RINGS OF SEMI-ALGEBRAIC FUNCTIONS ON THE LINE

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Abstract

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We look at real closed rings of semi-algebraic functions, particularly in one dimension. The overall aim is to provide results that lead towards a positive or negative result on the decidability of the ring $C_1^{\mathcal{R}}$ of continuous semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$ for a real closed ring $\mathcal{R}$.

We provide a complete classification of the rings between $B_1^{\mathcal{R}}$, the bounded, continuous definable functions $\mathcal{R} \longrightarrow \mathcal{R}$, and $D_1^{\mathcal{R}}$, all the definable functions. We use this to then describe the Zariski spectra of all such rings. We also give a schema of sentences that separates the isomorphism classes of finite extensions of $B_1^{\mathcal{R}}$.

We show an attempt to use the Feferman-Vaught technique to obtain a decidability result for a non-von Neumann regular real closed ring. The attempt fails, due to a problem with coheirs that we describe, as well as limitations of the Feferman-Vaught technique in decidability results with non-regular real closed rings.

A result had been thought to have been found in [GMP04] on vector lattices of definable functions. However, we discovered a flaw in the proof, which has been communicated to the authors. At the time of print there is no alternative proof. Indeed, we show that one of the two key results required in the paper is in fact false. From this period of research, however, we have obtained results on continuous definable choice functions for fields.

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Chapter 1

Introduction

The topics in the thesis live in the intersection of model theory and real algebraic geometry. The overall research interest pursued over the course of obtaining the results in the thesis was the decidability of the ring C_1 of continuous semialgebraic functions $\mathbb{R}_{\text{alg}} \longrightarrow \mathbb{R}_{\text{alg}}$. This is still an open question.

Following Tarski's quantifier elimination of real closed fields [Tar48], and van den Dries, Knight, Pillay and Steinhorn's work on o-minimal structures [vdD98], [PS86], [KPS86], [PS88], real closed fields are known to have a very tame model theory and geometry. It seems natural to try and extend such results for new constructions involving real closed fields. This has led to the investigation first of the valuation rings of real closed fields in Cherlin and Dickmann's paper [CD83], and then the creation of the more general **real closed rings** by Schwartz in [Sch89].

However, the theory of real closed rings is not model complete, complete or decidable. It's model completion (by [PS99]) is the theory of von Neumann regular real closed rings without minimal idempotents. While it is clearly not complete (models include real closed fields and products of real closed fields), any von Neumann regular real closed ring is decidable by [Ast08].

It therefore makes sense to look at non-von Neumann regular real closed rings. The most recognized von Neumann regular real closed rings are the rings $D_n^{\mathcal{R}}$, of definable functions $\mathcal{R}^n \longrightarrow \mathcal{R}$, for $n \in \mathbb{N}$ and $\mathcal{R}$ a real closed field. We provide a complete classification of the rings between $B_1^{\mathcal{R}}$ — the bounded, continuous definable functions $\mathcal{R} \longrightarrow \mathcal{R}$ — and $D_1^{\mathcal{R}}$. We use this to then describe the Zariski spectra of all such rings.

We show an attempt to use the Feferman-Vaught technique to obtain a decidability result for a non-von Neumann regular real closed ring. The attempt fails, due to a problem with coheirs that we describe, as well as limitations of the Feferman-Vaught technique in decidability results with non-regular real closed rings.

A result had been thought to have been found in [GMP04] on vector lattices of definable functions. However, we discovered a flaw in the proof, which has been communicated to the authors. At the time of print there is no alternative proof. Indeed, we show that one of the two key results required in the paper is in fact false. From this period of research, however, we have obtained results on continuous definable choice functions.

For the rest of this introductory chapter we establish some of the background material that is used extensively in the thesis. This is primarily:

- The basic ideas of model theory, real closed fields and o-minimality
- Decidability
- Real closed rings
- The link between the Zariski spectrum of rings of definable functions and type spaces

For readers who are familiar with any one of these topics, each section also establishes notation conventions throughout the text.

Following on from these background sections, we give a more detailed account of the broad themes of each of the subsequent chapters as well as the main results in them.

1.1 Model Theory, Real Closed Fields and o-minimality

For a text on model theory, see [Hod93]. By and large, notation in the thesis agrees with this text. The model theory of real closed fields is used throughout. For the basics of real closed fields, see [BCR87]; a source for the model theory of real closed fields is [Mar02].

Definition 1.1.1. Let $\mathcal{L}$ be a language. Then the set of formulas of $\mathcal{L}$ is denoted $Fml(\mathcal{L})$, and the set of sentences of $\mathcal{L}$ is denoted $Sent(\mathcal{L})$.

Definition 1.1.2. The **language of rings** is the language with symbols 0, 1, $-$, $+$, $.$, with the usual arities, and is denoted $\mathcal{L}_R$. The **language of ordered rings** is $\mathcal{L}_{OR} := \mathcal{L}_R \cup \{<\}$. The theory of real closed fields is **RCF**, it is in either the language $\mathcal{L}_R$ or $\mathcal{L}_{OR}$ and we specify which in the text.

Definition 1.1.3. A subset $S \subseteq M^k$ is **definable** if there is an $\mathcal{L}(M)$-formula (a formula with parameters from M) $\phi(x_1, \ldots, x_k)$ such that for all $\langle a_1, \ldots, a_k \rangle \in M^k$

$$M \models \phi(x_1, \ldots, x_k) \iff \langle a_1, \ldots, a_k \rangle \in S$$

Where we use no parameters, we say S is $\emptyset$–definable. A function $f : S \longrightarrow M^l$ is definable if its graph $\Gamma(f) \subseteq M^{k+l}$ is definable.

Definition 1.1.4. Let $\mathcal{R}$ be a real closed field. A subset $S \subseteq \mathcal{R}^k$ is called **semi-algebraic** if it is definable in the $\mathcal{L}_{OR}$-structure of $\mathcal{R}$ without using quantifiers, i.e. it has the form:

$$\bigvee_{i \in I} (\bigwedge_{j \in J_i} P_j(\overline{x}) = 0 \wedge \bigwedge_{j \in J_i} Q_j(\overline{x}) < 0)$$

where I is an index set as are the J_i for each $i \in I$, and P_j and Q_j are polynomials in $\mathcal{R}[x_1, \ldots, x_k]$. We abbreviate this to "s.a.".

Using the following famous theorem of Tarski, we use "definable" and "semi-algebraic" interchangeably, where appropriate.

Theorem 1.1.5. *[Mar02, Thm 3.3.15] The $\mathcal{L}_{OR}$-theory RCF is complete and has positive quantifier elimination.*

Proof. The theorem cited proves quantifier elimination. For positive quantifier elimination we note that $x \neq y$ is equivalent to $x < y \vee y < x$ and that $x \not< y$ is equivalent to $x = y \vee y < x$. $\square$

Definition 1.1.6. Let M be structure with a binary symbol $<$ which is interpreted as a dense linear order without end points. We say M is **o-minimal** if it satisfies either of the following equivalent conditions:

(a) every definable subset of M is the union of finitely many open intervals and a finite set

(b) every definable subset of M is definable in the structure $\langle M, <\rangle$

Theorem 1.1.7. *[KPS86, Section 4] If a structure M is o-minimal, then every elementary extension of M is also o-minimal.*

By 1.1.5, every RCF is o-minimal.

O-minimal theories have nice geometric/topological properties and [vdD98] is a standard text. The three papers [PS86], [KPS86], [PS88] are where the major concepts were first introduced. Cell decomposition as well as other properties of o-minimal theories are discussed in individual chapters of the thesis.

For a reference on decidability, see [Hod93]. We do not go into a rigorous definition involving Gödel numbers. In essence, an $\mathcal{L}$-theory T is decidable if there is an effective procedure to decide whether, for any $\phi \in Sent(\mathcal{L})$, $T \models \phi$ or $T \not\models \phi$. An $\mathcal{L}$-structure M is decidable if its complete first order theory is decidable.

We notice that the notion of decidability makes sense only in the case of countable languages.

1.2 Real Closed Rings

Real closed rings were first introduced in [Sch89] as certain rings of sections of sheaves over the real spectra of rings. We will use the definition given in [Tre07]. These two definitions are proven to be equivalent in [SM99]. Cherlin and Dickmann gave a definition of real closed ring in [CD83], this is now what is called a **real closed valuation ring**.

Throughout the thesis all rings are commutative with unit.

Definition 1.2.1. A ring A is **real closed** if it is commutative and unital and such that for every $n \in \mathbb{N}$ we have a collection of functions such that for every $f \in C_n^{\mathbb{R}_{\text{alg}}} := \{g : \mathbb{R}_{\text{alg}}^n \longrightarrow \mathbb{R}_{\text{alg}} \mid g \text{ is semi-algebraic and continuous}\}$ we have a function $f_A : A^n \longrightarrow A$ with the following properties:

1. If f is the constant function 0 or the constant function 1, then so is f_A; if $f = id_{\mathbb{R}_{\text{alg}}}$, then $f_A = id_A$; if f is addition, multiplication or subtraction in the ring $\mathbb{R}_{\text{alg}}$ then f_A is the respective function in the ring A.
2. If $k \in \mathbb{N}$ and $\{f_i\}_{1 \leq i \leq n} \subseteq C_k^{\mathbb{R}_{\text{alg}}}$, then

$$[f \circ (f_1, \ldots, f_n)]_A = f_R \circ (f_{1,A}, \ldots, f_{n,A})$$

The following equivalent definition is given in [PS99]:

1. A is a reduced, commutative unital ring.
2. The squares of A form the positive cone of a partial order; together with this order, A is an f-ring.
3. If $0 \leq a \leq b$, then $a^2 \in Ab$
4. For every prime ideal $\mathfrak{p} \in A$, the quotient field $qf(A/\mathfrak{p})$ is real closed and the ring $A/\mathfrak{p}$ is integrally closed.

An **f-ring** A is a ring which is also a lattice under $\vee$ and $\wedge$, such that, if $\leq$ is the lattice partial order, then $x \geq 0$ and $y \geq 0 \Longrightarrow x.y \geq 0$, and for all $z \in A, \quad x \geq y \Longrightarrow z + x \geq z + y$, and such that for all x, y, z, $x \wedge y = 0$ and $z \geq 0 \Longrightarrow zx \wedge y = xz \wedge y = 0$.

Remark 1.2.2. The functions f_A are not part of the structure A, i.e. we have not extended the language, the definition simply guarantees that such functions exist.

We can define a partial order $\leq$ on any real closed ring by

$$x \leq y \Longleftrightarrow \exists z \; y - x = z^2$$

Theorem 1.2.3. *[PS99, Section 2] The class of real closed rings in either of the languages $\mathcal{L}_R$ or $\mathcal{L}_R \cup \{\leq\}$ is elementary.*

Examples of real closed rings are: rings of the continuous semi-algebraic functions on expansions of real closed fields; real closed valuation rings in the sense of [CD83]; real closed fields themselves; the rings of n-ary semi-algebraic functions $\mathcal{R}^n \longrightarrow \mathcal{R}$ for a real closed field $\mathcal{R}$.

Theorem 1.2.4. *[Tre07, Proposition 2.11] Let A be a real closed ring. Then for each $n \in \mathbb{N}$, and each $f \in C_n^{\mathbb{R}_{\text{alg}}}$, $f_A : A^n \longrightarrow A$ is definable. In particular, there is some polynomial $P(\overline{x}, y, \overline{u}) \in \mathbb{Z}[\overline{x}, y, \overline{u}]$ such that $\exists \overline{u} \; P(\overline{x}, y, \overline{u}) = 0$ defines the graph of f in $\mathbb{R}_{\text{alg}}^{n+1}$ and f_A in A^{n+1}.*

Corollary 1.2.5. *The category of real closed rings is a full subcategory of **Ring**.*

We collect some properties of real closed rings, which can be found in [Tre07]. Let A be a real closed ring:

- The category RCR of real closed rings has arbitrary products and coproducts
- RCR is a variety in the sense of universal algebra
- For any radical ideal I, A/I is a real closed ring
- The real closed rings that are fields are precisely the real closed fields
- For any maximal ideal $\mathfrak{m}$, $A/\mathfrak{m}$ is a real closed field
- For any multiplicatively closed set $S \subseteq A$, the localization $S^{-1}A$ is real closed
- The Zariski spectrum $\operatorname{Spec} A$ of A is homeomorphic to the real spectrum $\mathrm{Sper}A$ of A, and the homeomorphism is given by the support map, which takes a positive cone P and maps it to $P \cap -P$, i.e. for each prime ideal $\mathfrak{p}$, there is a unique ordering on the quotient field $qf(A/\mathfrak{p})$

Any further properties are given in the text.

Definition 1.2.6. Let $\mathcal{R}$ be a real closed field, let $n \in \mathbb{N}$. Then $D_n^{\mathcal{R}}$ is the ring of semi-algebraic (definable) functions $\mathcal{R}^n \longrightarrow \mathcal{R}$, $C_n^{\mathcal{R}}$ is the ring of continuous definable functions $\mathcal{R}^n \longrightarrow \mathcal{R}$. If we fix the base field $\mathcal{R}$, then we may just write D_n and C_n.

These rings and extensions of them are the main object of study in Chapters 3 and 4.

1.3 The Zariski Spectrum and the Type Space

Definition 1.3.1. Let A be a ring. The **Zariski spectrum** of A is the set of prime ideals of A. It is a topological space with basis of open sets $\{\mathfrak{p} \mid a \notin \mathfrak{p}\}$ for each element $a \in A$.

Definition 1.3.2. Let $\mathcal{L}$ be a language, and M an $\mathcal{L}$-structure. A **type** p (of M) is a maximal, consistent set of $\mathcal{L}$-formulas with parameters from M and with the same number of free variables, such that if $\phi(\overline{x}) \in p$, then $M \models \exists \overline{x}\ \phi(\overline{x})$ (this is often called a complete type). An n**-type** is a type p such that every formula has exactly n free variables. The n**-type space** of M (often just "type space") is the set whose elements are the n-types of M and it is denoted $S_n(M)$, where n denotes the number of free variables in the formulas, and it has the basis of clopen sets $\|\phi\| = \{p \mid \phi \in p\}$.

We may wish to specify a parameter set other than the entire structure. Let $\Lambda \subseteq M$, not necessarily definable. An **n-type of M over Λ** is a maximal consistent set of $\mathcal{L}$-formulas with parameters from Λ. The type space $S_n(M, \Lambda)$ is the set of all such n-types. It has the basis of clopen sets $\|\phi\|_\Lambda = \{p \in S_n(M, \Lambda) \mid \phi \in p\}$.

Let $X \subseteq M^n$ be a definable subset with defining formula ϕ. Then we write $\|X\|_\Lambda = \|\phi\|_\Lambda$. This is well-defined since if ϕ and ψ both define a given subset $X \subseteq M^n$, then $M \models \forall \overline{x}\ \phi(\overline{x}) \longleftrightarrow \psi(\overline{x})$, and so $\|\phi\|_\Lambda = \|\psi\|_\Lambda$.

The following proposition will be used extensively, primarily in the case that $\Lambda = M$ and $X = M^n$. The proof below appears in [Tre99], we have altered it slightly to make more use of characteristic functions.

Proposition 1.3.3. *[Tre99, Theorem 2.1] Let M be an o-minimal expansion of a field, $\Lambda \subseteq M$ a set of parameters, $n \in \mathbb{N}$. Let $X \subseteq M^n$ be Λ-definable. Let D be the ring of Λ-definable functions $X \longrightarrow M$. Then the maps*

$$\begin{aligned} \Theta : \|X\|_\Lambda &\to \mathrm{Spec}(D) \\ p &\mapsto \{f \in D \mid f(\overline{x}) = 0 \in p\} \\ \Psi : \mathrm{Spec}(D) &\to \|X\|_\Lambda \\ \mathfrak{p} &\mapsto \{\phi \mid 1 - \chi_\phi \in \mathfrak{p}\} \end{aligned}$$

are homeomorphisms such that $\Theta \circ \Psi = id_{\mathrm{Spec}(D)}$ and $\Psi \circ \Theta = id_{\|X\|_\Lambda}$.

In particular, if $\phi(\overline{x})$ is a formula in n free variables, then

$$\phi \in p \iff 1 - \chi_\phi \in \Theta(p)$$

where χ_ϕ is the characteristic function of the set in M defined by $\phi(\overline{x})$.

Proof. If $p \in \|X\|_\Lambda$, then $\Theta(p) = \{f \in D \mid f(\overline{x}) = 0 \in p\}$ is prime by maximality of p and the map is well-defined.

Now suppose $\mathfrak{p} \in \mathrm{Spec}(D)$. Then we claim that

$$Th(M, \Lambda) \cup \{\overline{x} \in X\} \cup \{\chi(\overline{x}) = 0 \mid \chi^2 = \chi \in \mathfrak{p}\} \cup \{\chi(\overline{x}) \neq 0 \mid \chi^2 = \chi \notin \mathfrak{p}\}$$

is consistent (and so $\Psi(\mathfrak{p})$ is a maximal, consistent set of formulas). Let $\chi_1, \ldots, \chi_k \in \mathfrak{p}$ and $\chi'_1, \ldots, \chi'_l \notin \mathfrak{p}$. Then let $\chi = \sum \chi_i \in \mathfrak{p}$ and $\chi' = \prod \chi'_j \notin \mathfrak{p}$. So, for all $\overline{a} \in M^n$:

$$\begin{aligned} \chi(\overline{a}) = 0 &\iff \chi_1(\overline{a}) = 0 \wedge \ldots \wedge \chi_k(\overline{a}) = 0 \\ \chi'(\overline{a}) = 0 &\iff \chi'_1(\overline{a}) = 0 \vee \ldots \vee \chi'_k(\overline{a}) = 0 \end{aligned}$$

Since $\chi \cdot \chi' \in \mathfrak{p}$, then $\chi \cdot \chi' \neq \chi'$ and so there is some $\overline{a} \in M^n$ such that $\chi(\overline{a}) = 0$ and $\chi'(\overline{a}) \neq 0$. By compactness, the above set of formulas is consistent. By identifying a characteristic function χ with the set $\{\chi = 0\}$, we see that $\Psi(\mathfrak{p})$ is indeed an n-type and the map Ψ is well defined.

Let $p \in \|X\|_\Lambda$. Then

$$\begin{aligned} \Psi \circ \Theta(p) &= \Psi(\{f \mid f(\overline{x}) = 0 \in p\}) \\ &= \{\phi \mid 1 - \chi_\phi(\overline{x}) = 0 \in p\} \\ &= \{\phi \mid \phi(\overline{x}) \in p\} \\ &= p \end{aligned}$$

Now let $\mathfrak{p} \in \mathrm{Spec}(D)$. Remembering that $f(\overline{x}) = 0 \iff 1 - \chi_{\{f=0\}}(\overline{x}) = 0$, we have:

$$\begin{aligned} \Theta \circ \Psi(\mathfrak{p}) &= \Theta(\{\phi \mid 1 - \chi_\phi \in \mathfrak{p}\}) \\ &= \{f \mid f(\overline{x}) = 0 \in \{\phi \mid 1 - \chi_\phi \in \mathfrak{p}\}\} \\ &= \mathfrak{p} \end{aligned}$$

To show that both Ψ and Θ are homeomorphisms, we notice that for any $f \in D$, $\Psi(D(f)) = \{p \mid f(\overline{x}) \notin p\}$ and for any formula $\phi(\overline{x})$, $\Theta(\|\phi\|) = \{\mathfrak{p} \mid 1 - \chi_\phi \notin \mathfrak{p}\}$. □

1.4 Description of the Content of the Thesis

It is known from [Ast08] that for any real closed field and any n, $D_n^{\mathcal{R}}$ is decidable. This is proven using a sheaf-theoretic variation of the Feferman-Vaught technique [FV59], which is described in [Com74], [Mac73], [Ast08]. From [Tre08], we know that for any $n \geq 2$ and any real closed field $\mathcal{R}$, $C_n^{\mathcal{R}}$ is undecidable and in fact we can interpret $\mathbb{Z}$ in the ring. This also shows that the theory of real closed rings is undecidable. It is an open question whether $C_1^{\mathcal{R}}$ is decidable.

The original intent of the thesis was to investigate the rings $C_1^{\mathcal{R}}$ with an overall view to contributing to a proof of their decidability or undecidability. This has been done in three ways:

In Chapter 2 — The proof of the decidability of $D_n^{\mathcal{R}}$ involves constructing a sheaf representation over what is variously, the Stone space of the boolean algebra of idempotents of $D_n^{\mathcal{R}}$, the type space of $\mathcal{R}$, the Zariski/real spectrum. However, the ring $C_1^{\mathcal{R}}$ has only two idempotents: the functions that are constantly 0 or 1, so the Stone space is just a single point. We cannot give a sheaf representation of $C_1^{\mathcal{R}}$ over $S_1(\mathcal{R})$ since the space is boolean and so is not connected. Therefore the best sheaf representation of $C_1^{\mathcal{R}}$ is over the Zariski spectrum. This leads us to consider the Zariski spectrum of each ring between $C_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$. Chapter 2 contains a complete characterization of the rings between $C_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$. This is done by first considering rings between $C_1^{\mathcal{R}}$ and $V_1^{\mathcal{R}}$ (the ring of definable functions $\mathcal{R} \longrightarrow \mathcal{R}$ that are bounded by continuous functions). The main tool here is the use of characteristic functions; we first show that any finite extension of $C_1^{\mathcal{R}}$ by functions from $V_1^{\mathcal{R}}$ can be generated by extending by characteristic functions, and in fact that any finite extension is a simple extension. A modified version of this technique gives all rings between $C_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$. We prove that all such rings are real closed rings. Next we look at restricting the domain of functions in the ring to a semi-algebraic subset of $\mathcal{R}$, and in particular, by considering $\{f : [0,1] \longrightarrow \mathcal{R} \mid \exists g \in A,\ g\restriction_A = f\}$, for each $C_1^{\mathcal{R}} \subseteq A \subseteq D_1^{\mathcal{R}}$ we obtain a characterization of all rings between $B_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$, where $B_1^{\mathcal{R}}$ is the ring of bounded, continuous semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$. Additionally, we provide a schema of sentences that separates each isomorphism class of finite extensions of $B_1^{\mathcal{R}}$.

In Chapter 3 — Several attempts were made to use the sheaf theoretic Feferman-Vaught technique developed by [Com74], [Mac73], [Ast08] to prove the decidability of $bsa(\mathcal{R})$, the ring of bounded semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$. This chapter first provides an explanation of the sheaf theoretic Feferman-Vaught technique, where we represent a structure as the global sections over some étalé space, which is called a sheaf of $\mathcal{L}$-structures in the literature. In particular we provide an explicit construction of this étalé space from the usual functor definition of a sheaf. We then consider the ring

$$D_n^K\restriction_{\mathcal{R}} = \{f : \mathcal{R}^n \longrightarrow K \mid \exists g \in D_n^K,\ g\restriction_{\mathcal{R}^n} = f\}$$

i.e. the K-definable functions $K^n \longrightarrow K$ restricted to $\mathcal{R}^n$, where K is a tame extension of $\mathcal{R}$ and $n \in \mathbb{N}$. We extend the language of rings to include unary predicates V and $\mathfrak{m}$ which are meant to be interpreted as the valuation ring of K which is the convex hull of $\mathcal{R}$, and the unique maximal ideal of that valuation ring. The $\mathcal{L}_{V,\mathfrak{m}}$-theory of real closed valued fields is decidable and positively model complete. The interpretation of V in $D_n^K\restriction_{\mathcal{R}}$ is the convex hull of the constant functions $\mathcal{R}^n \longrightarrow V$, and is denoted W, similarly for $\mathfrak{m}$, the interpretation of which is denoted $\mathfrak{n}$. Then $W/\mathfrak{n} = bsa(\mathcal{R}^n)$. The sheaf construction over the Zariski spectrum then gives a sheaf of $\mathcal{L}_{V,\mathfrak{m}}$-structures $A_{\mathfrak{p}}$, each of which is a field, and such that the interpretation of V in $A_{\mathfrak{p}}$ is the convex hull of $\mathcal{R}$. However, the interpretation of $\mathfrak{m}$ is not the maximal ideal of the valuation ring V. This means that the normal method of the Feferman-Vaught technique does not work here. The reason behind this is due to a problem with the coheirs of types of $\mathcal{R}$ in K. We give a characterization of the Zariski spectrum of $D_n^K\restriction_{\mathcal{R}}$ and show how the problem arises due to the coheirs.

Chapter 4 — In the paper [GMP04], it is claimed that the free vector lattice (free Riesz space) on two generators is decidable. This vector lattice is isomorphic, by [BM02], to the vector lattice of semilinear functions $f : [0,1] \longrightarrow [0,1]$ such that $f(0) = f(1)$. Based on this approach, we began modifying the result to prove that the vector lattice of $\mathcal{L}_R(\mathbb{R})$-definable functions is decidable. However, we discovered a flaw in the proof, that we give in the thesis. This means that the decidability result cannot be reached by this method. The basic approach was to find an effective method to determine whether there was a continuous definable choice function from a 1-dimensional definable set X to a definable set $S \subseteq X \times M$. However, we would also require the ability to guarantee that our choice function h takes its values in certain subsets $T_1, \ldots, T_k$. This would allow us to remove existential quantifiers in the decidability proof. However, we do prove several results about definable choice functions. We hope that this will help lead to the construction of an effective procedure which decides whether there is a continuous definable choice function $X \longrightarrow S$, where $X \subseteq M^m$ is of arbitrary dimension and $S \subseteq X \times M^n$.

Chapter 2

Classification of rings between C_1 and D_1

2.1 Preliminaries

In this chapter we will classify all rings between $C_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$, i.e. all rings A such that $C_1^{\mathcal{R}} \subseteq A \subseteq D_1^{\mathcal{R}}$. In fact, we show that once this has been done, it is an easy step to classify all rings between $B_1^{\mathcal{R}}$ — the bounded, continuous semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$ — and $D_1^{\mathcal{R}}$. To begin, we look at rings between $C_1^{\mathcal{R}}$ and $V_1^{\mathcal{R}}$ — the ring of semi-algebraic functions bounded by continuous functions, this is also the convex hull of $C_1^{\mathcal{R}}$ in $D_1^{\mathcal{R}}$ under the pointwise ordering. Results from this section are used to classify rings between $C_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$. We provide a complete classification of the prime ideals of all rings between $C_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$, and of their Zariski spectra. This allows us to look at rings of the form $C^{\mathcal{R}}(I)$, the continuous semi-algebraic functions $I \longrightarrow \mathcal{R}$, where I is some interval. Knowing that $B_1^{\mathcal{R}} \cong C^{\mathcal{R}}([0,1])$ then allows to expand our results to include rings between $B_1^{\mathcal{R}}$ and $D_1^{\mathcal{R}}$. Additionally, we find sentences that separate $B_1^{\mathcal{R}}$ and $C_1^{\mathcal{R}}$.

Notation 2.1.1. Throughout we let $\mathcal{R}$ be a fixed real closed field. Consequently, we suppress the superscript $\mathcal{R}$ in the rings $C_1^{\mathcal{R}}$, $D_1^{\mathcal{R}}$, etc.

Definition 2.1.2. Let C_1 denote the ring of continuous semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$. Let D_1 denote the ring of semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$.

For a semi-algebraic set $X \subseteq \mathcal{R}$, let $C(X)$ be the ring of continuous semi-algebraic functions $X \longrightarrow \mathcal{R}$. Let $D(X)$ denote the ring of semi-algebraic functions $X \longrightarrow \mathcal{R}$.

Definition 2.1.3. Let $f : \mathcal{R} \longrightarrow \mathcal{R}$. Then f has a **point discontinuity** at $d \in \mathcal{R}$ if $\lim_{t \nearrow d} f(t) \neq f(d) \neq \lim_{t \searrow d} f(t)$. Furthermore, f has a **removable discontinuity** at $d \in \mathcal{R}$ if d is a point discontinuity and $\lim_{t \nearrow d} f(t) = \lim_{t \searrow d} f(t)$ and the limit is bounded; f has an **isolated discontinuity** at $d \in \mathcal{R}$ if d is a point discontinuity and $\lim_{t \nearrow d} f(t) \neq \lim_{t \searrow d} f(t)$ or if either limit is unbounded. We say that f has a **left essential discontinuity** at d if $\lim_{t \nearrow d} f(t) = \pm\infty$, it has a **right essential discontinuity** at d if $\lim_{t \searrow d} f(t) = \pm\infty$ and a **double essential discontinuity** at d if it has both a left and right essential discontinuity at d. We call right or left essential discontinuities **positive right essential** or **negative right essential** etc. depending on the sign of the limit. If $\lim_{t \nearrow d} f(t) = f(d) \neq \lim_{t \searrow d} f(t)$ we say that f is **left continuous and right discontinuous** at d, similarly for right continuous.

Definition 2.1.4. Let $f, g \in D_1$, let $U \subseteq \mathcal{R}$. Then we say $f = g$ on U if $f(r) = g(r)$ for all $r \in U$. We write this $f\restriction_U = g\restriction_U$.

Definition 2.1.5. Let $X \subseteq \mathcal{R}$ be a semi-algebraic interval $\langle a, b\rangle$, where $\langle , \rangle$ indicate it may be open or closed, and where we allow $a = -\infty$ or $b = +\infty$. The the **midpoint** of X is:

- $\frac{a+b}{2}$ if a and b are finite
- $b - 1$ if $a = -\infty$ and b is finite
- $a + 1$ if a is finite and $b = \infty$
- 0 if $a = -\infty$ and $b = \infty$

For a given interval $\langle a, b\rangle$, we write the midpoint as $e(\langle a, b\rangle)$.

Definition 2.1.6. We will often be dealing with an interval on which a function has removable discontinuities. For convenience, we define an **almost interval** to be either an interval (closed or open) or an interval with finitely many points removed.

Definition 2.1.7. Let $f : \mathcal{R} \longrightarrow \mathcal{R}$ be a semi-algebraic function. Then a non-empty $X \subseteq \mathcal{R}$ is an **f-set** if X is an almost interval such that f is continuous on X and wherever $d \in \operatorname{int}(\overline{X}) \setminus X$, d is a removable discontinuity of f. We say that X is a **maximal f-set** if it is not contained in a larger f-set.

Remark 2.1.8. Note that the closure of an f-set (and of an almost interval) is an interval.

Proposition 2.1.9. *Let $f \in D_1$ and let X be an f-set. Then X lies in a unique maximal f-set.*

Proof. Let Y be the union of the f-sets containing X. If Y is an f-set, then clearly it is maximal. Y is a union of almost intervals, in fact, the points removed must be discontinuity points of f, or else we could find a larger f-set containing such a point. Since f is semi-algebraic, there are only finitely many discontinuity points, and Y is a union of almost intervals all containing a point, so Y itself must be an almost interval. At any given point $a \in Y$, f must be continuous, since, by the construction of Y, there must be some f-set $X' \supseteq X$ with $a \in X'$. So Y is an f-set. □

2.2 Rings between C_1 and V_1

We look at extensions of C_1 by adding characteristic functions of semi-algebraic sets of $\mathcal{R}$. This will allow us to characterize all rings between C_1 and V_1. Results from this section are used to characterize all rings between C_1 and D_1 in the next section.

Notation 2.2.1. Characteristic functions are denoted $\chi, \chi_1, \chi_2, \ldots$, their supports, or base sets, are denoted by $X, X_1, X_2, \ldots$.

Definition 2.2.2. The ring V_1 is the set of all semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$ which are bounded above and below by continuous functions.

Remark 2.2.3. The ring V_1 contains all semi-algebraic functions without poles, alternatively it is the convex hull of C_1 in D_1. It contains functions which aren't bounded at $\pm\infty$, e.g. x^2, but all functions in V_1 are bounded on any bounded interval of $\mathcal{R}$.

Definition 2.2.4. Let $X \subseteq \mathcal{R}$ be definable. Then for any $a \in \mathcal{R}$, we say $a < X$ if for all $x \in X$, $a < x$. Similarly for $a > X$.

Proposition 2.2.5. *Each $f \in V_1$ can be written uniquely as $f = \sum_{i=1}^{i=n} f_i.\chi_i$, for some $n \in \mathbb{N}$, such that*

- *$\{X_i\}$ is a semi-algebraic partition of $\mathcal{R}$*

- *each* X_i *is a maximal* f*-set, so is either a point or an almost interval*
- $i < j$ *if and only if there is* $x \in X_i$ *such that* $x < X_j$
- $f_i \in C_1$*, for all* i
- $f_i \restriction_{X_i} = f \restriction_{X_i}$*, for all* i
- *if* $y \in \mathcal{R}$ *and* $y < X_i$*, then* $f_i(y) = f_i(\inf\{X_i\})$*, or if* $y > X_i$*, then* $f_i(y) = f_i(\sup\{X_i\})$*, for all* i

Proof. Let $f \in V_1$. Either f is continuous, in which case $X_1 = \mathcal{R}$ and we're done or, by o-minimality, f has finitely many discontinuities, let these be $d_1 < \ldots < d_k$, for some $k \in \mathbb{N}$. Each of these d_i is one of four distinct types: (1) it is removable, (2) it is isolated, (3) $\lim_{t \nearrow d_i} f(t) = f(d_i)$ (and so $\lim_{t \searrow d_i} f(t) \neq f(d_i)$), (4) $\lim_{t \searrow d_i} f(t) = f(d_i)$. We define sets $A_0 = (-\infty, d_1)$, then $A_i = (d_i, d_{i+1})$ for $1 \leq i < k$, and $A_k = (d_k, \infty)$.

Now if d_i is of type (4), set $A'_i := [d_i, d_i + 1)$, otherwise $A'_i := A_i$. Then if d_{i+1} is of type (3), set $A''_i := A'_i \cup \{d_{i+1}\}$, else $A''_i := A'_i$. If $d_i, d_{i+1}, \ldots, d_{i+j}$ are of type (1), then let $B_i = A''_i \cup \ldots \cup A''_{i+j}$, $B_{i+p} = \emptyset$, for $1 \leq p \leq j$, else $B_i = A''_i$. Now let $X_1, \ldots, X_n$ be the non-empty B_i, together with the discontinuity points of types (1) and (2), ordered such that there is $x \in X_i$ such that $x < y$ for all $y \in X_j$ if and only if $i < j$, which is possible by o-minimality.

We can define f_i as desired, and $X_1, \ldots, X_n$ are the maximal f-sets. This is unique since maximal f-sets are unique. □

Definition 2.2.6. Suppose we have $f \in V_1$ and we write $f = \sum f_i \chi_i$ satisfying the conditions of the statement of Proposition 2.2.5. Then we call $\sum f_i \chi_i$ the **standard representation** of f.

Proposition 2.2.7. *Let* $\chi_1, \ldots, \chi_n$ *be an arbitrary, finite set of characteristic functions, with semi-algebraic base sets* $X_1, \ldots, X_n$*. Let* $Y_1, \ldots, Y_m$ *be the atoms of the boolean algebra generated by* $X_1, \ldots, X_n$*. Then*

$$C_1[\chi_1, \ldots, \chi_n] = C_1[\chi_{Y_1}, \ldots, \chi_{Y_m}]$$

Proof. This holds since we can generate the characteristic function of any element in the boolean algebra, since for any sets A, B: $\chi_{\mathcal{R} \setminus A} = 1 - \chi_A$, $\chi_{A \cap B} = \chi_A . \chi_B$ and $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A . \chi_B$. Clearly $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ generate the same boolean algebra. □

Proposition 2.2.8. *Let χ be the characteristic function of a definable set X. Let $d_1, \dots, d_k$ be the removable discontinuity points of χ, i.e. $d_i \in \int(overlineX) \setminus X$ for all i. Then $C_1[\chi]$ is the set of $f \in V_1$ such that f is continuous wherever χ is, and for each i, $\lim_{t \nearrow d_i} f(t) = \lim_{t \searrow d_i} f(t)$.*

Proof. If $f \in C_1[\chi]$, then we can find $f_1, f_2 \in C_1$ such that $f = f_1 + f_2.\chi = (f_1 + f_2)\chi + f_1(1-\chi)$. Clearly the inclusion $\subseteq$ holds.

