (0,1)} |f'(r)|_{L^2}.$$

That means

$$|S_\varepsilon(t)g - e^{-tA}g|_{L^2} \quad \leq \quad C_3\varepsilon \cdot |g|_{H^2}, \quad \text{for } g \in D(A). \tag{4.19}$$

Combining inequalities (4.16), (4.18) and (4.19), we get our result (4.15). □

Proof of Theorem 4.2.1 We know that

$$\begin{aligned}
\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{L^2}] \quad \leq \quad & 2\mathbb{E}[|[(I+\varepsilon A)^{-(m+1)} - e^{-tA}]v_0|^2_{L^2}] \\
& +4\mathbb{E}[|\int_0^{t_m} [(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))dW_s|^2_{L^2}] \\
& +4\mathbb{E}[|\int_{t_m}^{t} [I - e^{-(t-s)A}]\sigma(y_\varepsilon(s))dW_s|^2_{L^2}] \\
& +4\mathbb{E}[|\int_0^{t} e^{-(t-s)A}[\sigma(y_\varepsilon(s)) - \sigma(v(s))]dW_s|^2_{L^2}].
\end{aligned}$$

We have below two inequalities

$$|[(I+\varepsilon A)^{-(m+1)} - e^{-tA}]v_0|_{L^2} \quad \leq \quad C \cdot \varepsilon \cdot |g|_{H^2}, \quad \text{for } g \in D(A);$$

$$\begin{aligned}
& |[(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))|^2_{Q,L^2} \\
= & \sum_i \alpha_i |[(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))e_i|_{L^2} \\
= & \sum_i \alpha_i |[(I+\varepsilon A)^{-(m-\frac{d(n,s)}{\varepsilon})} - e^{-(t-s)A}]\sigma(y_\varepsilon(s))e_i|_{L^2} \\
\leq & \sum_i \alpha_i [C\varepsilon |\sigma(y_\varepsilon(s))e_i|^2_{H^2}] \\
= & C\varepsilon \sum_i \alpha_i |\sigma(y_\varepsilon(s))e_i|^2_{H^2}, \ s \in [t_m, t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{L^2}] \\
\leq & 2C\varepsilon \mathbb{E}[|v_0|^2_{H^2}] + 4C\varepsilon \int_0^t b(1 + \mathbb{E}[|y_\varepsilon(s)|^2_{H^2})ds \\
& + 4CL \int_0^t \mathbb{E}[|y_\varepsilon(s) - v(s)|_{L^2}]ds.
\end{aligned}$$

Then Gronwall inequality implies that

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{L^2}] \leq \varepsilon \cdot Ce^{4C\varepsilon t},$$

which is the result (ii).

For $v_0 \in L^2$, we have

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{L^2}] \leq 2\tilde{C}(\varepsilon) + 4\tilde{C}(\varepsilon)t + 4CL \int_0^t \mathbb{E}[|y_\varepsilon(s) - v(s)|^2_{L^2}]ds,$$

where $\tilde{C}(\varepsilon) \to 0$ as $\varepsilon \to 0$ because of the exponential formula. So we apply Gronwall inequality and get the result (i):

$$\mathbb{E}[|y_\varepsilon(t) - v(t)|^2_{L^2}] \to 0, \text{ as } \varepsilon \to 0.$$

□

4.4 Future Study: Stochastic Navier-Stokes Equations

In the section, we would like to mention some future study for approximating stochastic Navier-Stokes equations.

Consider the following stochastic Navier-Stokes system:

$$\frac{\partial v}{\partial t} - \Delta v + (v \cdot \nabla)v + \nabla \pi = \tilde{\sigma}(v)\dot{W}_t \text{ in } D \times (0,T); \tag{4.20}$$

$$\nabla \cdot v = 0 \text{ in } D \times (0,T); \tag{4.21}$$

$$v = 0 \text{ on } \partial D \times [0,T]; \tag{4.22}$$

$$v(0) = v_0 \text{ in } D, \tag{4.23}$$

where v, π, $\tilde{\sigma}$and W has the same setting as that in Section 4.1.

This system can be split into the Euler equations with tangential boundary conditions and stochastic Stokes equations with no-slip boundary conditions more exactly

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \text{ in } D \times (0,T); \tag{4.24}$$

$$\nabla \cdot u = 0 \text{ in } D \times (0,T); \tag{4.25}$$

$$u(t) \cdot N = 0 \text{ on } \partial D \times [0,T] \tag{4.26}$$

and

$$\frac{\partial y}{\partial t} - \Delta y + \nabla \pi = \tilde{\sigma}(y)\dot{W}_t; \text{ in } D \times (0,T); \tag{4.27}$$

$$\nabla \cdot y = 0 \text{ in } D \times (0,T); \tag{4.28}$$

$$y = 0 \text{ on } \partial D \times [0,T]. \tag{4.29}$$

We present the following scheme that the approximation is continuous on $[0,T]$ which is different from that in section 4.2. To make the estimation clear, we write the details of each interval $((m-1)\varepsilon,\ m\varepsilon]$, $m = 1, 2, \cdots, n$.