Now suppose $f \in V_1$ and f is continuous wherever χ is, and for each i, $\lim_{t \nearrow d_i} f(t) = \lim_{t \searrow d_i} f(t)$. Let $A_1, \dots, A_n$ be the connected components of $\overline{X}$, and let a_i, b_i be the lower and upper bounds of each A_i, with the possibility that $a_1 = -\infty$ and $b_n = +\infty$ or $a_i = b_i$. Then let $g = f$ on X. For $d_i \notin X$, let $g(d_i) = \lim_{t \to d_i} f(t)$ and for $x \in \mathcal{R} \setminus \overline{X}$:

$$g = \frac{f(a_{i+1}) - f(b_i)}{a_{i+1} - b_i} x + \frac{f(b_i)a_{i+1} - f(a_{i+1})b_i}{a_{i+1} - b_i}$$

or where $(-\infty, a_1) \nsubseteq X$ or $(b_n, \infty) \nsubseteq X$, then g is constantly $f(a_1)$ or $f(b_n)$ on the respective interval. Thus g is the function equal to f on X, for $d_i \notin X$, $g(d_i) = \lim_{t \to d_i} f(t)$ and on $\mathcal{R} \setminus (X \cup \{d_1, \dots, d_k\})$ it is equal to the straight line segment between $f(b_i)$ and $f(a_{i+1})$, or constant on unbounded intervals. We similarly define h, letting $A'_1, \dots, A'_{n'}$ be the connected components of $\overline{\mathcal{R} \setminus X}$, with $h = f$ on $\mathcal{R} \setminus X$ etc.

Thus $f = g.\chi + h.(1-\chi) \in C_1[\chi]$. □

Proposition 2.2.9. *Let $X_1, \dots, X_n$ be semi-algebraic sets and let $\chi_1, \dots, \chi_n$ be their respective characteristic functions. Then there is a bounded, semi-algebraic function h, such that $C_1[h] = C_1[\chi_1, \dots, \chi_n]$.*

Proof. Using Proposition 2.2.7, it is sufficient to consider the atoms of the boolean algebra generated by $X_1, \dots, X_n$. Now we have a finite partition of $\mathcal{R}$, $Y_1, \dots, Y_m$, say, which are the atoms of this boolean algebra. Define h by $h = \sum_{k=1}^{k=m} k\chi_{Y_k}$. So

$$\prod_{k=1}^{k=m-1} (h-k)(x) = \begin{cases} (m-1)! & \text{if } x \in Y_m \\ 0 & \text{if } x \in Y_1, \dots, Y_{m-1} \end{cases}$$

Then by rescaling by $1/(m-1)!$, we obtain χ_{Y_m}. Now we have

$$\prod_{k=1}^{k=m-2} (h-k)(x) - (m-1)!\chi_{Y_m} = (m-2)!\chi_{Y_{m-1}}$$

to obtain $\chi_{Y_{m-1}}$.

So, for $l < m-1$, having obtained $\chi_{Y_m}, \ldots, \chi_{Y_{m-l}}$, we obtain $\chi_{Y_{m-l-1}}$ by:

$$\prod_{k=1}^{k=m-l-2} (h-k)(x) - \sum_{i=0}^{i=l} \frac{(m-l+i-2)!}{(i+1)!} \chi_{Y_{m-1+i}} = (m-l-2)! \chi_{Y_{m-l-1}}$$

□

Proposition 2.2.10. *Let $f \in V_1$. Then there are $\chi_1, \ldots, \chi_n$, for some n, such that $C_1[f] = C_1[\chi_1, \ldots, \chi_n]$. Furthermore, each of $X_1, \ldots, X_n$ is either a point, or an almost interval*

Proof. Since $f \in V_1$, it has a standard representation $f = \sum_{1 \le i \le n} f_i \chi_i$ for some $n \in \mathbb{N}$. We claim that these are the required sets and characteristic functions. Clearly $f \in C_1[\chi_1, \ldots, \chi_n]$ and so the inclusion $\subseteq$ holds.

We can obtain the characteristic functions of the isolated discontinuity as follows: let f have an isolated discontinuity at d, let $X_i = \{d\}$, and so X_{i-1}, X_{i+1} are the sets in the standard representation either side of d, with $\overline{X_{i-1}} \cap \overline{X_{i+1}} = \{d\}$ (if d is a removable discontinuity, then replace $i+1$ with $i-1$ throughout). Then $(f - f_{i-1})(f - f_{i+1})$ is non-zero at d, and zero on $X_{i-1} \cup X_{i+1}$. Furthermore, we may assume, by multiplication by a constant, that $(f - f_{i-1})(f - f_{i+1})(d) = 1$. So there is some open set $(d-\delta, d) \cup (d, d+\delta)$ on which $(f - f_{i-1})(f - f_{i+1}) = 0$. Now define:

$$\Lambda(t) = \begin{cases} 0 & \text{if } t \le d - \delta \\ \frac{1}{\delta}(t - (d-\delta)) & \text{if } d - \delta < t \le d \\ \frac{1}{\delta}(-t + d + \delta) & \text{if } d < t \le d + \delta \\ 0 & \text{if } d + \delta < t \end{cases}$$

so $\Lambda(t).(f - f_{i-1})(f - f_{i+1})(t) = \chi_{\{d\}}$. So $C_1[f]$ contains the characteristic functions of all the isolated discontinuities of f. Thus by adding appropriately scaled characteristic functions for each of these points, we create f', which is equal to f except at each isolated discontinuity, d, where $f'(d) = \lim_{t \nearrow d} f(t)$, so let the standard representation be $f' = \sum f_i \chi_{Y_i}$, say.

Let a_i be the supremum of each Y_i for each $i < n$, where n is the number of intervals in the standard representation of f'. Then we may assume, wlog, that for each i, $\lim_{t \searrow a_{i-1}} f_i(t) < f_{i+1}(a_i)$ — if not, we can take $f'' = f' + \lambda x$, where λ is a suitable constant. Then by the construction of the standard representation, $\lim_{t \searrow a_{i-1}} f_i(t) < f_{i+1}(a_{i-1}) = f_{i+1}(a_i)$. To obtain the characteristic functions for each of the Y_i, we do the same trick on the intervals: By construction, each Y_i is an interval with endpoints a and b. So $(f' - f_{i-1})^2(f' - f_{i+1})^2$ is nowhere negative, is zero on $Y_{i-1} \cup Y_{i+1}$ and is such that $\lim_{t \searrow a}(f' - f_{i-1})(f' - f_{i+1})(t) \neq 0 \neq \lim_{t \nearrow b}(f' - f_{i-1})(f' - f_{i+1})(t)$ (by the the first two sentences of this paragraph). So there is some $0 < \delta < \frac{b-a}{2}$ such that $(f' - f_{i-1})^2(f' - f_{i+1})^2$ is non-zero on $(a, a+2\delta)$ and $(b-2\delta, b)$. Now, letting $\kappa = \sup_{t \in Y_i}\{(f' - f_{i-1})^2(f' - f_{i+1})^2\} + 1$ define:

$$\Lambda_i(t) = \begin{cases} 0 & \text{if } t \leq a+\delta \\ \frac{\kappa}{\delta}(t-(a+\delta)) & \text{if } a+\delta < t \leq a+2\delta \\ \kappa & \text{if } a+2\delta \leq t < b-2\delta \\ \frac{\kappa}{\delta}(-t+b-\delta) & \text{if } b-2\delta \leq t < d+\delta \\ 0 & \text{if } b-\delta \leq t \end{cases}$$

Now $(f - f_{i-1})^2(f - f_{i+1})^2 + \Lambda_i$ is zero on $Y_{i-1} \cup Y_{i+1}$ and bounded below by some $\epsilon > 0$ on Y_i. Define:

$$g(t) = \begin{cases} 1/(\lim_{t \searrow a}((f - f_{i-1})^2(f - f_{i+1})^2 + \Lambda_i)) & \text{if } t \leq a \\ ((f - f_{i-1})^2(f - f_{i+1})^2 + \Lambda_i)^{-1} & \text{if } a < t < b \\ 1/(\lim_{t \searrow a}((f - f_{i-1})^2(f - f_{i+1})^2 + \Lambda_i)) & \text{if } b \leq t \end{cases}$$

Now g is continuous and $g.((f - f_{i-1})^2(f - f_{i+1})^2 + \Lambda_i) := h$ is 1 on Y_i and 0 on $Y_{i-1} \cup Y_{i+1}$. We now pick δ' such that $(a-\delta', a) \subseteq Y_{i-1}$ and $(b, b+\delta') \subseteq Y_{i+1}$ and define:

$$\Lambda_i'(t) = \begin{cases} 0 & \text{if } t \leq a-\delta' \\ \frac{1}{\delta}(t-(a-\delta')) & \text{if } a-\delta' < t \leq a \\ 1 & \text{if } a < t \leq b \\ \frac{1}{\delta}(-t+b+\delta') & \text{if } b < t \leq b+\delta' \\ 0 & \text{if } b+\delta' < t \end{cases}$$

Thus $h.\Lambda_i' = \chi_{Y_i}$.

In this way we obtain the characteristic functions of all Y_i. By subtracting the characteristic function of appropriate isolated discontinuities, we can obtain the characteristic functions for all X_i. □

Definition 2.2.11. We shall refer to a function of a similar form to Λ and Λ_i' as a **lambda function**, i.e. a continuous semi-algebraic function f that is zero except on some bounded interval $(d-\delta, d+\delta)$ where there is some $r > 0$ and

f is the linear interpolation between $\langle d-\delta, 0\rangle$ and $\langle d, r\rangle$ and between $\langle d, r\rangle$ and $\langle d+\delta, 0\rangle$; or there is some $0 < \delta' < \delta$ and f is the linear interpolation between $\langle d-\delta, 0\rangle$ and $\langle d-\delta', r\rangle$, $\langle d-\delta', r\rangle$ and $\langle d+\delta', r\rangle$ and between $\langle d+\delta', r\rangle$ and $\langle d+\delta, 0\rangle$. The name derives from the resemblance of the functions to the letter Λ.

Corollary 2.2.12. *Let $f_1, \dots, f_n \in V_1$. Then there is a bounded, semi-algebraic function h such that $C_1[f_1, \dots, f_n] = C_1[h]$. In other words, all finite extensions of C_1 by functions in V_1 are simple extensions.*

Proof. Just combine the previous two results. $\square$

Corollary 2.2.13. *Let $X_1, \dots, X_n$ be a finite collection of semi-algebraic sets. Then we can find $Y_1, \dots, Y_m$ such that $C_1[\chi_1, \dots, \chi_n] = C_1[\chi_{Y_1}, \dots, \chi_{Y_m}]$, where each of the Y_i is either a point or an almost interval, and $Y_1, \dots, Y_m$ partition $\mathcal{R}$.*

Proof. We notice that each characteristic function χ_i itself has a standard representation and use Proposition 2.2.10. $\square$

Proposition 2.2.14. *Let $X_1, \dots, X_n$ be a finite collection of semialgebraic sets of $\mathcal{R}$. Then there are Y_1, Y_2 such that $C_1[\chi_1, \dots, \chi_n] = C_1[\chi_{Y_1}, \chi_{Y_2}]$. In particular, in the boolean algebra generated by $X_1, \dots, X_n$, if at least one of the atoms X_i' is a point and there are X_j', X_k' such that $X_i' \subseteq \overline{X_j'} \cap \overline{X_k'}$, then there is no Y such that $C_1[\chi_1, \dots, \chi_n] = C_1[\chi_Y]$. Otherwise there is Y such that $C_1[\chi_1, \dots, \chi_n] = C_1[\chi_Y]$.*

Proof. Suppose, wlog, that $X_1, \dots, X_n$ partition $\mathcal{R}$ (so are the atoms of a boolean algebra of subsets of $\mathcal{R}$) and are indexed such that

$$\exists x_i \in X_i \forall x_j \in X_j \; x_i < x_j \iff i < j$$

this is always possible by o-minimality. We may also assume that each X_i is either a point or an almost interval, since we can take the standard representation of each χ_i.

We deal with the two cases separately: Suppose that there are X_j, X_k, X_l such that X_j is a point and $X_j \subseteq \overline{X_k} \cap \overline{X_l}$, in other words, X_j is the non-removable, isolated discontinuity of, say, $\chi_j + 2\chi_k + 3\chi_l$. By Proposition 2.2.8, if there is a single Y such that $C_1[\chi_1, \dots, \chi_n] = C_1[\chi_Y]$, then any function $f \in C_1[\chi_Y]$ has only removable discontinuities; this contradicts the hypothesis of the previous sentence, so there are at least two Y_1, Y_2 that generate the ring $C_1[\chi_1, \dots, \chi_n]$.

We let Y_1 be the union of all X_i that are points, so $\mathcal{R} \setminus Y_1$ is the union of all X_i that are almost intervals. Now consider the sets $X_{j_1}, \ldots, X_{j_m}$, with $k < l \Leftrightarrow j_k < j_l$, which are all X_i which are almost intervals, and let Y_2 be the union of all Y_{j_i} where i is odd, i.e. we take alternating intervals on the line. By construction, $\chi_{Y_1}, \chi_{Y_2} \in C_1[\chi_1, \ldots, \chi_n]$.

We now show that each $\chi_i \in C_1[\chi_{Y_1}, \chi_{Y_2}]$. To obtain the characteristic functions of points, we use appropriate Λ functions as in Proposition 2.2.10. For an almost interval, X_{j_i}, say, with i odd, then χ_{Y_2} has value 1 on X_{j_i} and is 0 on $X_{j_{i-1}}$, $X_{j_{i+1}}$. A similar trick obtains the characteristic function of X_{j_i}; similarly when i is even.

Now suppose that are no X_j, X_k, X_l such that X_j is a point and $X_j \subseteq \overline{X_k} \cap \overline{X_l}$, in other words, each X_j which is a point is the removable discontinuity of some χ_i. In a similar way to the previous case, consider the sets $X_{j_1}, \ldots, X_{j_m}$, with $k < l \Leftrightarrow j_k < j_l$, which are all the X_i which are almost intervals, and let Y' be the union of all Y_{j_i} where i is odd. Now let Y be the union of Y' together with all the X_k which are points and such that there is some $X_{j_i} \subseteq Y'$ with $X_i \subseteq \overline{X_{2j_i}}$. In other words Y contains alternate almost intervals; whenever an interval with finitely many points removed is contained in Y then the removed points aren't in Y; and whenever an interval with finitely many points removed is not contained in Y then the removed points are in Y. Again, we obtain the characteristic functions of the removable singularities by looking at χ_Y or $(1 - \chi_Y)$ and using appropriate Λ functions. □

Proposition 2.2.15. *Let $X_1, \ldots, X_n$ be a semi-algebraic partition of $\mathcal{R}$. Let*

$$\phi_i : C_1 \longrightarrow C(X_i), f \mapsto f\restriction_{X_i}$$

where $C(X_i)$ is the ring of continuous, semialgebraic maps $X_i \longrightarrow \mathcal{R}$, for each i. Then

$$C_1[\chi_1, \ldots, \chi_n] \cong \prod \phi_i(C_1) \cong \prod C(\overline{X_i})$$

and it is a real closed ring.

Proof. The homomorphism

$$\begin{aligned} C_1[\chi_1, \ldots, \chi_n] &\longrightarrow \prod \phi_i(C_1) \\ f &\mapsto (\phi_1(f), \ldots, \phi_n(f)) \end{aligned}$$

is clearly surjective. It is also injective, since if $(\phi_1(f), \ldots, \phi_n(f)) = 0$, then $f(x) = 0$ for all $x \in \bigcup X_i = \mathcal{R}$. This gives the left isomorphism.

For the isomorphism on the right-hand side, we see that $\phi_i(C_1) \cong C(\overline{X_i})$, for each i, since continuity preserves values at limits, and since we are working in the bounded case, these limits exist.

Each $C(\overline{X_i})$ is a real closed ring and since $C_1[\chi_1, \ldots, \chi_n]$ is isomorphic to a product of real closed rings, $C_1[\chi_1, \ldots, \chi_n]$ is also a real closed ring. □

Proposition 2.2.16. *Let $f \in V_1$. Then there exist unique $k, l \in \mathbb{N}$, $p \in \{0, 2\}$ and $q \in \{0, 1\}$ such that*

$$C_1[f] \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^p \times C_1^q$$

Proof. We can write f in its standard representation $f = \sum_{1 \leq i \leq n} \chi_i {\cdot} f$, where χ_i is the characteristic function of X_i and the $X_1, \ldots, X_n$ are ordered as before. Then by Prop 2.2.10, $C_1[f] = C_1[\chi_1, \ldots, \chi_n]$, and by Prop 2.2.15, we have that

$$C_1[f] \cong \prod C(\overline{X_i})$$

But each of the $\overline{X_i}$ is connected and is one of the following:

(1) a point, and $C(\overline{X_i}) \cong \mathcal{R}$

(2) a bounded interval $[a, b]$ for some $a, b \in \mathcal{R}$, and $C_1(\overline{X_i}) \cong C([0, 1])$

(3) a half-unbounded interval $(-\infty, b]$ or $[a, \infty)$ for some $a, b \in \mathcal{R}$, and $C(\overline{X_i}) \cong C([0, \infty))$

(4) the entire line $\mathcal{R}$, and $C(\overline{X_i}) \cong C_1$

If we have $\overline{X_1} = \mathcal{R}$, then the other X_i can only be points and f just has finitely many removable discontinuities, so

$$C_1[f] \cong \mathcal{R}^k \times C_1$$

for some $k \in \mathbb{N}$. If $\overline{X_1} \neq \mathcal{R}$, then we must have $\overline{X_1} = (-\infty, a]$ and $\overline{X_k} = [b, \infty)$ for some $a, b \in \mathcal{R}$, $k \leq n$. Then $p = 2$ and $q = 0$. So

$$C_1[f] \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^2$$

where k is the number of removable and isolated discontinuities of f and l is the number of maximal f-sets. $\square$

Proposition 2.2.17. *Let A be a ring such that $C_1 \subseteq A \subseteq V_1$. Then A is a real closed ring.*

Proof. We define the real closed ring structure on A as usual: $f_A(a_1, \dots, a_k) = f \circ (a_1, \dots, a_k)$ for $f : \mathbb{R}^k_{\text{alg}} \longrightarrow \mathbb{R}_{\text{alg}}$ and $a_1, \dots, a_n \in A$. Clearly $+, -, ., 0, 1$ take their appropriate values, and since V_1 is a real closed ring with similarly defined f_A, composition of functions is preserved. It only remains to show that f_A has range A. Let $a_1, \dots, a_n \in A$. Then $C_1[a_1, \dots, a_n] \subseteq A$ and, since $C_1[a_1, \dots, a_n]$ is a real closed ring, $f_A(a_1, \dots, a_n) \in C_1[a_1, \dots, a_n]$. $\square$

Proposition 2.2.18. *Let A be a ring such that $C_1 \subseteq A \subseteq V_1$. Then there exists a boolean subalgebra $\mathcal{X}$ of the boolean algebra of semi-algebraic subsets of $\mathcal{R}$, such that $A = C_1[\{\chi_X\}_{X \in \mathcal{X}}]$.*

Proof. Let $\bigcup_{f \in A} \bigcup \{X_i\} = \mathcal{X}$ be the set of all the base sets of the standard representation of each $f \in A$. This is a boolean algebra, since for any $X_1, X_2 \in \mathcal{X}$, we must have $f_1, f_2 \in A$ such that X_1, X_2 are base sets in their respective standard representations. So $\chi_1, \chi_2 \in A$, and we are able to generate the characteristic sets of $X_1 \cap X_2, X_1 \cup X_2, \mathcal{R} \setminus X_1$. $\square$

2.3 Rings between C_1 and D_1

Now let $f \in D_1$, the set of semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$. We first look at a way to characterize the ring $C_1[f]$.

Proposition 2.3.1. *[vdD86a, p.192] Let $f \in D_1$, such that f is not constantly* 0 *on any interval (a, ∞). Then there are constants $c \in \mathcal{R}$ and $q \in \mathbb{Q}$ such that f is asymptotically equivelent to $c \cdot x^q$, i.e. $\lim_{t \to \infty} \frac{f(x)}{cx^q} = 1$. We write this $f \sim cx^q$.*

By change of variables, let $d \in \mathcal{R}$, if $\lim_{t \searrow d} f(t) = \infty$, then there is $c \in \mathcal{R}$ and $q \in Q$ such that $\lim_{t \searrow d} \frac{f(x)}{c(x-d)^q} = 1$.

Proposition 2.3.2. *Let $f \in D_1$, then f can be written as $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i} \chi_i$, for some $n \in \mathbb{N}$, such that*

- *$\{X_i\}$ is a semi-algebraic partition of $\mathcal{R}$*

- *each X_i is a maximal f-set, and so is either a point, an interval or an interval with finitely many points removed*
- *$i < j$ if and only if there is $x \in X_i$ such that $x < y$ for all $y \in X_j$*
- *$g_i, h_i \in C_1$, for all i*
- *$\frac{g_i(x)}{h_i(x)} = f(x)$, for all $x \in X_i$, for all i*
- *$h_i(x) \neq 0$, for all $x \in X_i$*

Proof. Let $f \in D_1$. Either f is continuous, in which case $X_1 = \mathcal{R}$ and we're done or $f \in V_1$, in which case we're done by Proposition 2.2.5, or f has at least one essential discontinuity. We create $\{X_i\}$ as before, and where we have an essential discontinuity (at d, say), we treat it as though it were a discontinuity with $\lim_{t\nearrow d} f(t) \neq \lim_{t\searrow d} f(t)$, with $X_i = \{d\}$ for some i if d is an isolated discontinuity.

If f is bounded on X_i, then it has a standard representation, with $h_i = 1$, and g_i the same as f_i in the notation of Proposition 2.2.5. If we have a positive right essential discontinuity d, say, then by Proposition 2.3.1 we have some $0 \neq c \in \mathcal{R}$ and $0 < q \in \mathbb{Q}$ such that $\lim_{t\searrow d} f(x)c(x-d)^q = 1$. Then also there is some $\delta > 0$ such that $f(x) \cdot (x-d)^q$ is bounded on $(d, d+\delta)$. Let m be the midpoint of X_i, then $f(x) \cdot (x-d)^q$ is a bounded on all of (d, m) and (except possibly at removable discontinuities) is a continuous semi-algebraic function on (d, m). Now let $g_i = \frac{(x-d)^q}{(m-d)^q} f$ and $h_i = \frac{(x-d)^q}{(m-d)^q}$ on (d, m). Letting $d' = \sup(X_i)$, on (m, d'), let $h_i = 1$ and $g_i = f$, unless we have a left essential discontinuity at d', in which case we define g_i and h_i similarly to the case of the right essential discontinuity. For $x < X_i$, we define $g_i(x) = \lim_{t\searrow d} g_i(t) = \frac{1}{c}$, similarly for h_i and for $x > X_i$.

Then each $g_i, h_i \in C_1$ for each i, and $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i} \chi_i$. □

Definition 2.3.3. The representation of f given in the above proof is called the **standard representation** of f. By the construction given in the proof, it is unique.

Proposition 2.3.4. *Let $f \in D_1$. Then writing f in its standard representation $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i} \chi_i$, for some $n \in \mathbb{N}$, we have $C_1[f] = C_1[\chi_1, \ldots, \chi_n][\frac{1}{h}]$ for some $h \in V_1$.*

Proof. Let $X_1, \ldots, X_n$ be the maximal f-sets given by the standard representation. Let $h = \sum_{i=1}^{i=n} h_i \chi_i$ — and so $\frac{1}{h} = \sum_{i=1}^{i=n} \frac{1}{h_i} \chi_i$.

First we show: $C_1[f] \subseteq C_1[\chi_1, \ldots, \chi_n][\frac{1}{h}]$. It is sufficient to show $C_1[\frac{g_i}{h_i}\chi_i] \subseteq C_1[\chi_i][\frac{1}{h}]$. This is clearly true since $g_i \in C_1$ and $\frac{1}{h_i}\chi_i = \frac{1}{h}\chi_i$.

For the opposite inclusion, we must first show that $C_1[f] \supseteq C_1[\chi_1, \ldots, \chi_n]$. For this section with characteristic functions, we assume that $f \geq 1$ on $\mathcal{R}$, if not, with $d_1, \ldots, d_k$ denoting the discontinuities, let D be the maxima of: $\left|f(d_i)\right|$, and $\left|\lim_{t \nearrow d_i} f(t)\right|$ and $\left|\lim_{t \searrow d_i} f(t)\right|$ (where such limits are finite). Then $(f+D+1)^2+1$ is bounded below by 1, and is discontinuous at exactly $d_1, \ldots, d_k$, with the same type of discontinuity for each d_i (negative essential discontinuities are now positive discontinuities, but this does not matter). Using the same technique as in the proof of Proposition 2.2.10 we can generate the characteristic functions of the removable singularities (by using Λ functions on suitably small neighbourhoods of removable discontinuities we just restrict to the bounded case). So we may assume that $X_1, \ldots, X_n$ are intervals or points (by adding appropriately scaled characteristic functions of removable discontinuities).

By construction and assumption of $X_1, \ldots, X_n$, $X_1 = (-\infty, d_1)$ or $(-\infty, d_1]$. If d is neither a right nor a left essential discontinuity, then we can generate χ_1 as in Proposition 2.2.10. If d is a left essential discontinuity, then define $f' = \frac{1}{f}$ on X_1, $f' = 0$ on $\mathcal{R} \setminus X_1$, so $f' \in C_1$ and $f'.f = \chi_1$. If d is a right essential discontinuity but not a left essential discontinuity, then we can find δ such that f is continuous on $(d, d+\delta]$, now define

$$f'(t) = \begin{cases} 0 & \text{if } t \leq d \\ \frac{1}{f} & \text{if } d < t \leq d+\delta \\ \frac{1}{f(d+\delta)} & \text{if } d+\delta < t \end{cases}$$

so $f' \in C_1$ and define

$$\Lambda(t) = \begin{cases} 1 & \text{if } t \leq d \\ \frac{1}{\delta}(-t+d+\delta) & \text{if } d < t \leq d+\delta \\ 0 & \text{if } d+\delta < t \end{cases}$$

So $(1 - f'.f).\Lambda = \chi_{(-\infty,d]}$. If $d \in X_1$, we're done. If not, then d is an isolated discontinuity of $\chi_{(-\infty,d]}.f$ with finite left and right limits, and we can generate $\chi_{\{d\}}$ and χ_1.

Now suppose we have $\chi_1, \ldots, \chi_k$ and want χ_{k+1}. If X_{k+1} is an isolated discontinuity, then we can find χ_{k+1} by the same method as the final sentence of the previous paragraph. So we may assume that X_{k+1} is an interval and d_1 is its infimum and d_2 is its supremum (if $d_2 = \infty$, we're done since this is the same as the case for X_1). If d_1 is not an right essential discontinuity, then we're done. Suppose it is, then we simply use the function which is the inverse of f on X_{k+1} and zero everywhere else. If d_2 is a left essential discontinuity, then we're done, otherwise define Λ as before. In this way we generate the characteristic functions of all X_i.

We now show that $\frac{1}{h_i}\chi_i$ can be generated from $C_1[f]$. If X_i is a point, then the result is trivial, so assume that it is an almost interval with infimum d and supremum d'. If f is bounded on some $(d, d+\delta)$, once again the result is trivial. We suppose f has a right essential discontinuity at d. We know from the construction of h_i in the standard representation that $f \cdot h$ is a bounded, continuous semialgebraic function which is nowhere zero on some $(d, d+\delta)$. In which case $\frac{1}{fh}$ is likewise a bounded, continuous sem-algebraic function nowhere zero function on $(d, d+\delta)$. Thus we can extend $\frac{1}{fh}$ to a function in $C_1[\chi_i] \subseteq C_1[f]$ and so there is a function in $C_1[f]$ equal to $\frac{f}{fh}\chi_i = \frac{1}{h}\chi_i$ on $(d, d+\delta)$. Since $\frac{1}{h}$ and f are bounded on $X_i \setminus (d, d+\delta)$ (possibly removing some $(d'-\delta', d')$ if f has a left essential discontinuity at d'), we see that $\frac{1}{h_i}\chi_i \in C_1[f]$. □

Corollary 2.3.5. *Let $f \in D_1$. Then there is $h \in V_1$ such that $C_1[f] = C_1[\frac{1}{h}]$.*

Proof. By Proposition 2.3.2, f has standard form: $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i}\chi_i$. Combining Proposition 2.2.9 and Proposition 2.3.4, we can find $h_2 \in V_1$ such that $C_1[f] = C_1[\chi_1, \ldots, \chi_n][\frac{1}{h_2}]$ and we can find bounded, nowhere zero h_1 such that $C_1[h_1] = C_1[\chi_1, \ldots, \chi_n]$. So $C_1[f] = C_1[h_1][\frac{1}{h_2}]$. Let $h = \frac{h_2}{h_1}$. This is in V_1 since h_1 only takes strictly positive natural values so is bounded away from 0. We claim that $C_1[f] = C_1[\frac{1}{h}]$. Since $\frac{1}{h} = \frac{h_1}{h_2}$, the inclusion $\supseteq$ is clear.

The only problem would occur if at some discontinuity point d we had some combination of $\lim_{t \nearrow d} \frac{h_1(t)}{h_2(t)}$, $\frac{h_1(d)}{h_2(d)}$, $\lim_{t \searrow d} \frac{h_1(t)}{h_2(t)}$ pairwise equal which didn't occur for f, i.e. $\frac{h_1}{h_2}$ is (left-/right-) continuous where f isn't. Since $\frac{h_1}{h_2}$ is bounded away from zero at its discontinuity points, and by o-minimality there are only finitely many such points, then we can alter the values of h_1 to ensure that limits aren't equal where they shouldn't be. □

Remark 2.3.6. Note that Proposition 2.3.4 doesn't hold if instead of $h \in V_1$ we pick $h \in C_1$: If f has a right essential discontinuity at 0, but say $\lim_{t \nearrow 0} f(t) = 1$, then we need $\lim_{t\searrow 0} h(t) = 0$, but if h is continuous at 0, then $\frac{1}{h}$ has a left essential discontinuity, but there is no such function in $C_1[f]$.

However, we easily combine the ideas of Proposition 2.2.14 and Proposition 2.3.5 to obtain:

Corollary 2.3.7. *Let $f \in D_1$. Then there exist X_1, X_2 and $h_1, h_2 \in C_1$ such that $C_1[f] = C_1[\frac{\chi_1}{h_1}, \frac{\chi_2}{h_2}]$.*

Definition 2.3.8. Let $f \in D_1$. Then at a given point $d \in \mathcal{R}$, f satisfies one and only one of the following:

$\langle 1\rangle$ $\lim_{t\nearrow d} f(t)$ and $\lim_{t\searrow d} f(t)$ are unbounded

$\langle 2\rangle$ $\lim_{t\nearrow d} f(t)$ is unbounded, $f(d) \neq \lim_{t\searrow d} f(t)$ is bounded

$\langle 3\rangle$ $\lim_{t\nearrow d} f(t)$ is bounded and $\neq f(d)$, $\lim_{t\searrow d} f(t)$ is unbounded

$\langle 4\rangle$ $\lim_{t\nearrow d} f(t)$ is unbounded, and $f(d) = \lim_{t\searrow d} f(t)$

$\langle 5\rangle$ $f(d) \neq \lim_{t\nearrow d} f(t) \neq \lim_{t\searrow d} f(t) \neq f(d)$ and both limits are bounded

$\langle 6\rangle$ $\lim_{t\nearrow d} f(t) = f(d)$, and $\lim_{t\searrow d} f(t)$ is unbounded

$\langle 7\rangle$ $\lim_{t\nearrow d} f(t) = f(d) \neq \lim_{t\searrow d} f(t) \neq \pm\infty$

$\langle 8\rangle$ f has a removable discontinuity at d

$\langle 9\rangle$ $\pm\infty \neq \lim_{t\nearrow d} f(t) \neq f(d) = \lim_{t\searrow d} f(t)$

$\langle 10\rangle$ f is continuous at d

We say that f is of type $\langle i\rangle$ at d for appropriate i, where necessary, we write this $\left\langle i_d^f \right\rangle$.

Proposition 2.3.9. *Let $f \in D_1$ and $d \in \mathcal{R}$. Then if f is of type $\langle i\rangle$ at d, then*

$$C_1[f] = \{g \in D_1 \mid \forall d \in \mathcal{R},\ \langle i_d^g\rangle \leq \left\langle i_d^f \right\rangle\}$$

where the types of discontinuities are ordered by the Hasse diagram below:

$$\begin{tikzcd}
& & \langle 1\rangle & & \\
& \langle 2\rangle \arrow[ur] & & \langle 3\rangle \arrow[ul] & \\
\langle 4\rangle \arrow[ur] & & \langle 5\rangle \arrow[ul] \arrow[ur] & & \langle 6\rangle \arrow[ul] \\
\langle 7\rangle \arrow[u] \arrow[urr] & & \langle 8\rangle \arrow[u] & & \langle 9\rangle \arrow[u] \arrow[ull] \\
& & \langle 10\rangle \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Furthermore, if $g \in D_1$ *is of type* $\langle j\rangle \not\leq \langle i\rangle$ *at* d, *then* $g \notin C_1[f]$.

Proof. By o-minimality, f is of type $\langle 10\rangle$ at all but finitely many points (the discontinuity points of f). Let d be such a discontinuity point. We will first show that $C_1[f]$ contains all functions in D_1 that are of type less than or equal to $\left\langle i_d^f \right\rangle$ and of type $\langle 10\rangle$ at all other points. In the case that f is of type $\langle 5\rangle$, $\langle 7\rangle$, $\langle 8\rangle$ or $\langle 9\rangle$ the result follows immediately from Proposition 2.2.10.

Suppose that f has a right essential singularity at d. We will first show that $C_1[f]$ contains all functions of type $\langle 6\rangle$ at d. By Proposition 2.3.4, there is some $f' \in C_1[f]$ which is equal to $(x-d)^q$ on some $(d, d+\delta)$, such that $0 > q \in \mathbb{Q}$ and $\lim_{t \searrow d} \frac{f(t)}{f'(t)} \in \mathcal{R} \setminus \{0\}$. By taking powers of $(x-d)^q$ we obtain functions equal to $(x-d)^{-n}$ on $(d, d+\delta)$ for arbitrarily large $n \in \mathbb{N}$. Since $\overline{\mathcal{R}}$ is polynomially bounded, for any $g \in D_1$ with a right essential discontinuity at d, we can find a negative power of $(x-d)$ that dominates g on $(d, d+\delta)$. We use this as follows: let g be of type $\langle 6\rangle$ at d and of type $\langle 10\rangle$ on $\mathcal{R} \setminus \{d\}$. Since $\overline{\mathcal{R}}$ is polynomially bounded, there is some $n \in \mathbb{N}$ such that $g(x-d)^n$ is bounded on $(d, d+\delta)$, i.e. $g(x-d)^n \in C_1$, and such that $(x-d)^{-n} \in C_1[f]$. Now let h be given by

$$h(t) = \begin{cases} 1 & \text{if } t \leq d \\ (x-d)^n & \text{if } d < t \leq d+1 \\ 1 & \text{if } d+1 < t \end{cases}$$

Then $h \in C_1[f]$ and $g.h \in C_1[f]$, since this is bounded on some $(d, d+\delta)$. Also $\frac{1}{h} \in C_1[f]$, thus $g \cdot h \cdot \frac{1}{h} = g \in C_1[f]$. If f has an isolated discontinuity then

$\chi_{\{d\}} \in C_1[f]$ and thus all functions of type $\langle 3 \rangle$ at d are in $C_1[f]$. If f has a double essential singularity, then we similarly obtain all functions of type $\langle 1 \rangle$ at d. We use a similar technique when f has a left essential singularity at d.

We can then use multiplication by characteristic functions and addition to show that

$$C_1[f] \supseteq \{g \in D_1 \mid \forall d \in \mathcal{R},\ \langle i_d^g \rangle \leq \left\langle i_d^f \right\rangle\}$$

For the inclusion $\subseteq$, we notice that if f is (left-/right-) continuous at some $d \in \mathcal{R}$, then any addition or multiplication by continuous functions preserves this continuity. Similarly if f is bounded on a neighbourhood of d, then we cannot generate a function that is unbounded at d from the set $C_1 \cup \{f\}$. $\square$

Corollary 2.3.10. *Let* $A = C_1[f, g]$ *with* $f, g \in D_1$. *Then for each* $d \in \mathcal{R}$, A *contains a function of type* $\langle i \rangle \vee \langle j \rangle$ *at* d, *where* f *is of type* $\langle i \rangle$ *at* d, g *is of type* $\langle j \rangle$ *at* d *and* $\vee$ *denotes the join of two elements of the poset of types. Thus, additionally, each of* $f + g$ *and* $f.g$ *is of type* $\langle k \rangle \leq \langle i \rangle \vee \langle j \rangle$.