On $(0,\varepsilon]$, consider firstly,

$$\begin{cases} u'_\varepsilon(t) = -P(u_\varepsilon(t)\cdot\nabla)u_\varepsilon(t); \\ u_\varepsilon(0+) = v_0. \end{cases}$$

Then, $u_\varepsilon(t) = v_0 - \int_0^t P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds := E_u(t,0)v_0$.
Secondly, from equation $y'_\varepsilon(t) = -Ay_\varepsilon(t) + \sigma(y_\varepsilon(t))\dot{W}_t$, where $\sigma = P\tilde{\sigma}$. We define, for $t \in (0,\varepsilon]$,

$$\begin{aligned} y_\varepsilon(t) &= e^{-tA}u_\varepsilon(t) + \int_0^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s \\ &= e^{-tA}v_0 - \int_0^t e^{-tA}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds + \int_0^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s. \end{aligned}$$

On $(\varepsilon, 2\varepsilon]$ with the initial value $y_\varepsilon(\varepsilon)$,

$$\begin{cases} u'_\varepsilon(t) = -P(u_\varepsilon(t)\cdot\nabla)u_\varepsilon(t); \\ u_\varepsilon(\varepsilon+) = y_\varepsilon(\varepsilon). \end{cases}$$

We get $u_\varepsilon(t) = y_\varepsilon(\varepsilon) - \int_\varepsilon^t P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds := E_u(t,\varepsilon)y_\varepsilon(\varepsilon)$.

And, by the equation (4.27), defined $y_\varepsilon(t)$ by

$$\begin{aligned} y_\varepsilon(t) &= e^{-(t-\varepsilon)A}u_\varepsilon(t) + \int_\varepsilon^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s \\ &= e^{-(t-\varepsilon)A}y_\varepsilon(\varepsilon) - \int_\varepsilon^t e^{-(t-\varepsilon)A}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds + \int_\varepsilon^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s, \end{aligned}$$

so

$$\begin{aligned} y_\varepsilon(t) &= e^{-tA}v_0 - \int_0^\varepsilon e^{-tA}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds \\ &\quad - \int_\varepsilon^t e^{-(t-\varepsilon)A}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds + \int_0^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s, \end{aligned}$$

On $(m\varepsilon,(m+1)\varepsilon]$ with the initial value $y_\varepsilon(m\varepsilon)$, similarly, we have $u_\varepsilon(t) = y_\varepsilon(m\varepsilon) - \int_{m\varepsilon}^t P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds := E_u(t,m\varepsilon)y_\varepsilon(m\varepsilon)$. And define

$$\begin{aligned} y_\varepsilon(t) &= e^{-(t-m\varepsilon)A}u_\varepsilon(t) + \int_{m\varepsilon}^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s \\ &= e^{-(t-m\varepsilon)A}y_\varepsilon(\varepsilon) - \int_{m\varepsilon}^t e^{-(t-m\varepsilon)A}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds + \int_{m\varepsilon}^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s \\ &= e^{-tA}v_0 - \sum_{i=0}^{m-1}\int_{i\varepsilon}^{(i+1)\varepsilon} e^{-(t-i\varepsilon)A}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds \\ &\quad - \int_{m\varepsilon}^t e^{-(t-m\varepsilon)A}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds + \int_0^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s. \end{aligned}$$

If we use the notation $d(n,t)$, $y_\varepsilon(t)$ and $u_\varepsilon(t)$ are rewritten as

$$u_\varepsilon(t) = E_u(t,d(n,t))y_\varepsilon(d(n,t));$$

$$\begin{aligned} y_\varepsilon(t) &= e^{-tA}v_0 - \int_0^t e^{-(t-d(n,s))A}P(u_\varepsilon(s)\cdot\nabla)u_\varepsilon(s)ds \\ &\quad + \int_0^t e^{-(t-s)A}\sigma(y_\varepsilon(s))dW_s. \end{aligned} \tag{4.30}$$

Remark 4.4.1. *Even though the boundary conditions of above two equations are different, using the smoothness of the Stokes evolution, we will get the no-slip boundary* *condition in the Stokes steps.*

We consider the linear splitting up scheme which is quite similar as nonlinear case. $\tilde{u}_\varepsilon(t)(:= E_v(t, m\varepsilon)\tilde{y}_\varepsilon(m\varepsilon))$ is the solution of

$$\begin{cases} \tilde{u}_\varepsilon'(t) = -P(v(t)\cdot\nabla)\tilde{u}_\varepsilon(t), \text{ in } (m\varepsilon, (m+1)\varepsilon]; \\ \tilde{u}_\varepsilon(m\varepsilon+) = \tilde{y}_\varepsilon(m\varepsilon), \end{cases} \tag{4.31}$$

where $\tilde{y}_\varepsilon(m\varepsilon)$ is analogy with $y_\varepsilon(m\varepsilon)$ which are generated by linear Euler equations in each step. Then

$$\begin{aligned} \tilde{y}_\varepsilon(t) &= e^{-(t-m\varepsilon)A}\tilde{u}_\varepsilon(t) + \int_{m\varepsilon}^t e^{-(t-s)A}\sigma(\tilde{y}_\varepsilon(s))dW_s \\ &= e^{-(t-m\varepsilon)A}\tilde{y}_\varepsilon(m\varepsilon) - \int_{m\varepsilon}^t e^{-(t-m\varepsilon)A}P(v(s)\cdot\nabla)\tilde{u}_\varepsilon(s)ds \\ &\quad + \int_{m\varepsilon}^t e^{-(t-s)A}\sigma(\tilde{y}_\varepsilon(s))dW_s. \end{aligned} \tag{4.32}$$

We want to evaluate the difference between the approximation y_ε and the exact solution v under good assumptions on the solution v.

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