Proof. Let $\frac{1}{h_f}$ and $\frac{1}{h_g}$ be the functions given in Corollary 2.3.5. Then $\frac{1}{h_f} + \frac{1}{2}.\frac{1}{h_g}$ gives a function of type $\langle i \rangle \vee \langle j \rangle$ in all cases. $\square$

Definition 2.3.11. We say that A is of type $\langle i \rangle$ at d if it contains a function of type $\langle i \rangle$ at d and no function of greater type.

Proposition 2.3.9 and Corollary 2.3.10 ensure that this is well defined at each point.

Theorem 2.3.12. *There is a bijection between the set of all functions from* $\mathcal{R}$ *to* $\{1, \ldots, 10\}$ *and the set of rings between* C_1 *and* D_1, *which sends a function* ξ *to the ring of functions which is of type* $\left\langle \xi(d) \right\rangle$ *at* d, *for each* $d \in \mathcal{R}$.

Proof. Let $\xi : \mathcal{R} \longrightarrow \{1, \ldots, 10\}$. For each $d \in \mathcal{R}$, we can pick a canonical function $f_{(d,\xi(d))}$ which is of type $\left\langle \xi(d) \right\rangle$ at d and which is continuous otherwise as follows: Let g be any function that is of type $\langle i \rangle$ at d, we will use g simply for the purpose of describing the behaviour at d. If g has a left bounded discontinuity at d, then $f_{(d,\xi(d))} = 0$ on $(-\infty, d)$, if g has a left essential discontinuity at d then $f_{(d,\xi(d))}$ is equal to $(x - d)^{-1}$ on $(-\infty, d)$. If $\lim_{t \nearrow d} g(t) = g(d)$, then $f_{(d,\xi(d))}(d) = 0$, otherwise $f_{(d,\xi(d))}(d) = \frac{1}{2}$. If $\lim_{t \nearrow d} g(t) = \lim_{t \searrow d} g(t)$, then $f_{(d,\xi(d))} = 0$ on (d, ∞); if $g(d) = \lim_{t \searrow d} g(t)$, then $f_{(d,\xi(d))} = f_{(d,\xi(d))}(d)$ on (d, ∞); if $\lim_{t \nearrow d} g(t) \neq \lim_{t \searrow t} g(t) \neq g(d)$ and g does not have a right discontinuity at d, then $f_{(d,\xi(d))} = 1$

on (d, ∞); otherwise g has a right discontinuity at d and $f_{(d,\xi(d))} = (x-d)^{-1}$ on (d, ∞). We claim that the ring $A_\xi = C_1[\bigcup_{d \in \mathcal{R}} f_{(d,\xi(d))}] \subseteq D_1$ is of type $\big\langle \xi(d) \big\rangle$ for all $d \in \mathcal{R}$.

Clearly A_ξ is of at least type $\big\langle \xi(d) \big\rangle$ at d. Suppose it were of type $\langle i \rangle \geq \big\langle \xi(d) \big\rangle$ at d. Then we would have some $f \in A_\xi$ of type $\langle i \rangle$ at d, where f is a finite additive and multiplicative combination of C_1 functions and $\{f_{(d_1,\xi(d_1))}, \ldots, f_{(d_n,\xi(d_n))}\} \subseteq \bigcup_{d \in \mathcal{R}} f_{(d,\xi(d))}$, for some $n \in \mathbb{N}$. But, by Corollary 2.3.10 the multiplication or addition of two functions is simply a function whose type is at most the join of their types at each $d \in \mathcal{R}$. But $\bigvee_{i \leq n} \Big\langle i_{f_{(d_i,\xi(d_i))}} \Big\rangle$ is just $\langle 10 \rangle$ if $d \notin \{d_1, \ldots, d_n\}$ or $\big\langle \xi(d) \big\rangle$ if $d \in \{d_1, \ldots, d_n\}$. So $A_\xi = C_1[\bigcup_{d \in \mathcal{R}} f_{(d,\xi(d))}]$ is a ring of type $\big\langle \xi(d) \big\rangle$ at each $d \in \mathcal{R}$.

So the function Ξ which takes an element $\xi \in 10^{\mathcal{R}}$ and maps it to the ring A_ξ as described above is well-defined and total. The map Ξ is injective, since if $C_1 \subseteq A \subseteq D_1$ is of type $\langle i \rangle$ at d and $C_1 \subseteq B \subseteq D_1$ is of type $\langle j \rangle$ at d and, wlog, $\langle i \rangle \not\leq \langle j \rangle$, then there is $f \in A$ of type $\langle i \rangle$ at d and $f \notin B$, by the definition of the type of a ring at a point.

For surjectivity, let $C_1 \subseteq A \subseteq D_1$, and let A be of type $\big\langle \xi(d) \big\rangle$ at d for each $d \in \mathcal{R}$. Then for each $d \in \mathcal{R}$, A contains a function of type $\big\langle \xi(d) \big\rangle$ at d and so by Proposition 2.3.9 A contains all functions of type $\big\langle \xi(d) \big\rangle$ at d and of type $\langle 10 \rangle$ on $\mathcal{R} \setminus \{d\}$, including $f_{(d,\xi(d))}$. Therefore $\Xi(\xi) = A_\xi \subseteq A$. For the other inclusion, let $f \in A$, with maximal f-sets $X_1, \ldots, X_n$. Then $\chi_1, \ldots, \chi_n \in A$, and by the construction of A_ξ, we also have $\chi_1, \ldots, \chi_n \in A_\xi$. But now we only need that $f \cdot \chi_i \in A_\xi$ for each i. If X_i is a point the result is clear. Suppose that X_i is an almost interval with infimum d and supremum d' (the case for unbounded intervals is easier). By Proposition 2.3.9 there is some $\delta > 0$ and some $g \in A_\xi$ such that $f = g$ on $(d, d+\delta)$, similarly there is some $\delta' > 0$ and $g' \in A_\xi$ with $f = g'$ on $(d' - \delta, d')$. With m as the midpoint of X_i and supposing $d + \delta < m$, we define functions h and h' such that $h = g = f$ on $(d, d+\delta)$, h is the straight line interpolation between the point $\big\langle d+\delta, f(d+\delta) \big\rangle$ and the point $\langle m, 0 \rangle$ on $(d+\delta, m)$ and $h = 0$ on (m, d'); and $h' = g' = f$ on (m, d'), $h' = f - h$ on $(d+\delta, m)$ and $h' = 0$ on $(d, d+\delta)$. Then $h, h' \in A_\xi$ and $h + h' = f\chi_i$. Thus $A = A_\xi$ and the map Ξ is bijective. □

So all of the rings between C_1 and D_1 are uniquely determined by their type at each point.

Definition 2.3.13. We say that A is of type ξ, where $\xi : \mathcal{R} \longrightarrow \{1, \ldots, 10\}$ and

A is of type $\langle \xi(d) \rangle$ at d for each $d \in \mathcal{R}$.

Proposition 2.3.14. *Let* $f \in D_1$. *Then there exist unique* $k, l, p, q \in \mathbb{N}$, *such that*

$$C_1[f] \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^p \times C_1^q$$

Proof. Let $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i} \chi_i$ be the standard representation. Let $h \in V_1$ be the function obtained in Corollary 2.3.5 such that $C_1[f] = C_1[\frac{1}{h}]$. For each X_i we describe the possible behaviour of f on $\overline{X_i}$. This is to avoid having to write "possibly with finitely many points removed" for each item. The behaviours are as follows:

(1) X_i is a point

(2) $\overline{X_i}$ is a bounded interval $[a, b]$ and f is bounded on $\overline{X_i}$

(3) $\overline{X_i}$ is a bounded interval $[a, b]$ and $\lim_{t \searrow a} f(t) = \pm\infty$ and $\lim_{t \nearrow b} f(t)$ is bounded; or $\lim_{t \searrow a} f(t)$ is bounded and $\lim_{t \nearrow b} f(t) = \pm\infty$ (f is bounded at one end of the interval and unbounded at the other)

(4) $\overline{X_i}$ is a bounded interval $[a, b]$ and $\lim_{t \searrow a} f(t) = \pm\infty$ and $\lim_{t \nearrow b} f(t) = \pm\infty$ (f is unbounded at both ends of the interval)

(5) $\overline{X_i}$ is a half-bounded interval, $(-\infty, a]$ and $\lim_{t \searrow a} f(t)$ is bounded (similarly for $[b, \infty)$)

(6) $\overline{X_i}$ is a half-bounded interval, $(-\infty, a]$ and $\lim_{t \searrow a} f(t) = \pm\infty$ (similarly for $[b, \infty)$)

(7) $\overline{X_i}$ is $\mathcal{R}$

We claim that

$$C_1[f] \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^p \times C_1^q$$

where k is the number of X_i with behaviour (1), l is the number of X_i with behaviour (2), p is the number of X_i with behaviour (3) or (5), q is the number of X_i with behaviour (4), (6) or (7). We prove this now.

First, for ease of use, we define the following continuous (order-preserving or order-reversing) bijections for arbitrary $a, b \in \mathcal{R}$:

(1) $\alpha_1 : \{0\} \longrightarrow \{a\},\ x \mapsto a$

(2) $\alpha_2 : [0,1] \longrightarrow [a,b],\ x \mapsto (b-a)x+a$

(3) $\alpha_3^l : [0,\infty) \longrightarrow (a,b],\ x \mapsto a + \frac{b-a}{x+1}$; $\alpha_3^r : [0,\infty) \longrightarrow [a,b),\ x \mapsto b - \frac{b-a}{x+1}$

(4) $\alpha_4 : (-\infty,\infty) \longrightarrow (a,b),\ x \mapsto \begin{cases} \frac{bx+\frac{b+a}{2}}{x+1} & x \geq 0 \\ \frac{ax-\frac{b+a}{2}}{x-1} & x \leq 0 \end{cases}$

(5) $\alpha_5^l : [0,\infty) \longrightarrow (-\infty,b],\ x \mapsto b-x$; $\alpha_5^r : [0,\infty) \longrightarrow [a,\infty),\ x \mapsto a+x$

(6) $\alpha_6^l : (-\infty,\infty) \longrightarrow (-\infty,b),\ x \mapsto \begin{cases} b - \frac{1}{x+1} & x \geq 0 \\ b-1+x & x \leq 0 \end{cases}$;

$\alpha_6^r : (-\infty,\infty) \longrightarrow (a,\infty),\ x \mapsto \begin{cases} a + \frac{1}{1-x} & x \geq 0 \\ a+1+x & x \leq 0 \end{cases}$

(7) $\alpha_7 : (-\infty,\infty) \longrightarrow (-\infty,\infty),\ x \mapsto x$

Now let $\phi_i : C_1[f] \longrightarrow C(X_i)$ be given by $g \mapsto g\restriction_{X_i}$. Then

$$\Phi : C_1[f] \longrightarrow \prod \phi_i(C_1[f]), g \mapsto (g\restriction_{X_i})_{i\leq n}$$

is an isomorphism: It is clearly an injective homomophism. For surjectivity, let $(\widetilde{g}_i)_{i\leq n} \in \prod \phi_i(C_1[f])$, then each $\widetilde{g}_i$ can be extended to $g_i : \mathcal{R} \longrightarrow \mathcal{R}$ such that $g_i\restriction_{X_i} = \widetilde{g}_i$ and g_i is zero on $\mathcal{R} \setminus X_i$. Since $\chi_i \in C_1[f]$, then $g_i \in C_1[f]$ and so $g = \sum \chi_i \cdot g_i \in C_1[f]$ with $\Phi(g) = (\widetilde{g}_i)_{i\leq n}$.

Let $j(i)$ be the behaviour of each X_i. We now give an isomorphism $\phi_i(C_1[f]) \cong C(Y_i)$ for each $1 \leq i \leq n$, where Y_i depends on the behaviour of X_i as follows: (1) $Y_i = \{0\}$ and so $C(Y_1) \cong \mathcal{R}$; (2) $Y_i = [0,1]$; (3) or (5) $Y_i = [0,\infty)$; (4), (6) or (7) $Y_i = (\infty,\infty) = \mathcal{R}$ and so $C(Y_i) = C_1$. Then $\psi_i : \phi_i(C_1[f]) \longrightarrow C(Y_i)$ such that

$$\psi_i(g) : v \mapsto \begin{cases} g \circ \alpha^*_{j(i)}(v) & \text{if } \alpha^*_{j(i)}(v) \in X_i \\ \lim_{t\to v, t\in(\alpha^*_{j(i)})^{-1}(X_i)} g \circ \alpha^*_{j(i)}(t) & \text{if } \alpha^*_{j(i)}(v) \notin X_i \end{cases}$$

The two cases are just to deal with removable discontinuities of f in X_i or where $v = 0$ or 1 and f is bounded at an endpoint of X_i but the endpoint isn't in X_i. Therefore the limit given exists by the continuity and bijectivity of $\alpha^*_{j(i)}$. The asterisk α^* is used to signify that we may need $\alpha^l_{j(i)}$ or $\alpha^r_{j(i)}$ depending on where the bounded and unbounded endpoints of X_i are. $\square$

Proposition 2.3.15. *Let* $f \in D_1$*. Then* $A = C_1[f]$ *is a real closed ring.*

Proof. By Prop 2.3.14,

$$C_1[f] \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^p \times C_1^q$$

Each factor in this product is a real closed ring, so their product must be a real closed ring. □

Theorem 2.3.16. *Let* A *be a ring such that* $C_1 \subseteq A \subseteq D_1$*. Then* A *is a real closed ring.*

Proof. This follows the same proof as Proposition 2.2.17 □

Proposition 2.3.17. *Let* $f \in D_1$*, then* $C_1[f] \cong C(\overline{\Gamma(f)})$ *— the continuous definable functions from the closure of the graph of* f *(considered as a subset of* $\mathcal{R}^2$*) to* $\mathcal{R}$*.*

Remark 2.3.18. We take the closure of $\Gamma(f)$ in case f has any removable discontinuities.

Proof. By o-minimality, $\overline{\Gamma(f)}$ has finitely many connected components $U_1, \ldots, U_m$, say, and each of these must itself be closed in $\mathcal{R}^2$. Then

$$C(\overline{\Gamma(f)}) \cong \prod_{1 \leq i \leq n} C(U_i)$$

Each U_i must be homeomorphic to one of: a point, the closed interval $[0,1]$, the half-open interval $[0,\infty)$ or $\mathcal{R}$, by a similar argument as in the proof of Proposition 2.3.14. Let $X_1, \ldots, X_n$ be the projection of $U_1, \ldots, U_n$. We classify the behaviour of f on each X_i and then $C(U_i) \cong C((\alpha^*_{j(i)})^{-1}(X_i))$, where $\alpha^*_{j(i)}$ is defined in 2.3.14. Then $\prod_{1 \leq i \leq n} C(U_i)$ is isomorphic to the representation of $C_1[f]$ given in 2.3.14. □

Remark 2.3.19. We conjecture that for $n > 1$, and for any $f \in D_n$, that $C_n[f] \cong C(\overline{\Gamma(f)})$. This would prove that any finite extension of C_n by an element of D_n is real closed. However, we cannot obtain a nice decomposition into characteristic functions: let $f : \mathcal{R}^2 \longrightarrow \mathcal{R}$ be defined by:

$$f(x,y) = \begin{cases} 0 & \text{if } x^2 + y^2 \geq 1 \\ x & \text{if } x^2 + y^2 < 1 \end{cases}$$

Then $\Gamma(f)$ is connected since $f(0,1) = f(0,-1) = 0$. There is clearly no characteristic function such that $C_1[\chi] \cong C(\overline{\Gamma(f)})$. The best approach appears to be to classify the behaviour of f by looking at germs of paths at each point, i.e. for each $\overline{a}$, we look at $\lim t \nearrow 1 f(\gamma(t))$, where $\gamma : (0,1) \longrightarrow \mathcal{R}^n$, for each path γ, where $\lim_{t\nearrow} \gamma(t) = \overline{a}$.

2.4 The Zariski Spectrum

In this section we calculate the Zariski spectrum of each ring between C_1 and D_1. Throughout, we assume that A is a ring such that $C_1 \subseteq A \subseteq D_1$. We will use the notion of "irrational type" or "non-definable" type. We give a definition now that we will repeat in Chapter 4

Definition 2.4.1. A type $p \in S_1(\mathcal{R})$ is **irrational** if we can partition $\mathcal{R}$ into two convex sets C^- and C^+, such that C^- does not have a greatest element and C^+ does not have a least element, and p is determined by the cut

$$\{x > a \mid a \in C^-\} \cup \{x < a \mid a \in C^+\}$$

Otherwise a type is **rational**. Note that in the o-minimal context "rational type" is the same as "definable type".

Remark 2.4.2. In $\mathbb{R}_{\text{alg}}$, π is irrational, however, in $\mathbb{R}$ π is defined by a rational type.

We know that there is a canonical bijection $\Phi : \operatorname{Spec} D_1 \longrightarrow \operatorname{Spec} C_1$ given by $\mathfrak{p} \mapsto \mathfrak{p} \cap C_1$. So for any A, we can have the following commuting maps:

$$\begin{tikzcd}
\operatorname{Spec} D_1 \arrow[r] \arrow[rr, bend left, "\cong"] & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C_1 \\
\mathfrak{p} \arrow[r, mapsto] & \mathfrak{p} \cap A \arrow[r, mapsto] & \mathfrak{p} \cap C_1
\end{tikzcd}$$

Since the composed map is a bijection, we know that the map $\Phi_{D_1,A} : \operatorname{Spec} D_1 \longrightarrow \operatorname{Spec} A$ is an injection and the map $\Phi_{A,C_1} \operatorname{Spec} A \longrightarrow \operatorname{Spec} C_1$ is a surjection. We also know that $\operatorname{Spec} D_1$ is homeomorphic to $S_1(\mathcal{R})$ (see [Tre99]), which is $\mathcal{R}$ together with the set of Dedekind cuts of $\mathcal{R}$. Explicitly, the prime ideals of D_1 are of the following form, for each $d \in \mathcal{R}$:

- $\mathfrak{p}_d = \{f \in D_1 \mid f(d) = 0\}$
- $\mathfrak{p}_{d^-} = \{f \in D_1 \mid f\restriction_{(d-\delta,d)} = 0 \text{ for some } \delta > 0\}$
- $\mathfrak{p}_{d^+} = \{f \in D_1 \mid f\restriction_{(d,d+\delta)} = 0 \text{ for some } \delta > 0\}$

for $\tau \in S_1(\mathcal{R})$ non-definable:

- $\mathfrak{p}_\tau = \{f \in D_1 \mid f \text{ is zero in a neighbourhood of } \tau\}$

and finally

- $\mathfrak{p}_\infty = \{f \in D_1 \mid f\restriction_{(d,\infty)} = 0 \text{ for some } d \in \mathcal{R}\}$
- $\mathfrak{p}_{-\infty} = \{f \in D_1 \mid f\restriction_{(-\infty,d)} = 0 \text{ for some } d \in \mathcal{R}\}$

We will define two further ideals, which aren't in D_1, $\mathfrak{p}_{\nearrow d}$ and $\mathfrak{p}_{\searrow d}$ later. By basic properties of prime ideals, for $\mathfrak{p} \in \operatorname{Spec} D_1$, $\mathfrak{p} \cap A$ is a prime ideal, and these are distinct since the map $\operatorname{Spec} D_1 \longrightarrow \operatorname{Spec} A$ is an injection.
First, fix $d \in \mathcal{R}$. Let $F_{\mathfrak{p}_d}$ denote the preimage of $\mathfrak{p}_d$ in $\operatorname{Spec} A$ under Φ_{A,C_1}. We use $\mathfrak{q}$ to denote a member of $F_{\mathfrak{p}_d}$. We notice that if $f \in C_1$ and $f(d) \neq 0$, then $f \notin \mathfrak{q}$, since $\mathfrak{q} \cap A = \mathfrak{p}_d$.

Claim 2.4.3. *Let $g \in A$ be such that $g(d) = 0$ and continuous on an open neighbourhood $U \ni d$. Then $g \in \mathfrak{q}$.*

Proof. By hypothesis, g is semi-algebraic and continuous at d, so it is continuous on a small interval $(d-\delta, d+\delta) \subseteq U$. Now define

$$\Lambda(t) = \begin{cases} 0 & \text{if } t \leq d - \delta \\ t - (d-\delta) & \text{if } d - \delta < t \leq d \\ -t + d + \delta & \text{if } d < t \leq d + \delta \\ 0 & \text{if } d + \delta < t \end{cases}$$

So $\Lambda \notin \mathfrak{q}$, since it is continuous and $\Lambda(d) \neq 0$. Then $\Lambda.g$ is continuous and vanishes at d, so $\Lambda.g \in \mathfrak{q}$, and, by primality, $g \in \mathfrak{q}$. □

Claim 2.4.4. *Let $g \in A$ be such that $|g| > \epsilon > 0$ on some open interval $I = (d-\delta, d+\delta)$, for some $\epsilon, \delta > 0$. Then $g \notin \mathfrak{q}$.*

Proof. Suppose $g \in \mathfrak{q}$. There exists g' such that $g' = 1/g$ on I, and such that $g' = \lim_{t \searrow d-\delta}(g(t))^{-1}$ on $(-\infty, d-\delta]$ and $g' = \lim_{t \nearrow d+\delta}(g(t))^{-1}$ on $[d+\delta, \infty)$. We observe that $g' \in A$. Then $g.g' = 1$ on $(d-\delta, d+\delta)$. Additionally, $1 - g.g' = 0$ on I, so by the previous claim, $1 - g.g' \in \mathfrak{q}$. Thus $g.g' + 1 - g.g' = 1 \in \mathfrak{q}$, which contradicts the properness of $\mathfrak{q}$. □

Claim 2.4.5. *Let $f, g \in A$. Suppose $f \in \mathfrak{q}$ and $f = g$ on some open neighbourhood of d. Then $g \in \mathfrak{q}$.*

Proof. We have some $\delta > 0$ such that $f = g$ on $(d-\delta, d+\delta)$, and so $f - g$ is continuous on a neighbourhood of d. Also $(f-g)(d) = 0$. Then $f - g \in \mathfrak{q}$ and thus $g \in \mathfrak{q}$. □

Hence we need only consider the behaviour of functions near each point. Additionally, when defining functions, we will only define them on an interval I of d and assume that they are continuous on $\mathcal{R} \setminus I$.

Clearly if A is of type $\langle 10 \rangle$ at d (continuous on both sides at d), then $F_{\mathfrak{p}_d} = \{\{f \in A \mid f(d) = 0\}\}$, since any function $f \in A$ which is continuous at d must be continuous on an interval around d. Either $f(d)$=0 or $|f| > \epsilon > 0$ on some interval for some $\epsilon > 0$.

Now suppose A is not of type $\langle 10 \rangle$ at d. Since the preimage of $\mathfrak{p}_d$ can't contain a function bounded away from 0 at d, any element $f \in \mathfrak{q}$ must be such that at least one of the following properties is satisfied:

(i) $f(d) = 0$

(ii) $\lim_{t \nearrow d} f(t) = 0$

(iii) $\lim_{t \searrow d} f(t) = 0$

We first deal with the special case when A is of type $\langle 8 \rangle$ at d (i.e. $\lim_{t \nearrow d} f(t) = \lim_{t \searrow d} f(t)$). Here $\chi_{\{d\}}, \chi_{\mathcal{R}\setminus\{d\}} \in A$. Furthermore, since

$$\chi_{\{d\}}.\chi_{\mathcal{R}\setminus\{d\}} = 0 \in \mathfrak{p}_d$$

and $\chi_{\{d\}} + \chi_{\mathcal{R}\setminus\{d\}} = 1$, one and only of one of $\chi_{\{d\}}, \chi_{\mathcal{R}\setminus\{d\}}$ must be in $\mathfrak{q}$. If $\chi_{\{d\}} \in \mathfrak{q}$, then for any $f \in \mathfrak{p}_d$ and $r \in \mathcal{R}$, $f + r.\chi_{\{d\}} \in \mathfrak{q}$, i.e. all functions such that $\lim_{t \to d} f(t) = 0$ are in $\mathfrak{q}$. Now the set $\mathfrak{q} = \{f \in A \mid \lim_{t \to d} f(t) = 0\}$ is a maximal ideal, with $A/\mathfrak{q} = \mathcal{R}$. If $\chi_{\mathcal{R}\setminus\{d\}} \in \mathfrak{q}$, then for all $f \in A$, $f.\chi_{\mathcal{R}\setminus\{d\}} \in \mathfrak{q}$.

By inspection, $\{f \in A \mid f(d) = 0\}$ is also a maximal ideal. So these are the two preimages of $\mathfrak{p}_d$ in $\operatorname{Spec} A$.

We notice that if A is of type $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$, $\langle 5\rangle$ or $\langle 7\rangle$ at d, i.e. there is some $f \in A$ such that $\lim_{t\nearrow d} f(t) \neq f(d)$, then $\mathfrak{q}$ must contain one and only one of $\chi_{(-\infty,d)}$ or $\chi_{[d,\infty)}$, since $\chi_{(-\infty,d)}{\cdot}\chi_{[d,\infty)} = 0$ and $\chi_{(-\infty,d)} + \chi_{[d,\infty)} = 1$. Similarly when there is some $f \in A$ such that $\lim_{t\searrow d} f(t) \neq f(d)$.

Claim 2.4.6. *Let $\chi_{(-\infty,d)} \in A$. If $\chi_{[d,\infty)} \in \mathfrak{q}$, then all $f \in A$ such that $\lim_{t\nearrow d} f(t)$-$= 0$ and $f(d) = \lim_{t\searrow d} f(t)$ are in $\mathfrak{q}$. If $\chi_{(-\infty,d)} \in \mathfrak{q}$, then all $f \in A$ such that $f(d) = \lim_{t\searrow d} f(t) = 0$ are in $\mathfrak{q}$.*

Proof. Suppose $\chi_{[d,\infty)} \in \mathfrak{q}$ and let $f \in A$ be such that $\lim_{t\nearrow d} f(t) = 0$ and $f(d) = \lim_{t\searrow d} f(t)$. Then $\chi_{(-\infty,d)} \notin \mathfrak{q}$, however $\chi_{(-\infty,d)}{\cdot}f$ is continuous around d and 0 at d, so it is in $\mathfrak{q}$. By primality of $\mathfrak{q}$, $f \in \mathfrak{q}$. Similarly for $\chi_{(-\infty,d)} \in \mathfrak{q}$ and similarly for the case $\chi_{(-\infty,d]} \in A$. □

Claim 2.4.7. *Let A be of type $\langle 1\rangle$, $\langle 2\rangle$ or $\langle 4\rangle$ at $d \in \mathcal{R}$. Then $\chi_{[d,\infty)} \notin \mathfrak{q}$.*

Proof. Suppose it is, then all f such that $\lim_{t\nearrow d} f(t) = 0$ are in $\mathfrak{q}$. By the type of A at d, there is $g \in A$ such that $f.g = 1$ on an interval $(d-\delta, d)$. Then $f.g \in \mathfrak{q}$ and $f.g + \chi_{[d,\infty)} = 1$ on a neighbourhood of d, which is a contradiction. □

We now work through the various possible types of A.

Proposition 2.4.8. *Let A be of type ξ and let $d \in \mathcal{R}$. Then the preimages of $\mathfrak{p}_d$ in A are precisely the prime ideals of the form $\mathfrak{p}_d = \{f \in A \mid f(d) = 0\}$ together with the following ideals at each $d \in \mathcal{R}$, for appropriate $\big\langle\xi(d)\big\rangle$, where $\mathfrak{p}_{\nearrow d} = \{f \in A \mid \lim_{t\nearrow d} f(t) = 0\}$ and $\mathfrak{p}_{\searrow d} = \{f \in A \mid \lim_{t\searrow d} f(t) = 0\}$*

$\langle 1\rangle$ *nothing; $\mathfrak{p}_{\nearrow d}$,$\mathfrak{p}_{\searrow d}$ are not ideals*

$\langle 2\rangle$ $\mathfrak{p}_{\searrow d}$*; $\mathfrak{p}_{\nearrow d}$ is not an ideal*

$\langle 3\rangle$ $\mathfrak{p}_{\nearrow d}$*; $\mathfrak{p}_{\searrow d}$ is not an ideal*

$\langle 4\rangle$ *nothing; $\mathfrak{p}_{\nearrow d}$ is not an ideal, $\mathfrak{p}_{\searrow d} = \mathfrak{p}_d$*

$\langle 5\rangle$ $\mathfrak{p}_{\nearrow d}$,$\mathfrak{p}_{\searrow d}$

$\langle 6\rangle$ *nothing; $\mathfrak{p}_{\nearrow d} = \mathfrak{p}_d$, $\mathfrak{p}_{\searrow d}$ is not an ideal*

$\langle 7\rangle$ $\mathfrak{p}_{\nearrow d}$*; $\mathfrak{p}_{\searrow d} = \mathfrak{p}_d$*

⟨8⟩ $\mathfrak{p}_{\nearrow d} = \mathfrak{p}_{\searrow d}$

⟨9⟩ $\mathfrak{p}_{\searrow d}$*;* $\mathfrak{p}_{\nearrow d} = \mathfrak{p}_d$

⟨10⟩ *nothing;* $\mathfrak{p}_d = \mathfrak{p}_{\nearrow d} = \mathfrak{p}_{\searrow d}$

Proof. For all $d \in \mathcal{R}$, no matter what the type of A at d, $\mathfrak{p}_d$ is a maximal ideal. For type ⟨1⟩, neither $\chi_{(-\infty,d)}$ nor $\chi_{(d,\infty)} \in \mathfrak{q}$, so $\chi_{\{d\}} \in \mathfrak{q}$ (since $\chi_{(-\infty,d)}{\cdot}\chi_{(d,\infty)}{\cdot}\chi_{\{d\}} = 0 \in \mathfrak{q}$). Then $\chi_{\{d\}}.A = \mathfrak{p}_d$, which is maximal, so this is the only possible preimage.

If A is of type ⟨2⟩, then $\chi_{[d,\infty)} \notin \mathfrak{q}$, by Claim 2.4.7, and either $\chi_{(-\infty,d]}$ or $\chi_{(d,\infty)} \in \mathfrak{q}$. In the first case, since $\chi_{(-\infty,d]} \in \mathfrak{q}$, we have that all $f \in A$ such that $\lim_{t\searrow d} f(t) = 0$ and $f(d) = \lim_{t\nearrow d} f(t)$ are in $\mathfrak{q}$. But since $\chi_{(-\infty,d]}, \chi_{(-\infty,d)} \in \mathfrak{q}$, we must have $\chi_{\{d\}} \in \mathfrak{q}$, and for any $f \in \mathfrak{q}$, $f + \chi_{\{d\}}$ has a left discontinuity at d. Thus $\mathfrak{q}$ must contain all functions such that $\lim_{t\searrow d} f(t) = 0$, i.e. $\mathfrak{q} = \mathfrak{p}_{\searrow d}$. In the second case, $\chi_{(-\infty,d)}, \chi_{(d,\infty)} \in \mathfrak{q}$, so $1 - \chi_{\{d\}} \in \mathfrak{q}$, and $\mathfrak{q} = \mathfrak{p}_d$. Similarly for type ⟨3⟩.

For type ⟨4⟩ we see that $\mathfrak{p}_d = \mathfrak{p}_{\searrow d}$. Similarly for type ⟨6⟩.

For type ⟨5⟩, we claim we have one of three possibilities: either $\chi_{(-\infty,d]} \in \mathfrak{q}$ or $\chi_{(-\infty,d)\cup(d,\infty)} \in \mathfrak{q}$ or $\chi_{[d,\infty)} \in \mathfrak{q}$. This is because if $\chi_{(-\infty,d]} \in \mathfrak{q}$, then $\chi_{(-\infty,d]}{\cdot}\chi_{(-\infty,d)} = \chi_{(-\infty,d)} \in \mathfrak{q}$, so $\chi_{[d,\infty)} \notin \mathfrak{q}$; also $\chi_{(d,\infty)} \notin \mathfrak{q}$, so $\chi_{(-\infty,d)\cup(d,\infty)} \notin \mathfrak{q}$, since $\chi_{(-\infty,d)\cup(d,\infty)}{\cdot}\chi_{(d,\infty)} = \chi_{(d,\infty)}$.

For type ⟨7⟩, we treat this similarly to type ⟨5⟩, but notice that $\mathfrak{p}_d = \mathfrak{p}_{\nearrow d}$. Similarly for ⟨9⟩. □

Remark 2.4.9. For a given ring A, if A is of type ⟨1⟩, ⟨3⟩ or ⟨6⟩ at d, i.e. it contains all functions with a right essential discontinuity at d, then the set $\{f \in A \mid \lim_{t\searrow d} f(t)\}$ is not a prime ideal: For any f in this set, $\frac{1}{f} \in A$, and so it is not an ideal.

We now look at ideals of the form $\mathfrak{p}_{d^-} = \{f \in C_1 \mid \exists\delta > 0 \; f\restriction_{(d-\delta,d]} = 0\}$. As in the previous section, let $F_{\mathfrak{p}_{d-}}$ denote the preimage under the intersection map, and let $\mathfrak{q}$ denote a member of $F_{\mathfrak{p}_{d-}}$. If A is left continuous at d (of type ⟨6⟩, ⟨9⟩ or ⟨10⟩), then we must have $\chi_{(d,\infty)} \in \mathfrak{q}$ (as $\chi_{(-\infty,d]}$ can't be in $\mathfrak{q}$, since $(x-d).\chi_{(-\infty,d]} \in A$, but is not in $\mathfrak{p}_{d^-}$). When A is not of type ⟨6⟩, ⟨9⟩ or ⟨10⟩ at d, then by a similar technique it must the case that $\chi_{[d,\infty)} \in \mathfrak{q}$ and $\chi_{(-\infty,d)\cup(d,\infty)} \notin \mathfrak{q}$ (otherwise $\mathfrak{q} = \{f \in A \mid f(d) = 0\}$, in which case $\mathfrak{q} \cap C_1 \neq \mathfrak{p}_{d^-}$). So $\chi_{(-\infty,d)} \in \mathfrak{q}$. From this, we see that $\mathfrak{q} \supseteq \{f \in A \mid f\restriction_{(d-\delta,d)} = 0 \text{ for some } \delta > 0\}$.

Proposition 2.4.10. $\mathfrak{q} = \{f \in A \mid f\restriction_{(d-\delta,d)} = 0$ *for some* $\delta > 0\}$

Proof. Suppose not. Then $\mathfrak{q}$ contains a function, v such that either (1) $|v| > \epsilon$ for $\epsilon > 0$ on some interval $(d-\delta, d)$ or (2) $\lim_{t \nearrow d} v(t) = 0$. In case (1), v is invertible on $(d-\delta, d)$, and so $\chi_{(-\infty,d)} \in \mathfrak{q}$, and thus $1 \in \mathfrak{q}$. In case (2) then $\mathfrak{q} = \{f \in A \mid \lim_{t \nearrow d} v(t) = 0\}$, and once again $\mathfrak{q} \cap C_1 \neq \mathfrak{p}_{d^-}$. $\square$

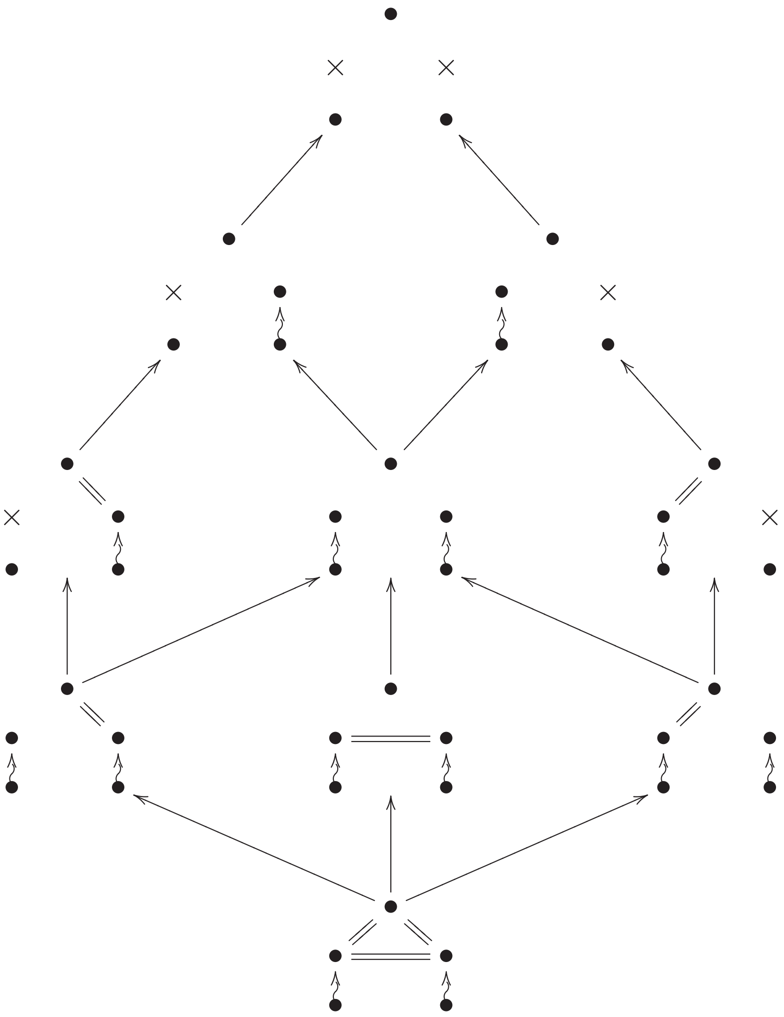
Clearly prime ideals of the form $\{f \in C_1 \mid f\restriction_{(\tau-\delta,\tau+\delta)} = 0 \text{ for some } \delta > 0\}$, where τ is non-definable, are just the same, i.e. their preimage in $\operatorname{Spec} A$ is just $\{f \in A \mid f\restriction_{(\tau-\delta,\tau+\delta)} = 0 \text{ for some } \delta > 0\}$. This gives us all of $\operatorname{Spec} A$.

We present the diagram below, which shows the prime ideals at a point d of rings of each type at d. Here each pentagon represents (starting bottom left and working round clockwise) the ideals $\mathfrak{p}_{d^-}$, $\mathfrak{p}_{\nearrow d}$, $\mathfrak{p}_d$, $\mathfrak{p}_{\searrow d}$, $\mathfrak{p}_{d^+}$.

$$\begin{tikzcd}
 & \mathfrak{p}_d & \\
\mathfrak{p}_{\nearrow d} & & \mathfrak{p}_{\searrow d} \\
\mathfrak{p}_{d^-} & & \mathfrak{p}_{d^+}
\end{tikzcd}$$

The arrow $\bullet \rightsquigarrow \bullet$ represents specialization, i.e. $\mathfrak{p}_{\nearrow d}$ or $\mathfrak{p}_{\searrow d}$ is a prime ideal. The arrow $\bullet = \bullet$ shows equality between two prime ideals. The symbol × appears when the set $\mathfrak{p}_{\nearrow d}$ or $\mathfrak{p}_{\searrow d}$ is not an ideal. The pentagons each represent the prime ideals at a point d which is of type $\langle i \rangle$, where each type assumes the same place as in the previous Hasse diagram, as given in the statement of

Proposition 2.3.9.



The ideals of the form $\mathfrak{p}_{\nearrow d}$ and $\mathfrak{p}_{\searrow d}$ are closed singletons in the Zariski topology on Spec A (where such sets are ideals): Let f be the function which is $(x-d)$ on $(-\infty, d)$ and 1 on $[d, \infty)$. Then $D(f) = \{\mathfrak{p}_{\nearrow d}\}$. However, such ideals are not open: if $g \in A$ and $g \notin \mathfrak{p}_{\nearrow d}$, then $\lim_{t \nearrow d} g(t) \neq 0$, in which case $g \notin \mathfrak{p}_{d^-}$. So any closed set containing $\mathfrak{p}_{\nearrow d}$ must also contain $\mathfrak{p}_{d^-}$.

2.5 Rings of functions on intervals

In this section we consider functions from a semi-algebraic subset X of $\mathcal{R}$ to $\mathcal{R}$. First we deal with intervals.

Definition 2.5.1. Let X be a definable set in $\mathcal{R}$. As usual, $C(X)$ denotes the ring of continuous definable functions $X \longrightarrow \mathcal{R}$. Let A be a ring between C_1 and D_1. Then $A\restriction_X$ is the ring obtained by restricting functions in A to the interval X, i.e.:

$$A\restriction_X := \{f : X \longrightarrow \mathcal{R} \mid \exists g \in A,\ g\restriction_X = f\}$$

This gives a canonical homomorphism $\pi_X^A : A \longrightarrow A\restriction_X$, which maps f to its restriction to X, $f\restriction_X$.

The following lemma is well known.

Lemma 2.5.2. *Let $f \in B_1$. Then $\lim_{t\to\infty} f(t)$ exists and is in $\mathcal{R}$ (similarly for $-\infty$).*

Proof. By the Monotonicity Theorem ([vdD98, Ch.3,Theorem 1.2]), there is some $r' \in \mathcal{R}$ such that on the interval (r', ∞), f is either constant, in which case the lemma is clear, or monotone increasing or monotone decreasing. We assume f is monotone increasing, wlog. Then the following sets are definable:

$$f^{\geq} := \{u \in \mathcal{R} \mid \exists r > r'\ \forall s > r,\ u \geq f(s)\}$$

$$f^{\leq} := \{u \in \mathcal{R} \mid \exists r > r'\ \forall s > r,\ u \leq f(s)\}$$

Since f is bounded and monotone increasing, $f^{\geq}$ and $f^{\leq}$ are both non-empty, convex sets which cover $\mathcal{R}$. Clearly $\sup(f^{\leq})$ and $\inf(f^{\geq})$ are both definable in $\overline{\mathcal{R}}$ and $\sup(f^{\leq}) \leq \inf(f^{\geq})$. Since $f^{\geq}$ and $f^{\leq}$ cover $\mathcal{R}$, we have

$$\sup(f^{\leq}) = \lim_{t\to\infty} f(t) = \inf(f^{\geq})$$

□

Corollary 2.5.3. $B_1 \cong C([0,1])$

Remark 2.5.4. Let $X = (0,1)$, then $C(X) \neq C_1\restriction_X$. This is because $C(X)$ is the ring of continuous definable functions $(0,1) \longrightarrow \mathcal{R}$, and is isomorphic to C_1 (since there is a definable homeomorphism $I \leftrightarrow \mathcal{R}$); however, $C_1\restriction_X$ is isomorphic to $C([0,1])$, since each function in $C_1\restriction_X$ has a unique extension to $[0,1]$ by taking limits at 0 and 1, thus $C_1\restriction_X \cong B_1$.

Definition 2.5.5. Let $I = (a,b) \subseteq \mathcal{R}$. Let $C_1 \subseteq A \subseteq D_1$. We define the subset $\widetilde{I} \subseteq \mathrm{Spec}(A)$ as follows. For a prime ideal $\mathfrak{p}$ of A we say that $\mathfrak{p} \in \widetilde{I}$ if and only if $\mathfrak{p}$ is of the form

- $\mathfrak{p}_{\nearrow d}$, $\mathfrak{p}_{d^-}$, $\mathfrak{p}_d$, $\mathfrak{p}_{\searrow}$ or $\mathfrak{p}_{d^+}$, for $d \in I$
- $\mathfrak{p}_\tau$ for some non-definable cut $\tau \in I$
- $\mathfrak{p}_{a^+}$ or $\mathfrak{p}_{\searrow a}$, or $\mathfrak{p}_{b^-}$ or $\mathfrak{p}_{\nearrow b}$

Combining this with $\widetilde{\{d\}} = \{\mathfrak{p}_d\}$ gives us the definition of $\widetilde{X}$ for an arbitrary definable set X. Note that $\widetilde{X}$ is always pro-constructible (closed in the constructible topology): For any open interval (a, b), let Λ be the lambda function whose support is (a, b). Then $V(\Lambda) = \{\mathfrak{p} \mid \Lambda \notin \mathfrak{p}\} = \widetilde{(a, b)}$ is closed in the Zariski topology. For each point $d \in \mathcal{R}$, $\widetilde{\{d\}} = \{\mathfrak{p}_d\}$ is open in the Zariski topology, and so closed in the constructible topology. Since any definable set is a finite union of open intervals and points, then $\widetilde{X}$ must be closed in the Zariski topology.

Remark 2.5.6. If A is of type $\langle 10\rangle$, say, then we will have $\mathfrak{p}_d \in \widetilde{(d, \infty)}$. This is because $\mathfrak{p}_{\searrow d} \in \widetilde{(d, \infty)}$ by definition and $\mathfrak{p}_d = \mathfrak{p}_{\searrow d}$ in A.

Lemma 2.5.7. *Let $X \subseteq \mathcal{R}$ be a definable set and let A be a ring between C_1 and D_1. Then π^A_X induces an injective, continuous map $\sigma^A_X : \mathrm{Spec}(A\upharpoonright_X) \longrightarrow \mathrm{Spec}(A)$, and $\mathrm{Spec}(A\upharpoonright_X)$ is homeomorphic to its image in $\mathrm{Spec}(A)$, which is $\widetilde{X} \subseteq \mathrm{Spec}(A)$.*

Proof. The map σ^A_X is given by:

$$\begin{aligned} \sigma^A_X : \mathrm{Spec}(A\upharpoonright_X) &\longrightarrow \mathrm{Spec}(A) \\ \mathfrak{p} &\mapsto \{f \in A \mid f\upharpoonright_X \in \mathfrak{p}\} \end{aligned}$$

From basic commutative algebra, we know that the map is injective (since π^A_X is surjective) and continuous. It is clear from the definition of $\widetilde{X}$ that $\sigma^A_X(\mathrm{Spec}(A\upharpoonright_X))$ is just $\widetilde{X}$. For each basic open set of the form $D(g\upharpoonright_X)$, it is clear that $\sigma^A_X(D(g\upharpoonright_X))$ is $D(f) \cap \widetilde{X}$, where f is any member of A such that $f\upharpoonright_I = g\upharpoonright_I$. □

We fix the interval $X = (0, 1)$, and study rings between $C_1\upharpoonright_X$ and $D_1\upharpoonright_X$. Let $C_1\upharpoonright_X \subseteq A \subseteq D_1\upharpoonright_X$. We obtain the following diagram:

$$\begin{tikzcd}
D_1 \arrow[r, two heads, "\pi_X"] & D_1\upharpoonright_X \\
 & A \arrow[u, hook] \\
 & C_1\upharpoonright_X \arrow[u, hook]
\end{tikzcd}$$

Using standard commutative algebra we can find pullbacks such that the diagram below commutes:

$$\begin{tikzcd}
D_1 \arrow[r, two heads, "\pi_X"] & D_1\upharpoonright_X \\
\pi_X^{-1}(A) \arrow[u] \arrow[r] & A \arrow[u, hook] \\
\pi_X^{-1}(C_1\upharpoonright_X) \arrow[u] \arrow[r] & C_1\upharpoonright_X \arrow[u, hook]
\end{tikzcd}$$

Lemma 2.5.8. *Let* $C_1\upharpoonright_X \subseteq A \subseteq D_1\upharpoonright_X$*. Then the pullback of the diagram*

$$\begin{tikzcd}
D_1 \arrow[r, two heads, "\pi_X"] & D_1\upharpoonright_X \\
 & A \arrow[u, hook]
\end{tikzcd}$$

is (up to isomorphism)

$$A_{D_1} := \{f \in D_1 \mid f\upharpoonright_X \in A\}$$

and the arrows are inclusion into D_1 *and* π_X^A *onto* A*.*

Proof. We know from commutative algebra that the pullback is just the fibre product. So:

$$\begin{aligned}
D_1 \times_{D_1\upharpoonright_X} A &= \{(f,g) \in D_1 \times A \mid \pi_X(f) = g\} \\
&= \{(f,g) \in D_1 \times A \mid f\upharpoonright_X = g\} \\
&\cong \{f \in D_1 \mid f\upharpoonright_X \in A\}
\end{aligned}$$

The result about arrows follows from the definition of fibre products. □

So the diagram is really:

$$\begin{tikzcd}
D_1 \arrow[r, two heads, "\pi_X"] & D_1\upharpoonright_X \\
A_{D_1} \arrow[u, hook] \arrow[r, two heads, "\pi_X"] & A \arrow[u, hook] \\
(C_1\upharpoonright_X)_{D_1} \arrow[u, hook] \arrow[r, two heads, "\pi_X"] & C_1\upharpoonright_X \arrow[u, hook]
\end{tikzcd}$$

This gives us the following commuting diagram of maps between the Zariski spectra:

$$\begin{tikzcd}
\mathrm{Spec}(D_1) \arrow[d, hook] & \mathrm{Spec}(D_1\upharpoonright_X) \arrow[l, hook'] \arrow[d, hook] \\
\mathrm{Spec}(A_{D_1}) \arrow[d] & \mathrm{Spec}(A) \arrow[l, hook'] \arrow[d] \\
\mathrm{Spec}((C_1\upharpoonright_X)_{D_1}) & \mathrm{Spec}(C_1\upharpoonright_X) \arrow[l, hook']
\end{tikzcd}$$

Recall, by Lemma 2.5.7, Spec(A) is homeomorphic to Spec(A_{D_1}) $\cap \widetilde{X}$, so the inclusion arrows here are clear. The inclusion arrows Spec(D_1) $\hookrightarrow$ Spec(A_{D_1}) are clear, since the ideals $\mathfrak{p}_{d^-}$, $\mathfrak{p}_d$ and $\mathfrak{p}_{d^-}$ are in Zariski spectra of all rings between C_1 and D_1, similarly for the inclusion for the restriction to X. The map Spec(A) $\longrightarrow$ Spec($C_1\upharpoonright_X$) is not necessarily surjective (whereas it always is when $X = \mathcal{R}$). For example, if $X = (0,1)$ and $A = C(X)$, then Spec($C_1\upharpoonright_X$) contains the ideal $\mathfrak{p}_{\searrow 0}$, whereas this set is not an ideal in A.

We now consider rings between B_1 and D_1.

Definition 2.5.9. Let $B_1 \subseteq E \subseteq D_1$. Let $\alpha : (0,1) \longrightarrow \mathcal{R}$ be a fixed semi-algebraic, order-preserving, continuous bijection (so it is a homeomorphism under the order topology). Then we define $E \circ \alpha$ to be the ring of semi-algebraic functions $(0,1) \longrightarrow \mathcal{R}$ of the form $f \circ \alpha$ for some $f \in E$.

We now fix the interval $I = (0,1)$.

Lemma 2.5.10. *Let $I = (0,1)$. Then $C_1\upharpoonright_I = B_1 \circ \alpha$ and $D_1\upharpoonright_I = D_1 \circ \alpha$. Furthermore, for any $B_1 \subseteq E \subseteq D_1$, $E \cong E \circ \alpha$.*

Proof. This follows from the fact that α is a definable homeomorphism. □

We thus have the following commuting diagram:

$$\begin{tikzcd}
D_1 \arrow[r, two heads, "\pi_I"] & D_1\upharpoonright_I \arrow[r, "\cong"] & D_1 \\
(E \circ \alpha)_{D_1} \arrow[u, hook] \arrow[r, two heads, "\pi_I"] & E \circ \alpha \arrow[u, hook] \arrow[r, "\cong"] & E \arrow[u, hook] \\
(C_1\upharpoonright_I)_{D_1} \arrow[u, hook] \arrow[r, two heads, "\pi_I"] & C_1\upharpoonright_I \arrow[u, hook] \arrow[r, "\cong"] & B_1 \arrow[u, hook]
\end{tikzcd}$$

The upward injections are simply inclusions, the isomorphisms are given by Lemma 2.5.10, the surjections are just the restriction maps.

This induces the following maps of the Zariski spectra, such that the diagram commutes:

$$\begin{tikzcd}
\mathrm{Spec}(D_1) \arrow[d, hook] & \mathrm{Spec}(D_1{\upharpoonright}_I) \arrow[l, hook'] \arrow[d, hook] & \mathrm{Spec}(D_1) \arrow[l, "\cong"'] \arrow[d, hook] \\
\mathrm{Spec}((A\circ\alpha)_{D_1}) \arrow[d] & \mathrm{Spec}(A\circ\alpha) \arrow[l, hook'] \arrow[d] & \mathrm{Spec}(A) \arrow[l, "\cong"'] \arrow[d] \\
\mathrm{Spec}((C_1{\upharpoonright}_I)_{D_1}) & \mathrm{Spec}(C_1{\upharpoonright}_I) \arrow[l, hook'] & \mathrm{Spec}(B_1) \arrow[l, "\cong"']
\end{tikzcd}$$

The right-left homeomorphisms and injections come simply from the ring isomorphisms and surjections in the ring homomorphism diagram. The arrow $\mathrm{Spec}\, D_1 \hookrightarrow \mathrm{Spec}(A)$ is an injection since the ideals $\mathfrak{p}_{d^-}$, $\mathfrak{p}_d$ and $\mathfrak{p}_{d^-}$ are in Zariski spectra of all rings between B_1 and D_1. The arrow $\mathrm{Spec}(D_1{\upharpoonright}_I) \hookrightarrow \mathrm{Spec}(A\circ\alpha)$ is an injection for similar reasons; alternatively it is injective since the top left square commutes and the map $\mathrm{Spec}(D_1) \cong \mathrm{Spec}(D_1{\upharpoonright}_I)$ is a surjection.

This allows us to obtain many results from previous sections, now for all rings between B_1 and D_1.

Proposition 2.5.11. *Let $f \in D_1$, then f can be written as $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i}\chi_i$, for some $n \in \mathbb{N}$, such that*

- *$\{X_i\}$ is a semi-algebraic partition of $\mathcal{R}$*
- *each X_i is a maximal f-set, and so is either a point, an interval or an interval with finitely many points removed*
- *$i < j$ if and only if there is $x \in X_i$ such that $x < y$ for all $y \in X_j$*
- *$g_i, h_i \in B_1$, for all i*
- *$\frac{g_i(x)}{h_i(x)} = f(x)$, for all $x \in X_i$, for all i*
- *$h_i(x) \neq 0$, for all $x \in X_i$*

Proof. Let $\alpha : (0,1) \longrightarrow \mathcal{R}$ be a definable homeomorphism. Then $f \circ \alpha \in D_1{\upharpoonright}_I$, and we define $f' \in D_1$, which equals $f \circ \alpha$ on I and is 0 otherwise. Then by Proposition 2.3.2, f' has a standard representation $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i}\chi_i$, then $f \circ \alpha =$

$\sum_{i=1}^{i=n} \frac{g_i\restriction_I}{h_i\restriction_I}\chi_i\restriction_I$, where $g_i\restriction_I$ and $h_i\restriction_I$ are bounded on $(0,1)$. Then

$$f = \sum_{i=1}^{i=n} \frac{g_i\restriction_I \circ \alpha^{-1}}{h_i\restriction_I \circ \alpha^{-1}}\chi_i\restriction_I \circ \alpha^{-1}$$

is an appropriate representation. □

We give a definition. Then by similar methods to the proof of Proposition 2.5.11 we obtain:

Definition 2.5.12. Let $bsa(\mathcal{R}) := \{f \in D_1 \mid f \text{ is bounded}\}$. Here, unlike in the definition of V_1 we mean that there is some $r \in \mathcal{R}$ such that for all $x \in \mathcal{R}$, $\big|f(x)\big| < r$. The ring $bsa(\mathcal{R})$ is the convex hull of B_1 in D_1.

Proposition 2.5.13. *Let $f \in D_1$. Then writing f in its standard representation $f = \sum_{i=1}^{i=n} \frac{g_i}{h_i}\chi_i$, given by Proposition 2.5.11, for some $n \in \mathbb{N}$, we have $C_1[f] = C_1[\chi_1, \ldots, \chi_n][\frac{1}{h}]$ for some $h \in bsa(\mathcal{R})$.*

Corollary 2.5.14. *Let $f \in D_1$. Then there is $h \in bsa(\mathcal{R})$ such that $C_1[f] = C_1[\frac{1}{h}]$.*

We recognize that in B_1, there is not only the prime ideal $\mathfrak{p}_\infty$ of functions that are eventually 0, but also the prime ideal

$$\mathfrak{p}_{\nearrow\infty} := \{f \in B_1 \mid \lim_{t\to\infty} f(t) = 0\}$$

This is of course corresponds to the ideal $\mathfrak{p}_{\nearrow 1}$ in the ring $C_1\restriction_I$. So our results about prime ideals also hold:

Definition 2.5.15. Let $B_1 \subseteq A \subseteq D_1$. Then we say that A is of type $\langle 1\rangle$ at $-\infty$ if there is a function $f \in A$ such that $\lim_{t\to-\infty} f(t) = \pm\infty$. We say A is of type $\langle 2\rangle$ at $-\infty$, if all functions of A are bounded at $-\infty$. Similarly for $+\infty$, however, if A is bounded at $= \infty$, we say A is of type $\langle 3\rangle$ at ∞.

Let Ξ be the set of all functions $\xi : \mathcal{R} \cup \{-\infty, +\infty\} \longrightarrow \{1, \ldots, 10\}$ such that $\xi(-\infty) \in \{1, 2\}$ and $\xi(+\infty) \in \{1, 3\}$.

Theorem 2.5.16. *There is a bijection between Ξ and the set of rings between B_1 and D_1, which sends a function ξ to the ring of functions which is of type $\big\langle\xi(d)\big\rangle$ at d, for each $d \in \mathcal{R} \cup \{-\infty, +\infty\}$.*

Proposition 2.5.17. *Let $f \in D_1$. Then there exist unique $k, l, p, q \in \mathbb{N}$, such that*

$$B_1[f] \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,1)))^p \times C(0,1)^q$$

Theorem 2.5.18. *Let A be a ring such that $B_1 \subseteq A \subseteq D_1$. Then A is a real closed ring.*

2.6 Sentences separating isomorphism classes of finite extensions of C_1

In this section we give a schema of sentences such that any one sentence holds true in precisely one isomorphism class of the finite extenions of B_1. First we find a sentence separating B_1 and C_1.

Definition 2.6.1. For any ring A, we define the **Jacobson radical of** A to be $\bigcap_{\mathfrak{m}\in\mathrm{Spec}(A)^{\max}} \mathfrak{m}$, the intersection of all maximal ideals of A. We denote it $\mathrm{Jac}(A)$.

For $g \in A$ we define the **Jacobson radical of** g to be $\bigcap_{\mathfrak{m}\in\mathrm{Spec}^{\max}, g\in\mathfrak{m}} \mathfrak{m}$, the intersection of all the maximal ideals containing g. We denote it $\mathrm{Jac}(g)$. Note that $\mathrm{Jac}(0) = \mathrm{Jac}(A)$.

Remark 2.6.2. By [Tre07, Section 4], the relation $f \in \mathrm{Jac}(g)$ is $\emptyset$-definable in any ring A by the formula

$$\forall u\ \exists v, w,\ 1 + wg = v(1 + uf)$$

and is denoted $f \prec g$.

We follow Section 4 of [Tre07] to obtain a sentence separating B_1 and C_1.

Definition 2.6.3. Let A be a ring of functions from a set S to a field K. Then S is **represented** in K if

- for all $f, g \in A$, $f \prec g \iff \{g = 0\} \subseteq \{f = 0\}$
- for all $x, y \in S$, there is $f \in A$ with $f(x) \neq f(y)$

where $\{g = 0\} := \{x \in S \mid f(x) = 0\}$.

Clearly the set $[0, 1]$ is represented in $C([0, 1])$.

We now let $\mathrm{pt}(x)$ be the formula

$$\mathrm{pt}(x) := \forall u\ ux \neq 1 \wedge (\forall y\ (\forall u\ uy \neq 1) \wedge x \prec y \longrightarrow y \prec x)$$

Then by [Tre07], for any $f \in C([0,1])$,

$$\begin{aligned} C([0,1]) \models \mathrm{pt}(f) &\iff f \text{ has exactly one zero in } [0,1] \\ &\iff \mathrm{Jac}(f) \ \text{ is a maximal ideal} \end{aligned}$$

We then let the sentence LINE be given by:

$$\forall x\ \mathrm{pt}(x) \longrightarrow \exists y\ (x \succ\prec y \wedge y \not\geq 0 \wedge y \not\leq 0)$$

and the formula $\mathrm{COMP}(x)$ (for "comparable") be given by:

$$\mathrm{pt}(x) \wedge \forall y\ (x \succ\prec y \longrightarrow y \leq 0 \vee y \geq 0)$$

Proposition 2.6.4. *There are sentences separating C_1, $C([0,\infty))$ and $C([0,1])$.*

Proof. We claim that the following sentences hold in precisely one of C_1, $C([0,\infty))$ and $C([0,1])$

$$\begin{aligned} C_1 &\models \mathrm{LINE} \\ C([0,\infty)) &\models \exists x\ (\mathrm{COMP}(x)\ \wedge \forall z\ \mathrm{COMP}(z) \longrightarrow x \succ\prec z) \\ C([0,1]) &\models \exists x_1, x_2\ (\mathrm{COMP}(x_1) \wedge \mathrm{COMP}(x_2) \wedge x_1 \not\prec x_2\ \wedge x_2 \not\prec x_1) \end{aligned}$$

By [Tre07] $C_1 \models \mathrm{LINE}$: if $C_1 \models \mathrm{pt}(f)$, then f has precisely one zero in $\mathcal{R}$, d say. Then the function $x \mapsto (x - d)$ is in C_1, has the same Jacobson radical as f and is incomparable with the 0 function.

In $C([0,\infty))$ the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \end{cases}$$

is in the maximal ideal $\mathfrak{m}_0 = \mathfrak{m}_{\searrow 0}$ and no other maximal ideal. But now for any $g \in C([0,\infty))$ such that $\mathrm{Jac}(f) = \mathrm{Jac}(g)$, then g similarly has no zeroes on $(0,\infty)$, in which case, by continuity and the intermediate value theorem, $g \leq 0$

or $g \geq 0$, so $C([0,\infty)) \models \mathrm{COMP}(f)$. Now suppose $C([0,\infty)) \models \mathrm{COMP}(f')$ for some f'. Then f' has no zeroes in $(0,\infty)$, thus the only maximal ideal it can lie inside is $\mathfrak{m}_0$, i.e. $\mathrm{Jac}(f) = \mathrm{Jac}(f')$.

In $C([0,1])$ the functions $f : x \mapsto x$ and $g : x \mapsto 1-x$ lie in precisely the ideals $\mathfrak{m}_0 := \{f \in C([0,1]) \mid f(0) = 0\}$ and $\mathfrak{m}_1$ respectively. Any f' with $\mathrm{Jac}(f') = \mathfrak{m}_0$, has only one zero (at 0) and by continuity and the intermediate value theorem, $f' \geq 0$ or $f' \leq 0$. Similarly for g and $\mathfrak{m}_1$.

Finally, the sentences given are clearly pairwise inconsistent. $\square$

Corollary 2.6.5. *There is a sentence separating C_1 and B_1. In other words:*

$$C_1 \not\equiv B_1$$

Theorem 2.6.6. *Let A, A' be non-isomorphic finite extensions of B_1 by elements of D_1. Then there is a sentence separating A and A', thus $A \not\equiv A'$.*

Proof. Let $A = B_1[f]$ and $A' = B_1[g]$ for appropriate $f, g \in D_1$. By Prop 2.5.17, we have

$$A \cong \mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^p \times C_1^q$$

and

$$A' \cong \mathcal{R}^{k'} \times (C([0,1]))^{l'} \times (C([0,\infty)))^{p'} \times C_1^{q'}$$

for some $k, k', l, l', p, p', q, q' \in \mathbb{N}$. By assumption of $A \ncong A'$, then one of $k \neq k'$, $l \neq l'$, $p \neq p'$, $q \neq q'$ holds. We construct a sentence which holds in A and not in A' which asserts the following:

- There are exactly k idempotent elements in A $\chi_1, \dots, \chi_k$, such that for $1 \leq i \leq k$, $\chi_i \cdot A$ is a field
- There are exactly l idempotent elements in A $\chi_1, \dots, \chi_l$, such that for $1 \leq i \leq l$, the sentence

  $$\exists x_1, x_2 \; (\mathrm{COMP}(x_1) \wedge \mathrm{COMP}(x_2) \wedge x_1 \not\prec x_2 \; \wedge x_2 \not\prec x_1)$$

  holds in $\chi_i \cdot A$
- There are exactly p idempotent elements in A $\chi_1, \dots, \chi_k$, such that for $1 \leq i \leq p$, the sentence

  $$\exists x \; (\mathrm{COMP}(x) \; \wedge \forall z \; \mathrm{COMP}(z) \longrightarrow x \succ\!\prec z)$$

holds in $\chi_i \cdot A$, and $\chi_i \cdot A$ is not a field

- There are exactly q idempotent elements in A $\chi_1, \ldots, \chi_k$, such that for $1 \leq i \leq q$, the sentence LINE holds in $\chi_i \cdot A$

In each case the necessary idempotents are the just the idempotents whose isomorphic copies in $\mathcal{R}^k \times (C([0,1]))^l \times (C([0,\infty)))^p \times C_1^q$ are the function that is 1 on one of the factors and 0 on all other factors. In fact, these are precisely the idempotents $\chi_1, \ldots, \chi_n$ in the standard representation $f = \sum \chi_i \cdot \frac{g_i}{h_i}$. □

Proposition 2.6.7. *We can separate B_1, C_1 and D_1 from all other rings between B_1 and D_1.*

Proof. Let SEP_{B_1,C_1} be the sentence

$$\forall x, y, z \ (x^2 = x \wedge y^2 = y \wedge z^2 = z) \longrightarrow (x = y \wedge y = z \wedge z = x)$$

which says "there are at most two idempotent elements". This is true in B_1, C_1, the ring of continuous semi-algebraic functions that are bounded at $-\infty$ and unbounded at $+\infty$, and the ring of continuous semi-algebraic functions that are unbounded at $-\infty$ and bounded at $+\infty$. In these cases the only idempotent elements are the functions which are constantly 0 or 1. Any proper finite extension has additional idempotents. We have sentences to separate B_1 and C_1 from Proposition 2.6.4.

Let SEP_{D_1} be the sentence

$$\forall x \exists y \ xyx = x$$

which says "the ring is a von Neumann regular ring". By Theorem 2.3.12, for any proper subring of $C_1 \subseteq A \subsetneq D_1$ there is a point $d \in \mathcal{R}$ such that for all $f \in A$, there is $\epsilon > 0$ such that f is bounded on an interval $(d-\epsilon, d)$ or on $(d, d+\epsilon)$. In which case $f : x \mapsto x - d$ cannot have a pseudo-inverse. □

Remark 2.6.8. It is not clear whether there is a sentence that separates $bsa(\mathcal{R})$ from all other rings between B_1 and D_1. Sentences that try and encode information about the Zariski spectrum seem to fail. We would like to say "A has prime ideals that look like $\mathfrak{p}_{\searrow d}$, and $\mathfrak{p}_{\nearrow d}$ at every d" or "for each $d \in \mathcal{R}$ there is a function with $\lim_{t \nearrow d} f(t) = 0$, and no pseudo-inverse". However, the discontinuity at all points of $bsa(\mathcal{R})$ means that we divorce the cut d^- from its physical point:

for example, there are automorphisms of V_1 and of D_1 that “swap” the intervals $[0, 1)$ and $[1, 2)$.

Chapter 3

Sheaves of Structures and the Ring $D_1^K\restriction_{\mathcal{R}}$

3.1 Sheaves

We assume that the reader is familiar with the basics of sheaf theory. For a reference see [MM92].

Sheaf theoretic techniques have been used to prove decidability results since the 1970's in papers by Comer [Com74] and Macintyre [Mac73], with useful surveys and further additions by Burris and Werner ([BW79]). Weispfenning uses a similar method to prove model completeness and quantifier elimination results [Wei75]. This method is an extension of the Feferman-Vaught technique [FV59], in which we view a sheaf (often called an étalé space in the literature) as a kind of product. A recent addition to this technique by Astier [Ast08] proves that D_n is decidable for all n, and gives a slightly more general set of conditions for the technique to be applied.

The results in this chapter are motivated by a 1980 paper of Cherlin [Che80], which was split into two parts: the first proved that if X is a non-discrete metric space, then the ring of all continuous functions $X \longrightarrow \mathbb{R}$ interprets second order arithmetic. Tressl in [Tre08] has proven an analogous result for continuous semi-algebraic functions: for any real closed field $\mathcal{R}$, the ring $C_2(\mathcal{R})$ of continuous semi-algebraic functions $\mathcal{R}^2 \longrightarrow \mathcal{R}$ interprets $(\mathbb{Z}, 0, 1, +, -, .)$. The second part of the Cherlin paper proves: if X is the Stone-Čech compactification of a discrete set, then the ring of all continuous functions $X \longrightarrow \mathbb{R}$ is decidable. The analogous result for semi-algebraic functions would therefore be that the ring of bounded semi-algebraic functions $\mathcal{R} \longrightarrow \mathcal{R}$ is decidable. We initially thought we had proven this result, however, the structure formed by the sheaf construction interprets certain relations in a surprising way. This sheds light on the difficulties surrounding this problem.

The ethos of the Feferman-Vaught technique in [FV59] is to take structures which are represented as certain kinds of products (direct, ordinal, etc.) and reduce questions about them to questions about the factors and index set (which has some kind of structure attached to it). In the sheaf theoretic version, we study a structure of global sections on an étalé space. In some ways this is more general, since many of the products in [FV59] can be thought of as simple kinds of sheaves. However, we often have to restrict ourselves to *positively model complete* theories, the definition and characterization of which we give later.

To begin, we give a brief overview of sheaves of $\mathcal{L}$-structures for a fixed language $\mathcal{L}$. First we look at sheaves in the traditional sense: contravariant functors from the open sets of a topological space X to the category of all $\mathcal{L}$-structures. This gives rise to an étalé space (or sheaf space). In the literature ([Com74], [Mac73], etc.), the étalé space construction is referred to as a sheaf of $\mathcal{L}$-structures. To avoid confusion, we call the functor definition of sheaf an **$\mathcal{L}$-sheaf**. It is well known that there is an equivalence between the category of sheaves of $\mathcal{L}$-structures and the category of $\mathcal{L}$-sheaves. We then devote a brief section to Astier's reformulation of Comer's Feferman-Vaught-like sheaf result.

Finally, we look at a ring $D_n^K\restriction_{\mathcal{R}}$ obtained by restricting the ring of functions semi-algebraic functions $K^n \longrightarrow K$ to a subfield $\mathcal{R}$, where K is a tame extension of $\mathcal{R}$. The Feferman-Vaught technique fails due to a problem related to coheirs of types of the smaller field $\mathcal{R}$ in K.

Notation 3.1.1. Throughout this section, we fix a countable first order language $\mathcal{L}$; we denote its constants by $c, c_1, c_2, \ldots, c_n, \ldots$, its operation or function symbols by $f, f_1, f_2, \ldots, f_n, \ldots$, and its predicate or relation symbols by $R, R_1, R_2, \ldots, R_n, \ldots$. We will denote by $Cat(\mathcal{L})$ or by $\mathtt{C}$ the category whose objects are all $\mathcal{L}$-structures and whose arrows are the $\mathcal{L}$-homomorphisms between them. We denote $\mathcal{L}$-structures by A or B and this will also denote the carrier/universe of the structure where no confusion arises; ϕ, ψ, χ will denote $\mathcal{L}$-homomorphisms.

We begin with a reminder of some definitions and then a theorem proving the existence of direct limits in $Cat(\mathcal{L})$. The important part of this theorem is the construction of the direct limit *as an $\mathcal{L}$-structure*, and particularly the treatment of relations.

Definition 3.1.2. A poset $(I, \leq)$ is a **directed set**, if any two elements have an (not necessarily least) upper bound. A **direct system** of $\mathcal{L}$-structures and $\mathcal{L}$-homomorphisms indexed by I and $I \times I$ is a pair (A_i, ϕ_{ij}), where each A_i is an $\mathcal{L}$-structure and for $i \leq j$, $\phi_{ij} : A_i \longrightarrow A_j$ is an $\mathcal{L}$-homomorphism, such that for each $i, j, k \in I$ with $i \leq j \leq k$: ϕ_{ii} is the identity isomorphism on A_i and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$.

Definition 3.1.3. Let (A_i, ϕ_{ij}) be a direct system indexed by I. Let $a_i \in A_i$ and $a_j \in A_j$. Then we say a_i **equals** a_j **eventually** if there is some $k \in I$ such that $\phi_{ik}(a_i) = \phi_{jk}(a_j)$.

Proposition 3.1.4 ([Ell74, p.165]). ***Cat(*$\mathcal{L}$*)*** *has direct limits and a direct limit in* ***Cat(*$\mathcal{L}$*)*** *of the direct system* (A_i, ϕ_{ij}) *has as its universe the direct limit of the universes of the* A_i *in* `Set`*, which is* $A := \varinjlim_{i\in I} A_i = \coprod_{i\in I} A_i/\sim$*, where, for* $a_i \in A_i$ *and* $a_j \in A_j$*,* $a_i \sim a_j \iff$ *there is some* $k \geq i, j$ *such that* $\phi_{ik}(a_i) = \phi_{jk}(a_j)$*, and the arrows* $\psi_i : A_i \longrightarrow A$ *simply map elements of* A_i *to their appropriate equivalence classes. Symbols are interpreted as follows:*

- *for a constant* c*,* $c = a \in A \iff \exists i \in I, u \in A_i$ *such that* $A_i \models u = c$ *and* $\psi_i(u) = a$
- *for an* n*-ary function* f*,* $f(\overline{a}) = b \iff \exists i \in I, \overline{u}, v \in A_i$ *such that* $A_i \models f(\overline{u}) = v$ *and* $\psi_i(\overline{u}) = \overline{a}$*,* $\psi(v) = b$
- *for an* n*-ary relation symbol* R*,* $A \models R(\overline{a}) \iff \exists i \in I, \overline{u} \in A_i$ *such that* $A_i \models R(\overline{u})$ *and* $\psi_i(\overline{u}) = \overline{a}$

Proof. The enlightening part of the proof is for relations. There are clearly many possibilities for the interpretation of the relation symbols on the set A, however, the interpretation of any relation symbol R in the direct limit must be, in a sense, minimal, i.e. $A \models R(\overline{a})$ only when it is necessary. We make this precise now.

First note that if we have $\overline{u} \in A_i^n$ for some $i \in I$ such that $[\overline{u}]_\sim = \overline{a}$ and $A_i \models R(\overline{u})$, then since ψ_i is a homomorphism, and $\psi_i(\overline{u}) = \overline{a}$, we must have $A \models R(\overline{a})$. Note if such an i exists, then, for any $j \geq i$, since ϕ_{ij} is an $\mathcal{L}$-homomorphism, we must have $A_j \models R(\phi_{ij}(\overline{u}))$. Now suppose we have some n-tuple $\overline{a} \in A^n$ such that $A \models R(\overline{a})$ and suppose that there is no $i \in I$ and $\overline{u} \in A_i^n$ such that $A_i \models R(\overline{u})$ and $[\overline{u}]_\sim = \overline{a}$. Then let (A', ψ_i) be the pair such that A' is the structure A except that $A' \not\models R(\overline{a})$ and ψ_i are the same functions $A_i \longrightarrow A'$ between universes (and ψ_i are clearly still $\mathcal{L}$-homomorphisms). Then there is no $\mathcal{L}$-homomorphism $\omega : A \longrightarrow A'$, thus A' does not factor through A and so A cannot be the direct limit. So for any direct limit (A, ψ_i) and n-tuple $\overline{a}$, if $A \models R(\overline{a})$, then there is A_i and $\overline{u} \in A_i^n$ such that $A_i \models R(\overline{u})$ and $\overline{u} \in \overline{a}$.

That this structure is the universal cocone follows from basic category theory.

□

Definition 3.1.5. Let X be a topological space, $\mathtt{C} := Cat(\mathcal{L})$ for fixed $\mathcal{L}$. An **$\mathcal{L}$-sheaf** is a triple $(X, \mathcal{L}, F)$ (or where X and $\mathcal{L}$ are fixed, simply F) such that:

(1) for each open set U in X, $F(U) \in Obj(\mathtt{C})$, i.e. is an $\mathcal{L}$-structure

(2) for each inclusion $V \subseteq U$ of open sets, there is an $\mathcal{L}$-homomorphism $res_{V,U} : F(U) \longrightarrow F(V)$ such that

- for every open set W, $res_{W,W} : F(W) \longrightarrow F(W)$ is the identity map
- for U, V, W with $W \subseteq V \subseteq U$, $res_{W,V} \circ res_{V,U} = res_{W,U}$

(3) (Local Identity) if $\{U_i\}$ is an open cover of an open set $U \subseteq X$ and $s, t \in F(U)$, such that $s\restriction_{U_i} = t\restriction_{U_i}$ for each U_i, then $s = t$, where $s\restriction_{U_i}$ means $res_{U_i,U}(s)$

(4) (Gluing) if $\{U_i\}$ is an open cover of an open set $U \subseteq X$, and, if for each i, there is a section s_i of F over U_i such that for each pair U_j, U_k we have $s_j\restriction_{U_j \cap U_k} = s_k\restriction_{U_j \cap U_k}$, then there is a section $s \in F(U)$ such that $s\restriction_{U_i} = s_i$ for each i

Consider $\mathcal{O}(X)$, the set of open sets of X, as a category with open sets as the objects and inclusion as arrows. Then F is a contravariant functor from $\mathcal{O}(X)$ to $\mathtt{C}$. Each element of $F(U)$ is called a **sheaf section**, or simply **section**, an element of $F(X)$ is called a **global section**. The restriction $res_{V,U}(s)$ can be written $s\restriction_V$. It should also be noted that $F(\emptyset)$ is terminal in the image category $F[\mathcal{O}(X)]$.

The following result allows us to extend a sheaf on a basis (a generating set of the topology closed under finite intersection) to a set on the whole space. The result is well known.

Theorem 3.1.6 (Basis Extension Theorem [Per08, Lemma 3.2.1])**.** *Let X be a topological space and suppose that $\mathcal{B}(X)$ is a topological basis for X. Let F be a contravariant functor from $\mathcal{B}(X)$ to $\mathtt{C}$, where $\mathcal{B}(X)$ is ordered by inclusion, and where the arrow $V \subseteq U$ is mapped to some $\mathcal{L}$-homomorphism which we call $res_{V,U}$. Additionally, suppose that for any $U \in \mathcal{B}$, if $\{U_i\}$ is a cover of basis sets of U and for $s, t \in F(U)$ we have $s\restriction_{U_i} = t\restriction_{U_i}$, then $s = t$; and, if for each U_i we have some $s_i \in F(U_i)$ such that $s_i\restriction_{U_i \cap U_j} = s_j\restriction_{U_i \cap U_j}$ for all i, j, then there is $s \in F(U)$ such that $s\restriction_{U_i} = s_i$ for all i. This is called a* ***sheaf on a basis****.*

Then we can extend F uniquely (up to isomorphism) to a sheaf on all the open sets of X.

Definition 3.1.7. Fix some point $x \in X$. We now consider the set

$$\coprod_{U \in \mathcal{O}(X), U \ni x} F(U)$$

and define a relation $\sim_x$ on this set such that if $s \in F(U)$ and $t \in F(V)$, then $s \sim_x t \iff$ there is an open neighbourhood $W \subseteq U \cap V$ of x such that $res_{W,U}(s) = res_{W,V}(t)$. This is clearly an equivalence relation. For a section $s \in F(U)$, we call $[s]_{\sim_x}$ the **germ** of s at x and denote it s_x (since this is independent of U). We denote the set of germs at a point x by F_x, and call this the **stalk** at x.

Proposition 3.1.8. *For each $x \in X$, $F_x = \varinjlim_{U \ni x} F(U)$ where we take the direct limit, in $\mathtt{C}$, over open neighbourhoods of x and where the set of open neighbourhoods is made directed by the ordering $U \leq V \iff V \subseteq U$.*

Proof. We can see that for each $x \in X$, F_x is the direct limit of the direct system $(F(U), res_{V,U})$ in **Set**. By Proposition 3.1.4, this is the universe for the direct limit in $\mathtt{C}$ and it also gives us the suitable homomorphisms, which we will denote $\phi^F_{U,x}$ for each open neighbourhood U of x. □

Definition 3.1.9. Let $(X, \mathtt{C}, F)$ be a sheaf. The **étalé space** of $(X, \mathtt{C}, F)$ is a triple (X, S, π) where:

- $S = \coprod_{x \in X} F_x$
- π is the projection of S onto X, taking $s_x \in F_x$ to $x \in X$

S is also equipped with the following topology: let $U \in \mathcal{O}(X)$, $s \in F(U)$, then the set $\{s_x \mid x \in U\}$ is declared open and the sets of this form a subbasis (in fact we prove that this is a basis).

We denote the étalé space of $(X, \mathtt{C}, F)$ by $Et((X, \mathtt{C}, F))$, or simply by $Et(F)$, when X and $\mathtt{C}$ are fixed or S_F.

Proposition 3.1.10. *The subbasis given in the definition of the étalé space of a sheaf F is in fact a basis (closed under finite intersection), π is a local homeomorphism and each stalk has the discrete subtopology.*

Proof. Let $U, V \subseteq X$ and let $s \in F(U)$, $t \in F(V)$. Define $Q_{s,t}$ by

$$Q_{s,t} := \{x \in U \cap V \mid \exists \text{ open nbd } W \subseteq U \cap V \text{ of } x \text{ s.t. } res_{W,U}(s) = res_{W,V}(t)\}$$

We claim that this set is open: take $x \in Q_{s,t}$, then there is some open neighbourhood W of x such that $res_{W,U}(s) = res_{W,V}(t)$. Now for any $y \in W$, W is an open neighbourhood of y, so $y \in Q_{s,t}$, and so $Q_{s,t}$ is open. Now, we see that

$$\{s_x \mid x \in U\} \cap \{t_x \mid x \in V\} = \{s_x \mid x \in Q_{s,t}\} = \{t_x \mid x \in Q_{s,t}\}$$

So we do indeed have a basis.

Now take $s_x \in S$ and pick some $s \in s_x$, such that $s \in F(U)$ for some $U \in \mathcal{O}(X)$. Then $T := \{s_y \mid y \in U\}$ is open in S, and $s_x \in T$. We can see that $\pi(T) = U$ and is open in X. Clearly $\pi\restriction_T$ is injective, since s is fixed. It is also surjective, since for any $y \in U$ we have $s_y \in T$ by definition. So $\pi\restriction_T$ is bijective. For any open $T' \subseteq T$ in the basis of the topology of S, i.e. $T' := \{s_y \mid y \in V\}$ for some open V in X, we can see that $\pi\restriction_T(T') = V$, so $\pi\restriction_T$ is an open map. Now take some open $V \subseteq U$, then $(\pi\restriction_T)^{-1}(V)$ is well-defined and is equal to $\{s_y \mid y \in V\}$, which is open in T, so $\pi\restriction_T$ is continuous. Thus $\pi\restriction_T$ is a homeomorphism and π is a local homeomorphism.

Take any $x \in X$ and let $s_x \in F_x$. By the direct limit construction, there is some open $U \subseteq X$ and some $s \in F(U)$ such that $\varinjlim_{U \ni x} s = s_x$. Then $\{s_y \mid y \in U\} \cap F_x = \{s_x\}$, and so the stalk has the discrete subtopology. □

We are ready to introduce the notion of a sheaf of $\mathcal{L}$-structures, as given in [Com74] et. al. We first define the notion of "sheaf of $\mathcal{L}$-structures", then show that these are precisely the étalé spaces of $\mathcal{L}$-sheaves. We are not aware of an explicit proof in the literature, and include such results for completeness.

Definition 3.1.11. For a fixed language $\mathcal{L}$, a **sheaf of $\mathcal{L}$-structures** is a triple (X, S, π), such that:

(1) X and S are a topological spaces

(2) π is a local homeomorphism from S onto X

(3) $S_x := \pi^{-1}(x)$ is an $\mathcal{L}$-structure for each $x \in X$

(4) for each constant symbol $c \in \mathcal{L}$ the map from X to S that sends $x \in X$ to the interpretation of c in S_x is continuous

(5) for each n-ary function symbol $f \in \mathcal{L}$ the map from $\bigcup_{x\in X}(S_x^n)$ to S sending $\overline{s} \in S_x$ to the interpretation of $f(s_1, \ldots, s_n)$ in S_x is continuous (where $\bigcup_{x\in X}(S_x^n)$ inherits the topology on $(\bigcup_{x\in X} S_x)^n = S^n$)

(6) for each n-ary relation symbol $R \in \mathcal{L}$, $\{\overline{s} \in \bigcup_{x\in X}(S_x^n) \mid R(\overline{s})\}$ is open in $\bigcup_{x\in X}(S_x^n)$

Comer and Macintyre ([Com74] and [Mac73]) require that the set in (6) is clopen, however Astier [Ast08] shows that this is not necessary for a Feferman-Vaught result, where our stalk theory is positively model complete.

A **local homeomorphism** $\pi : S \longrightarrow X$ is a function such that for each $y \in S$, there is an open neighbourhood T such that $\pi(T)$ is open and $\pi\restriction_T : T \longrightarrow \pi(T)$ is a homeomorphism. Clearly a local homeomorphism is continuous and open.

Proposition 3.1.12. *Let $(X, \mathcal{C}, F)$ be an $\mathcal{L}$-sheaf and let (X, S, π) be its étalé space. Then S is a sheaf of $\mathcal{L}$-structures in the sense of Definition 3.1.11.*

Proof. Clearly (1) and (3) are satisfied. (2) follows from Proposition 3.1.10. For (4), let c be a constant symbol of $\mathcal{L}$, let $\kappa_c : x \mapsto [c]_{S_x}$ and consider an open set $T \subseteq S$. We may assume, wlog, that T is of the form $\{s_x \mid x \in U\}$ for some open $U \subseteq X$ and $s \in F(U)$. So $\kappa_c^{-1}(T) = \{x \in U \mid s_x = [c]_{S_x}\}$. Pick arbitrary $y \in \kappa_c^{-1}(T)$, i.e. $s(y) = [c]_{S_y}$. So s equals $[c]_{F(U)}$ eventually and there is some open $V_y \subseteq U$ such that $res_{V_y,U}(s) = [c]_{F(V_y)}$, so $V_y \subseteq \kappa_c^{-1}(T)$, and so κ_c is continuous.

For (5), let f be an n-ary function and let $\kappa_f : \bigcup_{x\in X} S_x^n \longrightarrow S, (a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$. Let T be an open set in S and wlog, $T = \{s_x \mid x \in U\}$ for some open $U \subseteq X$ and $s \in F(U)$. Then $\kappa_f^{-1}(T) = \{(a_1, \ldots, a_n) \mid x \in U \text{ and } f(a_1, \ldots, a_n) = s_x\}$. For any tuple $(a_1, \ldots, a_n) \in T$, which is in S_x^n for some $x \in U$, we must have some $V_x \subseteq U$ such that for each a_i we have some $s_i \in F(V_x)$ such that $(s_i)_x = a_i$ and $f(s_1, \ldots, s_n) = res_{V_x,U}(s)$. Then for every $y \in V_x$ we must also have $f(s_1, \ldots, s_n) = res_{V_x,U}(s)$, so we can find $(b_1, \ldots, b_n) \in S_y^n$ such that $(s_i)_y = b_i$ and $\kappa_f(b_1, \ldots, b_n) = s_y$, so $y \in \kappa_f^{-1}(T)$, and thus κ_f is continuous.

For (6), let R be an n-ary relation symbol. Let

$$I_R := \bigcup_{x\in X}\{(a_1,\dots,a_n)\in S_x^n \mid S_x \models R(a_1,\dots,a_n)\}$$

and suppose for some $x \in X$ we have $(a_1,\dots,a_n) \in I_R \cap S_x$. Then there is some U_x such that there are $s_1,\dots,s_n$ with $F(U_x) \models R(s_1,\dots,s_n)$ and $(s_i)_x = a_i$ for each $i = 1,\dots,n$. Then for any $y \in U_x$ we must have $S_y \models R((s_1)_y,\dots,(s_n)_y)$ and so $\{((s_1)_y,\dots,(s_n)_y) \mid y \in U_x\} \subseteq I_R$ and this is an open neighbourhood of $(a_1,\dots,a_n)$, thus I_R is open. □

Definition 3.1.13. Let (X, S, π) be a sheaf of $\mathcal{L}$-structures and U an open subset of X. A **section** of (X, S, π) over U is a continuous map $\sigma : U \longrightarrow S$ such that $\pi \circ \sigma = id_U$.

Definition 3.1.14. Let (X, S, π) be a sheaf of $\mathcal{L}$-structures. For each open $U \subseteq X$, we let $\Gamma_S(U)$ be the set of sections over U. We furnish $\Gamma_S(U)$ with the constants, functions and relations of $\mathcal{L}$ as follows:

- for a constant c, the map $\kappa_c : x \mapsto [c]_{S_x}$ is the interpretation of c (this is continuous by the definition of étalé space)
- for an n-ary function symbol f, we let $f(\overline{\sigma}) : x \mapsto f(\sigma_1(x),\dots,\sigma_n(x))$
- for an n-ary relation symbol R, $\Gamma_S(U) \models R(\sigma_1,\dots,\sigma_n)$ if and only if for all $x \in U$, $S_x \models R(\sigma_1(x),\dots,\sigma_n(x))$

We define $\Gamma_S : \mathcal{O}(X) \longrightarrow \mathtt{C}$ to be the contravariant functor taking an open set U to $\Gamma_S(U)$ and which takes the arrow $U \supseteq V$ to the arrow $res_{V,U}$ which maps a section $\sigma : U \longrightarrow S$ to the restricted map $\sigma\restriction_V : V \longrightarrow S, x \mapsto \sigma(x)$.

Proposition 3.1.15. *Γ_S is an $\mathcal{L}$-sheaf.*

Proof. The only difficulty is in proving local identity and gluing. Let $U \subseteq X$ be open and let $\{U_i\}_{i\in I}$ be an open cover. Let σ and τ be sections over U such that for each $i \in I$, $\sigma\restriction_{U_i} = \tau\restriction_{U_i}$. Then $\sigma = \tau$ since equality of functions is defined pointwise. Now suppose we have $\{\sigma_i\}_{i\in I}$ such that for each i, σ_i is a section over U_i and that for any $i, j \in I$, $\sigma_i\restriction_{U_i\cap U_j} = \sigma_j\restriction_{U_i\cap U_j}$. Then we can define $\sigma : U \longrightarrow S$ pointwise and each point of U has an open neighbourhood such that σ is equal to a continuous function on that neighbourhood, so σ is continuous. □

Before we prove that each sheaf of $\mathcal{L}$-structures is the étalé space of an $\mathcal{L}$-sheaf, we require the following lemma:

Lemma 3.1.16 ([MM92, 88])**.** *Let (X, S, π) be a sheaf of $\mathcal{L}$-structures, then the following are all true:*

(1) π is an open map, and any section σ over open $U \subseteq X$ is an open, injective function

(2) Every element of S is in the range of some section over some open set

(3) The set of images of sections over all open sets of X forms a basis for the topology on S

(4) If σ and τ are two sections over some open set U, then $\{x \mid s(x) = t(x)\}$ is open in S

Proposition 3.1.17. *Let (X, S, π) be a sheaf of $\mathcal{L}$-structures. Let Γ be the associated sheaf of sections. Let $(X, S^\Gamma, \pi^\Gamma)$ be the étalé space of Γ. Then the following hold:*

- *For each $x \in X$, $S_x \cong S^\Gamma_x$*
- *S is homeomorphic to S^Γ*

Proof. For each $x \in X$, we define a map $\nu_x : S_x \longrightarrow S^\Gamma_x$ as follows: for each $a \in S_x$, let $U \subseteq X$ be an open set with $x \in U$ and let σ be a section of $\Gamma(U)$ such that $\sigma(x) = a$ (such a U and σ exist by Lemma 3.1.16), then $\nu_x(a) = \sigma_x$, the equivalence class of σ in S^Γ_x. This is well-defined, since if $\sigma \in \Gamma(U)$, $\tau \in \Gamma(V)$ for open $U, V \ni x$, and $s(x) = t(x) = a$, then

$$W_{\sigma,\tau} = \{x \mid \sigma\restriction_{U\cap V}(x) = \tau\restriction_{U\cap V}(x)\}$$

is an open set containing a, and so σ equals τ eventually.

To show ν_x is a homomorphism: Let c be a constant, then the constant section $\kappa_c \in \Gamma(X)$ is such that $\kappa_c(x) = [c]_{S_x}$. Then

$$\nu([c]_{S_x}) = (\kappa_c)_x = [c]_{S^\Gamma_x}$$

The result is immediate for functions. Let R be an n-ary relation, and suppose that $S_x \models R(\overline{a})$. Take some open neighbourhood $V \subseteq \bigcup(S_x)^n$ of $\overline{a}$ such that $\pi^n\restriction_V$

is a homeomorphism. Then $W = V^n \cap \bigcup R(S_x)$ is open in $\bigcup(S_x)^n$, and there must be some basic open subset of W defined by some $\overline{\sigma}$ over some $W' \subseteq S$ containing $\overline{a}$. Then $\Gamma(W') \models R(\overline{\sigma})$ and so $S_x^\Gamma \models R(\overline{\sigma}_x)$. Now let $\nu_x(\overline{a}) = \overline{\sigma}_x$ and suppose that $S_x^\Gamma \models R(\overline{\sigma}_x)$. Then there is an open set $U \ni x$ such that $\Gamma(U) \models R(\overline{\sigma}\restriction_U)$. Since $\overline{\sigma}\restriction_U(x) = \overline{a}$, we have $S_x \models R(\overline{a})$. So ν_x is a homomorphism.

That S is homeomorphic to S^Γ is just an application of Lemma 3.1.16 (3). (It is also just the standard result that the categories of sheaves of sets and étalé spaces where the stalks are pure sets are equivalent). □

Therefore sheaves of $\mathcal{L}$-structures are precisely the étalé spaces of $\mathcal{L}$-structures.

3.2 Feferman-Vaught Technique

Having defined sheaves of $\mathcal{L}$-structures, we now give a brief overview of the Feferman-vaught technique for sheaves as developed by Comer [Com74], Macintyre [Mac73] and Astier [Ast08]. Throughout, basic notions of decidability are assumed. We fix a first-order language $\mathcal{L}$, we denote $\mathbf{Cat(\mathcal{L})}$ by `C`; further languages are introduced later in the text. We write the set of sections of a sheaf over X by $F(U)$, for an open set $U \subseteq X$. We give some brief definitions to get us started.

Definition 3.2.1. A topological space X is **totally disconnected** if the only connected components are the singleton sets. A topological space X is **boolean** if it is compact, Hausdorff and totally disconnected.

Definition 3.2.2. A theory T is called **positively model complete** if each of its formulas is equivalent, modulo T, to an $\exists_1^+$ or **positive existential formula** (i.e. a prenex-normal formula whose only quantifier is existential and which has no occurrence of the $\neg$ symbol, and we allow both $\vee$ and $\wedge$ to occur).

We provide two useful semantic characterizations of positive model completeness. We cannot find these in the literature, but the results must be known.

Definition 3.2.3. Let ϕ be an $\mathcal{L}$-formula. Let M, N be $\mathcal{L}$-structures and let $f : M \longrightarrow N$ be an $\mathcal{L}$-homomorphism. Then f **preserves** ϕ if for all $\overline{a} \in M$, $M \models \phi(\overline{a}) \Longrightarrow N \models \phi(f(\overline{a}))$.

Let T be an $\mathcal{L}$-theory. Then ϕ is **preserved under homomorphisms between models of** T if for all $M, N \models T$ and $\mathcal{L}$-homomorphisms $f : M \longrightarrow N$, f preserves ϕ.

Proposition 3.2.4. *Let T be a theory in a language $\mathcal{L}$. Then the following are equivalent:*

(a) T is positively model complete

(b) Every formula is preserved under any homomorphism between models of T

(c) Let $M, N \models T$, then $f : M \longrightarrow N$ is an $\mathcal{L}$-homomorphism if and only if it is an elementary embedding.

(d) Every injective homomorphism between models of T is an elementary embedding and the formula "$x \neq y$" is equivalent modulo T to an $\exists_1^+$ formula

Proof. $[(a) \iff (b)]$ A formula ϕ is preserved under homomorphisms between models of T if and only if it is equivalent modulo T to an $\exists_1^+$ formula, [Hod93, Section 2.4].

$[(b) \implies (c)]$ Since every formula is preserved, then $x \neq y$ and $\neg R(\overline{x})$ are preserved and so every homomorphism between models of T must be an embedding. Since all other formulas (with parameters) are preserved, the only homomorphisms are elementary embeddings.

$[(c) \implies (b)]$ Elementary embeddings preserve all formulas.

$[(c) \implies (d)]$ Obvious from previous results.

$[(d) \implies (c)]$ We need only show that every homomorphism is injective. The formula $x \neq y$ is equivalent to an $\exists_1^+$ formula, ψ, which must be preserved under homomorphisms of models of T. $\square$

We now fix a $\mathtt{C}$-valued sheaf F over a topological space X. Its étalé space is denoted S, and sections are typically denoted s or t (remembering that we can identify the members of $F(U)$ with the continuous sections over U).

Definition 3.2.5. Let $\theta(\overline{u})$ be an $\mathcal{L}$-formula. Let $U \subseteq X$ be open and let $\overline{s} \in F(U)^{\overline{u}}$ be a $|\overline{u}|$-tuple of sections over U. Then we let

$$\left\|\theta(\overline{s})\right\|_U := \{x \in U \mid S_x \models \theta(\overline{s}(x))\}$$

Let L_B be a fixed language expanding the language $\mathcal{L}_{BA} = \langle \cap, \cup, ', \top, \bot \rangle$ of boolean algebras. We let $\mathfrak{C}(X)$ be the boolean algebra of clopen subsets of X and assume that it is an L_B structure.

Definition 3.2.6. An **acceptable sequence** is a sequence

$$\zeta := \left\langle \Phi(U_1, \dots, U_m), \theta_1(\overline{z}), \dots, \theta_m(\overline{z}) \right\rangle$$

where $m \in \mathbb{N}$, Φ is an $\mathcal{L}_B$-formula, with m free variables, and $\theta_1(\overline{z}), \dots, \theta_m(\overline{z})$ are $\mathcal{L}$-formulas with the same free variables $\overline{z}$.

An acceptable sequence is called **partitioning** if $\vdash_{\mathcal{L}} \bigvee_{1 \le i \le m} \theta_i$ and $\vdash_{\mathcal{L}} \neg(\theta_i \wedge \theta_j)$ for $i < j \le m$.

The language $\mathcal{L}_{gp}$ is the **language of generalized products** of $\mathfrak{C}(X)$, its symbols are exactly all the n-ary relations of the form R_ζ for each acceptable sequence $\zeta := \left\langle \Phi(U_1, \dots, U_m), \theta_1(\overline{z}), \dots, \theta_m(\overline{z}) \right\rangle$ and for each n-tuple $\overline{z}$ and each $n \in \mathbb{N}$. For each open $U \subseteq X$ we interpret R_ζ by:

$$F(U) \models R_\zeta(\overline{s}) \iff \overline{s} \in F(U) \text{ and } \mathfrak{C}(X) \models \Phi(\left\|\theta_1(\overline{s})\right\|_U, \dots, \left\|\theta_m(\overline{t})\right\|_U)$$

This definition clearly requires that the sets $\left\|\theta_i(\overline{t})\right\|_U$ are clopen in X. Positive model completeness gives us this:

Lemma 3.2.7. *[Ast08, Lemma 1, p330] Let F be a sheaf, let $\theta(\overline{z})$ be an $\mathcal{L}$-formula, let $U \subseteq X$ be open and let $\overline{s} \in F(U)$. Then*

1. *If $\theta(\overline{z})$ is positive existential, then $\left\|\theta(\overline{s})\right\|_U$ is open in U*
2. *If $Th_{\mathcal{L}}(\{S_x \mid x \in N\})$ is positively model complete, then $\left\|\theta(\overline{s})\right\|_U$ is clopen in U.*

Definition 3.2.8. We call $Th_{\mathcal{L}}(\{S_x\}_{x \in X})$ the **stalk theory** of the étalé space (X, S, π). We call $Th_{\mathcal{L}_{gp}}(F(X))$ the **section theory**.

Lemma 3.2.9. *[FV59, Lemma 2.2, p63] Let $\zeta = \langle \Phi, \theta_1, \dots, \theta_m \rangle$ be an acceptable partitioning sequence. Then for any $\overline{s} \in F(X)$*

(1) $\bigcup_{1 \le i \le m} \left\|\theta_i(\overline{s})\right\|_X = X$

(2) $\left\|\theta_i(\overline{s})\right\|_X \cap \left\|\theta_j(\overline{s})\right\|_X = \emptyset$ *for all* $i \neq j$

Proof. (1) We use the result that for any two formulas $\theta_1(\overline{z})$ and $\theta_2(\overline{y})$ and appropriate $\overline{s}$ and $\overline{t}$, $\left\|\theta_1(\overline{s})\right\|_X \cup \left\|\theta_2(\overline{t})\right\|_X = \left\|\theta_1(\overline{s}) \vee \theta_2(\overline{t})\right\|_X$. Thus

$$\bigcup_{1 \le i \le m} \left\|\theta_i(\overline{s})\right\|_X = \left\| \bigvee_{1 \le i \le m} \theta_i(\overline{s}) \right\|_X$$

But the formula on the right hand side is logically valid, since ζ is partitioning, so it holds in S_x for every $x \in X$, and so the result is proven.

(2) By a similar argument, $\theta_i(\overline{s}) \wedge \theta_j(\overline{s})$ cannot hold in any stalk. □

Theorem 3.2.10. *[Ast08, Proposition 2] or [Com74, Theorem 1.1] There is an effective procedure which assigns to each $\mathcal{L}_{gp}$ formula $\phi(\overline{z})$ an acceptable (partitioning) sequence $\langle \Phi(U_1, \dots, U_m), \theta_1(\overline{z}), \dots, \theta_m(\overline{z}) \rangle$ such that for any étalé space of $\mathcal{L}$-structures (X, S, π), with X a boolean topological space and $Th_{\mathcal{L}}(\{S_x\}_{x\in X})$ positively model complete, any L_B structure on $\mathfrak{C}(X)$ and any $\overline{s} \in F(X)^{|\overline{z}|}$*

$$F(X) \models \phi(\overline{s}) \iff \mathfrak{C}(X) \models \Phi(\left\|\theta_1(\overline{s})\right\|_X, \dots, \left\|\theta_m(\overline{s})\right\|_X)$$

where F is the $\mathcal{L}$-sheaf associated with S.

Burris and Werner [BW79] show that this is a generalization of the original Feferman-Vaught paper.

This leads us immediately to the main tool for proving decidability for sheaves of $\mathcal{L}$-structures.

Theorem 3.2.11. *[Ast08, Corollary 3] Let (X, S, π) be an étalé space of $\mathcal{L}$-structures such that X is boolean and $Th_{\mathcal{L}}(\{S_x\}_{x\in X})$ is positively model complete. Then, if the stalk theory and $Th_{L_B}(\mathfrak{C}(X))$ are both decidable, then the section theory is decidable.*

3.3 Sheaves of Rings

Throughout this section A will denote a commutative unital ring. $\mathcal{L}_R$ will denote the language of rings which has constants $0, 1$, and operations $+, -, .$ and no relations. Multiplication may be written without the . symbol where no confusion arises. We denote the categories of rings in the language $\mathcal{L}_R$ by `Ri`.

Definition 3.3.1. For a ring A, the set of all idempotents form a boolean algebra where, for idempotents a, b, $a \cap b = a.b$ and $a \cup b = a + b - a.b$. We call this the **boolean algebra of idempotents of** A and we denote it B_A or simply B.

We define the set X_A to be the **Stone space** of B, i.e. the space whose sets are the maximal ideals of the boolean algebra B and such that a set in the basis of open sets is of the form $\{x \in X \mid e \notin x\} =: X_e$ for $e \in B_A$.

The topological space X is in fact **boolean**, i.e. it is compact, Hausdorff and totally disconnected (the only connected components are the singleton sets), this is just the Stone Representation Theorem, [Sto36, Theorem 67, p106].

Lemma 3.3.2. *(1) All sets in the basis are clopen.*

(2) The basis of open sets as given above is closed under finite intersection and finite union and $X_{e_1 \cap e_2} = X_{e_1} \cap X_{e_2}$ *and* $X_{e_1 \cup e_2} = X_{e_1} \cup X_{e_2}$.

(3) $e_1 \cap e_2 = e_1 \iff e_1 \cup e_2 = e_2$*, if either side holds we write* $e_1 \leq e_2$

(4) $Ae_1 \subseteq Ae_2 \iff e_1.e_2 = e_1$ *and thus there is a canonical homomorphism* $Ae_2 \longrightarrow Ae_1, ae_2 \mapsto ae_1e_2$

Proof. (1)-(3) is just Stone duality.

(4) If $e_1e_2 = e_1$, then any element a of Ae_1 can be written $a'e_1 = a'e_1e_2 \in Ae_2$, for some $a' \in A$. If $Ae_1 \subseteq Ae_2$, then $e_1 \in Ae_2$, and since e_2 is the multiplicative unit in Ae_2, we have $e_1e_2 = e_1$. □

We can define a contravariant functor F from the set of basis sets, ordered by inclusion, to `Ri` by sending X_e to $A \cdot e$, and the arrow $X_{e_i} \subseteq X_{e_j}$ to the map $\lambda_{j,i} : Ae_j \longrightarrow Ae_i, ae_j \mapsto ae_ie_j = ae_i$.

In fact, Ae is isomorphic to A_e, the localization of A at e and this isomorphism is canonical by $f \cdot e \mapsto \frac{f}{1}$. We denote the equivalence relation induced by the localization by $\sim$. We note that $\frac{f}{1} \sim \frac{f}{e}$ if and only if $e(f - fe) = 0$, which is always true so $\frac{f}{1} \sim \frac{f}{e}$ for any $f \in A$. The map is well-defined and injective since $f \sim g \iff e(f - g) = 0$. It is clearly surjective.

Proposition 3.3.3. *The functor F from the set of basis sets of X to* `Ri` *can be extended to a functor* $A^* : \mathcal{O}(X) \longrightarrow$ `Ri` *and A^* is a sheaf.*

Notation 3.3.4. For a basis set X_{e_i}, we notice that $A^*(X_{e_i}) = Ae_i$.

Proof. This is just Theorem 3.1.6. The functor F is a sheaf on a basis: Let $U = X_e$ be a basis set with cover $\{U_i\}$. All basis sets of X are clopen and so are compact and we can choose a finite subcover to partition U, $X_{e_1}, \ldots, X_{e_n}$ say. This means that $\sum e_i = \bigcup e_i = e$. Suppose we have $s, t \in F(U)$ such that for each i, $se_i = te_i$. Then $\sum se_i = \sum te_i$, thus $se = te$, and $s = t$ (since e is the multiplicative inverse in Ae). So F satisfies local identity. If we have $s_i \in Ae_i$ for each i, then letting s_i' be a representative of s_i in Ae, we have $s = \sum s_i'e_i$ and $s\restriction_{X_{e_i}} = s \cdots e_i = s_i$. So F satisfies gluing. □

We thus have, for each $x \in X$, a direct system of all rings Ae, where $e \notin x$ with the ordering $e_i \leq e_j \iff Ae_i \supseteq Ae_j$. We can then form the direct limit at x by:

$$A_x := \varinjlim_{e \notin x} Ae$$

and from here we obtain the étalé space (X, S, π), where $S = \coprod_{x \in X} A_x$.

We notice that since 1 is not in any ideal in X, then the ring of global sections is just A localized at 1, which is simply A.

Proposition 3.3.5. *Let A be a ring, let X be its Stone space of maximal ideals of idempotents, and let (X, S, π) be the étalé space constructed as above. Then if the stalk theory is positively model complete and decidable and $Th_{\mathcal{L}_{BA}}(\mathfrak{C}(X))$ is decidable, then $Th_{\mathcal{L}_R}(A)$ is decidable.*

Proof. This is an immediate consequence of Theorem 3.2.11. □

Lemma 3.3.6. *[Ast08, Lemma 2] Let A be a ring. Let $x \in X_A$ and let (x) be the ring ideal generated by the elements of x. Then*

(1) $(x) = \bigcup_{e \in x} Ae$

(2) $A_x \cong A/(x)$

(3) A_x is isomorphic to A localized at $\{e \notin x\}$

(4) If A is von Neumann regular, then (x) is maximal and A_x is a field

Theorem 3.3.7 ([Ast08, Theorem 1])**.** *Let A be a von Neumann regular real closed ring. Then A is decidable, i.e. $Th_{\mathcal{L}_R}(A)$ is decidable. In particular, let $\mathcal{R}$ be a real closed field, and let $n \in \mathbb{N}$, then $D_n(\mathcal{R})$ the ring of semi-algebraic functions $\mathcal{R}^n \longrightarrow \mathcal{R}$ is decidable.*

Proof. Let (X_A, S, π) be an étalé space such that $F(X) = A$. By Lemma 3.3.6, $S_x = A/(x)$ and, since (x) is maximal, this is a real closed field. Now the stalk theory $Th_{\mathcal{L}_R}(\{S_x\}_{x \in X_A})$ is just the theory of real closed fields, which is decidable and positively model complete. By [Kop89, Proposition 18.8], $Th_{\mathcal{L}_{BA}}(\mathfrak{C}(X))$ is decidable (its theory is just that determined by its elementary invariant and this is decidable). So from Corollary 3.2.11, $Th_{\mathcal{L}_{gp}}(A)$ is decidable. Thus $Th_{\mathcal{L}_R}(A)$ is decidable. □

3.4 Coheirs and sheaf constructions

In this section we let M and N denote general o-minimal expansions of fields. We fix two real closed fields $\mathcal{R}$ and K, where K is a tame extension of $\mathcal{R}$, which we define below.

This section consists of a particular sheaf construction together with results about coheirs of types. By restricting the ring of definable functions $K^n \longrightarrow K$ we obtain a new ring, which we denote B. We show that B is von Neumann regular and its Zariski spectrum is canonically homeomorphic to the Zariski spectrum of $D_n(\mathcal{R})$. It was hoped that by interpreting predicates W and $\mathfrak{n}$ in B for those functions which are the restrictions of functions taking values in V, the convex hull of $\mathcal{R}$ in K and $\mathfrak{m}$, its maximal ideal, respectively, we would be able to obtain a decidability result for the ring of bounded semi-algebraic functions $\mathcal{R}^n \longrightarrow \mathcal{R}$. However, the interpretation of the predicate for $\mathfrak{m}$ in the stalks at prime ideals of the form $\mathfrak{p}_{d^-}$ and $\mathfrak{p}_{d^+}$ is not the maximal ideal of the convex hull of $\mathcal{R}$.

Definition 3.4.1. Let M be an o-minimal structure. Let $p \in S_n(M)$ be a n-type. Then the structure $M\langle p\rangle$ denotes the smallest elementary extension of M that realizes p.

A **cut** of M is a maximal consistent set of formulas of the form $x < a$ or $x > a$ or $x = a$, where $a \in M$. By [PS86], the cuts of M are just the 1-types of M in the reduct $(M, <)$ (so there is a canonical bijection between $S_1(M)$ and the cuts of M).

A type $p \in S_1(M)$ is **irrational** if we can partition M into two convex sets C^- and C^+, such that C^- does not have a greatest element and C^+ does not have a least element, and p is determined by the cut

$$\{x > a \mid a \in C^-\} \cup \{x < a \mid a \in C^+\}$$

Otherwise a type is **rational**. Note that in the o-minimal context "rational type" is the same as "definable type".

For $n \geq 2$ we need a different, but compatible definition. We say that a type $p(\overline{x})$ is rational if for every $\mathcal{L}$-formula $\phi(\overline{x}, \overline{y})$ there is $\mathrm{d}\,\phi(\overline{y})$ such that for any $\overline{a} \in M$,

$$\phi(\overline{x}, \overline{a}) \in p \Longleftrightarrow M \models \mathrm{d}\,\phi(\overline{a})$$

Remark 3.4.2. The rational 1-types are types of the form d or $d^{\pm}$ for $d \in M$ or

$\pm\infty$.

We recall from Theorem 1.3.3 that the Zariski spectrum of $D_n^{\mathcal{R}}$ is homeomorphic to the n-type space $S_n(\mathcal{R})$.

Definition 3.4.3. Let $M \prec N$ be two o-minimal structures. Then we say that N is a **tame extension** of M if M is dedekind complete in N, i.e. every 1-type of M realized in N is rational.

Definition 3.4.4. Let $M \prec N$. Then an n-type $q \in S_n(N)$ is a **coheir** over M if q is finitely realizable in M, i.e. for each $\phi(\overline{x}) \in q$, there is some $\overline{a} \in M^n$ such that $N \models \phi(\overline{a})$.

Let $p \in S_n(M)$. Then we say that $q \in S_n(N)$ is a **coheir of** p if q is a coheir over M and $p = q \cap Fml(\mathcal{L}, M)$, i.e. p is the set of formulas in q which take parameters only from M.

Let $coh_n(N/M)$ denote the set of coheirs over M in $S_n(N)$.

Remark 3.4.5. For a description of the coheirs over M, for a structure M with tame extension N see [Poi00, p245].

Definition 3.4.6. Let N be a real closed field, and let $Y \subseteq N^n$ be any subset. Then we define

$$D_n^N\restriction_Y := \{f : Y \longrightarrow N \mid \exists g \in D_n^N \; g\restriction_Y = f\}$$

i.e. $D_n^N\restriction_Y$ is just the restriction of all functions of D_n^N to the subset Y.

When we have N, a tame extension of M, then we simply write $D_n^N\restriction_M$ to mean $D_n^N\restriction_{M^n}$.

Proposition 3.4.7. *Let M and N be o-minimal expansions of fields, and let $M \prec N$ be a tame extension. Let $A = D_n^N\restriction_M$ and let $\varrho : D_n^N \longrightarrow A$ be the restriction map. Then:*

(i) A is von Neumann regular, ϱ is a homomorphism and its kernel is

$$I = \{f \in D_n^N \mid f(M^n) = \{0\}\}$$

(ii) Let $\mathfrak{p} \in \mathrm{Spec}(D_n^N)$ with associated type $p \in S_n(N)$. Then $I \subseteq \mathfrak{p}$ if and only if p is a coheir over M.

(iii) $\mathrm{Spec}(\varrho):\mathrm{Spec}(A)\longrightarrow\mathrm{Spec}(D_n^N)$ *is a homeomorphism onto the subspace of* $\mathrm{Spec}(D_n^N)$ *of all coheirs over* M *(identifying the Zariski spectrum with the type space by Proposition 1.3.3).*

Proof. (i) is clear.

(ii) ($\Longrightarrow$) Assume $I\subseteq\mathfrak{p}$. Let $\phi(\overline{x})\in p$, the type associated with $\mathfrak{p}$. Then $1-\chi_\phi$ is in $\mathfrak{p}$. Thus $\chi_\phi\notin\mathfrak{p}$. By the assumption $I\subseteq\mathfrak{p}$, there must be some $\overline{a}\in M^n$ such that $\chi_\phi(\overline{a})=1$ and so $1-\chi_\phi(\overline{a})=0$. By Proposition 1.3.3 $N\models\phi(\overline{a})$ and so p is a coheir over M.

($\Longleftarrow$) Assume that p is a coheir over M. Also let $f\in D_n^N$ be such that $f\upharpoonright_{M^n}=0$. Suppose $f\notin\mathfrak{p}$. Then $f(\overline{x})=0\notin p$, so $f(\overline{x})\neq 0\in\mathfrak{p}$ and so there is some $\overline{a}\in M^n$ such that $f(\overline{a})\neq 0$, which contradicts $f\in I$.

(iii) By (i), $A\cong D_n^N/I$, combined with (ii) gives the result. □

Proposition 3.4.8. *Let* $M\prec N$ *be tame. Then for any* $p\in S_n(M)$ *there is a unique coheir in* $S_n(N)$*. The map* $\Omega:S_n(M)\longrightarrow coh_n(N/M)\subseteq S_n(N)$ *taking a type to its unique coheir is a homeomorphism.*

Proof. The first part is just [MS94, Theorem 4.2]. That the map Ω is a homeomorphism follows since it is a continuous bijection, and since the spaces are boolean, and have a basis of clopen sets, the map is open as well. □

Proposition 3.4.9. *Let* $M\prec N$ *be a tame extension. Let* $A=D_n^N\upharpoonright_M$*. Then* $\mathrm{Spec}(A)\cong\mathrm{Spec}(D_n^M)$ *and the diagram:*

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r, "\mathfrak{q}\mapsto D_n^M\cap\mathfrak{q}"] \arrow[d, "\mathrm{Spec}(\varrho)"', "\cong"] & \mathrm{Spec}(D_n^M) \arrow[d, "\Theta^{-1}"', "\cong"] \\
coh_n(N/M) \arrow[r, "\cong", "\Omega^{-1}"'] & S_n(M)
\end{tikzcd}$$

commutes.

Proof. We know that the maps $\mathrm{Spec}(\varrho)$, Θ and Ω are homeomorphisms by Propositions 3.4.7, 3.4.8 and 1.3.3 respectively. Then we only need to show that $\Theta\circ\Omega^{-1}\circ\mathrm{Spec}(\varrho)(\mathfrak{q})=\mathfrak{q}\cap D_n^M$.

Explicitly, the maps in the diagram are:

$$\begin{tikzcd}
\mathfrak{q} \arrow[d, mapsto] & \mathfrak{p} = \{f \in D_n^M \mid f(\overline{x}) = 0 \in p\} \\
q = \{\phi(\overline{x}) \in Fml(\mathcal{L}, N) \mid 1 - \chi_\phi\restriction_M \in \mathfrak{q}\} \arrow[r, mapsto] & p = q \cap Fml(\mathcal{L}, M) \arrow[u, mapsto]
\end{tikzcd}$$

Suppose $f \in \mathfrak{q} \cap D_n^M \subseteq A$. Then $f(\overline{x}) = 0 \in q$ and uses only parameters from M, so $f(\overline{x}) = 0 \in p$, thus $f \in \mathfrak{p}$. Now suppose that $f \in D_n^M \setminus \mathfrak{q} \subseteq A$, similarly $f(\overline{x}) \neq 0 \in p$ and so $f \notin \mathfrak{p}$. ☐

We now fix a real closed field $\mathcal{R}$, and $n \in \mathbb{N}$, and define K, a tame extension of $\mathcal{R}$ as follows:

Definition 3.4.10. Let $\mathcal{R}$ be a real closed field. Then we define the field $K := (D_1^{\mathcal{R}})_{\mathfrak{p}_{0^+}}$ — the ring D_1 localized at $\mathfrak{p}_{0^+}$.

For convenience, let $A_n := D_n^K\restriction_{\mathcal{R}}$.

Clearly, we could replace $\mathfrak{p}_{0^+}$ in the definition of K by $\mathfrak{p}_{d^+}$, $\mathfrak{p}_{d^-}$, $\mathfrak{p}_\infty$ or $\mathfrak{p}_{-\infty}$.

Lemma 3.4.11. *[Tre99] Let $\mathcal{R}$ be a real closed field. Then K is a tame extension of $\mathcal{R}$ and is isomorphic to $\mathcal{R}\langle p\rangle$, the smallest real closed field realizing the type $p = \Theta^{-1}(\mathfrak{p}_{0^+})$.*

Definition 3.4.12. We now extend the language of rings to include two new predicates V and $\mathfrak{m}$ which are interpreted in K: V is the convex hull of $\mathcal{R}$ in K, and is a valuation ring and $\mathfrak{m}$ is its unique maximal ideal. We call this new structure $\overline{K}$ and the new extended language is called $\mathcal{L}_{V,\mathfrak{m}} := \mathcal{L}_R \cup \{V, \mathfrak{m}\}$.

We want to expand both K and A_n to be $\mathcal{L}_{V,\mathfrak{m}}$-structures. To avoid confusion, we label the interpretation of the predicates V and $\mathfrak{m}$ in A_n by W and $\mathfrak{n}$ respectively and the interpretation is as follows:

$$A_n \models f \in W \iff \exists g \in D_n^K\ \forall \overline{r} \in K^n,\ g(\overline{r}) \in V\ \&\ g\restriction_{\mathcal{R}} = f$$
$$A_n \models f \in \mathfrak{n} \iff \exists g \in D_n^K\ \forall \overline{r} \in K^n,\ g(\overline{r}) \in \mathfrak{m}\ \&\ g\restriction_{\mathcal{R}} = f$$

The expanded $\mathcal{L}_{V,\mathfrak{m}}$-structure $(A_n, W, \mathfrak{n})$ is denoted $\overline{A_n}$.

Remark 3.4.13. The set W is precisely the convex hull of $\mathcal{R}$ (interpreted as the constant functions taking values in $\mathcal{R}$) in A_n: If $f \in W$, then $Im(f) \subsetneq V$. By o-minimality, V isn't definable in the language of rings, so there must be some $r \in V$ such that $Im(f) \subseteq (-r, r)$. Similarly, $\mathfrak{n}$ is the convex hull of $\mathfrak{m}$ in A_n.

Remark 3.4.14. We haven't changed the language used to define the elements (functions) in A_n. Clearly there are many functions which are K-definable in the language $\mathcal{L}_{V,\mathfrak{m}}$ that aren't K-definable in the pure language of rings, and $(K, V, \mathfrak{m})$ is not an o-minimal structure.

Theorem 3.4.15. *The $\mathcal{L}_{V,\mathfrak{m}}$-theory of real closed valued fields is complete and positively model complete.*

Proof. From Cherlin and Dickmann [CD83], the theory of real closed valued fields with a predicate for the valuation ring is complete. Cherlin and Dickmann also prove quantifier elimination for the theory of real closed valuation rings in the language $\mathcal{L}_R \cup \{|\}$, where

$$K \models x|y \iff K \models \exists z \in V \; xz = y$$

It follows that the $\mathcal{L}_R \cup \{V\}$ theory of real closed valued fields is model complete. We now show that the negations of atomic formulas or $\mathcal{L}_{V,\mathfrak{m}}$ are $\exists_1^+$ definable.

We know that $x \neq y$ is equivalent to $\exists z \; z(x-y) = 1$. The formula $x \notin V$ is equivalent to $\exists z \in \mathfrak{m} \; xz = 1$. Similarly $x \notin \mathfrak{m}$ is equivalent to $\exists z \in V \; xz = 1$. □

Theorem 3.4.16. *[vdD86b, p70] Let $M \prec N$ be tame. Let $V := \mathrm{conv}_N(M)$, the convex hull of M in N, and let $\lambda : N \longrightarrow M \cup \{\infty\}$ map elements to their standard parts in M or ∞. Furthermore, let $f : N^n \longrightarrow N$ be N-definable. Then the composed map*

$$\lambda(f) : \{x \mid f(x) \in V\} \cap M^n \hookrightarrow N^n \xrightarrow{f} N \xrightarrow{\lambda} M \cup \{\infty\}$$

is M-definable. The function $\lambda(f)$ is the unique map $\{f \in V\} \cap M^n \longrightarrow M$ such that $(\lambda(f))(\overline{a}) = \lambda(f(\overline{a}))$ for all $\overline{a} \in V^n$ with $f(\overline{a}) \in V$.

Proposition 3.4.17. *The ring $W/\mathfrak{n}$ is isomorphic to the ring $bsa_n(\mathcal{R})$ of bounded $\mathcal{R}$-definable functions $\mathcal{R}^n \longrightarrow \mathcal{R}$.*

Proof. We define a homomorphism $\alpha : bsa_n(\mathcal{R}) \longrightarrow W/\mathfrak{n}$ as follows: Let $f \in bsa_n(\mathcal{R})$. Since f is $\mathcal{R}$-definable, there is some $f' \in D_n^K$ such that for all $\overline{r} \in \mathcal{R}^n$, $f(\overline{r}) = f'(\overline{r})$. Then $f'\restriction_{\mathcal{R}} \in A_n$ is more particularly in W and $f'\restriction_{\mathcal{R}}$ is independent of the choice of f'. Now let $\alpha(f) = f'\restriction_{\mathcal{R}}/\mathfrak{n}$. This is clearly a homomorphism.

To show injectivity, let $0 \neq f \in bsa_n(\mathcal{R})$. Then

$$\exists \overline{r} \in \mathcal{R}^n,\ s \in \mathcal{R}^\times\ f(\overline{r}) = s$$

So $f'\restriction_{\mathcal{R}}(\overline{r}) = s \notin \mathfrak{m}$ and there is no $\epsilon \in \mathfrak{n}$ such that $f'\restriction_{\mathcal{R}} + \epsilon = 0$.

For surjectivity, let $g/\mathfrak{n} \in W/\mathfrak{n}$ and let $g \in W$ be a representative. Then also $g \in A_n$, and so there is $g^K \in D_n^K$ such that for all $\overline{r} \in \mathcal{R}^n$, $g(\overline{r}) = g^K(\overline{r})$. By Theorem 3.4.16, $\lambda(g^K)$ is an $\mathcal{R}$-definable map $\mathcal{R}^n \longrightarrow \mathcal{R} \cup \{\infty\}$. Since $g \in W$, $\lambda(g^K) : \mathcal{R}^n \longrightarrow \mathcal{R}$ and for all $\overline{r} \in \mathcal{R}^n$, $\lambda(g)(\overline{r}) = \lambda(g(\overline{r}))$. The function $\epsilon := g - \lambda(g)$ is $\mathcal{R}$-definable and is in $\mathfrak{n}$. Thus $\alpha(\lambda(g)) = \alpha(g) = g/\mathfrak{n}$. $\square$

For convenience let $X := \operatorname{Spec}(A_n)$. We now represent $\overline{A_n}$ (the structure $(A_n, W, \mathfrak{n})$) as the continuous global sections over its Zariski spectrum. The construction follows the standard construction of the Zariski sheaf, (see [Har77, p70]), with the additional task of interpreting the relation symbols V and $\mathfrak{m}$.

Let F_0 be a contravariant functor from the basic open sets of X to $\operatorname{Cat}(\mathcal{L}_{V,\mathfrak{m}})$, which maps a basic open set $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$, represented by some $f \in A_n$, to the ring $(A_n)_f$, and there is a canonical $\mathcal{L}_R$-homomorphism $\phi_f : A_n \longrightarrow (A_n)_f$, $f \mapsto \frac{f}{1}$. We use $\sim$ to denote the equivalence relation induced by the localization. We expand $(A_n)_f$ to an $\mathcal{L}_{V,\mathfrak{m}}$-structure $(\overline{A_n})_f$ and ϕ_f to an $\mathcal{L}_{V,\mathfrak{m}}$-homomorphism where:

$$(\overline{A_n})_f \models \frac{a}{g} \in W_{(\overline{A_n})_f} \iff \exists h \in W,\ \frac{h}{1} \sim \frac{a}{g}$$
$$(\overline{A_n})_f \models \frac{a}{g} \in \mathfrak{n}_{(\overline{A_n})_f} \iff \exists h \in \mathfrak{n},\ \frac{h}{1} \sim \frac{a}{g}$$

The subsets $W_{(\overline{A_n})_f}$ and $\mathfrak{n}_{(\overline{A_n})_f}$ are the smallest possible interpretations of V and $\mathfrak{m}$ such that ϕ_f can be expanded to an $\mathcal{L}_{V,\mathfrak{m}}$-homomorphism. We can then extend F_0 uniquely to all open sets of the Zariski spectrum by Theorem 3.1.6. This expanded functor is now denoted F.

Lemma 3.4.18. *For each $\mathfrak{p} \in X$, the $\mathcal{L}_R$-reduct of the localization $(\overline{A_n})_{\mathfrak{p}}$ is a field. The interpretation of V and $\mathfrak{m}$ in $(\overline{A_n})_{\mathfrak{p}}$, for each $\mathfrak{p} \in X$ is:*

- *V is the convex hull of $\mathcal{R}$ in $(A_n)_{\mathfrak{p}}$*
- *If $\mathfrak{p}$ is the ideal corresponding to either a type realized in M or an irrational type, then $\mathfrak{m}$ is the maximal ideal of V*

- *If $\mathfrak{p}$ is corresponds to a rational type not realized in $\mathcal{R}^n$, then $\mathfrak{m}$ is a strict subset of the maximal ideal of V*

Proof. The localization $(A_n)_\mathfrak{p}$ is a field and in particular by [Ber71, Sections 2 & 3] or [Ast08, Lemma 2], and by [Tre99, Corollary 3.3]

$$(A_n)_\mathfrak{p} \cong A_n/\mathfrak{p} \cong K\langle p\rangle$$

In particular, at a type p representing a point $\overline{d} \in \mathcal{R}^n$, the localization is simply $(A_n)_\mathfrak{p} = K$ and $V_\mathfrak{p}$ and $\mathfrak{m}_\mathfrak{p}$ are the convex hull of $\mathcal{R}$ in K and its unique maximal ideal.

Claim. Let $p \in S_n(\mathcal{R})$ be an ideal not realized in $\mathcal{R}$, then any open set $p \in U \subseteq S_n(\mathcal{R})$ contains some ideal q which is realized in $\mathcal{R}$.

It suffices to show the result for basic open sets. Let $p \in \{p' \mid \phi(\overline{x}) \in p'\}$ for some formula ϕ. Then $\mathcal{R}\langle p\rangle \models \exists\overline{x}\phi(\overline{x})$, so $\mathcal{R} \models \exists\overline{x}\phi(\overline{x})$. Let q be the type of an element of $\mathcal{R}$ satisfying ϕ, then $q \in \{p' \mid \phi(\overline{x}) \in p'\}$ and the claim is proven.

Now let p be an ideal not realized in $\mathcal{R}$. The interpretations of V and $\mathfrak{m}$ must be convex. We show that there is no $a \in V_\mathfrak{p}$ such that $a > \mathcal{R}$. Suppose we have some $a \in (A_n)_\mathfrak{p}$ such that $a > \mathcal{R}$. We will show that any $f \in A_n$ such that $f_\mathfrak{p} = a$ cannot be in W. Suppose that $f \in W$ and that $f_\mathfrak{p} = a$. Then f is bounded on the points of $\mathcal{R}$ by some $r \in \mathcal{R}$, i.e. $\forall\overline{x} \in \mathcal{R}, \left|f(\overline{x})\right| < r$. By assumption, $a = f(\mathfrak{p}) > r$. Then the formula $f(\overline{x}) > r$ holds in $(A_n)_\mathfrak{p}$, and so $p \in \{p' \mid f(\overline{x}) > r \in p'\}$. This an open set, so it must contain the type of some $\overline{d} \in \mathcal{R}$, and so $K \models f(\overline{d}) > r$, which contradicts the boundedness of f.

If p is irrational, then $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal of V: Let p be a type realized by some point $\overline{\tau}$. Let $f \in \mathfrak{n} \subseteq A_n$, then f is bounded by some $\mu \in \mathfrak{m}$ on a neighbourhood of $\overline{\tau}$. If $f_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$, then $\frac{1}{f(\overline{\tau})}$ must be greater than V, otherwise we could find a neighbourhood U of $\overline{\tau}$ and some $r \in \mathcal{R}^n \cap U$ such that $\frac{1}{f(r)} < V$.

Now let $\mathfrak{p}$ be a rational type not realized in $\mathcal{R}^n$. Then p is realized by some $\overline{d} + \overline{\mu}$, with $d \in M$ and μ an infinitessimal in a tame extension of $\mathcal{R}$. Let f be a bounded function (bounded in M) that is equal to the identity function on a bounded neighbourhood (bounded in M) U of d. Then $f_\mathfrak{p}$ is not invertible in $V_\mathfrak{p}$, since its inverse is $\frac{1}{x}$. But $f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}$, since f takes no values in $\mathfrak{m}$ on U. So the interpretation of $\mathfrak{m}$ in $(\overline{A_n})_\mathfrak{p}$ is a strict subset of the maximal ideal of $V_\mathfrak{p}$. □

We now construct a sheaf of $\mathcal{L}_{V,\mathfrak{m}}$-structures as the étalé space of the sheaf

$(X, Cat(\mathcal{L}_{V,\mathfrak{m}}), F)$ as in Definition 3.1.9. The $\mathcal{L}_R$ reduct of each stalk will be the same as the stalks in the standard Zariski sheaf construction. Explicitly:

- For each prime $\mathfrak{p}$, $\varinjlim_{D(f)\ni\mathfrak{p}} F(D(f)) = (\overline{A_n})_{\mathfrak{p}}$
- For an open set U of X, the set of sections $\Gamma(U, F)$ is the set of maps $s : U \longrightarrow \coprod_{\mathfrak{p}\in U}(\overline{A_n})_{\mathfrak{p}}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in (\overline{A_n})_{\mathfrak{p}}$, and such that for each $\mathfrak{p} \in U$ there is some $f \in A_n \setminus \mathfrak{p}$ and some $a \in A_n$ such that for all $\mathfrak{q} \in D(f)$, $s(\mathfrak{q}) = \frac{a}{f}$ in $(\overline{A_n})_{\mathfrak{q}}$. Note, this is the Zariski sheaf construction, but it is the same as the construction given in Definition 3.1.9:

  It is sufficient to show for a given $s : U \longrightarrow S$ satisfying the above conditions, the preimage of a basic open set in S is open in X. Let $N = \{b(\mathfrak{p}) \mid \mathfrak{p} \in U'\}$ be a basic open set in S, where $b \in A_n$ and $U' \subseteq U$ is open. Then

  $$\begin{aligned} s^{-1}(N) &= \{\mathfrak{p} \in X \mid (\mathfrak{p}, s(\mathfrak{p})) \in N\} \\ &= \{\mathfrak{p} \in U' \mid s(\mathfrak{p}) = b(\mathfrak{p})\} \end{aligned}$$

  But any $\mathfrak{p} \in U'$, such that $s(\mathfrak{p}) = b(\mathfrak{p})$, has some open set $D(f)$ around it such that for all for all $\mathfrak{q} \in D(f)$, $s(\mathfrak{q}) = b(\mathfrak{q})$, and so $s^{-1}(N)$ is open.

  The interpretations of V and $\mathfrak{m}$, which we denote W_U and $\mathfrak{n}_U$ respectively, are given by:

  $$\begin{aligned} \Gamma(U, F) \models s \in W_U &\iff \forall\mathfrak{p} \in X\ (\overline{A_n})_{\mathfrak{p}} \models s(\mathfrak{p}) \in V \\ \Gamma(U, F) \models s \in \mathfrak{n}_U &\iff \forall\mathfrak{p} \in X\ (\overline{A_n})_{\mathfrak{p}} \models s(\mathfrak{p}) \in \mathfrak{m} \end{aligned}$$

- We have an étalé space $S := \coprod_{\mathfrak{p}\in X}(\overline{A_n})_{\mathfrak{p}}$, with basis of sets of the form $U_s := \{s(x) \mid x \in U\}$ for each open $U \subseteq X$ and each $s \in \Gamma(U, F)$.
- $(\overline{A_n}) \cong \Gamma(X, F)$

We know from Theorem 3.1.4 that the direct limit construction will give us a sheaf of $\mathcal{L}_{V,\mathfrak{m}}$-structures.

Now remember that we have $X := \mathrm{Spec}(A_n)$, and an étalé space $S := \coprod_{\mathfrak{p}\in X}(A_n)_{\mathfrak{p}}$, with basis of sets of the form $U_s := \{s(x) \mid x \in U\}$ for each open $U \subseteq X$ and each $s \in \Gamma(U, F)$, and the projection map $\pi : S \longrightarrow X$, $a \in (A_n)_{\mathfrak{p}} \mapsto \mathfrak{p}$. Then by Theorem 3.1.4, this is a sheaf of $\mathcal{L}_{V,\mathfrak{m}}$-structures.

By [Pie67, p40], a ring is von Neumann regular if and only if it has a sheaf representation as the ring of global sections over a sheaf of $\mathcal{L}_R$-structures (X, S, π), where X is boolean and for each $x \in X$, the stalk S_x is a field. In this case, to obtain a result for a ring $B_1 \subseteq A \subsetneq D_1$ which is not D_1 (and so not von Neumann regular), we must either have a sheaf representation where some of the stalks are not fields, or have an interpretation of A in some von Neumann regular ring.

Note that if we have two stalks x and y such that $S_x \models$ RCF and such that $S_y \models$ RCR, but $S_y \not\models$ RCF, then the stalk theory is not positively model complete: If S_y is a real closed ring that is not a field, then it has a non-trivial maximal ideal $\mathfrak{m}_y$, and $S_y \longrightarrow S_y/\mathfrak{m}_y$ is a non-injective homomorphism from S_y to a real closed field. By Lemma 3.2.4, the combined stalk theory cannot be positively model complete.

We know that the theory of valuation rings in the language $\mathcal{L}_R \cup \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of the valuation ring is positively model complete by [CD83]. However, attempts here also fail because the interpretation of the ideal in the stalk S_x is not always the maximal ideal of S_x.

Chapter 4

Continuous Definable Choice Functions

The contents of this section were originally motivated by [GMP04], which looks at the free vector lattices on n generators. For a reference on vector lattices (also known as "Riesz spaces") see [Bir48] or [LZ71].

Definition 4.0.1. Let K be an ordered field and let V be a K-vector space. Then V is a **K-vector lattice** when it is a structure $\left\langle V; 0, +, -, \vee, \wedge, \{\lambda\}_{\lambda \in K} \right\rangle$ which is a distributive lattice under the operations $\wedge$ and $\vee$ and which satisfies the following:

VL1 $\forall x, y, z \; (x+y) \vee (x+z) = x + (y \vee z)$

VL2 $\forall x, y \; x \wedge y = x + y - x \vee y$

VL3 for all $0 \leq \lambda \in K$, $\forall x, y \; \lambda(x \vee y) = \lambda x \vee \lambda y$

VL4 for all $0 \geq \lambda \in K$, $\forall x, y \; \lambda(x \wedge y) = \lambda x \vee \lambda y$

We call $\mathcal{L}_{KVL} := \{0, +, -, \vee, \wedge, \{\lambda\}_{\lambda \in K}\}$ the **language of K-vector lattices**.

In particular in [GMP04], it is claimed that for recursively enumerable K, the free vector lattice on two generators $FVL_K(2)$ is decidable, whereas for all $n \geq 3$, $FVl_K(n)$ interprets $\mathbb{Z}$. Note that the base field must be countable for decidability results to make sense. The free vector lattice on 2 generators is in fact isomorphic to:

- the vector lattice of functions $K^2 \longrightarrow K$ generated in K^{K^2} by co-ordinate projections [Ble73]

- the vector lattice of continuous $\emptyset$-definable continuous functions $K^2 \longrightarrow K$
- the vector lattice of continuous definable functions $f : [0,1] \longrightarrow K$ (in the language $\mathcal{L}_{KVL}$ with parameters from K) such that $f(0) = f(1)$ [BM02]
- the vector lattice of continuous definable functions $f : X \longrightarrow K$ (in the language $\mathcal{L}_{KVL}$ with parameters from K) where X is the unit square, centred at the origin, i.e. the union of the straight line segments $\{\langle x, y\rangle \in K \mid x = 1,\ -1 < y < 1\}$, $\{\langle x, y\rangle \in K \mid -1 < x < 1\ y = 1\}$ etc.

In the third representation we see that this may give insight into the decidability of C_1. For the proofs we need a notion of connectedness; from now on, by $FVl_K(2)$ we mean the vector lattice of continuous definable functions $f : X \longrightarrow K$ (in the language $\mathcal{L}_{KVL}$ with parameters from K) where X is the unit square, centred at the origin.

We need a definition and some notation before going further:

Definition 4.0.2. Let M be an o-minimal structure. Let $X \subseteq M^m$ be definable and let $S \subseteq X \times M^n$ be definable. Then $f : X \longrightarrow M^n$ is a **definable choice function for** S **above** X if it is definable and $\Gamma(f) \subseteq S$. The function f is a **continuous definable choice function** if it is a definable choice function and it is continuous.

Notation 4.0.3. To avoid confusion with closed and open intervals, for an $(m+n)$-tuple in M^{m+n} whose first m coordinates are $\overline{u} \in M^m$ and whose final n coordinates are $\overline{v} \in M^n$ we write $\langle \overline{u}, \overline{v}\rangle \in M^{m+n}$.

There is a flaw in the proof given in [GMP04]. We discuss this in the first section. The paper attempts to remove existential quantifiers from certain formulas and this is equivalent to finding a particular kind of continuous definable choice functions. In particular, if the result in [GMP04] were to be modified to use for C_1, we would also need to be able to ensure that, if we had a continuous definable choice function $f : X \longrightarrow M^n$ for S, then f would “hit” a subsets $T_i \subseteq S$ (that is that there would exist $\overline{x}_i$ such that $\langle \overline{x}_i, f(\overline{x}_i)\rangle \in T_i$). We call such a T_i a “red box”. We provide a proof for the case $n = 1$.

4.1 The Glass, Macintyre, Point Paper

Notation 4.1.1. For this section only, let K be a recursively enumerable field. We drop the subscript in $FVl_K(2)$. Additionally for this section only, since we are dealing with lattices, we use $\vee$ and $\wedge$ for the lattice operations of join and meet and $\mathbb{V}$ and $\mathbb{\Lambda}$ for the logical operators of disjunction and conjunction.

Definition 4.1.2. Let $\mathcal{L}_K = \{0, 1, +, -, <, \{\lambda\}_{\lambda \in K}\}$ be the language of totally ordered K-vector spaces.

The two major theorems in [GMP04] are as follows (emphasis added):

Proposition 4.1.3. *[GMP04, Proposition 3.2]* ***Let*** *$g_1, \ldots, g_m \in FVl(2)$, and let θ, θ_j $(j \in J)$ be open $\mathcal{L}_K$-formulae. Then one can effectively construct open $\mathcal{L}_K$-formulae θ', θ'_j $(j \in J)$ such that*

$$\exists f \in FVl(2)((\forall u \in X\ K \models \theta(f(u), \overline{g}(u))) \mathbin{\mathbb{\Lambda}} \bigwedge\nolimits_{j \in J} \exists u_j \in X\ K \models \theta_j(f(u_j), \overline{g}(u_j)))$$

if and only if

$$\forall u \in X\ K \models \theta'(\overline{g}(u)) \mathbin{\mathbb{\Lambda}} \bigwedge\nolimits_{j \in J} \exists u_j \in X\ K \models \theta'_j(\overline{g}(u_j))$$

Proposition 4.1.4. *[GMP04, Proposition 3.3] There is an effective procedure such that for any $\mathcal{L}_{VL}$-formula ϕ, we can construct open $\mathcal{L}_K$-formulae θ_i $(i \in I)$, and θ'_j $(j \in J)$ such that* ***for any tuple*** *$\overline{g} \in FVl(2)^{|\overline{g}|}$, $FVl(2) \models \phi(\overline{g})$ if and only if*

$$\bigvee\nolimits_I ((\forall u \in X\ K \models \theta_i(\overline{g}(u))) \mathbin{\mathbb{\Lambda}} \bigwedge\nolimits_J (\exists u_j \in K^2\ K \models \theta'_j(\overline{g})))$$

They lead immediately to:

Corollary 4.1.5. *[GMP04, Corollary 3.4] For any $\mathcal{L}_{KVL}$ sentence ϕ, there is an $\mathcal{L}_K$ sentence θ such that $FVl(2) \models \phi \iff K_{\mathcal{L}_K} \models \theta$. Therefore, $FVl(2)$ is decidable in the language $\mathcal{L}_{KVL}$.*

From Proposition 4.1.3 to Proposition 4.1.4 the quantifier on $g_1, \ldots, g_m \in FVl(2)$ has switched places with the statement "there is an effective procedure". In Proposition 4.1.3 we can construct an effective procedure for given $g_1, \ldots, g_m \in FVl(2)$, we can determine whether there exists a continuous definable choice function for $\Gamma(\overline{g}) \subseteq M^n$ above M. Whereas Proposition 4.1.4 requires a given effective procedure to remove existential quantifiers.

We show that in fact Proposition 4.1.3 is false by extending $\mathcal{L}_{KVL}$ by a predicate $\mathrm{NST}(x, y)$ which says the zero set of x is a subset of the zero set of y. We prove that the expanded structure on $FVl(2)$ is not model complete by giving an explicit $\mathcal{L}_{KVL} \cup \{\mathrm{NST}\}$-formula that cannot be written as a universal formula. We derive a contradiction by showing that Proposition 4.1.3 implies that the structure is model complete.

We use some facts from [GMP04] first.

Definition 4.1.6. Let $f \in FVl(2)$. Then $\{f = 0\} := \{x \in X \mid f(x) = 0\}$. We define $\{f > 0\}$, $\{f \geq 0\}$ etc. similarly.

Proposition 4.1.7. *[GMP04, Propositions 4.1 and 4.4] The following hold in $FVl(2)$, for all $f, g \in FVl(2)$:*

(i) $\{f \neq 0\} \cap \{g \neq 0\} = \{|f| \wedge |g| \neq 0\}$

(ii) $\{f \neq 0\} \cup \{g \neq 0\} = \{|f| \vee |g| \neq 0\}$

(iii) $\{f = 0\} \cap \{g = 0\} = \{|f| \vee |g| = 0\}$

(iv) $\{f = 0\} \cup \{g = 0\} = \{|f| \wedge |g| = 0\}$

(v) $\{f = 0\}$ *has nonempty interior* $\Longleftrightarrow \exists h \ h \neq 0 \doublewedge |f| \wedge |h| = 0$

(vi) $\{f \neq 0\}$ *is definably connected* $\Longleftrightarrow \forall h, h' \ \ |f| = |h| \vee |h'| \longrightarrow |h| \wedge |h'| \neq 0$

(vii) $\{f = 0\} = \emptyset \iff \{f = 0\}$ *has empty interior and* $\{f \neq 0\}$ *is definably connected*

Definition 4.1.8. Let

$$U_X = \{f \in FVl(2) \mid \{f = 0\} = \emptyset\}$$
$$P_X = \{f \in FVl(2) \mid \{f = 0\} \text{ is a single point}\}$$

Here U stands for unit and P stands for point.

By 4.1.7(vii) we can define U_X in $FVl(2)$ by a quantifier free $\mathcal{L}_{KVL}$-formula. Then we can also define P_X with a formula that says "$f \notin U_X$ and the interior of $\{f = 0\}$ is empty and $\{f \neq 0\}$ is definably connected (remembering that we are considering the domain of functions in $FVl(2)$ to be X, the unit quare). Finally we can define an equivalence relation $\sim$ on P_X, such that $f \sim g$ if and only if f and g have the same zero by $|f| \vee |g| \in P_X$.

Proposition 4.1.9. *There is an* $\mathcal{L}_{KVL}$*-formula* $\eta(x,y)$*, that contains no scalars from* K*, such that for all* $f,g \in FVL(2)$*:*

$$FVl(2) \models \eta(f,g) \iff \{f=0\} \subseteq \{g=0\}$$

Proof. It suffices to consider the case $f \in P_X$, since for general f

$$\{f=0\} \subseteq \{g=0\} \iff \forall p \in P_X(\{p=0\} \subseteq \{f=0\} \longrightarrow \{p=0\} \subseteq \{g=0\})$$

For $p \in P_X$ we have

$$\{p=0\} \subseteq \{g=0\} \iff \exists q \in P_X(p \sim q \Wedge |g| \leq |q|)$$

The $\Longleftarrow$ direction is clear. For the $\Longrightarrow$ direction, assume that $\{x\} = \{p=0\} \subseteq \{g=0\}$. Since p and g are both semilinear functions, we can find $\lambda \in K$ and an open neighbourhood $U \ni x$ such that $|g| \leq |\lambda p|$ on U. Now set $q = |g| \vee |\lambda p|$. $\square$

Definition 4.1.10. Let $\mathrm{NST}(x,y)$ (for "Nullstellensatz") be a binary predicate, let $\mathcal{L}_{\mathrm{NST}} = \mathcal{L}_{KVL} \cup \{\mathrm{NST}\}$. We expand $FVl(2)$ to an $\mathcal{L}_{\mathrm{NST}}$-structure such that

$$FVl(2) \models \forall x,y,\ \mathrm{NST}(x,y) \longleftrightarrow \eta(x,y)$$

and denote this expansion V_{NST}.

Proposition 4.1.11. *For any quantifier free* $\mathcal{L}_K$ *formula* $\theta(\overline{x})$ *there is a quantifier free* $\mathcal{L}_{\mathrm{NST}}$*-formula* $\phi(\overline{x})$ *such that for all* $\overline{g} \in V_{\mathrm{NST}}^{|\overline{x}|}$

$$\forall u \in X,\ K \models \theta(\overline{g}(u)) \iff V_{\mathrm{NST}} \models \phi(\overline{g})$$

Proof. Any quantifier free $\mathcal{L}_K$-formula is equivalent in K to a disjunction of atomic formulas, since $x \neq y$ is equivalent to $x < y \Vee y < x$, and $x \not< y$ is equivalent to $x = y \Vee y < x$. Thus we may assume that θ is of the form

$$\bigVee_i (\mu_i(\overline{x}) > 0) \Vee \bigVee_j (\nu_j(\overline{x}) = 0)$$

where μ_i and ν_j are k-definable (semilinear) maps $K^{|\overline{x}|}$. Now θ my be rewritten;

$$(\bigvee_i \mu_i(\overline{x}) \vee 0) = 0 \longrightarrow (\bigwedge_j \big|\nu_j(\overline{x})\big|) = 0$$

Now let $\phi(\overline{x})$ be

$$\mathrm{NST}(\bigvee_i \mu_i(\overline{x}) \vee 0, \bigwedge_j |\nu_j(\overline{x})|)$$

Then ϕ just says that

$$\{(\bigvee_i \mu_i(\overline{x}) \vee 0) = 0\} \subseteq \{\bigwedge_j |\nu_j(\overline{x})|\}$$

$\square$

Proposition 4.1.12. *V_{NST} is not model complete in the language $\mathcal{L}_{\mathrm{NST}}$. In particular, the formula*

$$\exists y \ (y \neq 0 \mathbin{\wedge\!\!\!\wedge} y \geq 0 \mathbin{\wedge\!\!\!\wedge} y \wedge z = 0)$$

is not equivalent in V_{NST} to a universal $\mathcal{L}_{\mathrm{NST}}$-formula.

Proof. Let $\sigma : [-2, 2] \longrightarrow X$ be a continuous semilinear surjection such that $\sigma(-2) = \sigma(2)$ and the restriction $\sigma_{[-2,2)}$ is a bijection, see [BM02, Proposition 3.1]. That such a map can be found is what is used to show (in [BM02]) that $FVl(2)$ is isomorphic to the vector lattice of continuous definable functions $f : [-2, 2] \longrightarrow K$ (in the language $\mathcal{L}_{KVL}$ with parameters from K) such that $f(-2) = f(2)$. Let $L = \{\sigma(t) \mid |t| \geq 1\}$.

Let $\tau : [-2, 2] \longrightarrow X$ be defined by

$$\tau(t) = \begin{cases} \sigma(-2) & \text{if } |t| \geq 1 \\ \sigma(2t) & \text{if } |t| < 1 \end{cases}$$

So τ stretches the interval $[-1, 1]$ to fill all of X. We define the composition $\omega(\overline{x}) = \tau(\sigma^{-1}(\overline{x}))$, $\omega : X \longrightarrow X$ which is continuous and injective.

Now let $F : FVl(2) \longrightarrow FVl(2)$ be defined by $F(f) = f \circ \omega$, and F is an injective and we have an isomorphism of K-vector lattices onto

$$V' = \{g \in FVl(2) \mid g \text{ is constant on } L\}$$

and $V' \equiv FVl(2)$. We expand V' to be an $\mathcal{L}_{\mathrm{NST}}$-structure V'_{NST} and F is still an $\mathcal{L}_{\mathrm{NST}}$-isomorphism.

We notice that V'_{NST} is the $\mathcal{L}_{\mathrm{NST}}$ structure induced by the inclusion $V' \hookrightarrow$

$FVl(2)$, i.e. for all $g, h \in V'$,

$$V'_{\mathrm{NST}} \models \mathrm{NST}(g, h) \Longleftrightarrow V_{\mathrm{NST}} \models \mathrm{NST}(F^{-1}(g), F^{-1}(h))$$

Now let g be the distance function from L (so it is very much like a Λ function as defined in Chapter 2). Then g is 0 on L, and so $g \in V'$. Then taking f as the distance function from $X \setminus L$, then f satisfies

$$y \neq 0 \mathbin{\wedge\!\!\!\wedge} y \geq 0 \mathbin{\wedge\!\!\!\wedge} y \wedge g = 0$$

However, there is no such $f \in V'$: If $f \in V'$ then f is constant on L. If f is constantly 0 on L, then in order that $f \wedge g = 0$ we must have $f = 0$. If f is a constant $k \neq 0$, then there is a non-zero neighbourhood of L such that f is non-zero and $f \wedge g \neq 0$. □

Proposition 4.1.13. *For any recursively enumerable K, [GMP04, Proposition 3.3] (Proposition 4.1.4) fails.*

Proof. Suppose that the Proposition holds and let ϕ be the $\mathcal{L}_{KVL}$-formula

$$\exists y \ (y \neq 0 \mathbin{\wedge\!\!\!\wedge} y \geq 0 \mathbin{\wedge\!\!\!\wedge} y \wedge z = 0)$$

Then by [GMP04, Proposition 3.3], we have open $\mathcal{L}_K$-formulas θ_i $(i \in I)$, and θ'_j $(j \in J)$ such that for any tuple $\overline{g} \in FVl(2)^{|\overline{g}|}$, $FVl(2) \models \phi(\overline{g})$ if and only if

$$\bigvee\!\!\!\!\bigvee_I ((\forall u \in X \ K_{\mathcal{L}_K} \models \theta_i(\overline{g}(u))) \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\!\bigwedge_J (\exists u_j \in K^2 \ K_{\mathcal{L}_K} \models \theta'_j(\overline{g})))$$

By Proposition 4.1.11, we can replace open $\mathcal{L}_K$-formulas by open $\mathcal{L}_{\mathrm{NST}}$ formulas. Let ψ_i, ψ_j be $\mathcal{L}_{\mathrm{NST}}$-formulas such that for all $\overline{g} \in FVl(2)$ and all i and j

$$\begin{aligned} \forall t \in X, K \models \theta_i(\overline{t}) &\Longleftrightarrow V_{\mathrm{NST}} \models \psi_i(\overline{g}) \\ \forall t \in X, K \models \neg\theta'_j(\overline{t}) &\Longleftrightarrow V_{\mathrm{NST}} \models \neg\psi'_j(\overline{g}) \end{aligned}$$

Then $FVl(2) \models \phi(\overline{g})$ if and only if

$$\bigvee\!\!\!\!\bigvee_I ((V_{\mathrm{NST}} \models \psi_i(\overline{g})) \mathbin{\wedge\!\!\!\wedge} \bigwedge\!\!\!\!\bigwedge_J (V_{\mathrm{NST}} \models \psi'_j(\overline{g})))$$

which is a quantifier free formula, contradicting Proposition 4.1.12. □

4.2 Cell-decomposition

Throughout, let M be an o-minimal $\mathcal{L}$-structure for some language $\mathcal{L}$ which contains the language of ordered groups.

We begin with a definition from [vdD98]:

Definition 4.2.1. Let $(i_1, \ldots, i_n)$ be a sequence of 0's and 1's of length n. We define a $(i_1, \ldots, i_n)$-cell $C \subseteq M^n$ by induction as follows:

1. a (0)-cell is a point in M, a (1)-cell is an open interval (a, b), where we allow $a = -\infty$ or $b = +\infty$

2. suppose we have already defined an $(i_1, \ldots, i_n)$-cell, then an $(i_1, \ldots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a continuous definable function $f : X \longrightarrow M$, where X is an $(i_1, \ldots, i_n)$-cell. An $(i_1, \ldots, i_n, 1)$-cell is a band

$$(f, g) := \{\langle \overline{x}, y\rangle \in X \times M \mid f(\overline{x}) < y < g(\overline{x})\}$$

   where X is an $(i_1, \ldots, i_n)$-cell, and where f and g can be continuous definable functions $X \longrightarrow M$ or $\pm\infty$ and $f < g$.

Definition 4.2.2. Let $m < n \in \mathbb{N}$. Then by $p^n_m : M^n \longrightarrow M^m$ we mean the map $\langle x_1, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_n\rangle \mapsto \langle x_1, \ldots, x_m\rangle$, the projection onto the first m coordinates. More generally, let $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, with $i_j \leq i_k$ if and only if $j \leq k$, then $p^n_I : \langle x_1, \ldots, x_n\rangle \mapsto \langle x_{i_1}, \ldots, x_{i_m}\rangle$.

More simply, let $\pi^n_m : \langle x_1, \ldots, x_n\rangle \mapsto x_m$. Where there is no confusion as to the value of n, we write just π_m.

Definition 4.2.3. The **dimension** of an $(i_1, \ldots, i_n)$-cell C is $\Sigma_j i_j$.

Definition 4.2.4. Let C be an $(i_1, \ldots, i_n)$-cell and let i_m be the last 1 in the sequence. Then we call C an m-cell in M^n. An m-cell is at most m-dimensional.

Definition 4.2.5. Let $X \subseteq M^n$ be a definable subset. Then $\overline{X}$ denotes the closure of X. The **frontier** of X is denoted $\partial X := \overline{X} \setminus X$.

Notation 4.2.6. For a point $\overline{u} \in M^{m+n}$ we write $\overline{u} = \langle \overline{x}, \overline{y}\rangle$, where $\overline{x} \in M^m$ and $\overline{y} \in M^n$.

Remark 4.2.7. Here are some facts about cells ([vdD98, p50 onwards]):

- Each cell is locally open (open in its closure).

- Each $(1, 1, \dots, 1)$-cell is open in M^n.
- Let C be an $(i_1, \dots, i_n)$-cell and let $I \subseteq \{1, \dots, n\}$ be the set of all the 1's in $(i_1, \dots, i_n)$. Then $p^n_I(C)$ is open in $M^{|I|}$ and p^n_I is a homeomorphism.
- Each cell is definable connected.
- For each cell $C \subseteq M^{m+n}$ there is a continuous definable map $f : p^{m+n}_m(C) \longrightarrow M^m$ with $\Gamma(f) \subseteq C$.

Definition 4.2.8. A **cell decomposition** of M^n is a partition with very nice properties. It is defined inductively as follows:

- A **decomposition** $\mathcal{C}$ of M is a finite set of cells $C_1, \dots, C_l$ for some $l \in \mathbb{N}$ which partition M. Thus $\mathcal{C}$ is of the form

$$\{(-\infty, a_1), \{a_1\}, (a_1, a_2), \dots, \{a_{l'}\}, (a_{l'}, \infty)\}$$

- A **decomposition** $\mathcal{C}$ of M^{n+1} is a finite set of cells $C_1, \dots, C_l$ for some $l \in \mathbb{N}$ which partition M^{n+1} and are such that for any $m < n+1$ the set of projections $p^{n+1}_m(\mathcal{C})$ is a decomposition of M^m.

Theorem 4.2.9 (Cell Decomposition, [vdD98, p52, Chapter 3, Theorem 2.11])**.**

(I_m) *Let $S_1, \dots, S_l \subseteq M^m$ be definable sets. Then there is a cell decomposition of M^m that partitions each of $S_1, \dots, S_l$.*

(II_m) *Let $S \subseteq M^m$ be definable and let $f : S \longrightarrow M$ be definable. There is a cell decomposition $\mathcal{C}$ of M^m that partitions S and such that for each $\mathcal{C}$-cell $C \subseteq S$, $f \restriction_C$ is continuous.*

Definition 4.2.10. If $\mathcal{C}$ is a decomposition and $C \in \mathcal{C}$, then we call C a **$\mathcal{C}$-cell**.

A decomposition is **cylindrical** in the sense that for a decomposition $\mathcal{C}$ of $M^{m_1+m_2} = M^n$, there is a decomposition $\mathcal{D}$ of M^{m_1}, such that each $C \in \mathcal{C}$ is contained in $D \times M^{m_2}$ for some $D \in \mathcal{D}$. We can then consider $\mathcal{C}_D := \{C \in \mathcal{C} \mid p^n_{m_1}(C) = D\}$. Then in fact $\{\mathcal{C}_D\}_{D \in \mathcal{D}}$ gives us a partition of $\mathcal{C}$ as a finite set of cells. In such a case we write $\mathcal{D} = p^n_{m_1}(\mathcal{C})$, and say that $\mathcal{C}$ is a **decomposition over** $\mathcal{D}$, or that $\mathcal{D}$ is a **projection decomposition** of $\mathcal{C}$, $\mathcal{D}$ is **the projection decomposition onto** M^{m_1} of $\mathcal{C}$.

We call a decomposition $\mathcal{C}$ a **partition** of a definable set S if each cell in $\mathcal{C}$ is either contained in S or disjoint from S. In other words, S is a (finite) union of cells in $\mathcal{C}$.

More generally, for a finite set $T_1, \ldots, T_l$ of definable sets (not necessarily disjoint), we say that a cell decomposition $\mathcal{C}$ **partitions each of** $T_1, \ldots, T_l$ if every T_i is a union of cells of $\mathcal{C}$.

Let $\mathcal{C}$ be a cell decomposition of M^n. Then a cell decomposition $\mathcal{C}^*$ of M^n is a **refinement** of $\mathcal{C}$ if $\mathcal{C}^*$ partitions $\mathcal{C}$.

Proposition 4.2.11. *Let $\mathcal{C}$ be a decomposition of M^{m+n} and let $\mathcal{D} = p_m^{m+n}(\mathcal{C})$ be the projection decomposition. Let $\mathcal{D}^*$ be a refinement of $\mathcal{D}$. Then for each $C \in \mathcal{C}$, the set $C^* := C \cap (D \times M^n)$ is a cell and the collection of such cells $\mathcal{C}^*$ is a cell decomposition of M^{m+n} which refines $\mathcal{C}$.*

Proof. By the definition of cell and induction on n. □

Definition 4.2.12. We call $\mathcal{C}^*$ the **refinement** of $\mathcal{C}$ above $\mathcal{D}^*$.

Definition 4.2.13. Let $X \subseteq M$ be the interval (a, b) (the definition also holds for closed intervals and half-open intervals), and where we allow $a = -\infty$ or $b = +\infty$. We suppose that we have fixed some constant > 0, which we call 1 for convenience. Then we define the $e(X)$, the **midpoint** of X as follows:

$$e(X) := \begin{cases} 0 & \text{if } a = -\infty \text{ and } b = +\infty \\ b-1 & \text{if } a = -\infty \text{ and } b \in M \\ a+1 & \text{if } a \in M \text{ and } b = +\infty \\ \frac{a+b}{2} & \text{if } a, b \in M \end{cases}$$

When $S \subseteq M^{n+1}$ is a definable set such that for each $\overline{a} \in M^n$, the fibre $S_{\overline{a}}$ is a point or an interval, then we can define the function $e : M^n \longrightarrow M, \overline{a} \mapsto e(S_{\overline{a}})$. When C is a cell, then $e : p_n^{n+1} : (C) \longrightarrow M$ is a continuous function.

Theorem 4.2.14 (Definable Choice [vdD98, p94, Chapter6, Theorem 1.2])**.** *Let $S \subseteq M^{m+n}$ be definable. Then there is a definable map $f : p_m^{m+n}(S) \longrightarrow M^n$ with $\Gamma(f) \subseteq S$.*

Corollary 4.2.15. *Let $C \subseteq M^{m+n}$ be a cell, and let $\left\langle \overline{a}, \overline{b} \right\rangle \in C$. Then there is a continuous definable map $f : p_m^{m+n}(C) \longrightarrow M^n$ with $\Gamma(f) \subseteq C$ and $f(\overline{a}) = \overline{b}$. In* *particular, when C is an m-cell, C is the graph of a continuous definable function $p_m^{m+n}(C) \longrightarrow M^n$.*

Proof. Follows from the construction of the function f in Theorem 4.2.14. $\square$

Theorem 4.2.16 (Curve Selection [vdD98, p94, Chapter 6, Corollary 1.5])**.** *Let $X \subseteq M^n$ be a definable set and let $\overline{u} \in \partial X$. Then there is a continuous definable map $\gamma : (0,1) \longrightarrow X$ such that $\lim_{t \nearrow 1} \gamma(t) = \overline{u}$.*

Definition 4.2.17. The **dimension** of a definable set S is the maximum dimension of the cells contained in S. This is equivalent to taking any cell decomposition $\mathcal{C}$ of S and finding the maximum of the dimensions of the cells of $\mathcal{C}$, see [vdD98, Chapter 4]. We define

$$\dim(\emptyset) = -\infty$$

The **local dimension** of a definable set S at a point $\overline{u}$ is

$$\min\{\dim(U \cap S) \mid U \text{ is an open nbd of } \overline{u}\}$$

and is written $\dim_{\overline{u}}(S)$. This is well defined since if $V \subseteq U$, then $\dim(V \cap S) \leq \dim(U \cap S)$ and since $\dim_{\overline{u}}(S) = -\infty \Longrightarrow \overline{u} \notin \overline{S}$, see [vdD98, p.69].

Definition 4.2.18. To measure the distance between two points $\langle x_1, \ldots, x_n \rangle = \overline{x}$ and $\langle y_1, \ldots, y_n \rangle = \overline{y}$, we use the metric:

$$|\overline{x} - \overline{y}| = \sum_{1 \leq i \leq n} |x_i - y_i|$$

Lemma 4.2.19. *Let $C \subseteq M^{m+n}$ be an m-cell and let $\overline{x} \in M^m, \overline{y} \in M^n$ with $\langle \overline{x}, \overline{y} \rangle \in C$. Then $\overline{C}_{\overline{x}} = \{\overline{y}\}$.*

Proof. Suppose there is some $\overline{z} \in M^n$ with $\langle \overline{x}, \overline{z} \rangle \in \overline{C}$ and $\overline{y} \neq \overline{z}$. By Theorem 4.2.16, there is a definable continuous curve $\gamma : (0,1) \longrightarrow C$ with $\lim_{t \nearrow 1} \gamma(t) = \langle \overline{x}, \overline{z} \rangle$.

By definition of an m-cell, C is the graph of a continuous definable function $f : p_m^{m+n}(C) \longrightarrow M^n$. Taking $0 < \epsilon < \frac{|\overline{y} - \overline{z}|}{4}$, we can find $\delta > 0$ such that for all $\overline{u} \in p_m^{m+n}(C)$, $|\overline{u} - \overline{x}| < \delta \Longrightarrow \big|f(\overline{u}) - \overline{y}\big| < \epsilon$.

By taking such ϵ and δ, we see that for all $\langle \overline{u}, \overline{v} \rangle \in C$, if $|\overline{u} - \overline{x}| < \delta$, then

$$\begin{aligned} |\overline{y} - \overline{z}| &\leq |\overline{y} - \overline{v}| + |\overline{v} - \overline{z}| \\ 4\epsilon &\leq \epsilon + |\overline{v} - \overline{z}| \\ 3\epsilon &\leq |\overline{v} - \overline{z}| \end{aligned}$$

and so for all $\langle \overline{u}, \overline{v} \rangle \in C$, $\big| \langle \overline{u}, \overline{v} \rangle - \langle \overline{x}, \overline{z} \rangle \big| \geq \min(\epsilon, \delta)$. So $\langle \overline{x}, \overline{z} \rangle$ cannot be the limit of a path in C, which is a contradiction. $\square$

Proposition 4.2.20. *Let $D \subseteq M^m$ be a cell and let $C \subseteq M^{m+1}$ be a cell defined by the graph of some continuous definable function $f : D \longrightarrow M$. Then for each $\overline{x} \in p_m^{m+1}(\overline{C})$, the fibre*

$$\overline{C}_{\overline{x}} := \{ y \in M \mid \langle \overline{x}, y \rangle \in \overline{C} \}$$

is either a single point or a closed interval.

Proof. Clearly, for $\overline{x} \in p_m^{m+1}(\overline{C})$, $\overline{C}_{\overline{x}}$ is non-empty. We also know that if $\overline{x} \in p_m^{m+1}(C) = D$, then $\overline{C}_{\overline{x}}$ is a single point. So we concern ourselves with $\overline{x} \in p_m^{m+1}(\partial C)$. Suppose $\overline{C}_{\overline{x}}$ is neither a point nor an interval, then there are at least two points $\langle \overline{x}, y \rangle, \langle \overline{x}, z \rangle \in \partial C$ and a third point $\langle \overline{x}, u \rangle \notin \partial C$ such that $y < u < z$.

Since $\langle \overline{x}, u \rangle \notin \overline{C}$, we know that

$$\exists \epsilon > 0 \, \forall \langle \overline{x}', v \rangle \in C \ \left| \langle \overline{x}', v \rangle - \langle \overline{x}, u \rangle \right| \geq \epsilon$$

Take such an ϵ and let $C' = \{ \langle \overline{x}', v \rangle \in M^{m+1} \mid |\overline{x}' - \overline{x}| < \frac{\epsilon}{2} \} \cap C$. Then C' is the intersection of a cell with a cylinder in the first m co-ordinates, so we can take smaller ϵ if necessary to obtain a cell. Then we must have a continuous definable path $\gamma : (0,1) \longrightarrow C'$ with $\lim_{t \searrow 0} \gamma(t) = \langle \overline{x}, y \rangle$ and $\lim_{t \nearrow 1} \gamma(t) = \langle \overline{x}, z \rangle$. But by the intermediate value theorem applied to $\pi_{m+1} \circ \gamma$, $\pi_{m+1} \circ \gamma$ must take values in $(u - \frac{\epsilon}{2}, u + \frac{\epsilon}{2})$ and so there is some $\langle \overline{x}', v \rangle \in \Gamma(\gamma) \subseteq C'$ with $\big| \langle \overline{x}', v \rangle - \langle \overline{x}, u \rangle \big| < \epsilon$, which is a contradiction.

So $\overline{C}_{\overline{x}}$ is convex. By o-minimality it is an interval. Clearly it must be closed. $\square$

4.3 Connected Components and Pathways

Definition 4.3.1. A **box** in M^n is a subset that is a product of intervals in M; the intervals may be open or closed and may have $\pm\infty$ as end points. We call a box open or closed if each of its component intervals in the product are open or closed (alternatively, it is open or closed in the product topology of M^n). For a given subset $S \subseteq M^n$, the **closed box around** S, $\overline{B}(S)$, is the smallest closed box containing S. When $S = \{\overline{a}_1, \overline{a}_2\}$, with $\overline{a}_i = \langle a_{1,i}, \ldots, a_{k,i} \rangle$ for some $k \in \mathbb{N}$ and $i = 1, 2$, then we write $B(\overline{a}_1, \overline{a}_2)$ for the box $I_1 \times \ldots \times I_k$, where I_j is: the interval $(a_{j,1}, a_{j,2})$ if $a_{j,1} < a_{j,2}$, the interval $(a_{j,2}, a_{j,1})$ if $a_{j,2} < a_{j,1}$ or the point $a_{j,1}$ if $a_{j,1} = a_{j,2}$. Note that $B(\overline{a}_1, \overline{a}_2)$ is open if and only if $a_{j,1} \neq a_{j,2}$ for all j. We write $\overline{B}(\overline{a}_1, \overline{a}_2)$ for the closed box around $\overline{a}_1$ and $\overline{a}_2$; this is the closure of $B(\overline{a}_1, \overline{a}_2)$. For the box $B(\overline{a} - \overline{\epsilon}, (\overline{a} + \overline{\epsilon}))$ we write $B_{\overline{\epsilon}}(\overline{a})$, where $\overline{\epsilon} \in M^n$.

Definition 4.3.2. Let $S \subseteq M^{m+n}$ be a definable, definably connected set. Let $\mathcal{C}$ be a cell-decomposition that partitions S and let $\mathcal{D} = p_m^{m+n}(\mathcal{C})$ be the projection decomposition. For each $D \in \mathcal{D}$, let S_D be the subset of S above D, i.e. $S_D := S \cap (D \times M^n)$. Now S_D is no longer necessarily definably connected. We let $S_D^1, \ldots, S_D^{k_D}$ be the connected components of S_D. Now let σ be a choice function assigning to each cell D_i a connected component $S_{D_i}^{\sigma(i)}$ and set $S^\sigma = \bigcup S_{D_i}^{\sigma(i)}$. If S^σ is definably connected, then we say it is an $(m, \mathcal{C})$**-pathway** of S. We use $\mathcal{P}(m, \mathcal{C}, S)$ to denote the set of $(m, \mathcal{C})$-pathways of S.

We define $\#cc(S)$ to be the number of connected components of S.

Proposition 4.3.3. *Let $S \subseteq M^{m+1}$ be a definable, definably connected set. Let $\mathcal{C}$ be a cell decomposition that partitions S and let $\mathcal{D} = p_m^{m+1}(\mathcal{C})$. Let $\langle \overline{a}, b \rangle \in C$ for some $C \in \mathcal{C}$, and let $\overline{a} \in D = p_m^{m+1}(C)$. Then $\#cc(S_D) = \#cc(S_{\overline{a}})$. Furthermore, each connected component of S_D contains precisely one connected component of $S_{\overline{a}}$.*

Proof. Let $\#cc(S_D) = q$ and $\#cc(S_{\overline{a}}) = r$, and let $S_D^1, \ldots, S_D^q$ be the connected components of S_D and let $S_{\overline{a}}^1, \ldots, S_{\overline{a}}^r$ be the connected components of $S_{\overline{a}}$. Each cell in $\mathcal{C}_D$ is either a function or a band over D. Thus for any $C \in \mathcal{C}_D$ with $C \subseteq S$, C must be in one and only one connected component of S_D and $C_{\overline{a}}$ must be in exactly one connected component of $S_{\overline{a}}$. We also note that definably connected and definably path connected are equivalent conditions.

It is clear that $q \leq r$: Suppose that $C_1 \cap S_{\overline{a}}$ and $C_2 \cap S_{\overline{a}}$ are in the same connected component of $S_{\overline{a}}$. Then for any two points $\overline{x} \in C_1$, $\overline{y} \in C_2$, we can find paths from $\overline{x}$ to $C_1 \cap S_{\overline{a}}$ (since cells have continuous definable choice functions), from $C_1 \cap S_{\overline{a}}$ to $C_2 \cap S_{\overline{a}}$, and then to $\overline{y}$. So C_1 and C_2 must be in the same connected component of S.

We now consider $T = M^{m+1} \setminus S$ and $T^1_{\overline{a}}, \dots, T^u_{\overline{a}}$, the connected components of $T_{\overline{a}}$. Since $S_{\overline{a}}, T_{\overline{a}}$ partition M we can order their connected components (which are open intervals or points) and call them $X^1 < \dots < X^{r+u}$, where $X^i \in S_{\overline{a}} \Longrightarrow X^{i+1} \subseteq T_{\overline{a}}$ and vice versa. We pick a point in each connected component X^i, y_i, say, and let $C_i \in \mathcal{C}$ be the cell containing the point $\langle \overline{a}, y_i \rangle$. Then there is some $f_i : D \longrightarrow M$, where $f_i(\overline{a}) = \overline{y}_i$, and $\Gamma(f) \subseteq C_i$. Since the cells are disjoint and are coverings of D, we have $f_i < f_j$ whenever $X^i < X^j$. But now, if $q < r$, then we must have some $S^{i'}_{\overline{a}} = X_i \neq X_j = S^{j'}_{\overline{a}}$ such that $\overline{y}_i, \overline{y}_j$ are definably path connected by some path γ in S. Since $X_i, X_j \subseteq S_{\overline{a}}$, we must have $i < i+1 < j$ and so $Im(\gamma) \cap \Gamma(f_{i+1}) \neq \emptyset$, in which case $\Gamma(\gamma)$ intersects T and so it can't be contained in S, which is a contradiction. □

Corollary 4.3.4. *Let $S \subseteq M^{m+1}$. Let $\mathcal{C}$ and $\mathcal{C}'$ be cell decompositions of M^{m+1} that partition S. Then $\mathcal{P}(m, \mathcal{C}, S) = \mathcal{P}(m, \mathcal{C}', S)$.*

Proof. It suffices to take a common refinement $\mathcal{C}''$ of $\mathcal{C}$ and $\mathcal{C}'$ and prove the result for $\mathcal{C}$ and $\mathcal{C}''$; so, wlog, assume that $\mathcal{C}'$ is a refinement of $\mathcal{C}$, with projection decompositions $\mathcal{D}'$ and $\mathcal{D}$ respectively. Then by Proposition 4.3.3, for each cell $D \in \mathcal{D}$, each $D' \in \mathcal{D}'$ such that $D' \subseteq D$ and each $\overline{a} \in D'$, we have $\#cc(S_D) = \#cc(S_{\overline{a}}) = \#cc(S_{D'})$. Suppose that there are l connected components of each of S_D, $S_{D'}$ and $S_{\overline{a}}$, and label them $S^1_D, \dots, S^l_D$, $S^1_{D'}, \dots, S^l_{D'}$ and $S'^1_{\overline{a}}, \dots, S^l_{\overline{a}}$ respectively, such that $i < j \iff S^i_D < S^j_D$ etc. Then by Proposition 4.3.3, for each $1 \leq i \leq l$, $S^i_{\overline{a}} \subseteq S^i_D \cap S^i_{D'}$. Since $S^i_{D'}$ is connected and has non-empty intersection with S^i_D, we must have $S^i_{D'} \subseteq S^i_D$. So letting $D'_1, \dots, D'_k$ be the refinement of D in $\mathcal{D}'$, $S^i_D = \bigcup_j S^i_{D'_j}$. So for any $(m, \mathcal{C}')$-pathway $S^{\sigma'}$, $S^i_{D'} \subseteq S^{\sigma'} \iff S^i_D \subseteq S^{\sigma'}$. □

Example 4.3.5. Proposition 4.3.3 is not true if we try and generalize for $S \subseteq M^{m+n}$. In the case that $m = 1$ and $n = 2$, then we let

$$\begin{aligned} A =& \{\langle x, y, z \rangle \in M^3 \mid x \in M, y > 0, z > 1\} \\ B =& \{\langle x, y, z \rangle \in M^3 \mid x \in M, y < 0, z < 0\} \\ C =& \{\langle x, y, z \rangle \in M^3 \mid x = z, y = 0\} \\ S =& A \cup B \cup C \end{aligned}$$

We can then take cell decomposition as follows: let $D = M$, decompose M^2 above D into E_1, E_2, E_3 defined by the formulas $y > 0, y < 0, y = 0$ (the upper half-plane, lower half-plane and x-axis respectively), then decompose $E_i \times M$ in the obvious way. For $a = \frac{1}{2}$, $\#cc(S_a) = 3$, but $\#cc(S_D) = 1$.

Definition 4.3.6. A **stratification** $\mathcal{S}$ of M^n is a finite set of cells $C_1, \ldots, C_k$, called **strata** of $\mathcal{S}$, which partition M^n and such that for any $1 \leq i \leq k$, $\overline{C_i}$ is a union of strata of $\mathcal{S}$.

A cell decomposition that is also a stratification is a **stratified cell decomposition**.

Definition 4.3.7. Let $X \subseteq M^{m+n}$ and $Y \subseteq M^m$ be definable. Then the **closure of X above Y**, written $\mathrm{cl}_Y(X)$ is given by:

$$\mathrm{cl}_Y(X) := \overline{X} \cap (Y \times M^n)$$

Proposition 4.3.8. *[vdD98, p. 65] Let $S \subseteq M^{m+n}$ be definable. Then the set*

$$S_m(d) := \{\overline{a} \in M^m \mid \dim(S_{\overline{a}}) = d\}$$

is definable. Additionally,

$$\dim(\bigcup_{\overline{a} \in S_m(d)} \{\overline{a}\} \times S_{\overline{a}}) = \dim(S_m(d)) + d$$

Proposition 4.3.9. *[vdD98, p. 67, Exercises] Let $S \subseteq M^{m+n}$ be definable. Then the set*

$$S_m^{\#cc}(d) := \{\overline{a} \in M^m \mid \#cc(S_{\overline{a}}) = d\}$$

is definable.

Proposition 4.3.10. *Let $S \subseteq M^{m+n}$. Then there is a stratified cell decomposition $\mathcal{C}$ partitioning S such that, if $\mathcal{D} = p_m^{m+n}(\mathcal{C})$, then for each $D \in \mathcal{D}$ and all $\overline{a} \in D$, $\#cc(S_{\overline{a}}) = \#cc(S_D)$. Furthermore each connected component of S_D contains precisely one connected component of $S_{\overline{a}}$.*

Proof. Let $\mathcal{C}'$ be a cell decomposition partitioning S and let k be the number of cells of $\mathcal{C}$ contained in S. Then for $1 \leq d \leq k$, we can define $S_m^{\#cc}(d)$, and these partition $p_m^{m+n}(S)$. We now take a stratified cell decomposition $\mathcal{C}$ partitioning S

and each of the $S_m^{\#cc}(d)$, and let $\mathcal{D} = p_m^{m+n}(\mathcal{C})$. By construction, for each $D \in \mathcal{D}$ and all $\overline{a}, \overline{b} \in D$, $\#cc(S_{\overline{a}}) = \#cc(S_{\overline{b}})$.

Let $D \in \mathcal{D}$ and let $C, C' \in \mathcal{C}$ be in the same connected component of S_D. Let $\overline{a} \in D$. Then C, C' are in the same connected component if and only if there is some sequence $C = C_1, \dots, C_l = C'$ such that for each i, $\mathrm{cl}_D(C_i) \cap C_{i+1} \neq \emptyset$ or $\mathrm{cl}_D(C_{i+1}) \cap C_i \neq \emptyset$. Suppose, wlog, that $\mathrm{cl}_D(C_i) \cap C_{i+1} \neq \emptyset$. Since $\mathcal{C}$ is stratified, then $\mathrm{cl}_D(C_i) \cap C_{i+1} = C_{i+1}$. Thus $\mathrm{cl}_{\overline{a}}(C_i) \cap (C_{i+1})_{\overline{a}} = (C_{i+1})_{\overline{a}}$, which is non-empty since $C_1, \dots, C_l$ are all cells over $D \ni \overline{a}$. So we have a connected chain $C_{\overline{a}} = (C_1)_{\overline{a}}, \dots, (C_l)_{\overline{a}} = C'_{\overline{a}}$ and $C_{\overline{a}}, C'_{\overline{a}}$ are in the same connected component of $S_{\overline{a}}$. The same argument gives the "Furthermore..." statement. □

Definition 4.3.11. Let $S \subseteq M^{m+n}$ be definable. Let $\mathcal{C}$ be a stratified cell decomposition of M^{m+n} partitioning S, with projection decomposition $p_m^{m+n}(\mathcal{C}) = \mathcal{D}$. Let $\mathcal{D}$ be such that for each $D \in \mathcal{D}$ and all $\overline{a} \in D$, $\#cc(S_{\overline{a}}) = \#cc(S_D)$. Then we say that $\mathcal{D}$ has **uniform connected components of** S.

Corollary 4.3.12. *Let $S \subseteq M^{m+n}$ be definable and definably connected. Let $\mathcal{C}$ be a stratified cell decomposition partitioning S. Let $\mathcal{D} = p_m^{m+n}(\mathcal{C})$ have uniform connected components of S. Then for any cell decomposition $\mathcal{C}'$, each $(m, \mathcal{C})$-pathway, considered as a subset of M^{m+n} is contained in some $(m, \mathcal{C}')$-pathway. In particular, for any stratified refinement $\mathcal{C}^*$ of $\mathcal{C}$, $\mathcal{P}(m, \mathcal{C}, S) = \mathcal{P}(m, \mathcal{C}^*, S)$.*

Proof. It suffices to show that for any stratified refinement (a refinement that is itself stratified) $\mathcal{C}^*$ of $\mathcal{C}$, $\mathcal{P}(m, \mathcal{C}, S) = \mathcal{P}(m, \mathcal{C}^*, S)$: since for any cell decomposition $\mathcal{C}'$ of S, we can take a common stratified refinement $\mathcal{C}^*$ of $\mathcal{C}$ and $\mathcal{C}'$, and clearly each $(m, \mathcal{C}^*)$-pathway is a subset of some $(m, \mathcal{C}')$-pathway.

Let $\mathcal{C}^*$ be a stratified refinement of $\mathcal{C}$, with projection decomposition $\mathcal{D}^*$. Then for each $D' \in \mathcal{D}^*$ and each $\overline{a} \in D'$ we must again have $\#cc(S_{\overline{a}}) = \#cc(S_{D'})$ and each connected component of $S_{D'}$ contains precisely one connected component of $S_{\overline{a}}$. But now each connected component of $S_{D'}$ is contained in precisely one component of S_D, where D is the cell of $\mathcal{D} = p_m^{m+n}(\mathcal{C})$ containing D'.

Now let $D'_1, \dots D'_k$ be cells partitioning D in the refinement $\mathcal{D}^*$. Then for each i,j, each connected component of S_D^i of S_D contains precisely once connected component of $S_{D'_j}$, which we label $S_{D'_j}^i$, and so $S_D^i = \bigcup_j S_{D'_j}^i$. So an $(m, \mathcal{C}^*)$-pathway must be equal to some $(m, \mathcal{C})$-pathway. □

4.4 The Formula NBD

Lemma 4.4.1. *Let $X \subseteq M^m$ and let $S \subseteq X \times M^n$. Let $\langle \overline{u}, \overline{x} \rangle \in S$. Then if $\langle \overline{u}, \overline{x} \rangle$ satisfies*

$$\exists \epsilon > 0 \; \forall \delta > 0 \; p_m^{m+n}(S \cap B_{\overline{\delta},\overline{\epsilon}}(\langle \overline{u}, \overline{x} \rangle)) \subsetneq X \cap B_{\overline{\delta}}(\overline{u})$$

then there is no continuous function $f : X \longrightarrow M^n$ such that $\Gamma(f) \subseteq S$ and such that $f(\overline{u}) = \overline{x}$ (where $\overline{\delta}, \overline{\epsilon}$ are respectively the m-tuple $\langle \delta, \delta, \dots, \delta \rangle$ and the n-tuple $\langle \epsilon, \epsilon, \dots, \epsilon \rangle$).

Proof. Suppose that there is such a continuous function, then under the usual ϵ-δ definition of continuity

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall \overline{v} \in X \; \bigwedge_{1 \le i \le m} |u_i - v_i| < \delta \longrightarrow \bigwedge_{1 \le j \le n} \left| f(\overline{u})_j - f(\overline{v})_j \right| < \epsilon$$

It is easy to check that this implies the negation of the formula in the statement of the lemma. □

Definition 4.4.2. Let $X \subseteq M^m$ and $S \subseteq X \times M^n$ be definable sets. Let $\mathrm{NBD}_S(\overline{u}, \overline{x})$ be the formula

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall \overline{v} \in X \; (\bigwedge_{1 \le i \le m} |u_i - v_i| < \delta \longrightarrow$$
$$(\exists \overline{y} \; (\bigwedge_{1 \le j \le n} \left| x_j - y_j \right| < \epsilon) \wedge \langle \overline{v}, \overline{y} \rangle \in S)) \wedge \langle \overline{u}, \overline{x} \rangle \in S$$

This is equivalent to

$$\forall \epsilon > 0 \; \exists \delta > 0 \; X \cap B_{\overline{\delta}}(\overline{u}) = p_m^{m+n}(S \cap B_{\overline{\delta},\overline{\epsilon}}(\langle \overline{u}, \overline{x} \rangle)) \wedge \langle \overline{u}, \overline{x} \rangle \in S$$

We assume that m and n are given.

Remark 4.4.3. We would like NBD_S to define all the points at which local continuous choice functions exist, i.e. if $\langle \overline{u}, \overline{x} \rangle$ satisfies

$$\forall \epsilon > 0 \; \exists \delta > 0 \; p_m^{m+n}(S \cap B_{\overline{\delta},\overline{\epsilon}}(\langle \overline{u}, \overline{x} \rangle)) \supseteq X \cap B_{\overline{\delta}}(\overline{u})$$

then there is a region U of $\overline{u}$ such that there is a continuous function $f : U \cap X \longrightarrow M^n$ with $\Gamma(f) \subseteq S$ and $f(\overline{u}) = \overline{x}$. However, this is not the case.

Consider the set which is the union of the following definable sets in M^{2+1}:

- $A := M \times M^{>0} \times \{0\}$
- $B := M \times M^{<0} \times \{0\}$
- $C := M \times \{0\} \times M^{>0}$
- $\langle 0, 0, 0\rangle$

Let $X = M^2$ and let $S := A \cup B \cup C \cup D$. Then $M \models \mathrm{NBD}_S(0, 0, 0)$, but there is clearly no continuous definable choice function f in an open neighbourhood of $\langle 0, 0, 0\rangle$ with $f(\langle 0, 0\rangle) = 0$.

We notice that if we define $\mathrm{NBD}^1_S(\overline{u}, \overline{x})$ by

$$\forall \epsilon > 0\ \exists \delta > 0\ X \cap B_{\overline{\delta}}(\overline{u}) = p^{m+n}_m(S \cap \mathrm{NBD}_S(M^{m+n}) \cap B_{\overline{\delta},\overline{\epsilon}}(\langle \overline{u}, \overline{x}\rangle)) \wedge \langle \overline{u}, \overline{x}\rangle \in S$$

then in the above case, $\mathrm{NBD}^1_S(M^{2+1}) = A \cup B$ and for each point $\langle \overline{u}, \overline{x}\rangle \in S$, $M \models \mathrm{NBD}^1_S(\overline{u}, \overline{x})$ if and only if there is a region U of $\overline{u}$ such that there is a continuous function $f : U \cap X \longrightarrow M^n$ with $\Gamma(f) \subseteq S$ and $f(\overline{u}) = \overline{x}$. This leads us to the following definition.

Definition 4.4.4. Let $X \subseteq M^m$ and $S \subseteq X \times M^n$ be definable sets. Then we define the formula $\mathrm{NBD}^k_S(\overline{u}, \overline{x})$ inductively as follows (for $1 \leq k \leq m$):

- $\mathrm{NBD}^0_S(\overline{u}, \overline{x})$ is $\mathrm{NBD}(\overline{u}, \overline{x})$
- $\mathrm{NBD}^{k+1}_S(\overline{u}, \overline{x})$ is

$$\begin{aligned}\forall \epsilon > 0\ \exists \delta > 0\ X \cap B_{\overline{\delta}}(\overline{u}) \subseteq p^{m+n}_m(S \cap \mathrm{NBD}^k_S(M^{m+n}) \cap B_{\overline{\delta},\overline{\epsilon}}(\langle \overline{u}, \overline{x}\rangle))\\ \wedge \langle \overline{u}, \overline{x}\rangle \in S \cap \mathrm{NBD}^k_S(M^{m+n})\end{aligned}$$

- $\mathrm{NBD}^*_S(\overline{u}, \overline{x}) := \mathrm{NBD}^m_S(\overline{u}, \overline{x})$

Note that we do not need to explicitly show X as a subscript in NBD_S, since it is the projection of S onto the first $|\overline{u}|$ coordinates, and so given S we assume that X is also given.

Lemma 4.4.5. *Let $X \subseteq M^m$ and $S \subseteq X \times M^n$ be definable sets. Let $\mathcal{C}$ be a stratified cell decomposition of S. Then for every cell $C \in \mathcal{C}$ and every $1 \leq k \leq m$,* *either*

$$\forall \langle \overline{u}, \overline{x} \rangle \in C,\ M \models \mathrm{NBD}^k_S(\overline{u}, \overline{x}) \ \textit{or}\ \ \forall \langle \overline{u}, \overline{x} \rangle \in C,\ M \models \neg\, \mathrm{NBD}^k_S(\overline{u}, \overline{x})$$

Proof. Let $C \in \mathcal{C}$ and let $\langle \overline{u}, \overline{x} \rangle, \langle \overline{u}', \overline{x}' \rangle \in C$. Let $\mathcal{D} = p^{m+n}_m(\mathcal{C})$ and let $D = p^{m+n}_m(C) \in \mathcal{D}$. Suppose that $M \models \neg\, \mathrm{NBD}_S(\overline{u}, \overline{x})$. We will prove that $M \models \neg\, \mathrm{NBD}_S(\overline{u}', \overline{x}')$. If $\langle \overline{u}, \overline{x} \rangle \notin S$, then clearly the statement of the result holds.

Suppose that $\langle \overline{u}, \overline{x} \rangle \in S$. If D is open in X, then there is clearly a function $f : D \longrightarrow M^n$ with $\Gamma(f) \subseteq S$ and $f(\overline{u}) = \overline{x}$. Then by Lemma 4.4.1, and taking δ and ϵ sufficiently small, we see that $M \models \mathrm{NBD}_S(\overline{u}, \overline{x})$. So D can't be open in X.

We know from the definition of a cell that

$$\forall \epsilon > 0\ \exists \delta > 0\ D \cap B_{\overline{\delta}}(\overline{u}) = p^{m+n}_m(C \cap B_{\overline{\delta},\overline{\epsilon}}(\langle \overline{u}, \overline{x} \rangle))$$

So if $M \not\models \mathrm{NBD}_S(\overline{u}, \overline{x})$, then taking appropriate ϵ and fixing δ, we have $\overline{v} \in X \setminus D$ such that

$$\bigwedge_{1 \le i \le m} |u_i - v_i| < \delta \wedge (\forall \overline{y}\ (\bigwedge_{1 \le j \le n} |x_j - y_j| < \epsilon) \longrightarrow \langle \overline{v}, \overline{y} \rangle \notin S)$$

We can take δ increasingly smaller to obtain a sequence $\overline{v}^l \in X$, each term of which satisfies the above formula, and such that $\lim_{l \to \infty} \overline{v}^l = \overline{u}$. We must therefore have some $D' \in \mathcal{D}$ containing an infinite subsequence $\overline{w}^l$ and such that $\overline{u} \in \overline{D'}$. By construction $D' \subseteq X$. Thus for any the cell C' above D' such that $\langle \overline{u}, \overline{x} \rangle \in \overline{C'}$, it must be that $C' \cap S = \emptyset$. Also, since $\mathcal{C}$ is a stratification, then for any cell above D',

$$\langle \overline{u}, \overline{x} \rangle \in \overline{C'} \Longleftrightarrow \overline{C'} \cap S \supseteq C$$

and in particular $\langle \overline{u}', \overline{x}' \rangle \in \overline{C'}$.

Therefore, for sufficiently small $\epsilon > 0$ there is $\delta > 0$ such that we have

$$B_{\delta,\epsilon}(\langle \overline{u}', \overline{x}' \rangle) \cap (D' \times M^n) \cap S = \emptyset$$

So there is $\overline{v} \in D' \cap B_\delta(\overline{u}')$ such that there is no $\overline{y}$ with $\langle \overline{v}, \overline{y} \rangle \in S \cap B_{\delta,\epsilon}(\langle \overline{u}', \overline{x}' \rangle)$, i.e. $M \models \neg\, \mathrm{NBD}_S(\overline{u}', \overline{x}')$.

Now suppose the lemma holds for k. Since the formula NBD^{k+1}_S is just NBD^k_S but with $S \cap \mathrm{NBD}^{k-1}_S$ replaced by $S \cap \mathrm{NBD}^k_S(M^{m+n})$, and it is assumed that NBD^k_S is a union of cells, then the result holds for NBD^{k+1}_S. □

Lemma 4.4.6. *Let $X \subseteq M^m$ and $S \subseteq X \times M^n$ be definable sets. Then for each $\overline{u} \in X$,*

$$\dim_{\overline{u}}(p_m^{m+n}(S \cap \neg\,\mathrm{NBD}_S)) < \dim_{\overline{u}}(X)$$

More generally, for $1 \leq k \leq m$,

$$\dim_{\overline{u}}(p_m^{m+n}(\neg\,\mathrm{NBD}_S^k \cap \mathrm{NBD}_S^{k-1})) < \dim_{\overline{u}}(p_m^{m+n}(S \cap \neg\,\mathrm{NBD}_S^{k-1})) < \dim_{\overline{u}}(X) - k$$

Thus in particular, $\mathrm{NBD}_S^{m-1} \cap \neg\,\mathrm{NBD}_S^* = \emptyset$ *and* $\dim_{\overline{u}}(S \setminus \mathrm{NBD}_S^*) < \dim_{\overline{u}}(X)$.

Proof. Let $\mathcal{C}$ be a stratified cell decomposition partitioning S and X, and let $\mathcal{D} = p_m^{m+n}(\mathcal{C})$. For the case $k = 0$, suppose that we have some $\overline{u} \in M^m$ such that

$$\dim_{\overline{u}}(p_m^{m+n}(S \cap \neg\,\mathrm{NBD}_S)) = \dim_{\overline{u}}(X) = d$$

Then there is some neighbourhood $U \subseteq M^m$ of $\overline{u}$ such that

$$\dim(U \cap p_m^{m+n}(S \cap \neg\,\mathrm{NBD}_S)) = \dim_{\overline{u}}(U \cap X) = d$$

There is therefore some $D \in \mathcal{D}$ such that

$$\dim(D \cap U \cap p_m^{m+n}(S \cap \neg\,\mathrm{NBD}_S)) = d$$

and $\overline{u} \in \overline{D}$ (and $D \subseteq X$). By Lemma 4.4.5, there is some cell C above D such that $C \subseteq S \cap \neg\,\mathrm{NBD}_S$. For arbitrary $\langle \overline{u}', \overline{x}' \rangle \in C$, we can pick $\delta > 0$ such that $B_\delta(\overline{u}') \cap X \subseteq D \cap U$. Now we can find a choice function $f : B_\delta(\overline{u}') \cap X \longrightarrow M^n$ such that $\Gamma(f) \subseteq C$ and such that $f(\overline{u}') = \overline{x}'$, which contradicts $\langle \overline{u}', \overline{x}' \rangle \in \neg\,\mathrm{NBD}_S$.

Now suppose that the result is true for $k-1$. Take $\langle \overline{u}, \overline{x} \rangle \in \mathrm{NBD}_S^{k-1} \cap \neg\,\mathrm{NBD}_S^k$ and let C be the cell containing $\langle \overline{u}, \overline{x} \rangle$. By the construction of $\mathrm{NBD}_S^{k-1} \cap \neg\,\mathrm{NBD}_S^k$, we have $C \subseteq \overline{\mathrm{NBD}_S^{k-2} \cap \neg\,\mathrm{NBD}_S^{k-1}}$. Let $C_1, \ldots, C_l$ be the cells of $\mathrm{NBD}_S^{k-2} \cap \neg\,\mathrm{NBD}_S^{k-1}$ such that $C \subseteq \overline{C_i}$. Since $\mathcal{C}$ is a stratification, these are all the cells contained in $\mathrm{NBD}_S^{k-2} \cap \neg\,\mathrm{NBD}_S^{k-1}$ whose closures intersect with C. We let $D = p_m^{m+n}(C)$ and $D_1 \ldots, D_l$ be the projections of $C_1, \ldots, C_l$ (with possible repetitions). So for each cell D_i, since the projection decomposition $\mathcal{D}$ is stratified, then either $\overline{u} \in D_i$ or $\overline{u} \in \partial D_i$. Then we pick a cell D' from $D_1, \ldots, D_l$ of highest dimension, which gives $\dim(D') = \dim_{\overline{u}}(p_m^{m+n}(\mathrm{NBD}_S^{k-2} \cap \neg\,\mathrm{NBD}_S^{k-1}))$. If $\overline{u} \in \partial D'$, then since $\dim(\partial Y) < \dim(Y)$ for any definable Y, we have $\dim(D) < \dim(D')$, which proves the result.

Otherwise we have $\overline{u} \in D'$, so $D = D'$. Since for all $1 \leq i \leq l$, $\dim(D_i) \leq \dim(D)$, then we must have $D \not\subseteq \overline{D_i}$ if $D \neq D_i$, in which case, since the projection decomposition $\mathcal{D}$ is stratified, then $D \cap \overline{D_i} = \emptyset$ and so D is the only cell of $p_m^{m+n}(\mathrm{NBD}_S^{k-1} \cap \neg\, \mathrm{NBD}_S^k)$ containing $\overline{u}$ in its closure. So we can find an open neighbourhood U of M^m such that $U \cap D_i = \emptyset$ for all $D_i \neq D$. Now fix ϵ and pick δ so that $B_\delta(\overline{u}) \subseteq U$. Now we want to know whether for all $\overline{v} \in B_\delta(\overline{u}) \cap X$ there is some $\overline{y} \in M^n$ such that $\langle \overline{v}, \overline{y} \rangle \in S \cap \mathrm{NBD}_S^{k-1} \cap B_{\delta,\epsilon}(\overline{u}, \overline{x})$. If we take $\overline{v} \notin D$, then $\overline{v} \notin p_m^{m+n}(\mathrm{NBD}_S^k \cap \neg\, \mathrm{NBD}_S^{k-1})$ and so such a $\overline{y}$ exists. If $\overline{v} \in D$, then there is some $\langle \overline{v}, \overline{y} \rangle \in C \subseteq \mathrm{NBD}_S^{k-1} \cap \neg\, \mathrm{NBD}_S^k$. So $M \models \mathrm{NBD}_S^k(\overline{u}, \overline{x})$, which contradicts the assumption $M \models \neg\, \mathrm{NBD}_S^k(\overline{u}, \overline{x})$.

□

Lemma 4.4.7. *Let $X \subseteq M^m$ and $S \subseteq X \times M$ be definable sets. Let $\mathcal{C}$ be a stratified cell decomposition of S. Let $C \in \mathcal{C}$ be an m-cell, and let $D = p_m^{m+n}(C)$ be such that $M \models \mathrm{NBD}_S^*(\overline{C})$. Then for any $\overline{u} \in \partial D$, $\overline{C}_{\overline{u}}$ is either empty or a single point.*

Proof. Take $\overline{u} \in \partial D$, then by Lemma 4.2.20, $\overline{C}_{\overline{u}}$ is either empty, a point or a closed interval. We suppose the fibre $\overline{C}_{\overline{u}}$ is a closed interval $[a, b]$ and derive a contradiction.

Since C is an m-cell, we suppose that it is the graph of some continuous, definable $f : D \longrightarrow M$. Let $y = \frac{a+b}{2}$ be the midpoint of $[a, b]$ and let $\epsilon = \frac{b-a}{4}$. Take any $\delta > 0$. Since $b \in \overline{C}_{\overline{u}}$, then by curve selection there is some continuous definable $\gamma : (0, 1) \longrightarrow C$ such that $\lim_{t \nearrow 1} \gamma(t) = \langle \overline{u}, b \rangle$. There must be some $\langle \overline{v}, z \rangle \in Im(\gamma)$ with $|b - z| < \epsilon$ and $|\overline{u} - \overline{v}| < \delta$ and so $\langle \overline{v}, z \rangle \notin B_{\overline{\delta},\epsilon}(\overline{u}, y)$ but $\overline{v} \in B_{\overline{\delta}}(\overline{u})$. Since C is an m-cell, there is no z' such that $\langle \overline{v}, z' \rangle \in B_{\overline{\delta},\epsilon}(\overline{u}, y)$, which contradicts $M \models \mathrm{NBD}_S^*(\overline{C})$. □

4.5 Continuous Definable Choice Functions

In this section we let M be an o-minimal expansion of fields.

Proposition 4.5.1. *Let $S \subseteq M^2$. Let $\mathcal{C}$ be a stratified cell decomposition partitioning S, let $\mathcal{D} = p_1^2(\mathcal{C})$ have uniform connected components of S. Let $D \in \mathcal{D}$ be a bounded 1-dimensional cell with end points u and v. Let S'_D be a uniform connected component of S above D. Let $\langle u, a \rangle, \langle v, b \rangle \in \overline{S'_D}$. Let $\langle w, c \rangle \in S'_D$.*

Then there is a continuous definable choice function $h : D \longrightarrow M$ *of* $\overline{S'_D}$ *such that* $h(u) = a$*,* $h(w) = c$ *and* $h(v) = b$*.*

Proof. For convenience, let $D = (0, 1)$, so $u = 0$ and $v = 1$. Additionally, let $w = \frac{1}{2}$. Since S'_D is a uniform connected component above D, then it can be written as the union of cells $C_1 < C_2 < \ldots < C_k$ above D. Let f be the function defining the bottom of C_1 and let g be the cell defining the top of C_k, where we allow $f = -\infty$ or $g = \infty$. Then the band (f, g) is a cell above D, which we call C and S'_D is some union of C with possibly $\Gamma(f)$ or $\Gamma(g)$. Let $e : D \longrightarrow M$ be the midpoint function, and so e is continuous on D.

We transform the second coordinate axis using the map

$$\alpha(y) = \pm\sqrt{1 + \frac{1}{4y^2}} - \frac{1}{2y}$$

taking the values in the interval $(-1, 1)$, which is the inverse of the bijective mapping $y \mapsto \frac{y}{1-y^2}$, $(0, 1) \longrightarrow M$. Then the mapping $\langle x, y\rangle \mapsto \big\langle x, \alpha(y)\big\rangle$ squashes $[0, 1] \times M$ into the rectangle $[0, 1] \times (-1, 1)$. In this way by considering $f' = \alpha \circ f$ and $g' = \alpha \circ g$ (where $f' = -1$ if $f = -\infty$, similarly for g), then we may assume that S'_D is a bounded subset of M^2. We can then map $\alpha(S'_D)$ to $(0, 1) \times (-1, 1)$ by:

$$\beta(\langle x, y\rangle) = \begin{cases} \left\langle x, \frac{y-e(x)}{g(x)-e(x)} \right\rangle & \text{if } y \geq e(x) \\ \left\langle x, \frac{y-e(x)}{e(x)-f(x)} \right\rangle & \text{if } y < e(x) \end{cases}$$

Depending on whether $\Gamma(f), \Gamma(g) \subseteq S'_D$ and by restricting the codomain, $\beta(x, \alpha(y))$ is a homeomorphism from S'_D to $(0, 1)$ times the open, closed or a half-open interval from -1 to 1. We denote the unary map on the second coordinate at a fixed x by β_x.

On the interval $(0, \frac{1}{2})$ there are now two cases: (1) $\lim_{t\searrow 0} f(t) \neq \lim_{t\searrow 0} g(t)$, in which case $a \in [\lim_{t\searrow 0} f(t), \lim_{t\searrow 0} g(t)]$ the map $\beta(x, \alpha(y))$ extends to $\{0\} \times [\lim_{t\searrow 0} f(t), \lim_{t\searrow 0} g(t)]$; (2) $a = \lim_{t\searrow 0} f(t) = \lim_{t\searrow 0} g(t)$. In the first case we simply take a linear interpolation h' from $\beta(0, \alpha(a))$ to $\beta(\frac{1}{2}, \alpha(c))$. Then the map $h = \alpha^{-1} \circ \beta_x^{-1} \circ h \circ \beta_x \circ \alpha$ defined on $[0, \frac{1}{2}]$ is a continuous definable choice function with $\Gamma(h) \subseteq S'_D \cup \langle 0, a\rangle$. In the second case, we let h' be the linear interpolation from $\langle 0, 0\rangle$ to $\beta(\frac{1}{2}, \alpha(c))$. Then the map $h = \alpha^{-1} \circ \beta_x^{-1} \circ h \circ \beta_x \circ \alpha$ defined on $(0, \frac{1}{2}]$ is a continuous definable choice function with $\Gamma(h) \subseteq S'_D$ and since $f < h < g$ on $(0, \frac{1}{2}]$, then we set $h(0) = a$ to obtain a continuous map on $[0, \frac{1}{2}]$. We deal with the interval $(\frac{1}{2}, 1]$ similarly to obtain the required definable choice function. $\square$

Proposition 4.5.2 (Green and Red Boxes Theorem 1)**.** *Let $\mathcal{L}$ be a language extending fields and let M be an o-minimal $\mathcal{L}$-structure with decidable theory. Then there is an effective procedure such that for any $\mathcal{L}$-formula ϕ defining S a definably connected subset of M^2, with $X = p_1^2(S)$, and for any $\mathcal{L}$-formulas $\psi_1, \dots, \psi_n$ defining $T_1, \dots, T_n \subseteq M^2$ definable subsets of S, the procedure determines whether there is a continuous definable function $f : X \longrightarrow M$ such that $\Gamma(f) \subseteq S$ and such that for each i, there exist $u_i \in M$ with $\big\langle u_i, f(u_i)\big\rangle \in T_i$.*

Proof. If we consider S as a green box and $T_1, \dots, T_n$ as red boxes, then the theorem says that we can decide whether there is a continuous definable choice function that remains within the green box and hits every red box. Let $S, T_1, \dots, T_n$ be given. Let $\mathcal{C}$ be a stratified cell decomposition of S, and let $\mathcal{D} = p_1^2(\mathcal{C})$ have uniform connected components of S; by inspection of the proofs of their existence, there is an effective procedure to find such a decomposition. If we have a definable choice function, then it must be contained in some $(m, \mathcal{C}, S)$-pathway. We therefore check each pathway in turn and see whether it has a continuous definable choice function. There is clearly an effective procedure for obtaining formulas defining each pathway. We let S' be such a pathway. We make some assumptions about S' before proceeding with the proof.

- We assume that $p_1^2(S') = X$
- By Lemma 4.4.1, we may assume that $S' = \mathrm{NBD}^*_{S'}(M^2)$
- For each i, $T_i \cap S' \neq \emptyset$; in fact, for convenience, we assume that $T_i \subseteq S'$

If any of these conditions fail then there is clearly no continuous choice function. Furthermore, we must check that we do not have situations where $p_1^2(T_i) = \{u\} = p_1^2(T_j)$ with $T_i \cap T_j$, i.e. there must be distinct u_i for each T_i in M such that $u_i \in p_1^2(T_i \cap S')$, with suitable combinatorial exceptions where $T_i \cap T_j \neq \emptyset$ etc.

Suppose wlog that for some $n' \leq n$, $T_1, \dots, T_{n'}$ are precisely the red boxes whose projections are finite sets, i.e. their projections contain no open intervals. We clearly require $\langle u_1, a_1\rangle, \dots, \langle u_K, a_K\rangle \in S'$ such that for each $i \leq n'$, $T_i \cap \bigcup \langle u_k, a_k\rangle \neq \emptyset$, note we may have $K < n'$ if we have some $\langle u_k, a_k\rangle \in T_i \cap T_j$. The condition is easily checked by looking at all possible combinations of inclusions-exclusions of red boxes on each fibre S'_{u_k}. We claim that this condition is also sufficient to ensure that there is a continuous definable choice function.

Suppose that the condition is fulfilled. We want to pick out a point in M^2 for each remaining red box, so we assume that $T_i \cap S'_{u_k} = \emptyset$ for each $i > n'$ and each k. We have $T_{n'+1}, \dots, T_n$ each of whose projections $p_1^2(T_i)$ contains an open interval. Let $U_1, \dots, U_L$ be all the connected components of each possible combination of projections of $T_{n'+1}, \dots, T_n$, i.e. the connected components of $p_1^2(T_i) \cap p_1^2(T_j)$, $p_1^2(T_i) \cap p_1^2(T_j) \cap p_1^2(T_k)$ etc. We order the U_l such that $l < l' \iff U_l < U_{l'}$. Starting with $T_{n'+1}$ let l be the first index such that $U_l \subseteq p_1^2(T_{n'+1})$. Let $u_{n'+1} = e(U_l)$, and let $a_{n'+1} = e(S'_{u_{n'+1}})$, we then set $U'_l = U_l \cap (-\infty, u_i)$. We then repeat the process for $T_{n'+2}$ and so on. In this way we pick out $\langle u_{n'+1}, a_{n'+1} \rangle, \dots, \langle u_n, a_n \rangle$ such that $\langle u_i, a_i \rangle \in T_i$ and the u_i are distinct.

Let $\{d_1\}, \dots, \{d_m\}$ be the singleton elements of $\mathcal{D}$ that are not equal to some u_i and which are in X. Define $\langle u_{n+1}, a_{n+1} \rangle, \dots, \langle u_{n+m}, a_{n+m} \rangle$ by $\langle u_{n+i}, a_{n+i} \rangle = \left\langle d_i, e(S'_{d_i}) \right\rangle$. Now reorder the u_i such that $i < j \implies u_i < u_j$. So each interval (u_i, u_{i+1}) is contained in some $D \subseteq \mathcal{D}$. Let $\mathcal{D}'$ be refinement of $\mathcal{D}$ such that each $(u_i, u_{i+1}) \in \mathcal{D}'$. By the assumption that $S' = \mathrm{NBD}^*_{S'}(M^2)$, we have that each $\langle u_i, a_i \rangle$ is in the closure of the cell to the right and to the left of it. We may therefore apply Proposition 4.5.1 to obtain a function $h : X \longrightarrow M$ such that $\Gamma(h) \subseteq S'$, and such that for each T_j there is some u_i such that $h(u_i) \in T_j$. $\square$

Conclusion

We have seen in Chapter 2 a complete classification of rings between B_1 and D_1 together with their Zariski spectra. Additionally we have separated isomorphism classes of finite extensions of B_1 by elements of D_1. The most obvious area of future research here is to classify all the rings between B_n and D_n. We conjecture that for $n > 1$, and for any $f \in D_n$, that $C_n[f] \cong C(\overline{\Gamma(f)})$. This would prove that any finite extension of C_n by an element of D_n is real closed. However, we cannot obtain a nice decomposition into characteristic functions: let $f : \mathcal{R}^2 \longrightarrow \mathcal{R}$ be defined by:

$$f(x,y) = \begin{cases} 0 & \text{if } x^2 + y^2 \geq 1 \\ x & \text{if } x^2 + y^2 < 1 \end{cases}$$

Then $\Gamma(f)$ is connected since $f(0,1) = f(0,-1) = 0$. There is clearly no characteristic function such that $C_1[\chi] \cong C(\overline{\Gamma(f)})$. The best approach appears to be to classify the behaviour of f by looking at germs of paths at each point, i.e. for each $\overline{a}$, we look at $\lim t \nearrow 1 f(\gamma(t))$, where $\gamma : (0,1) \longrightarrow \mathcal{R}^n$, for each path γ, where $\lim_{t \nearrow} \gamma(t) = \overline{a}$.

In Chapter 3, we show the problems over coheirs for sheaf constructions. We have modified the Feferman-Vaught construction used to prove the decidability of D_n by extending the language of rings to include unary predicates V and $\mathfrak{m}$ which are meant to be interpreted as the valuation ring of K which is the convex hull of $\mathcal{R}$, and the unique maximal ideal of that valuation ring. The $\mathcal{L}_{V,\mathfrak{m}}$-theory of real closed valued fields is decidable and positively model complete. The interpretation of V in $D_n^K\upharpoonright_{\mathcal{R}}$ is the convex hull of the constant functions $\mathcal{R}^n \longrightarrow V$, and is denoted W, similarly for $\mathfrak{m}$, the interpretation of which is denoted $\mathfrak{n}$. Then $W/\mathfrak{n} = bsa(\mathcal{R})$. The sheaf construction over the Zariski spectrum then gives a sheaf of $\mathcal{L}_{V,\mathfrak{m}}$-structures $A_\mathfrak{p}$, each of which is a field, and such that the interpretation of V in $A_\mathfrak{p}$ is the convex hull of $\mathcal{R}$. However, the interpretation of $\mathfrak{m}$ is not the maximal ideal of the valuation ring V. This means that the normal method of the Feferman-Vaught technique does not work here. The reason behind this is due to a problem with the coheirs of types of $\mathcal{R}$ in K. We give a characterization of the Zariski spectrum of $D_n^K\upharpoonright_{\mathcal{R}}$ and show how the problem arises due to the coheirs.

Chapter 4 — In the paper [GMP04], it is claimed that the free vector lattice (free Riesz space) on two generators is decidable. This vector lattice is isomorphic, by [BM02], to the vector lattice of semilinear functions $f : [0,1] \longrightarrow [0,1]$ such that $f(0) = f(1)$. Based on this approach, we began modifying the result to prove that the vector lattice of $\mathcal{L}_R(\mathbb{R})$-definable functions is decidable. However, we discovered a flaw in the proof, that we give in the thesis. This means that the decidability result cannot be reached by this method. The basic approach was to find an effective method to determine whether there was a continuous definable choice function from a 1-dimensional definable set X to a definable set $S \subseteq X \times M$. However, we would also require the ability to guarantee that our choice function h takes its values in certain subsets $T_1, \ldots, T_k$. This would allow us to remove existential quantifiers in the decidability proof. However, we do prove several results about definable choice functions. We hope that this will help lead to the construction of an effective procedure which decides whether there is a continuous definable choice function $X \longrightarrow S$, where $X \subseteq M^m$ is of arbitrary dimension and $S \subseteq X \times M^n$.

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